ΠΟΛΥΤΕΧΝΕΙΟ ΚΡΗΤΗΣ
TECHNICAL UNIVERSITY OF CRETE

Ion Trap Quantum Computation

Quantum Technology

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Introduction

In principle, quantum computers hold clear promise in addressing problems not generally tractable with classical simulation techniques, and are capable of solving certain exponentially hard problems in polynomial time, taking advantage of quantum phenomena such as superposition and entanglement to achieve parallelism. In practice though, quantum computers still lack in several aspects when compared to their classical counterparts. In contrast to digital computers, where semiconductor technologies have replaced every pre-existing hardware, the case for quantum computers is hardly similar. There are many approaches to constructing a quantum computer. In addition to the numerous underlying technologies that govern their implementation and determine their function, a plethora of different computational architectures also exist.

In order for quantum computing to be viable, a number of obstacles need to be overcome. Decoherence and systematic errors when performing unitary transformations are the most important problems, since they are limiting the accuracy of the computations. In order to account for these problems, a substantial external active control system to achieve fault-tolerant computing is required.

In the current report, we will analyze one of the numerous state-of the-art technologies for implementing quantum computers, the Ion Trap Quantum Computer, first devised by Cirac and Zoller [1]. We will describe in detail how the ion trapping technology works, as well as discuss different methods of trapping, what types of particles are suitable for this model, as well as how these particles are cooled in order to become isolated from the environment. As an extent, the fundamental operations on ion-trapped qubits are going to be shown, as well as how they are translated to the quantum circuit model.

Chapter
TWO

Description of the physical system

In the current chapter, the basic techniques which make quantum computation using cold, confined ions feasible, are being analyzed. We are going to list the atoms that are appropriate for the implementation of the trapped-ion model, as well as the properties that allow us to create and measure efficiently two-level quantum systems. In addition, we discuss cooling and trapping techniques, which are critical for initialization and controlling of the atoms. In order to approach these topics in a methodical and precise manner, it is relevant to recall DiVincenzo's criteria[2] for a working quantum computer, which are listed in the following section.

2.1 DiVincenzo's five criteria for a working quantum computer

1. The system must have well-defined qubits, and be scalable

2. The system must have long coherence times, to permit the quantum mechanical phase of the system wavefunction to be preserved through the many steps of computation so that the error-rate must be sufficiently low.

3. Universal quantum gates must be possible to perform all elementary logical operations.

4. There must be an efficient means to measure the states of the qubits. At the end of a calculation, the state of each qubit state must be measured.,

5. The system must be initialized reliably. Initialization sets the quantum system to a reproducible reference state.

2.2 Preparation of the physical system

2.2.1 Choosing the atoms

According to the 1st Divincenzo's criterion, the system must have well-defined qubits, and also it must be scalable. In order for this statement to be satisfied, we have to find atoms or groups of atoms that are appropriate for the implementation of a system that has at least two possible states: a ground state ($|g\rangle \equiv |0\rangle$) and an excited state ($|e\rangle \equiv |1\rangle$).

In principle, any atom that has at least one electron can satisfy this condition. In practice though, since light-matter interaction is central to atomic physics systems, and interaction between light and the valence electron of the atom is of specific interest, it is important that we choose groups of atoms that have a single electron in the valence band, in order to create a quantum two-level system. Therefore, we are led to choose atoms from the group IA of the periodic table (i.e. alkali metals) or atoms that after ionization have one electron in the valence band. Those atoms belong in the groups IIA (alkaline earth metals), and IIIB (Transition metals).

It is significant to distinguish the advantages and liabilities of each group as per the different methods they require for control, cooling, the energy that is required for the excitation of the valence electron, and other potentially useful properties.

Alkali metals

In order to confine Alkali metals (neutral atoms) magnetic and optical forces need to be exerted upon them.[4] The typical depth of these traps is $\sim 1K$, meaning that in order to trap neutral atoms effectively, we need to create Bose-Einstein condensates of those atoms, in addition to generating strong magnetic fields. The cooling techniques used in neutral atoms are, Doppler cooling and evaporative cooling, the former of which we are going to describe in detail later.

