Homework 1 of Dynamic Programming and Optimal Control

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2019 年 11 月 13 日

1 Exercise 1.1

Consider the system

$$x_{k+1} = x_k + u_k + w_k, \qquad k = 0, 1, 2, 3$$

with initial state $x_0 = 5$, and the cost function

$$\sum_{k=0}^{3}(x_k^2 + u_k^2)$$

Apply the DP algorithm for the following three cases:

(a) The control constraint set $U_k(x_k)$ is $\{u | 0 \le x_k + u \le 5, u : \text{ integer}\}$ for all x_k and k, and the disturbance w_k is equal to zero for all k.

(c) The control constraint is as in part (a) and the disturbance w_k takes the values -1 and 1 with equal probability 1/2 for all x_k and u_k, except if $x_k + u_k$ is equal to 0 or 5, in which case $w_k = 0$ with probability 1.

Solutions:

(a) 由题可知，$N = 4$ 且 $g_N(x_N) = 0$。由于 $w_k = 0$ for all k, $x_{k+1} = x_k + u_k + w_k = x_k + u_k$，且

$$\begin{aligned}
&\min_{u_k \in U_k(x_k)} \underset{w_k}{E} \left\{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \right\} \\
&= \min_{u_k \in U_k(x_k)} g_k(x_k, u_k) + J_{k+1}(f_k(x_k, u_k)) \\
&= \min_{u_k \in U_k(x_k)} g_k(x_k, u_k) + J_{k+1}(x_k + u_k)
\end{aligned} \tag{1}$$

另外，代价函数 $\sum_{k=0}^{3}(x_k^2 + u_k^2) = \sum_{k=0}^{3} g_k(x_k, u_k)$ 可以明显地通过最小化每个子问题 $g_k(x_k, u_k)$ 来最小化本身，符合动态规划的使用场景。因此我们使用动态规划的方法来解决本问题。

从约束 $\{u|0 \leq x_k + u_k \leq 5, u: \text{ integer}\}$ 中我们可以得到 u 的取值集合为 $U_k(x_k) = \{-x_k, 1-x_k, 2-x_k, 3-x_k, 4-x_k, 5-x_k\}$。

则，当 $N = 4$ 时

$$J_4(x_4) = g_4(x_4) = 0 \tag{2}$$

当 $k = 3$ 时，

$$\begin{aligned} J_3(x_3) &= \min_{u_3 \in U_3(x_3)} g_3(x_3, u_3) + J_4(x_3 + u_3) \\ &= \min_{u_3 \in U_3(x_3)} x_3^2 + u_3^2 \end{aligned} \tag{3}$$

令 $h_3(u_3) = x_3^2 + u_3^2$, 并对其求导可得

$$h_3(u_3)' = 2u_3 \tag{4}$$

由于存在约束 $\{u|0 \leq x_3 + u_3 \leq 5, u: \text{ integer}\}$，我们要分情况讨论。

(a) 当 $x_3 < 0$ 时, $u_3 > 0$ 且 $h_3(u^3)' = 2u_3 > 0$。因此 u_3 的取值越小越好, 即 $u_3^* = -x_3$。

(b) 当 $x_3 > 5$ 时, $u_3 < 0$ 且 $h_3(u^3)' = 2u_3 < 0$。因此 u_3 的值越大越好, 即 $u_3^* = 5 - x_3$。

(c) 当 $0 \leq x_3 \leq 5$ 时，最小值为 $h_3^*(u_3) = 0$ 且 $u_3^* = 0$。

将上述结果代回去可以得到

$$J_3(x_3) = \begin{cases} x_3^2 + u_3^2 & , x_3 \leq 0 \\ x_3^2 & , 0 \leq x_3 \leq 5 \\ 2x_3^2 - 10x_3 + 25 & , x_3 > 5 \end{cases} \tag{5}$$

当 $k = 2$ 时，$0 \leq x_3 = x_2 + u_2 \leq 5$，因此

$$\begin{aligned} J_2(x_2) &= \min_{u_2 \in U(x_2)} g_2(x_2, u_2) + J_3(x_2 + u_2) \\ &= \min_{u_2 \in U(x_2)} x_2^2 + u_2^2 + (x_2 + u_2)^2 \end{aligned} \tag{6}$$

