CK-12 Trigonometry Concepts

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CHAPTER **1**

Introduction to Trigonometry

Chapter Outline

Introduction

The study of trignometry begins with triangles. While some properties of triangles are covered in a Geometry book, Trigonometry extends the ideas found in Geometry and shows more properties, ideas, and functions related to the study of triangles.

In this Chapter, students will be introduced to important properties of triangles, special types of triangles, as well as related topics such as functions involving triangles. This important Chapter sets the stage for later Chapters with more in depth information about trigonometry and its applications.

1.1 Lengths of Triangle Sides Using the Pythagorean Theorem

Here you'll learn what the Pythagorean Theorem is and how to use it to find the length of an unknown side of a right triangle.

You've just signed up to be an architect's assistant in a new office downtown. You're asked to draw a scale model of a sculpture for a business plaza. The sculpture has a large triangular piece where one of the angles between the sides is ninety degrees. This type of triangle is called a "right triangle." The architect you're working for comes into the room and tells you that the sides of the triangle that form the right angle are 9 feet and 12 feet. Can you tell how long the third side is? When you've completed this Concept, you'll be able to find the length of an unknown side of a right triangle by using the lengths of the other two sides.

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James Sousa:ThePythagoreanTheorem

Guidance

From Geometry, recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$ where a and b are the legs of a right triangle and c is the hypotenuse. Also, the side opposite the angle is lower case and the angle is upper case. For example, angle A is opposite side a.

The Pythagorean Theorem is used to solve for the sides of a right triangle.

Example A

Use the Pythagorean Theorem to find the missing side.

Solution: $a = 8$, $b = 15$, we need to find the hypotenuse.

$$8^2 + 15^2 = c^2$$
$$64 + 225 = c^2$$
$$289 = c^2$$
$$17 = c$$

Notice, we do not include -17 as a solution because a negative number cannot be a side of a triangle.

Example B

Use the Pythagorean Theorem to find the missing side.

Solution: Use the Pythagorean Theorem to find the missing leg.

$$
\begin{aligned}
\left(5\sqrt{7}\right)^2 + x^2 &= \left(5\sqrt{13}\right)^2 \\
25 \cdot 7 + x^2 &= 25 \cdot 13 \\
175 + x^2 &= 325 \\
x^2 &= 150 \\
x &= 5\sqrt{6}
\end{aligned}
$$

Example C

Use the Pythagorean Theorem to find the missing side.

Solution: Use the Pythagorean Theorem to find the missing leg.

$$
\begin{aligned}
10^2 + x^2 &= \left(10\sqrt{2}\right)^2 \\
100 + x^2 &= 100 \cdot 2 \\
100 + x^2 &= 100 \\
x^2 &= 100 \\
x &= 10
\end{aligned}
$$

Vocabulary

Pythagorean Theorem: The **Pythagorean Theorem** is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a, b, and c are lengths of the triangle.

Guided Practice

1. Use the Pythagorean Theorem to find the missing side of the following triangle:

2. Use the Pythagorean Theorem to find the missing side of the following triangle:

3. Find the missing side of the right triangle below. Leave the answer in simplest radical form.

Solutions:

1. $a = 1$, $b = 8$, we need to find the hypotenuse.

$$
\begin{aligned}
1^2 + 8^2 &= c^2 \\
1 + 64 &= c^2 \\
65 &= c^2 \\
\left(\sqrt{65}\right) &= c
\end{aligned}
$$

2. $a = 3$, $c = 11$, we need to find the length of side b.

$$3^2 + b^2 = 11^2$$
$$9 + b^2 = 11^2$$
$$121 - 9 = b^2$$
$$112 = b^2$$
$$\left(\sqrt{112} \right) = b$$

3. 2. $a = 7$, $c = 18$, we need to find the length of side b.

$$7^2 + b^2 = 18^2$$
$$49 + b^2 = 18^2$$
$$324 - 49 = b^2$$
$$275 = b^2$$
$$\left(\sqrt{275} \right) = b$$

Concept Problem Solution

With your knowledge of the Pythagorean Theorem, you can see that the triangle has sides with lengths 9 feet and 12 feet. You work to find the hypotenuse:

$$a^2 + b^2 = c^2$$
$$9^2 + 12^2 = c^2$$
$$81 + 144 = c^2$$
$$225 = c^2$$
$$\left(\sqrt{225} \right) = 15 = c$$

With the knowledge that the length of the third side of the triangle is 15 feet, you are able to construct your scale model with ease.

Practice

Find the missing sides of the right triangles. Leave answers in simplest radical form.

1. If the legs of a right triangle are 3 and 4, then the hypotenuse is _____.
2. If the legs of a right triangle are 6 and 8, then the hypotenuse is _____.
3. If the legs of a right triangle are 5 and 12, then the hypotenuse is _____.
4. If the sides of a square are length 6, then the diagonal is _____.
5. If the sides of a square are 9, then the diagonal is _____.
6. If the sides of a square are x, then the diagonal is _____.
7. If the legs of a right triangle are 3 and 7, then the hypotenuse is _____.

8. If the legs of a right triangle are $2\sqrt{5}$ and 6, then the hypotenuse is _____.
9. If one leg of a right triangle is 4 and the hypotenuse is 8, then the other leg is _____.
10. If one leg of a right triangle is 10 and the hypotenuse is 15, then the other leg is _____.
11. If one leg of a right triangle is $4\sqrt{7}$ and the hypotenuse is $10\sqrt{6}$, then the other leg is _____.
12. If the legs of a right triangle are x and y, then the hypotenuse is _____.

Pythagorean Theorem Proof

Use the picture below to answer the following questions.

13. Find the area of the square in the picture with sides $(a+b)$.
14. Find the sum of the areas of the square with sides c and the right triangles with legs a and b.
15. Explain why the areas found in the previous two problems should be the same value. Then, set the expressions equal to each other and simplify to get the Pythagorean Theorem.

1.2 Identifying Sets of Pythagorean Triples

Here you'll learn about Pythagorean Triples and how to identify whether a set of numbers counts as one of them.

While working as an architect's assistant, you're asked to utilize your knowledge of the Pythagorean Theorem to determine if the lengths of a particular triangular brace support qualify as a Pythagorean Triple. You measure the sides of the brace and find them to be 7 inches, 24 inches, and 25 inches. Can you determine if the lengths of the sides of the triangular brace qualify as a Pythagorean Triple? When you've completed this Concept, you'll be able to answer this question with certainty.

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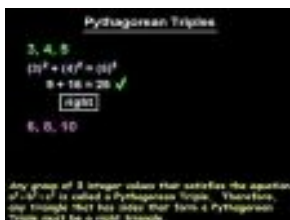
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YourTeacher.com:PythagoreanTriples

Guidance

Pythagorean Triples are sets of whole numbers for which the Pythagorean Theorem holds true. The most well-known triple is 3, 4, 5. This means that 3 and 4 are the lengths of the legs and 5 is the hypotenuse. The largest length is always the hypotenuse

Example A

Determine if the following lengths are Pythagorean Triples.

7, 24, 25

Solution: Plug the given numbers into the Pythagorean Theorem.

$$7^2 + 24^2 \overset{?}{=} 25^2$$
$$49 + 576 = 625$$
$$625 = 625$$

Yes, 7, 24, 25 is a Pythagorean Triple and sides of a right triangle.

Example B

Determine if the following lengths are Pythagorean Triples.

9, 40, 41

Solution: Plug the given numbers into the Pythagorean Theorem.

$$9^2 + 40^2 \overset{?}{=} 41^2$$
$$81 + 1600 = 1681$$
$$1681 = 1681$$

Yes, 9, 40, 41 is a Pythagorean Triple and sides of a right triangle.

Example C

Determine if the following lengths are Pythagorean Triples.

11, 56, 57

Solution: Plug the given numbers into the Pythagorean Theorem.

$$11^2 + 56^2 \overset{?}{=} 57^2$$
$$121 + 3136 = 3249$$
$$3257 \neq 3249$$

No, 11, 56, 57 do not represent the sides of a right triangle.

Vocabulary

Pythagorean Triple: A **Pythagorean Triple** is a set of three whole numbers that satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

Guided Practice

1. Determine if the following lengths are Pythagorean Triples.

5, 10, 13

2. Determine if the following lengths are Pythagorean Triples.

8, 15, 17

3. Determine if the following lengths are Pythagorean Triples.

11, 60, 61

Solutions:

1. Plug the given numbers into the Pythagorean Theorem.

$$5^2 + 10^2 \overset{?}{=} 13^2$$
$$25 + 100 = 169$$
$$125 \neq 169$$

No, 5, 10, 13 is not a Pythagorean Triple and not the sides of a right triangle.

2. Plug the given numbers into the Pythagorean Theorem.

$$8^2 + 15^2 \stackrel{?}{=} 17^2$$
$$64 + 225 = 289$$
$$289 = 289$$

Yes, 8, 15, 17 is a Pythagorean Triple and sides of a right triangle.

3. Plug the given numbers into the Pythagorean Theorem.

$$11^2 + 60^2 \stackrel{?}{=} 61^2$$
$$121 + 3600 = 3721$$
$$3721 = 3721$$

Yes, 11, 60, 61 is a Pythagorean Triple and sides of a right triangle.

Concept Problem Solution

Since you know that the sides of the brace have lengths of 7, 24, and 25 inches, you can substitute these values in the Pythagorean Theorem. If the Pythagorean Theorem is satisfied, then you know with certainty that these are indeed sides of a triangle with a right angle:

$$7^2 + 24^2 \stackrel{?}{=} 25^2$$
$$49 + 576 = 625$$
$$625 = 625$$

The Pythagorean Theorem is satisfied with these values as a lengths of sides of a right triangle. Since each of the sides is a whole number, this is indeed a set of Pythagorean Triples.

Practice

1. Determine if the following lengths are Pythagorean Triples: 9, 12, 15.
2. Determine if the following lengths are Pythagorean Triples: 10, 24, 36.
3. Determine if the following lengths are Pythagorean Triples: 4, 6, 8.
4. Determine if the following lengths are Pythagorean Triples: 20, 99, 101.
5. Determine if the following lengths are Pythagorean Triples: 21, 99, 101.
6. Determine if the following lengths are Pythagorean Triples: 65, 72, 97.
7. Determine if the following lengths are Pythagorean Triples: 15, 30, 62.
8. Determine if the following lengths are Pythagorean Triples: 9, 39, 40.
9. Determine if the following lengths are Pythagorean Triples: 48, 55, 73.
10. Determine if the following lengths are Pythagorean Triples: 8, 15, 17.
11. Determine if the following lengths are Pythagorean Triples: 13, 84, 85.
12. Determine if the following lengths are Pythagorean Triples: 15, 16, 24.
13. Explain why it might be useful to know some of the basic Pythagorean Triples.
14. Prove that any multiple of 5, 12, 13 will be a Pythagorean Triple.
15. Prove that any multiple of 3, 4, 5 will be a Pythagorean Triple.

1.3 Pythagorean Theorem to Triangles

Here you'll learn how to determine if a triangle is a right triangle, an acute triangle, or an obtuse triangle using a relationship between the lengths of the triangle's sides.

While painting a wall in your home one day, you realize that the wall you are painting seems "tilted", as though it might fall over. You realize that if the wall is standing upright, the angle between the wall and the floor is ninety degrees. After a few careful measurements, you find that the distance from the bottom of the ladder to the wall is 3 feet, the top of the ladder is at a point 10 feet up on the wall, and the ladder is 12 feet long. Can you determine if the wall is still standing upright, or if it is starting to lean? When you've completed this Concept, you'll know for certain how to determine if the wall is standing upright.

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James Sousa:ThePythagoreanTheoremandtheConverse ofthePythagoreanTheorem

Guidance

We can use the Pythagorean Theorem to help determine if a triangle is a right triangle, if it is acute, or if it is obtuse. To help you visualize this, think of an equilateral triangle with sides of length 5. We know that this is an acute triangle. If you plug in 5 for each number in the Pythagorean Theorem we get $5^2 + 5^2 = 5^2$ and $50 > 25$. Therefore, if $a^2 + b^2 > c^2$, then lengths a, b, and c make up an acute triangle. Conversely, if $a^2 + b^2 < c^2$, then lengths a, b, and c make up the sides of an obtuse triangle. *It is important to note that the length c is always the longest.*

Example A

Determine if the following lengths make an acute, right or obtuse triangle.

5, 6, 7

Solution: Plug in each set of lengths into the Pythagorean Theorem.

a.

$$5^2 + 6^2 \ ? \ 7^2$$
$$25 + 36 \ ? \ 49$$
$$61 > 49$$

Because $61 > 49$, this is an acute triangle.

Example B

Determine if the following lengths make an acute, right or obtuse triangle.

5, 10, 14

Solution: Plug in each set of lengths into the Pythagorean Theorem.

b.

$$5^2 + 10^2 \ ? \ 14^2$$
$$25 + 100 \ ? \ 196$$
$$125 < 196$$

Because $125 < 196$, this is an obtuse triangle.

Example C

Determine if the following lengths make an acute, right or obtuse triangle.

12, 35, 37

Solution: Plug in each set of lengths into the Pythagorean Theorem.

c.

$$12^2 + 35^2 \ ? \ 37^2$$
$$144 + 1225 \ ? \ 1369$$
$$1369 = 1369$$

Because the two sides are equal, this is a right triangle.

NOTE: All of the lengths in the above examples represent the lengths of the sides of a triangle. Recall the Triangle Inequality Theorem from geometry which states: The length of a side in a triangle is less than the sum of the other two sides. For example, 4, 7 and 13 cannot be the sides of a triangle because $4 + 7$ is not greater than 13.

Vocabulary

Acute Triangle: An **acute triangle** is a triangle where the angle opposite the hypotenuse is less than ninety degrees.

Right Triangle: A **right triangle** is a triangle where the angle opposite the hypotenuse is equal to ninety degrees.

Obtuse Triangle: An **obtuse triangle** is a triangle where the angle opposite the hypotenuse is greater than ninety degrees.

Guided Practice

1. Determine if the following lengths make an acute, right or obtuse triangle.

8, 15, 20

2. Determine if the following lengths make an acute, right or obtuse triangle.

15, 22, 25

Answers:

1. Plug in each set of lengths into the Pythagorean Theorem.

$$8^2 + 15^2 \text{ ? } 20^2$$
$$64 + 225 \text{ ? } 400$$
$$289 < 400$$

Because $289 < 400$, this is an obtuse triangle.

2. Plug in each set of lengths into the Pythagorean Theorem.

$$15^2 + 22^2 \text{ ? } 25^2$$
$$225 + 484 \text{ ? } 625$$
$$709 > 625$$

Because $709 > 625$, this is an acute triangle.

Concept Problem Solution

The ladder is making a triangle with the floor as one side, the wall as another, and the ladder itself serves as the hypotenuse. To see if the wall is leaning, you can determine the type of triangle that is made with these lengths (right, acute, or obtuse). If the triangle is a right triangle, then the wall is standing upright. Otherwise, it is leaning.

Plugging the lengths of the sides into the Pythagorean Theorem:

$$3^2 + 10^2 \text{ ? } 12^2$$
$$9 + 100 \text{ ? } 144$$
$$109 < 144$$

Yes, you were right. Because 109

Practice

Determine if each of the following lengths make a right triangle.

1. 9, 40, 41.
2. 12, 24, 26.
3. 5, 10, 14.
4. 3, $3\sqrt{3}$, 6.

Determine if the following lengths make an acute, right or obtuse triangle.

5. 10, 15, 18.
6. 4, 20, 21.
7. 15, 16, 17.
8. 15, 15, $15\sqrt{2}$.
9. 12, 17, 19.
10. 3, 4, 5.

11. 12, 12 $\sqrt{3}$, 24.
12. 2, 4, 5.
13. 3, 5, 7.
14. Explain why if $a^2 + b^2 < c^2$ then the triangle is obtuse.
15. Explain why if $a^2 + b^2 > c^2$ then the triangle is acute.

1.4 Pythagorean Theorem to Determine Distance

Here you'll learn how to use the Pythagorean Theorem to determine the distance between two points.

While walking to school one day, you decide to use your knowledge of the Pythagorean Theorem to determine how far it is between your home and school. You know that you walk 3 blocks east, and then turn and walk 7 blocks north to get to school. Is it possible to use the Pythagorean Theorem to help you determine the "straight line" distance between your home and school? By the end of this Concept, you'll be able to use the Pythagorean Theorem to do just that.

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James Sousa:Determine the Distance Between Two Points

Guidance

An application of the Pythagorean Theorem is to find the distance between two points. Consider the points (-1, 6) and (5, -3). If we plot these points on a grid and connect them, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$
\begin{aligned}
9^2 + (-6)^2 &= d^2 \\
81 + 36 &= d^2 \\
117 &= d^2 \\
\sqrt{117} &= d \\
3\sqrt{13} &= d
\end{aligned}
$$

Notice, that the $x-$values were subtracted from each other to find the horizontal distance and the $y-$values were subtracted from each other to find the vertical distance. If this process is generalized for two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, the Distance Formula is derived.

$$
(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2
$$

This is the Pythagorean Theorem with the vertical and horizontal differences between $(x_1,\ y_1)$ and $(x_2,\ y_2)$. Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

This is the Distance Formula. The following example shows how to apply the distance formula.

Example A

Find the distance between the two points.

(4, 2) and (-9, 5)

Solution: Plug each pair of points into the distance formula.

$$
\begin{aligned}
d &= \sqrt{(4 - (-9))^2 + (2 - 5)^2} \\
&= \sqrt{13^2 + (-3)^2} \\
&= \sqrt{169 + 9} \\
&= \sqrt{178}
\end{aligned}
$$

Example B

Find the distance between the two points.

(-10, 3) and (0, -15)

Solution: Plug each pair of points into the distance formula.

$$
\begin{aligned}
d &= \sqrt{(-10 - 0)^2 + (3 - (-15))^2} \\
&= \sqrt{(-10)^2 + (18)^2} \\
&= \sqrt{100 + 324} \\
&= \sqrt{424} = 2\sqrt{106}
\end{aligned}
$$

Example C

Find the distance between the two points.

(3, 1) and (2, -7)

Solution: Plug each pair of points into the distance formula.

$$
\begin{aligned}
d &= \sqrt{(3 - 2)^2 + (1 - (-7))^2} \\
&= \sqrt{(1)^2 + (8)^2} \\
&= \sqrt{1 + 64} \\
&= \sqrt{65}
\end{aligned}
$$

Vocabulary

Distance Formula: The **Distance Formula** is a formula using the Pythagorean Theorem to find the distance between two points on a coordinate system using the differences between the "x" and "y" coordinates of the two points as side lengths.

Guided Practice

Find the distance between the two points.

1. (3, 1) and (2, -7)

2. (5, -8) and (0, 3)

3. (2, 6) and (2, 9)

Solutions:

1. Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(3-2)^2 + (1-(-7))^2} \\ &= \sqrt{(1)^2 + (8)^2} \\ &= \sqrt{1+64} \\ &= \sqrt{65} \end{aligned}$$

2. Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(5-0)^2 + (-8-(3))^2} \\ &= \sqrt{(5)^2 + (-11)^2} \\ &= \sqrt{25+121} \\ &= \sqrt{146} \end{aligned}$$

3. Plug each pair of points into the distance formula.

$$\begin{aligned} d &= \sqrt{(2-2)^2 + (6-9)^2} \\ &= \sqrt{(0)^2 + (-3)^2} \\ &= \sqrt{9} = 3 \end{aligned}$$

Concept Problem Solution

Since you know that the trip to school involves walking 3 blocks east followed by 7 blocks north, you can construct a triangle on a coordinate system out of these lengths, like this:

Since you went three blocks to the east, the school has an "x" coordinate of 3. Likewise, since you went 7 blocks north, the school has a "y" coordinate of 7. To find the straight-line distance to school, you can use the Distance Formula:

$$d = \sqrt{(3-0)^2 + (7-0)^2}$$
$$= \sqrt{(3)^2 + (7)^2}$$
$$= \sqrt{58}$$

This is a straight line distance of approximately 7.6 blocks.

Practice

Find the distance between each pair of points. Round each answer to the nearest tenth.

1. (2, 4) and (5, 10)
2. (1, 5) and (8, 9)
3. (-2, 3) and (6, 4)
4. (5, 7) and (5, 10)
5. (8, 12) and (15, 12)
6. (1, -4) and (25, -2)
7. (5, -6) and (3, 7)
8. (12, -9) and (-1, 5)
9. (-3, 14) and (8, 10)
10. (-11, 3) and (-5, 1)
11. (5, 2) and (11, 13)
12. (8, 10) and (9, -6)

Find the perimeter of each triangle. Round each answer to the nearest tenth.

13. $A(3, -5), B(-5, -8), C(-2, 7)$
14. $A(5, 3), B(2, -7), C(-1, 5)$
15. $A(1, 2), B(1, 5), C(4, 5)$

1.5 Lengths of Sides in Isosceles Right Triangles

Here you'll learn what an isosceles right triangle is, the relationships between the sides of an isosceles right triangle, and how to find the length of an unknown side.

"Paper Football" is a game that people often play involving a piece of paper folded into a triangle. To score a touchdown, you have to slide your "football" across a desk to your opponent's side. However, the football must be partway past the edge of the table, but not so far past that it falls. This is called the "end zone". If more than half of the football is past the edge of the table, it will fall off and you won't get a touchdown. So a diagram of the farthest the football can be over the edge of the table so that you can score would look like this:

You have decided to play a game of paper football with your friends, and proceed to create your football by repeatedly folding a piece of paper. When you are finished, the football is triangle that has a right angle and two other angles that are the same as each other. You decide to figure out what the maximum distance is that the football can be over the edge without falling over. This is half the length of the hypotenuse. You measure the length of one of the shorter sides of the triangle and find that it is 2 cm long.

Can you figure out what the length of half of the hypotenuse is with just this information?

When you've completed this Concept, you'll know how to find just such as length, and the special type of triangle that your football's shape is.

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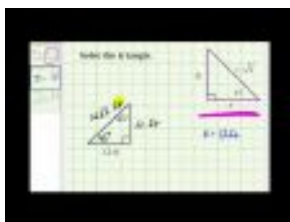
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James Sousa Examples: Solve a 4545 Right Triangle

Guidance

An isosceles right triangle is an isosceles triangle and a right triangle. This means that it has two congruent sides and one right angle. Therefore, the two congruent sides must be the legs.

Because the two legs are congruent, we will call them both a and the hypotenuse c. Plugging both letters into the Pythagorean Theorem, we get:

$$a^2 + a^2 = c^2$$
$$2a^2 = c^2$$
$$\sqrt{2a^2} = \sqrt{c^2}$$
$$a\sqrt{2} = c$$

From this we can conclude that the hypotenuse length is the length of a leg multiplied by $\sqrt{2}$. Therefore, we only need one of the three lengths to determine the other two lengths of the sides of an isosceles right triangle. The ratio is usually written $x : x : x\sqrt{2}$, where x is the length of the legs and $x\sqrt{2}$ is the length of the hypotenuse.

Example A

Find the lengths of the other two sides of the isosceles right triangle below.

Solution: If a leg has length 8, by the ratio, the other leg is 8 and the hypotenuse is $8\sqrt{2}$.

Example B

Find the lengths of the other two sides of the isosceles right triangle below.

Solution: If the hypotenuse has length $7\sqrt{2}$, then both legs are 7.

Example C

Find the lengths of the other two sides of the isosceles right triangle below.

Solution: Because the leg is $10\sqrt{2}$, then so is the other leg. The hypotenuse will be $10\sqrt{2}$ multiplied by an additional $\sqrt{2}$.

$$10\sqrt{2} \cdot \sqrt{2} = 10 \cdot 2 = 20$$

Vocabulary

Isosceles Right Triangle: An **isosceles right triangle** is a triangle with one angle equal to ninety degrees and each of the other two angles equal to forty five degrees.

Guided Practice

1. Find the length of the other two sides of the isosceles right triangle below:

2. Find the length of the other two sides of the isosceles right triangle below:

3. Find the length of the other two sides of the isosceles right triangle below:

Solutions:

1. Since we know the length of the given leg is 12, and it isn't the hypotenuse, that means the other side that isn't opposite the right angle also has a length of 12. We can then determine from the relationships for an isosceles right triangle that the length of the hypotenuse is $12\sqrt{2}$.

2. Since we know the length of the hypotenuse is $\sqrt{8}$, we can determine the lengths of the other two sides. Because the length of the hypotenuse is $\sqrt{2}$ times the length of the other sides, we can construct the following:

$$x\sqrt{2} = \sqrt{8}$$
$$x = \frac{\sqrt{8}}{\sqrt{2}}$$
$$x = \sqrt{4} = 2$$

3. Since we know the length of the given leg is $\sqrt{2}$, the length of the hypotenuse is then $\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$. The length of the other side is the same as the given side, so it is $\sqrt{2}$.

Concept Problem Solution

With your knowledge of the ratios of an isosceles right triangle, you know that the hypotenuse is equal to $\sqrt{2}$ times the length of each of the other sides. Since it is known that the length of the other side is 2 cm, you therefore know that the length of the hypotenuse is $2\sqrt{2}$ cm. However, since the football will fall off of the table if it is more than halfway over the edge, the farthest the football can go off of the table is $\frac{2\sqrt{2}}{2} = \sqrt{2} \approx 1.41$ cm.

Practice

1. In an isosceles right triangle, if a leg is 3, then the hypotenuse is _____.
2. In an isosceles right triangle, if a leg is 7, then the hypotenuse is _____.
3. In an isosceles right triangle, if a leg is x, then the hypotenuse is _____.
4. In an isosceles right triangle, if the hypotenuse is $16\sqrt{2}$, then each leg is _____.
5. In an isosceles right triangle, if the hypotenuse is $12\sqrt{2}$, then each leg is _____.
6. In an isosceles right triangle, if the hypotenuse is 22, then each leg is _____.
7. In an isosceles right triangle, if the hypotenuse is x, then each leg is _____.
8. A square has sides of length 16. What is the length of the diagonal?
9. A square's diagonal is $28\sqrt{2}$. What is the length of each side?
10. A square's diagonal is 28. What is the length of each side?
11. A square has sides of length $3\sqrt{2}$. What is the length of the diagonal?
12. A square has sides of length $6\sqrt{2}$. What is the length of the diagonal?
13. A square has sides of length $4\sqrt{3}$. What is the length of the diagonal?
14. A baseball diamond is a square with 80 foot sides. What is the distance from home base to second base? (HINT: It's the length of the diagonal).
15. Four isosceles triangles are formed when both diagonals are drawn in a square. If the length of each side in the square is s, what are the lengths of the legs of the isosceles triangles?

1.6 Relationships of Sides in 30-60-90 Right Triangles

Here you'll learn what a 30-60-90 triangle is, the relationship between the lengths of the sides, and how to find the length of an unknown side.

You are working on a project in your Industrial Arts class. You have been instructed to design a building. You have a plastic triangular piece that helps you with straight edges and designs. This triangle has interior angles of 30°, 60°, and 90°. Laying the triangle flat on the paper, you decide to use it to help you with the right angle at the building's base.

You trace out the bottom and left edge of the triangle on the paper to serve as the side and bottom of the structure. Looking at the length of the base of the building, you see that it is 7 inches long. Can you determine what the height of the building is from this information?

Read on, and at the conclusion of this Concept you'll be able to do just that.

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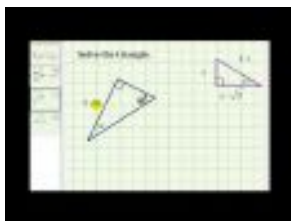
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James Sousa Examples: Solve a 30-60 Right Triangle

Guidance

$30 - 60 - 90$ refers to each of the angles in this special right triangle. To understand the ratios of the sides, start with an equilateral triangle with an altitude drawn from one vertex.

Recall from geometry that an altitude, h, cuts the opposite side directly in half. So we know that one side, the hypotenuse, is $2s$ and the shortest leg is s. Also, recall that the altitude is a perpendicular and angle bisector, which is why the angle at the top is split in half. To find the length of the longer leg, use the Pythagorean Theorem:

$$s^2 + h^2 = (2s)^2$$
$$s^2 + h^2 = 4s^2$$
$$h^2 = 3s^2$$
$$h = s\sqrt{3}$$

From this we can conclude that the length of the longer leg is the length of the short leg multiplied by $\sqrt{3}$ or $s\sqrt{3}$. Just like the isosceles right triangle, we now only need one side in order to determine the other two in a $30 - 60 - 90$ triangle. The ratio of the three sides is written $x : x\sqrt{3} : 2x$, where x is the shortest leg, $x\sqrt{3}$ is the longer leg and $2x$ is the hypotenuse.

Notice, that the shortest side is always*always* opposite 90°.

Example A

Find the lengths of the two missing sides in the $30 - 60 - 90$ triangle.

Solution: Determine which side in the $30 - 60 - 90$ ratio is given and solve for the other two.

$4\sqrt{3}$ is the longer leg because it is opposite the $60°$. So, in the $x : x\sqrt{3} : 2x$ ratio, $4\sqrt{3} = x\sqrt{3}$, therefore $x = 4$ and $2x = 8$. The short leg is 4 and the hypotenuse is 8.

Example B

Find the lengths of the two missing sides in the $30 - 60 - 90$ triangle.

Solution: Determine which side in the $30 - 60 - 90$ ratio is given and solve for the other two.

17 is the hypotenuse because it is opposite the right angle. In the $x : x\sqrt{3} : 2x$ ratio, $17 = 2x$ and so the short leg is $\frac{17}{2}$ and the long leg is $\frac{17\sqrt{3}}{2}$.

Example C

Find the lengths of the two missing sides in the $30 - 60 - 90$ triangle.

Solution: Determine which side in the $30 - 60 - 90$ ratio is given and solve for the other two.

15 is the long leg because it is opposite the $60°$. Even though 15 does not have a radical after it, we can still set it equal to $x\sqrt{3}$.

$$x\sqrt{3} = 15$$
$$x = \frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3} \quad \text{So, the short leg is } 5\sqrt{3}.$$

Multiplying $5\sqrt{3}$ by 2, we get the hypotenuse length, which is $10\sqrt{3}$.

Vocabulary

30-60-90 Triangle: A **30-60-90 Triangle** is a triangle with one angle equal to ninety degrees, one angle equal to thirty degrees, and one angle equal to sixty degrees.

Guided Practice

1. Find the lengths of the two missing sides in the $30 - 60 - 90$ triangle.

2. Find the lengths of the two missing sides in the 30-60-90 triangle below.

3. Find the angles in the $30 - 60 - 90$ triangle.

Solutions:

1. Since the given side has a length of 8 and is the side which is opposite the right angle, we know that this is the "2x" side of the triangle. Therefore, the short side of the triangle is $\frac{8}{2} = 4$ and the third side of the triangle is $4\sqrt{3}$.

2. We can see that the shortest side of the triangle has a length of 3 while the longest side has a length of 6 and the other side has a length of $3\sqrt{3}$. This means that the $30°$ angle is opposite the side with length 3, the $60°$ angle is opposite the side with length $3\sqrt{3}$, and the $90°$ angle is opposite the side with length 6.

3. The length of the given side is $3\sqrt{3}$, and is opposite the 60° angle. This means that the side opposite the 30° angle is 3, and the length of the side opposite the 90° angle is 6

Concept Problem Solution

Since you know the ratios of sides of a 30-60-90 triangle, you know that since the bottom side has a length of 7 cm, the left side must have a length of $7\sqrt{3} \approx 12.12$ cm.

