Verjetnostne metode v računalništvu: kolokvij

April 26, 2018

You have 120 minutes. Every solution must be justified. Good luck!

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Seat (2.02)

Name and surname

Student ID

Problem 1 (25 points)

Let T be a tree with root r and each vertex has 5 children, except the leaves at depth $2d$, i.e. there are $n = 5^{2d}$ leaves, $d \in \mathbb{N}$. Each leaf is assigned a value 0 or 1. Inductively then each vertex obtains a value 0 or 1 based on the following rule:

- If the distance from the vertex to the root is even: it gets its value by the majority rule of the values of his children.
- If the distance is odd: it gets value 1 iff exactly 2 or 3 children have value 1.

Construct a randomized sub-linear algorithm for which the expected value of the leaves that it needs to check is in $O(n^c)$ for some $c < 1$ as small as you can.

Problem 2 (25 points)

Consider the MAX-E3SAT problem in which you are given k clauses $C_1,\ldots,C_k$ and m Boolean variables $x_1,\ldots,x_m$, such that each C_i is of the form $y_1^i \vee y_2^i \vee y_3^i$, where each y_i^j is either a variable x_l or its negation $\bar{x_l}$ for some $1 \leq l \leq m$. Each clause has *exactly* three distinct literals. The task is to find an assignment of 0 or 1 to the variables so that the number of clauses with value 1 is as large as possible. Find a randomized algorithm that returns an assignment of the variables such that $E(X) \geq \frac{7k}{8}$, where X is the number of clauses having value 1, in polynomial time.

Problem 3 (25 points)

Let G be a random graph on vertices $\{1,\dots,n\}$, meaning that for all distinct $i,j \in \{1,\dots,n\}$ we include the edge ij in G with the same fixed probability p for all the edges. The choices for all edges are mutually independent, but the parameter p is always the same.

a) For distinct vertices i,j find the expected number of 2-paths (paths of length 2) in G with endpoints in i and j.

b) Let $p = 10^{-2}$. Show that for $n > 100$ large enough the graph G is 100-connected with high probability. Find n as small as possible for which the probability is at least $1 - 10^{-3}$. (A graph is 100-connected if the removal of any 99 vertices leaves always a connected graph.) You can give n as a solution of some equation, but argue that the solution exists.

Problem 4 (25 points)

Consider a random walk on a complete graph K_n as a random process.

a) For vertices $i, j \in V(K_n)$ determine the expected hitting time $h_{i,j}$.

b) Assume that n is even. Let Y be the time until the random walk covers half of the vertices of K_n. Find $E(Y)$. Using Chernoff bound, show that the probability $\Pr\left[Y > 2(E(Y))\right]$ is very small for n sufficiently large.

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Verjetnostne metode v računalništvu: izpit

June 15, 2018

You have 180 minutes. Every solution must be justified. Good luck!

Seat (2.02)

Name and surname

Student ID

Problem 1 (25 points)

Let G be a $n \times n$ cylindrical grid graph, i.e. vertices of G are $\{0, 1, \ldots n-1\}^2$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if:

- $x_1 = y_1$ and $(|x_2 - y_2| = 1$ or $\{x_2, y_2\} = \{0, n-1\})$, or
- $x_2 = y_2$ and $|x_1 - y_1| = 1$.

We consider a random walk on G starting from vertex $(0, 0)$.

a) Show that the expected number of steps before the random walk hits vertex of $(n-1, n-1)$ is in $O(n^3)$.

b) Show that the expected number of steps before the random walk hits the first vertex of the form $(n-1, x_2)$ is in $O(n^2)$.

Problem 2 (25 points)

Let $A \in \mathbb{Z}_2^{m\times n}$ be a $m \times n$ matrix, $n > m$, with entries in $\mathbb{Z}_2$ and rank m (i.e. full rank). Consider the following simple algorithm for sampling random elements of the kernel of A (possibly with repeats), where $k > 0$. Recall that the kernel (slo. jedro) of A is $\{x \in \mathbb{Z}_2^n \mid Ax = 0\}$.

```
Sample(A):
s = []
for i = 0...k do
    chose x1 uniformly at random from Z_2^n
    chose x2 uniformly at random from Z_2^n
    if Ax1 = 0 then
        add x1 to s
    end
    else
        if Ax2 = 0 then
            add x2 to s
        end
        else
            if Ax1 = Ax2 then
                add x1 - x2 to s
            end
        end
    end
end
return s
```

When solving an item, you can assume that previous items are solved, even if you did not solve them.

a) Argue that choosing $x \in \mathbb{Z}_2^n$ uniformly at random and calculating $y = Ax$ gives y uniformly at random.

b) What is $E(X)$, where X is the expected number of returned elements (also counting doubled)?

c) Show that for $k = m2^m$, there is a high probability (exponential in m) that the number of returned elements is greater than m.

Problem 3

In this exercise we consider hash functions hashing an n-element set S to n values. We call a family H of hash functions k-universal if for any $x_1,\dots,x_k$ from S and for a hash function $h \in H$ chosen uniformly at random it holds $\Pr\big(h(x_1) = \dots = h(x_k)\big) \le 1/n^{k-1}$.

Let H be a 2-universal family of hash functions, hashing an n-element set S to n values.

(a) Show that with probability at most 1/2 the number of collisions of a randomly chosen hash function is more than n. Hint: Markov inequality.

(b) Show that with probability at most 1/2 there exist $x \in S$ such that more than $\sqrt{2n}$ other elements are hashed in the same value as x (for a randomly chosen h).