令 $h_2(u_2) = x_2^2 + u_2^2 + (x_2 + u_2)^2$，并对其求导可得

$$h_2(u_2)' = 4u_2 + 2x_2 = 4(x_2 + u_2) - 2x_2 \tag{7}$$

由于存在约束 $\{u|0 \leq x_2 + u_2 \leq 5, u: \text{ integer}\}$，我们要分情况讨论。

(a) 当 $x_2 > 10$ 时，$h_2(u_2)' < 0$，因此 u_2 的值越大越好，即 $u_2^* = 5 - x_2$

(b) 当 $x_2 < 0$ 时，$h_2(u_2)' > 0$，因此 u_2 的取值越小越好，即 $u_2^* = -x_2$

(c) 当 $0 \leq x_2 \leq 10$ 时，最小值出现在 $h' = 0$ 的时候

  i. 当 x_2 为偶数的时候，$u_2^* = -\frac{x_2}{2}$

  ii. 当 x_3 为奇数的时候，$u_2^* = -\frac{x_2+1}{2}$ 或者 $u_2^* = -\frac{x_2-1}{2}$，因为 h 是偶函数。

将上述结果代回去可以得到

$$J_2(x_2)=\begin{cases}2x_2^2 & ,x_2<0\\ \frac{3}{2}x_2^2+\frac{1}{2}\cdot\mathbf{1}_{x_2\text{ is odd}} & ,0\le x_2\le 10\\ 2x_2^2-10x_2+50 & ,x>10\end{cases}\tag{8}$$

当 $k=1$ 时，$0\le x_2=x_1+u_1\le 5$，因此

$$\begin{aligned}J_1(x_1)&=\min_{u_1\in U(x_1)} g_1(x_1,u_1)+J_2(x_1+u_1)\\&=\min_{u_1\in U(x_1)} x_1^2+u_1^2+\frac{3}{2}(x_1+u_2)^2+\frac{1}{2}\cdot\mathbf{1}_{x_1+u_1\text{ is odd}}\end{aligned}\tag{9}$$

令 $h_1(u_1)=x_1^2+u_1^2+\frac{3}{2}(x_1+u_2)^2+\frac{1}{2}\cdot\mathbf{1}_{x_1+u_1\text{ is odd}}$，并对其求导可得

$$h_1(u_1)'=5u_1+3x_1=5(x_1+u_1)-2x_1\tag{10}$$

由于存在约束 $\{u|0\le x_1+u_1\le 5,u:\text{ integer}\}$，且 h_1' 有点难以直接求解。因此，直接列出可能的取值以求解。

$$\begin{aligned}h_1(-x_1)&=2x_1^2\\h_1(1-x_1)&=2x_1^2-2x_1+3\\h_1(2-x_1)&=2x_1^2-4x_1+10\\h_1(3-x_1)&=2x_1^2-6x_1+23\\h_1(4-x_1)&=2x_1^2-8x_1+40\\h_1(5-x_1)&=2x_1^2-10x_1+63\end{aligned}\tag{11}$$

对上述式子联立求区间可得最终结果

$$J_1(x_1)=\begin{cases}2x_1^2 & ,u_1^*=-x_1\text{ and }x_1\le\frac{3}{2}\\ 2x_1^2-2x_1+3 & ,u_1^*=1-x_1\text{ and }\frac{3}{2}<x_1\le\frac{7}{2}\\ 2x_1^2-4x_1+10 & ,u_1^*=2-x_1\text{ and }\frac{7}{2}<x_1\le\frac{13}{2}\\ 2x_1^2-6x_1+23 & ,u_1^*=3-x_1\text{ and }\frac{13}{2}<x_1\le\frac{17}{2}\\ 2x_1^2-8x_1+40 & ,u_1^*=4-x_1\text{ and }\frac{17}{2}<x_1\le\frac{23}{2}\\ 2x_1^2-10x_1+63 & ,u_1^*=5-x_1\text{ and }x_1>\frac{23}{2}\end{cases}\tag{12}$$

当 $k=0$ 时，$0\le x_1=x_0+u_0\le 5$ 且 $x_0=5,-5\le u_0\le 0$，因此

$$\begin{aligned}J_0(x_0)&=\min_{u_0\in U_{(x_0)}} g_0(x_0,u_0)+J_1(x_0+u_0)\\&=25+u_0^2+J_1(5+u_0)\end{aligned}\tag{13}$$

