Homework 1 of Stochastic Process

姓名：林奇峰　　学号：19110977

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1 Prove (1.2)-(1.12) in the textbook and describe the relationship among these formulae.

The three fundamental probability axioms are defined as follows:

1. $\Pr\{\Omega\} = 1$.
2. For every event A, $\Pr\{A\} \geq 0$.
3. The probability of the union of any sequence $A_1, A_2, \dots$ of disjoint events is given by

$$\Pr\big\{ \cup_{n=1}^{\infty} A_n \big\} = \sum_{n=1}^{\infty} \Pr\{A_n\}, \tag{1}$$

where $\sum_{n=1}^{\infty} \Pr\{A_n\}$ is shorthand for $\lim_{m\to\infty} \sum_{n=1}^{m} \Pr\{A_n\}$.

Prove the following axioms.

$$\Pr\{\emptyset\} = 0. \tag{2}$$

$$\Pr\big\{ \cup_{n=1}^{m} A_n \big\} = \sum_{n=1}^{m} \Pr\{A_n\} \qquad \text{for } A_1, \dots, A_m \text{ disjoint.} \tag{3}$$

$$\Pr\{A^c\} = 1 - \Pr\{A\} \qquad \text{for all } A. \tag{4}$$

$$\Pr\{A\} \leq \Pr\{B\} \qquad \text{for all } A \subseteq B. \tag{5}$$

$$\Pr\{A\} \leq 1 \qquad \text{for all } A. \tag{6}$$

$$\sum_{n} \Pr\{A_n\} \leq 1 \qquad \text{for } A_1, A_2, \dots \text{ disjoint.} \tag{7}$$

$$\Pr\big\{ \cup_{n=1}^{\infty} A_n \big\} = \lim_{m\to\infty} \Pr\{\cup_{n=1}^{m} A_n\} \tag{8}$$

$$\Pr\big\{ \cup_{n=1}^{\infty} A_n \big\} = \lim_{n\to\infty} \Pr\{A_n\} \qquad \text{for } A_1 \subseteq A_2 \subseteq \cdots. \tag{9}$$

$$\Pr\big\{ \cap_{n=1}^{\infty} A_n \big\} = \lim_{n\to\infty} \Pr\{A_n\} \qquad \text{for } A_1 \supseteq A_2 \supseteq \cdots. \tag{10}$$

$$\Pr\{\cup_{n=1}^{m} A_n\} = \sum_{n=1}^{m} \Pr\{B_n\} \qquad B_1 = A_1, \text{ for each } n > 1,\ B_n = A_n - \cup_{m=1}^{n-1} A_m \tag{11}$$

$$\Pr\{\cup_n A_n\} \leq \sum_{n} \Pr\{A_n\} \tag{12}$$

Solutions:

2. To verify formula (2), consider the sequence of events $\{A_1, A_2, \ldots, A_n\}$ where $A_i = \emptyset, \forall_i$. These events has no common outcome since each event has no outcome and ae therefore disjoint. According to (1),

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \Pr\{\emptyset\} = \lim_{m\to\infty} \sum_{n=1}^{m} \Pr\{A_n\} = \lim_{m\to\infty} m\Pr\{\emptyset\}$$

$\Pr\{\emptyset\} = \lim_{m\to\infty} m\Pr\{\emptyset\}$. If $\Pr\{\emptyset\} \neq 0$, the the right side will diverge and the equation does not hold. Therefore, (2) is proved.

3. To verify (3), consider the sequence of events $\{A_1, \ldots, A_m, \emptyset, \ldots\}$. These events are disjoint since $A_1, \ldots, A_m$ are disjoint and $\emptyset$ has no common outcome with other events. Then, according to (1) and (2)

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \Pr\{\cup_{n=1}^{m} A_n\} = \sum_{n=1}^{m} \Pr\{A_n\} + \sum_{n=m+1}^{\infty} \Pr\{\emptyset\} = \sum_{n=1}^{m} \Pr\{A_n\}$$

(3) is proved.