(c) Assume now that H is k-universal. Find a bound $C(n,k)$ as small as possible such that with probability at most 1/2 there exist $x \in S$ such that more than $C(n,k)$ other elements are hashed in the same value as x (for a randomly chosen h).

Problem 4

Let E be a collection of n regions in the plane, where each region is an ellipse of the form $c_i(x-a_i)^2+d_i(y-b_i)^2 \leq 1$, for known parameters $a_i, b_i, c_i, d_i > 0$. Give an algorithm that finds a point in the intersection of all the regions in E with maximum y coordinate in expected linear time. Thus, the input to the algorithm is the sequence of parameters $((a_i, b_i, c_i, d_i))_{i=1,\ldots n}$. You can assume that you can calculate the intersection of two ellipses in constant time, and that no three ellipses intersect in a common point.

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Verjetnostne metode v računalništvu: izpit

July 3, 2018

You have 180 minutes. Every solution must be justified. Good luck!

Seat (2.02)

Name and surname

Student ID

Problem 1 (25 points)

Suppose that we have n jobs to be completed by m processors, say m divides n. Each job on a processor is finished in 1 step with probability 1/2 and in 9 steps also with probability 1/2. The duration of each job is independent of the duration of the other. We run on each processor n/m jobs, all processors working in parallel.

a) Show that the expected finishing time of all the jobs is not always $5n/m$. It is enough to prove this for some chosen n, m.

b) Let $n = m^2$. Show that with high probability the finishing time is less than $6n/m$, for a big m.

Problem 2 (25 points)

Let G be a graph on $2n$ vertices obtained in the following way. Start with a complete graph K_n on n vertices and connect to each vertex a pendant edge. Consider a random walk on G starting from an arbitrary vertex.

a) Show that the expected covering time of the random walk is in $\Theta(n^3)$.

b) Show that the expected covering time of all the vertices in the complete subgraph is in $O(n\log(n))$.

Problem 3 (25 points)

Let G_1, G_2 be graphs both on vertices $\{0, 1, \ldots, n-1\}$ given by their sets of edges. Additionally assume that both of them have edges colored in such a way that each color appears at most 4 times. We will say that G_1 and G_2 are equivalent if the underlying graphs are the same (really the same, not just isomorphic), and we can get the coloring of G_1 from coloring of G_2 by permuting the colors.

a) Construct polynomials $p(G_i)$ such that $p(G_1) = p(G_2)$ if and only if the graphs are equivalent.

b) Write an algorithm of type Monte Carlo, that answers if the graphs are equivalent in linear time in the number of edges of both graphs.

Problem 4 (25 points)

Let S be a set of halfspaces in the plane given by equations of the form $a_i y + b_i x + c_i \geq 0$ for $a_i, b_i, c_i \in \mathbb{Z}$.

a) Give an algorithm that in expected linear time (in the number of halfspaces) answers if there exists a vertical segment of length 1, that lies in all of the halfspaces.

b) Give an algorithm that in expected linear time (in the number of halfspaces) answers if there exists a unit axis-parallel square, that lies in all of the halfspaces.

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Verjetnostne metode v računalništvu: izpit

August 28, 2018

You have 180 minutes. Every solution must be justified. Good luck!

Seat (2.02)

Name and surname

Student ID

Problem 1 (25 points)

Let $G = K_{n_1,n_2}$ be a complete bipartite graph with bipartition A, B, such that $|A| = n_1, |B| = n_2$.

a) Let $n_2 = 1$ and let n_1 be arbitrary. Consider a random walk on G starting from a vertex in A. Compute the exact expected covering time of such a walk on G.

b) Let both n_1 and n_2 be arbitrary. Show that the covering time of G of a random walk starting in an arbitrary vertex is in $\Theta(n \log n)$, where $n = \max(n_1, n_2)$.

Problem 2 (25 points)

Assume that Alice has n coins in her bag, each coin has a value of 1 or 5 cents. She wants to find a way to say:

- "True", if she has more than $4n$ cents in the bag.
- "False", if she has less than $3n$ cents in the bag.
- Both "True"or "False" are a good answer, if she has between $3n$ and $4n$ cents.

Derive a sublinear (in n) Monte Carlo algorithm with bounded error that solves the above question. Assume that you can uniformly at random pick coins from the bag.

Problem 3 (25 points)

We model a queue in a store with the following discrete process. Assume that after each minute with probability p a customer in front of the line leaves the queue if such customer is in the line, and with probability q a customer arrives at the end of the line.

a) Let $p = 1/3$ and $q = 1/2$. Assume that at time 0 the queue is empty. What is the expected time until there are 3 people in the queue for the first time?

b) Let $p \geq q > 0$ and assume that there are n customers in the queue at time 0. Additionally assume that if there are $2n$ customers in the line, then no new customer can arrive at the end of the line. Show that the line will be empty for the first time in expected time in $O(n^2)$ (with some constant depending on p, q.

c) Let $p = 1/4$ and $q = 3/4$ and at time 0 the queue is empty. Show that with high probability at time $2n$ there are at least n customers in the line. Hint: calculate the probability that the queue extends at each step and use the Chernoff bound.

Problem 4 (25 points)

Assume we have a set of n bins, each containing two notes, on each note there are written 2 numbers. The bins are indistinguishable, and two notes are distinguishable only if they have different numbers written on them. Assume Alice changed the content of some bins. Use polynomials to derive a liner time algorithm (in n) of type Monte Carlo that checks the bins before and after to see if Alice altered the setting, i.e. if one can find a difference between the setting before and after. Assume that each arithmetic operation and comparison between two numbers takes constant time. Prove the correctness of the algorithm.