同样，令 $h_0(u_0) = 25 + u_0^2 + J_1(5 + u_0)$，并对 u_0 进行取值可得

$$\begin{aligned} h_0(-5) &= 50 \\ h_0(-4) &= 43 \\ h_0(-3) &= 41 \\ h_0(-2) &= 44 \\ h_0(-1) &= 52 \\ h_0(0) &= 65 \end{aligned} \tag{14}$$

则 $u_0^* = -3$ 且 $J_0(x_0) = 41$。

(c) 由题可知，此时 w_k 不总是为 0。因此，

$$\begin{aligned} &\min_{u_k \in U_k(x_k)} \underset{w_k}{E} \left\{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \right\} \\ =& \min_{u_k \in U_k(x_k)} g_k(x_k, u_k) + \underset{w_k}{E} \left\{ J_{k+1}(f_k(x_k, u_k, w_k)) \right\} \\ =& \min_{u_k \in U_k(x_k)} g_k(x_k, u_k) + \underset{w_k}{E} \left\{ J_{k+1}(x_k + u_k + w_k) \right\} \end{aligned} \tag{15}$$

当 $k = 4$ 时，

$$J_4(x_4) = g_4(x_4) = 0 \tag{16}$$

当 $k = 3$ 时，

$$\begin{aligned} J_3(x_3) &= \min_{u_3 \in U_3(x_3)} g_3(x_3, u_3) + \underset{w_3}{E} \left\{ J_4(x_3 + u_3 + w_3) \right\} \\ &= \min_{u_3 \in U_3(x_3)} x_3^2 + u_3^2 \end{aligned} \tag{17}$$

令 $h_3(u_3) = x_3^2 + u_3^2$，列出所有的 u_3 取值可得

$$\begin{aligned} h_3(-x_3) &= 2x_3^2 \\ h_3(1 - x_3) &= 2x_3^2 - 2x_3 + 1 \\ h_3(2 - x_3) &= 2x_3^2 - 4x_3 + 4 \\ h_3(3 - x_3) &= 2x_3^2 - 6x_3 + 9 \\ h_3(4 - x_3) &= 2x_3^2 - 8x_3 + 16 \\ h_3(5 - x_3) &= 2x_3^2 - 10x_3 + 25 \end{aligned} \tag{18}$$

则可得最后的结果

$$J_3(x_3) = \begin{cases} 2x_3 & ,u_1^* = -x_3 \text{ and } x_3 \leq \frac{1}{2} \\ 2x_3^2 - 2x_3 + 1 & ,u_1^* = 1 - x_3 \text{ and } \frac{1}{2} < x_3 \leq \frac{3}{2} \\ 2x_3^2 - 4x_3 + 4 & ,u_1^* = 2 - x_3 \text{ and } \frac{3}{2} < x_3 \leq \frac{5}{2} \\ 2x_3^2 - 6x_3 + 9 & ,u_1^* = 3 - x_3 \text{ and } \frac{5}{2} < x_3 \leq \frac{7}{2} \\ 2x_3^2 - 8x_3 + 16 & ,u_1^* = 4 - x_3 \text{ and } \frac{7}{2} < x_3 \leq \frac{9}{2} \\ 2x_3^2 - 10x_3 + 25 & ,u_1^* = 5 - x_3 \text{ and } x_3 > \frac{9}{2} \end{cases} \tag{19}$$

当 $k = 2$ 时

$$\begin{aligned} J_2(x_2) &= \min_{u_2 \in U_2(x_2)} g_2(x_2, u_2) + \mathop{E}_{w_2} \left\{ J_3(x_2 + u_2 + w_2) \right\} \\ &= \min_{u_2 \in U_2(x_2)} x_2^2 + u_2^2 + \mathop{E}_{w_2} \left\{ J_3(x_2 + u_2 + w_2) \right\} \end{aligned} \tag{20}$$