4. To verify (4), since $\Omega = A \cup A^c$, according to (3)

$$\Pr\{\Omega\} = \Pr\{A \cup A^c\} = \Pr\{A\} + \Pr\{A^c\} = 1$$

Therefore, $\Pr\{A^c\} = 1 - \Pr\{A\}$ and (4) is proved.

5. To verify (5), since $A \subseteq B$, it can be rewritten as $B = A \cup (B - A)$. According to (3)

$$\Pr\{B\} = \Pr\{A \cup (B - A)\} = \Pr\{A\} + \Pr\{(B - A)\}$$

Since $(B - A)$ is also an event and thus $\Pr\{B - A\} \geq 0$, (5) is proved.

6. To verify (6), since each event A is a subset of Ω, that is $A \subseteq \Omega$. According to (5)

$$\Pr\{A\} \leq \Pr\{\Omega\} = 1$$

(6) is proved

7. To verify (7), according to (3)

$$\sum_n \Pr\{A_n\} = \Pr\{\cup A_n\} \leq 1$$

since $\cup A_n$ is also a subset of Ω. (7) is proved.

8. For two arbitrary events A_1 and A_2, we can obtain the following formula:

$$A_1 \cup A_2 = A_1 \cup (A_2 - A_1) \quad \text{where} \quad A_2 - A_1 = A_2 \cap A_1^c$$

A_1 and $A_2 \cap A_1^c$ are disjoint. Suppose that there exists an event e inside A_1, which means that e is outside A_1^c. If A_1 and $A_2 \cap A_1^c$ are not disjoint, it means that there exists an e inside A_1^c and A_1 at the same time, which is impossible.

Defining $B_n = A_n - \cup_{m=1}^{n-1} A_m$ and $B_1 = A_1$, we can obtain that $\cup_{m=1}^{n} B_m = \cup_{m=1}^{n} A_m$ and $B_1, B_2, \ldots$ are disjoint. We can use induction to prove the result.

Firstly, it is obvious that $\cup_{m=1}^{2} A_m = A_1 \cup A_2 = A_1 \cup (A_2 - A_1) = B_1 \cup B_2 = \cup_{m=1}^{2} B_m$. Therefore, it holds for 2. Suppose that it also holds for $n-1$ and $n \geq 3$, then $\cup_{m=1}^{n} A_m = (\cup_{m=1}^{n-1} A_m) \cup A_n = (\cup_{m=1}^{n-1} A_m) \cup (A_n - (\cup_{m=1}^{n-1} A_m)) = (\cup_{m=1}^{n-1} B_m) \cup B_n = \cup_{m}^{n} B_m$. If $n = \infty$, the equation is also holds. Because if $e \in \cup_{n=1}^{\infty} A_n$, it means that for some n, e is inside A_n and therefore inside $\cup_{m=1}^{n} B_m$ and also inside $\cup_{n=1}^{\infty} B_n$.

Secondly, $(\cup_{m=1}^{n-1} A_m)$ and $(A_n - (\cup_{m=1}^{n-1} A_m))$ are disjoint, which means $\cup_{m=1}^{n-1} B_m$ and B_n are disjoint. Therefore, B_n is disjoint from $B_1, B_2, \ldots, B_{n-1}$. For any n it holds.

Then, since $B_1, B_2, \ldots$ are disjoint, according to (1) and (3)

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \Pr\{\cup_{n=1}^{\infty} B_n\} = \lim_{m\to\infty} \sum_{n=1}^{m} \Pr\{B_n\} = \lim_{m\to\infty} \Pr\{\cup_{n=1}^{m} B_n\} = \lim_{m\to\infty} \Pr\{\cup_{n=1}^{m} A_n\}$$

(8) is proved

9. To verify (9), since $A_1 \supseteq A_2 \ldots$, it results in $\cup_{n=1}^{m} A_n = A_m$ and according to (8):

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \lim_{m\to\infty} \Pr\{\cup_{n=1}^{m} A_n\} = \lim_{m\to\infty} \Pr\{A_m\}$$

(9) is proved.