Alkaline earth metals

Alkaline earth metals (charged ions) can be trapped by generating coulomb forces. The generated electric field (typically 100 V/mm) creates a trap with typical depth $\sim 10^4K$, meaning that charged ions can be trapped effectively at room temperatures.

Transition Metals, Ytterbium-171

The cooling and trapping methods for the case of transition metals are analogous to that of the alkaline earth metals. The major drawback of transition metals is that the energy required for the excitation of the valence electron is in the order of $\sim$ 10 eV, meaning that in order to produce the excitations a laser that produces photons of appropriate wavelength ($\sim$ 100 nm) coherently and with a low margin of error, is required. In order to create an efficient quantum computing device, it is relevant to minimize the required energy used for the valence electron excitations, and this property of transition metals is a limiting factor.

Next, we will be considering the case for Ytterbium ions (and specifically Ytterbium-171), since it has properties that are appealing for the ion trap configuration. A significant advantage Ytterbium atoms have over other transition metals is that the excitation energy of the valence electron is relatively lower, and laser light in the visible spectrum can be used for the excitations. This property allows for efficient initialization of all qubits by utilizing currently available technologies for light transmission (i.e. optical fibers). Another useful property is that it has a hyperfine structure, which is defined by small shifts in otherwise degenerate energy levels and the resulting splitting in those energy levels of ions, due to electromagnetic multipole interaction between the nucleus and electron clouds. These properties make Ytterbium 171 ions an excellent qubit candidate.[5, 6]

2.2.2 Cooling particles using Lasers

Cooling particles is essential to most physics applications. In this case, cooling is the enemy of decoherence. Also, it can lead to trapping, which will be expanded upon in later chapters. In this chapter, we will analyze some important theories and methods to cooling subatomic particles.

To understand how we can use lasers to cool particles, firstly we need to expand on Wave–particle duality, particle resonance frequency and the Doppler effect.

Wave–particle duality

Wave particle duality is a concept in Quantum mechanics stating that quantum entities exhibit both wave and particle properties. Important wave properties: Wavelength, Frequency. Important particle properties: mass, momentum, position.

De Broglie equation:

$$\lambda_D = \frac{\hbar}{mv} \tag{2.1}$$

where λ_D is the wave length of the particle, $\hbar$ is h-bar and m, v are the mass and velocity of the particle respectively.

From thermodynamics, we know that

$$v^2 \propto T \tag{2.2}$$

where T is the temperature in absolute scale.
Combining eqs. (2.1) and (2.2), we have:

$$\lambda_D \propto \frac{1}{\sqrt{T}} \tag{2.3}$$

In conclusion, the de Broglie wavelength of a particle is inversely proportional to its absolute Temperature. This is very important, as a lower temperature means a bigger wavelength, so the particle (molecule or ion) starts behaving as a wave even at relatively large scale (measurable in a Lab).

Particle resonance frequency

Particles exhibit wave like properties, i.e., frequency. Quantum particles absorb and emit quantized amounts of energy; to absorb an incoming photon, said photon must have a certain quantized frequency (energy). Absorbing the photon changes the energy of the particle.

The Doppler Effect

The Doppler Effect (or Doppler Shift) was proposed in 1842 by Christian Doppler and states the following:
There is an apparent change in the frequency of a wave in relation to a moving observer.

Why is this important?

Wave frequency is related to wave length, which-as shown above-is inversely proportional to the root of the Temperature. The Doppler effect is widely used in a plethora of temperature-related applications, ranging from astrophysics to quantum mechanics. We will expand upon the latter in the following sections.

2.2.3 Doppler Cooling

Doppler cooling can be achieved using 6 lasers carrying the same frequency, which is equal to a slightly lesser frequency than that of the resonance frequency of the particle system (Off resonant slower frequency). These are used on a gas which is pre-cooled using a different method (chirp , Zeeman e.t.c.)

