Άσκηση 1

$$P(\vec{X}|w_1) = N(\mu_1, \Sigma_1) = f_{X_1, X_2 | w = w_1}(x_1, x_2)$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad f_{X_1,X_2|w=w_1}(x_1,x_2) = f_{\vec{X}|w=w_1}(\vec{y})$$

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f_{\vec{x}|w=w_1}(\vec{y}) = \frac{1}{\sqrt{(2\pi)^2|\Sigma_1|}} e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_1)^T \Sigma_1^{-1}(\vec{y}-\vec{\mu}_1)}$$

$$\vec{\mu}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \Sigma_1 = \begin{pmatrix} 1.2 & -0.4 \\ -0.4 & 1.2 \end{pmatrix} \qquad (1)$$

ομοίως, $$f_{\vec{x}|w=w_2}(\vec{y}) = \frac{1}{\sqrt{(2\pi)^2|\Sigma_1|}} e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1}(\vec{y}-\vec{\mu}_2)}$$

Πιθανότητα Σφάλματος:

$$P_e = P(\hat{w} \neq w) = P(w=w_1)\,p(\hat{w}\neq w | w=w_1) + P(w=w_2)\,p(\hat{w}\neq w|w=w_2)$$

Δηλαδή, η πιθανότητα να εκτιμήσω λάθος είναι:

Πιθανότητα $w=w_1$ × Πιθανότητα λάθους ($\hat{w}=w_2$), δεδομένου ότι $w=w_1$ ⊕ Πιθανότητα $w=w_2$ × Πιθανότητα $\hat{w}=w_1 | w=w_2$

Πιθανότητα λάθους δεδομένου $w=w_1$: $p(\hat{w}=w_2 | w=w_1)$

— || — $w=w_2$: $p(\hat{w}=w_1 | w=w_2)$

$$P(\hat{w} \neq w_{1} | w=w_1) = 1 - P(\hat{w}=w_{1} | w=w_1)$$

Άρα: $P_e = P(\omega=\omega_1)\left[1 - P(\hat{\omega}=\omega \mid \omega=\omega_1)\right] + P(\omega=\omega_2)\, P(\hat{\omega}=\omega_1 \mid \omega=\omega_2)$

$= P(\omega=\omega_1) - P(\omega=\omega_1) P(\hat{\omega}=\omega \mid \omega=\omega_1) + P(\omega=\omega_2) P(\hat{\omega}=\omega_1 \mid \omega=\omega_2)$

$= P(\omega=\omega_1) - P(\omega=\omega_1)\int_{R_1} f_{\vec{Z}\mid\omega_1}(\vec{y})\, d\vec{y} + P(\omega=\omega_2)\int_{R_1} f_{\vec{Z}\mid\omega_2}(\vec{y})\, d\vec{y} =$

$P(\omega=\omega_1) + \int_{R_1}\left[f_{\vec{Z}\mid\omega_2}(\vec{y})\, P(\omega=\omega_2) - f_{\vec{Z}\mid\omega_1}(\vec{y})\, P(\omega=\omega_1)\right] d\vec{y} =$

$P(\omega=\omega_1) + \int_{R_1}\left[P(\omega=\omega_2 \mid \vec{Z}=\vec{y}) - P(\omega=\omega_1 \mid \vec{Z}=\vec{y})\right] f_{\vec{Z}}(\vec{y})\, d\vec{y}$

$R_1 = \{(x_1, x_2) \in \mathbb{R}^2 : P(\omega=\omega_2 \mid \vec{Z}=\vec{y}) < P(\omega=\omega_1 \mid \vec{Z}=\vec{y})\}$

$R_2 = \{(x_1, x_2) \in \mathbb{R}^2 : P(\omega=\omega_2 \mid \vec{Z}=\vec{y}) > P(\omega=\omega_2 \mid \vec{Z}=\vec{y})\}$

$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

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Κανόνας Απόφασης:

$$P\left(\omega=\omega_2 \mid \vec{Z}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \underset{\omega_1}{\overset{\omega_2}{\gtrless}} P\left(\omega=\omega_1 \mid \vec{Z}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$P\left(\omega=\omega_2 \mid \vec{Z}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) f_{\vec{Z}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = P(\omega=\omega_2)\, f_{\vec{Z}\mid\omega_2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \Rightarrow$$

$$P\left(\omega=\omega_2 \mid \vec{Z}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{P(\omega=\omega_2)\, f_{\vec{Z}\mid\omega_2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{f_{\vec{Z}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}$$