10. Since $A_1 \supseteq A_1 \supseteq \ldots$, according to De Morgan's euqalities, we can obtain that

$$\cap_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} A_n^c$$

where A_n^c is the corresponding complementary set of A_n and $A_1^c \subseteq A_2^c \subseteq \ldots$ Then, in terms of (8),

$$\Pr\{\cap_{n}^{\infty} A_n\} = \Pr\{\cup_{n}^{\infty} A_n^c\} = \lim_{m\to\infty} \Pr\{\cup_{n}^{m} A_n^c\} = \lim_{m\to\infty} \Pr\{A_m\}$$

since $\cup_{n}^{m} A_n^c = \cap_{n}^{m} A_n = A_n$. (10) is proved.

11. To verify (11), from example 9 we can know that $B_1, B_n, \ldots$ are disjoint and $\cup_{n=1}^{m} A_n = \cup_{n=1}^{m} B_n$. Accroding to (3),

$$\Pr\{\cup_{n=1}^{m} A_n\} = \Pr\{\cup_{n=1}^{m} B_n\} = \sum_{n=1}^{m} \Pr\{B_n\}$$

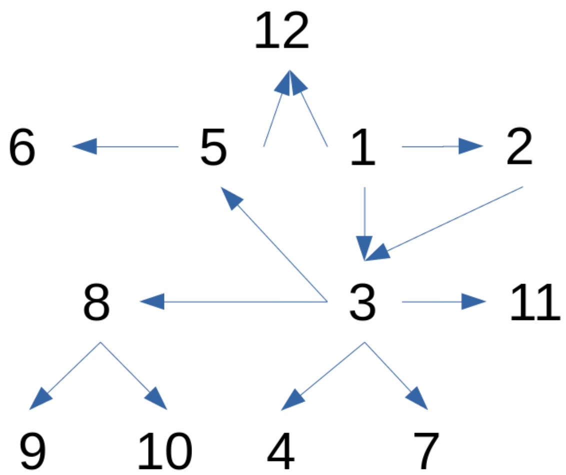
(11) is proved.

12. To verify (12), from example 9, we can know that $B_1, B_n, \ldots$ are disjoint and $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$. Since $B_n = A_n - \cup_{m=1}^{n-1} A_m$ which means $B_n \subseteq A_n$ and then according to (5),

$\Pr\{B_n\} \leq \Pr\{A_n\}$. Therefore, in terms of (1) and (5)

$$\Pr\{\cup_n A_n\} = \Pr\{\cup_n B_n\} = \sum_n \Pr\{B_n\} \leq \sum_n \Pr\{A_n\}$$

The relationship among these formulae can be showed as follow:



2 Verify the means, variances and MGFs of the rv.s given in Table 1.1 and Table 1.2 according to their PDFs/PMFs.

表 1: The PDF, mean, variance and MGF for some common continuous rv s

Name	PDF $f_X(x)$	Mean	Variance	MGF $g_X(r)$
Exponjential:	$\lambda \exp(-\lambda x); \quad x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - r}; \quad$ for $r < \lambda$
Erlang:	$\frac{\lambda^n x^{n-1} \exp(-\lambda x)}{(n-1)!}; \quad x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - r}\right)^n; \quad$ for $r < \lambda$
Gaussian:	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-a)^2}{2\sigma^2}\right)$	a	σ^2	$\exp(ra + r^2\sigma^2/2)$
Uniform	$\frac{1}{a}; \quad 0 \leq x \leq a$	$\frac{a}{2}$	$\frac{a^2}{12}$	$\frac{\exp(ra)-1}{ra}$

Solutions:

1. Exponential

$$
\begin{aligned}
E[X] &= \int_0^\infty \frac{\lambda x}{e^{\lambda x}} dx \\
&= \frac{1}{\lambda} \int_0^\infty \frac{t}{e^t} dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{\lambda} \int_0^\infty t d(\frac{1}{e^t}) \\
&= \frac{-1}{\lambda} \left(\frac{t}{e^t} \Big|_0^\infty - \int_0^\infty \frac{1}{e^t} dt \right) \\
&= \frac{-1}{\lambda} [(0-0)-1] \\
&= \frac{1}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \frac{1}{\lambda^2} \int_0^\infty \frac{t^2}{e^t} dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{\lambda^2} \int_0^\infty t^2 d(\frac{1}{e^t}) \\
&= \frac{-1}{\lambda^2} \left(\frac{t^2}{e^t} \Big|_0^\infty - 2 \int_0^\infty \frac{t}{e^t} dt \right) \\
&= \frac{-1}{\lambda^2} [(0-0)-2] \quad \text{from equations above} \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

Therefore,

$$
\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{\lambda^2}
$$

$$
\begin{aligned}
g_X(r) &= \lambda \int_0^\infty e^{(r-\lambda)x} dx \\
&= \frac{\lambda}{r-\lambda} e^{(r-\lambda)x} \Big|_0^\infty \\
&= \frac{\lambda}{r-\lambda} (0-1), \quad \text{with } r < \lambda \text{, otherwise } \lim_{x \to \infty} e^{(r-\lambda)x} = \infty \\
&= \frac{\lambda}{\lambda - r}, \quad \text{with } r < \lambda
\end{aligned}
$$

The mean, varaince and MGF of the exponential distribution function has been derived.

2. Erlang

$$\begin{aligned}
E[X] &= \int_0^\infty \frac{(\lambda x)^n}{(n-1)!e^{\lambda x}}dx \\
&= \frac{1}{(n-1)!\lambda}\int_0^\infty \frac{t^n}{e^t}dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{(n-1)!\lambda}\int_0^\infty t^n d(\frac{1}{e^t}) \\
&= \frac{-1}{(n-1)!\lambda}\left[\frac{t^n}{e^t}\bigg|_0^\infty - n\int_0^\infty \frac{t^{n-1}}{e^t}dt\right] \\
&= \frac{-1}{(n-1)!\lambda}\left[(0-0) - n\int_0^\infty \frac{t^{n-1}}{e^t}dt\right] \\
&= \frac{n}{(n-1)!\lambda}\int_0^\infty \frac{t^{n-1}}{e^t}dt \\
&= \vdots \quad \text{from the third equation above} \\
&= \frac{n}{\lambda}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_0^\infty \frac{\lambda^n x^{n+1}}{(n-1)!e^{\lambda x}}dx \\
&= \frac{1}{(n-1)!\lambda^2}\int_0^\infty \frac{t^{n+1}}{e^t}dt \\
&= \frac{1}{(n-1)!\lambda^2}\cdot(n+1)! \\
&= \frac{n^2+n}{\lambda^2}
\end{aligned}$$

Therefore,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{n^2+n}{\lambda^2} - \frac{n^2}{\lambda^2} = \frac{n}{\lambda^2}$$

$$
\begin{aligned}
g_X(r) &= \int_0^\infty \frac{\lambda^n x^{n-1} e^{rx}}{(n-1)! e^{\lambda x}} dx \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty x^{n-1} e^{(r-\lambda)x} dx \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot \int_0^\infty x^{n-1} de^{(r-\lambda)x} \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot \left[x^{n-1} e^{(r-\lambda)x} \Big|_0^\infty - \int_0^\infty e^{(r-\lambda)x} d(x^{n-1}) \right] \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot [(0-0) - (n-1) \int_0^\infty x^{n-2} e^{(r-\lambda)x} dx] \\
&\qquad\qquad \text{with } r < \lambda \text{, otherwise } \lim_{x\to\infty} x^{n-1} e^{(r-\lambda)x} = \infty \\
&= \frac{\lambda^n}{\lambda - r} \cdot \frac{1}{(n-2)! \cdot} \int_0^\infty x^{n-2} e^{(r-\lambda)x} dx \\
&= \vdots, \quad \text{from the third equation above} \\
&= \left(\frac{\lambda}{\lambda - r} \right)^n, \quad \text{with } r < \lambda
\end{aligned}
$$

The mean, varaince and MGF of the Erlang distribution function has been derived.