We shall derive what happens on one of the three axes, as follows:

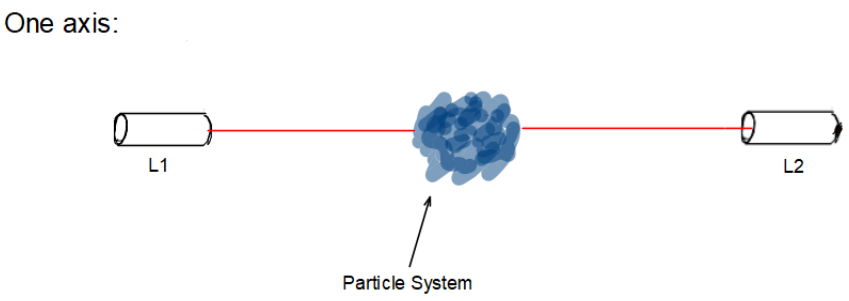


Figure 2.1

Suppose a particle in the system is stationary. The lasers have a non-resonant frequency, so the particle does not absorb the electromagnetic radiation of the lasers. Suppose now that the particle is moving toward laser 1 (L1 in the figure). According to the Doppler effect, the frequency of L2 is red shifted, moving its value further away from resonance; it is not absorbed. On the other hand, the frequency of L1 is blue-shifted, so it appears resonant. The particle absorbs the energy of the e.m. pulse, reverting it back to its original state/speed. A completely analogous thing happens for a particle moving toward L2.

A "kind-of" trapping: Optical molasses

Having 6 lasers -or 3 lasers and mirrors respectively- forces the system of particles to a steady state, each axis operates as shown above. The particles become trapped in a specific cloud-like space (forced to perform random walks). The particles are not forced together by an external force (like a magnetic field) ; their directional motion is being constantly controlled by the lasers. For more information, see polarization gradient cooling.

Doppler limit

The Doppler temperature limit refers to the minimum achievable temperature through Doppler cooling. When an atom absorbs a photon while moving in the opposite direction to the light source, its velocity decreases due to the conservation of momentum. Subsequently, when the excited atom spontaneously emits the absorbed photon, it receives a random momentum kick in a random direction. Since these spontaneous emissions occur in all directions equally, the average momentum kicks cancel out for the mean velocity. However, the mean squared velocity is non-zero, resulting in the supply of heat to the atom. At equilibrium, the rates of heating and cooling equalize, thereby creating a limit on the extent to which the atom can be cooled. Given that the transitions used in Doppler cooling have broad natural γ rad/sec, this lower limit defines the lowest bound:

$$T_{\text{Doppler}} = \frac{\hbar \cdot \gamma}{2 \cdot k_\beta} \tag{2.4}$$

where k_β is the Boltzmann constant and $\hbar$ is the reduced Planck constant, which is the lowest possible absolute temperature one can reach using Doppler cooling.

2.2.4 Sideband Cooling

Cooling beyond the Doppler limit

Usually, the system on which sideband cooling is performed is a two-level system including a sufficiently cooled atom, or in our use case an ion. Modeling the system as a harmonic oscillator interacting with a classical electromagnetic field result s-after applying the rotating wave approximation- in the following familiar Hamiltonian:

$$\hat{H} = \hat{H}_{H.O.} + \hat{H}_{al} \tag{2.5}$$

with:

$$\hat{H}_{H.O.} = \hbar v(n + \frac{1}{2}) \tag{2.6}$$

$$\hat{H}_{al} = -\hbar\Delta \left|e\right\rangle \left\langle e\right| + \hbar\frac{\Omega}{2}(\left|e\right\rangle \left\langle g\right| \exp(ik \cdot r) + \left|g\right\rangle \left\langle e\right| \exp(-ik \cdot r)) \tag{2.7}$$

where:
n is the number operator
v is the frequency spacing of the oscillator
Ω is the Rabi frequency due to the atom-light interaction
Δ is the laser detuning from ω_0

k is the laser wave vector

That is the Jaynes-Cummings Hamiltonian used to describe the phenomenon of an atom coupled to a cavity in quantum electrodynamics (skipping a lot of math for the sake of simplicity. For more, check Turchette et al.)

