Homework 8&9 of Stochastic Processes

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1 Exercise 5.6

A town starts a mosquito control program and rv Z_n is the number of mosquitos at the end of the nth year ($n = 0, 1, 2, \dots$). Let X_n be the growth rate of the mosquito population in year n; i.e., $Z_n = X_n Z_{n-1}$; $n \geq 1$. Assume that $\{X_n; n \geq 1\}$ is a sequence of IID rv s with the PMF $\Pr\{X = 2\} = 1/2$; $\Pr\{X = 1/2\} = 1/4$; $\Pr\{1/4\} = 1/4$. Suppose that Z_0, the initial number of mosquitos, is some known constant and assume for simplicity and consistency that Z_n can take on non-integer values.

(a) Find $\mathrm{E}[Z_n]$ as a function of n and find $\lim_{n\to\infty}\mathrm{E}[Z_n]$.

(b) Let $W_n = \log_2 X_n$. Find $\mathrm{E}[W_n]$ and $\mathrm{E}[\log_2(Z_n/Z_0)]$ as a function of n.

(c) There is a constant α such that $\lim_{n\to\infty}(1/n)[\log_2(Z_n/Z_0)] = \alpha$ WP1. Find α and explain how this follows from the SLLN.

(d) Using (c), show that $\lim_{n\to\infty} Z_n = \beta$ WP1 for some β and evaluate β.

(e) Explain carefully how the result in (a) and the result in (d) are compatible. What you should learn from this problem is that the expected value of the log of a product of IID rv s might be significant than the expected value of the product itself.

Solutions:

(a)

$$\begin{aligned}
\mathrm{E}[Z_n] &= \mathrm{E}[Z_0 \prod_{i=1}^{n} X_i] \\
&= Z_0 \prod_{i=1}^{n} \mathrm{E}[X_i] \qquad \text{from the fact that } \{X_n; n \geq 1\} \text{ is a sequence of IID rv s} \\
&= Z_0 \cdot \left(\frac{19}{16}\right)^n
\end{aligned}$$

since each rv X_i is IID and $\mathrm{E}[X_i] = 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} = \frac{19}{16}$.

$$\lim_{n\to\infty} \mathrm{E}[Z_n] = \lim_{n\to\infty} Z_0 \cdot \left(\frac{19}{16}\right)^n = \infty$$

(b)

$$\begin{aligned}\mathrm{E}[W_n] &= \frac{1}{2}\cdot\log_2 2 + \frac{1}{4}\cdot\log_2\frac{1}{2} + \frac{1}{4}\cdot\log_2\frac{1}{4}\\ &= -\frac{1}{4}\end{aligned}$$

$$\begin{aligned}Z_n/Z_0 = X_n\cdot X_{n-1}\cdots X_1\cdot Z_0/Z_0 &\quad = \prod_{i=1}^{n} X_i\\ &\Downarrow\\ \log_2(Z_n/Z_0) &= \sum_{i=1}^{n}\log_2 X_i\\ &= \sum_{i=1}^{n} W_i\\ &\Downarrow\\ \mathrm{E}[\log_2(Z_n/Z_0)] &= \sum_{i=1}^{n}\mathrm{E}[W_i]\\ &= -\frac{n}{4}\end{aligned}$$

From the fact that $\{X_n; n\geq 1\}$ is a sequence of IID rv s and thus $W_n; n\geq 1$ is also a sequence of IID rv s.

(c) Since

$$\begin{aligned}\lim_{n\to\infty}(1/n)[\log_2(Z_n/Z_0)] &= \lim_{n\to\infty}\frac{1}{n}\cdot-\frac{n}{4}\\ &= \lim_{n\to\infty}-\frac{1}{4}\\ &= -\frac{1}{4}\end{aligned}$$

We can obtain that $\alpha = -\frac{1}{4}$.

According to Theorem 5.2.3 (SLLN), *For each integer* $n\geq 1$, *let* $S_n = X_1+\cdots+X_n$, *where* $X_1, X_2, \ldots$ *are IID rv s satisfying* $\mathrm{E}[|X|]<\infty$. *Then*

$$\Pr\left\{\omega : \lim_{n\to\infty}\frac{S_n(\omega)}{n} = \overline{X}\right\} = 1$$

We can see that, under this case, $S_n = \sum_{i=1}^{n} W_i$ where W_i is IID rv and it satisfies $\mathrm{E}[|W_i|] = -\frac{1}{4} < \infty$. The result shows that

$$\Pr\left\{\omega : \lim_{n\to\infty}\frac{S_n(\omega)}{n} = \overline{X} = -\frac{1}{4}\right\} = 1$$

holds.

(d) TO-DO

(e) TO-DO

2 Exercise 5.11

Let $Y(t) = S_{N(t)+1} - t$ be the residual life at time t of a renewal process. First consider a renewal process in which the interarrival time has density $f_X(x) = e^{-x}; x \geq 0$, and next consider a renewal process with density

$$f_X(x) = \frac{3}{(x+1)^4}, \quad x \geq 0$$

For each of the above densities, use renewal-reward theory to find:

(a) the time average of $Y(t)$;

(b) the time average of $Y^2(t)$(i.e.,$\lim_{Y\to\infty}(1/T)\int_0^T Y^2(t)dt$). For the exponential density, verify your answers by finding E$[Y(t)]$ and E$[Y^2(t)]$ directly.

3 Exercise 5.14

Let $F_Z(z)$ be the fraction of time over the limiting interval $(0, \infty)$ that the age of a renewal process is at most z. Show that $F_Z(z)$ satisfying

$$F_Z(z) = \frac{1}{\overline{X}} \int_{x=0}^{z} \Pr\{X > x\} dx \qquad \text{WP1.}$$

Hint: Follow the argument in Example 5.4.7.