3. Gaussian

$$
\begin{aligned}
E[X] &= \int_{-\infty}^\infty \frac{x}{\sigma\sqrt{2\pi}} e^{\frac{-(x-a)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-(\frac{x-a}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\sqrt{2}\sigma t + a}{e^{t^2}} dt, \quad \text{with } t = \frac{x-a}{\sqrt{2}\sigma} \\
&= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^\infty \frac{\sqrt{2}\sigma t}{e^{t^2}} dt + \int_{-\infty}^\infty \frac{a}{e^{t^2}} dt \right] \\
&= \frac{a}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{1}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{\sqrt{2}\sigma t}{e^{t^2}} \text{ is an odd function} \\
&= \frac{2a}{\sqrt{\pi}} \int_0^\infty \frac{1}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{1}{e^{t^2}} \text{ is an even function} \\
&= \frac{2a}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \\
&= a
\end{aligned}
$$

Note:

$$
(\int_0^\infty e^{-x^2} dx)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy
$$

let $x = r\cos\theta$ and $y = r\sin\theta$, then

$$
\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-r^2} r dr = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi}{4}
$$

Therefore, $(\int_0^\infty e^{-x^2}dx)^2 = \frac{\pi}{4}$ and $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{\frac{-(x-a)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} x^2 e^{-(\frac{x-a}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2\sigma^2 t^2 + 2\sqrt{2}\sigma a t + a^2}{e^{t^2}} dt \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t^2}{e^{t^2}} dt + \frac{2\sqrt{2}\sigma a}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t}{e^{t^2}} dt + \frac{a^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{e^{t^2}} dt \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t^2}{e^{t^2}} dt + 0 + a^2, \quad \text{since } f(t) = \frac{t}{e^{t^2}} \text{ is an odd function and use the fact before} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t^2}{e^{t^2}} dt + a^2
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_{\infty}^{\infty} \frac{t^2}{e^{t^2}} dt &= 2\int_0^{\infty} \frac{t^2}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{t^2}{e^{t^2}} \text{ is an even function} \\
&= \int_0^{\infty} u^{\frac{1}{2}} e^{-u} du, \quad \text{with } u = t^2 \\
&= \int_0^{\infty} u^{\frac{3}{2}-1} e^{-u} du \\
&= \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t^2}{e^{t^2}} dt + a^2 \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + a^2 \\
&= \sigma^2 + a^2
\end{aligned}
$$

And

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \sigma^2 + a^2 - a^2 = \sigma^2$$

$$
\begin{aligned}
g_X(r) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{rx} e^{\frac{-(x-a)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx-(\frac{x-a}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t^2-\sqrt{2}r\sigma t-ra)} dt, \quad \text{with } t = \frac{x-a}{\sqrt{2}\sigma} \\
&= \frac{e^{ra}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t^2-\sqrt{2}r\sigma t+\frac{r^2\sigma^2}{2})+\frac{r^2\sigma^2}{2}} dt \\
&= \frac{e^{ra+\frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-\frac{\sqrt{2}}{2}r\sigma)} dt \\
&= \frac{e^{ra+\frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du, \quad \text{with } u = t - \frac{\sqrt{2}}{2} r\sigma \\
&= \frac{e^{ra+\frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \cdot \sqrt{\pi}, \quad \text{use the fact before} \\
&= e^{ra+\frac{r^2\sigma^2}{2}}
\end{aligned}
$$

Therefore, the mean, varaince and MGF of the Gaussian distribution function has been derived.