2.2.5 Trapping Ions

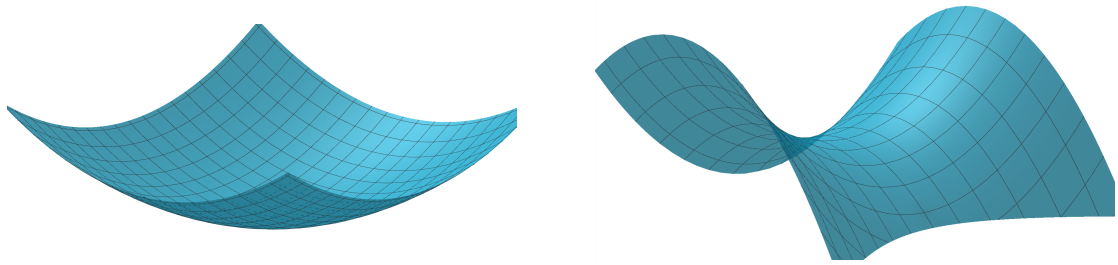
In the ion-trap model, independent manipulation of each individual qubit is accomplished by directing different laser beams to each of the ions. The coupling of the motion of the ions is provided by the Coulomb repulsion, which is much stronger than any other interaction for typical separations between the ions, and allows for creating traps that have a depth $\sim 10^4$ Kelvin. The charge has major differences when confining and manipulating particles. Comparing them, the force exerted on neutral ions are magnetic and optical forces, typically the traps that are created have a depth ~ 1 Kelvin.

Penning Ion Trap

The Penning ion trap uses a quadrupole potential in order to confine a charged particle axially, with a 3-dimensional minimum potential

$$\phi(x, y, z) = Ax^2 + By^2 + Cz^2 \tag{2.8}$$

where A, B, C are all positive. However, the Laplace equation $\nabla^2\phi = 0$ requires that $A + B + C = 0$, rendering a 3-dimensional trapping with a static field not possible.



(a) (b)

Figure 2.2: The confining potential well (a) and the potential well of a static electric field (b)

In order to solve this problem, a uniform magnetic field is introduced, the Lorentz forces of which could be used for confinement of the ion in the radial plane. The Lorentz forces transform the radial motion of the ion into a cyclotron motion. The motion of the charges in the joint potential is basically harmonic, having well-defined frequencies, satisfying the Cirac-Zoller (and DiVincenzo) conditions for a quantum computer. However, the uniform magnetic field for effective ion confinement, and must not charge in space-time over the region in space where the ions are confined to preserve coherence. The stability of a magnetic field, which affects the energies of the magnetic sublevels of an ion electronic state by the Zeeman effect, may compromise wavefunction coherence unless carefully controlled. These conditions can be difficult to meet.

Radio Frequency Trap

The radio frequency ion trap (or Paul trap) is chosen for quantum computation for several reasons. It provides a stable 3-dimensional minimum in space of the effective electric potential using only radio frequency and DC electric fields. A magnetic field may additionally be imposed, but is not required for confinement, and can be kept small. Since the electric fields vanish at the potential minimum, the effects on the energies of the ion electronic levels by the Stark effect[7] is very small, when the ions are cold. The motion of very cold ions in the trap is harmonic, i.e., the restoring force toward the equilibrium is linear in space.

