Homework 10 of Stochastic Processes

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1 Exercise 9.6

Define $\gamma(r)$ as $\ln[\mathrm{g}(r)]$ where $\mathrm{g}(r) = \mathrm{E}[\exp(rX)]$. Assume that X is discrete with possible outcomes $\{a_i; i \geq 1\}$, let p_i denote $\Pr\{X = a_i\}$, and assume that $\mathrm{g}(r)$ exists in some open interval $(r\_, r_+)$ containing $r = 0$. For any given $r, r\_ < r < r_+$, define a rv X_r with the same set of possible outcomes $\{a_i; i \geq 1\}$ as X, but with a PMF $q_i = \Pr\{X_r = a_i\} = p_i \exp[a_i r - \gamma(r)]$. Note that X_r is not a function of X and is not even to be viewed as in the same probability space as X; it is of interest simply because of the behavior of its defined probability mass function. It is called a tilted rv relative to X, and this exercise, along with Exercise 9.11, will justify our interest in it.

(a) Verify that $\sum_i q_i = 1$.

(b) Verify that $\mathrm{E}[X_r] = \sum_i a_i q_i$ is equal to $\gamma'(r)$.

(c) Verify that $\mathrm{VAR}[X_r] = \sum_i a_i^2 q_i - (\mathrm{E}[X_r])^2$ is equal to $\gamma''(r)$.

(d) Argue that $\gamma''(r) \geq 0$ for all r such that $\mathrm{g}(r)$ exists, and that $\gamma''(r) > 0$ if $\gamma''(0) > 0$.

Solutions:

(a)

$$\sum_i q_i = \sum_i p_i \exp[a_i r - \gamma(r)] = \sum_i \frac{p_i \exp[a_i r]}{\mathrm{g}(r)} = \frac{\sum_i p_i \exp[a_i r]}{\mathrm{g}(r)} = \frac{\mathrm{g}(r)}{\mathrm{g}(r)} = 1$$

(b)

$$\begin{aligned}
\mathrm{E}[X_r] &= \sum_i a_i q_i \\
&= \sum_i a_i p_i \exp[a_i r - \gamma(r)] \\
&= \frac{\sum_i p_i a_i \exp[a_i r]}{\mathrm{g}(r)} \\
&= \frac{1}{\mathrm{g}(r)} \frac{d \sum_i p_i \exp[a_i r]}{dr} \\
&= \frac{\mathrm{g}'(r)}{\mathrm{g}(r)} = \gamma'(r)
\end{aligned}$$

(c) Since

$$
\begin{aligned}
\sum_i a_i^2 q_i &= \sum_i a_i^2 p_i \exp[a_i r - \gamma(r)] \\
&= \frac{1}{\mathrm{g}(r)} \sum_i p_i a_i^2 \exp[a_i r] \\
&= \frac{1}{\mathrm{g}(r)} \frac{d \sum_i p_i a_i \exp[a_i r]}{dr} \\
&= \frac{1}{\mathrm{g}(r)} \frac{d\mathrm{g}'(r)}{dr} \\
&= \frac{\mathrm{g}''(r)}{\mathrm{g}(r)}
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathrm{VAR}[X_r] &= \sum_i a_i^2 q_i - (\mathrm{E}[X_r])^2 \\
&= \frac{\mathrm{g}''(r)}{\mathrm{g}(r)} - \frac{[\mathrm{g}'(r)]^2}{[\mathrm{g}(r)]^2} \\
&= \frac{\mathrm{g}''(r) - [\mathrm{g}(r)]^2}{[\mathrm{g}(r)]^2} \\
&= \gamma''(r)
\end{aligned}
$$

(d) Since $\gamma''(r) = \mathrm{VAR}[X_r]$, it is nonnegative and $\gamma''(r) \geq 0$. If $\gamma''(0) > 0$, then $\mathrm{VAR}[X] > 0$, which means that X is non-atomic. Thus X_r is non-atomic and $\gamma''(r) > 0$.