4. Uniform

$$
\begin{aligned}
E[X] &= \int_0^a \frac{x}{a} dx \\
&= \frac{1}{2a} \cdot x^2 \big|_0^a \\
&= \frac{a}{2}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \int_0^a \frac{x^2}{a} dx \\
&= \frac{1}{3a} \cdot x^3 \big|_0^a \\
&= \frac{a^2}{3}
\end{aligned}
$$

Therefore,

$$
\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}
$$

$$
\begin{aligned}
g_X(r) &= \int_0^a \frac{e^{rx}}{a} dx \\
&= \frac{1}{ra} \cdot e^{rx} \big|_0^a \\
&= \frac{e^{ra}-1}{ra}
\end{aligned}
$$

The mean, varaince and MGF of the uniform distribution function has been derived.

表 2: The PMF, mean, variance and MGF for some common discrete rv s

Name	PMF $\mathrm{p}_M(m)$	Mean	Variance	MGF $\mathrm{g}_M(r)$
Binary:	$\mathrm{p}_M(1) = p; \mathrm{p}_M(0) = 1-p$	p	$p(1-p)$	$1-p+pe^r$
Binomial	$\binom{n}{m} p^m (1-p)^{n-m}; 0 \leq m \leq n$	np	$np(1-p)$	$[1-p+pe^r]^n$
Geometric	$p(1-p)^{m-1}; m \geq 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^r}{1-(1-p)e^r}$; for $r < \ln \frac{1}{1-p}$
Poisson:	$\frac{\lambda \exp(-\lambda)}{n!}; n \geq 0$	λ	λ	$\exp[\lambda(e^r - 1)]$

Solutions:

1. Binary

$$\begin{aligned} E[X] &= 1 \cdot p + 0 \cdot (1-p) \\ &= p \end{aligned}$$

$$\begin{aligned} E[X^2] &= 1^2 \cdot p + 0^2 \cdot (1-p) \\ &= p \end{aligned}$$

Then,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = p(1-p)$$

$$\begin{aligned} g_X(r) &= e^{r \cdot 1} * p + e^{r \cdot 0}(1-p) \\ &= 1 - p + pe^r \end{aligned}$$

2. Binomial

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{[(n-1)-(k-1)]!(k-1)!} p^{k-1} (1-p)^{[(n-1)-(k-1)]} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{[(n-1)-(k-1)]} \\
&= np \sum_{v=0}^{n-1} \binom{n-1}{v} p^{v} (1-p)^{(n-1)-v}, \quad \text{with } v = k-1 \\
&= np \sum_{v=0}^{u} \binom{u}{v} p^{v} (1-p)^{u-v}, \quad \text{with } u = n-1 \\
&= np(p+1-p)^u, \quad \text{according to the binomial theorem} \\
&= np
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k^2 \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{(n-k!)(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{[(n-1)-(k-1)]!(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{k=1}^{n} k \cdot \binom{n-1}{k-1} p^{k} - 1 (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{v=0}^{u} (v+1) \binom{u}{v} p^{v} (1-p)^{u-v}, \quad \text{with } u = n-1, v = k-1 \\
&= np \left[\sum_{v=0}^{u} v \binom{u}{v} p^{v} (1-p)^{u-v} + \sum_{v=0}^{u} \binom{u}{v} p^{v} (1-p)^{u-v} \right] \\
&= np(up + (p+1-p)^u), \quad \text{use the facts before} \\
&= np[(n-1)p+1] \\
&= n^2p^2 + np(1-p)
\end{aligned}
$$

Then,

$$
\sigma_X^2 = E[X^2] - (E[X])^2 = n^2p^2 + np(1-p) - n^2p^2 = np(1-p)
$$

$$
\begin{aligned}
g_X(r) &= \sum_{k=0}^{n} e^{rk} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (pe^r)^k (1-p)^{n-k} \\
&= (1-p+pe^r)^n, \quad \text{according to the binomial theorem}
\end{aligned}
$$