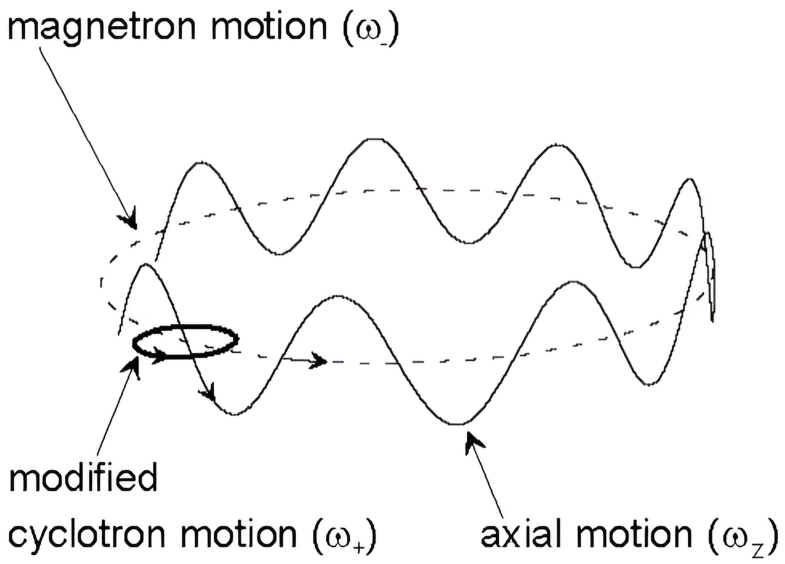


Figure 2.3: Characteristic ion motions in a Penning Trap. Source: LMU

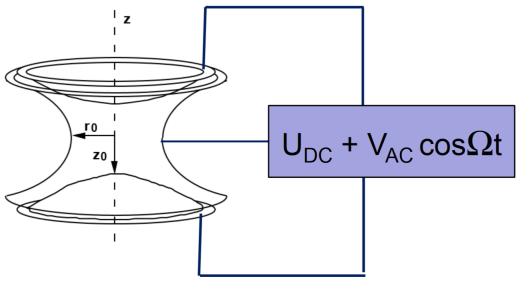


Figure 2.4: The Paul trap consists of a ring and 2 endcaps of hyperboloid shape. An oscillating voltage between the electrodes is applied in order to effectively produce a rotating potential well that confines the ion at the minimax point of the potential well. The resulting potential is $\phi(x, y, z, t) = (U_{DC} + V_{AC} \cos \Omega_{\text{rf}} t)(x^2 + y^2 + 2z^2)/(2r_0^2)$. Source: Martina Knoop, CNRS/Université d'Aix-Marseille

In order to accomplish the confinement of multiple qubits we could use a linear Paul trap design, which is an analogue of fig. 2.4. With the ions confined in a cylindrical space within four linear conducting electrodes oriented along the z-axis, the z component of the potential is suppressed. As a result, the time-varying electric quadrupole potential within the volume defined by the rods, produces a spatially non-uniform time varying electric field which is described by the following equation:

$$\phi(x, y, t) = (U - V \cos \Omega_{\text{rf}} t) \frac{(x^2 - y^2)}{2r_0^2} \tag{2.9}$$

A time-varying potential $\frac{V}{2} \cos \Omega_{\text{rf}} t$ is applied to two opposite electrodes, and $-\frac{V}{2} \cos \Omega_{\text{rf}} t$ to the other opposite pair, or alternatively $V \cos \Omega_{\text{rf}} t$ is applied to one electrode pair and the other pair is grounded, an electrically simpler equivalent arrangement.

The motion of the ions in the linear Paul trap along the z axis can be confined, by using a positive DC potential that increases the positive ion moves in either direction along the z axis away from the linear trap center. Since the Laplace equation

$$\nabla^2 V_{DC}(x, y, z) = 0 \tag{2.10}$$

must also be satisfied, there is additionally a modification in the transverse frequency ω due to the DC potential. The result is overal harmonic confinement of the ions in all three dimensions but with different frequencies in different dimensions.

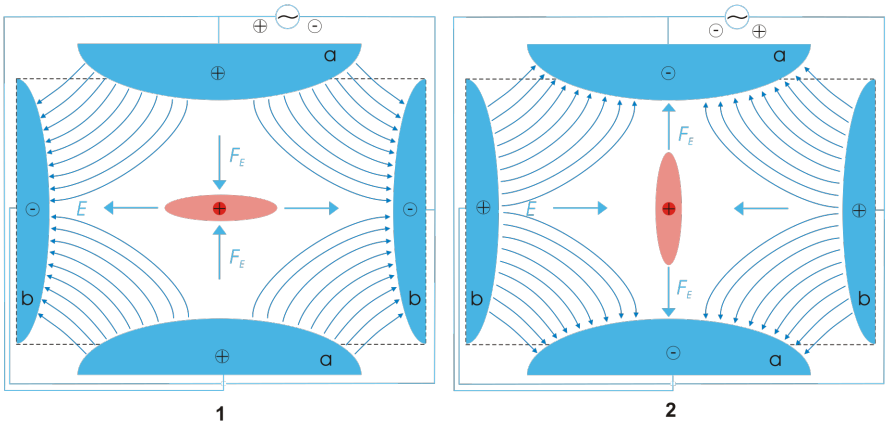


Figure 2.5: Scheme of a quadrupole ion trap of classical setup with a particle of positive charge (dark red), surrounded by a cloud of similarly charged particles (light red). The electric field E (blue) is generated by a quadrupole of endcaps (a, positive) and a ring electrode (b). Picture 1 and 2 show two states during an AC cycle. Source: Quadrupole Ion Trap