令 $h_2(u_2) = x_2^2 + u_2^2 + E_{w_2} \left\{ J_3(x_2 + u_2 + w_2) \right\}$，并列出所有的 u_2 的取值，可得

$$\begin{aligned} h_2(-x_2) &= x_2^2 + (-x_2)^2 + 1 \cdot J_3(0) && = 2x_2^2 \\ h_2(1 - x_2) &= x_2^2 + (1 - x_2)^2 + \frac{1}{2} \cdot J_3(0) + \frac{1}{2} \cdot J_3(2) && = 2x_2^2 - 2x_2 + 3 \\ h_2(2 - x_2) &= x_2^2 + (2 - x_2)^2 + \frac{1}{2} \cdot J_3(1) + \frac{1}{2} \cdot J_3(3) && = 2x_2^2 - 4x_2 + 9 \\ h_2(3 - x_2) &= x_2^2 + (3 - x_2)^2 + \frac{1}{2} \cdot J_3(2) + \frac{1}{2} \cdot J_3(4) && = 2x_2^2 - 6x_2 + 19 \\ h_2(4 - x_2) &= x_2^2 + (4 - x_2)^2 + \frac{1}{2} \cdot J_3(3) + \frac{1}{2} \cdot J_3(5) && = 2x_2^2 - 8x_2 + 33 \\ h_2(5 - x_2) &= x_2^2 + (5 - x_2)^2 + 1 \cdot J_3(5) && = 2x_2^2 - 10x_2 + 50 \end{aligned} \tag{21}$$

则可得最后的结果

$$J_2(x_2) = \begin{cases} 2x_2^2 & ,u_2^* = -x_2 \text{ and } x_2 \leq \frac{3}{2} \\ 2x_2^2 - 2x_2 + 3 & ,u_2^* = 1 - x_2 \text{ and } \frac{3}{2} < x_2 \leq 3 \\ 2x_2^2 - 4x_2 + 9 & ,u_2^* = 2 - x_2 \text{ and } 3 < x_2 \leq 5 \\ 2x_2^2 - 6x_2 + 19 & ,u_2^* = 3 - x_2 \text{ and } 5 < x_2 \leq 7 \\ 2x_2^2 - 8x_2 + 33 & ,u_2^* = 4 - x_2 \text{ and } 7 < x_2 \leq \frac{17}{2} \\ 2x_2^2 - 10x_2 + 50 & ,u_2^* = 5 - x_2 \text{ and } x_2 > \frac{17}{2} \end{cases} \tag{22}$$

当 $k = 1$ 时，

$$\begin{aligned} J_1(x_1) &= \min_{u_1 \in U_1(x_1)} g_1(x_1, u_1) + \mathop{E}_{w_1} \left\{ J_2(x_1 + u_1 + w_1) \right\} \\ &= \min_{u_1 \in U_1(x_1)} x_1^2 + u_1^2 + \mathop{E}_{w_1} \left\{ J_2(x_1 + u_1 + w_1) \right\} \end{aligned} \tag{23}$$

令 $h_1(u_1) = x_1^2 + u_1^2 + E_{w_1}\left\{J_2(x_1+u_1+w_1)\right\}$，并列出所有的 u_2 的取值，可得

$$
\begin{aligned}
h_1(-x_1) &= x_1^2 + (-x_1)^2 + 1\cdot J_2(0) && = 2x_1^2 \\
h_1(1-x_1) &= x_1^2 + (1-x_1)^2 + \frac{1}{2}\cdot J_2(0) + \frac{1}{2}\cdot J_2(2) && = 2x_2^2 - 2x_2 + \frac{9}{2} \\
h_1(2-x_1) &= x_1^2 + (2-x_1)^2 + \frac{1}{2}\cdot J_2(1) + \frac{1}{2}\cdot J_2(3) && = 2x_2^2 - 4x_2 + \frac{25}{2} \\
h_1(3-x_1) &= x_1^2 + (3-x_1)^2 + \frac{1}{2}\cdot J_2(2) + \frac{1}{2}\cdot J_2(4) && = 2x_2^2 - 6x_2 + 25 \\
h_1(4-x_1) &= x_1^2 + (4-x_1)^2 + \frac{1}{2}\cdot J_2(3) + \frac{1}{2}\cdot J_2(5) && = 2x_2^2 - 8x_2 + 43 \\
h_1(5-x_1) &= x_1^2 + (5-x_1)^2 + 1\cdot J_2(5) && = 2x_2^2 - 10x_2 + 64
\end{aligned}
\tag{24}
$$