ομοίως $$P\left(\omega=\omega_1 \mid \vec{Z}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{P(\omega=\omega_1)\, f_{\vec{Z}\mid\omega_2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{f_{\vec{Z}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}$$

$P(\omega=\omega_1) = P(\omega_1)$

$P(\omega=\omega_2) = P(\omega_2)$

άρα $P(\omega_2)\, f_{\vec{Z}|\omega_2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \underset{\omega_1}{\overset{\omega_2}{\gtrless}} P(\omega_1)\, f_{\vec{Z}|\omega_1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \Rightarrow$

$$P(\omega_2)\frac{1}{\sqrt{(2\pi)^2|\Sigma_2|}} e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1}(\vec{y}-\vec{\mu}_2)} \underset{\omega_1}{\overset{\omega_2}{\gtrless}} P(\omega_1)\frac{1}{\sqrt{(2\pi)^2|\Sigma_1|}} e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_1)^T \Sigma_1^{-1}(\vec{y}-\vec{\mu}_1)}$$

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$$\frac{P(\omega_2)}{P(\omega_1)}\sqrt{\frac{|\Sigma_1|}{|\Sigma_2|}} = k \in \mathbb{R}$$

$$\vec{y}-\vec{\mu}_2 = \begin{bmatrix} x_1-6 \\ x_2-6 \end{bmatrix}, \quad (\vec{y}-\vec{\mu}_2)^T = \begin{bmatrix} x_1-6 & x_2-6 \end{bmatrix}$$

$$\Sigma_2^{-1} = \frac{1}{(1.2)^2-(0.4)^2}\begin{bmatrix} 1.2 & -0.4 \\ -0.4 & 1.2 \end{bmatrix} = \begin{bmatrix} 0.9375 & -0.3125 \\ -0.3125 & 0.9375 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{15}{16} & -5/16 \\ -5/16 & 15/16 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 15 & -5 \\ -5 & 15 \end{bmatrix}$$

$$(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1} = \underset{1\times 2}{\begin{bmatrix} x_1-6 & x_2-6 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} \frac{15}{16} & -\frac{5}{16} \\ -\frac{5}{16} & \frac{15}{16} \end{bmatrix}} =$$

$$\left[(x_1-6)\frac{15}{16} - \frac{5}{16}(x_2-6),\ (x_2-6)\frac{15}{16} - \frac{5}{16}(x_1-6) \right]$$

$$(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1}(\vec{y}-\vec{\mu}_2) = \left[(x_1-6)\frac{15}{16} - \frac{5}{16}(x_2-6),\ (x_2-6)\frac{15}{16} - \frac{5}{16}(x_1-6)\right]\begin{bmatrix} x_1-6 \\ x_2-6 \end{bmatrix} =$$

$$\frac{15}{16}(x_1-6)^2 - \frac{5}{16}(x_1-6)(x_2-6) - \frac{5}{16}(x_1-6)(x_2-6) + \frac{15}{16}(x_2-6)^2 =$$

$$\frac{15}{16}\left[(x_1-6)^2 + (x_2-6)^2\right] - \frac{10}{16}(x_1-6)(x_2-6)$$

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ομοίως : $\Sigma_1^{-1} = \frac{1}{16}\begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$

και $(\vec{y}-\vec{\mu}_1)^T \Sigma_1^{-1} (\vec{y}-\vec{\mu}_1) = \frac{15}{16}\left[(x_1-3)^2+(x_2-3)^2\right] + \frac{10}{16}(x_1-3)(x_2-3)$

