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CS 577 - Homework 11
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1 Graded written problem

Part a: Two Travelling Turkey Problem (TTTP) as mentioned in the written problem has the following contraints:

- Each city is visited at least once
- Number of distinct routes has to be as small as possible
- Each city can be visited any number of times
- Path starts and ends at the same city
- Maximum of the total effort of the two turkeys is minimized

To formally describe the problem we will consider a complete graph $G = (V, E)$ with 2 edge-weight functions $w_a : E \to \mathbb{R}$ and $w_h : E \to \mathbb{R}$. Where $w_a(e)$ represents the effort taken by *Abe* to travel along the edge e from a city on one end of the edge to the other, while $w_h(e)$ represents the effort required by *Honest*

With the above definition for the graph G we can describe the decision version of the problem as:

For the graph G and $k \in \mathbb{R}$, report *yes* if there exists a spanning tree S such that $MAX(\sum_{\forall e \in S} w_a(e), \sum_{\forall e \in S} w_h(e))$ is at-most k, else report *no*.

Part b: Decision problem is NP-complete

To prove that TTTP decision problem is NP-complete, we will show that Partition Set(PS) that is a known NP-complete problem, can be reduced to TTTP problem. Given a set S of Real numbers and the sum of all elements in S as $2 * w$, we compute a graph G, such that G has a spanning tree ST with $MAX(\sum_{\forall e \in ST} w_a(e), \sum_{\forall e \in ST} w_h(e)) \leq w$ only if we can Partition the set using PS algorithm.

To transform the elements of S to the graph G we create a root node, and for every element i in S and add two vertices with edge weight as ($w_a = S[i]$, $w_h = 0$) and ($w_a = 0$, $w_h = S[i]$) to the root note. The edge weight between these two vertices is (0,0). Construct this gadget with all the elements in S and make it a complete graph by connecting all the unconnected vertices with edge weights ($w_a = \infty$,$w_h = \infty$).

Now, if the TTTP-decision problem returns that *yes* we claim that we can partition the set S in 2 subsets such that each subset has sum w. This is possible when the TTTP-decision problem can find a spanning tree in the graph G such that $\sum_{\forall e \in ST} w_a(e) = \sum_{\forall e \in ST} w_h(e) = w$. All the weights returned by $w_a(e)$ can be considered elements of one subset and the weights returned by $w_h(e)$ the elements of the other subset for the *Partition* problem on set S, and we can see that these subsets contain all the elements of set S and their sums are equal and so the PS-decision problem on set will also return *yes*.

If the TTTP-decision problem returns *no*, then consider a spanning tree made of edges whose weights are not ∞. Such a spanning tree will have minimum total edge weight for w_e and w_h summed together of $2 * w$ when all the elements in the set S either as the edge weights for one turkey or the other. Now since we cannot find a spanning tree, wherein $\sum_{\forall e \in ST} w_a(e) \leq w$ and $\sum_{\forall e \in ST} w_h(e) \leq w$, because if one of them is less than w then the other one will be greater than w. And hence these set of edge weights cannot be used to create a solution for the PS problem, so PS-decision problem will also return *no*.

Thus, PS-decision can be reduced to TTTP-decision problem and so TTTP-decision problem is NP-complete.