则可得最后的结果

$$
J_1(x_1) = \begin{cases}
2x_1^2 & , u_1^* = -x_1 \text{ and } x_1 \le \frac{9}{4} \\
2x_2^2 - 2x_2 + \frac{9}{2} & , u_1^* = 1 - x_1 \text{ and } \frac{9}{4} < x_1 \le 4 \\
2x_2^2 - 4x_2 + \frac{25}{2} & , u_1^* = 2 - x_1 \text{ and } 4 < x_1 \le \frac{25}{4} \\
2x_2^2 - 6x_2 + 25 & , u_1^* = 3 - x_1 \text{ and } \frac{25}{4} < x_1 \le 9 \\
2x_2^2 - 8x_2 + 43 & , u_1^* = 4 - x_1 \text{ and } 9 < x_1 \le \frac{21}{2} \\
2x_2^2 - 10x_2 + 64 & , u_1^* = 5 - x_1 \text{ and } x_1 > \frac{21}{2}
\end{cases}
\tag{25}
$$

当 $k=0$ 时，$x_0 = 5$，则

$$
\begin{aligned}
J_0(x_0) &= \min_{u_0 \in U_0(x_0)} g_1(x_0, u_0) + \mathop{E}_{w_0}\left\{J_1(x_0+u_0+w_0)\right\} \\
&= \min_{u_0 \in U_0(x_0)} x_0^2 + u_0^2 + \mathop{E}_{w_0}\left\{J_1(x_0+u_0+w_0)\right\} \\
&= \min_{u_0 \in U_0(x_0)} 25 + u_0^2 + \mathop{E}_{w_0}\left\{J_1(5+u_0+w_0)\right\}
\end{aligned}
\tag{26}
$$

令 $h_0(u_0) = 25 + u_0^2 + E_{w_0}\left\{J_1(5+u_0+w_0)\right\}$，并列出所有的取值可得

$$
\begin{aligned}
h_0(-5) &= 25 + 25 + 1\cdot J_1(0) && = 50 \\
h_0(-4) &= 25 + 16 + \frac{1}{2}\cdot J_1(0) + \frac{1}{2}\cdot J_1(2) && = 45 \\
h_0(-3) &= 25 + 9 + \frac{1}{2}\cdot J_1(1) + \frac{1}{2}\cdot J_1(3) && = 43\frac{1}{4} \\
h_0(-2) &= 25 + 4 + \frac{1}{2}\cdot J_1(2) + \frac{1}{2}\cdot J_1(4) && = 47\frac{1}{4} \\
h_0(-1) &= 25 + 1 + \frac{1}{2}\cdot J_1(3) + \frac{1}{2}\cdot J_1(5) && = 54\frac{1}{2} \\
h_0(0) &= 25 + 0 + 1\cdot J_1(5) = 67\frac{1}{2}
\end{aligned}
\tag{27}
$$

则 $u_0^* = -3$ 且 $J_0(x_0) = 43\frac{1}{4}$

2 Exercise 2.4 (Dijkstra's Algorithm for Shortest Paths)

Consider the best-first version of the label correcting algorithm of Section 2.3.1. Here at each iteration we remove from OPEN a node that has minimum label over all nodes in OPEN.

(a) Show that each node j will enter OPEN at most once, and show that at the time it exits OPEN, its label d_j is equal to the shortest distance from s to j. *Hint:* Use the nonnegative arc length assumption to argue that in the label correcting algorithm, in order for the node i that exist OPEN to reenter, there must exist another node k in OPEN with $d_k + a_{ki} < d_i$.

(b) Show that the number of arithmetic operations required for termination is bounded by cN^2 where N is the number of nodes and c is some constant.

Solutions:

(a) If there exists a node i that exits OPEN to reenter, then there must exist another node k in OPEN with $d_k + a_{ki} < d_i$. Since k is directly connected to i, the algorithm should choose $d_k + a_{ki}$ as the value of d_i but not other values, which is contradictory to the current value of d_i.

(b) For 1th node, it requires at most $N - 1$ operations to compute the distance. For 2th node, it requires at most $N - 2$ operations to compute the distance. Similarly, for $N - 1$th node, it requires at most 1 operations to compute the distance. Therefore, to compute distance, it required at most $1 + 2 + \cdots + (N - 1) = \frac{N(N-1)}{2}$ operations. Since it also need to update and store, the value of each label, we can rewrite it as cN^2.