Άρα :

$$k\, e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1}(\vec{y}-\vec{\mu}_2)} \underset{\omega_1}{\overset{\omega_2}{\gtrless}} e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_1)^T \Sigma_1^{-1}(\vec{y}-\vec{\mu}_1)} \Rightarrow$$

$$\ln\left(\left[k\, e^{-\frac{1}{2}(\vec{y}-\vec{\mu}_2)^T \Sigma_2^{-1}(\vec{y}-\vec{\mu}_2)}\right)\right. \underset{\omega_1}{\overset{\omega_2}{\gtrless}} \ln\left(e^{-1/2(\vec{y}-\vec{\mu}_1)^T \Sigma_1^{-1}(\vec{y}-\vec{\mu}_1)}\right) \Rightarrow$$

$$\ln(k) - \frac{1}{2}\left[\frac{15}{16}\left[(x_1-6)^2+(x_2-6)^2\right] - \frac{10}{16}(x_1-6)(x_2-6)\right] \underset{\omega_1}{\overset{\omega_2}{\gtrless}} -\frac{1}{2}\left[\frac{15}{16}\left[(x_1-3)^2+(x_2-3)^2\right] + \frac{(x_1-3)(x_2-3)}{16}\right.$$

$$\ln(k) + \left[-\frac{15}{32}\left[(x_1-6)^2+(x_2-6)^2\right] + \frac{10}{32}(x_1-6)(x_2-6)\right] \underset{\omega_1}{\overset{\omega_2}{\gtrless}} -\frac{15}{32}\left[(x_1-3)^2+(x_2-3)^2\right] - \frac{10}{32}(x_1-3)(x_2-3) \Rightarrow$$

$$\ln(k) - \frac{15}{32}(x_1^2-12x_1+36) - \frac{15}{32}(x_2^2-12x_2+36) + \frac{10}{32}(x_1x_2-6x_1-6x_2+36) \underset{\omega_1}{\overset{\omega_2}{\gtrless}} -\frac{15}{32}(x_1^2-6x_1+9) - \frac{15}{32}(x_2^2-6x_2+9) - \frac{10}{32}(x_1x_2-3x_1-3x_2+9) \Rightarrow$$

$$\ln(k) + \frac{15\cdot12}{32}x_1 + \frac{15\cdot12}{32}x_2 - \frac{15\cdot36}{16} + \frac{10x_1x_2}{32} - \frac{6\cdot10}{32}x_1 - \frac{6\cdot10}{32}x_2 + \frac{10\cdot36}{32} \underset{\omega_1}{\overset{\omega_2}{\gtrless}} +\frac{6\cdot15x_1}{32} + \frac{6\cdot15x_2}{32} - \frac{9\cdot15}{16} - \frac{10}{32}x_1x_2 + \frac{3\cdot10}{32}x_1 + \frac{3\cdot10x_2}{32} - \frac{9\cdot10}{32} \Rightarrow$$

$$\ln(k) + \cancel{\frac{120}{32}x_1} + \cancel{\frac{120}{32}x_2} + \frac{5\cdot36}{16} - \frac{15\cdot36}{16} + \frac{10}{32}x_1x_2 \underset{\omega_1}{\overset{\omega_2}{\gtrless}} +\cancel{\frac{120x_1}{32}} + \cancel{\frac{120}{32}x_2} - \frac{9\cdot5}{16} - \frac{9\cdot15}{16} - \frac{10}{32}x_1x_2 \Rightarrow$$

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$$\ln(k) - \frac{10\cdot36}{16} + \frac{10}{32}x_1x_2 \underset{\omega_1}{\overset{\omega_2}{\gtrless}} -\frac{20\cdot9}{16} - \frac{10}{32}x_1x_2 \Rightarrow$$

$$\frac{20}{32}x_1x_2 \underset{\omega_1}{\overset{\omega_2}{\gtrless}} \frac{10\cdot36-20\cdot9}{16} - \ln(k) \Rightarrow \frac{5}{8}x_1x_2 \underset{\omega_1}{\overset{\omega_2}{\gtrless}} 11.25 - \ln(k) \Rightarrow$$

$$x_1x_2 \underset{\omega_1}{\overset{\omega_2}{\gtrless}} 18 - \frac{8}{5}\ln(k)$$

για $\Sigma_1 = \Sigma_2 = \begin{pmatrix} 1.2 & 0.4 \\ 0.4 & 1.2 \end{pmatrix} = \Sigma$