Practice

1. In a 30-60-90 triangle, if the shorter leg is 8, then the longer leg is _____ and the hypotenuse is _____.
2. In a 30-60-90 triangle, if the shorter leg is 12, then the longer leg is _____ and the hypotenuse is _____.
3. In a 30-60-90 triangle, if the longer leg is 10, then the shorter leg is _____ and the hypotenuse is _____.
4. In a 30-60-90 triangle, if the shorter leg is 16, then the longer leg is _____ and the hypotenuse is _____.
5. In a 30-60-90 triangle, if the longer leg is 3, then the shorter leg is _____ and the hypotenuse is _____.
6. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is _____ and the hypotenuse is _____.
7. In a 30-60-90 triangle, if the longer leg is x, then the shorter leg is _____ and the hypotenuse is _____.
8. A rectangle has sides of length 7 and $7\sqrt{3}$. What is the length of the diagonal?
9. Two (opposite) sides of a rectangle are 15 and the diagonal is 30. What is the length of the other two sides?
10. What is the height of an equilateral triangle with sides of length 6 in?
11. What is the area of an equilateral triangle with sides of length 10 ft?
12. A regular hexagon has sides of length 3 in. What is the area of the hexagon?
13. The area of an equilateral triangle is $36\sqrt{3}$. What is the length of a side?
14. If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel 3 miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?
15. If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel x miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?

1.7 Special Triangle Ratios

Here you'll learn how to determine if a given triangle is a 30-60-90 or a 45-45-90 triangle by examining the relationship between the lengths of the sides.

While working in your Industrial Arts class one day, your Instructor asks you to use your 45-45-90 triangle to make a scale drawing. Unfortunately, you have two differently shaped triangles to use at your drafting table, and there aren't labels to tell you which triangle is the correct one to use.

You turn the triangles over and over in your hands, trying to figure out what to do, when you spot the ruler at your desk. Taking one of the triangles, you measure two of its sides. You determine that the first side is 7 inches long, and the second side is just a little under 9.9 inches. Can you determine if this is the correct triangle for your work?

At the completion of this Concept, you'll be able to verify if this is the correct triangle to use.

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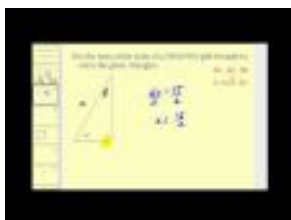
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James Sousa Solving Special Right Triangles

Guidance

Special right triangles are the basis of trigonometry. The angles $30°$, $45°$, $60°$ and their multiples have special properties and significance in the unit circle (which you can read about in other Concepts). Students are usually required to memorize the ratios of sides in triangles with these angles because of their importance.

First, let's compare the two ratios, so that we can better distinguish the difference between the two. For a $45 - 45 - 90$ triangle the ratio is $x : x : x\sqrt{2}$ and for a $30 - 60 - 90$ triangle the ratio is $x : x\sqrt{3} : 2x$. An easy way to tell the difference between these two ratios is the isosceles right triangle has two congruent sides, so its ratio has the $\sqrt{2}$, whereas the $30 - 60 - 90$ angles are all divisible by 3, so that ratio includes the $\sqrt{3}$. Also, if you are ever in doubt or forget the ratios, you can always use the Pythagorean Theorem. The ratios are considered a short cut.

Example A

Determine if the set of lengths below represents a special right triangle. If so, which one?

$8\sqrt{3} : 24 : 16\sqrt{3}$

Solution: Yes, this is a $30 - 60 - 90$ triangle. If the short leg is $x = 8\sqrt{3}$, then the long leg is $8\sqrt{3} \cdot \sqrt{3} = 8 \cdot 3 = 24$ and the hypotenuse is $2 \cdot 8\sqrt{3} = 16\sqrt{3}$.

Example B

Determine if the set of lengths below represents a special right triangle. If so, which one?

$\sqrt{5} : \sqrt{5} : \sqrt{10}$

Solution: Yes, this is a $45 - 45 - 90$ triangle. The two legs are equal and $\sqrt{5} \cdot \sqrt{2} = \sqrt{10}$, which would be the length of the hypotenuse.

Example C

Determine if the set of lengths below represents a special right triangle. If so, which one?

$6\sqrt{7} : 6\sqrt{21} : 12$

Solution: No, this is not a special right triangle. The hypotenuse should be $12\sqrt{7}$ in order to be a $30 - 60 - 90$ triangle.

Vocabulary

Special Triangle: A **special triangle** is a triangle that has particular internal angles that cause the sides to have a certain length relationship with each other. Examples include a 45-45-90 triangle and a 30-60-90 triangle.

Guided Practice

1. Determine if the set of lengths below represents a special right triangle. If so, which one?

$3\sqrt{2} : 3\sqrt{2} : 6$

2. Determine if the set of lengths below represents a special right triangle. If so, which one?

$4 : 2 : 2\sqrt{3}$

3. Determine if the set of lengths below represents a special right triangle. If so, which one?

$13 : 84 : 85$

Solutions:

1. The sides are the same length. This means that if the triangle is one of the special triangles at all, it must be a 45-45-90 triangle. To test this, we take either of the sides that are equal and multiply it by $\sqrt{2}$:

$3\sqrt{2} \times \sqrt{2} = 3 \times \sqrt{4} = 3 \times 2 = 6$

Yes, this triangle is a special triangle. It is a 45-45-90 triangle.

2. It can immediately be seen that the second side is one half the length of the first side. This means that if it is a special triangle, it must be a 30-60-90 triangle. To see if it is indeed such a triangle, look at the relationship between the shorter side and the final side. The final side is $\sqrt{3}$ times the short side. So yes, this fulfills the criteria for a 30-60-90 triangle.

3. It can be seen immediately that the lengths of sides given aren't a special triangle, since 84 is so close to 85. Therefore it can't be a 45-45-90 triangle, which would require $84\sqrt{2}$ to be a side or a 30-60-90 triangle, where a one of these two sides would have a relationship of multiplying/dividing by 2 or by $\sqrt{3}$.

Concept Problem Solution

Since you know the ratios of lengths of sides for special triangles, you can test to see if the triangle in your hand is the correct one by testing the relationship:

hypotenuse $= \sqrt{2}x$

where "x" is the length of the shorter sides. If you test this relationship with the triangle you are holding:

hypotenuse $= 7\sqrt{2} = 9.87$ in

Yes, you are holding the correct triangle.

Practice

For each of the set of lengths below, determine whether or not they represent a special right triangle. If so, which one?

1. $2 : 2 : 2\sqrt{2}$
2. $3 : 3 : 6$
3. $3 : 3\sqrt{3} : 6$
4. $4\sqrt{2} : 4\sqrt{2} : 8\sqrt{2}$
5. $5\sqrt{2} : 5\sqrt{2} : 10$
6. $7 : 7\sqrt{2} : 14$
7. $6\sqrt{5} : 18\sqrt{5} : 12\sqrt{5}$
8. $4\sqrt{6} : 12\sqrt{2} : 8\sqrt{6}$
9. $8\sqrt{15} : 24\sqrt{5} : 16$
10. $7\sqrt{6} : 7\sqrt{6} : 14\sqrt{3}$
11. $5\sqrt{7} : 5\sqrt{14} : 5\sqrt{7}$
12. $9\sqrt{6} : 27\sqrt{2} : 18\sqrt{6}$

13. Explain why if you cut any square in half along its diagonal you will create two 45-45-90 triangles.
14. Explain how to create two 30-60-90 triangles from an equilateral triangle.
15. Could a special right triangle ever have all three sides with integer lengths?

1.8 Sine, Cosine, and Tangent Functions

Here you'll learn the definition of sine, cosine, and tangent functions and how to apply them.

You are helping your grandfather with some repairs around the house when he mentions that he could use some help painting some boards on the staircase of his front porch. When you go out to see what he means, you notice that the stairs are supported by a set of boards that are glued together so that they are shaped like a triangle, with one extra board placed over top of them for decoration.

As you are looking around for paint, you think about the situation and realize that this reminds you a lot of your math class. In your study of triangles, you are about to start a unit working with the relationships between the sides of a triangle. You begin to wonder: "How many possible relationships are there between sides of triangles, anyway?"

By the end of this Concept, you'll have studied three of these important relationships, as well as know how many relationships there are total.

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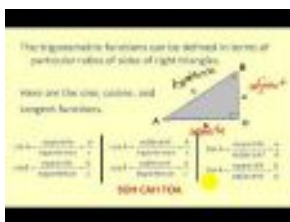
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James Sousa:TheTrigonometric Functionsin Terms ofRight Triangles

Guidance

The first three trigonometric functions we will work with are the sine, cosine, and tangent functions. The elements of the domains of these functions are angles. We can define these functions in terms of a right triangle: The elements of the range of the functions are particular ratios of sides of triangles.

We define the sine function as follows: For an acute angle x in a right triangle, the $sinx$ is equal to the ratio of the side opposite of the angle over the hypotenuse of the triangle. For example, using this triangle, we have: $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Since all right triangles with the same acute angles are similar, this function will produce the same ratio, no matter which triangle is used. Thus, it is a well-defined function.

Similarly, the cosine of an angle is defined as the ratio of the side adjacent (next to) the angle over the hypotenuse of the triangle. Using this triangle, we have: $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Finally, the tangent of an angle is defined as the ratio of the side opposite the angle to the side adjacent to the angle. In the triangle above, we have: $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these functions. First, keep in mind that the abbreviations $sinx, cosx$, and $tanx$ are just like $f(x)$. They simply stand for specific kinds of functions. Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $sinx$ it

is still pronounced *sine*, with a long *i*. When we write *cosx*, we still say co-sine. And when we write *tanx*, we still say tangent.

We can use these definitions to find the sine, cosine, and tangent values for angles in a right triangle.

Example A

Find the sine, cosine, and tangent of $\angle A$:

Solution:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{3}$$

Example B

Find $\sin B$ using $\triangle ABC$ and $\triangle NAP$.

Solution:

Using $\triangle ABC : \sin B = \frac{3}{5}$

Using $\triangle NAP : \sin B = \frac{6}{10} = \frac{3}{5}$

Example C

Find $\sin B$ and $\tan A$ in the triangle below:

Solution:

$$\sin B = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{12}{13}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{5}{12}$$

Vocabulary

Sine: The **sine** of an angle in a right triangle is a relationship found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Cosine: The **cosine** of an angle in a right triangle is a relationship found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.

Tangent: The **tangent** of an angle in a right triangle is a relationship found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.

Guided Practice

Using the triangle shown here:

Find

1. The sine of angle $\angle A$

2. The cosine of angle $\angle A$

3. The tangent of angle $\angle A$

Solution: 1. The sine is equal to the opposite divided by the hypotenuse.

$\sin A = \frac{opposite}{hypotenuse} = \frac{11}{61} \approx .18$

2. The cosine is equal to the adjacent divided by the hypotenuse.

$\cos A = \frac{adjacent}{hypotenuse} = \frac{60}{61} \approx .98$

3. The tangent is equal to the opposite divided by the adjacent.

$\tan A = \frac{opposite}{adjacent} = \frac{61}{60} \approx 1.01$

Concept Problem Solution

Looking at a triangle like the shape of your grandfather's staircase:

We can see that there are several ways to make relationships between the sides. In this case, we are only interested in ratios between the sides, which means one side will be divided by another. If we assume that dividing a side by itself doesn't count (since the answer would always be equal to one), let's look at the number of possible combinations:

If we use the angle labelled above, there is: 1) The side opposite the angle divided by the hypotenuse (the sine function) 2) The side adjacent the angle divided by the hypotenuse (the cosine function) 3) The side opposite the angle divided by adjacent side (the tangent function)

You can also imagine taking the same sides, except reversing the numerator and denominator: 4) The hypotenuse divided by the side opposite the angle 5) The hypotenuse divided by the side adjacent to the angle 6) The side adjacent to the angle divided by the side opposite to the angle

The first three functions are what we introduced in this Concept. The last three are other functions you'll learn about in a different Concept.

Practice

Use the diagram below for questions 1-3.

1. Find $\sin A$ and $\sin C$.
2. Find $\cos A$ and $\cos C$.
3. Find $\tan A$ and $\tan C$.

Use the diagram to fill in the blanks below.

4. $\tan A = \frac{?}{?}$
5. $\sin C = \frac{?}{?}$
6. $\tan C = \frac{?}{?}$
7. $\cos C = \frac{?}{?}$
8. $\sin A = \frac{?}{?}$

9. $\cos A = \frac{?}{?}$

From questions 4-9, we can conclude the following. Fill in the blanks.

10. $\cos$____ $= \sin A$ and $\sin$____ $= \cos A$.
11. $\tan A$ and $\tan C$ are _____ of each other.
12. Explain why the cosine of an angle will never be greater than 1.
13. Use your knowledge of 45-45-90 triangles to find the sine, cosine, and tangent of a 45 degree angle.
14. Use your knowledge of 30-60-90 triangles to find the sine, cosine, and tangent of a 30 degree angle.
15. Use your knowledge of 30-60-90 triangles to find the sine, cosine, and tangent of a 60 degree angle.
16. As the degree of an angle increases, will the tangent of the angle increase or decrease? Explain.

1.9 Secant, Cosecant, and Cotangent Functions

Here you'll learn the definition of secant, cosecant, and cotangent functions and how to apply them.

While working to paint your grandfather's staircase you are looking at the triangular shape made by the wall that support the stairs. The staircase looks like this:

You are thinking about all of the possible relationships between sides. You already know that there are three common relationships, called sine, cosine, and tangent.

How many others can you find?

By the end of this Concept, you'll know the other important relationships between sides of a triangle.

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James Sousa:Graphing Cosecant and secant

Guidance

We can define three more functions also based on a right triangle. They are the reciprocals of sine, cosine and tangent.

If $\sin A = \frac{a}{c}$, then the definition of cosecant, or csc, is $\csc A = \frac{c}{a}$.

If $\cos A = \frac{b}{c}$, then the definition of secant, or sec, is $\sec A = \frac{c}{b}$.

If $\tan A = \frac{a}{b}$, then the definition of cotangent, or cot, is $\cot A = \frac{b}{a}$.

Example A

Find the secant, cosecant, and cotangent of angle B.

Solution:

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$5^2 + 12^2 = H^2$$
$$25 + 144 = H^2$$
$$169 = H^2$$
$$H = 13$$

Now we can find the secant, cosecant, and cotangent of angle B:

$$\sec B = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$

$$\cot B = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

Example B

Find the secant, cosecant, and cotangent of angle A

Solution:

$$\sec A = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{41}{40}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{41}{9}$$

$$\cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{40}{9}$$

Example C

Find the sine, cosine, and tangent of angle A, and then use this to construct the secant, cosecant, and cotangent of the angle

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{7}{25}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{24}{25}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{7}{24}$$

Since we know that cosecant is the reciprocal of sine, secant is the reciprocal of sine, and cotangent is the reciprocal of tangent, we can construct these functions as follows:

$$\sec A = \frac{1}{\cos A} = \frac{25}{24}$$

$$\csc A = \frac{1}{\sin A} = \frac{25}{7}$$

$$\cot A = \frac{1}{\tan A} = \frac{24}{7}$$

Vocabulary

Cosecant: The **cosecant** of an angle in a right triangle is a relationship found by dividing the length of the hypotenuse by the length of the side opposite to the given angle. This is the reciprocal of the sine function.

Secant: The **secant** of an angle in a right triangle is a relationship found by dividing length of the hypotenuse by the length of the side adjacent the given angle. This is the reciprocal of the cosine function.

Cotangent: The **cotangent** of an angle in a right triangle is a relationship found by dividing the length of the side adjacent to the given angle by the length of the side opposite to the given angle. This is the reciprocal of the tangent function.

Guided Practice

Find the

1. secant

2. cosecant

3. cotangent

of $\angle A$

Solutions:

1. The secant function is defined to be $\frac{1}{\cos}$. Since $\cos = \frac{adjacent}{hypotenuse}$, $\sec = \frac{hypotenuse}{adjacent}$.

$\sec = \frac{hypotenuse}{adjacent} = \frac{37}{12} \approx 3.08$

2. The cosecant function is defined to be $\frac{1}{\sin}$. Since $\sin = \frac{opposite}{hypotenuse}$, $\csc = \frac{hypotenuse}{opposite}$.

$\csc = \frac{hypotenuse}{opposite} = \frac{37}{35} \approx 1.06$

3. The cotangent function is defined to be $\frac{1}{\tan}$. Since $\tan = \frac{opposite}{adjacent}$, $\cot = \frac{adjacent}{opposite}$.

$\cot = \frac{adjacent}{opposite} = \frac{12}{35} \approx .34$

Concept Problem Solution

Looking at a triangle-like the shape of the wall supporting your grandfather's staircase:

We can see that there are several ways to make relationships between the sides. In this case, we are only interested in ratios between the sides, which means one side will be divided by another. We've already seen some functions, such as:

1) The side opposite the angle divided by the hypotenuse (the sine function)

2) The side adjacent the angle divided by the hypotenuse (the cosine function)

3) The side opposite the angle divided by adjacent side (the tangent function)

In this section we introduced the reciprocal of the above trig functions. These are found by taking ratios between the same sides shown above, except reversing the numerator and denominator:

4) The hypotenuse divided by the side opposite the angle (the cosecant function)

5) The hypotenuse divided by the side adjacent to the angle (the secant function)

6) The adjacent side divided by the opposite side (the cotangent function)

Practice

Use the diagram below for questions 1-3.

1. Find $\csc A$ and $\csc C$.
2. Find $\sec A$ and $\sec C$.
3. Find $\cot A$ and $\cot C$.

Use the diagram to fill in the blanks below.

4. $\cot A = \frac{?}{?}$
5. $\csc C = \frac{?}{?}$
6. $\cot C = \frac{?}{?}$
7. $\sec C = \frac{?}{?}$
8. $\csc A = \frac{?}{?}$
9. $\sec A = \frac{?}{?}$

From questions 4-9, we can conclude the following. Fill in the blanks.

10. $\sec$ ____ $= \csc A$ and $\csc$ ____ $= \sec A$.
11. $\cot A$ and $\cot C$ are _____ of each other.
12. Explain why the csc of an angle will always be greater than 1.
13. Use your knowledge of 45-45-90 triangles to find the cosecant, secant, and cotangent of a 45 degree angle.
14. Use your knowledge of 30-60-90 triangles to find the cosecant, secant, and cotangent of a 30 degree angle.
15. Use your knowledge of 30-60-90 triangles to find the cosecant, secant, and cotangent of a 60 degree angle.
16. As the degree of an angle increases, will the cotangent of the angle increase or decrease? Explain.

1.10 Pythagorean Theorem for Solving Right Triangles

Here you'll learn how to use the Pythagorean Theorem along with trig functions to solve for the unknown lengths and angles of a right triangle.

You are out on the playground with friends playing a game of tetherball. In this game, a ball is attached by a rope to the top of a pole. Each person is trying to hit the ball in a different direction until it wraps the rope completely around the pole. The first person to get the ball wrapped around the pole in their direction is the winner. You can see an example of a tetherball game on the right hand side of the picture shown here:

You notice that the rope attached to the tetherball is 1 meter long, and that the angle between the rope and the pole is $35°$ degrees. Can you use that information to find out how far the ball is from the pole?

At the end of this Concept, you'll know how to solve this problem.
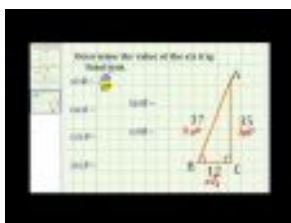
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James Sousa Example:Determine Trig FunctionValue Givena RightTriangle

Guidance

You can use your knowledge of the Pythagorean Theorem and the six trigonometric functions to solve a right triangle. Because a right triangle is a triangle with a 90 degree angle, solving a right triangle requires that you find the measures of one or both of the other angles. How you solve for these other angles, as well as the lengths of the triangle's sides, will depend on how much information is given.

Example A

Solve the triangle shown below.

Solution:

We need to find the lengths of all sides and the measures of all angles. In this triangle, two of the three sides are given. We can find the length of the third side using the Pythagorean Theorem:

$$8^2 + b^2 = 10^2$$
$$64 + b^2 = 100$$
$$b^2 = 36$$
$$b = \pm 6 \Rightarrow b = 6$$

(You may have also recognized that this is a "Pythagorean Triple," 6, 8, 10, instead of using the Pythagorean Theorem.)

You can also find the third side using a trigonometric ratio. Notice that the missing side, b, is adjacent to $\angle A$, and the hypotenuse is given. Therefore we can use the cosine function to find the length of b:

$$\cos 53.13° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{10}$$
$$0.6 = \frac{b}{10}$$
$$b = 0.6(10) = 6$$

Example B

Solve the triangle shown below.

Solution:

In this triangle, we need to find the lengths of two sides. We can find the length of one side using a trig ratio. Then we can find the length of the third side by using a trig function with the information given originally and a different trig function. Because the side we found is an approximation*Only use the given information when solving right triangles.*

We are given the measure of $\angle A$, and the length of the side adjacent to $\angle A$. If we want to find the length of the hypotenuse, c, we can use the cosine ratio:

$$\cos 40° = \frac{adjacent}{hypotenuse} = \frac{6}{c}$$
$$\cos 40° = \frac{6}{c}$$
$$c \cos 40° = 6$$
$$c = \frac{6}{\cos 40°} \approx 7.83$$

If we want to find the length of the other leg of the triangle, we can use the tangent ratio. This will give us the most accurate answer because we are not using approximations.

$$\tan 40° = \frac{opposite}{adjacent} = \frac{a}{6}$$
$$a = 6 \tan 40° \approx 5.03$$

Example C

Solve the triangle shown below.

Solution:

In this triangle, we have the length of one side and one angle. Therefore, we need to find the length of the other two sides. We can start with a trig function:

$$\tan 30° = \frac{opposite}{adjacent} = \frac{b}{7}$$
$$\tan 30° = \frac{b}{7}$$
$$7\tan 30° = b$$
$$b = 7\tan 30° \approx 4.04$$

We can then use another trig relationship to find the length of the hypotenuse:

$$\sin 30° = \frac{opposite}{hypotenuse} = \frac{4.04}{c}$$
$$\sin 30° = \frac{4.04}{c}$$
$$c\sin 30° = 4.04$$
$$c = \frac{4.04}{\sin 30°} \approx 8.08$$

Vocabulary

Sine: The **sine** of an angle in a right triangle is a relationship found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Cosine: The **cosine** of an angle in a right triangle is a relationship found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.

Tangent: The **tangent** of an angle in a right triangle is a relationship found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.

Guided Practice

1. Solve the triangle shown below:

2. Solve the triangle shown below:

3. Solve the triangle shown below:

Solutions:

1. Since the angle given is $40°$, and the length of the side opposite the angle is 9, we can use the tangent function to determine the length of the side adjacent to the angle:

$$\tan 40° = \frac{9}{a}$$
$$a = \frac{9}{\tan 40°}$$
$$a = \frac{9}{.839}$$
$$a = 10.73$$

We can then use another trig function to find the length of the hypotenuse:

$$\sin 40° = \frac{9}{c}$$
$$c = \frac{9}{\sin 40°}$$
$$c = \frac{9}{.643}$$
$$c = 13.997$$

Finally, the other angle in the triangle can be found either by a trigonometric relationship, or by recognizing that the sum of the internal angles of the triangle have to equal 180°:

$$90° + 40° + \theta = 180°$$
$$\theta = 180° - 90° - 40°$$
$$\theta = 50°$$

2. Since this triangle has two sides given, we can start with the Pythagorean Theorem to find the length of the third side:

$$a^2 + b^2 = c^2$$
$$8^2 + b^2 = 17^2$$
$$b^2 = 17^2 - 8^2$$
$$b^2 = 289 - 64 = 225$$
$$b = 15$$

With this knowledge, we can work to find the other two angles:

$$\tan \angle B = \frac{15}{8}$$
$$\tan \angle B = 1.875$$
$$\angle B = \tan^{-1} 1.875 \approx 61.93°$$

And the final angle is:

$$180° - 90° - 61.93° = 28.07°$$

3. There are a number of things known about this triangle. Since we know all of the internal angles, there are a few different ways to solve for the unknown sides. Here let's use the 60° angle to find the unknown sides:

$$\tan 60° = \frac{a}{4}$$
$$a = 4 \tan 60°$$
$$a = (4)(1.73) = 6.92$$

and

$$\cos 60° = \frac{4}{h}$$
$$h = \frac{4}{\cos 60°}$$
$$h = \frac{4}{.5}$$
$$h = 8$$

So we have found that the lengths of the sides are 4, 8, and 6.92.

Concept Problem Solution

From our knowledge of how to solve right triangles, we can set up a triangle with the rope and the pole, like this:

From this, it is straightforward to set up a trig relationship for sine that can help:

$$\sin 35° = \frac{opposite}{1}$$
$$(1) \sin 35° = opposite$$
$$opposite \approx .5736$$

Practice

Use the picture below for questions 1-3.

1. Find $m\angle A$.
2. Find $m\angle B$.
3. Find the length of AC.

Use the picture below for questions 4-6.

4. Find $m\angle A$.
5. Find $m\angle C$.
6. Find the length of AC.

Use the picture below for questions 7-9.

7. Find $m\angle A$.
8. Find $m\angle B$.
9. Find the length of BC.

Use the picture below for questions 10-12.

10. Find $m\angle A$.
11. Find $m\angle B$.
12. Find the length of AB.

Use the picture below for questions 13-15.

13. Find $m\angle A$.
14. Find $m\angle C$.
15. Find the length of BC.
16. Explain when to use a trigonometric ratio to find missing information about a triangle and when to use the Pythagorean Theorem.
17. Is it possible to have a triangle that you must use cosecant, secant, or cotangent to solve?
18. What is the minimum information you need about a triangle in order to solve it?

1.11 Inverse Trigonometric Functions

Here you'll learn how to use the inverses of trig functions to solve for unknown angles.

One day after school you are trying out for the track team. Your school has a flag pole at the very end of its football field. The pole is 50 feet tall. While standing at the end of the track, you know that the distance between you and the flag pole is 350 feet. Being curious about such things, you decide to find the angle between the ground and the top of the flag pole from where you are standing.

Can you solve this problem?

At the end of this Concept, you'll be able to do this using what is called an "inverse trigonometric function".

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James Sousa:Introduction toInverseSine, Inverse Cosine, and Inverse Tangent

Guidance

Consider the right triangle below.

From this triangle, we know how to determine all six trigonometric functions for both $\angle C$ and $\angle T$. From any of these functions we can also find the value of the angle, using our graphing calculators. You might recall that $\sin 30° = \frac{1}{2}$. If you type 30 into your graphing calculator and then hit the SIN button, the calculator yields 0.5. (Make sure your calculator's mode is set to degrees.)

Conversely, with the triangle above, we know the trig ratios, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. These functions are located above the SIN, COS, and TAN buttons on the calculator. To access one of these functions, press 2nd and the appropriate button and the measure of the angle appears on the screen.

$\cos T = \frac{24}{25} \rightarrow \cos^{-1} \frac{24}{25} = T$ from the calculator we get

Example A

Find the angle measure for the trig function below.

$\sin x = 0.687$

Solution: Plug into calculator.

$\sin^{-1} 0.687 = 43.4°$

Example B

Find the angle measure for the trig function below.

$\tan x = \frac{4}{3}$

Solution: Plug into calculator.

$\tan^{-1} \frac{4}{3} = 53.13°$

Example C

You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy and to move it manually down a ladder would take too much time and effort. You decide to devise a make shift conveyor belt made of bed sheets that you will attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?

Solution:

From the picture, we need to use the inverse sine function.

$$\sin \theta = \frac{25 \; feet}{30 \; feet}$$
$$\sin \theta = 0.8333$$
$$\sin^{-1}(\sin \theta) = \sin^{-1} 0.8333$$
$$\theta = 56.4°$$

The sheets should be anchored at an angle of $56.4°$.

Vocabulary

Inverse Trigonometric Function: An **inverse trigonometric function** is a function that cancels out a trigonometric function, leaving the argument of the original trigonometric function as a result.

Guided Practice

1. Find the angle measure for the trig function below.

$\sin x = 0.823$

2. Find the angle measure for the trig function below.

$\cos x = -0.112$

3. Find the angle measure for the trig function below.

$\tan x = 0.2$

Solutions:

1. Plug into calculator.

$\sin^{-1} 0.823 \approx 55.39°$

2. Plug into calculator.

$\cos^{-1} -0.112 \approx 96.43°$

3. Plug into calculator.

$\tan^{-1} 0.2 \approx 11.31°$

Concept Problem Solution

Using your knowledge of inverse trigonometric functions, you can set up a tangent relationship to solve for the angle:

$$\tan\theta = \frac{50}{350}$$
$$\theta = \tan^{-1} \frac{50}{350}$$
$$\theta \approx 8.13°$$

Practice

Use inverse trigonometry to find the angle measure of angle A for each angle below.

1. $\sin A = 0.839$
2. $\cos A = 0.19$
3. $\tan A = 0.213$
4. $\csc A = 1.556$
5. $\sec A = 2.063$
6. $\cot A = 2.356$
7. $\csc A = 8.206$
8. $\sin A = 0.9994$
9. $\cot A = 1.072$
10. $\cos A = 0.174$
11. $\tan A = 1.428$
12. $\csc A = 2.92$
13. A 70 foot building casts an 100 foot shadow. What is the angle that the sun hits the building?
14. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
15. Whitney is sailing and spots a shipwreck 100 feet below the water. She jumps from the boat and swims 250 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?

1.12 Alternate Formula for the Area of a Triangle

Here you'll learn how to find the area of a triangle using the length of two sides of the triangle, the angle between them, and the sine function.

You are studying the Gulf of Mexico in your Geography class. Your Instructor brings up the idea of the Bermuda Triangle. This is a place where, according to some, many planes get lost. Here is a picture of it:

photocredit, www.earthspots.com

The first thing this makes you think of is your math class, since that class is your favorite. You would like to know just how big the Bermuda Triangle is. Unfortunately, the Bermuda Triangle isn't a right triangle. However, you do know that the lengths of one of the sides is 950 miles, the other side is 975 miles, and the angle between them is $60°$.

Is there any way to use this information to help you find out just how big the Bermuda Triangle is?

Read on, and at the end of this Concept, you'll be able to use the information presented here to calculate the area of the Bermuda Triangle.

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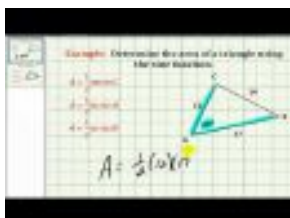
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James Sousa Example:Determine the Area of aTriangle Using the SineFunction

Guidance

In Geometry, you learned that the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height, or altitude. Now that you know the trig ratios, this formula can be changed around, using sine.

Looking at the triangle above, you can use sine to determine $h, \sin C = \frac{h}{a}$. So, solving this equation for h, we have $a \sin C = h$. Substituting this for h, we now have a new formula for area.

$$A = \frac{1}{2}ab \sin C$$

What this means is you do not need the height to find the area anymore. All you now need is two sides and the angle between the two sides, called the included angle.

Example A

Find the area of the triangle.

Solution: Using the formula, $A = \frac{1}{2} ab \sin C$, we have

$$
\begin{aligned}
A &= \frac{1}{2} \cdot 8 \cdot 13 \cdot \sin 82° \\
&= 4 \cdot 13 \cdot 0.990 \\
&= 51.494
\end{aligned}
$$

Example B

Find the area of the parallelogram.

Solution: Recall that a parallelogram can be split into two triangles. So the formula for a parallelogram, using the new formula, would be: $A = 2 \cdot \frac{1}{2} ab \sin C$ or $A = ab \sin C$.

$$
\begin{aligned}
A &= 7 \cdot 15 \cdot \sin 65° \\
&= 95.162
\end{aligned}
$$

Example C

Find the area of the triangle.

