

Homework 1 of Stochastic Process

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1 Prove (1.2)-(1.12) in the textbook and describe the relationship among these formulae.

The three fundamental probability axioms are defined as follows:

1. $\Pr\{\Omega\} = 1$.
2. For every event A , $\Pr\{A\} \geq 0$.
3. The probability of the union of any sequence A_1, A_2, \dots of disjoint events is given by

$$\Pr\left\{\bigcup_{n=1}^{\infty} A_n\right\} = \sum_{n=1}^{\infty} \Pr\{A_n\}, \quad (1)$$

where $\sum_{n=1}^{\infty} \Pr\{A_n\}$ is shorthand for $\lim_{m \rightarrow \infty} \sum_{n=1}^m \Pr\{A_n\}$.

Prove the following axioms.

$$\Pr\{\emptyset\} = 0. \quad (2)$$

$$\Pr\left\{\bigcup_{n=1}^m A_n\right\} = \sum_{n=1}^m \Pr\{A_n\} \quad \text{for } A_1, \dots, A_m \text{ disjoint.} \quad (3)$$

$$\Pr\{A^c\} = 1 - \Pr\{A\} \quad \text{for all } A. \quad (4)$$

$$\Pr\{A\} \leq \Pr\{B\} \quad \text{for all } A \subseteq B. \quad (5)$$

$$\Pr\{A\} \leq 1 \quad \text{for all } A. \quad (6)$$

$$\sum_n \Pr\{A_n\} \leq 1 \quad \text{for } A_1, A_2, \dots \text{ disjoint.} \quad (7)$$

$$\Pr\left\{\bigcup_{n=1}^{\infty} A_n\right\} = \lim_{m \rightarrow \infty} \Pr\left\{\bigcup_{n=1}^m A_n\right\} \quad (8)$$

$$\Pr\left\{\bigcup_{n=1}^{\infty} A_n\right\} = \lim_{n \rightarrow \infty} \Pr\{A_n\} \quad \text{for } A_1 \subseteq A_2 \subseteq \dots \quad (9)$$

$$\Pr\left\{\bigcap_{n=1}^{\infty} A_n\right\} = \lim_{n \rightarrow \infty} \Pr\{A_n\} \quad \text{for } A_1 \supseteq A_2 \supseteq \dots \quad (10)$$

$$\Pr\left\{\bigcup_{n=1}^m A_n\right\} = \sum_{n=1}^m \Pr\{B_n\} \quad B_1 = A_1, \text{ for each } n > 1, B_n = A_n - \bigcup_{m=1}^{n-1} A_m \quad (11)$$

$$\Pr\left\{\bigcup_n A_n\right\} \leq \sum_n \Pr\{A_n\} \quad (12)$$

Solutions:

2. To verify formula (2), consider the sequence of events $\{A_1, A_2, \dots, A_n\}$ where $A_i = \emptyset, \forall_i$. These events has no common outcome since each event has no outcome and ae therefore disjoint. According to (1),

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \Pr\{\emptyset\} = \lim_{m \rightarrow \infty} \sum_{n=1}^m \Pr\{A_n\} = \lim_{m \rightarrow \infty} m \Pr\{\emptyset\}$$

$\Pr\{\emptyset\} = \lim_{m \rightarrow \infty} m \Pr\{\emptyset\}$. If $\Pr\{\emptyset\} \neq 0$, the the right side will diverge and the equation does not hold. Therefore, (2) is proved.

3. To verify (3), consider the sequence of events $\{A_1, \dots, A_m, \emptyset, \dots\}$. These events are disjoint since A_1, \dots, A_m are disjoint and \emptyset has no common outcome with other events. Then, according to (1) and (2)

$$\Pr\{\cup_{n=1}^{\infty} A_n\} = \Pr\{\cup_{n=1}^m A_n\} = \sum_{n=1}^m \Pr\{A_n\} + \sum_{n=m+1}^{\infty} \Pr\{\emptyset\} = \sum_{n=1}^m \Pr\{A_n\}$$

(3) is proved.