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$$\Sigma^{-1} = \frac{1}{16}\begin{bmatrix} 15 & -5 \\ -5 & 15 \end{bmatrix}$$

$$(\vec{y}-\vec{\mu}_1)^T \Sigma^{-1} (\vec{y}-\vec{\mu}_2) = \frac{15}{16}\left[(x_1-3)^2+(x_2-3)^2\right] - \frac{10}{16}(x_1-3)(x_2-3)$$

έχουμε:

$$\ln(k) - \frac{15}{32}\left[(x_1-6)^2+(x_2-6)^2\right] + \frac{10}{32}(x_1-6)(x_2-6) \underset{\omega_1}{\overset{\omega_2}{\gtrless}} -\frac{15}{32}\left[(x_1-3)^2+(x_2-3)^2\right] + \frac{10}{32}(x_1-3)(x_2-3) \Rightarrow$$

$$32\ln(k) - 15(x_1^2 - 12x_1 + x_2^2 - 12x_2 + 72) + 10(x_1x_2 - 6x_1 - 6x_2 + 36) \underset{w_1}{\overset{w_2}{\gtrless}} -15(x_1^2 - 6x_1 - 6x_2 + x_2^2 + 18) + 10(x_1x_2 - 3x_1 - 3x_2 \ldots$$

$$\Rightarrow 32\ln(k) + 15\cdot 12x_1 + 15\cdot 12x_2 - 15\cdot 72 + 10x_1x_2 - 60x_1 - 60x_2 + 360 \underset{w_1}{\overset{w_2}{\gtrless}} 15\cdot 6x_1 + 15\cdot 6x_2 - 18\cdot 15 + 10x_1x_2 - 30x_1 - 30x_2 + \ldots$$

$$\Rightarrow 32\ln(k) + 120x_1 + 120x_2 - 720 \underset{w_1}{\overset{w_2}{\gtrless}} 60x_1 + 60x_2 - 180 \Rightarrow$$

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$$32\ln(k) + 60x_1 + 60x_2 \underset{w_1}{\overset{w_2}{\gtrless}} 540 \Rightarrow$$

$$60(x_1 + x_2) \underset{w_1}{\overset{w_2}{\gtrless}} 540 - 32\ln(k) \Rightarrow$$

$$x_1 + x_2 \underset{w_1}{\overset{w_2}{\gtrless}} 9 - \frac{32}{60}\ln(k) \Rightarrow$$

$$x_1 + x_2 \underset{w_1}{\overset{w_2}{\gtrless}} 9 - \frac{8}{15}\ln(k)$$

Άσκηση 3

$$P(\omega_1) = \pi = P(\omega_2)$$

$$P(x|\omega_i) = \begin{cases} \dfrac{x}{\sigma_i^2} e^{-\frac{x^2}{2\sigma_i^2}}, & x \geq 0 \\ 0, & \text{αλλιώς} \end{cases}$$

$$\sigma_1 = 1, \quad \sigma_2 = 2 \qquad L = \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} r_1 &= \lambda_{21} P(\omega_2|x) \\ r_2 &= \lambda_{12} P(\omega_1|x) \\ P(\omega_2|x) &= P(\omega_2) P(x|\omega_2) \\ P(\omega_1|x) &= P(\omega_1) P(x|\omega_1) \end{aligned}\right\} \Rightarrow \begin{aligned} r_1 &= \pi P(x|\omega_2) \\ r_2 &= \frac{\pi}{2} P(x|\omega_1) \end{aligned}$$

$$r_2 \underset{\omega_2}{\overset{\omega_1}{\gtrless}} r_1 \Rightarrow \frac{\cancel{\pi}}{2} P(x|\omega_1) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \cancel{\pi} P(x|\omega_2) \Rightarrow$$

$$\frac{1}{2} e^{-\frac{x^2}{2}} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{4} e^{-\frac{x^2}{8}} \Rightarrow e^{-\frac{x^2}{2}} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{2} e^{-\frac{x^2}{8}} \rightarrow$$

$$-\frac{x^2}{2} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \ln\left(\frac{1}{2} e^{-\frac{x^2}{8}}\right) \Rightarrow -\frac{x^2}{2} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \ln(1/2) + \ln\left(e^{-x^2/8}\right)$$

$$\Rightarrow -\frac{x^2}{2} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} -\frac{x^2}{8} - \ln(2) \Rightarrow \ln(2) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{x^2}{2} - \frac{x^2}{8} \Rightarrow$$

$$\ln(2) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} 3\frac{x^2}{8} \Rightarrow x \underset{\omega_1}{\overset{\omega_2}{\gtrless}} \sqrt{\frac{8\ln(2)}{3}} \quad (x<0 \text{ απορρίπτονται})$$

