Homework 4 of Stochastic Processes

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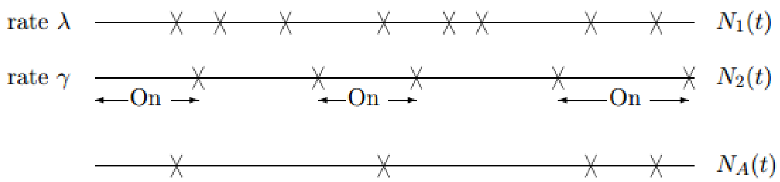
1 Exercise 2.23

Exercise 2.23 Let $\{N_1(t);\ t > 0\}$ be a Poisson counting process of rate λ. Assume that the arrivals from this process are switched on and off by arrivals from a second independent Poisson process $\{N_2(t); t > 0\}$ of rate γ.



Let $\{N_A(t); t\geq 0\}$ be the switched process, i.e., $N_A(t)$ includes the arrivals from $\{N_1(t);\ t > 0\}$ during periods when $N_2(t)$ is even and excludes the arrivals from $\{N_1(t);\ t > 0\}$ while $N_2(t)$ is odd.

(a) Find the PMF for the number of arrivals of the first process, $\{N_1(t);\ t > 0\}$, during the nth period when the switch is on.

(b) Given that the first arrival for the second process occurs at epoch τ, find the conditional PMF for the number of arrivals of the first process up to τ.

(c) Given that the number of arrivals of the first process, up to the first arrival for the second process, is n, find the density for the epoch of the first arrival from the second process.

(d) Find the density of the interarrival time for $\{N_A(t); t \geq 0\}$. Note: This part is quite messy and is done most easily via Laplace transforms.

Soultions

a) The combined process $\{N_1(t) + N_2(t)\}$ is also a Poisson process with rate $\lambda+\gamma$. During the nth period when the switch is on, once there is an arrival of process $\{N_2(t); t > 0\}$, the nth period will end. Therefore, the PMF for the number of arrivals of the first process $\{N_1(t); t > 0\}$ is

$$\mathrm{p}_{N_1(t)}(k) = \left(\frac{\lambda}{\lambda+\gamma}\right)^k \frac{\gamma}{\lambda+\gamma}$$

where k denotes the number of arrivals of the first process $\{N_1(t); t > 0\}$ and n denotes the nth period when the switch is on.

b) According to the Theorem 2.2.10, *for a Poisson of rate λ, and for any $t > 0$, the PMF for $N(t)$, i.e., the number of arrivals in $(0, t]$, is given by the Poisson PMF,*

$$\mathrm{p}_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}.$$

Since $\{N_1(t); t > 0\}$ and $\{N_2(t); t > 0\}$ are independent,given that the first arrival for process $\{N_2(t); t > 0\}$ at epoch τ, the conditional PMF for the number of arrivals of $\{N_1(t); t > 0\}$ up to τ is

$$\mathrm{p}_{N_1(\tau)}(n) = \frac{(\lambda \tau)^n \exp(-\lambda \tau)}{n!}.$$

c) Suupose that the first arrival of process $\{N_2(t); t > 0\}$ is at epcoh τ, according to the Bayes' Law, we can obtain that

$$\mathrm{p}_{N_1(\tau)}(n)\mathrm{f}_{S_1^2|N_1(\tau)}(\tau|n) = \mathrm{f}_{S_1^2}(\tau)\mathrm{p}_{N_1(\tau)|S_1^2}(n|\tau)$$

where S_1^2 denotes the 1rd arrival of process 2.

From a) we can know that

$$\mathrm{p}_{N_1(\tau)}(n) = (\frac{\lambda}{\lambda + \gamma})^n \frac{\gamma}{\lambda + \gamma}$$

From b) we can know that

$$\mathrm{p}_{N_1(\tau)|S_1^2}(n|\tau) = \frac{(\lambda \tau)^n \exp(\lambda \tau)}{n!}$$

And from the equation (2.13) $\mathrm{f}_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!}$ in the textbook we can konw that

$$\mathrm{f}_{S_1^2}(\tau) = \gamma \exp(-\gamma \tau)$$

Therefore,

$$\begin{aligned}
\mathrm{f}_{S_1^2|N_1(\tau)}(\tau|n) &= \mathrm{f}_{S_1^2}(\tau)\frac{\mathrm{p}_{N_1(\tau)|S_1^2}(n|\tau)}{\mathrm{p}_{N_1(\tau)}(n)} \\
&= \gamma e^{-\gamma\tau} \cdot \frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!} \cdot \frac{(\lambda+\gamma)^{n+1}}{\lambda^n \gamma} \\
&= \frac{(\lambda+\gamma)^{n+1}\tau^n e^{-(\lambda+\gamma)\tau}}{n!}
\end{aligned}$$

d) Since $N_1(t); t > 0$ and $N_2(t); t > 0$ are both independent and both renewal process, $N_A(t); t > 0$ should be also a renewal process. Therefore, each interval of $N_A(t); t > 0$ is independent and identically distributed.