Solution: Using the formula, $A = \frac{1}{2} ab \sin C$, we have

$$
\begin{aligned}
A &= \frac{1}{2} \cdot 16.45 \cdot 19 \cdot \sin 30° \\
&= 8.225 \cdot 19 \cdot 0.5 \\
&= 78.14
\end{aligned}
$$

Vocabulary

Triangle Area Formula: The **triangle area formula** is a formula to find the area of a triangle involving the lengths of two sides of the triangle and the sine of the angle between them.

Guided Practice

1. Find the area of the triangle.

2. Find the area of the triangle.

3. Find the area of the triangle.

Solutions:

1. Using the formula, $A = \frac{1}{2} ab \sin C$, we have

$$
\begin{aligned}
A &= \frac{1}{2} \cdot 4 \cdot 10 \cdot \sin 15° \\
&= 2 \cdot 10 \cdot 0.2589 \\
&= 5.178
\end{aligned}
$$

2. Using the formula, $A = \frac{1}{2}\,ab\sin C$, we have

$$A = \frac{1}{2} \cdot 8 \cdot 15 \cdot \sin 25^\circ$$
$$= 4 \cdot 15 \cdot 0.4226$$
$$= 25.356$$

3. Using the formula, $A = \frac{1}{2}\,ab\sin C$, we have

$$A = \frac{1}{2} \cdot 10 \cdot 11 \cdot \sin 32^\circ$$
$$= 5 \cdot 11 \cdot 0.53$$
$$= 29.15$$

Concept Problem Solution

Now that you know the equation for the area of a triangle in terms of two of the sides and the included angle, we can use that to solve for the area of the Bermuda Triangle:

$$A = \frac{1}{2}ab\sin\theta$$
$$A = \frac{1}{2}(950)(975)\sin 60^\circ$$
$$A = \frac{1}{2}(950)(975)(.866)$$
$$A = 401066.25$$

The area of the triangle is approximately 401,006 square miles.

Practice

Use the following picture for questions 1 and 2.

1. Find the values of a, b, and C needed for the formula to find the area of the triangle.
2. Now find the area of the triangle.

Use the following picture for questions 3 and 4.

3. Find the values of a, b, and C needed for the formula to find the area of the triangle.
4. Now find the area of the triangle.

Find the area of each triangle below.

5.
6.
7.

8.
9.
10.
11.

Find the area of each parallelogram.

12.
13.
14. Describe another way you could have found the area of the parallelogram in the previous problem.
15. When you first learned about sine, you learned how it worked for right triangles. Explain why this method for calculating area uses sine, but works for non-right triangles.

1.13 Angles of Elevation and Depression

Here you'll learn how to use a triangle to find a distance if you know the angle of elevation or depression.

You have decided to go camping with some friends. While out on a hike, you reach the top of a ridge and look down at the trail behind you. In the distance, you can see your camp. You're thinking about how far you've traveled, and wonder if there is a way to determine it.

By using a small device called a clinometer, you're able to measure the angle between your horizontal line of sight and the camp as $37°$, and you know that the hill you just hiked up has a height of 300 m. Is it possible to find out how far away your camp is using this information? (Assume that the trail you hiked is slanted like the side of a triangle.)

Keep reading and at the conclusion of this Concept, you'll know how to solve this problem.

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James Sousa Example:Determine What TrigFunction Relates Specific Sidesof a Right Triangle

Guidance

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.

The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight down to

Example A

You are standing 20 feet away from a tree, and you measure the angle of elevation to be $38°$. How tall is the tree?

Solution:

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

$$\tan 38° = \frac{opposite}{adjacent} = \frac{T}{20}$$

$$\tan 38° = \frac{T}{20}$$

$$T = 20\tan 38° \approx 15.63$$

$$\text{Height of tree} \approx 20.63 \ ft$$

Example B

You are standing on top of a building, looking at a park in the distance. The angle of depression is $53°$. If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Solution:

If we ignore the height of the person, we solve the following triangle:

Given the angle of depression is $53°$, $\angle A$ in the figure above is $37°$. We can use the tangent function to find the distance from the building to the park:

$$\tan 37° = \frac{opposite}{adjacent} = \frac{d}{100}$$

$$\tan 37° = \frac{d}{100}$$

$$d = 100\tan 37° \approx 75.36 \ ft.$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

$$\tan 37° = \frac{opposite}{adjacent} = \frac{d}{105}$$

$$\tan 37° = \frac{d}{105}$$

$$d = 105\tan 37° \approx 79.12 \ ft.$$

If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Example C

You are on a long trip through the desert. In the distance you can see mountains, and a quick measurement tells you that the angle between the mountaintop and the ground is $13.4°$. From your studies, you know that one way to define a mountain is as a pile of land having a height of at least 2,500 meters. If you assume the mountain is the minimum possible height, how far are you away from the center of the mountain?

Solution:

$$\tan 13.4° = \frac{opposite}{adjacent} = \frac{2500}{d}$$
$$\tan 13.4° = \frac{2500}{d}$$
$$d = \frac{2500}{\tan 13.4°} \approx 10,494 \; meters.$$

Vocabulary

Angle of Depression: The **angle of depression** is the angle formed by a horizontal line and the line of sight down to an object when the image of an object is located beneath the horizontal line.

Angle of Elevation: The **angle of elevation** is the angle formed by a horizontal line and the line of sight up to an object when the image of an object is located above the horizontal line.

Guided Practice

1. You are six feet tall and measure the angle between the horizontal and a bird in the sky to be 40°. You can see that the shadow of the bird is directly beneath the bird, and 200 feet away from you on the ground. How high is the bird in the sky?

2. While out swimming one day you spot a coin at the bottom of the pool. The pool is ten feet deep, and the angle between the top of the water and the coin is 15°. How far away is the coin from you along the bottom of the pool?

3. You are hiking and come to a cliff at the edge of a ravine. In the distance you can see your campsite at the base of the cliff, on the other side of the ravine. You know that the distance across the ravine is 500 meters, and the angle between your horizontal line of sight and your campsite is 25°. How high is the cliff? (Assume you are five feet tall.)

Solutions:

1. We can use the tangent function to find out how high the bird is in the sky:

$$\tan 40° = \frac{height}{200}$$
$$height = 200 \tan 40°$$
$$height = (200)(.839)$$
$$height = 167.8$$

We then need to add your height to the solution for the triangle. Since you are six feet tall, the total height of the bird in the sky is 173.8 feet.

2. Since the distance along the bottom of the pool to the coin is the same as the distance along the top of the pool to the coin, we can use the tangent function to solve for the distance to the coin:

$$\tan 15° = \frac{opposite}{adjacent}$$
$$\tan 15° = \frac{10}{x}$$
$$x = \frac{10}{\tan 15°}$$
$$x \approx 37.32°$$

3. Using the information given, we can construct a solution:

$$\tan 25° = \frac{opposite}{adjacent}$$
$$\tan 25° = \frac{height}{500}$$
$$height = 500 \tan 25°$$
$$height = (500)(.466)$$
$$height = 233 \text{ meters}$$

This is the total height from the bottom of the ravine to your horizontal line of sight. Therefore, to get the height of the ravine, you should take away five feet for your height, which gives an answer of 228 meters.

Concept Problem Solution

Since you know the angle of depression is $37°$, you can use this information, along with the height of the hill, to create a trigonometric relationship:

Since the unknown side of the triangle is the hypotenuse, and you know the opposite side, you should use the sine relationship to solve the problem:

$$\sin 37° = \frac{300}{hypotenuse}$$
$$hypotenuse = \frac{300}{\sin 37°}$$
$$hypotenuse \approx 498.5$$

You have traveled approximately 498.5 meters up the hill.

Practice

1. A 70 foot building casts an 50 foot shadow. What is the angle that the sun hits the building?
2. You are standing 10 feet away from a tree, and you measure the angle of elevation to be $65°$. How tall is the tree? Assume you are 5 feet tall up to your eyes.
3. Kaitlyn is swimming in the ocean and notices a coral reef below her. The angle of depression is $35°$ and the depth of the ocean, at that point is 350 feet. How far away is she from the reef?
4. The angle of depression from the top of a building to the base of a car is $60°$. If the building is 78 ft tall, how far away is the car?

5. The Leaning Tower of Pisa currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?
6. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?
7. You are standing 15 feet away from a tree, and you measure the angle of elevation to be 35°. How tall is the tree? Assume you are 5 feet tall up to your eyes.
8. Bill spots a tree directly across the river from where he is standing. He then walks 18 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 55°. How wide is the river?
9. A 50 foot building casts an 50 foot shadow. What is the angle that the sun hits the building?
10. Eric is flying his kite one afternoon and notices that he has let out the entire 100 ft of string. The angle his string makes with the ground is 60°. How high is his kite at this time?
11. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?
12. Upon descent an airplane is 15,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
13. Tara is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 10 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
14. Over 3 miles (horizontal), a road rises 1000 feet (vertical). What is the angle of elevation?
15. Over 4 miles (horizontal), a road rises 1000 feet (vertical). What is the angle of elevation?

1.14 Right Triangles, Bearings, and other Applications

Here you'll learn how to relate right triangle relationships to course bearings.

While on a camping trip with your friends, you take an orienteering trip. You end up on a course which results in you hiking $30°$ toward the South of the direction of East. This is represented as $E30°S$. You hike until you are 5 miles from where you started. Is it possible to determine how far South you are from where you started?

Read on, and when you have completed this Concept, you'll be able to make this calculation.

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Click image to the left for more content.

GCSEMath Bearings

Guidance

We can also use right triangles to find distances using angles given as bearings. In navigation, a bearing is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north. The graph below shows an angle of 70 degrees:

It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are measured in trigonometry. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70°E$ refers to an angle from the north, towards the east, while $N70°W$ refers to an angle from the north, towards the west. $N70°E$ is the same as the angle shown in the graph above. $N70°W$ would result in an angle in the second quadrant.

Example A

A ship travels on a $N50°E$ course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?

Solution: The angle between d and 10 nm is the complement of $50°$, which is $40°$. Therefore we can find d using the cosine function:

$$\cos 40° = \frac{ad\,jacent}{hypotenuse} = \frac{10}{d}$$

$$\cos 40° = \frac{10}{d}$$

$$d\cos 40° = 10$$

$$d = \frac{10}{\cos 40°} \approx 13.05 \; nm$$

Example B

An airplane flies on a course of $S30°E$, for 150 km. How far south is the plane from where it originated?

Solution: We can construct a triangle using the known information, and then use the cosine function to solve the problem:

$$\cos 30° = \frac{ad\,jacent}{hypotenuse} = \frac{y}{150}$$

$$\cos 30° = \frac{y}{150}$$

$$150\cos 30° = y$$

$$y = 150\cos 30° \approx 130km$$

Example C

Jean travels to school each day by walking 200 meters due East, and then turning left and walking 100 meters due North. If she had walked in a straight line, what would the angle between her home and the school be if the beginning of the angle is taken from due East? What would be two different ways to describe the direction to take walking there in a straight line, using what we've learned in this Concept?

Solution: From the triangle given above, we can use the tangent function to determine the angle if she had walked in a straight line.

$$\tan\theta = \frac{opposite}{ad\,jacent} = \frac{100}{200}$$

$$\tan\theta = \frac{100}{200}$$

$$\theta = 26.57°$$

One way of describing her straight line path is how far north of east she is: $E26.57°N$

Also, since we know the angle between the East and the North directions is $90°$, her motion can also be described by how far east of north she is: $N63.43°E$

Vocabulary

Bearing: In navigation, a **bearing** is the direction taken to get from one place to another.

Guided Practice

1. Plot a course of $S30°W$ on a rectangular coordinate system.

2. Scott is boating on a course of $N15°E$. What course would he need to take to return to where he came from?

3. Adam hikes on a course of $N47°E$ for 7 km. How far East is Adam from where he started?

Solutions:

1.

2. The opposite direction would return him to his starting point. This would be $S15°W$.

3.

We can find the length of the triangle above (which is how far he traveled East) by using the sine function:

$$\sin 47° = \frac{x}{7}$$
$$x = 7\sin 47°$$
$$x = (7)(.7313)$$
$$x = 5.1191$$

He is 5.1191 km East of where he started.

Concept Problem Solution

From your knowledge of how to construct a triangle using bearings, you can draw the following:

This shows that the opposite side of the triangle is what's not known. Therefore, you can use the sine function to solve the problem:

$$\sin 30° = \frac{opposite}{5}$$
$$opposite = 5\sin 30°$$
$$opposite = (5)(.5) = 2.5$$

You are 2.5 miles South of where you started.

Practice

1. Plot a course of $N40°E$ on a rectangular coordinate system.
2. Plot a course of $E30°N$ on a rectangular coordinate system.
3. Plot a course of $S70°W$ on a rectangular coordinate system.
4. Plot a course of $W85°S$ on a rectangular coordinate system.
5. Plot a course of $N42°W$ on a rectangular coordinate system.
6. You are on a course of $E35°N$. What course would you need to take to return to where you came from?
7. You are on a course of $W56°S$. What course would you need to take to return to where you came from?
8. You are on a course of $N72°W$. What course would you need to take to return to where you came from?
9. You are on a course of $S10°E$. What course would you need to take to return to where you came from?

10. You are on a course of $W65°N$. What course would you need to take to return to where you came from?
11. You are on a course of $N47°E$ for 5 km. How far East are you from where you started?
12. You are on a course of $S32°E$ for 8 km. How far East are you from where you started?
13. You are on a course of $N15°W$ for 10 km. How far West are you from where you started?
14. You are on a course of $S3°W$ for 12 km. How far West are you from where you started?
15. You are on a course of $S67°E$ for 6 km. How far East are you from where you started?

1.15 Angles of Rotation in Standard Positions

Here you'll learn how to express angles of rotation.

While playing a game with friends, you are using a spinner. You know that the best number to land on is 7. The spinner looks like this:

Can you determine how to represent the angle of the spinner if it lands on the 7? Read on to answer this question by the end of this Concept.

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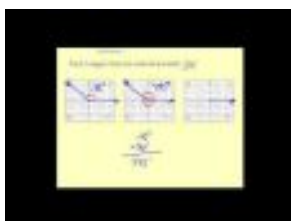
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James Sousa:Angles in Standard Position

Guidance

Consider our game that is played with a spinner. When you spin the spinner, how far has it gone? You can answer this question in several ways. You could say something like "the spinner spun around 3 times." This means that the spinner made 3 complete rotations, and then landed back where it started.

We can also measure the rotation in degrees. In the previous lesson we worked with angles in triangles, measured in degrees. You may recall from geometry that a full rotation is 360 degrees, usually written as $360°$. Half a rotation is then $180°$ and a quarter rotation is $90°$. Each of these measurements will be important in this Concept. We can use our knowledge of graphing to represent any angle. The figure below shows an angle in what is called **standard position**.

The initial side of an angle in standard position is always on the positive $x-$axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a **counterclockwise** direction. This means that if we rotate **clockwise**, we will generate a negative angle. Below are several examples of angles in standard position.

The 90 degree angle is one of four **quadrantal** angles. A quadrantal angle is one whose terminal side lies on an axis. Along with $90°$, $0°$, $180°$ and $270°$ are quadrantal angles.

These angles are referred to as quadrantal because each angle defines a quadrant. Notice that without the arrow indicating the rotation, $270°$ looks as if it is a $-90°$, defining the fourth quadrant. Notice also that $360°$ would look just like $0°$.

Example A

Identify what the angle is in this graph:

Solution:

The angle drawn out is 135°.

Example B

Identify what the angle is in this graph:

Solution:

The angle drawn out is 0°.

Example C

Identify what the angle is in this graph:

Solution:

The angle drawn out is 30°

Vocabulary

Standard Position: A **standard position** is the usual method of drawing an angle, where the measurement begins at the positive 'x' axis and is drawn counter-clockwise.

Quadrantal Angle: A **quadrantal angle** is an angle whose terminal side lies along either the positive or negative 'x' axis or the positive or negative 'y' axis.

Guided Practice

1. Identify what the angle is in this graph, using negative angles:

2. Identify what the angle is in this graph, using negative angles:

3. Identify what the angle is in this graph, using negative angles:

Solutions:

1. The angle drawn out is $-135°$.

2. The angle drawn out is $-180°$.

3. The angle drawn out is $-225°$.

Concept Problem Solution

Since you know that the angle between the horizontal and vertical directions is 90°, each number on the spinner takes up 30°. Therefore, since you are on the 7, you know that you are $\frac{2}{3}$ of the way to the vertical. Therefore, the angle of the spinner when it lands on 7 is 60°.

Practice

1. Draw an angle of 90°.
2. Draw an angle of 45°.
3. Draw an angle of $-135°$.
4. Draw an angle of $-45°$.
5. Draw an angle of $-270°$.

6. Draw an angle of 315°.

For each diagram, identify the angle. Write the angle using positive degrees.

7.
8.
9.

For each diagram, identify the angle. Write the angle using negative degrees.

10.
11.
12.
13. Explain how to convert between angles that use positive degrees and angles that use negative degrees.
14. At what angle is the 7 on a standard 12-hour clock? Use positive degrees.
15. At what angle is the 2 on a standard 12-hour clock? Use positive degrees.

1.16 Coterminal Angles

Here you'll learn how to identify coterminal angles.

While playing a game with friends, you use a spinner that looks like this:

As you can see, the angle that the spinner makes with the horizontal is $60°$. Is it possible to represent the angle any other way?

At the completion of this Concept, you'll know more than one way to represent this angle.

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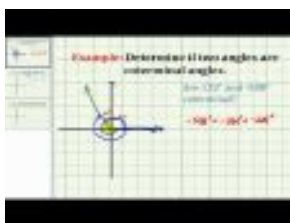
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James Sousa Example:Determine if Two AnglesareCoterminal

Guidance

Consider the angle $30°$, in standard position.

Now consider the angle $390°$. We can think of this angle as a full rotation $(360°)$, plus an additional 30 degrees.

Notice that $390°$ looks the same as $30°$. Formally, we say that the angles share the same terminal side. Therefore we call the angles **co-terminal**. Not only are these two angles co-terminal, but there are infinitely many angles that are co-terminal with these two angles. For example, if we rotate another $360°$, we get the angle $750°$. Or, if we create the angle in the negative direction (clockwise), we get the angle $-330°$. Because we can rotate in either direction, and we can rotate as many times as we want, we can continuously generate angles that are co-terminal with $30°$.

Example A

Is the following angle co-terminal with $45°$?

$-45°$

Solution: No, it is not co-terminal with $45°$

Example B

Is the following angle co-terminal with $45°$?

Solution: $405°$ Yes, $405°$ is co-terminal with $45°$.

Example C

Is the following angle co-terminal with $45°$?

$-315°$

Solution: Yes, $-315°$ is co-terminal with $45°$.

Vocabulary

Coterminal Angles: A set of **coterminal angles** are angles with the same terminal side but expressed differently, such as a different number of complete rotations around the unit circle or angles being expressed as positive versus negative angle measurements.

Guided Practice

1. Find a coterminal angle to $23°$

2. Find a coterminal angle to $-90°$

3. Find two coterminal angles to $70°$ by rotating in the positive direction around the circle.

Solutions:

1. A coterminal angle would be an angle that is at the same terminal place as $23°$ but has a different value. In this case, $-337°$ is a coterminal angle.

2. A coterminal angle would be an angle that is at the same terminal place as $-90°$ but has a different value. In this case, $270°$ is a coterminal angle.

3. Rotating once around the circle gives a coterminal angle of $430°$. Rotating again around the circle gives a coterminal angle of $790°$.

Concept Problem Solution

You can either think of $60°$ as $420°$ if you rotate all the way around the circle once and continue the rotation to where the spinner has stopped, or as $-300°$ if you rotate clockwise around the circle instead of counterclockwise to where the spinner has stopped.

Practice

1. Is $315°$ co-terminal with $-45°$?
2. Is $90°$ co-terminal with $-90°$?
3. Is $350°$ co-terminal with $-370°$?
4. Is $15°$ co-terminal with $1095°$?
5. Is $85°$ co-terminal with $1880°$?

For each diagram, name the angle in 3 ways. At least one way should use negative degrees.

6.
7.
8.
9.
10.

11. Name the angle of the 8 on a standard clock two different ways.
12. Name the angle of the 11 on a standard clock two different ways.
13. Name the angle of the 4 on a standard clock two different ways.
14. Explain how to determine whether or not two angles are co-terminal.
15. How many rotations is $4680°$?

1.17 Trigonometric Functions and Angles of Rotation

Here you'll learn how to use trig functions on angles of rotation about a circle.

You've been working hard in your math class, and are getting to be quite the expert on trig functions. Then one day your friend, who is a year ahead of you in school, approaches you.

"So, you're doing pretty well in math? And you're good with trig functions?" he asks with a smile.

"Yes," you reply confidently. "I am."

"Alright, then what's the sine of $150°$?" he asks.

"What? That doesn't make sense. No right triangle has an angle like that, so there's no way to define that function!" you say.

Your friend laughs. "As it turns out, it is quite possible to have trig functions of angles greater than $90°$."

Is your friend just playing a joke on you, or does he mean it? Can you actually calculate $\sin 150°$?

At the conclusion of this Concept, you'll be able to answer this question.

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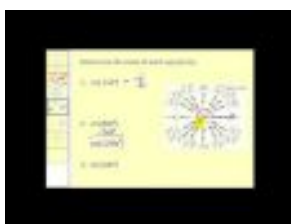
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James Sousa:Determine TrigonometricFunction ValuesUsing the Unit Circle

Guidance

Just as it is possible to define the six trigonometric functions for angles in right triangles, we can also define the same functions in terms of angles of rotation. Consider an angle in standard position, whose terminal side intersects a circle of radius r. We can think of the radius as the hypotenuse of a right triangle:

The point (x,y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of x, y, and r:

$$\cos\theta = \frac{x}{r} \qquad\qquad \sec\theta = \frac{r}{x}$$
$$\sin\theta = \frac{y}{r} \qquad\qquad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \qquad\qquad \cot\theta = \frac{x}{y}$$

And, we can extend these functions to include non-acute angles.

Consider an angle in standard position, such that the point (x, y) on the terminal side of the angle is a point on a circle with radius 1.

This circle is called the **unit circle**. With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad\qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad\qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$

$$\tan\theta = \frac{y}{x} \qquad\qquad \cot\theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the y and x coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.

We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $\sin 90° = y = 1$.

Example A

The point (-3, 4) is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Solution:

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

$$\cos\theta = \frac{-3}{5} \qquad\qquad \sec\theta = \frac{5}{-3}$$

$$\sin\theta = \frac{4}{5} \qquad\qquad \csc\theta = \frac{5}{4}$$

$$\tan\theta = \frac{4}{-3} \qquad\qquad \cot\theta = \frac{-3}{4}$$

Notice that the value of r depends on the coordinates of the given point. You can always find the value of r using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you can see, doing this allows us to simplify the definitions of the trig functions.

Example B

Use the unit circle above to find the value of $\cos 90°$

Solution: $\cos 90° = 0$

The ordered pair for this angle is (0, 1). The cosine value is the x coordinate, 0.

Example C

Use the unit circle above to find the value of $\cot 180°$

Solution: $\cot 180°$ is undefined

The ordered pair for this angle is (-1, 0). The ratio $\frac{x}{y}$ is $\frac{-1}{0}$, which is undefined.

Vocabulary

Quadrantal Angle: A **quadrantal angle** is an angle that has its terminal side on one of the four lines of axis: positive "x", negative "x", positive "y", or negative "y".

Guided Practice

Use this figure:

to answer the following questions.

1. Find $\cos\theta$ on the circle above.

2. Find $\cot\theta$ on the circle above.

3. Find $\csc\theta$ on the circle above.

Solutions:

1. We can see from the "x" and "y" axes that the "x" coordinate is $\frac{-\sqrt{3}}{2}$, the "y" coordinate is $\frac{1}{2}$, and the hypotenuse has a length of 1. This means that the cosine function is:

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{\frac{-\sqrt{3}}{2}}{1} = \frac{-\sqrt{3}}{2}$$

2. We know that $\cot = \frac{1}{\tan} = \frac{1}{\frac{opposite}{adjacent}} = \frac{adjacent}{opposite}$. The adjacent side to θ in the circle is $\frac{-\sqrt{3}}{2}$ and the opposite side is $\frac{1}{2}$. Therefore,

$$\cot\theta = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

3. We know that $\csc = \frac{1}{\sin} = \frac{1}{\frac{opposite}{hypotenuse}} = \frac{hypotenuse}{opposite}$. The opposite side to θ in the circle is $\frac{1}{2}$ and the hypotenuse is 1. Therefore,

$$\csc\theta = \frac{hypotenuse}{opposite} = \frac{1}{\frac{1}{2}} = 2$$

Concept Problem Solution

Since you now know that it is possible to apply trigonometric functions to angles greater than $90°$, you can calculate the $\sin 150°$. The easiest way to do this without difficulty is to consider that an angle of $150°$ is in the same position as $30°$, except it's in the second quadrant. This means that it has the same "x" and "y" values as $30°$, except that the "x" value is negative.

Therefore,

$\sin 150° = \frac{1}{2}$

Practice

Find the values of the six trigonometric functions for each angle below.

1. $0°$
2. $90°$
3. $180°$
4. $270°$
5. Find the sine of an angle that goes through the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
6. Find the cosine of an angle that goes through the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
7. Find the tangent of an angle that goes through the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
8. Find the secant of an angle that goes through the point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$.
9. Find the cotangent of an angle that goes through the point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$.
10. Find the cosecant of an angle that goes through the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$.
11. Find the sine of an angle that goes through the point $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.
12. Find the cosine of an angle that goes through the point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$.
13. The sine of an angle in the first quadrant is 0.25. What is the cosine of this angle?
14. The cosine of an angle in the first quadrant is 0.8. What is the sine of this angle?
15. The sine of an angle in the first quadrant is 0.15. What is the cosine of this angle?

1.18 Reference Angles and Angles in the Unit Circle

Here you'll learn the definition of reference angles and how to express angles on the unit circle.

When you walk into math class one day, your teacher has a surprise for the class. You're going to play series of games related to the things you've been learning about in class. For the first game, your teacher hands each group of students a spinner with an "x" and "y" axis marked. The game is to see how many angles you identify correctly. However, in this game, you are supposed to give what is called the "reference angle". You spin your spinner three times. Each picture below shows one of the spins:

Can you correctly identify the reference angles for these pictures?

At the end of this Concept, you'll know what reference angles are and be able to identify them in the pictures above.

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James Sousa:DeterminingTrig Function Values Using Reference AnglesandReference Triangles

Guidance

Consider the angle $150°$. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of $30°$, across the $y-$axis.

Notice that $150°$ makes a $30°$ angle with the negative $x-$axis. Therefore we say that $30°$ is the **reference angle** for $150°$. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the $x-$axis. Notice that $30°$ is the reference angle for many angles. For example, it is the reference angle for $210°$ and for $-30°$.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.

Example A

Graph each angle and identify its reference angle.

a. $140°$

b. $240°$

c. $380°$

Solution:

a. $140°$ makes a $40°$ angle with the negative $x-$axis. Therefore the reference angle is $40°$.

b. 240° makes a 60° with the negative $x-$axis. Therefore the reference angle is 60°.

c. 380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

Example B

Find the ordered pair for 240° and use it to find the value of $\sin 240°$.

Solution: $\sin 240° = \frac{-\sqrt{3}}{2}$

As we found in Example A, the reference angle for 240° is 60°. The figure below shows 60° and the three other angles in the unit circle that have 60° as a reference angle.

The terminal side of the angle 240° represents a reflection of the terminal side of 60° over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The $y-$coordinate is the sine value, so $\sin 240° = -\frac{\sqrt{3}}{2}$.

Just as the figure above shows 60° and three related angles, we can make similar graphs for 30° and 45°.

Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

Example C

Find the value of $\cot 300°$

Solution: $\cot 300° = -\frac{1}{\sqrt{3}}$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is $\cot 300° = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Vocabulary

Reference Angle: A **reference angle** is the angle formed between the terminal side of an angle and the closest of either the positive or negative 'x' axis.

Guided Practice

1. Graph 210° and identify its reference angle.

2. Graph 315° and identify its reference angle.

3. Find the ordered pair for 150° and use it to find the value of cos 150°.

Solutions:

1. The graph of 210° looks like this:

and since the angle makes a 30° angle with the negative "x" axis, the reference angle is 30°.

2. The graph of 315° looks like this:

and since the angle makes a 45° angle with the positive "x" axis, the reference angle is 45°.

3. Since the reference angle is $30°$, we know that the coordinates for the point on the unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. This is the same as the value for $30°$, except the "x" coordinate is negative instead of positive. Knowing this,

$$\cos 150° = \frac{adjacent}{hypotenuse} = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$$

Concept Problem Solution

Since you know how to measure reference angles now, upon examination of the spinners, you know that the first angle is $30°$, the second angle is $45°$, and the third angle is $60°$.

Practice

1. Graph $100°$ and identify its reference angle.
2. Graph $200°$ and identify its reference angle.
3. Graph $290°$ and identify its reference angle.

Calculate each value using the unit circle and special right triangles.

4. $\sin 120°$
5. $\cos 120°$
6. $\csc 120°$
7. $\cos 135°$
8. $\sin 135°$
9. $\tan 135°$
10. $\sin 210°$
11. $\cos 210°$
12. $\cot 210°$
13. $\sin 225°$
14. $\cos 225°$
15. $\sec 225°$

Here you'll learn how to find the results of trigonometric functions for negative angles.

While practicing for the track team, you regularly stop to consider the values of trig functions for the angle you've covered as you run around the circular track at your school. Today, however, is different. To keep things more interesting, your coach has decided to have you and your teammates run the opposite of the usual direction on the track. From your studies at school, you know that this is the equivalent of a "negative angle".

You have run $-45°$ around the track, and want to fine the value of the cosine function for this angle. Is it still possible to find the values of trig functions for these new types of angles?

At the completion of this Concept, you'll be able to calculate the values of trig functions for negative angles, and find the value of cosine for the $-45°$ you have traveled.

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Coterminal and Negative Angles

Guidance

Recall that graphing a negative angle means rotating clockwise. The graph below shows $-30°$.

Notice that this angle is coterminal with $330°$. So the ordered pair is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right)$. We can use this ordered pair to find the values of any of the trig functions of $-30°$. For example, $\cos(-30°) = x = \frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of $30°$, $45°$, or $60°$, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.

Example A

Find the value of the expression: $\sin(-45°)$

Solution:

$\sin(-45°) = -\frac{\sqrt{2}}{2}$

$-45°$ is in the 4th quadrant, and has a reference angle of $45°$. That is, this angle is coterminal with $315°$. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

Example B

Find the value of the expression: $\sec(-300°)$

Solution:

$\sec(-300°) = 2$

The angle $-300°$ is in the 1st quadrant and has a reference angle of $60°$. That is, this angle is coterminal with $60°$. Therefore the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ and the secant value is $\frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$.

Example C

Find the value of the expression: $\cos(-90°)$

Solution:

$\cos(-90°) = 0$

The angle $-90°$ is coterminal with $270°$. Therefore the ordered pair is (0, -1) and the cosine value is 0.

We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Vocabulary

Negative Angle: A **negative angle** is an angle measured by rotating clockwise (instead of counter-clockwise) from the positive 'x' axis.

Guided Practice

1. Find the value of the expression: $\cos -180°$

2. Find the value of the expression: $\sin -90°$

3. Find the value of the expression: $\tan -270°$

Solutions:

1. The angle $-180°$ is coterminal with $180°$. Therefore the ordered pair of points is (-1, 0). The cosine is the "x" coordinate, so here it is -1.

2. The angle $-90°$ is coterminal with $270°$. Therefore the ordered pair of points is (0, -1). The sine is the "y" coordinte, so here it is -1.

