Homework 7 of Stochastic Processes

姓名：林奇峰　　学号：19110977

2019 年 10 月 27 日

1 Exercise 4.22

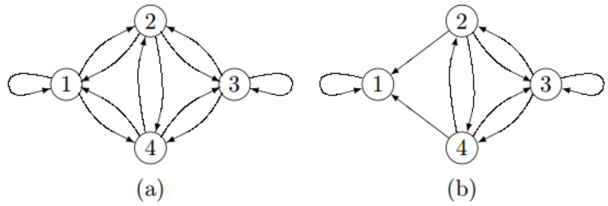


Figure 4.6 The conversion of the four-state recurrent Markov chain in (a) into the chain in (b) for which state 1 is a trapping state, i.e., the outgoing arcs from node 1 have been removed.

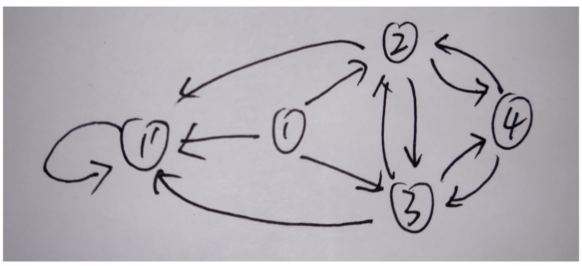
Section 4.5.1 showed how to find the expected first-passage times to a fixed state, say 1, from all other states. It is often desirable to include the expected first recurrence time from state 1 to return to state 1. This can be done by splitting state 1 into two states, first an initial state with no transitions coming to it but the original transitions going out, and, second, a final trapping state with the original transitions coming in.

(a) For the chain on the left side of Figure 4.6, draw the graph for the modified chain with five states where state 1 has been split into two states.

(b) Suppose one has found the expected first-passage times ν_j for states $j = 2, \ldots, 4$ (or in general from 2 to M). Find an expression for ν_1, the expected first recurrence time for state 1 in terms of $\nu_2, \nu_3, \ldots, \nu_{\mathrm{M}}$ and $P_{12}, \ldots, P_{1\mathrm{M}}$.

Solutions:

(a) Picture:



(b)

$$\nu_1 = 1 + \sum_{j=2}^{4} P_{1j}\nu_j$$

2 Exercise 4.28

Consider finding the expected time until a given string appears in a IID binary sequence with $\Pr\{X_n = 1\} = p_1, \Pr\{X_n = 0\} = p_0 = 1 - p_1$.

(a) Following the procedure in Example 4.5.1, draw the three-state Markov chain for the string (0,1). Find the expected number of trials unitl the first occurence of the string.

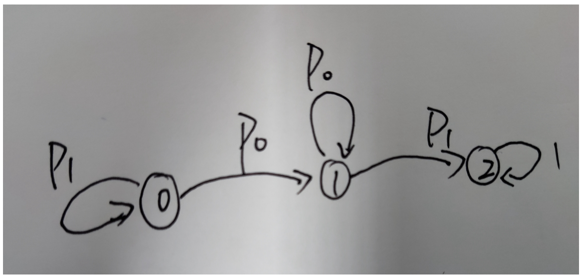
(b) For (b) and (c), let $(a_1, a_2, a_3, \ldots, a_k) = (0, 1, 1, \ldots, 1)$, i.e., zero followed by $k-1$ ones. Draw the corresponding Markov chain for $k = 4$.

(c) Let $\nu_i, 1 \le i \le k$ be the expected first-passage time from state i to state k. Note that $\nu_k = 0$. For each $i, 1 \le i \le k$, show that $\nu_i = \alpha_i + \nu_{i+1}$ and $\nu_0 = \beta_i + \nu_{i+1}$, where α_i and β_i are each expressed as a product of powers of p_0 and p_1. Hint: Use induction on i taking $i = 1$ as the base. For the inductive step, first find β_{i+1} as a function of β_i starting with $i = 1$ and using the equation $\nu_0 = 1/p_0 + \nu_1$.

(d) Let $\boldsymbol{a} = (0, 1, 0)$. Draw the corresponding Markov chain for this string. Evaluate ν_0, the expected time for (0,1,0) to occur.