4. To verify (4), since $\Omega = A \cup A^c$, according to (3)

$$\Pr\{\Omega\} = \Pr\{A \cup A^c\} = \Pr\{A\} + \Pr\{A^c\} = 1$$

Therefore, $\Pr\{A^c\} = 1 - \Pr\{A\}$ and (4) is proved.

5. To verify (5), since $A \subseteq B$, it can be rewritten as $B = A \cup (B - A)$. According to (3)

$$\Pr\{B\} = \Pr\{A \cup (B - A)\} = \Pr\{A\} + \Pr\{(B - A)\}$$

Since $(B - A)$ is also an event and thus $\Pr\{B - A\} \geq 0$, (5) is proved.

6. To verify (6), since each event A is a subset of Ω , that is $A \subseteq \Omega$. According to (5)

$$\Pr\{A\} \leq \Pr\{\Omega\} = 1$$

(6) is proved

7. To verify (7), according to (3)

$$\sum_n \Pr\{A_n\} = \Pr\{\cup A_n\} \leq 1$$

since $\cup A_n$ is also a subset of Ω . (7) is proved.

8. For two arbitrary events A_1 and A_2 , we can obtain the following formula:

$$A_1 \cup A_2 = A_1 \cup (A_2 - A_1) \quad \text{where} \quad A_2 - A_1 = A_2 \cap A_1^c$$

A_1 and $A_2 \cap A_1^c$ are disjoint. Suppose that there exists an event e inside A_1 , which means that e is outside A_1^c . If A_1 and $A_2 \cap A_1^c$ are not disjoint, it means that there exists an e inside A_1^c and A_1 at the same time, which is impossible.

Defining $B_n = A_n - \cup_{m=1}^{n-1} A_m$ and $B_1 = A_1$, we can obtain that $\cup_{m=1}^n B_m = \cup_{m=1}^n A_m$ and B_1, B_2, \dots are disjoint. We can use induction to prove the result.

Firstly, it is obvious that $\cup_{m=1}^2 A_m = A_1 \cup A_2 = A_1 \cup (A_2 - A_1) = B_1 \cup B_2 = \cup_{m=1}^2 B_m$. Therefore, it holds for 2. Suppose that it also holds for $n-1$ and $n \geq 3$, then $\cup_{m=1}^n A_m = (\cup_{m=1}^{n-1} A_m) \cup A_n = (\cup_{m=1}^{n-1} A_m) \cup (A_n - (\cup_{m=1}^{n-1} A_m)) = (\cup_{m=1}^{n-1} B_m) \cup B_n = \cup_{m=1}^n B_m$. If $n = \infty$, the equation is also holds. Because if $e \in \cup_{n=1}^\infty A_n$, it means that for some n , e is inside A_n and therefore inside $\cup_{m=1}^n B_m$ and also inside $\cup_{n=1}^\infty B_n$.

Secondly, $(\cup_{m=1}^{n-1} A_m)$ and $(A_n - (\cup_{m=1}^{n-1} A_m))$ are disjoint, which means $\cup_{m=1}^{n-1} B_m$ and B_n are disjoint. Therefore, B_n is disjoint from B_1, B_2, \dots, B_{n-1} . For any n it holds.

Then, since B_1, B_2, \dots are disjoint, according to (1) and (3)

$$\Pr\{\cup_{n=1}^\infty A_n\} = \Pr\{\cup_{n=1}^\infty B_n\} = \lim_{m \rightarrow \infty} \sum_{n=1}^m \Pr\{B_n\} = \lim_{m \rightarrow \infty} \Pr\{\cup_{n=1}^m B_n\} = \lim_{m \rightarrow \infty} \Pr\{\cup_{n=1}^m A_n\}$$

(8) is proved

9. To verify (9), since $A_1 \supseteq A_2 \dots$, it results in $\cup_{n=1}^m A_n = A_m$ and according to (8):

$$\Pr\{\cup_{n=1}^\infty A_n\} = \lim_{m \rightarrow \infty} \Pr\{\cup_{n=1}^m A_n\} = \lim_{m \rightarrow \infty} \Pr\{A_m\}$$

(9) is proved.