3. The angle $-270°$ is coterminal with $90°$. Therefore the ordered pair of points is (0, 1). The tangent is the "y" coordinate divided by the "x" coordinate. Since the "x" coordinate is 0, the tangent is undefined.

Concept Problem Solution

What you want to find is the value of the expression: $\cos(-45°)$

Solution:

$\cos(-45°) = \frac{\sqrt{2}}{2}$

$-45°$ is in the 4th quadrant, and has a reference angle of $45°$. That is, this angle is coterminal with $315°$. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$ and the cosine value is $\frac{\sqrt{2}}{2}$.

Practice

Calculate each value.

1. $\sin -120°$
2. $\cos -120°$
3. $\tan -120°$
4. $\csc -120°$
5. $\sec -120°$
6. $\cot -120°$
7. $\csc -45°$
8. $\sec -45°$
9. $\tan -45°$
10. $\cos -135°$
11. $\csc -135°$
12. $\sec -135°$
13. $\tan -210°$
14. $\sin -270°$
15. $\cot -90°$

1.20 Trigonometric Functions of Angles Greater than 360 Degrees

Here you'll learn how to find the values of trigonometric functions for angles exceeding 360 degrees.

While out at the local amusement park with friends, you take a ride on the Go Karts. You ride around a circular track in the carts three and a half times, and then stop at a "pit stop" to rest. While waiting for your Go Kart to get more fuel, you are talking with your friends about the ride. You know that one way of measuring how far something has gone around a circle (or the trig values associated with it) is to use angles. However, you've gone more than one complete circle around the track.

Is it still possible to find out what the values of sine and cosine are for the change in angle you've made?

When you complete this Concept, you'll be able to answer this question by computing the trig values for angles greater than $360°$

Watch This

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The UnitCircle

Guidance

Consider the angle $390°$. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore $390°$ is coterminal with $30°$. As you saw above with negative angles, this means that $390°$ has the same ordered pair as $30°$, and so it has the same trig values. For example,

$$\cos 390° = \cos 30° = \frac{\sqrt{3}}{2}$$

In general, if an angle whose measure is greater than $360°$ has a reference angle of $30°$, $45°$, or $60°$, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. Again, determine the reference angle first.

Example A

Find the value of the expression: $\sin 420°$

Solution:

$\sin 420° = \frac{\sqrt{3}}{2}$

$420°$ is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with $60°$, and so it shares the same ordered pair, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The sine value is the $y-$coordinate.

Example B

Find the value of the expression: $\tan 840°$

Solution:

$\tan 840° = -\sqrt{3}$

$840°$ is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore $840°$ is coterminal with $120°$, so the ordered pair is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan 840° = \tan 120° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

Example C

Find the value of the expression: $\cos 540°$

Solution:

$\cos 540° = -1$

$540°$ is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with $180°$, and the ordered pair is (-1, 0). So the cosine value is -1.

Vocabulary

Coterminal: Two angles are **coterminal** if they are drawn in the standard position and both have terminal sides that are at the same location.

Guided Practice

1. Find the value of the expression: $\sin 570°$

2. Find the value of the expression: $\cos 675°$

3. Find the value of the expression: $\sin 480°$

Solutions:

1. Since $570°$ has the same terminal side as $210°$, $\sin 570° = \sin 210° = \frac{\frac{-1}{2}}{1} = \frac{-1}{2}$

2. Since $675°$ has the same terminal side as $315°$, $\cos 675° = \cos 315° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$

3. Since $480°$ has the same terminal side as $120°$, $\sin 480° = \sin 120° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

Concept Problem Solution

Since you've gone around the track 3.5 times, the total angle you've traveled is $360° \times 3.5 = 1260°$. However, as you learned in this unit, this is equivalent to $180°$. So you can use that value in your computations:

$$\sin 1260° = \sin 180° = 0$$
$$\cos 1260° = \cos 180° = -1$$

Practice

Find the value of each expression.

1. $\sin 405°$
2. $\cos 810°$
3. $\tan 630°$
4. $\cot 900°$
5. $\csc 495°$
6. $\sec 510°$
7. $\cos 585°$
8. $\sin 600°$
9. $\cot 495°$
10. $\tan 405°$
11. $\cos 630°$
12. $\sec 810°$
13. $\csc 900°$
14. $\tan 600°$
15. $\sin 585°$
16. $\tan 510°$
17. Explain how to evaluate a trigonometric function for an angle greater than $360°$.

1.21 Reciprocal Identities

Here you'll learn what the reciprocal trig functions are, and how they relate to the sine, cosine, and tangent functions.

You are already familiar with the trig identities of sine, cosine, and tangent. As you know, any fraction also has an inverse, which is found by reversing the positions of the numerator and denominator.

Can you list what the ratios would be for the three trig functions (sine, cosine, and tangent) with the numerators and denominators reversed?

At the end of this Concept, you'll be able to list these ratios, as well as know what they are called.

Watch This

The first portion of this video will help you understand reciprocal functions.

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Click image to the left for more content.

James Sousa:TheReciprocal,Quotient, and Pythagorean Identities

Guidance

A **reciprocal** of a fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{or} \qquad \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \qquad \text{or} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Example A

Find the value of the expression using a reciprocal identity.

$\cos\theta = .3, \sec\theta = ?$

Solution: $\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = \frac{10}{3}$.

Example B

Find the value of the expression using a reciprocal identity.

$\cot\theta = \frac{4}{3}, \tan\theta =?$

Solution: These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

Example C

Find the value of the expression using a reciprocal identity.

$\sin\theta = \frac{1}{2}, \csc\theta =?$

Solution: These functions are reciprocals, and the reciprocal of $\frac{1}{2}$ is 2.

Vocabulary

Domain: The **domain** of a function is the set of 'x' values for which the function is defined.

Range: The **range** of a function is the set of 'y' values for which the function is defined.

Reciprocal Trig Function: A **reciprocal trig function** is a relationship that is the reciprocal of a typical trig function. For example, since $\sin x = \frac{opposite}{hypotenuse}$, the reciprocal function is $\csc x = \frac{hypotenuse}{opposite}$

Guided Practice

1. State the reciprocal function of cosecant.

2. Find the value of the expression using a reciprocal identity.

$\sec\theta = \frac{2}{\pi}, \cos\theta =?$

3. Find the value of the expression using a reciprocal identity.

$\csc\theta = 4, \cos\theta =?$

Solutions:

1. The reciprocal function of cosecant is sine.

2. These functions are reciprocals, and the reciprocal of $\frac{2}{\pi}$ is $\frac{\pi}{2}$.

3. These functions are reciprocals, and the reciprocal of 4 is $\frac{1}{4}$.

Concept Problem Solution

Since the three regular trig functions are defined as:

$$\sin = \frac{opposite}{hypotenuse}$$
$$\cos = \frac{adjacent}{hypotenuse}$$
$$\tan = \frac{opposite}{adjacent}$$

then the three functions - called "reciprocal functions" are:

$$\csc = \frac{hypotenuse}{opposite}$$
$$\sec = \frac{hypotenuse}{adjacent}$$
$$\cot = \frac{adjacent}{opposite}$$

Practice

1. State the reciprocal function of secant.
2. State the reciprocal function of cotangent.
3. State the reciprocal function of sine.

Find the value of the expression using a reciprocal identity.

4. $\sin\theta = \frac{1}{2}, \csc\theta = ?$
5. $\cos\theta = -\frac{\sqrt{3}}{2}, \sec\theta = ?$
6. $\tan\theta = 1, \cot\theta = ?$
7. $\sec\theta = \sqrt{2}, \cos\theta = ?$
8. $\csc\theta = 2, \sin\theta = ?$
9. $\cot\theta = -1, \tan\theta = ?$
10. $\sin\theta = \frac{\sqrt{3}}{2}, \csc\theta = ?$
11. $\cos\theta = 0, \sec\theta = ?$
12. $\tan\theta = $ undefined, $\cot\theta = ?$
13. $\csc\theta = \frac{2\sqrt{3}}{3}, \sin\theta = ?$
14. $\sin\theta = \frac{-1}{2}$ and $\tan\theta = \frac{\sqrt{3}}{3}, \cos\theta = ?$
15. $\cos\theta = \frac{\sqrt{2}}{2}$ and $\tan\theta = 1, \sin\theta = ?$

1.22 Domain, Range, and Signs of Trigonometric Functions

Here you'll learn the domain and range, as well as the sign in different quadrants, for six trig functions.

You are doodling in art class one day when you draw a circle. Then you draw a few lines extending outward from the center to the edge of the circle. You draw a triangle with the "x" axis, and realize that you're thinking about your math class again.

You notice that the relationship for the sine function involves the length of the side opposite the angle divided by the length of the hypotenuse. But while the hypotenuse is always a positive number, the sign of the opposite side can be different, depending on what quadrant the angle is drawn in.

Can you determine what the sign of the sine function will be in each of the four quadrants, based on the knowledge of the ratio that defines the sine function?

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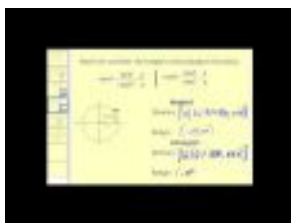
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James Sousa:Domain,Range, and Signs ofTrigonometric Functions

Guidance

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, R. We can determine the range of the functions if we think about the fact that the sine of an angle is the $y-$coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $x-$coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.

In this right triangle, x and y are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative $y-$coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative $x-$coordinate.

In either case, the minimum value is -1. For example, $\cos 180° = -1$ and $\sin 270° = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

TABLE 1.1:

	Domain	Range
Sine	$\theta = R$	$-1 \le y \le 1$
Cosine	$\theta = R$	$-1 \le y \le 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are $0°$, $180°$, $360°$, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

TABLE 1.2:

	Domain	Range
Cosecant	$\theta \varepsilon R, \theta \ne 0, 180, 360\dots$	$\csc\theta \le -1$ or $\csc\theta \ge 1$
Secant	$\theta \varepsilon R, \theta \ne 90, 270, 450\dots$	$\sec\theta \le -1$ or $\sec\theta \ge 1$

Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan\theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an $x-$coordinate of $0 : 90°, 270°$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a $y-$coordinate of $0 : 0°$, $180°$, $360°$, etc.

TABLE 1.3:

Function	Domain	Range
Tangent	$\theta \varepsilon R, \theta \ne 90, 270, 450\dots$	All reals
Cotangent	$\theta \varepsilon R, \theta \ne 0, 180, 360\dots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.

An easy way to remember this is "All Students Take Calculus." Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which trig functions are positive and where.

Example A

State the sign of $\cos 100°$

Solution: The angle $100°$ is in the second quadrant. Therefore the $x-$coordinate is negative and so $\cos 100°$ is negative.

Example B

State the sign of $\csc 220°$

Solution: The angle $220°$ is in the third quadrant. Therefore the $y-$coordinate is negative. So the sine, and the cosecant are negative.

Example C

State the sign of $\tan 370°$

Solution: The angle $370°$ is in the first quadrant. Therefore the tangent value is positive.

Vocabulary

Domain: The **domain** of a function is the set of 'x' values for which the function is defined.

Range: The **range** of a function is the set of 'y' values for which the function is defined.

Guided Practice

1. State the sign of $\cos 70°$

2. State the sign of $\sin 130°$

3. State the sign of $\tan 250°$

Solutions:

1. The angle $70°$ is in the first quadrant. Cosine is defined to be the adjacent side divided by the hypotenuse. Since the hypotenuse of the unit circle is one and the adjacent side is the "x" coordinate, the sign of the cosine function is determined by the sign of the "x" coordinate. Since $70°$ is in the first quadrant, the "x" value is positive. Therefore the cosine value is positive.

2. The angle $130°$ is in the second quadrant. Sine is defined to be the opposite side divided by the hypotenuse. Since the hypotenuse of the unit circle is one and the opposite side is the "y" coordinate, the sign of the sine function is determined by the sign of the "y" coordinate. Since $130°$ is in the second quadrant, the "y" value is positive. Therefore the sine value is positive.

3. The angle $250°$ is in the third quadrant. Tangent is defined to be the opposite side divided by the adjacent side. In the third quadrant, the "x" values are negative, and the "y" values are negative. A negative divided by a negative equals a positive. Therefore the tangent of $250°$ is positive.

Concept Problem Solution

Since the sine function is defined to be the length of the opposite side divided by the length of the hypotenuse, the sign of the sine function is the sign of the "y" coordinate for whatever quadrant is being considered. In quadrants 1 and 2, the "y" coordinate is positive, so the sine function is positive. In quadrants 3 and 4, the "y" coordinate is negative, so the sine function is negative as well.

Practice

1. In what quadrants is the sine function positive?
2. In what quadrants is the cotangent function negative?

3. In what quadrants is the cosine function negative?
4. In what quadrants is the tangent function positive?
5. For what angles is the cosecant function undefined?
6. State the sign of $\sin 510°$.
7. State the sign of $\cos 315°$.
8. State the sign of $\tan 135°$.
9. State the sign of $\cot 220°$.
10. State the sign of $\csc 40°$.
11. State the sign of $\cos 330°$.
12. State the sign of $\sin 60°$.
13. State the sign of $\sec -45°$.
14. Explain why the cosecant function is never equal to 0.
15. Using your knowledge of domain and range, make a possible sketch for $y = \sin x$.

1.23 Quotient Identities

Here you'll learn what a quotient identity is and how to derive it.

You are working in math class one day when your friend leans over and asks you what you got for the sine and cosine of a particular angle.

"I got $\frac{1}{2}$ for the sine, and $\frac{\sqrt{3}}{2}$ for the cosine. Why?" you ask.

"It looks like I'm supposed to calculate the tangent function for the same angle you just did, but I can't remember the relationship for tangent. What should I do?" he says.

Do you know how you can help your friend find the answer, even if both you and he don't remember the relationship for tangent?

Keep reading, and by the end of this Concept, you'll be able to help your friend.

Watch This

The middle portion of this video reviews the Quotient Identities.

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Click image to the left for more content.

James Sousa:TheReciprocal,Quotient, and Pythagorean Identities

Guidance

The definitions of the trig functions led us to the reciprocal identities, which can be seen in the Concept about that topic. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example A

If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

Solution: $\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example B

Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solution:

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

Example C

If $\cos\theta = \frac{7}{25}$ and $\sin\theta = \frac{24}{25}$, what is the value of $\cot\theta$?

Solution: $\cot\theta = \frac{7}{24}$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}$$

Vocabulary

Quotient Identity: The **quotient identity** is an identity relating the tangent of an angle to the sine of the angle divided by the cosine of the angle.

Guided Practice

1. If $\cos\theta = \frac{17}{145}$ and $\sin\theta = \frac{144}{145}$, what is the value of $\tan\theta$?

2. If $\sin\theta = \frac{63}{65}$ and $\cos\theta = \frac{16}{65}$, what is the value of $\tan\theta$?

3. If $\tan\theta = \frac{40}{9}$ and $\cos\theta = \frac{9}{41}$, what is the value of $\sin\theta$?

Solutions:

1. $\tan\theta = \frac{144}{17}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{144}{145}}{\frac{17}{145}} = \frac{144}{145} \times \frac{145}{17} = \frac{144}{17}$$

2. $\tan\theta = \frac{63}{16}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{65} \times \frac{65}{16} = \frac{63}{16}$$

3. $\sin\theta = \frac{40}{41}$. We can see this from the relationship for the tangent function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin\theta = (\tan\theta)(\cos\theta)$$
$$\sin\theta = \frac{40}{9} \times \frac{9}{41}$$
$$\sin\theta = \frac{40}{41}$$

Concept Problem Solution

Since you now know that:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

you can use this knowledge to help your friend with the sine and cosine values you measured for yourself earlier:

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

Practice

Fill in each blank with a trigonometric function.

1. $\tan\theta = \frac{\sin\theta}{?}$
2. $\cos\theta = \frac{\sin\theta}{?}$
3. $\cot\theta = \frac{?}{\sin\theta}$
4. $\cos\theta = (\cot\theta) \cdot (?)$
5. If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{1}{13}$, what is the value of $\tan\theta$?
6. If $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$, what is the value of $\tan\theta$?
7. If $\cos\theta = \frac{7}{25}$ and $\sin\theta = \frac{24}{25}$, what is the value of $\tan\theta$?
8. If $\sin\theta = \frac{12}{37}$ and $\cos\theta = \frac{35}{37}$, what is the value of $\tan\theta$?
9. If $\cos\theta = \frac{20}{29}$ and $\sin\theta = \frac{21}{29}$, what is the value of $\tan\theta$?
10. If $\sin\theta = \frac{39}{89}$ and $\cos\theta = \frac{80}{89}$, what is the value of $\tan\theta$?
11. If $\cos\theta = \frac{48}{73}$ and $\sin\theta = \frac{55}{73}$, what is the value of $\tan\theta$?
12. If $\sin\theta = \frac{65}{97}$ and $\cos\theta = \frac{72}{97}$, what is the value of $\tan\theta$?
13. If $\cos\theta = \frac{1}{2}$ and $\cot\theta = \frac{\sqrt{3}}{3}$, what is the value of $\sin\theta$?
14. If $\tan\theta = 0$ and $\cos\theta = -1$, what is the value of $\sin\theta$?
15. If $\cot\theta = -1$ and $\sin\theta = -\frac{\sqrt{2}}{2}$, what is the value of $\cos\theta$?

1.24 Cofunction Identities and Reflection

Here you'll learn about the four cofunction identities and how to apply them to solve for the values of trig functions.

While toying with a triangular puzzle piece, you start practicing your math skills to see what you can find out about it. You realize one of the interior angles of the puzzle piece is $30°$, and decide to compute the trig functions associated with this angle. You immediately want to compute the cosine of the angle, but can only remember the values of your sine functions.

Is there a way to use this knowledge of sine functions to help you in your computation of the cosine function for $30°$?

Read on, and by the end of this Concept, you'll be able to apply knowledge of the sine function to help determine the value of a cosine function.

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Cofunctions

Guidance

In a right triangle, you can apply what are called "cofunction identities". These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90° - \theta) \qquad\qquad \cos\theta = \sin(90° - \theta)$$
$$\tan\theta = \cot(90° - \theta) \qquad\qquad \cot\theta = \tan(90° - \theta)$$

Example A

Find the value of $\cos 120°$.

Solution: Because this angle has a reference angle of $60°$, the answer is $\cos 120° = -\frac{1}{2}$.

Example B

Find the value of $\cos(-120°)$.

Solution: Because this angle has a reference angle of $60°$, the answer is $\cos(-120°) = \cos 240° = -\frac{1}{2}$.

Example C

Find the value of $\sin 135°$.

Solution: Because this angle has a reference angle of $45°$, the answer is $\sin 135° = \frac{\sqrt{2}}{2}$

Vocabulary

Cofunction Identity: A **cofunction identity** is a relationship between one trig function of an angle and another trig function of the complement of that angle.

Guided Practice

1. Find the value of $\sin 45°$ using a cofunction identity.

2. Find the value of $\cos 45°$ using a cofunction identity.

3. Find the value of $\cos 60°$ using a cofunction identity.

Solutions:

1. The sine of $45°$ is equal to $\cos(90° - 45°) = \cos 45° = \frac{\sqrt{2}}{2}$.

2. The cosine of $45°$ is equal to $\sin(90° - 45°) = \sin 45° = \frac{\sqrt{2}}{2}$.

3. The cosine of $60°$ is equal to $\sin(90° - 60°) = \sin 30° = .5$.

Concept Problem Solution

Since you now know the cofunction relationships, you can use your knowledge of sine functions to help you with the cosine computation:

$$\cos 30° = \sin(90° - 30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

Practice

1. Find a value for θ for which $\sin \theta = \cos 15°$ is true.
2. Find a value for θ for which $\cos \theta = \sin 55°$ is true.
3. Find a value for θ for which $\tan \theta = \cot 80°$ is true.
4. Find a value for θ for which $\cot \theta = \tan 30°$ is true.
5. Use cofunction identities to help you write the expression $\tan 255°$ as the function of an acute angle of measure less than $45°$.
6. Use cofunction identities to help you write the expression $\sin 120°$ as the function of an acute angle of measure less than $45°$.
7. Use cofunction identities to help you write the expression $\cos 310°$ as the function of an acute angle of measure less than $45°$.
8. Use cofunction identities to help you write the expression $\cot 260°$ as the function of an acute angle of measure less than $45°$.
9. Use cofunction identities to help you write the expression $\cos 280°$ as the function of an acute angle of measure less than $45°$.
10. Use cofunction identities to help you write the expression $\tan 60°$ as the function of an acute angle of measure less than $45°$.

11. Use cofunction identities to help you write the expression $\sin 100°$ as the function of an acute angle of measure less than $45°$.
12. Use cofunction identities to help you write the expression $\cos 70°$ as the function of an acute angle of measure less than $45°$.
13. Use cofunction identities to help you write the expression $\cot 240°$ as the function of an acute angle of measure less than $45°$.
14. Use a right triangle to prove that $\sin \theta = \cos(90° - \theta)$.
15. Use the sine and cosine cofunction identities to prove that $\tan(90° - \theta) = \cot \theta$.

1.25 Pythagorean Identities

Here you'll learn what a Pythagorean Identity is and how to use it to solve for unknown values of trig functions.

What if you were working on a problem using the unit circle and had the value of one trig function (such as sine), but wanted instead to find the value of another trig function (such as cosine)? Is this possible?

Try it with $\sin\theta = \frac{1}{2}$

Keep reading, and when this Concept is finished, you'll know how to use this information to help you find $\cos\theta$.

Watch This

The final portion of this video reviews the Pythagorean Identities.

MEDIA
Click image to the left for more content.

James Sousa:TheReciprocal,Quotient, and Pythagorean Identities

Guidance

One set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In other Concepts we used the Pythagorean Theorem to find the sides of right triangles. Consider the way that the trig functions are defined. Let's look at the unit circle:

The legs of the right triangle are x and y. The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example A

If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{1}{4}\right)^2 + \sin^2\theta = 1$$

$$\frac{1}{16} + \sin^2\theta = 1$$

$$\sin^2\theta = 1 - \frac{1}{16}$$

$$\sin^2\theta = \frac{15}{16}$$

$$\sin\theta = \pm\sqrt{\frac{15}{16}}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example B

Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that $\cot^2\theta + 1 = \csc^2\theta$

Solution:

$$\cos^2\theta + \sin^2\theta = 1$$ Divide both sides by $\sin^2\theta$.

$$\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$ $\frac{\sin^2\theta}{\sin^2\theta} = 1$

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} + 1 = \frac{1}{\sin\theta} \times \frac{1}{\sin\theta}$$ Write the squared functions in terms

of their factors.

$$\cot\theta \times \cot\theta + 1 = \csc\theta \times \csc\theta$$ Use the quotient and reciprocal

identities.

$$\cot^2\theta + 1 = \csc^2\theta$$ Write the functions as squared

functions.

Example C

If $\sin\theta = \frac{1}{2}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\cos\theta = \sqrt{\frac{3}{4}}$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$
$$\frac{1}{4} + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{1}{4}$$
$$\cos^2\theta = \frac{3}{4}$$
$$\cos\theta = \pm\sqrt{\frac{3}{4}}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the cosine value is positive, so $\cos\theta = \sqrt{\frac{3}{4}}$.

Vocabulary

Pythagorean Identity: A **pythagorean identity** is a relationship showing that the sine of an angle squared plus the cosine of an angle squared is equal to one.

Guided Practice

1. If $\cos\theta = \frac{1}{2}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

2. If $\sin\theta = \frac{1}{8}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

3. If $\sin\theta = \frac{1}{3}$ what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.

Solutions:

1. The solution is $\sin\theta = \sqrt{\frac{3}{4}}$. We can see this from the Pythagorean Identity:

$$\cos^2\theta + \sin^2\theta = 1$$
$$\left(\frac{1}{2}\right)^2 + \sin^2\theta = 1$$
$$\frac{1}{4} + \sin^2\theta = 1$$
$$\sin^2\theta = 1 - \frac{1}{4}$$
$$\sin^2\theta = \frac{3}{4}$$
$$\sin\theta = \pm\sqrt{\frac{3}{4}}$$

2. The solution is $\cos\theta = \sqrt{\frac{63}{64}}$. We can see this from the Pythagorean Identity:

$$\cos^2\theta + \sin^2\theta = 1$$
$$\left(\frac{1}{8}\right)^2 + \cos^2\theta = 1$$
$$\frac{1}{64} + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{1}{64}$$
$$\cos^2\theta = \frac{63}{64}$$
$$\cos\theta = \pm\sqrt{\frac{63}{64}}$$

3. The solution is $\cos\theta = \sqrt{\frac{8}{9}}$. We can see this from the Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$
$$\frac{1}{9} + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{1}{9}$$
$$\cos^2\theta = \frac{8}{9}$$
$$\cos\theta = \pm\sqrt{\frac{8}{9}}$$

Concept Problem Solution

Since we now know that:

$\sin^2\theta + \cos^2\theta = 1$

we can use this to help us compute the cosine of the angle from the problem at the beginning of this Concept. It was given at the beginning of this Concept that:

$\sin\theta = \frac{1}{2}$

Therefore, $\sin^2\theta = \frac{1}{4}$

If we use this to solve for cosine:

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$\cos^2 \theta = 1 - \sin^2 \theta$$
$$\cos^2 \theta = 1 - \frac{1}{4}$$
$$\cos^2 \theta = \frac{3}{4}$$
$$\cos \theta = \frac{\sqrt{3}}{2}$$

Practice

1. If you know $\sin \theta$, what other trigonometric value can you determine using a Pythagorean Identity?
2. If you know $\sec \theta$, what other trigonometric value can you determine using a Pythagorean Identity?
3. If you know $\cot \theta$, what other trigonometric value can you determine using a Pythagorean Identity?
4. If you know $\tan \theta$, what other trigonometric value can you determine using a Pythagorean Identity?

For questions 5-14, assume all angles are in the first quadrant.

5. If $\sin \theta = \frac{1}{2}$, what is the value of $\cos \theta$?
6. If $\cos \theta = \frac{\sqrt{2}}{2}$, what is the value of $\sin \theta$?
7. If $\tan \theta = 1$, what is the value of $\sec \theta$?
8. If $\csc \theta = \sqrt{2}$, what is the value of $\cot \theta$?
9. If $\sec \theta = 2$, what is the value of $\tan \theta$?
10. If $\cot \theta = \sqrt{3}$, what is the value of $\csc \theta$?
11. If $\cos \theta = \frac{1}{4}$, what is the value of $\sin \theta$?
12. If $\sec \theta = 3$, what is the value of $\tan \theta$?
13. If $\sin \theta = \frac{1}{5}$, what is the value of $\cos \theta$?
14. If $\tan \theta = \frac{\sqrt{3}}{3}$, what is the value of $\sec \theta$?
15. Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ to show that $\tan^2 \theta + 1 = \sec^2 \theta$

Summary

This chapter introduced properties and functions related to the study of triangles. The Pythagorean Theorem was presented as a way to find the length of unknown sides of a right triangle. "Special Triangles" were introduced as being right triangles with certain internal angles that lead to well-known properties. Also introduced were functions involving angles in a triangle. referred to as "trigonmetric functions". These functions are relationships between the sides of a triangle as a ratio of one side to another. This was followed by lessons on how to apply trigonometry to rotation in a circle by using one axis as a basis and the angle of rotation from that axis as the argument of a function. Finally, derivation of other trigonometric functions and identities from the known trigonometric functions were presented.

CHAPTER **2**

Graphs of Trigonometric Functions

Chapter Outline

Introduction

This chapter deals with how to graph trigonometric functions. To do this effectively, you'll be dealing with not only the trigonometric functions themselves, but with related topics.

You'll learn how to measure angles in a new way, called "radians". This will introduce you to how to think of angles as a relationship among lengths instead of as an arbitrary number of "degrees". In this vein, you'll learn about applications of this type of measure when dealing with circles.

While dealing with trigonometric functions and how to graph them, you'll also learn how to represent changes to graphs of functions, such as horizontal and vertical shifts, as well as "stretches" and "shrinks" of the graph.

2.1 Radian Measure

Here you'll learn what radian measure is, and how to find the radian values for common angles on the unit circle.

While working on an experiment in your school science lab, your teacher asks you to turn up a detector by rotating the knob $\frac{\pi}{2}$ radians. You are immediately puzzled, since you don't know what a radian measure is or how far to turn the knob.

Read this Concept, and at its conclusion, you will be able to turn the knob by the amount your teacher requested.

Watch This

MEDIA
Click image to the left for more content.

James Sousa:Radian Measure

Guidance

Until now, we have used degrees to measure angles. But, what exactly is a degree? A **degree** is $\frac{1}{360^{th}}$ of a complete rotation around a circle. **Radians** are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the *radius* of a circle. One **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics.

What if we were to rotate all the way around the circle? Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.

Recall from geometry that the arc length of a complete rotation is the circumference, where the formula is equal to 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

$$360 \text{ degrees} = 2\pi \text{ radians}$$

With this as our starting point, we can find the radian measure of other angles. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be $\frac{1}{2}\pi$, written $\frac{\pi}{2}$.

Extending the radian measure past the first quadrant, the quadrantal angles have been determined, except $270°$. Because $270°$ is halfway between $180°(\pi)$ and $360°(2\pi)$, it must be 1.5π, usually written $\frac{3\pi}{2}$.

For the $45°$ angles, the radians are all multiples of $\frac{\pi}{4}$. For example, $135°$ is $3 \cdot 45°$. Therefore, the radian measure should be $3 \cdot \frac{\pi}{4}$, or $\frac{3\pi}{4}$. Here are the rest of the multiples of $45°$, in radians:

Notice that the additional angles in the drawing all have reference angles of 45 degrees and their radian measures are all multiples of $\frac{\pi}{4}$. All of the even multiples are the quadrantal angles and are reduced, just like any other fraction.

Example A

Find the radian measure of these angles.

TABLE 2.1:

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	
30	

Solution: Because 45 is half of 90, half of $\frac{1}{2}\pi$ is $\frac{1}{4}\pi$. 30 is one-third of a right angle, so multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

and because 60 is twice as large as 30:

$$2 \times \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Here is the completed table:

TABLE 2.2:

Angle in Degrees	Angle in Radians
90	$\frac{\pi}{2}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$

There is a formula to convert between radians and degrees that you may already have discovered while doing this example. However, many angles that are commonly used can be found easily from the values in this table. For example, most students find it easy to remember 30 and 60. 30 is π over **6** and 60 is π over **3**. Knowing these angles, you can find any of the special angles that have reference angles of 30 and 60 because they will all have the same denominators. The same is true of multiples of $\frac{\pi}{4}$ (45 degrees) and $\frac{\pi}{2}$ (90 degrees).

Example B

Complete the following radian measures by counting in multiples of $\frac{\pi}{3}$ and $\frac{\pi}{6}$:

Solution:

Notice that all of the angles with 60-degree reference angles are multiples of $\frac{\pi}{3}$, and all of those with 30-degree reference angles are multiples of $\frac{\pi}{6}$. Counting in these terms based on this pattern, rather than converting back to degrees, will help you better understand radians.

Example C

Find the radian measure of these angles.

TABLE 2.3:

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	
240	
270	
300	

Solution: Because 30 is one-third of a right angle, multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

adding this to the known value for ninety degrees of $\frac{\pi}{2}$:

$$\frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

Here is the completed table:

TABLE 2.4:

Angle in Degrees	Angle in Radians
120	$\frac{2\pi}{3}$
180	π
240	$\frac{4\pi}{3}$
300	$\frac{5\pi}{3}$

Vocabulary

Radian: A **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle.