3. Geometric let $q = 1-p$, then

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kpq^{k-1} \\
&= p\sum_{k=1}^{\infty} kq^{k-1} \\
&= p\frac{d(\sum_{k=1}^{\infty} q^k)}{dq} \\
&= p\frac{d(\sum_{t=0}^{\infty} q^t)}{dq} \\
&= p\frac{d}{dq}(\frac{1}{1-q}) \\
&= p\frac{1}{(1-q)^2} \\
&= \frac{1}{p}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \sum_{k=1}^{\infty} k^2pq^{k-1} \\
&= p\Big[\sum_{k=1}^{\infty} k(k-1)q^{k-1} + \sum_{k=1}^{\infty} kq^{k-1}\Big] \\
&= p\Big[\sum_{k=1}^{\infty} k(k-1)q^{k-1} + \frac{1}{p^2}\Big], \quad \text{from the fact before}
\end{aligned}
$$

Since,

$$
\begin{aligned}
\sum_{k=1}^{\infty} k(k-1)q^{k-1} &= q\sum_{k=1}^{\infty} k(k-1)q^{k-2} \\
&= q\frac{d^2}{dq^2}\sum_{k=0}^{\infty} q^k \\
&= \frac{2q}{p^3}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[X^2] &= p(\frac{2q}{p^3} + \frac{1}{p^2}) \\
&= \frac{2q}{p^2} + \frac{1}{p} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

And

$$
\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{2-p}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}
$$

$$
\begin{aligned}
g_M(r) &= \sum_{k=1}^{\infty} pe^{rk}(1-p)^{k-1} \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty} e^{rk}(1-p)^k \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty}[(1-p)e^r]^k \\
&= \frac{p}{1-p} \cdot \lim_{n\to\infty} \frac{(1-p)e^r\big(1-[(1-p)e^r)]\big)^n}{1-\ 1-(1-p)e^r} \\
&= \frac{p}{1-p}\frac{(1-p)e^r}{1-(1-p)e^r}, \quad \text{with } (1-p)e^r < 1 \text{ to be convergent} \\
&= \frac{pe^r}{1-(1-p)e^r}, \quad \text{with } r < \ln\frac{1}{1-p}
\end{aligned}
$$

4. Poisson

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!e^{\lambda}} \\
&= \frac{1}{e^{\lambda}}\sum_{k=1}^{\infty}\frac{k\lambda^k}{k!} \\
&= \frac{\lambda}{e^{\lambda}}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\
&= \frac{\lambda}{e^{\lambda}}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} \\
&= \frac{\lambda}{e^{\lambda}} \cdot e^{\lambda} \\
&= \lambda
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k! e^{\lambda}} \\
&= \frac{1}{e^{\lambda}} \sum_{k=1}^{\infty} \frac{k^2 \lambda^k}{k!} \\
&= \frac{\lambda}{e^{\lambda}} \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} \\
&= \frac{\lambda}{e^{\lambda}} \sum_{n=0}^{\infty} \frac{(n+1) \lambda^n}{n!} \\
&= \frac{\lambda}{e^{\lambda}} \left[\sum_{n=0}^{\infty} \frac{n \lambda^n}{n!} + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \right] \\
&= \frac{\lambda}{e^{\lambda}} [\lambda e^{\lambda} + e^{\lambda}], \quad \text{use the facts before} \\
&= \lambda^2 + \lambda
\end{aligned}
$$

Then,

$$
\sigma_X^2 = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
$$

$$
\begin{aligned}
g_X(r) &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{rk}}{k! e^{\lambda}} \\
&= \frac{1}{e^{\lambda}} \sum_{k=0}^{\infty} \frac{(\lambda e^r)^k}{k!} \\
&= \frac{1}{e^{\lambda}} \cdot e^{\lambda e^r} \\
&= \exp[\lambda(e^r - 1)]
\end{aligned}
$$