Solutions:

(a) Picture:



We can obtain the following equations,

$$\nu_0 = 1 + p_1\nu_0 + p_0\nu_1$$
$$\nu_1 = 1 + p_0\nu_1 + p_1\nu_2$$
$$\nu_2 = 0$$

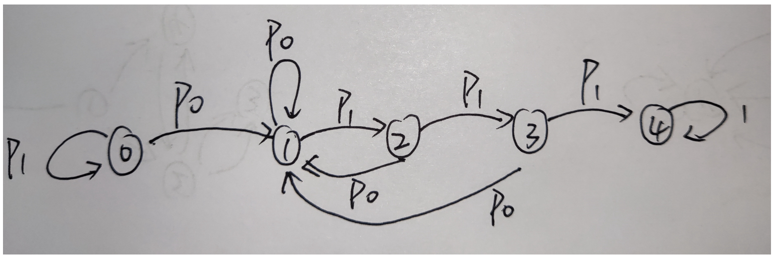
Solve the equal equations,

$$\nu_0 - \nu_1 = \frac{1}{p_0}$$
$$\nu_1 - \nu_2 = \frac{1}{p_1}$$
$$\nu_2 = 0$$

Finally,

$$\nu_0 = \frac{1}{p_0 p_1}$$

(b) Picture:



(c) From (a), we can know that

$$\alpha_1 = \frac{1}{p_1}$$
$$\beta_1 = \frac{1}{p_0 p_1}$$

Assume that $\nu_i = \alpha_i + \nu_{i+1}$ and $\nu_0 = \beta_i + \nu_{i+1}$ hold for each $i, 1 \leq i \leq k$, then

$$\begin{aligned}
\nu_{i+1} &= 1 + p_0\nu_1 + p_1\nu_{i+2} \quad \text{which can be seen from (b)} \\
&= p_0\nu_0 + p_1\nu_{i+2} \\
&= p_0(\beta_i + \nu_{i+1}) + p_1\nu_{i+2} \\
&= p_0\beta_i + p_0\nu_{i+1} + p_1\nu_{i+2} \\
&\Downarrow \\
\nu_{i+1} &= \frac{p_0\beta_i}{p_1} + \nu_{i+2}
\end{aligned}$$

Therefore

$$\alpha_{i+1} = \frac{p_0\beta_i}{p_1}$$

and

$$\nu_0 = \beta_i + \nu_{i+1} = \beta_i + \frac{p_0\beta_i}{p_1} + \nu_{i+2} = \frac{\beta_i}{p_1} + \nu_{i+2}$$

We can see that

$$\beta_{i+1} = \frac{\beta_i}{p_1}$$

and solve the following equations

$$\begin{aligned}
\beta_1 &= \frac{1}{p_0 p_1} \\
\beta_2 &= \beta_1 \frac{1}{p_1} \\
&\vdots \\
\beta_n &= \beta_{n-1} \frac{1}{p_1}
\end{aligned}$$

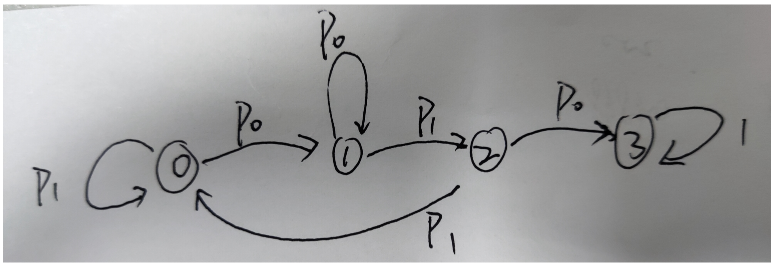
Thus,

$$\beta_n = \frac{1}{p_0 p_1^n}$$

and

$$\alpha_n = \frac{1}{p_1^n}$$

(d) Picture: Since,



$$\nu_0 = 1 + p_1\nu_0 + p_0\nu_1$$
$$\nu_1 = 1 + p_0\nu_1 + p_1\nu_2$$
$$\nu_2 = 1 + p_0\nu_3 + p_1\nu_0$$
$$\nu_3 = 0$$

Then,

$$\nu_0 = \frac{1}{p_0} + \frac{1}{p_1 p_0^2}$$