Guided Practice

1. Give the radian measure of $60°$

2. Give the radian measure of $75°$

3. Give the radian measure of $180°$

Solutions:

1. 30 is one-third of a right angle. This means that since $90° = \frac{\pi}{2}$, then $30° = \frac{\pi}{6}$. Therefore, multiplying gives:

$\frac{\pi}{6} \times 2 = \frac{\pi}{3}$

2. 15 is one-sixth of a right triangle. This means that since $90° = \frac{\pi}{2}$, then $15° = \frac{\pi}{12}$. Therefore, multiplying gives:

$\frac{\pi}{12} \times 5 = \frac{5\pi}{12}$

3. Since $90° = \frac{\pi}{2}$, then $180° = \frac{2\pi}{2} = \pi$

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Concept Problem Solution

Since $45° = \frac{\pi}{4} rad$, then $2 \times \frac{\pi}{4} = \frac{\pi}{2} = 2 \times 45°$. Therefore, a turn of $\frac{\pi}{2}$ is equal to $90°$, which is $\frac{1}{4}$ of a complete rotation of the knob.

Practice

Find the radian measure of each angle.

1. $90°$
2. $120°$
3. $300°$
4. $330°$
5. $-45°$
6. $135°$

Find the degree measure of each angle.

7. $\frac{3\pi}{2}$
8. $\frac{5\pi}{4}$
9. $\frac{7\pi}{6}$
10. $\frac{\pi}{6}$
11. $\frac{5\pi}{3}$
12. π
13. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.
14. Explain why if you are given an angle in radians and you multiply it by $\frac{180}{\pi}$ you will get the same angle in degrees.
15. Explain in your own words why it makes sense that there are 2π radians in a circle.

2.2 Conversion between Degrees and Radians

Here you'll learn how to convert degrees to radians, and vice versa.

You are hard at work in the school science lab when your teacher asks you to turn a knob on a detector you are using $75°$ degrees. Unfortunately, you have been working in radians for a while, and so you're having trouble remembering how far to turn the knob. Is there a way to translate the instructions in degrees to radians?

Read this Concept, and at the conclusion you'll be able to accomplish this task and turn the knob the appropriate amount.

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James Sousa Example:Converting Angles in DegreeMeasureto Radian Measure

Guidance

Since degrees and radians are different ways of measuring the distance moved around the circumference of a circle, it is reasonable to suppose that there is a conversion formula between these two units. This formula works for all degrees and radians. Remember that: π radians $= 180°$. If you divide both sides of this equation by π, you will have the conversion formula:

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

If we have a degree measure and wish to convert it to radians, then manipulating the equation above gives:

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$

Example A

Convert $\frac{11\pi}{3}$ to degree measure.

From the last section, you should recognize that this angle is a multiple of $\frac{\pi}{3}$ (or 60 degrees), so there are 11, $\frac{\pi}{3}$'s in this angle, $\frac{\pi}{3} \times 11 = 60° \times 11 = 660°$.

Here is what it would look like using the formula:

radians $\times \frac{180}{\pi} =$ degrees

Example B

Convert $-120°$ to radian measure. Leave the answer in terms of π.

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$
$$-120° \times \frac{\pi}{180} = \frac{-120°\pi}{180}$$

and reducing to lowest terms gives us $-\frac{2\pi}{3}$

You could also have noticed that 120 is 2×60. Since $60°$ is $\frac{\pi}{3}$ radians, then 120 is 2, $\frac{\pi}{3}$'s, or $\frac{2\pi}{3}$. Make it negative and you have the answer, $-\frac{2\pi}{3}$.

Example C

Express $\frac{11\pi}{12}$ radians terms of degrees.

$\text{radians} \times \frac{180}{\pi} = \text{degrees}$

Note: Sometimes students have trouble remembering if it is $\frac{180}{\pi}$ or $\frac{\pi}{180}$. It might be helpful to remember that radian measure is almost always expressed in terms of π. If you want to convert from radians to degrees, you want the π to cancel out when you multiply, so it must be in the denominator.

Vocabulary

Radian: A **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle.

Degree: A **degree** is a unit for measuring angles in a circle. There are 360 of them in a circle.

Guided Practice

1. Convert the following degree measures to radians. All answers should be in terms of π.

$240°, 270°, 315°, -210°, 120°$

2. Convert the following degree measures to radians. All answers should be in terms of π.

$15°, -450°, 72°, 720°, 330°$

3. Convert the following radian measures to degrees

$\frac{\pi}{2}, \frac{11\pi}{5}, \frac{2\pi}{3}, 5\pi, \frac{7\pi}{2}$

Solutions:

1. $\frac{4\pi}{3}, \frac{3\pi}{2}, \frac{7\pi}{4}, -\frac{7\pi}{6}, \frac{2\pi}{3}$

2. $\frac{\pi}{12}, -\frac{5\pi}{2}, \frac{\pi}{5}, 4\pi, \frac{11\pi}{6}$

3. $90°, 396°, 120°, 540°, 630°$

Concept Problem Solution

Since you now know that the conversion for a measurement in degrees to radians is

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$

you can find the solution to convert $75°$ to radians:

$$75° \times \frac{\pi}{180} = \frac{75\pi}{180} = \frac{5\pi}{12}$$

Practice

Convert the following degree measures to radians. All answers should be in terms of π.

1. $90°$
2. $360°$
3. $50°$
4. $110°$
5. $495°$
6. $-85°$
7. $-120°$

Convert the following radian measures to degrees.

8. $\frac{5\pi}{12}$
9. $\frac{3\pi}{5}$
10. $\frac{8\pi}{15}$
11. $\frac{7\pi}{10}$
12. $\frac{5\pi}{2}$
13. 3π
14. $\frac{7\pi}{2}$
15. Why do you think there are two different ways to measure angles? When do you think it might be more convenient to use radians than degrees?

2.3 Six Trigonometric Functions and Radians

Here you'll learn what the values of trig functions are when angles are expressed in radians.

While working in your math class one day, you are given a sheet of values in radians and asked to find the various trigonometric functions of them, such as sine, cosine, and tangent. The first question asks you to find the $\sin \frac{\pi}{6}$. You are about to start converting the measurements in radians into degrees when you wonder if it might be possible to just take the values of the functions directly.

Do you think this is possible? As it turns out, it is indeed possible to apply trig functions to measurements in radians. Here you'll learn to do just that.

At the end of this Concept, you'll be able to compute $\sin \frac{\pi}{6}$ directly.

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James Sousa:Determine ExactTrig Function Values With the Anglein Radians Using the Unit Circle

Guidance

Even though you are used to performing the trig functions on degrees, they still will work on radians. The only difference is the way the problem looks. If you see $\sin \frac{\pi}{6}$, that is still $\sin 30°$ and the answer is still $\frac{1}{2}$.

Most scientific and graphing calculators have a **MODE** setting that will allow you to either convert between the two, or to find approximations for trig functions using either measure. It is important that if you are using your calculator to estimate a trig function that you know which mode you are using. Look at the following screen:

If you entered this expecting to find the sine of 30 degrees$\frac{1}{2}$. In fact, as you may have suspected, the calculator is interpreting this as 30 **radians**. In this case, changing the mode to degrees and recalculating will give the expected result.

Scientific calculators will usually have a 3-letter display that shows either **DEG** or **RAD** to tell you which mode the calculator is in.

Example A

Find $\tan \frac{3\pi}{4}$.

Solution: If needed, convert $\frac{3\pi}{4}$ to degrees. Doing this, we find that it is $135°$. So, this is $\tan 135°$, which is -1.

Example B

Find the value of $\cos \frac{11\pi}{6}$.

Solution: If needed, convert $\frac{11\pi}{6}$ to degrees. Doing this, we find that it is $330°$. So, this is $\cos 330°$, which is $\frac{\sqrt{3}}{2}$.

Example C

Convert 1 radian to degree measure.

Solution: Many students get so used to using π in radian measure that they incorrectly think that **1** radian means 1π radians. While it is more convenient and common to express radian measure in terms of π, don't lose sight of the fact that π radians is a number. It specifies an angle created by a rotation of approximately 3.14 radius lengths. So 1 radian is a rotation created by an arc that is only a single radius in length.

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

So 1 radian would be $\frac{180}{\pi}$ degrees. Using any scientific or graphing calculator will give a reasonable approximation for this degree measure, approximately $57.3°$.

Vocabulary

Radian: A **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle.

Guided Practice

1. Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians: $\frac{6\pi}{7}$

2. Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians: $\frac{20\pi}{11}$

3. Gina wanted to calculate the $\sin 210°$ and got the following answer on her calculator:

Fortunately, Kylie saw her answer and told her that it was obviously incorrect.

1. Write the correct answer, in simplest radical form.
2. Explain what she did wrong.

Solutions:

1. $154.3°$

2. $327.3°$

3. The correct answer is $-\frac{1}{2}$. Her calculator was is the wrong mode and she calculated the sine of 210 *radians*.

Concept Problem Solution

As you have learned in this Concept, the $\sin \frac{\pi}{6}$ is the same as $\sin 30°$, which equals $\frac{1}{2}$. You could find this either by converting $\frac{\pi}{6}$ to degrees, or by using your calculator with angles entered in radians.

Practice

Using a calculator, find the approximate degree measure (to the nearest tenth) of the angle expressed in radians.

1. $\frac{4\pi}{7}$
2. $\frac{5\pi}{6}$
3. $\frac{8\pi}{11}$
4. $\frac{5\pi}{3}$
5. $\frac{8\pi}{3}$
6. $\frac{7\pi}{4}$
7. $\frac{12\pi}{5}$

Find the value of each using your calculator.

8. $\sin\frac{3\pi}{2}$
9. $\cos\frac{\pi}{2}$
10. $\tan\frac{\pi}{6}$
11. $\sin\frac{5\pi}{6}$
12. $\tan\frac{4\pi}{3}$
13. $\cot\frac{7\pi}{3}$
14. $\sec\frac{11\pi}{6}$
15. Do you think radians will always be written in terms of π? Is it possible to have, for example, exactly 2 radians?

2.4 Rotations in Radians

Here you'll learn how to express the rotation of clock hands as an angle in radians.

In your math class one morning you finish a quiz early. While you are waiting, you watch the clock as it ticks off five minutes. The time on the clock reads 9:00. Your recent lessons have taught you that one way to measure the position of something on a circle is to use an angle. Suddenly it occurs to you that this can be applied to clocks. Can you determine the angle between the two hands of the clock?

Read on, and at the completion of this Concept, you'll be able to determine the angle between the hands of a clock to answer this question.

Watch This
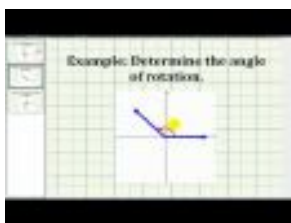
MEDIA
Click image to the left for more content.

James Sousa:Determine Angles ofRotation

Guidance

A lot of interesting information about rotations and how to measure them can come from looking at clocks. We are so familiar with clocks in our daily lives that we don't often stop to think about these little devices, with hands continually rotating. Let's take a few minutes in this Concept for a closer look at these examples of rotational motion.

Example A

The hands of a clock show 11:20. Express the obtuse angle formed by the hour and minute hands in radian measure.

Solution: The following diagram shows the location of the hands at the specified time.

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or $30°$). So, the angle between the 12 and the 4 is $4 \times \frac{\pi}{6} = \frac{2\pi}{3}$ (or $120°$). Because the rotation from 12 to 4 is one-third of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-third of the distance between the 11 and the 12. This means that the angle between the hour hand and the 12 is two-thirds of the distance between the 11 and the 12. So, $\frac{2}{3} \times \frac{\pi}{6} = \frac{2\pi}{18} = \frac{\pi}{9}$, and the total measure of the angle is therefore $\frac{\pi}{9} + \frac{2\pi}{3} = \frac{\pi}{9} + \frac{6\pi}{9} = \frac{7\pi}{9}$.

Example B

The hands of a clock show 4:15. Express the acute angle formed by the hour and minute hands in radian measure.

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or $30°$). So, the angle between the 3 (which is where the minute hand is located when it is 15 minutes after the hour) and the

4 is $\frac{\pi}{6}$ (or 30°). Further, since the minute hand has moved one quarter of the way around the hour, we can infer that the hour hand has moved one quarter of the way between four and five, which is $\frac{1}{4} \times \frac{\pi}{6} = \frac{\pi}{24}$. Adding these numbers gives: $\frac{\pi}{6} + \frac{\pi}{24} = \frac{4\pi}{24} + \frac{\pi}{24} = \frac{5\pi}{24}$.

Example C

The hands of a clock show 2:30. Express the acute angle formed by the hour and minute hands in radian measure.

Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\frac{2\pi}{12} = \frac{\pi}{6}$ (or 30°). So, the angle between the 3 and the 6 (which is where the minute hand is at 30 minutes after the hour) is $3 \times \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$ (or 90°). Because the rotation from 12 to 6 is one-half of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-half of the distance between the 2 and the 3. This means that the angle between the hour hand and the 3 is one-half of the distance between the 2 and the 3. So, $\frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$, and the total measure of the angle is therefore $\frac{\pi}{12} + \frac{\pi}{2} = \frac{\pi}{12} + \frac{6\pi}{12} = \frac{7\pi}{12}$.

Vocabulary

Radian: A **radian** (abbreviated rad) is the angle created by bending the radius length around the arc of a circle.

Guided Practice

The following image shows a 24-hour clock in Curitiba, Paraná, Brasil.

FIGURE 2.1

1. What is the angle between each number of the clock expressed in exact radian measure in terms of π ?

2. What is the angle between each number of the clock expressed to the nearest tenth of a radian? What about in degree measure?

3. Estimate the measure of the angle between the hands at the time shown to the nearest whole degree. And then in radian measure in terms of π.

Solutions:

1. Since there are 2π radians in a circle, and there are 24 separate increments, the answer is $\frac{2\pi}{24} = \frac{\pi}{12}$

2. Since there are 2π radians in a circle, the number of radians in each of 24 different divisions is $\frac{2\pi}{24} \approx .3$. In degrees we can do the same by taking the number of degrees in a circle and dividing it by 12: $\frac{360}{24} = 15°$.

3. $20°$. Answers may vary, anything above $15°$ and less than $25°$ is reasonable. In radians, this is $\frac{\pi}{9}$. Again, answers may vary.

Concept Problem Solution

Since you now know that the angle between the hours on a clock is $\frac{\pi}{6} = 30°$, you can use this information to construct an answer. There are three hours between the 9 and the 12 on a clock, so the answer is:

$3 \times \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} = 90°$

So there are $90°$ degrees between the 9 and 12 on the clock.

Practice

Use the clock below to help you find the angle between the hour hand and minute hand at each of the following times. Express your answer in degrees less than $180°$. Then express your answer in radian measure in terms of π.

1. 3:30
2. 5:15
3. 4:45
4. 6:30
5. 6:15
6. 2:30
7. 12:30
8. 9:30
9. 10:15
10. 11:30
11. 3:45
12. 2:15
13. 7:15
14. How many times in 12 hours will the hour and minute hands overlap?
15. When is the first time after 12:00 that the hour and minute hands will overlap exactly?

2.5 Length of an Arc

Here you'll learn how to find the length of a portion of the circumference of a circle using an angle in radians and the radius of the circle.

You have taken your little cousin to the amusement park for the day. While there, she decides she would like a ride on the carousel. After the ride, she excitedly bounces over to you. She is amazed that she went around, but in a way "didn't go anywhere", since she ended up where she started.

"How far did I go when I was halfway around the turn?", she asks.

You know that the radius of the carousel is 7 meters. Can you tell your little cousin how far she went in one half of a turn around the ride?

At the end of this Concept, you'll be able to do just that.

Watch This
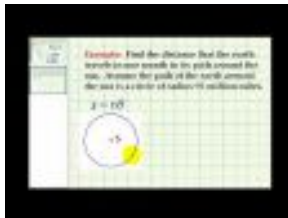
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James Sousa Example:ArcLength and Application of Arc Length

Guidance

The length of an arc on a circle depends on both the angle of rotation and the radius length of the circle. If you recall from the last lesson, the measure of an angle in radians is defined as the length of the arc cut off by one radius length. What if the radius is 4 cm? Then, the length of the half-circle arc would be π multiplied by the radius length, or 4π cm in length.

This results in a formula that can be used to calculate the length of any arc.

$$s = r\theta,$$

where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.

Solving this equation for θ will give us a formula for finding the radian measure given the arc length and the radius length:

Example A

The free-throw line on an NCAA basketball court is 12 ft wide. In international competition, it is only about 11.81 ft. How much longer is the half circle above the free-throw line on the NCAA court?

Solution: Find both arc lengths.

$$\begin{array}{cc}
\text{NCAA} & \text{INTERNATIONAL} \\
s_1 = r\theta & s_2 = r\theta \\
s_1 = \dfrac{12}{2}(\pi) & s_2 \approx \dfrac{11.81}{2}(\pi) \\
s_1 = 6\pi & s_2 \approx 5.905\pi
\end{array}$$

So the answer is approximately $6\pi - 5.905\pi \approx 0.095\pi$

This is approximately 0.3 ft, or about 3.6 inches longer.

Example B

Two connected gears are rotating. The smaller gear has a radius of 4 inches and the larger gear's radius is 7 inches. What is the angle through which the larger gear has rotated when the smaller gear has made one complete rotation?

Solution: Because the blue gear performs one complete rotation, the length of the arc traveled is:

$$s = r\theta$$
$$s = 4 \times 2\pi$$

So, an 8π arc length on the larger circle would form an angle as follows:

$$\theta = \frac{s}{r}$$
$$\theta = \frac{8\pi}{7}$$
$$\theta \approx 3.6$$

So the angle is approximately 3.6 radians.

$$3.6 \times \frac{180}{\pi} \approx 206°$$

Example C

The radius of a standard car tire is 27.94 cm. How far does a car go in one revolution of the tire?

Solution: Since the distance traveled by the tire is equal to the distance around the tire, we can use the circumference of the tire to answer the question.

$$s = r\theta$$
$$s = (27.94)(2\pi)$$
$$s = 175.46$$

Vocabulary

Arc: An **arc** is a length segment measured around the circumference of a circle.

Guided Practice

1. You are trying to push your car after it has broken down. Unfortunately, you aren't very strong, and so the car is just rocking back and forth instead of rolling as you push. If the radius of your car's tire is 14 inches, and the change in the tire's angle is $\frac{\pi}{2}$ radians, how far did the tire move?

2. If an object with a radius of 10 cm spins so that its arc covers 54 cm, what is the change in angle of the object?

3. If your DVD has a radius of 4.5 inches, how far does a point on the disk spin if the player turns it $\frac{\pi}{2}$ radians?

Solutions:

1. Since the distance the tire moved is equal to the length of the arc the tire rolled, you can use the equation $s = r\theta$ to determine how far the tire went:

$$s = r\theta$$
$$s = (14)(\frac{\pi}{2})$$
$$s = 7\pi$$
$$s \approx 21.98 in$$

2. You can again use the equation $s = r\theta$ to solve this problem:

$$s = r\theta$$
$$\theta = \frac{s}{r} = \frac{54}{10} = 5.4$$

The disk moves 5.4 radians, which is a little less than a complete rotation, since a complete rotation is approximately 6.28 radians.

3. Using $s = r\theta$,

$$s = r\theta$$
$$s = (4.5)(\frac{\pi}{2})$$
$$s = 2.25\pi \approx 7.065$$

A point on the disk turns 7.065 inches.

Concept Problem Solution

Since you now know that you can measure an arc length using $s = r\theta$, you can use this to find a solution to your cousin's question. Since your cousin wants to know how far she went around when she went $\frac{1}{2}$ of a rotation, and the radius of the ride is 7 meters, you can calculate her arc length:

$s = r\theta = 7\pi \approx 21.98 meters$

Practice

The radius of a carousel is 8 meters. Use this information to answer questions 1-3.

1. You are half way around the carousel. How far did you travel?
2. You are all the way around the carousel. How far did you travel?
3. You have now traveled all the way around the carousel twice. How far did you travel?

A pizza has a radius of 10in. Use this information to answer questions 4-6.

4. A slice is removed. The length of the crust of the missing slice is 3in. What is the central angle of the missing slice?
5. You eat three pieces with a central angle of $\frac{4\pi}{5}$. What is the length of the crust you ate?
6. A large pizza has a radius of 12in. What is the length of the crust of half of the large pizza?

The diameter of a tire is 35in. Use this information to answer questions 7-10.

7. What is the length around the whole tire?
8. The tire travels one mile (5280 ft). How many revolutions did the tire make?
9. You roll the tire so it rotates 7π radians. How far did it move?
10. The tire travels half a mile. How many radians did the tire rotate?

Consider a standard 12 hour clock like the one below with a radius of 5 inches. Use this to answer questions 11-15.

11. What is the length of the arc between the 3 and the 7?
12. What is the length of the arc between the 3 and the 2?
13. It is 12:30. What is the length of the arc between the minute and hour hands?
14. It is 7:20. What is the length of the arc between the minute and hour hands?
15. It is 1:25. What is the length of the arc between the minute and hour hands?

2.6 **Area of a Sector**

Here you'll learn to find the area of a portion of a circle using the radius of the circle and an angle in radians.

While eating lunch with your friends one day you decide to get some pie. The school cafeteria has a sale on pies if you buy the whole pie instead of an individual slice. Since you and your friends plan to eat the whole pie anyway, you are happy to make the purchase.

You decide to cut yourself a piece. When you have taken your section of pie out, one of your friends objects that you have cut out more than your fair share. If the radius of the pie is six inches, and the angle formed by the end of your pie wedge is $30°$, what is the area of your pie piece? Is it more than your fair share if there are five people (including you) splitting the pie?

Watch This

In the second part of this video, you'll see how to find the area of a sector in a circle.

MEDIA
Click image to the left for more content.

James Sousa:ArcLength and Areaof a Sector

Guidance

One of the most common geometric formulas is the area of a circle:

$$A = \pi r^2$$

In terms of angle rotation, this is the area created by 2π radians.

$$2\pi \text{ rad} = \pi r^2 \text{ area}$$

A half-circle, or π radian rotation would create a section, or **sector** of the circle equal to half the area or:

$$\frac{1}{2}\pi r^2$$

So an angle of 1 radian would define an area of a sector equal to:

$1 = \frac{1}{2}r^2$

From this we can determine the area of the sector created by any angle, θ radians, to be:

$$A = \frac{1}{2}r^2\theta$$

Example A

Crops are often grown using a technique called center pivot irrigation that results in circular shaped fields.

FIGURE 2.2

Here is a satellite image taken over fields in Kansas that use this type of irrigation system.

If the irrigation pipe is 450 m in length, what is the area that can be irrigated after a rotation of $\frac{2\pi}{3}$ radians?

Solution: Using the formula:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(450)^2\left(\frac{2\pi}{3}\right)$$

The area is approximately 212,058 square meters.

Example B

A doughnut has a hole in the middle with a radius of 1 cm, and the distance from the center of the hole to the outer edge of the doughnut is 3 cm. What is the area of a sector of $\frac{1}{4}$ of the doughnut?

Solution: The formula for the area of a sector is

$$A = \frac{1}{2}r^2\theta$$

. Using this formula to find the area of the sector from the center outward gives:
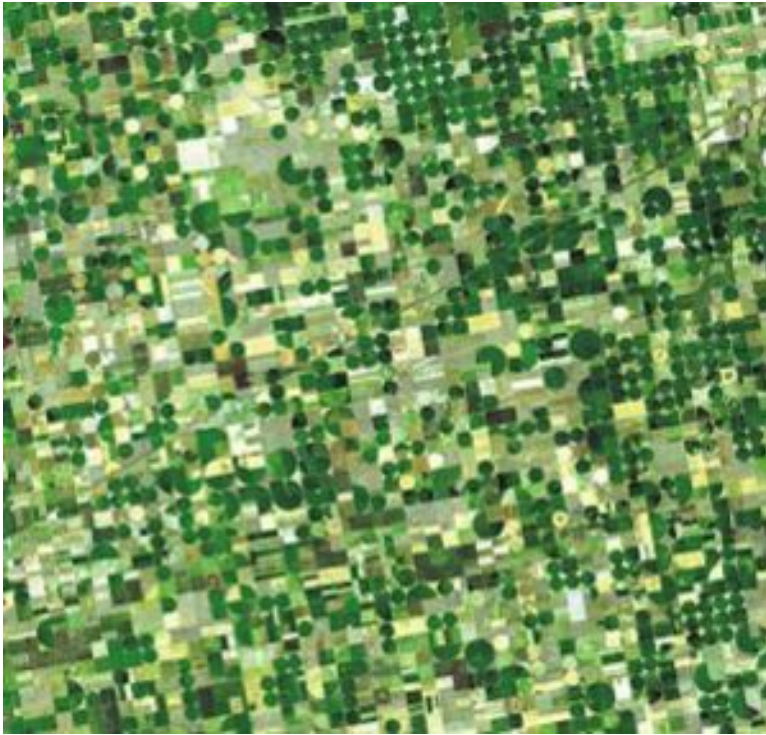
FIGURE 2.3

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}3^2\frac{\pi}{2}$$

$$A = \frac{9\pi}{4}$$

Now it is necessary to subtract the area of the sector that is part of the hole, and therefore not part of the doughnut:

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(1)^2\frac{\pi}{2}$$

$$A = \frac{\pi}{4}$$

Area of the sector of doughnut:

$$A = \frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi$$

Example C

A driver is traveling around a circular track that has radius of 70 meters. If the angle from the starting line to her current position is $\frac{\pi}{3}$ radians, what is the area of the sector traced out by her car?

Solution: The area of a sector is:

$$A = \frac{1}{2}r^2\theta$$

. This leads us to:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(70)^2\frac{\pi}{3}$$
$$A \approx 73.3$$

meters

Vocabulary

Sector: A **sector** is the portion of a circle between two lines from the origin to the circle's edge.

Guided Practice

1. If the radius of a sector is 5 feet, and the sector sweeps out an angle of $43°$, find the area of the sector.

2. If a pie wedge has an area of 15 square in inches, and the pie has a radius of 9 inches, find the angle swept out by the sector.

3. If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of $25°$, find the radius of the cake.

Solutions:

1. Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the area (don't forget to convert the degrees of the angle to radians):

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(25)(.75)$$
$$A = 9.375$$

2. Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the angle swept out by the sector (don't forget that the angle will be measured in radians):

$$A = \frac{1}{2}r^2\theta$$
$$15 = \frac{1}{2}(81)(\theta)$$
$$\theta = \frac{(2)(15)}{81}$$
$$\theta = .37$$

3. Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the radius of the cake (don't forget to convert the degrees of the angle to radians):

$$A = \frac{1}{2}r^2\theta$$
$$20 = \frac{1}{2}(r^2)(.436)$$
$$r^2 = \frac{(2)(20)}{.436}$$
$$r^2 = 91.743$$
$$r = 9.58$$

Concept Problem Solution

You now know that the equation for the area of a circle swept out by some angle is:

$$A = \frac{1}{2}r^2\theta$$

Applying this to your pie slice, you have an area of:

$$A = \frac{1}{2} \times 6^2 \times \frac{\pi}{6} = 3\pi$$

And the total area of the pie is:

$\pi r^2 = 36\pi$

To find out what your fair portion of the pie is, multiply the total area by your fraction:

$\frac{1}{5} \times 36\pi = 7.2\pi$

And since the piece you took is only 3π in size, you are definitely not taking too much!

Practice

1. If the radius of a sector is 8 inches, and the central angle of the sector is $40°$, find the area of the sector.
2. If the radius of a sector is 12 inches, and the central angle of the sector is $\frac{\pi}{6}$ radians, find the area of the sector.
3. If the radius of a sector is 6 inches, and the central angle of the sector is $140°$, find the area of the sector.
4. If the radius of a sector is 5 inches, and the central angle of the sector is $\frac{5\pi}{3}$ radians, find the area of the sector.
5. If the radius of a sector is 10 inches, and the central angle of the sector is $100°$, find the area of the sector.
6. If a pie wedge has an area of 10 square inches, and the pie has a radius of 6 inches, find the angle swept out by the sector.
7. If a pie wedge has an area of 15 square inches, and the pie has a radius of 4 inches, find the angle swept out by the sector.
8. If a pie wedge has an area of 12 square inches, and the pie has a radius of 3 inches, find the angle swept out by the sector.
9. If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of $\frac{\pi}{3}$ radians, find the radius of the cake.

10. If you have a piece of round cake that has an area of 100 square inches, and you know the piece sweeps out an angle of $50°$, find the radius of the cake.
11. If you have a piece of round cake that has an area of 35 square inches, and you know the piece sweeps out an angle of $\frac{2\pi}{5}$ radians, find the radius of the cake.
12. If you have a piece of round cake that has an area of 20 square inches, and you know the piece sweeps out an angle of $30°$, find the radius of the cake.

A pizza has a radius of 10in. Use this information to answer questions 13-15.

13. A slice is removed. The length of the crust of the missing slice is 3in. What is the area of the missing slice?
14. You eat three pieces with a central angle of $\frac{4\pi}{5}$. What is the area of the pizza you ate?
15. A large pizza has a radius of 12in. What is the area of half of the large pizza?

2.7 Length of a Chord

Here you'll learn what a chord is and how to find its length using the radius of the circle and the angle the chord creates.

You have been asked to help the younger students at your school with their Physical Education class. While working one afternoon, you are asked to take out a parachute that the students can play with. As the students are playing, one of them walks across a small portion of the parachute instead of under it like she is supposed to. If the chute is shaped like a circle with a radius of 6 meters, and the path the student walked across the chute covered an angle of 50°, what is the length of the path she walked across the parachute?

Read on, and at the completion of this Concept, you'll be able to answer this question.

Watch This

The first part of this video will help you understand what a chord is:
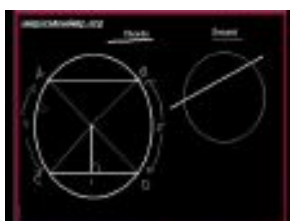
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Click image to the left for more content.

Chords

Guidance

You may recall from your Geometry studies that a chord is a segment that begins and ends on a circle.

$\overline{AB}$ is a chord in the circle.

We can calculate the length of any chord if we know the angle measure and the length of the radius. Because each endpoint of the chord is on the circle, the distance from the center to A and B is the same as the radius length.

Next, if we bisect the angle, the angle bisector must be perpendicular to the chord and bisect it (we will leave the proof of this to your Geometry class). This forms a right triangle.

We can now use a simple sine ratio to find half the chord, called c here, and double the result to find the length of the chord.

$$\sin\frac{\theta}{2} = \frac{c}{r}$$
$$c = r \times \sin\frac{\theta}{2}$$

So the length of the chord is:

$$2c = 2r\sin\frac{\theta}{2}$$

Example A

Find the length of the chord of a circle with radius 8 cm and a central angle of 110°. Approximate your answer to the nearest mm.

Solution: We must first convert the angle measure to radians:

$$110 \times \frac{\pi}{180} = \frac{11\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{11\pi}{18} \times \frac{1}{2} = \frac{11\pi}{36}$$
$$8 \times \sin \frac{11\pi}{36}$$

Multiply this result by 2.

So, the length of the chord is approximately 13.1 cm.

Example B

Find the length of the chord of a circle with a radius of 2 m that has a central angle of 90°.

Solution: First convert the angle to radians:

$$90 \times \frac{\pi}{180} = \frac{\pi}{2}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{\pi}{2} \times \frac{1}{2} = \frac{\pi}{4}$$
$$2 \times \sin \frac{\pi}{4}$$

Multiply this result by 2.

So the answer is approximately 2.83 meters.

Example C

Find the length of the chord of a circle with radius 1 m and a central angle of 170°.