άρα το όριο είναι $x_0 = \sqrt{\dfrac{8\ln(2)}{3}}$

Exercise 4

(1)

$$X = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Finding the eigenvalues :

$$det(s\mathbf{1} - X^T X) = 0 \implies \begin{vmatrix} s-2 & -1 & 1 \\ -1 & s-1 & 0 \\ 1 & 0 & s-1 \end{vmatrix} = 0 \implies s^3 - 4s^2 + 3s = 0 \implies$$

$$s(s^2 - 4s + 3) = 0 \implies s = 0 \ \ or \ \ s = 1 \ \ or \ \ s = 3$$

Non-zero (and normalized) eigenvectors :

$$\text{for } s = 1 : V_1 = \begin{bmatrix} 0 \\ 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\text{for } s = 3 : V_2 = \begin{bmatrix} 0.8165 \\ -0.4082 \\ 0.4082 \end{bmatrix}$$

(2)

$$\sigma_1 = 1, \ \ \sigma_2 = \sqrt{3}$$

(3)

$$X X^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$det(s\mathbf{1} - X X^T) = 0 \implies \begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix} = 0 \implies s^2 - 4s + 3 = 0 \implies$$

$$s = 1 \ \ or \ \ s = 3$$

$$\text{for } s_1 = 1 : \ v_1 = \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix}$$

$$\text{for } s_2 = 3 : \ v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

(4)

$$\hat{X} = v_2 \sigma_2 V_2^T = \begin{bmatrix} 1 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \end{bmatrix}$$

Exercise 6

$$P(\overrightarrow{X}|\omega_i) = \frac{1}{\sqrt{(2\pi^2|\Sigma_i|)}} e^{-\frac{1}{2}(x-\mu_i)^T\Sigma_i^{-1}(x-\mu_i)}$$

$$m_1 = E\{\overrightarrow{y}\} = E\{\overrightarrow{w^T}\overrightarrow{X}\} = \overrightarrow{w^T}E\{\overrightarrow{X}\} = \overrightarrow{w^T}\overrightarrow{\mu_1}$$

$$m_2 = \overrightarrow{w^T}\overrightarrow{\mu_2}$$

$$S_1^2 = E\{(\overrightarrow{y} - \overrightarrow{m_1})(\overrightarrow{y} - \overrightarrow{m_1})^T\} = E\{\overrightarrow{w^T}(\overrightarrow{x} - \overrightarrow{\mu_1})(\overrightarrow{x} - \overrightarrow{\mu_1})\overrightarrow{w}\} = \overrightarrow{w^T}\Sigma_1\overrightarrow{w}$$

$$S_2^2 = \overrightarrow{w^T}\Sigma_2\overrightarrow{w}$$

$$J(w) = \frac{(m_1-m_2)^2}{S_1^2+S_2^2} = \frac{\overrightarrow{w^T}S_b\overrightarrow{w}}{\overrightarrow{w^T}S_w\overrightarrow{w}} \implies \overrightarrow{w} = \beta S_w^{-1}S_b$$

$$S_w = \Sigma_1 + \Sigma_2, \quad S_b = \overrightarrow{\mu_1} - \overrightarrow{\mu_2}$$

We want to minimize the function J(w) :

$\overrightarrow{w} = \beta(\Sigma_1 + \Sigma_2)^{-1}(\overrightarrow{\mu_1} - \overrightarrow{\mu_2})$, for $b > 0$

$$\overrightarrow{w} = \beta \begin{bmatrix} 13 & 9 \\ 9 & 13 \end{bmatrix}^{-1} \begin{bmatrix} -15 \\ -10 \end{bmatrix}$$

choose $\beta = ad - bc = 13 * 13 - 9 * 9 = 88$

$$\overrightarrow{w} = \begin{bmatrix} -114 \\ 18 \end{bmatrix}$$