Homework 5 of Stochastic Processes

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1 Exercise 4.2

Show that every Markov chain with $\mathrm{M} < \infty$ states contains at least one recurrent set of states. Explaining each of the following statements is suffcient.

(a) if state i_1 is transient, then there is some other state i_2 such that $i_1 \to i_2$ and $i_2 \nrightarrow i_1$.

(b) if the i_2 of (a) is also transient, there is a third state i_3 such that $i_2 \to i_3$, $i_3 \nrightarrow i_2$; that state must satisfy $i_3 \neq i_2, i_3 \neq i_1$.

(c) Continue iteratively to repeat (b) for successive states i_3 such that $i_1, i_2, \ldots$. That is, if $i_1, \ldots, i_k$ are generated as above and are all transient, generate i_{k+1} such that $i_k \to i_{k+1}$ and $i_{k+1} \nrightarrow i_k$. Then $i_{k+1} \neq i_j$ for $1 \leq j \leq k$.

(d) Show that for some $k \leq \mathrm{M}$, k is not transient, i.e., it is recurrent, so a recurrent exists.

Solutions:

(a) According to the **Definition 4.2.5**, if there is no such state i_2, i_1 is recurrent, which is a contradiction with the fact that i_1 is transient. Also, $i_2 \neq i_1$ since otherwise $i_1 \to i_2, i_2 \to i_1$, which is recurrent.

(b) Firstly, $i_2 \to i_3, i_3 \nrightarrow i_2$ and $i_3 \neq i_2$ can be proved by (a).

Secondly, $i_1 \to i_2, i_2 \to i_3$ implies that $i_1 \to i_3$. if $i_3 \to i_1$, it implies that $i_3 \to i_1, i_1 \to i_2$ and $i_3 \to i_2$, which is a contradiction with the fact $i_3 \nrightarrow i_2$. Thus, $i_3 \nrightarrow i_1$ and $i_3 \neq i_1$ since otherwise $i_1 \to i_3, i_3 \to i_1$, which is recurrent.

(c) Firstly, the reason for which i_{k+1} exists with $i_k \to i_{k+1}, i_{k+1} \nrightarrow i_k$ and $i_{k+1} \neq i_k$ can be proved by (a).

Secondly, for $1 \leq j \leq k$, if $i_{k+1} = i_j$, it implies that $i_j \to i_{k+1}, i_{k+1} \to i_j, i_j \to i_k$ and $i_{k+1} \to i_k$, which is a contradiction with the fact $i_{k+1} \nrightarrow i_k$. Thus, $i_{k+1} \neq i_j$ for $1 \leq j \leq k$.

(d) If all states are transient, it means that $k = \mathrm{M}$ and there exists a state $i_{\mathrm{M}+1}$ with $i_{\mathrm{M}+1} \neq i_j$ for $1 \leq j \leq \mathrm{M}$, which is a contradiction with the fact there exists only M states. Therefore, there mus be some $k \leq \mathrm{M}$, i_k is not transient. Thus, a recurrent class exists.

2 Exercise 4.3

Consider a finite-state Markov chain in which some given state, say state 1, is accessible from every other state. Show that the chain has exactly one recurrent class $\mathcal{R}$ of states and state $1 \in \mathcal{R}$. (Note that the chain is then a unichain.)

Solutions:

Firstly, there is no state i such that $1 \to i$ and $i \nrightarrow 1$ since 1 is accessible from every other states. Therefore, the state 1 is recurrent.

Secondly, for any given state i, if $1 \nrightarrow i$, then i must be transient since $i \to 1$; if $1 \to i$, then $1 \leftrightarrow i$ and i must be in the same recurrent class as 1.

Thus, each state is either transient or in the same recurrent class as 1.

3 Exercise 4.8

A transition probability matrix [P] is said to be doubly stochastic if

$$\sum_j P_{ij} = 1 \quad \text{for all } i, \quad \sum_i P_{ij} = 1 \quad \text{for all j.}$$

That is, the row sum and the column sum each equal 1. If a doubly stochastic chain has M states and is ergodic (i.e., has a single class of states and aperiodic), calculate its steady-state probabilities.

Solutions:

TO-DO