10. Since $A_1 \supseteq A_1 \supseteq \dots$, according to De Morgan's euqalities, we can obtain that

$$\cap_{n=1}^\infty A_n = \cup_{n=1}^\infty A_n^c$$

where A_n^c is the corresponding complementary set of A_n and $A_1^c \subseteq A_2^c \subseteq \dots$. Then, in terms of (8),

$$\Pr\{\cap_{n=1}^\infty A_n\} = \Pr\{\cup_{n=1}^\infty A_n^c\} = \lim_{m \rightarrow \infty} \Pr\{\cup_{n=1}^m A_n^c\} = \lim_{m \rightarrow \infty} \Pr\{A_m^c\}$$

since $\cup_{n=1}^m A_n^c = \cap_{n=1}^m A_n = A_n$. (10) is proved.

11. To verify (11), from example 9 we can know that B_1, B_n, \dots are disjoint and $\cup_{n=1}^m A_n = \cup_{n=1}^m B_n$. Accroding to (3),

$$\Pr\{\cup_{n=1}^m A_n\} = \Pr\{\cup_{n=1}^m B_n\} = \sum_{n=1}^m \Pr\{B_n\}$$

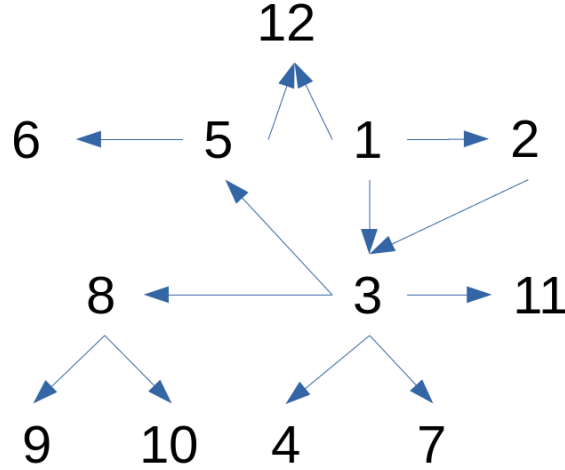
(11) is proved.

12. To verify (12), from example 9, we can know that B_1, B_n, \dots are disjoint and $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$. Since $B_n = A_n - \cup_{m=1}^{n-1} A_m$ which means $B_n \subseteq A_n$ and then according to (5),

$\Pr\{B_n\} \leq \Pr\{A_n\}$. Therefore, in terms of (1) and (5)

$$\Pr\{\cup_n A_n\} = \Pr\{\cup_n B_n\} = \sum_n \Pr\{B_n\} \leq \sum_n \Pr\{A_n\}$$

The relationship among these formulae can be showed as follow:



- 2 Verify the means, variances and MGFs of the rv.s given in Table 1.1 and Table 1.2 according to their PDFs/PMFs.

表 1: The PDF, mean, variance and MGF for some common continuous rv s

Name	PDF $f_X(x)$	Mean	Variance	MGF $g_X(r)$
Exponential:	$\lambda \exp(-\lambda x); \quad x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-r}; \quad \text{for } r < \lambda$
Erlang:	$\frac{\lambda^n x^{n-1} \exp(-\lambda x)}{(n-1)!}; \quad x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-r}\right)^n; \quad \text{for } r < \lambda$
Gaussian:	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-a)^2}{2\sigma^2}\right)$	a	σ^2	$\exp(ra + r^2\sigma^2/2)$
Uniform	$\frac{1}{a}; \quad 0 \leq x \leq a$	$\frac{a}{2}$	$\frac{a^2}{12}$	$\frac{\exp(ra)-1}{ra}$

Solutions:

1. Exponential

$$\begin{aligned}
E[X] &= \int_0^{\infty} \frac{\lambda x}{e^{\lambda x}} dx \\
&= \frac{1}{\lambda} \int_0^{\infty} \frac{t}{e^t} dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{\lambda} \int_0^{\infty} t d\left(\frac{1}{e^t}\right) \\
&= \frac{-1}{\lambda} \left(\frac{t}{e^t} \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{e^t} dt \right) \\
&= \frac{-1}{\lambda} [(0 - 0) - 1] \\
&= \frac{1}{\lambda}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \frac{1}{\lambda^2} \int_0^{\infty} \frac{t^2}{e^t} dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{\lambda^2} \int_0^{\infty} t^2 d\left(\frac{1}{e^t}\right) \\
&= \frac{-1}{\lambda^2} \left(\frac{t^2}{e^t} \Big|_0^{\infty} - 2 \int_0^{\infty} \frac{t}{e^t} dt \right) \\
&= \frac{-1}{\lambda^2} [(0 - 0) - 2] \quad \text{from equations above} \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

Therefore,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{\lambda^2}$$

$$\begin{aligned}
g_M(r) &= \int_{-\infty}^{\infty} \frac{\lambda e^{rx}}{e^{tx}} dx \\
&= \lambda \int_0^{\infty} e^{(r-\lambda)x} dx \\
&= \frac{\lambda}{r-\lambda} e^{(r-\lambda)x} \Big|_0^{\infty} \\
&= \frac{\lambda}{r-\lambda} (0 - 1), \quad \text{with } r < \lambda, \text{ otherwise } \lim_{x \rightarrow \infty} e^{(r-\lambda)x} = \infty \\
&= \frac{\lambda}{\lambda - r}, \quad \text{with } r < \lambda
\end{aligned}$$

The mean, variance and MGF of the exponential distribution function has been derived.

2. Erlang

$$\begin{aligned}
E[X] &= \int_0^\infty \frac{(\lambda x)^n}{(n-1)!e^{\lambda x}} dx \\
&= \frac{1}{(n-1)!\lambda} \int_0^\infty \frac{t^n}{e^t} dt, \quad \text{with } t = \lambda x \\
&= \frac{-1}{(n-1)!\lambda} \int_0^\infty t^n d\left(\frac{1}{e^t}\right) \\
&= \frac{-1}{(n-1)!\lambda} \left[\frac{t^n}{e^t} \Big|_0^\infty - n \int_0^\infty \frac{t^{n-1}}{e^t} dt \right] \\
&= \frac{-1}{(n-1)!\lambda} \left[(0-0) - n \int_0^\infty \frac{t^{n-1}}{e^t} dt \right] \\
&= \frac{n}{(n-1)!\lambda} \int_0^\infty \frac{t^{n-1}}{e^t} dt \\
&= \vdots \quad \text{from the third equation above} \\
&= \frac{n}{\lambda}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_0^\infty \frac{\lambda^n x^{n+1}}{(n-1)!e^{\lambda x}} dx \\
&= \frac{1}{(n-1)!\lambda^2} \int_0^\infty \frac{t^{n+1}}{e^t} dt \\
&= \frac{1}{(n-1)!\lambda^2} \cdot (n+1)! \\
&= \frac{n^2 + n}{\lambda^2}
\end{aligned}$$

Therefore,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{n^2 + n}{\lambda^2} - \frac{n^2}{\lambda^2} = \frac{n}{\lambda^2}$$

$$\begin{aligned}
g_M(r) &= \int_0^\infty \frac{\lambda^n x^{n-1} e^{rx}}{(n-1)! e^{\lambda x}} dx \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty x^{n-1} e^{(r-\lambda)x} dx \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot \int_0^\infty x^{n-1} d e^{(r-\lambda)x} \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot \left[x^{n-1} e^{(r-\lambda)x} \Big|_0^\infty - \int_0^\infty e^{(r-\lambda)x} d(x^{n-1}) \right] \\
&= \frac{\lambda^n}{(n-1)!} \cdot \frac{1}{r-\lambda} \cdot [(0-0) - (n-1) \int_0^\infty x^{n-2} e^{(r-\lambda)x} dx] \\
&\quad \text{with } r < \lambda, \text{ otherwise } \lim_{x \rightarrow \infty} x^{n-1} e^{(r-\lambda)x} = \infty \\
&= \frac{\lambda^n}{\lambda-r} \cdot \frac{1}{(n-2)!} \int_0^\infty x^{n-2} e^{(r-\lambda)x} dx \\
&= \vdots, \quad \text{from the third equation above} \\
&= \left(\frac{\lambda}{\lambda-r} \right)^n, \quad \text{with } r < \lambda
\end{aligned}$$