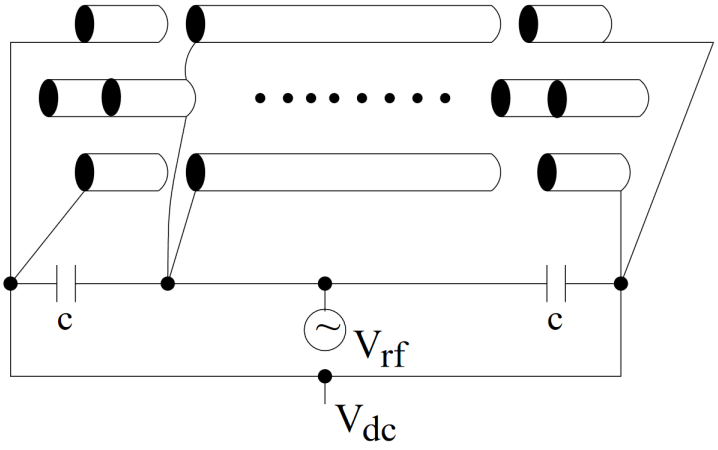


Figure 2.6: [3] Illustration of axial confinement in a linear RF trap. The gap in the rod structure permits a DC voltage on the ends of the rods, while capacitors C permit the RF voltage on all parts of the structure. A similar bias arrangement holds for the two rods not shown. A string of cold confined ions is shown as black dots.

Overall, the ions in the linear array are separated axially by a balance between the Coulomb repulsion of the charges, and the force exerted by the confining axial electric field. The equilibrium ion spacing of the ions along the z axis tends to decrease near the trap center. Nevertheless, ion separations in the array are measured in micrometers, while zero point oscillation amplitudes of cold ions are measured in nanometers, and ion internal state wave function distributions having significant probability in space are still smaller. This means that each ion is physically isolated from the others in the array unless a deliberate coupling is externally initiated.

Chapter

THREE

Implementing Quantum Gates in the Ion Trap Model

In this chapter, we are going to analyze the way gate operations can be performed in the ion trap configuration. Single qubit gates can be implemented by performing arbitrary phase transitions through external driving, while two qubit gates can be performed by coupling qubits to a harmonic oscillator.

3.1 Single Qubit Gates

3.1.1 Optical Qubit

The qubit in a trapped ion quantum computer encoded through the difference in the electronic energy levels. Basically, a two level system (Hydrogen atom) having two energy levels. Which orbitals (permitted states) are used to encode information depends on the element. The optimal orbitals used depend on the lifetime of the excited state (time before decoherance) and the energy (laser frequency) needed to drive the system to those states. Logic gates are usually implemented by driving it into resonance with a laser. This causes Rabi oscillations, making gate logic possible.

In the ion trap configuration, we can perform phase transitions through the means of stimulated Raman transitions. By manipulating the frequency of an external electromagnetic field and exposing the ions to the field for specific amounts of time, we are able to perform arbitrary qubit rotations. The following operator, describes this arbitrary rotation on a qubit, and in this case ϕ is the external drive phase and θ is the drive time (Ωt), Ω being the Rabi frequency.

$$R(\theta, \phi) = \begin{pmatrix} \cos\theta/2 & ie^{i\phi}\sin\theta/2 \\ ie^{-i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{3.1}$$

3.2 The Cirac-Zoller Gate

Cirac and Zoller, in their seminal paper [1], specified the operations necessary to implement a universal phase gate. In this section, we are going to describe the process of implementing the controlled phase gate $\hat{U}_{ROT}$. It is noted that by adding appropriate rotations to either control or the target qubit respectively, the CNOT gate can be implemented. In order to understand how to perform the Cirac-Zoller gate, it is relevant that we explain the center of mass mode (CM) of the trapped ions in a linear trap. Appropriate stimulation with a laser beam in the z-direction of an ion in the ion trap creates vibrational motion in which the ion lattice oscillates uniformly. For the CM motion, the interaction Hamiltonian coupling the internal qubit states of each ion (j = 1,..., N) to the CM motion can be written as:

$$\hat{H}_j = \frac{i\hbar\eta\Omega}{\sqrt{N}} e^{i\phi}\sigma_j^+ \otimes a + h.c. \tag{3.2}$$

where *h.c.* denotes the Hermitian conjugate of the first operator. $a^\dagger$ and a are the creation and annihilation operators of the CM phonons, respectively, Ω is the Rabi frequency, ϕ is the laser phase, and $\eta = [\hbar k_\theta^2/(2M\nu_x)]^{1/2}$ is the Lamb-Dicke Limit (LDL).