Let X_A denote the interval of process $\{N_A(t); t \geq 0\}$, X denote the interval of the combined process $\{N_1(t)+N_2(t); t > 0\}$, and X_2 denote the interval of process $\{N_2(t); t > 0\}$.

There exists two cases:

1. If the next arrival belongs to process $\{N_1(t); t > 0\}$, $X_A = X$
2. If the next arrivals belongs to process $\{N_2(t); t > 0\}$, then X_A is the sum of three rv's: X, X_2 and X_A。

Therefore, the density of the interarrival time for $\{N_A(t); t \geq 0\}$ is

$$
\begin{aligned}
\mathrm{f}_{X_A}(x) &= \frac{\lambda}{\lambda+\gamma}\mathrm{f}_X(x) + (\frac{\gamma}{\lambda+\gamma}\mathrm{f}_X(x)) \otimes \mathrm{f}_{X_2}(x) \otimes \mathrm{f}_{X_A}(x) \\
&= \frac{\lambda}{\lambda+\gamma} \cdot (\lambda+\gamma)e^{-(\lambda+\gamma)x} + [\frac{\gamma}{\lambda+\gamma} \cdot (\lambda+\gamma)e^{-(\lambda+\gamma)x}] \otimes (\gamma e^{-\gamma x}) \otimes \mathrm{f}_{X_A}(x) \\
&= \lambda e^{-(\lambda+\gamma)x} + (\gamma e^{-(\lambda+\gamma)x}) \otimes (\gamma e^{-\gamma x}) \otimes \mathrm{f}_{X_A}(x)
\end{aligned}
$$

where $\otimes$ denotes the convolution operator anbd all functions satisfy $f(x) = 0$ for $x < 0$.

Let

$$
\begin{aligned}
f_1(x) &= \lambda e^{-(\lambda+\gamma)x} \\
f_2(x) &= \gamma e^{-(\lambda+\gamma)x} \\
f_3(x) &= \gamma e^{-\gamma x}
\end{aligned}
$$

Then we can obtain that:

$$
\mathrm{f}_{X_A}(x) = f_1(x) + f_2(x) \otimes f_3(x) \otimes \mathrm{f}_{X_A}(x)
$$

Apply the Laplace transforms

$$
\begin{aligned}
\mathcal{L}[\mathrm{f}_{X_A}] &= \mathcal{L}[f_1] + \mathcal{L}[f_2] \cdot \mathcal{L}[f_3] \cdot \mathcal{L}[\mathrm{f}_{X_A}] \\
&\Downarrow \\
\mathcal{L}[\mathrm{f}_{X_A}] &= \frac{\mathcal{L}[f_1]}{1 - \mathcal{L}[f_2]\mathcal{L}[f_3]}
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathcal{L}[f_1] &= \int_0^\infty \lambda e^{-(\lambda+\gamma)t} \cdot e^{-st} dt &&= \frac{\lambda}{\lambda+\gamma+s} \\
\mathcal{L}[f_2] &= \int_0^\infty \gamma e^{-(\lambda+\gamma)t} \cdot e^{-st} dt &&= \frac{\gamma}{\lambda+\gamma+s} \\
\mathcal{L}[f_3] &= \int_0^\infty \gamma e^{-\gamma t} \cdot e^{-st} dt &&= \frac{\gamma}{\gamma+s}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}[\mathrm{f}_{X_A}] &= \frac{\lambda}{\lambda+\gamma+s} \cdot \frac{1}{1 - \frac{\gamma}{\lambda+\gamma+s} \cdot \frac{\gamma}{\gamma+s}} \\
&= \frac{\lambda}{(\lambda+\gamma+s) - \frac{\gamma^2}{\gamma+s}} \\
&= \frac{\lambda(\gamma+s)}{(\lambda+\gamma+s)(\gamma+s) - \gamma^2} \\
&= \frac{\lambda(\gamma+s)}{\lambda\gamma + (\lambda+2\gamma)s + s^2} \\
&= \frac{F_1(s)}{F_2(s)}
\end{aligned}
$$

To obatin f_{X_A}, it needs to perform inverse Laplace transforms.