The mean, varaince and MGF of the Erlang distribution function has been derived.

3. Gaussian

$$\begin{aligned}
E[X] &= \int_{-\infty}^\infty \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-\left(\frac{x-a}{\sqrt{2}\sigma}\right)^2} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\sqrt{2}\sigma t + a}{e^{t^2}} dt, \quad \text{with } t = \frac{x-a}{\sqrt{2}\sigma} \\
&= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^\infty \frac{\sqrt{2}\sigma t}{e^{t^2}} dt + \int_{-\infty}^\infty \frac{a}{e^{t^2}} dt \right] \\
&= \frac{a}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{1}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{\sqrt{2}\sigma t}{e^{t^2}} \text{ is an odd function} \\
&= \frac{2a}{\sqrt{\pi}} \int_0^\infty \frac{1}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{1}{e^{t^2}} \text{ is an even function} \\
&= \frac{2a}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \\
&= a
\end{aligned}$$

Note:

$$\left(\int_0^\infty e^{-x^2} dx \right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

let $x = r \cos \theta$ and $y = r \sin \theta$, then

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-r^2} r dr = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi}{4}$$

Therefore, $(\int_0^\infty e^{-x^2} dx)^2 = \frac{\pi}{4}$ and $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^\infty \frac{x^2}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty x^2 e^{-\left(\frac{x-a}{\sqrt{2}\sigma}\right)^2} dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{2\sigma^2 t^2 + 2\sqrt{2}\sigma a t + a^2}{e^{t^2}} dt \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{t^2}{e^{t^2}} dt + \frac{2\sqrt{2}\sigma a}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{t}{e^{t^2}} dt + \frac{a^2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{1}{e^{t^2}} dt \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{t^2}{e^{t^2}} dt + 0 + a^2, \quad \text{since } f(t) = \frac{t}{e^{t^2}} \text{ is an odd function and use the fact before} \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{t^2}{e^{t^2}} dt + a^2 \end{aligned}$$

Then,

$$\begin{aligned} \int_{-\infty}^\infty \frac{t^2}{e^{t^2}} dt &= 2 \int_0^\infty \frac{t^2}{e^{t^2}} dt, \quad \text{since } f(t) = \frac{t^2}{e^{t^2}} \text{ is an even function} \\ &= \int_0^\infty u^{\frac{1}{2}} e^{-u} du, \quad \text{with } u = t^2 \\ &= \int_0^\infty u^{\frac{3}{2}-1} e^{-u} du \\ &= \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{t^2}{e^{t^2}} dt + a^2 \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + a^2 \\ &= \sigma^2 + a^2 \end{aligned}$$

And

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \sigma^2 + a^2 - a^2 = \sigma^2$$

$$\begin{aligned}
g_M(r) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{rx} e^{-\frac{(x-a)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx - (\frac{x-a}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t^2 - \sqrt{2}rst - ra)} dt, \quad \text{with } t = \frac{x-a}{\sqrt{2}\sigma} \\
&= \frac{e^{ra}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t^2 - \sqrt{2}rst + \frac{r^2\sigma^2}{2}) + \frac{r^2\sigma^2}{2}} dt \\
&= \frac{e^{ra + \frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t - \frac{\sqrt{2}}{2}r\sigma)^2} dt \\
&= \frac{e^{ra + \frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du, \quad \text{with } u = t - \frac{\sqrt{2}}{2}r\sigma \\
&= \frac{e^{ra + \frac{r^2\sigma^2}{2}}}{\sqrt{\pi}} \cdot \sqrt{\pi}, \quad \text{use the fact before} \\
&= e^{ra + \frac{r^2\sigma^2}{2}}
\end{aligned}$$

Therefore, the mean, variance and MGF of the Gaussian distribution function has been derived.