Solution: We must first convert the angle measure to radians:

$$170 \times \frac{\pi}{180} = \frac{17\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{17\pi}{18} \times \frac{1}{2} = \frac{17\pi}{36}$$

$$1 \times \sin\frac{17\pi}{36} = .996$$

Multiply this result by 2.

So, the length of the arc is approximately 1.992

Notice that the length of the chord is almost 2 meters, which would be the diameter of the circle. If the angle had been 180 degrees, the chord would have just been the distance all the way across the circle going through the middle, which is the diameter.

Vocabulary

Chord: A **chord** is a straight line across a circle, intersecting the circle in two places, but not passing through the circle's center.

Guided Practice

1. If you run a piece of string across a doughnut you are eating, and the radius between the endpoints of the string to the center of the doughnut is 4 inches, how long is the string if the angle swept out by the chord is $20°$?

2. You are eating dinner one night with your family at the local Italian restaurant. A piece of spaghetti makes a chord across your plate. You know that the length of the spaghetti strand is 5 inches, and the radius of the plate is 7 inches. What is the angle swept out by the chord?

3. If you draw a chord across a circle and make a chord across it that has a length of 15 inches, sweeping out an angle of π radians, what is the radius of the circle you drew?

Solutions:

1. You can use the equation $C = 2r\sin\left(\frac{\theta}{2}\right)$ to solve this problem: (Don't forget to convert angles to radians)

$$C = 2r\sin\left(\frac{\theta}{2}\right)$$

$$C = (2)(4)\sin\left(\frac{.349}{2}\right)$$

$$C = 8(.1736)$$

$$C = 1.388$$

$$inches$$

2. Since the radius of the plate and the length of the chord are known, you can solve for the angle:

$$C = 2r\sin\left(\frac{\theta}{2}\right)$$

$$\frac{C}{2r} = \sin\left(\frac{\theta}{2}\right)$$

$$\sin^{-1}\left(\frac{c}{2r}\right) = \frac{\theta}{2}$$

$$\sin^{-1}\left(\frac{5}{14}\right) = \frac{\theta}{2}$$

$$.365 = \frac{\theta}{2}$$

$$\theta = .73$$

The angle spanned by the spaghetti is .73 radians.

3. Using the equation for the length of a chord:

$$c = 2r\sin\left(\frac{\theta}{2}\right)$$

$$15 = (2r)\sin\left(\frac{\pi}{2}\right)$$

$$r = 7.5$$

As you can see, the radius of the circle is 7.5 inches. This is what you should expect, since the chord sweeps out an angle of π. This means that it sweeps out half of the circle, so that the chord is actually going across the whole diameter of the circle. So if the chord is going across the diameter and has a length of 15 inches, then the radius of the circle should be 7.5 inches.

Concept Problem Solution

With the equation for the length of a chord in hand, you can calculate the distance the student ran across the parachute:

First convert the measure in degrees to radians:

$50 \times \frac{\pi}{180} \approx .27\pi$

$2r\sin\frac{\theta}{2} = (2)(6)\sin\frac{.27\pi}{2} = 12\sin .135\pi \approx 4.94 meters$

Practice

1. Find the length of the chord of a circle with radius 1 m and a central angle of $100°$.
2. Find the length of the chord of a circle with radius 8 km and a central angle of $130°$.
3. Find the length of the chord of a circle with radius 4 in and a central angle of $45°$.
4. Find the length of the chord of a circle with radius 3 ft and a central angle of $32°$.
5. Find the length of the chord of a circle with radius 2 cm and a central angle of $112°$.
6. Find the length of the chord of a circle with radius 7 in and a central angle of $135°$.

Solve for the missing variable in each circle.

7.
8.
9.
10.
11.
12.

Use the picture below for questions 13-15.

13. Suppose you knew the length of the chord, the length of the radius, and the central angle of the above circle. Describe one way to find the length of the red segment using the Pythagorean Theorem.
14. Suppose you knew the length of the chord, the length of the radius, and the central angle of the above circle. Describe one way to find the length of the red segment using cosine.
15. What would you need to know in order to find the area of the segment (the portion of the circle between the chord and the edge of the circle)? Describe how to find the area of this region.

2.8 Angular Velocity

Here you'll learn how to calculate linear and angular velocities for an object moving in a circle.

To find a particular song on your Ipod, you use the scroll wheel. This involves moving your finger around the wheel in a circular motion. Unfortunately for you, the song you want is near the very bottom of your songs list. And since an Ipod can hold over 1,000 songs, you have to scroll fast! As you are moving your finger in a circle, you wonder if you could measure how fast your finger is covering the distance around the circle.

Watching your finger, you realize that your finger is moving around the circle twice every second. If the radius of the Ipod wheel is 2 cm, what is the angular velocity of your finger as you scroll through your songs list? What is the linear velocity?

At the end of this Concept, you'll know how to answer these questions.

Watch This

James Sousa:Linear VelocityandAngularVelocity

Guidance

You may already be familiar with the measurement of speed as the relationship of an object's distance traveled to the time it has been in motion. However, this relationship is for objects that are moving in a straight line. What about objects that are traveling on a circular path?

Do you remember playing on a merry-go-round when you were younger?

If two people are riding on the outer edge, their velocities should be the same. But, what if one person is close to the center and the other person is on the edge? They are on the same object, but their speed is actually not the same.

Look at the following drawing.

Imagine the point on the larger circle is the person on the edge of the merry-go-round and the point on the smaller circle is the person towards the middle. If the merry-go-round spins exactly once, then both individuals will also make one complete revolution in the same amount of time.

However, it is obvious that the person in the center did not travel nearly as far. The circumference*linear velocity* (recall that linear velocity is found using distance = rate · time). If you have ever actually ridden on a merry-go-round, you know this already because it is much more fun to be on the edge than in the center! But, there is something about the two individuals traveling around that is the same. They will both cover the same rotation in the same period of time. This type of speed, measuring the angle of rotation over a given amount of time is called the **angular velocity.**

The formula for angular velocity is:

FIGURE 2.4

$$\omega = \frac{\theta}{t}$$

ω is the last letter in the Greek alphabet, omega, and is commonly used as the symbol for angular velocity. θ is the angle of rotation expressed in radian measure, and t is the time to complete the rotation.

In this drawing, θ is exactly one radian, or the length of the radius bent around the circle. If it took point A exactly 2 seconds to rotate through the angle, the *angular velocity* of A would be:

$$\omega = \frac{\theta}{t}$$
$$\omega = \frac{1}{2} \text{ radians per second}$$

In order to know the linear speed

If linear velocity is $v = \frac{d}{t}$ then, $v = \frac{5}{2}$ or 2.5 cm per second.

If the angle were not exactly 1 radian, then the distance traveled by the point on the circle is the length of the arc, $s = r\theta$, or, the radius length times the measure of the angle in radians.

Substituting into the formula for linear velocity gives: $v = \frac{r\theta}{t}$ or $v = r \cdot \frac{\theta}{t}$.

Look back at the formula for angular velocity. Substituting ω gives the following relationship between linear and angular velocity, $v = r\omega$. So, the *linear velocity* is equal to the radius times the *angular velocity*.

Remember in a unit circle, the radius is 1 unit, so in this case the linear velocity is the same as the angular velocity.

$$v = r\omega$$
$$v = 1 \times \omega$$
$$v = \omega$$

Here, the distance traveled around the circle is the same for a given unit of time as the angle of rotation, measured in radians.

Example A

Lindsay and Megan are riding on a Merry-go-round. Megan is standing 2.5 feet from the center and Lindsay is riding on the outside edge 7 feet from the center. It takes them 6 seconds to complete a rotation. Calculate the linear and

Solution: We are told that it takes 6 seconds to complete a rotation. A complete rotation is the same as 2π radians. So the angular velocity is:

$\omega = \frac{\theta}{t} = \frac{2\pi}{6} = \frac{\pi}{3}$ radians per second, which is slightly more than 1 (about 1.05), radian per second. Because both girls cover the same angle of rotation in the same amount of time, their *angular speed* is the same. In this case they rotate through approximately 60 degrees of the circle every second.

As we discussed previously, their linear velocities are different. Using the formula, Megan's linear velocity is:

$$v = r\omega = (2.5)\left(\frac{\pi}{3}\right) \approx 2.6 \text{ ft per sec}$$

Lindsay's linear velocity is:

$$v = r\omega = (7)\left(\frac{\pi}{3}\right) \approx 7.3 \text{ ft per sec}$$

Example B

A bug is standing near the outside edge of a compact disk (so that his radius from the center of the disc is 6 cm) that is rotating. He notices that he has traveled π radians in two seconds. What is his angular velocity? What is his linear velocity?

Solution: We know that the equation for angular velocity is

$\omega = \frac{\theta}{t} = \frac{\pi}{2}$ radians per second.

We can use the given equation to find his linear velocity:

$$v = r\omega = (6)\left(\frac{\pi}{2}\right) \approx 9.42 \text{ cm per sec}$$

Example C

How long does it take the bug in Example B to go through two complete turns?

Solution: Since the angular velocity of the bug is $\frac{\pi}{2}$ radians per second, we can use the equation for angular velocity and solve for time:

$$\omega = \frac{\theta}{t}$$

$$t = \frac{\theta}{\omega}$$

Since there are 4π radians in two complete turns of the disc, we can use this for the value of θ:

$$t = \frac{4\pi}{\frac{\pi}{2}} = 4\pi \times \frac{2}{\pi} = 8 \, seconds$$

Vocabulary

Angular Velocity: The **angular velocity** of a rotating object is the change in angle of an object divided by the change in time.

Linear Velocity: The **linear velocity** of an object is the change in position of an object divided by the change in time.

Guided Practice

1. Doris and Lois go for a ride on a carousel. Doris rides on one of the outside horses and Lois rides on one of the smaller horses near the center. Lois' horse is 3 m from the center of the carousel, and Doris' horse is 7 m farther away from the center than Lois'. When the carousel starts, it takes them 12 seconds to complete a rotation.

Calculate the linear velocity of each girl. Calculate the angular velocity of the horses on the carousel.

2. The Large Hadron Collider near Geneva, Switerland began operation in 2008 and is designed to perform experiments that physicists hope will provide important information about the underlying structure of the universe. The LHC is circular with a circumference of approximately 27,000 m. Protons will be accelerated to a speed that is very close to the speed of light ($\approx 3 \times 10^8$ meters per second).

How long does it take a proton to make a complete rotation around the collider? What is the approximate (to the nearest meter per second) angular speed of a proton traveling around the collider? Approximately how many times would a proton travel around the collider in one full second?

3. Ted is standing 2 meters from the center of a merry go round. If his linear velocity is 6 m/s, what is his angular velocity?

Solutions: 1. It is actually easier to calculate the angular velocity first. $\omega = \frac{2\pi}{12} = \frac{\pi}{6}$, so the angular velocity is $\frac{\pi}{6} \, rad$, or 0.524. Because the linear velocity depends on the radius,each girl has her own.

Lois: $v = r\omega = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$ or $1.57 \, m/sec$

Doris: $v = r\omega = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3}$ or $5.24 \, m/sec$

2. $v = \frac{d}{t} \rightarrow 3 \times 10^8 = \frac{27,000}{t} \rightarrow t = \frac{2.7 \times 10^4}{3 \times 10^8} = 0.9 \times 10^{-4} = 9 \times 10^{-5}$ or 0.00009 seconds. $\omega = \frac{\theta}{t} = \frac{2\pi}{0.00009} \approx$ 69,813 rad/sec The proton rotates around once in 0.00009 seconds. So, in one second it will rotate around the LHC $1 \div 0.00009 = 11,111.11$ times, or just over 11,111 rotations.

3. Since the equation relating linear and angular velocity is given by $v = r\omega$, we can solve for omega: $\omega = \frac{v}{r} = \frac{6}{2} = 3$

Concept Problem Solution

As you found out in this Concept, the angular velocity is the change in angle divided by the change in time. Since you sweep around the circle twice in a second, this becomes:

$\omega = \frac{4\pi}{1} = 4\pi$ rad/sec

Further, you can find the linear velocity with the equation:

$v = r\omega = (2)(4\pi) = 8\pi \approx 25.132 cm/s$

Practice

Beth and Steve are on a carousel. Beth is 7 ft from the center and Steve is right on the edge, 7 ft further from the center than Beth. Use this information and the following picture to answer questions 1-6.

1. The carousel makes a complete revolution in 12 seconds. How far did Beth go in one revolution? How far did Steve go in one revolution?
2. If the carousel continues making revolutions every 12 seconds, what is the angular velocity of the carousel?
3. What are Beth and Steve's linear velocities?
4. How far away from the center would Beth have to be in order to have a linear velocity of π ft per second.
5. The carousel changes to a new angular velocity of $\frac{\pi}{3}$ radians per second. How long does it take to make a complete revolution now?
6. With the carousel's new velocity, what are Beth and Steve's new linear velocities?
7. Beth and Steve go on another carousel that has an angular velocity of $\frac{\pi}{8}$ radians per second. Beth's linear velocity is 2π feet per second. How far is she standing from the center of the carousel?
8. Steve's linear velocity is only $\frac{\pi}{3}$ feet per second. How far is he standing from the center of the carousel?
9. What is the angular velocity of the minute hand on a clock? (in radians per minute)
10. What is the angular velocity of the hour hand on a clock? (in radians per minute)
11. A certain clock has a radius of 1 ft. What is the linear velocity of the tip of the minute hand?
12. On the same clock, what is the linear velocity of the tip of the hour hand?
13. The tip of the minute hand on another clock has a linear velocity of 2 inches per minute. What is the radius of the clock?
14. What is the angular velocity of the second hand on a clock? (in radians per minute)
15. The tip of the second hand on a clock has a linear velocity of 2 feet per minute. What is the radius of the clock?

2.9 Sine and Cosecant Graphs

Here you'll learn how to draw the graphs of the sine and cosecant functions.

Imagine for a moment that you have a clock that has only one hand - that rotates counterclockwise! However, the hand is very slim all the way until the tip, where there is a ball on the end. In fact, the hand is so slim you won't notice it. You only notice the ball on the end of the rotating hand. This hand is rotating faster than normal. Here is a picture of the clock:

Consider what it would be like if you put a light next to the clock and let the shadow of the hands fall on the far wall.

What pattern would that shadow trace out? If you think about it, you might realize that the shadow would make an up and down motion, over and over as the hand of the clock rotated. Now imagine that instead of a wall, there was a large piece of paper for the shadow to fall on. And wherever the shadow fell, there would be a mark on the paper. Finally, imagine moving the paper as the clock rotates. Can you imagine sort of pattern this would trace out?

By the end of this Concept, you'll understand just how this relates to trigonometric functions in general, and the sine graph in particular.

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James Sousa Animation: Graphing the SineFunction Using the UnitCircle

Guidance

By now, you have become very familiar with the specific values of sine, cosine, and tangents for certain angles of rotation around the coordinate grid. In mathematics, we can often learn a lot by looking at how one quantity changes as we consistently vary another. We will be looking at the sine value **as a function** of the angle of rotation around the coordinate plane. We refer to any such function as a **circular function**, because they can be defined using the unit circle. Recall from earlier sections that the sine of an angle in standard position is the ratio of $\frac{y}{r}$, where y is the $y-$coordinate of any point on the circle and r is the distance from the origin to that point.

Because the ratios are the same for a given angle, regardless of the length of the radius r, we can use the **unit circle** as a basis for all calculations.

The denominator is now 1, so we have the simpler expression, $\sin x = y$. The advantage to this is that we can use the $y-$coordinate of the point on the unit circle to trace the value of $\sin \theta$ through a complete rotation. Imagine if we start at 0 and then rotate counter-clockwise through gradually increasing angles. Since the $y-$coordinate is the sine value, watch the height of the point as you rotate.

Through Quadrant I that height gets larger, starting at 0, increasing quickly at first, then slower until the angle reaches $90°$, at which point, the height is at its maximum value, 1.

As you rotate into the second quadrant, the height starts to decrease towards zero.

When you start to rotate into the third and fourth quadrants, the length of the segment increases, but this time in a negative direction, growing to -1 at 270° and heading back toward 0 at 360°.

After one complete rotation, even though the angle continues to increase, the sine values will repeat themselves. The same would have been true if we chose to rotate clockwise to investigate negative angles, and this is why the sine function is a periodic function. The period is 2π because that is the angle measure before the sine of the angle will repeat its values.

Let's translate this circular motion into a graph of the sine value vs. the angle of rotation. The following sequence of pictures demonstrates the connection. These pictures plot $(\theta, \sin\theta)$ on the coordinate plane as (x, y).

After we rotate around the circle once, the values start repeating. Therefore, the sine curve, or "wave," also continues to repeat. The easiest way to sketch a sine curve is to plot the points for the quadrant angles. The value of $\sin\theta$ goes from 0 to 1 to 0 to -1 and back to 0. Graphed along a horizontal axis, it would look like this:

Filling in the gaps in between and allowing for multiple rotations as well as negative angles results in the graph of $y = \sin x$ where x is any angle of rotation, in radians.

As we have already mentioned, $\sin x$ has a period of 2π. You should also note that the $y-$values never go above 1 or below -1, so the **range** of a sine curve is $\{-1 \le y \le 1\}$. Because angles can be any value and will continue to rotate around the circle infinitely, there is no restriction on the angle x, so the **domain** of $\sin x$ is all reals.

Cosecant is the reciprocal of sine, or $\frac{1}{y}$. Therefore, whenever the sine is zero, the cosecant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the sine function in the same quadrants. Here is the graph.

The period of the function is 2π, just like sine. The domain of the function is all real numbers, except multiples of $\pi\{\ldots - 2\pi, -\pi, 0, \pi, 2\pi \ldots\}$. The range is all real numbers greater than or equal to 1, as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where sine is defined (other than the points at the top and bottom of the sine curve).

Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Many students are reminded of parabolas when they look at the half-period of the cosecant graph. While they are similar in that they each have a local minimum or maximum and they have the same beginning and ending behavior, the comparisons end there. Parabolas are not restricted by asymptotes, whereas the cosecant curve is.

Example A

Graph the following function:

$g(x) = \frac{1}{2}\sin(3x).$

Solution:

As you can see from the graph, the $\frac{1}{2}$ in front of the function reduces the function's height, while the 3 inside the argument of the function makes the function "squished" along the "x" axis.

Example B

Graph the following function:

$f(x) = \frac{1}{3}\csc\left(\frac{1}{2}x\right).$

Example C

Graph $f(x) = 5\sin\left(2(x + \frac{\pi}{3})\right)$.

Vocabulary

Circular Function: A **circular function** is a function that is measured by examining the angle of rotation around the coordinate plane.

Guided Practice

1. Graph $g(x) = 5\csc\left(\frac{1}{4}(x + \pi)\right)$.

2. Determine the function creating this graph:

3. Graph $h(x) = 3\sin\left(\frac{1}{2}(x + \frac{\pi}{2})\right)$.

Solutions:

1.

2.

This could be either a secant or cosecant function. We will use a cosecant model. First, the vertical shift is -1. The period is the difference between the two given $x-$values, $\frac{7\pi}{4} - \frac{3\pi}{4} = \pi$, so the frequency is $\frac{2\pi}{\pi} = 2$. The horizontal shift incorporates the frequency, so in $y = \csc x$ the corresponding $x-$value to $\left(\frac{3\pi}{4}, 0\right)$ is $\left(\frac{\pi}{2}, 1\right)$. The difference between the $x-$values is $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} - \frac{2\pi}{4} = \frac{\pi}{4}$ and then multiply it by the frequency, $2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$. The equation is $y = -1 + \csc\left(2(x - \frac{\pi}{2})\right)$.

3.

Concept Problem Solution

As you have seen in this Concept, the shadow of a light applied vertically to a rotating clock hand would trace out a sine graph. The graph would begin at zero when the hand is lying flat along the positive "x" axis. It would then increase until the hand was vertical. It would then decrease until the rotating hand was pointing straight down. Finally, the graph would increase again to zero when the hand is returning to the positive "x" axis.

Practice

Graph each of the following functions.

1. $f(x) = \sin(x)$.
2. $h(x) = \sin(2x)$.
3. $k(x) = \sin(2x + \pi)$.
4. $m(x) = 2\sin(2x + \pi)$.
5. $g(x) = 2\sin(2x + \pi) + 2$.
6. $f(x) = \csc(x)$.
7. $h(x) = \csc(2x)$.
8. $k(x) = \csc(2x + \pi)$.
9. $m(x) = 2\csc(2x + \pi)$.
10. $g(x) = 2\csc(2x + \pi) + 2$.
11. $h(x) = \sin(3x)$.

12. $k(x) = \sin(3x + \frac{\pi}{2})$.
13. $m(x) = 3\sin(3x + \frac{\pi}{2})$.
14. $g(x) = 3\sin(3x + \frac{\pi}{2}) + 3$.
15. $h(x) = \csc(3x)$.
16. $k(x) = \csc(3x + \frac{3\pi}{2})$.
17. $m(x) = 4\csc(3x + \frac{3\pi}{2})$.
18. $g(x) = 4\csc(3x + \frac{3\pi}{2}) - 3$.

2.10 Cosine and Secant Graphs

Here you'll learn to draw the graphs of the cosine and secant functions.

Imagine for a moment that you have a clock that has only one hand - that rotates counterclockwise!. However, the hand is very slim all the way until the tip, where there is a ball on the end. In fact, the hand is so slim you won't notice it. You only notice the ball on the end of the rotating hand. This hand is rotating faster than normal.

Consider what it would be like if you put a light above the clock and let the shadow of the hands fall on the wall under the clock. What pattern would that shadow trace out? If you think about it, you might realize that the shadow would make an left and right motion, over and over as the hand of the clock rotated. Now imagine that instead of a wall, there was a large piece of paper for the shadow to fall on. And wherever the shadow fell, there would be a mark on the paper. Finally, imagine moving the paper as the clock rotates. Can you imagine sort of pattern this would trace out?

By the end of this Concept, you'll understand just how this relates to trigonometric functions in general, and the cosine graph in particular.

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James Sousa Animation: Graphing the Cosine Function Using the Unit Circle

Guidance

If you have read other Trigonometry Concepts in this course, you may have learned that sine and cosine are very closely related. The cosine of an angle is the same as the sine of its complementary angle. So, it should not be a surprise that sine and cosine waves are very similar in that they are both periodic with a period of 2π, a range from -1 to 1, and a domain of all real angles.

The cosine of an angle is the ratio of $\frac{x}{r}$, so in the unit circle, the cosine is the $x-$coordinate of the point of rotation. If we trace the $x-$coordinate through a rotation, notice the change in the distance of $\cos x$ starts at one. The $x-$coordinate at $0°$ is 1 and the $x-$coordinate for $90°$ is 0, so the cosine value is decreasing from 1 to 0 through the 1st quadrant.

Here is a sequence of rotations. Compare the $x-$ coordinate of the point of rotation with the height of the point as it traces along the horizontal. These pictures plot $(\theta, \cos\theta)$ on the coordinate plane as (x, y).

Plotting the quadrant angles and filling in the in-between values shows the graph of $y = \cos x$

The graph of $y = \cos x$ has a period of 2π. The **range** of a cosine curve is $\{-1 \leq y \leq 1\}$ and the **domain** of $\cos x$ is all reals. If you've studied the sine function, you may notice that the shape of the curve is exactly the same, but *shifted* by $\frac{\pi}{2}$.

Secant is the reciprocal of cosine, or $\frac{1}{x}$. Therefore, whenever the cosine is zero, the secant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the cosine function in the same quadrants. Here is the graph.

The period of the function is 2π, just like cosine. The domain of the function is all real numbers, except multiples of π starting at $\frac{\pi}{2}$. $\left\{ \ldots \frac{-\pi}{2}, \frac{\pi}{2}, 0, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots \right\}$. The range is all real numbers greater than or equal to 1 as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where cosine is defined (other than the tops and bottoms of the cosine curve).

Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Again, this graph looks parabolic, but it is not.

Example A

Sketch a graph of $h(x) = 5 + \frac{1}{2} \sec 4x$ over the interval $[0, 2\pi]$.

Solution: If you compare this example to $f(x) = \sec x$, it will be translated 5 units up, with an amplitude of $\frac{1}{2}$ and a frequency of 4. This means in our interval of 0 to 2π, there will be 4 secant curves.

Example B

Find the equation for the graph below.

Solution: First of all, this could be either a secant or cosecant function. Let's say this is a secant function. Secant usually intersects the $y-$axis at $(0, 1)$ at a minimum. Now, that corresponding minimum is $\left(\frac{\pi}{2}, -2 \right)$. Because there is no amplitude change, we can say that the vertical shift is the difference between the two $y-$values, -3. It looks like there is a phase shift and a period change. From minimum to minimum is one period, which is $\frac{9\pi}{2} - \frac{\pi}{2} = \frac{8\pi}{2} = 4\pi$ and $B = \frac{2\pi}{4\pi} = \frac{1}{2}$. Lastly, we need to find the horizontal shift. Since secant usually intersects the $y-$axis at $(0, 1)$ at a minimum, and now the corresponding minimum is $\left(\frac{\pi}{2}, -2 \right)$, we can say that the horizontal shift is the difference between the two $x-$values, $\frac{\pi}{2}$. Therefore, our equation is $f(x) = -3 + \sec \left(\frac{1}{2} \left(x - \frac{\pi}{2} \right) \right)$.

Example C

Graph the function $h(x) = 2 - 3 \cos 4x$

Vocabulary

Circular Function: A **circular function** is a function that is measured by examining the angle of rotation around the coordinate plane.

Guided Practice

1. Graph $y = -2 + \frac{1}{2} \sec(4(x-1))$.

2. Determine the function creating this graph:

3. Graph $h(x) = \frac{1}{3} \cos 2x$

Solutions:

1.

2. This could be either a secant or cosecant function. We will use a cosecant model. First, the vertical shift is -1. The period is the difference between the two given $x-$values, $\frac{7\pi}{4} - \frac{3\pi}{4} = \pi$, so the frequency is $\frac{2\pi}{\pi} = 2$. The horizontal

shift incorporates the frequency, so in $y = \csc x$ the corresponding $x-$value to $\left(\frac{3\pi}{4}, 0\right)$ is $\left(\frac{\pi}{2}, 1\right)$. The difference between the $x-$values is $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} - \frac{2\pi}{4} = \frac{\pi}{4}$ and then multiply it by the frequency, $2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$. The equation is $y = -1 + \csc\left(2(x - \frac{\pi}{2})\right)$.

3.

Concept Problem Solution

As you have learned in this Concept, a light shining down on the rotating hand would create a shadow in the pattern of a cosine function, starting at a maximum value as the hand is lying along the "x" axis, going through zero to a maximum negative value when the hand is lying along the negative "y" axis. It would then begin to increase until it returned to a maximum value when the rotating hand was again lying along the positive "x" axis.

Practice

Graph each of the following functions.

1. $f(x) = \cos(x)$.
2. $h(x) = \cos(2x)$.
3. $k(x) = \cos(2x + \pi)$.
4. $m(x) = -2\cos(2x + \pi)$.
5. $g(x) = -2\cos(2x + \pi) + 1$.
6. $f(x) = \sec(x)$.
7. $h(x) = \sec(3x)$.
8. $k(x) = \sec(3x + \pi)$.
9. $m(x) = 2\sec(3x + \pi)$.
10. $g(x) = 3 + 2\sec(3x + \pi)$.
11. $h(x) = \cos(\frac{x}{2})$.
12. $k(x) = \cos(\frac{x}{2} + \frac{\pi}{2})$.
13. $m(x) = 2\cos(\frac{x}{2} + \frac{\pi}{2})$.
14. $g(x) = 2\cos(\frac{x}{2} + \frac{\pi}{2}) - 3$.
15. $h(x) = \sec(\frac{x}{4})$.
16. $k(x) = \sec(\frac{x}{4} + \frac{3\pi}{2})$.
17. $m(x) = -3\sec(\frac{x}{4} + \frac{3\pi}{2})$.
18. $g(x) = 2 - 3\sec(\frac{x}{4} + \frac{3\pi}{2})$.

2.11 Tangent and Cotangent Graphs

Here you'll learn to draw the graphs of the tangent and cotangent functions.

What if your instructor gave you a set of graphs like these:

and asked you to identify which were the graphs of the tangent and cotangent functions?

After completing this Concept, you'll be able to identify the graphs of tangent and cotangent.

Watch This
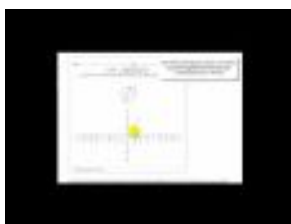
James Sousa Animation: Graphing the Tangent FunctionUsing the Unit Circle

Guidance

The name of the tangent function comes from the tangent line of a circle. This is a line that is perpendicular to the radius at a point on the circle so that the line touches the circle at exactly one point.

If we extend angle θ through the unit circle so that it intersects with the tangent line, the tangent function is defined as the length of the red segment.

The dashed segment is 1 because it is the radius of the unit circle. Recall that the $\tan\theta = \frac{y}{x}$, and it can be verified that this segment is the tangent by using similar triangles.

$$\tan\theta = \frac{y}{x} = \frac{t}{1} = t$$
$$\tan\theta = t$$

So, as we increase the angle of rotation, think about how this segment changes. When the angle is zero, the segment has no length. As we rotate through the first quadrant, it will increase very slowly at first and then quickly get very close to one, but never actually touch it.

As we get **very** close to the $y-$axis the segment gets infinitely large, until when the angle really hits $90°$, at which point the extension of the angle and the tangent line will actually be parallel and therefore never intersect.

This means there is no finite length of the tangent segment, or the tangent segment is infinitely large

Let's translate this portion of the graph onto the coordinate plane. Plot $(\theta, \tan\theta)$ as (x, y).

In fact as we get infinitely close to $90°$, the tangent value increases without bound, until when we actually reach $90°$, at which point the tangent is undefined. Recall there are some angles ($90°$ and $270°$, for example) for which the tangent is not defined. Therefore, at these points, there are going to be vertical asymptotes.

Rotating past $90°$, the intersection of the extension of the angle and the tangent line is actually below the $x-$axis. This fits nicely with what we know about the tangent for a 2nd quadrant angle being negative. At first, it will have very large negative values, but as the angle rotates, the segment gets shorter, reaches 0, then crosses back into the positive numbers as the angle enters the 3rd quadrant. The segment will again get infinitely large as it approaches $270°$. After being undefined at $270°$, the angle crosses into the 4th quadrant and once again changes from being infinitely negative, to approaching zero as we complete a full rotation.

The graph $y = \tan x$ over several rotations would look like this:

Notice the $x-$axis is measured in radians. Our asymptotes occur every π radians, starting at $\frac{\pi}{2}$. The period of the graph is therefore π radians. The domain is all reals except for the asymptotes at $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, etc.$ and the range is all real numbers.

Cotangent is the reciprocal of tangent, $\frac{x}{y}$, so it would make sense that where ever the tangent had an asymptote, now the cotangent will be zero. The opposite of this is also true. When the tangent is zero, now the cotangent will have an asymptote. The shape of the curve is generally the same, so the graph looks like this:

When you overlap the two functions, notice that the graphs consistently intersect at 1 and -1. These are the angles that have $45°$ as reference angles, which always have tangents and cotangents equal to 1 or -1. It makes sense that 1 and -1 are the only values for which a function and it's reciprocal are the same. Keep this in mind as we look at cosecant and secant compared to their reciprocals of sine and cosine.

The cotangent function has a domain of all real angles except multiples of $\pi \{\ldots - 2\pi, -\pi, 0, \pi, 2\pi \ldots\}$ The range is all real numbers.

Example A

Sketch the graph of $g(x) = -2 + \cot \frac{1}{3}x$ over the interval $[0, 6\pi]$.