For $F_2(s)$, $(\lambda+2\gamma)^2-4\lambda\gamma=\lambda^2+4\gamma^2>0$ and it has two roots with real value. Then we can rewrite $F_2(s)$ as

$$F_2(s)=\left(s-\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2}\right)\left(s-\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2}\right)=(s-s_1)(s-s_2)$$

And $\mathcal{L}[\mathrm{f}_{X_A}]$ can be rewritten as

$$\mathcal{L}[\mathrm{f}_{X_A}]=\frac{k_1}{s-s_1}+\frac{k_2}{s-s_2}$$

The value of k_1 and k_2 can be solved by:

$$\begin{aligned}
k_1&=(s-s_1)\mathcal{L}[\mathrm{f}_{X_A}]\Big|_{s=s_1}\\
&=\frac{\lambda(\gamma+s)}{s-s_2}\Bigg|_{s=s_1}\\
&=\frac{\lambda(\gamma+\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2})}{\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2}-\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2}}\\
&=\frac{\lambda\frac{-\lambda-\sqrt{\lambda^2+4\gamma^2}}{2}}{-\sqrt{\lambda^2+4\gamma^2}}\\
&=\frac{\lambda(\lambda+\sqrt{\lambda^2+4\gamma^2})}{2\sqrt{\lambda^2+4\gamma^2}}\\
&=\frac{\lambda}{2}\left(1+\frac{\lambda}{\sqrt{\lambda^2+4\gamma^2}}\right)\\
k_2&=(s-s_2)\mathcal{L}[\mathrm{f}_{X_A}]\Big|_{s=s_2}\\
&=\frac{\lambda(\gamma+s)}{s-s_1}\Bigg|_{s=s_2}\\
&=\frac{\lambda(\gamma+\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2})}{\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2}-\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2}}\\
&=\frac{\lambda\frac{-\lambda+\sqrt{\lambda^2+4\gamma^2}}{2}}{\sqrt{\lambda^2+4\gamma^2}}\\
&=\frac{\lambda(-\lambda+\sqrt{\lambda^2+4\gamma^2})}{2\sqrt{\lambda^2+4\gamma^2}}\\
&=\frac{\lambda}{2}\left(1-\frac{\lambda}{\sqrt{\lambda^2+4\gamma^2}}\right)
\end{aligned}$$

According to the formula of inverse Laplace tranform:

$$f(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\frac{k_1}{s-s_1}+\frac{k_2}{s-s_2}+\cdots+\frac{k_n}{s-s_n}]=\sum_{i=1}^{n}k_ie^{s_it}$$

we can obtain f_{X_A} as

$$\begin{aligned}
f_{X_A}(x) &= k_1 \cdot e^{\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2}x} + k_2 \cdot e^{\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2}x} \\
&= k_1 \cdot \exp\left[\frac{-(\lambda+2\gamma)-\sqrt{\lambda^2+4\gamma^2}}{2}x\right] + k_2 \cdot \exp\left[\frac{-(\lambda+2\gamma)+\sqrt{\lambda^2+4\gamma^2}}{2}x\right]
\end{aligned}$$

2 Exercise 2.25

Exercise 2.25 **(a)** Find the conditional density of S_{i+1}, conditional on $N(t) = n$ and $S_i = s_i$.

(b) Use (a) to find the joint density of $S_1, \ldots, S_n$ conditional on $N(t) = n$. Verify that your answer agrees with (2.38).

Solutions

a) From the equation (2.41) in the textbook,

$$\Pr\{S_1 > \tau | N(t) = n\} = \left[\frac{t-\tau}{t}\right]^n \qquad \text{for } 0 < \tau \le t$$

we can obtain that

$$\begin{aligned}
f_{S_1|N(t)}(\tau|n) &= (1 - \Pr\{S_1 > \tau | N(t) = n\})' \\
&= \left(1 - \left[\frac{t-\tau}{t}\right]^n\right)' \\
&= -n \cdot \left[\frac{t-\tau}{t}\right]^{n-1} \cdot -\frac{1}{t} \\
&= \frac{n(t-\tau)^{n-1}}{t^n}
\end{aligned}$$

Then, given that $N(t) = n$ and $S_i = s_i$, we can obtain that

$$\begin{aligned}
f_{S_{i+1}|N(t),S_i}(s_{i+1}|n, s_i) &= f_{X_{i+1}|N(t),S_i}(s_{i+1} - s_i | n, s_i) \\
&= f_{X_{i+1}}|\widetilde{N}(s_i, t)(s_{i+1} - s_i | n - i) \\
&= \frac{(n-i)[t - s_i - (s - i + 1 - s_i)]^{n-i-1}}{(t - s_i)^{n-i}} \\
&= \frac{(n-i)(t - s_{i+1})^{n-i-1}}{(t - s_i)^{n-i}}
\end{aligned}$$

b) For S_i, the conditional probability is independent of $S_1, S_2, \ldots, S_{i-2}$, and thus

$$\begin{aligned}
f_{S^{(n)}|N(t)=n}(s^{(n)}|n) &= f_{S_1|N(t)} \cdot f_{S_2|N(t),S_1} \cdots f_{S_n|N(t),\ldots,S_{n-1}} \\
&= f_{S_1|N(t)} \cdot f_{S_2|N(t),S_1} \cdots f_{S_n|N(t),S_{n-1}} \\
&= \frac{n(t-s_1)^{n-1}}{t^n} \cdot \frac{(n-1)(t-s_2)^{n-2}}{(t-s_1)^{n-1}} \cdots \frac{(t-s_n)^0}{t-s_{n-1}} \\
&= \frac{n!}{t^n}
\end{aligned}$$