4. Uniform

$$\begin{aligned}
E[X] &= \int_0^a \frac{x}{a} dx \\
&= \frac{1}{2a} \cdot x^2 \Big|_0^a \\
&= \frac{a}{2} \\
E[X^2] &= \int_0^a \frac{x^2}{a} dx \\
&= \frac{1}{3a} \cdot x^3 \Big|_0^a \\
&= \frac{a^2}{3}
\end{aligned}$$

Therefore,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

$$\begin{aligned}
g_M(r) &= \int_0^a \frac{e^{rx}}{a} dx \\
&= \frac{1}{ra} \cdot e^{rx} \Big|_0^a \\
&= \frac{e^{ra} - 1}{ra}
\end{aligned}$$

The mean, variance and MGF of the uniform distribution function has been derived.

表 2: The PMF, mean, variance and MGF for some common discrete rv s

Name	PMF $p_M(m)$	Mean	Variance	MGF $g_M(r)$
Binary:	$p_M(1) = p; p_M(0) = 1 - p$	p	$p(1 - p)$	$1 - p + pe^r$
Binomial	$\binom{n}{m} p^m (1 - p)^{n-m}; 0 \leq m \leq n$	np	$np(1 - p)$	$[1 - p + pe^r]^n$
Geometric	$p(1 - p)^{m-1}; m \geq 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^r}{1-(1-p)e^r}; \text{for } r < \ln \frac{1}{1-p}$
Poisson:	$\frac{\lambda \exp(-\lambda)}{n!}; n \geq 0$	λ	λ	$\exp[\lambda(e^r - 1)]$

Solutions:

1. Binary

$$\begin{aligned} E[X] &= 1 \cdot p + 0 \cdot (1 - p) \\ &= p \end{aligned}$$

$$\begin{aligned} E[X^2] &= 1^2 \cdot p + 0^2 \cdot (1 - p) \\ &= p \end{aligned}$$

Then,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = p(1 - p)$$

$$\begin{aligned} g_M(r) &= e^{r \cdot 1} \cdot p + e^{r \cdot 0} (1 - p) \\ &= 1 - p + pe^r \end{aligned}$$

2. Binomial

$$\begin{aligned}
E[X] &= \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^n k \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^n \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=1}^n \frac{(n-1)!}{[(n-1)-(k-1)]!(k-1)!} p^{k-1} (1-p)^{[(n-1)-(k-1)]} \\
&= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{[(n-1)-(k-1)]} \\
&= np \sum_{v=0}^{n-1} \binom{n-1}{v} p^v (1-p)^{(n-1)-v}, \quad \text{with } v = k-1 \\
&= np \sum_{v=0}^u \binom{u}{v} p^v (1-p)^{u-v}, \quad \text{with } u = n-1 \\
&= np(p+1-p)^u, \quad \text{according to the binomial theorem} \\
&= np
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^n k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^n k^2 \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^n k \cdot \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=1}^n k \cdot \frac{(n-1)!}{[(n-1)-(k-1)]!(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{k=1}^n k \cdot \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{v=0}^u (v+1) \binom{u}{v} p^v (1-p)^{u-v}, \quad \text{with } u = n-1, v = k-1 \\
&= np \left[\sum_{v=0}^u v \binom{u}{v} p^v (1-p)^{u-v} + \sum_{v=0}^u \binom{u}{v} p^v (1-p)^{u-v} \right] \\
&= np(up + (p+1-p)^u), \quad \text{use the facts before} \\
&= np[(n-1)p + 1] \\
&= n^2 p^2 + np(1-p)
\end{aligned}$$