The action of the interaction Hamiltonian on the state $|0\rangle_j |k+1\rangle_{CM}$ is:

$$\begin{aligned} H_j |0\rangle_j |k+1\rangle_{CM} &= \frac{\hbar\eta\Omega}{\sqrt{N}} \left[|1\rangle_j \langle 0|0\rangle_j e^{-i\phi} \otimes a |k+1\rangle_{CM} + |0\rangle_j \langle 1|0\rangle_j e^{i\phi} \otimes a^\dagger |k+1\rangle_{CM} \right] \\ &= \frac{\hbar\eta\Omega}{\sqrt{N}} e^{-i\phi} \sqrt{k+1} |1\rangle_j |k\rangle_{CM} \end{aligned} \tag{3.3}$$

and similarly:

$$\begin{aligned} H_j |1\rangle_j |k\rangle_{CM} &= \frac{\hbar\eta\Omega}{\sqrt{N}} \left[|1\rangle_j \langle 0|1\rangle_j e^{-i\phi} \otimes a |k\rangle_{CM} + |0\rangle_j \langle 1|1\rangle_j e^{i\phi} \otimes a^\dagger |k\rangle_{CM} \right] \\ &= \frac{\hbar\eta\Omega}{\sqrt{N}} e^{i\phi} \sqrt{k+1} |0\rangle_j |k+1\rangle_{CM} \end{aligned} \tag{3.4}$$

And the unitary evolution of the Hamiltonian is:

$$\hat{U}_j(t) = \exp\left\{ -i\hat{H}_j t/\hbar \right\} \tag{3.5}$$

The interaction Hamiltonian, with respect to the ordered basis $\{|0\rangle_j |k+1\rangle_{CM}, |1\rangle_j |k\rangle_{CM}\}$, $H_{j,k}$ admits the following matrix representation:

$$H_{j,k} = \hbar\eta\Omega \sqrt{\frac{k+1}{N}} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \tag{3.6}$$

Thus, its time evolution operator is given by:

$$U_{j,k}(t,\phi) = e^{-iH_j t/\hbar} = \begin{pmatrix} \cos\mathcal{E}_k t & ie^{i\phi}\sin\mathcal{E}_k t \\ ie^{-i\phi}\sin\mathcal{E}_k t & \cos\mathcal{E}_k t \end{pmatrix}, \ \mathcal{E}_k = \hbar\eta\Omega\sqrt{\frac{k+1}{N}} \tag{3.7}$$

Using the CM mode as the bus, we can realize the 2-bit quantum phase gate as follows. First, a qubit in logic state $|1\rangle$ is excited into a superposition of logic states $|1\rangle$ and $|0\rangle$, while remaining in motional state $|1\rangle_{CM}$, Then, the $|1\rangle |1\rangle_{CM}$ state is excited through an internal auxiliary and back by a 2π pulse, changing the sign of the wave function. Finally, the superposition is recombined by a rotation to the qubit state $|0\rangle |1\rangle_{CM}$. These steps are expressed below:

$$U \equiv U_1(3T,0) U_2^{aux}(2T,0) U_1(T,0), \ T \equiv \frac{\pi}{2\eta\Omega}\sqrt{N} \tag{3.8}$$

We can verify for each state:

$$\begin{aligned} U |0\rangle_1 |0\rangle_2 |0\rangle_{CM} &= U_1(3T,0) U_2^{aux}(2T,0) U_1(T,0) |0\rangle_1 |0\rangle_2 |0\rangle_{CM} \\ &= U_1(3T,0) U_2^{aux}(2T,0) U_1(T,0) [|0\rangle_1 |0\rangle_{CM}] |0\rangle_2 \\ &= U_1(3T,0) U_2^{aux}(2T,0) [|0\rangle_1 |0\rangle_{CM}] |0\rangle_2 \\ &= U_1(3T,0) U_2^{aux}(2T,0) [|0\rangle_2 |0\rangle_{CM}] |0\rangle_1 \\ &= U_1(3T,0) [|0\rangle_1 |0\rangle_{CM}] |0\rangle_2 \\ &= |0\rangle_1 |0\rangle_{CM} |0\rangle_2 \\ &= |0\rangle_1 |0\rangle_2 |0\rangle_{CM} \end{aligned} \tag{3.9}$$