Solution: Starting with $y = \cot x$, $g(x)$ would be shifted down two and frequency is $\frac{1}{3}$, which means the period would be 3π, instead of π. So, in our interval of $[0, 6\pi]$ there would be two complete repetitions. The red graph is $y = \cot x$.

Example B

Sketch the graph of $y = -3 \tan \left(x - \frac{\pi}{4}\right)$ over the interval $[-\pi, 2\pi]$.

Solution: If you compare this graph to $y = \tan x$, it will be stretched and flipped. It will also have a phase shift of $\frac{\pi}{4}$ to the right. The red graph is $y = \tan x$.

Example C

Sketch the graph of $h(x) = 4 \tan \left(x + \frac{\pi}{2}\right) + 3$ over the interval $[0, 2\pi]$.

Solution: The constant in front of the tangent function will cause the graph to be stretched. It will also have a phase shift of $\frac{\pi}{2}$ to the left. Finally, the graph will be shifted up three. Here you can see both graphs, where the red graph is $y = \tan x$.

Vocabulary

Circular Function: A **circular function** is a function that is measured by examining the angle of rotation around the coordinate plane.

Guided Practice

1. Graph $y = -1 + \frac{1}{3}\cot 2x$.

2. Graph $f(x) = 4 + \tan(0.5(x - \pi))$.

3. Graph $y = -2\tan 2x$.

Solutions:

1.

2.

3.

Concept Problem Solution

As you can tell after completing this Concept, when presented with the graphs:

1 2 3

4 5 6

The tangent and cotangent graphs are the third and sixth graphs.

Practice

Graph each of the following functions.

   1. $f(x) = \tan(x)$.
   2. $h(x) = \tan(2x)$.
   3. $k(x) = \tan(2x + \pi)$.
   4. $m(x) = -\tan(2x + \pi)$.
   5. $g(x) = -\tan(2x + \pi) + 3$.
   6. $f(x) = \cot(x)$.
   7. $h(x) = \cot(2x)$.
   8. $k(x) = \cot(2x + \pi)$.
   9. $m(x) = 3\cot(2x + \pi)$.
  10. $g(x) = -2 + 3\cot(2x + \pi)$.
  11. $h(x) = \tan(\frac{x}{2})$.
  12. $k(x) = \tan(\frac{x}{2} + \frac{\pi}{4})$.
  13. $m(x) = 3\tan(\frac{x}{2} + \frac{\pi}{4})$.
  14. $g(x) = 3\tan(\frac{x}{2} + \frac{\pi}{4}) - 1$.
  15. $h(x) = \cot(\frac{x}{2})$.
  16. $k(x) = \cot(\frac{x}{2} + \frac{3\pi}{2})$.
  17. $m(x) = -3\cot(\frac{x}{2} + \frac{3\pi}{2})$.
  18. $g(x) = 2 - 3\cot(\frac{x}{2} + \frac{3\pi}{2})$.

2.12 Vertical Translations

Here you'll learn how to express vertical translations of graphs algebraically.

You are working on a graphing project in your math class, where you are supposed to graph several functions. You are working on graphing a cosine function, and things seem to be going well, until you realize that there is a bold, horizontal line two units above where you placed your "x" axis! As it turns out, you've accidentally shifted your entire graph. You didn't notice that your instructor had placed a bold line where the "x" axis was supposed to be. And now, all of the points for your graph of the cosine function are two points lower than they are supposed to be along the "y" axis.

You might be able to keep all of your work, if you can find a way to rewrite the equation so that it takes into account the change in your graph.

Can you think of a way to rewrite the function so that the graph is correct the way you plotted it?

Keep reading, and at the end of this Concept, you'll know how to do exactly that using a "vertical shift".

Watch This

In the second portion of this video you will learn how to perform vertical translations of the sine and cosine functions.

MEDIA
Click image to the left for more content.

James Sousa:Horizontal and VerticalTranslations ofSine and Cosine

Guidance

When you first learned about vertical translations in a coordinate grid, you started with simple shapes. Here is a rectangle:

To translate this rectangle vertically, move all points and lines up by a specified number of units. We do this by adjusting the $y-$coordinate of the points. So to translate this rectangle 5 units up, add 5 to every $y-$coordinate.

This process worked the same way for functions. Since the value of a function corresponds to the $y-$value on its graph, to move a function up 5 units, we would increase the value of *the function* by 5. Therefore, to translate $y = x^2$ up five units, you would increase the $y-$value by 5. Because y is equal to x^2, then the equation $y = x^2 + 5$, will show this translation.

Hence, for any graph, adding a constant to the equation will move it up, and subtracting a constant will move it down. From this, we can conclude that the graphs of $y = \sin x$ and $y = \cos x$ will follow the same rules. That is, the graph of $y = \sin(x) + 2$ will be the same as $y = \sin x$, only it will be translated, or shifted, 2 units up.

To avoid confusion, this translation is usually written in front of $y = 2 + \sin x$.

Various texts use different notation, but we will use D as the constant for vertical translations. This would lead to the following equations: $y = D \pm \sin x$ and $y = D \pm \cos x$ where D is the vertical translation. *D can be positive or negative*.

Another way to think of this is to view sine or cosine curves "wrapped" around a horizontal line. For $y = \sin x$ and $y = \cos x$, the graphs are wrapped around the $x-$axis, or the horizontal line, $y = 0$.

For $y = 3 + \sin x$, we know the curve is translated up 3 units. In this context, think of the sine curve as being "wrapped" around the line, $y = 3$.

Either method works for the translation of a sine or cosine curve. Pick the thought process that works best for you.

Example A

Find the minimum and maximum of $y = -6 + \cos x$

Solution: This is a cosine wave that has been shifted down 6 units, or is now wrapped around the line $y = -6$. Because the graph still rises and falls one unit in either direction, the cosine curve will extend one unit above the "wrapping line" and one unit below it. The minimum is -7 and the maximum is -5.

Example B

Graph $y = 4 + \cos x$.

Solution: This will be the basic cosine curve, shifted up 4 units.

Example C

Find the minimum and maximum of $y = \sin x + 3$

Solution: This is a sine wave that has been shifted up 3 units, so now instead of going up and down around the 'x' axis, it will go up and down around the line y = 3. Since the sine function rises and falls one unit in each direction, the new minimum is 2 and the new maximum is 4.

Vocabulary

Vertical Translation: A **vertical translation** is a shift in a graph up or down along the "y" axis, generated by adding a constant to the original function.

Guided Practice

1. Which of the following is true for the equation: $y = \sin\left(x - \frac{\pi}{2}\right)$

The minimum value is 0.

The maximum value is 3.

The $y-$intercept is -2.

The $y-$intercept is -1.

This is the same graph as $y = \cos(x)$.

2. Which of the following is true for the equation: $y = 1 + \sin x$

The minimum value is 0.

The maximum value is 3.

The $y-$intercept is -2.

The $y-$intercept is -1.

This is the same graph as $y = \cos(x)$.

3. Which of the following is true for the equation: $y = 2 + \cos x$

The minimum value is 0.

The maximum value is 3.

The $y-$intercept is -2.

The $y-$intercept is -1.

This is the same graph as $y = \cos(x)$.

Solutions:

1. "This is the same graph as $y = \cos(x)$." is the answer to this question, since the $\frac{\pi}{2}$ is a shift to the graph which makes it the same as a cosine graph.

2. "The minimum value is 0." is the answer to this question, since this graph is the same as a regular sine graph, which ranges from -1 to 1, but shifted upward one unit on the "y" axis, so it ranges from 0 to 2.

3. "The maximum value is 3." is the answer to this question, since the is a cosine graph (which normally ranges between -1 and 1) shifted upward by two units. Therefore its new range is from 1 to 3.

Concept Problem Solution

Since you now know how to shift a graph vertically by adding or removing a constant after the function, you can keep your graph by changing the equation to $y = \cos(x) - 2$

Practice

Use vertical translations to graph each of the following functions.

1. $y = x^3 + 4$
2. $y = x^2 - 3$
3. $y = \sin(x) - 4$
4. $y = \cos(x) + 7$
5. $y = \sec(x) - 3$
6. $y = \tan(x) + 2$
7. $y = 3 + \sin(x)$
8. $y = \cos(x) + 1$
9. $y = 6 + \sec(x)$
10. $y = \tan(x) - 4$

Find the minimum and maximum value of each of the following functions.

11. $y = \sin(x) + 6$
12. $y = \cos(x) - 1$
13. $y = \sin(x) - 4$
14. $y = -3 + \cos(x)$
15. $y = 2 + \cos(x)$

16. Give an example of a sine function with a y-intercept of 6.
17. Give an example of a cosine function with a maximum of -1.
18. Give an example of a sine function with a minimum of 0.

2.13 Horizontal Translations or Phase Shifts

Here you'll learn how to express horizontal translations of graphs algebraically.

You are working on a graphing project in your math class, where you are supposed to graph several functions. Things seem to be going well, until you realize that there is a bold, vertical line three units to the left of where you placed your "y" axis! As it turns out, you've accidentally shifted your entire graph. You didn't notice that your instructor had placed a bold line where the "y" axis was supposed to be. And now, all of the points for your graph of the cosine function are three points farther to the right than they are supposed to be along the "x" axis.

You might be able to keep all of your work, if you can find a way to rewrite the equation so that it takes into account the change in your graph.

Can you think of a way to rewrite the function so that the graph is correct the way you plotted it?

Keep reading, and at the end of this Concept, you'll know how to do exactly that using a "horizontal shift".

Watch This

In the first part of this video you'll learn how to perform horizontal translations of sine and cosine graphs.

MEDIA
Click image to the left for more content.

James Sousa:Horizontal and VerticalTranslations ofSine and Cosine

Guidance

Horizontal translations involve placing a constant inside the argument of the trig function being plotted. If we return to the example of the parabola, $y = x^2$, what change would you make to the equation to have it move to the right or left? Many students guess that if you move the graph vertically by adding to the $y-$value, then we should add to the $x-$value in order to translate horizontally. This is correct, but the graph itself behaves in the opposite way than what you may think.

Here is the graph of $y = (x+2)^2$.

Notice that adding$x-$value shifted the graph 2 units *to the left*, or in the *negative* direction.

To compare, the graph $y = (x-2)^2$ moves the graph 2 units *to the right* or in the *positive* direction.

We will use the letter C to represent the horizontal shift value. Therefore, **subtracting** C from the $x-$value will shift the graph to the **right** and **adding** C will shift the graph C units to the **left**.

Adding to our previous equations, we now have $y = D \pm \sin(x \pm C)$ and $y = D \pm \cos(x \pm C)$ where D is the vertical translation and C is the *opposite sign* of the horizontal shift.

Example A

Sketch $y = \sin\left(x - \frac{\pi}{2}\right)$

Solution: This is a sine wave that has been translated $\frac{\pi}{2}$ units to the *right*.

Horizontal translations are also referred to as **phase shifts**. Two waves that are identical, but have been moved horizontally are said to be "out of phase" with each other. Remember that cosine and sine are really the same waves with this phase variation.

$y = \sin x$ can be thought of as a cosine wave shifted horizontally to the right by $\frac{\pi}{2}$ radians.

Alternatively, we could also think of cosine as a sine wave that has been shifted $\frac{\pi}{2}$ radians to the left.

Example B

Draw a sketch of $y = 1 + \cos(x - \pi)$

Solution: This is a cosine curve that has been translated up 1 unit and π units to the right. It may help you to use the quadrant angles to draw these sketches. Plot the points of $y = \cos x$ at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (as well as the negatives), and then translate those points before drawing the translated curve. The blue curve below is the final answer.

Example C

Graph $y = -2 + \sin\left(x + \frac{3\pi}{2}\right)$

Solution: This is a sine curve that has been translated 2 units down and moved $\frac{3\pi}{2}$ radians to the left. Again, start with the quadrant angles on $y = \sin x$ and translate them down 2 units.

Then, take that result and shift it $\frac{3\pi}{2}$ to the left. The blue graph is the final answer.

Vocabulary

Phase Shift: A **phase shift** is a horizontal translation.

Guided Practice

1. Draw a sketch of $y = 3 + \cos\left(x - \frac{\pi}{2}\right)$
2. Draw a sketch of $y = \sin\left(x + \frac{\pi}{4}\right)$
3. Draw a sketch of $y = 2 + \cos(x + 2\pi)$

Solutions:

1.

As we've seen, the 3 shifts the graph vertically 3 units, while the $-\frac{\pi}{2}$ shifts the graph to the right by $\frac{\pi}{2}$ units.

2.

The $\frac{\pi}{4}$ shifts the graph to the left by $\frac{\pi}{4}$.

3.

The 2 added to the function shifts the graph up by 2 units, and the 2π added in the argument of the function brings the function back to where it started, so the cosine graph isn't shifted horizontally at all.

Concept Problem Solution

As you've now seen by reading this Concept, it is possible to shift an entire graph to the left or the right by changing the argument of the graph.

So in this case, you can keep your graph by changing the function to $y = \cos(x - 3)$

Practice

Graph each of the following functions.

1. $y = \cos(x - \frac{\pi}{2})$
2. $y = \sin(x + \frac{3\pi}{2})$
3. $y = \cos(x + \frac{\pi}{4})$
4. $y = \cos(x - \frac{3\pi}{4})$
5. $y = -1 + \cos(x - \frac{\pi}{4})$
6. $y = 1 + \sin(x + \frac{\pi}{2})$
7. $y = -2 + \cos(x + \frac{\pi}{4})$
8. $y = 3 + \cos(x - \frac{3\pi}{2})$
9. $y = -4 + \sec(x - \frac{\pi}{4})$
10. $y = 3 + \csc(x - \frac{\pi}{2})$
11. $y = 2 + \tan(x + \frac{\pi}{4})$
12. $y = -3 + \cot(x - \frac{3\pi}{2})$
13. $y = 1 + \cos(x - \frac{3\pi}{4})$
14. $y = 5 + \sec(x + \frac{\pi}{2})$
15. $y = -1 + \csc(x + \frac{\pi}{4})$
16. $y = 3 + \tan(x - \frac{3\pi}{2})$

2.14 Amplitude

Here you'll learn how to find the amplitude of a trig function from either the graph or the algebraic equation.

While working on a sound lab assignment in your science class, your instructor assigns you an interesting problem. Your lab partner is assigned to speak into a microphone, and you are to record how "loud" the sound is using a device that plots the sound wave on a graph. Unfortunately, you don't know what part of the graph to read to understand "loudness". Your instructor tells you that "loudness" in a sound wave corresponds to "amplitude" on the graph, and that you should plot the values of the amplitude of the graph that is being produced.

Here is a picture of the graph:

Can you accomplish this task? By the end of this Concept, you'll understand what part of a graph the amplitude is, as well as how to find it.

Watch This

In the first part of this video you'll learn about the amplitude of trigonometric functions.

MEDIA
Click image to the left for more content.

James Sousa:Amplitude and Period ofSine and Cosine

Guidance

The **amplitude** of a wave is basically a measure of its height. Because that height is constantly changing, amplitude can be different from moment to moment. If the wave has a regular up and down shape, like a cosine or sine wave, the amplitude is defined as the *farthest* distance the wave gets from its center. In a graph of $f(x) = \sin x$, the wave is centered on the $x-$axis and the farthest away it gets (in either direction) from the axis is 1 unit.

So the amplitude of $f(x) = \sin x$ (and $f(x) = \cos x$) is 1.

Recall how to transform a linear function, like $y = x$. By placing a constant in front of the x value, you may remember that the slope of the graph affects the steepness of the line.

The same is true of a parabolic function, such as $y = x^2$. By placing a constant in front of the x^2, the graph would be either wider or narrower. So, a function such as $y = \frac{1}{8} x^2$, has the same parabolic shape but it has been "smooshed," or looks wider, so that it increases or decreases at a lower rate than the graph of $y = x^2$.

No matter the basic function; linear, parabolic, or trigonometric, the same principle holds. To dilate (flatten or steepen, wide or narrow) the function, multiply the function by a constant. Constants greater than 1 will stretch the graph vertically and those less than 1 will shrink it vertically.

Look at the graphs of $y = \sin x$ and $y = 2\sin x$.

Notice that the amplitude of $y = 2\sin x$ is now 2. An investigation of some of the points will show that each $y-$value

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is twice as large as those for $y = \sin x$. Multiplying values less than 1 will decrease the amplitude of the wave as in this case of the graph of $y = \frac{1}{2}\cos x$:

Example A

Determine the amplitude of $f(x) = 10\sin x$.

Solution: The 10 indicates that the amplitude, or height, is 10. Therefore, the function rises and falls between 10 and -10.

Example B

Graph $g(x) = -5\cos x$

Solution: Even though the 5 is negative, the amplitude is still positive 5. The amplitude is always the absolute value of the constant A. However, the negative changes the appearance of the graph. Just like a parabola, the sine (or cosine) is flipped upside-down. Compare the blue graph, $g(x) = -5\cos x$, to the red parent graph, $f(x) = \cos x$.

So, in general, the constant that creates this stretching or shrinking is the amplitude of the sinusoid. Continuing with our equations from the previous section, we now have $y = D \pm A\sin(x \pm C)$ or $y = D \pm A\cos(x \pm C)$. Remember, if $0 < |A| < 1$, then the graph is shrunk and if $|A| > 1$, then the graph is stretched. And, if A is negative, then the graph is flipped.

Example C

Graph $h(x) = -\frac{1}{4}\sin(x)$

Solution:

As you can see from the graph, the negative inverts the graph, and the $\frac{1}{4}$ makes the maximum height the function reaches reduced from 1 to $\frac{1}{4}$.

Vocabulary

Amplitude: The **amplitude** of a wave is a measure of the wave's height.

Guided Practice

1. Identify the minimum and maximum values of $y = \cos x$.

2. Identify the minimum and maximum values of $y = 2\sin x$

3. Identify the minimum and maximum values of $y = -\sin x$

Solutions:

1. The cosine function ranges from -1 to 1, therefore the minimum is -1 and the maximum is 1.

2. The sine function ranges from -1 to 1, and since there is a two multiplied by the function, the minimum is -2 and the maximum is 2.

3. The sine function ranges between -1 and 1, so the minimum is -1 and the maximum is 1.

Concept Problem Solution

Since you now know what the amplitude of a graph is and how to read it, it is straightforward to see from this graph of the sound wave the distance that the wave rises or falls at different times. For this graph, the amplitude is 7.

Practice

Determine the amplitude of each function.

1. $y = 3\sin(x)$
2. $y = -2\cos(x)$
3. $y = 3 + 2\sin(x)$
4. $y = -1 + \frac{2}{3}\sin(x)$
5. $y = -4 + \cos(3x)$

Graph each function.

6. $y = 4\sin(x)$
7. $y = -\cos(x)$
8. $y = \frac{1}{2}\sin(x)$
9. $y = -\frac{3}{4}\sin(x)$
10. $y = 2\cos(x)$

Identify the minimum and maximum values of each function.

11. $y = 5\sin(x)$
12. $y = -\cos(x)$
13. $y = 1 + 2\sin(x)$
14. $y = -3 + \frac{2}{3}\sin(x)$
15. $y = 2 + 2\cos(x)$
16. How does changing the constant k change the graph of $y = k\tan(x)$?
17. How does changing the constant k change the graph of $y = k\sec(x)$?

2.15 Period and Frequency

Here you'll learn how to find the period and frequency of a trig function from either the graph or the algebraic equation.

While working on an assignment about sound in your science class, your Instructor informs you that what you know as the "pitch" of a sound is, in fact, the frequency of the sound waves. He then plays a note on a musical instrument, and the pattern of the sound wave on a graph looks like this:

He then tells you to find the frequency of the sound wave from the graph? Can you do it?

Don't worry. By the end of this Concept, you'll understand what frequency is and be able to find it from a plot like this one.

Watch This

In the second part of this video you'll learn about the period of trigonometric functions.

MEDIA
Click image to the left for more content.

James Sousa:Amplitude and Period ofSine and Cosine

Guidance

The **period** of a trigonometric function is the horizontal distance traversed before the $y-$values begin to repeat. For both graphs, $y = \sin x$ and $y = \cos x$, the period is 2π. As you may remember, after completing one rotation of the unit circle, these values are the same.

Frequency is a measurement that is closely related to period. In science, the frequency of a sound or light wave is the number of complete waves for a given time period (like seconds). In trigonometry, because all of these periodic functions are based on the unit circle, we usually measure frequency as the number of complete waves every 2π units. Because $y = \sin x$ and $y = \cos x$ cover exactly one complete wave over this interval, their frequency is 1.

Period and frequency are inversely related. That is, the higher the frequency (more waves over 2π units), the lower the period (shorter distance on the $x-$axis for each complete cycle).

After observing the transformations that result from multiplying a number in front of*inside* the argument of the function, or in other words, by the x value. In general, the equation would be $y = \sin Bx$ or $y = \cos Bx$. For example, look at the graphs of $y = \cos 2x$ and $y = \cos x$.

Notice that the number of waves for $y = \cos 2x$ has increased, in the same interval as $y = \cos x$. There are now **2 waves** over the interval from 0 to 2π. Consider that you are doubling each of the x values because the function is $2x$. When π is plugged in, for example, the function becomes 2π. So the portion of the graph that normally corresponds to 2π units on the $x-$axis, now corresponds to *half* that distance—so the graph has been "scrunched" horizontally. The frequency of this graph is therefore 2, or the same as the constant we multiplied by in the argument. The period (the length for each complete wave) is π.

Example A

What is the frequency and period of $y = \sin 3x$?

Solution: If we follow the pattern from the previous example, multiplying the angle by 3 should result in the sine wave completing a cycle **three times** as often as $y = \sin x$. So, there will be three complete waves if we graph it from 0 to 2π. The frequency is therefore 3. Similarly, if there are 3 complete waves in 2π units, one wave will be a third of that distance, or $\frac{2\pi}{3}$ radians. Here is the graph:

This number that is multiplied by x, called B, will create a horizontal dilation. The larger the value of B, the more compressed the waves will be horizontally. To stretch out the graph horizontally, we would need to *decrease* the frequency, or multiply by a number that is less than 1. Remember that this dilation factor is *inversely* related to the period of the graph.

Adding, one last time to our equations from before, we now have: $y = D \pm A \sin(B(x \pm C))$ or $y = D \pm A \cos(B(x \pm C))$, where B is the frequency, the period is equal to $\frac{2\pi}{B}$, and everything else is as defined before.

Example B

What is the frequency and period of $y = \cos \frac{1}{4}x$?

Solution: Using the generalization above, the frequency must be $\frac{1}{4}$ and therefore the period is $\frac{2\pi}{\frac{1}{4}}$, which simplifies to: $\frac{2\pi}{\frac{1}{4}} = \frac{2\pi}{\frac{1}{4}} \cdot \frac{4}{1} = \frac{8\pi}{1} = 8\pi$

Thinking of it as a transformation, the graph is stretched horizontally. We would only see $\frac{1}{4}$ of the curve if we graphed the function from 0 to 2π. To see a complete wave, therefore, we would have to go four times as far, or all the way from 0 to 8π.

Example C

What is the frequency and period of $y = \sin \frac{1}{2}x$?

Solution:

Like the previous two examples, we can see that the frequency is $\frac{1}{2}$, and so the period is $\frac{2\pi}{\frac{1}{2}}$, which becomes $2\pi \times \frac{2}{1} = 4\pi$

Vocabulary

Period: The **period** of a wave is the horizontal distance traveled before the 'y' values begin to repeat.

Frequency: The **frequency** of a wave is number of complete waves every 2π units.

Guided Practice

1. Draw a sketch of $y = 3 \sin 2x$ from 0 to 2π.

2. Draw a sketch of $y = 2.5 \cos \pi x$ from 0 to 2π.

3. Draw a sketch of $y = 4 \sin \frac{1}{2}x$ from 0 to 2π.

Solutions:

1. The "2" inside the sine function makes the function "squashed" by a factor of 2 in the horizontal direction.

2. The π inside the sine function makes the function "squashed" by a factor of π in the horizontal direction.

3. The $\frac{1}{2}$ inside the sine function makes the function "stretched" by a factor of $\frac{1}{2}$ in the horizontal direction.

Concept Problem Solution

By inspecting the graph

You can see that the wave takes about 6.2 seconds to make one complete cycle. This means that the frequency of the wave is approximately 1 cycle per second (since 2π is approximately 6.28). (You should note that in a real wave of sound, you would need to use the speed of the wave and so the calculation would be different. But if you read the graph the same way you read trigonometric functions to find the frequency, this is the result you would find.)

Practice

Find the period and frequency of each function below.

1. $y = \sin(4x)$
2. $y = \cos(2x)$
3. $y = \cos(\frac{1}{2}x)$
4. $y = \sin(\frac{3}{4}x)$
5. $y = \sin(3x)$

Draw a sketch of each function from 0 to 2π.

6. $y = \sin(3x)$
7. $y = \cos(5x)$
8. $y = 3\cos(\frac{2}{5}x)$
9. $y = \frac{1}{2}\sin(\frac{3}{4}x)$
10. $y = -\sin(2x)$
11. $y = \tan(3x)$
12. $y = \sec(2x)$
13. $y = \csc(4x)$

Find the equation of each function.

14.
15.
16.
17.

2.16 Amplitude and Period

Here you'll learn how to solve problems that involve both the amplitude and period of a trig function.

You are working in science lab one afternoon when your teacher asks you to do a little more advanced work with her on sound. Excited to help, you readily agree. She gives you a device that graphs sound waves as they come in through a microphone. She then gives you a "baseline" graph of what the sound wave's graph would look like:

She then asks you to plot the sound wave she's about to generate. However, she tells you that the sound wave will be twice as loud and twice as high in pitch as the baseline sound wave she gave you.

Can you determine how large the graph needs to be to plot the new sound wave? What about the spacing of numbers on the "x" axis?

At the conclusion of this Concept, you'll know how to determine the required properties of the coordinate system and plot you're going to draw.

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James Sousa:Amplitude and Period ofSine and Cosine

Guidance

In other Concepts you have dealt with how find the amplitude of a wave, or the period of a wave. Here we'll take a few minutes to work problems that involve both the amplitude and period, giving us two variables to work with when thinking about sinusoidal equations.

Example A

Find the period, amplitude and frequency of $y = 2\cos\frac{1}{2}x$ and sketch a graph from 0 to 2π.

Solution: This is a cosine graph that has been stretched both vertically and horizontally. It will now reach up to 2 and down to -2. The frequency is $\frac{1}{2}$ and to see a complete period we would need to graph the interval $[0, 4\pi]$. Since we are only going out to 2π, we will only see half of a wave. A complete cosine wave looks like this:

So, half of it is this:

This means that this half needs to be stretched out so it finishes at 2π, which means that at π the graph should cross the $x-$axis:

The final sketch would look like this:

amplitude $= 2$, frequency $= \frac{1}{2}$, period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

Example B

Identify the period, amplitude, frequency, and equation of the following sinusoid:

Solution: The amplitude is 1.5. Notice that the units on the $x-$axis are not labeled in terms of π. This appears to be a sine wave because the $y-$intercept is 0.

One wave appears to complete in 1 unit (not 2π units? In previous examples, you were given the frequency and asked to find the period using the following relationship:

$$p = \frac{2\pi}{B}$$

Where B is the frequency and p is the period. With just a little bit of algebra, we can transform this formula and solve it for B:

$$p = \frac{2\pi}{B} \to Bp = 2\pi \to B = \frac{2\pi}{p}$$

Therefore, the frequency is:

$$B = \frac{2\pi}{1} = 2\pi$$

If we were to graph this out to 2π we would see 2π (or a little more than 6) complete waves.

Replacing these values in the equation gives: $f(x) = 1.5\sin 2\pi x$.

Example C

Find the period, amplitude and frequency of $y = 3\sin 2x$ and sketch a graph from 0 to 6π.

Solution: This is a sine graph that has been stretched both vertically and horizontally. It will now reach up to 3 and down to -3. The frequency is 2 and so we will see the wave repeat twice over the interval from 0 to 2π.

amplitude $= 3$, frequency $= 2$, period $= \frac{2\pi}{2} = \pi$

Vocabulary

Amplitude: The **amplitude** of a wave is a measure of the wave's height.

Period: The **period** of a wave is the horizontal distance traveled before the 'y' values begin to repeat.

Frequency: The **frequency** of a wave is number of complete waves every 2π units.

Guided Practice

1. Identify the amplitude, period, and frequency of $y = \cos 2x$

2. Identify the amplitude, period, and frequency of $y = 3\sin x$

3. Identify the amplitude, period, and frequency of $y = 2\sin \pi x$

Solutions:

1. period: π, amplitude: 1, frequency: 2

2. period: 2π, amplitude: 3, frequency: 1

3. period: 2, amplitude: 2, frequency: π

Concept Problem Solution

You know that the amplitude of the wave is the maximum height it makes above zero. You also know that the frequency is the number of cycles in a second. The scale of the graph you make should be able to take into account a maximum height of the wave that has been doubled, as well as a frequency that is twice as high. Your graph should look like this:

Practice

Find the period, amplitude, and frequency of the following functions.

1. $y = 2\sin(3x)$
2. $y = 5\cos(\frac{3}{4}x)$
3. $y = 3\cos(2x)$
4. $y = -2\sin(\frac{1}{2}x)$
5. $y = -\sin(2x)$
6. $y = \frac{1}{2}\cos(4x)$

Identify the equation of each of the following graphs.

7.
8.
9.
10.

Graph each of the following functions from 0 to 2π.

11. $y = 2\cos(4x)$
12. $y = 3\sin(\frac{5}{4}x)$
13. $y = -\cos(2x)$
14. $y = -2\sin(\frac{1}{2}x)$
15. $y = 4\sec(3x)$
16. $y = \frac{1}{2}\cos(3x)$
17. $y = 4\tan(3x)$
18. $y = \frac{1}{2}\csc(3x)$

2.17 Trigonometric Identities and Equations

Here you'll learn the general form of equations for trig functions and how to graph them.

Your math teacher has decided to give you a quiz to see if you recognize how to combine changes to graphs of sine and cosine functions. You recall that you've learned about shifting graphs, as well as stretching/dilating them. But now your teacher wants to see if you know how to combine both of these effects into one graph. She gives you the equation:

$$f(x) = 3 + 7\sin(4(x + \tfrac{\pi}{2}))$$

and asks you to plot the equation, and then identify what each part of the above equation does to change the graph.

Can you accomplish this task?

Read on, and at the conclusion of this Concept, you'll know how to plot this equation and identify which parts of it make changes to the graph.

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James Sousa Example:Graphing a Transformation of Sine and Cosine

Guidance

In other Concepts, you learned how to translate and dilate sine and cosine waves both horizontally and vertically. Combining all the information learned, the general equations are: $y = D \pm A\cos(B(x \pm C))$ or $y = D \pm A\sin(B(x \pm C))$, where A is the amplitude, B is the frequency, C is the horizontal translation, and D is the vertical translation.

Recall the relationship between period, p, and frequency, B.

$$p = \frac{2\pi}{B} \text{ and } B = \frac{2\pi}{p}$$

With this knowledge, we should be able to sketch any sine or cosine function as well as write an equation given its graph.

Example A

Given the function: $f(x) = 1 + 2\sin(2(x + \pi))$

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from -2π to 2π.

Solution: a. From the equation, the amplitude is 2 and the frequency is also 2. To find the period we use:

$$p = \frac{2\pi}{B} \rightarrow p = \frac{2\pi}{2} = \pi$$

So, there are two complete waves from $[0, 2\pi]$ and each individual wave requires π radians to complete.

b. $D = 1$ and $C = -\pi$, so this graph has been translated 1 unit up, and π units to the left.

c. To sketch the graph, start with the graph of $y = \sin(x)$

Translate the graph π units to the left (the C value).