Then,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = n^2 p^2 + np(1-p) - n^2 p^2 = np(1-p)$$

$$\begin{aligned} g_M(r) &= \sum_{k=0}^n e^{rk} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} (pe^r)^k (1-p)^{n-k} \\ &= (1-p + pe^r)^n, \quad \text{according to the binomial theorem} \end{aligned}$$

3. Geometric let $q = 1 - p$, then

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} k p q^{k-1} \\ &= p \sum_{k=1}^{\infty} k q^{k-1} \\ &= p \frac{d(\sum_{k=1}^{\infty} q^k)}{dq} \\ &= p \frac{d(\sum_{t=0}^{\infty} q^t)}{dq} \\ &= p \frac{d}{dq} \left(\frac{1}{1-q} \right) \\ &= p \frac{1}{(1-q)^2} \\ &= \frac{1}{p} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \sum_{k=1}^{\infty} k^2 p q^{k-1} \\ &= p \left[\sum_{k=1}^{\infty} k(k-1) q^{k-1} + \sum_{k=1}^{\infty} k q^{k-1} \right] \\ &= p \left[\sum_{k=1}^{\infty} k(k-1) q^{k-1} + \frac{1}{p^2} \right], \quad \text{from the fact before} \end{aligned}$$

Since,

$$\begin{aligned} \sum_{k=1}^{\infty} k(k-1) q^{k-1} &= q \sum_{k=1}^{\infty} k(k-1) q^{k-2} \\ &= q \frac{d^2}{dq^2} \sum_{k=0}^{\infty} q^k \\ &= \frac{2q}{p^3} \end{aligned}$$

Therefore,

$$\begin{aligned} E[X^2] &= p\left(\frac{2q}{p^3} + \frac{1}{p^2}\right) \\ &= \frac{2q}{p^2} + \frac{1}{p} \\ &= \frac{2-p}{p^2} \end{aligned}$$

And

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{2-p}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

$$\begin{aligned} g_M(r) &= \sum_{k=1}^{\infty} p e^{rk} (1-p)^{k-1} \\ &= \frac{p}{1-p} \sum_{k=1}^{\infty} e^{rk} (1-p)^k \\ &= \frac{p}{1-p} \sum_{k=1}^{\infty} [(1-p)e^r]^k \\ &= \frac{p}{1-p} \cdot \lim_{n \rightarrow \infty} \frac{(1-p)e^r (1 - [(1-p)e^r])^n}{1 - 1 - (1-p)e^r} \\ &= \frac{p}{1-p} \frac{(1-p)e^r}{1 - (1-p)e^r}, \quad \text{with } (1-p)e^r < 1 \text{ to be convergent} \\ &= \frac{pe^r}{1 - (1-p)e^r}, \quad \text{with } r < \ln \frac{1}{1-p} \end{aligned}$$

4. Poisson

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!e^\lambda} \\ &= \frac{1}{e^\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{k!} \\ &= \frac{\lambda}{e^\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \frac{\lambda}{e^\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\ &= \frac{\lambda}{e^\lambda} \cdot e^\lambda \\ &= \lambda \end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k! e^\lambda} \\
&= \frac{1}{e^\lambda} \sum_{k=1}^{\infty} \frac{k^2 \lambda^k}{k!} \\
&= \frac{\lambda}{e^\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} \\
&= \frac{\lambda}{e^\lambda} \sum_{n=0}^{\infty} \frac{(n+1) \lambda^n}{n!} \\
&= \frac{\lambda}{e^\lambda} \left[\sum_{n=0}^{\infty} \frac{n \lambda^n}{n!} + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \right] \\
&= \frac{\lambda}{e^\lambda} [\lambda e^\lambda + e^\lambda], \quad \text{use the facts before} \\
&= \lambda^2 + \lambda
\end{aligned}$$

Then,

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\begin{aligned}
g_M(r) &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{rk}}{k! e^\lambda} \\
&= \frac{1}{e^\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^r)^k}{k!} \\
&= \frac{1}{e^\lambda} \cdot e^{\lambda e^r} \\
&= \exp[\lambda(e^r - 1)]
\end{aligned}$$