$$\begin{aligned} U |0\rangle_1 |1\rangle_2 |0\rangle_{CM} &= U_1(3T,0) U_2^{aux}(2T,0) U_1(T,0) |0\rangle_1 |1\rangle_2 |0\rangle_{CM} \\ &= U_1(3T,0) U_2^{aux}(2T,0) U_1(T,0) [|0\rangle_1 |0\rangle_{CM}] |1\rangle_2 \\ &= U_1(3T,0) U_2^{aux}(2T,0) [|0\rangle_1 |0\rangle_{CM}] |1\rangle_2 \\ &= U_1(3T,0) U_2^{aux}(2T,0) [|1\rangle_2 |0\rangle_{CM}] |0\rangle_1 \\ &= U_1(3T,0) [|0\rangle_1 |0\rangle_{CM}] |1\rangle_2 \\ &= |0\rangle_1 |0\rangle_{CM} |1\rangle_2 \\ &= |0\rangle_1 |1\rangle_2 |0\rangle_{CM} \end{aligned} \tag{3.10}$$

$$\begin{aligned}
U\left|1\right\rangle_1\left|0\right\rangle_2\left|0\right\rangle_{CM} &= U_1(3T,0)U_2^{aux}(2T,0)U_1(T,0)\left|1\right\rangle_1\left|0\right\rangle_2\left|0\right\rangle_{CM} \\
&= U_1(3T,0)U_2^{aux}(2T,0)U_1(T,0)[\left|1\right\rangle_1\left|0\right\rangle_{CM}]\left|0\right\rangle_2 \\
&= U_1(3T,0)U_2^{aux}(2T,0)[-i\left|0\right\rangle_1\left|1\right\rangle_{CM}]\left|0\right\rangle_2 \\
&= -iU_1(3T,0)U_2^{aux}(2T,0)[\left|0\right\rangle_2\left|1\right\rangle_{CM}]\left|0\right\rangle_1 \\
&= -iU_1(3T,0)[\left|0\right\rangle_1\left|1\right\rangle_{CM}]\left|0\right\rangle_2 \\
&= (-i)i\left|1\right\rangle_1\left|0\right\rangle_{CM}\left|0\right\rangle_2 \\
&= \left|1\right\rangle_1\left|0\right\rangle_{CM}\left|0\right\rangle_2 \\
&= \left|1\right\rangle_1\left|0\right\rangle_2\left|0\right\rangle_{CM}
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
U\left|1\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_{CM} &= U_1(3T,0)U_2^{aux}(2T,0)U_1(T,0)\left|1\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_{CM} \\
&= U_1(3T,0)U_2^{aux}(2T,0)U_1(T,0)[\left|1\right\rangle_1\left|0\right\rangle_{CM}]\left|1\right\rangle_2 \\
&= U_1(3T,0)U_2^{aux}(2T,0)[-i\left|0\right\rangle_1\left|1\right\rangle_{CM}]\left|1\right\rangle_2 \\
&= -iU_1(3T,0)U_2^{aux}(2T,0)[\left|1\right\rangle_2\left|1\right\rangle_{CM}]\left|0\right\rangle_1 \\
&= iU_1(3T,0)[\left|0\right\rangle_1\left|1\right\rangle_{CM}]\left|1\right\rangle_2 \\
&= i\cdot i\left|1\right\rangle_1\left|0\right\rangle_{CM}\left|1\right\rangle_2 \\
&= -\left|1\right\rangle_1\left|0\right\rangle_{CM}\left|1\right\rangle_2 \\
&= -\left|1\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_{CM}
\end{aligned} \tag{3.12}$$

We can observe that the eq. (3.8) makes for a successful Controlled Phase gate. We can construct the Controlled NOT gate by performing two Hadamard operations on the target qubit, one before applying the Controlled Phase gate and one after.

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