Next, move the graph 1 unit up (D value)

Now we can add the dilations. Remember that the "starting point" of the wave is $-\pi$ because of the horizontal translation. A normal sine wave takes 2π units to complete a cycle, but this wave completes one cycle in π units. The first wave will complete at 0, then we will see a second wave from 0 to π and a third from π to 2π. Start by placing points at these values:

Using symmetry, each interval needs to cross the line $y = 1$ through the center of the wave.

One sine wave contains a "mountain" and a "valley". The mountain "peak" and the valley low point must occur halfway between the points above.

Extend the curve through the domain.

Finally, extend the minimum and maximum points to match the amplitude of 2.

Example B

Given the function: $f(x) = 3 + 3\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from -2π to 2π.

Solution: a. From the equation, the amplitude is 3 and the frequency is $\frac{1}{2}$. To find the period we use:

$$period = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

So, there is only one half of a cosine curve from 0 to 2π and each individual wave requires 4π radians to complete.

b. $D = 3$ and $C = \frac{\pi}{2}$, so this graph has been translated 3 units up, and $\frac{\pi}{2}$ units to the right.

c. To sketch the graph, start with the graph of $y = \cos(x)$

Adjust the amplitude so the cosine wave reaches up to 3 and down to negative three. This affects the maximum points, but the points on the $x-$axis remain the same. These points are sometimes called **nodes.**

According to the period, we should see one of these shapes every 4π units. Because the interval specified is $[-2\pi, 2\pi]$ and the cosine curve "starts" at the $y-$axis, at (0, 3) and at 2π the value is -3. Conversely, at -2π, the function is also -3.

Now, shift the graph $\frac{\pi}{2}$ units to the right.

Finally, we need to adjust for the vertical shift by moving it up 3 units.

Example C

Find the equation of the sinusoid graphed here.

Solution: First of all, remember that either sine or cosine could be used to model these graphs. However, it is usually easier to use cosine because the horizontal shift is easier to locate in most cases. Therefore, the model that we will be using is $y = D \pm A\cos(B(x \pm C))$.

First, if we think of the graph as a cosine function, it has a horizontal translation of zero. The maximum point is also the $y-$intercept of the graph, so there is no need to shift the graph horizontally and therefore, $C = 0$. The amplitude is the height from the center of the wave. If you can't find the center of the wave by sight, you can calculate it. The center should be halfway between the highest and the lowest points, which is really the **average** of the maximum and minimum. This value will actually be the vertical shift, or D value.

$$D = center = \frac{60 + -20}{2} = \frac{40}{2} = 20$$

The amplitude is the height from the center line, or vertical shift, to either the minimum or the maximum. So, $A = 60 - 20 = 40$.

The last value to find is the frequency. In order to do so, we must first find the period. The period is the distance required for one complete wave. To find this value, look at the horizontal distance between two consecutive maximum points.

On our graph, from maximum to maximum is 3.

Therefore, the period is 3, so the frequency is $B = \frac{2\pi}{3}$.

We have now calculated each of the four parameters necessary to write the equation. Replacing them in the equation gives:

$$y = 20 + 40\cos\frac{2\pi}{3}x$$

If we had chosen to model this curve with a sine function instead, the amplitude, period and frequency, as well as the vertical shift would all be the same. The only difference would be the horizontal shift. The sine wave starts in the middle of an upward sloped section of the curve as shown by the red circle.

This point intersects with the vertical translation line and is a third of the distance back to -3. So, in this case, the sine wave has been translated 1 unit to the left $y = 20 + 40\sin\left(\frac{2\pi}{3}(x+1)\right)$

Vocabulary

Trigonometric General Equations: The **trigonometric general equations** are equations allowing for trig functions with arbitrary constants added or subtracted both inside the argument of the function as well as outside the function, along with allowing for the trig function and/or its argument to be multiplied by a constant.

Guided Practice

1. Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = 2 + 3\sin(2(x-1))$.

2. Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = -1 + \sin\left(\pi(x + \frac{\pi}{3})\right)$.

3. Identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift of $y = \cos(40(x - 120)) + 5$.

Solutions:

1. This is a sine wave that has been translated 1 unit to the right and 2 units up. The amplitude is 3 and the frequency is 2. The period of the graph is π. The function reaches a maximum point of 5 and a minimum of -1.

2. This is a sine wave that has been translated 1 unit down and $\frac{\pi}{3}$ radians to the left. The amplitude is 1 and the period is 2. The frequency of the graph is π. The function reaches a maximum point of 0 and a minimum of -2.

3. This is a cosine wave that has been translated 5 units up and 120 radians to the right. The amplitude is 1 and the frequency is 40. The period of the graph is $\frac{\pi}{20}$. The function reaches a maximum point of 6 and a minimum of 4.

Concept Problem Solution

With your advanced knowledge of sinusoidal equations, you can identify in the equation:

$f(x) = 3 + 7\sin(4(x + \frac{\pi}{2}))$

The vertical shift of the graph is 3 units up. The amplitude of the graph is 7. The horizontal shift of the graph is $\frac{\pi}{2}$ units to the left. The frequency is 4.

Since the frequency is 4, the period can be calculated:

$$p = \frac{2\pi}{f}$$
$$p = \frac{2\pi}{4}$$
$$p = \frac{\pi}{2}$$

This means that the graph takes $\frac{\pi}{2}$ units to make one complete cycle.

The graph of this equation looks like this:

Practice

For each equation below, identify the period, amplitude, frequency, and any vertical/horizontal translations.

1. $y = 2 - 4\cos(\frac{2}{3}(x - 3))$
2. $y = 3 + \frac{1}{2}\sin(\frac{1}{2}(x - \pi))$
3. $y = 1 + 5\cos(4(x + \frac{\pi}{2}))$
4. $y = 4 - \cos(2(x + 1))$
5. $y = 3 + 2\sin(x - 4)$

Graph each of the following equations from -2π to 2π.

6. $y = 1 - 3\sin(\frac{1}{3}(x - \pi))$
7. $y = 5 + \frac{1}{2}\sin(\frac{1}{2}(x - 2))$
8. $y = 2 + \cos(4(x + \frac{\pi}{2}))$

9. $y = 4 + 2\cos(2(x+3))$
10. $y = 2 - 3\sin(x - \frac{3\pi}{2})$

Find the equation of each sinusoid.

11.
12.
13.
14.
15.

Summary

This chapter covered how to graph trigonometric functions. To do this, it first introduced radian measure and how to apply radian measure to find quantities related to circles, such as the length of a chord, the area of a sector, the length of an arc, and measurements of angular velocity.

The chapter also covered how to graph and represent translations of trigonometric functions, such as stretches and shrinks, vertical translations, and horizontal translations. The functions discussed included the sine, cosine, and tangent functions, as well as the secant, cosecant, and cotangent functions.

Finally, properties of graphs of functions such as amplitude, period, and frequency were explained.

2.18 References

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CHAPTER **3**

Trigonometric Identities

Chapter Outline

Introduction

By now you are familiar with trigonometric functions and how to compute them in straight forward situations. However, many complex combinations of trigonometric functions are possible, including combinations involving multiplication of trig functions by each other, dividing trig functions by each other, adding and subtracting combinations of trig functions, and computing the result of a trig function of half of a given angle, double a given angle, etc.

In this Chapter, you'll learn identities and equations that will make it easier to compute results for these sorts of complex combinations of functions.

3.1 Even and Odd Identities

Here you'll learn what even and odd functions are and how you can use them in solving for values of trig equations.

You and your friend are in math class together. You enjoy talking a lot outside of class about all of the interesting topics you cover in class. Lately you've been covering trig functions and the unit circle. As it turns out, trig functions of certain angles are pretty easy to remember. However, you and your friend are wishing there was an easy way to "shortcut" calculations so that if you knew a trig function for an angle you could relate it to the trig function for another angle; in effect giving you more reward for knowing the first trig function.

You're examining some notes and starting writing down trig functions at random. You eventually write down:

$\cos\left(\frac{\pi}{18}\right)$

Is there any way that if you knew how to compute this, you'd automatically know the answer for a different angle?

As it turns out, there is. Read on, and by the time you've finished this Concept, you'll know what other angle's value of cosine you already know, just by knowing the answer above.

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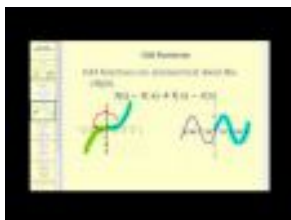
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James Sousa:Even and Odd Trigonometric Identities

Guidance

An even function is a function where the value of the function acting on an argument is the same as the value of the function when acting on the negative of the argument. Or, in short:

$f(x) = f(-x)$

So, for example, if f(x) is some function that is even, then f(2) has the same answer as f(-2). f(5) has the same answer as f(-5), and so on.

In contrast, an odd function is a function where the negative of the function's answer is the same as the function acting on the negative argument. In math terms, this is:

$-f(x) = f(-x)$

If a function were negative, then f(-2) = -f(2), f(-5) = -f(5), and so on.

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the $y-$axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions and therefore the parabola is not symmetric about the $y-$axis. What

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about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

<div align="center">

Even Function	Odd Function
$y = (-x)^2 = x^2$	$y = (-x)^3 = -x^3$

</div>

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30°) = \sin 330° = -\frac{1}{2} = -\sin 30°$$

$$\sin(-135°) = \sin 225° = -\frac{\sqrt{2}}{2} = -\sin 135°$$

From this we see that sine is **odd**. Therefore, $\sin(-x) = -\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30°) = \cos 330° = \frac{\sqrt{3}}{2} = \cos 30°$$

$$\cos(-135°) = \cos 225° = -\frac{\sqrt{2}}{2} = \cos 135°$$

This tells us that the cosine is **even**. Therefore, $\cos(-x) = \cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\sec(-x) = \sec x$$
$$\cot(-x) = -\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.

Example A

If $\cos(-x) = \frac{3}{4}$ and $\tan(-x) = -\frac{\sqrt{7}}{3}$, find $\sin x$.

Solution: We know that sine is odd. Cosine is even, so $\cos x = \frac{3}{4}$. Tangent is odd, so $\tan x = \frac{\sqrt{7}}{3}$. Therefore, sine is positive and $\sin x = \frac{\sqrt{7}}{4}$.

Example B

If $\sin(x) = .25$, find $\sin(-x)$

Solution: Since sine is an odd function, $\sin(-\theta) = -\sin(\theta)$.

Therefore, $\sin(-x) = -\sin(x) = -.25$

Example C

If $\cos(x) = .75$, find $\cos(-x)$

Solution:

Since cosine is an even function, $\cos(x) = \cos(-x)$.

Therefore, $\cos(-x) = .75$

Vocabulary

Even Function: An **even function** is a function with a graph that is symmetric with respect to the 'y' axis and has the property that f(-x) = f(x)

Odd Function: An **odd function** is a function with the property that f(-x) = -f(x)

Guided Practice

1. What two angles have a value for cosine of $\frac{\sqrt{3}}{2}$?

2. If $\cos\theta = \frac{\sqrt{3}}{2}$, find $\sec(-\theta)$

3. If $\cot\theta = -\sqrt{3}$ find $\cot-\theta$

Solutions:

1. On the unit circle, the angles $30°$ and $330°$ both have $\frac{\sqrt{3}}{2}$ as their value for cosine. $330°$ can be rewritten as $-30°$

2. There are 2 ways to think about this problem. Since $\cos\theta = \cos-\theta$, you could say $\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos(\theta)}$ Or you could leave the cosine function the way it is and say that $\sec(-\theta) = \sec(\theta) = \frac{1}{\cos\theta}$. But either way, the answer is $\frac{2}{\sqrt{3}}$

3. Since $\cot(-\theta) = -\cot(\theta)$, if $\cot\theta = -\sqrt{3}$ then $-\cot(-\theta) = -\sqrt{3}$. Therefore, $\cot(-\theta) = \sqrt{3}$.

Concept Problem Solution

Since you now know that cosine is an even function, you get to know the cosine of the negative of an angle automatically if you know the cosine of the positive of the angle.

Therefore, since $\cos\left(\frac{\pi}{18}\right) = .9848$, you automatically know that $\cos\left(-\frac{\pi}{18}\right) = \cos\left(\frac{17\pi}{18}\right) = .9848$.

Practice

Identify whether each function is even or odd.

1. $y = \sin(x)$
2. $y = \cos(x)$
3. $y = \cot(x)$
4. $y = x^4$
5. $y = x$
6. If $\sin(x) = .3$, what is $\sin(-x)$?
7. If $\cos(x) = .5$, what is $\cos(-x)$?
8. If $\tan(x) = .1$, what is $\tan(-x)$?

9. If $\cot(x) = .3$, what is $\cot(-x)$?
10. If $\csc(x) = .3$, what is $\csc(-x)$?
11. If $\sec(x) = 2$, what is $\sec(-x)$?
12. If $\sin(x) = -.2$, what is $\sin(-x)$?
13. If $\cos(x) = -.25$, what is $\sec(-x)$?
14. If $\csc(x) = 4$, what is $\sin(-x)$?
15. If $\tan(x) = -.2$, what is $\cot(-x)$?
16. If $\sin(x) = -.5$ and $\cos(x) = -\frac{\sqrt{3}}{2}$, what is $\cot(-x)$?
17. If $\cos(x) = -.5$ and $\sin(x) = \frac{\sqrt{3}}{2}$, what is $\tan(-x)$?
18. If $\cos(x) = -\frac{\sqrt{2}}{2}$ and $\tan(x) = -1$, what is $\sin(-x)$?

3.2 Proofs of Trigonometric Identities

Here you'll learn four different methods to use in proving trig identities to be true.

What if your instructor gave you two trigonometric expressions and asked you to prove that they were true. Could you do this? For example, can you show that

$\sin^2 \theta = \frac{1-\cos 2\theta}{2}$

Read on, and in this Concept you'll learn four different methods to help you prove identities. You'll be able to apply them to prove the above identity when you are finished.

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> Click image to the left for more content.

Educator.comTrigonometric Identities

Guidance

In Trigonometry you will see complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process for showing two trigonometric expressions to be equivalent (regardless of the value of the angle) is known as validating or proving trigonometric identities.

There are several options a student can use when proving a trigonometric identity.

Option One:

Option Two:

Option Three:

Option Four: $\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ can be *factored* to $\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ and in this situation, the factors cancel each other.

Example A

Prove the identity: $\csc\theta \times \tan\theta = \sec\theta$

Solution: Reducing each side separately. It might be helpful to put a line down, through the equals sign. Because we are proving this identity, we don't know if the two sides are equal, so wait until the end to include the equality.

$$
\begin{array}{c}
\csc x \times \tan x \\
\frac{1}{\sin x} \times \frac{\sin x}{\cos x} \\
\frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x} \\
\frac{1}{\cos x}
\end{array}
\quad\Bigg|\quad
\begin{array}{c}
\sec x \\
\frac{1}{\cos x} \\
\frac{1}{\cos x} \\
\frac{1}{\cos x}
\end{array}
$$

At the end we ended up with the same thing, so we know that this is a valid identity.

Notice when working with identities, unlike equations, conversions and mathematical operations are performed only on one side of the identity. In more complex identities sometimes both sides of the identity are simplified or expanded. The thought process for establishing identities is to view each side of the identity separately, and at the end to show that both sides do in fact transform into identical mathematical statements.

Example B

Prove the identity: $(1 - \cos^2 x)(1 + \cot^2 x) = 1$

Solution: Use the Pythagorean Identity and its alternate form. Manipulate $\sin^2\theta + \cos^2\theta = 1$ to be $\sin^2\theta = 1 - \cos^2\theta$. Also substitute $\csc^2 x$ for $1 + \cot^2 x$, then cross-cancel.

$$
\begin{array}{c}
(1 - \cos^2 x)(1 + \cot^2 x) \\
\sin^2 x \cdot \csc^2 x \\
\sin^2 x \cdot \frac{1}{\sin^2 x} \\
1
\end{array}
\quad\Bigg|\quad
\begin{array}{c}
1 \\
1 \\
1 \\
1
\end{array}
$$

Example C

Prove the identity: $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$.

Solution: Combine the two fractions on the left side of the equation by finding the common denominator: $(1 + \cos\theta) \times \sin\theta$, and the change the right side into terms of sine.

$$
\begin{array}{c}
\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} \\
\frac{\sin\theta}{\sin\theta} \cdot \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
\frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}
\end{array}
\quad\Bigg|\quad
\begin{array}{c}
2\csc\theta \\
2\csc\theta \\
2\csc\theta
\end{array}
$$

Now, we need to apply another algebraic technique, FOIL. (FOIL is a memory device that describes the process for multiplying two binomials, meaning multiplying the First two terms, the Outer two terms, the Inner two terms, and then the Last two terms, and then summing the four products.) Always leave the denominator factored, because you might be able to cancel something out at the end.

$$
\frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} \quad\Bigg|\quad 2\csc\theta
$$

Using the second option, substitute $\sin^2\theta + \cos^2\theta = 1$ and simplify.

$$
\begin{array}{c|c}
\frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\[2mm]
\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\[2mm]
\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\[2mm]
\frac{2}{\sin\theta} & \frac{2}{\sin\theta}
\end{array}
$$

Option Four: $\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ can be *factored* to $\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ and in this situation, the factors cancel each other.

Vocabulary

FOIL: FOIL is a memory device that describes the process for multiplying two binomials, meaning multiplying the First two terms, the Outer two terms, the Inner two terms, and then the Last two terms, and then summing the four products.

Trigonometric Identity: A **trigonometric identity** is an expression which relates one trig function on the left side of an equals sign to another trig function on the right side of the equals sign.

Guided Practice

1. Prove the identity: $\sin x \tan x + \cos x = \sec x$

2. Prove the identity: $\cos x - \cos x \sin^2 x = \cos^3 x$

3. Prove the identity: $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$

Solutions:

1. Step 1: Change everything into sine and cosine

$$
\sin x \tan x + \cos x = \sec x
$$
$$
\sin x \cdot \frac{\sin x}{\cos x} + \cos x = \frac{1}{\cos x}
$$

Step 2: Give everything a common denominator, $\cos x$.

$$
\frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} = \frac{1}{\cos x}
$$

Step 3: Because the denominators are all the same, we can eliminate them.

$$
\sin^2 x + \cos^2 x = 1
$$

We know this is true because it is the Trig Pythagorean Theorem

2. Step 1: Pull out a $\cos x$

$$
\cos x - \cos x \sin^2 x = \cos^3 x
$$
$$
\cos x(1 - \sin^2 x) = \cos^3 x
$$

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Step 2: We know $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$ is also true, therefore $\cos x(\cos^2 x) = \cos^3 x$. This, of course, is true, we are finished!

3. Step 1: Change everything in to sine and cosine and find a common denominator for left hand side.

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2\csc x$$

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{2}{\sin x} \leftarrow \text{LCD}: \ \sin x(1 + \cos x)$$

$$\frac{\sin^2 x + (1 + \cos x)^2}{\sin x(1 + \cos x)}$$

Step 2: Working with the left side, FOIL and simplify.

$$\frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x(1 + \cos x)} \qquad\qquad \rightarrow \text{FOIL } (1 + \cos x)^2$$

$$\frac{\sin^2 x + \cos^2 x + 1 + 2\cos x}{\sin x(1 + \cos x)} \qquad\qquad \rightarrow \text{move } \cos^2 x$$

$$\frac{1 + 1 + 2\cos x}{\sin x(1 + \cos x)} \qquad\qquad \rightarrow \sin^2 x + \cos^2 x = 1$$

$$\frac{2 + 2\cos x}{\sin x(1 + \cos x)} \qquad\qquad \rightarrow \text{add}$$

$$\frac{2(1 + \cos x)}{\sin x(1 + \cos x)} \qquad\qquad \rightarrow \text{fator out 2}$$

$$\frac{2}{\sin x} \qquad\qquad \rightarrow \text{cancel } (1 + \cos x)$$

}}

Concept Problem Solution

The original question was to prove that:

$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

First remember the Pythagorean Identity:

$\sin^2 \theta + \cos^2 \theta = 1$

Therefore,

$\sin^2 \theta = 1 - \cos^2 \theta$

From the Double Angle Identities, we know that

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$\cos^2 \theta = \cos 2\theta + \sin^2 \theta$$

Substituting this into the above equation for $\sin^2$,

$$\sin^2\theta = 1 - (\cos 2\theta + \sin^2\theta)$$
$$\sin^2\theta = 1 - \cos 2\theta - \sin^2\theta$$
$$2\sin^2\theta = 1 - \cos 2\theta$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Practice

Use trigonometric identities to simplify each expression as much as possible.

1. $\tan(x)\cos(x)$
2. $\cos(x) - \cos^3(x)$
3. $\frac{1-\cos^2(x)}{\sin(x)}$
4. $\cot(x)\sin(x)$
5. $\frac{1-\sin^2(x)}{\cos(x)}$
6. $\sin(x)\csc(x)$
7. $\tan(-x)\cot(x)$
8. $\frac{\sec^2(x)-\tan^2(x)}{\cos^2(x)+\sin^2(x)}$

Prove each identity.

9. $\tan(x) + \cot(x) = \sec(x)\csc(x)$
10. $\sin(x) = \frac{\sin^2(x)+\cos^2(x)}{\csc(x)}$
11. $\frac{1}{\sec(x)-1} + \frac{1}{\sec(x)+1} = 2\cot(x)\csc(x)$
12. $(\cos(x))(\tan(x) + \sin(x)\cot(x)) = \sin(x) + \cos^2(x)$
13. $\sin^4(x) - \cos^4(x) = \sin^2(x) - \cos^2(x)$
14. $\sin^2(x)\cos^3(x) = (\sin^2(x) - \sin^4(x))(\cos(x))$
15. $\frac{\sin(x)}{\csc(x)} = 1 - \frac{\cos(x)}{\sec(x)}$

3.3 Simpler Form of Trigonometric Equations

Here you'll learn how to rewrite trig expressions in a simpler form using trig identities.

Sometimes things are simpler than they look. For example, trigonometric identities can sometimes be reduced to simpler forms by applying other rules. For example, can you find a way to simplify

$\cos^3 \theta = \frac{3\cos\theta + \cos 3\theta}{4}$

Keep reading, and during this Concept you'll learn ways to break down complex trigonometric equations into simpler forms. You'll be able to apply this information to the equation above.

Watch This

In the first part of this video, you'll learn how to use trigonometric substitution to simplify equations.
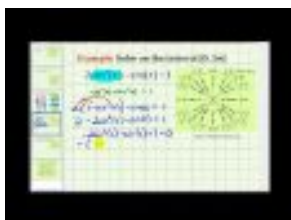
| MEDIA |
| Click image to the left for more content. |

James Sousa Example:Solvinga TrigonometricEquation Using a Trig Substitution and Factoring

Guidance

By this time in your school career you have probably seen trigonometric functions represented in many ways: ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with graphs. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

In order to do this, look for parts of the complex trigonometric expression that might be reduced to fewer trig functions if one of the identities you already know is applied to the expression. As you apply identities, some complex trig expressions have parts that can be cancelled out, others can be reduced to fewer trig functions. Observe how this is accomplished in the examples below.

Example A

Simplify the following expression using the basic trigonometric identities: $\frac{1+\tan^2 x}{\csc^2 x}$

Solution:

$$\frac{1+\tan^2 x}{\csc^2 x} \dots (1+\tan^2 x = \sec^2 x)\text{Pythagorean Identity}$$

$$\frac{\sec^2 x}{\csc^2 x} \dots (\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x})\text{Reciprocal Identity}$$

$$\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right)$$

$$\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{\sin^2 x}{1}\right) = \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x \rightarrow \text{Quotient Identity}$$

Example B

Simplify the following expression using the basic trigonometric identities: $\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$

Solution:

$$\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x} \dots (\sin^2 x + \cos^2 x = 1)\text{Pythagorean Identity}$$

$$\frac{1+\tan^2 x}{\sec x} \dots (1+\tan^2 x = \sec^2 x)\text{Pythagorean Identity}$$

$$\frac{\sec^2 x}{\sec x} = \sec x$$

Example C

Simplify the following expression using the basic trigonometric identities: $\cos x - \cos^3 x$

Solution:

$$\cos x - \cos^3 x$$

$$\cos x(1 - \cos^2 x) \qquad \dots \text{Factor out } \cos x \text{ and } \sin^2 x = 1 - \cos^2 x$$

$$\cos x(\sin^2 x)$$

Vocabulary

Trigonometric Identity: A **trigonometric identity** is an expression which relates one trig function on the left side of an equals sign to another trig function on the right side of the equals sign.

Guided Practice

1. Simplify $\tan^3(x)\csc^3(x)$

2. Show that $\cot^2(x) + 1 = \csc^2(x)$

3. Simplify $\frac{\csc^2(x)-1}{\csc^2(x)}$

Solutions:

1.

$$\tan^3(x)\csc^3(x)$$
$$= \frac{\sin^3(x)}{\cos^3(x)} \times \frac{1}{\sin^3(x)}$$
$$= \frac{1}{\cos^3(x)}$$
$$= \sec^3(x)$$

2. Start with $\sin^2(x) + \cos^2(x) = 1$, and divide everything through by $\sin^2(x)$:

$$\sin^2(x) + \cos^2(x) = 1$$
$$= \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} = \frac{1}{\sin^2(x)}$$
$$= 1 + \cot^2(x) = \csc^2(x)$$

3.

$$\frac{\csc^2(x) - 1}{\csc^2(x)}$$

Using $\cot^2(x) + 1 = \csc^2(x)$ that was proven in #2, you can find the relationship: $\cot^2(x) = \csc^2(x) - 1$, you can substitute into the above expression to get:

$$\frac{\cot^2(x)}{\csc^2(x)}$$
$$= \frac{\frac{\cos^2(x)}{\sin^2(x)}}{\frac{1}{\sin^2(x)}}$$
$$= \cos^2(x)$$

Concept Problem Solution

The original problem is to simplify

$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$

The easiest way to start is to recognize the triple angle identity:

$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$

Substituting this into the original equation gives:

$\cos^3\theta = \frac{3\cos\theta + (\cos^3\theta - 3\sin^2\theta\cos\theta)}{4}$

Notice that you can then multiply by four and subtract a $\cos^3\theta$ term:

$3\cos^3\theta = 3\cos\theta - 3\sin^2\theta\cos\theta$

And finally pulling out a three and dividing:

$\cos^3\theta = \cos\theta - \sin^2\theta\cos\theta$

Then pulling out a $\cos\theta$ and dividing:

$\cos^2\theta = 1 - \sin^2\theta$

Practice

Simplify each trigonometric expression as much as possible.

1. $\sin(x)\cot(x)$
2. $\cos(x)\tan(x)$
3. $\frac{1+\tan(x)}{1+\cot(x)}$
4. $\frac{1-\sin^2(x)}{1+\sin(x)}$
5. $\frac{\sin^2(x)}{1+\cos(x)}$
6. $(1+\tan^2(x))(\sec^2(x))$
7. $\sin(x)(\tan(x)+\cot(x))$
8. $\frac{\sec(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}$
9. $\frac{\sin(x)}{\cot^2(x)} - \frac{\sin(x)}{\cos^2(x)}$
10. $\frac{1+\sin(x)}{\cos(x)} - \sec(x)$
11. $\frac{\sin^2(x)-\sin^4(x)}{\cos^2(x)}$
12. $\frac{\tan(x)}{\csc^2(x)} + \frac{\tan(x)}{\sec^2(x)}$
13. $\sqrt{1-\cos^2(x)}$
14. $(1-\sin^2(x))(\cos(x))$
15. $(\sec^2(x)+\csc^2(x)) - (\tan^2(x)+\cot^2(x))$

Here you'll learn how to factor trig equations and then solve them using the factored form.

Solving trig equations is an important process in mathematics. Quite often you'll see powers of trigonometric functions and be asked to solve for the values of the variable which make the equation true. For example, suppose you were given the trig equation

$2\sin x \cos x = \cos x$

Could you solve this equation? (You might be tempted to just divide both sides by $\cos x$, but that would be incorrect because you would lose some solutions.) Instead, you're going to have to use factoring. Read this Concept, and at its conclusion, you'll be ready to factor the above equation and solve it.

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James Sousa Example:Solve aTrig Equation by Factoring

Guidance

You have no doubt had experience with factoring. You have probably factored equations when looking for the possible values of some variable, such as "x". It might interest you to find out that you can use the same factoring method for more than just a variable that is a number. You can factor trigonometric equations to find the possible values the function can take to satisfy an equation.

Algebraic skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers.

Example A

Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 < x \leq 2\pi$.

Solution:

$$2\sin^2 x - 3\sin x + 1 = 0 \quad \text{Factor this like a quadratic equation}$$
$$(2\sin x - 1)(\sin x - 1) = 0$$
$$\downarrow \qquad\qquad \searrow$$
$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$2\sin x = 1 \qquad\qquad \sin x = 1$$
$$\sin x = \frac{1}{2} \qquad\qquad x = \frac{\pi}{2}$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}$$

Example B

Solve $2\tan x \sin x + 2\sin x = \tan x + 1$ for all values of x.

Solution:

Pull out $\sin x$

There is a common factor of $(\tan x + 1)$

Think of the $-(\tan x + 1)$ as $(-1)(\tan x + 1)$, which is why there is a -1 behind the $2\sin x$.

Example C

Solve $2\sin^2 x + 3\sin x - 2 = 0$ for all $x, [0, \pi]$.

Solution:

$$2\sin^2 x + 3\sin x - 2 = 0 \rightarrow \text{Factor like a quadratic}$$
$$(2\sin x - 1)(\sin x + 2) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$2\sin x - 1 = 0 \qquad \sin x + 2 = 0$$
$$\sin x = \frac{1}{2} \qquad\qquad \sin x = -2$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} \quad \text{There is no solution because the range of } \sin x \text{ is } [-1, 1].$$

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Vocabulary

Factoring: Factoring is a way to solve trigonometric equations by separating the equation into two terms which, when multiplied together, give the original expression. Since the product of the two factors is equal to zero, each of the factors can be equal to zero to make the original expression true. This leads to solutions for the original expression.

Guided Practice

1. Solve the trigonometric equation $4\sin x\cos x + 2\cos x - 2\sin x - 1 = 0$ such that $0 \le x < 2\pi$.

2. Solve $\tan^2 x = 3\tan x$ for x over $[0, \pi]$.

3. Find all the solutions for the trigonometric equation $2\sin^2\frac{x}{4} - 3\cos\frac{x}{4} = 0$ over the interval $[0, 2\pi)$.

Solutions:

1. Use factoring by grouping.

$$\begin{array}{ll}
2\sin x + 1 = 0 \quad \text{or} & 2\cos x - 1 = 0 \\
2\sin x = -1 & 2\cos x = 1 \\
\sin x = -\dfrac{1}{2} & \cos x = \dfrac{1}{2} \\
x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6} & x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}
\end{array}$$

2.

$$\tan^2 x = 3\tan x$$
$$\tan^2 x - 3\tan x = 0$$
$$\tan x(\tan x - 3) = 0$$

$$\begin{array}{lll}
\tan x = 0 & \text{or} & \tan x = 3 \\
x = 0, \pi & & x = 1.25
\end{array}$$

3.

$$2\sin^2\tfrac{x}{4} - 3\cos\tfrac{x}{4} = 0$$

$$2\left(1 - \cos^2\frac{x}{4}\right) - 3\cos\frac{x}{4} = 0$$
$$2 - 2\cos^2\frac{x}{4} - 3\cos\frac{x}{4} = 0$$
$$2\cos^2\frac{x}{4} + 3\cos\frac{x}{4} - 2 = 0$$
$$\left(2\cos\frac{x}{4} - 1\right)\left(\cos\frac{x}{4} + 2\right) = 0$$

$$\begin{array}{ll}
2\cos\dfrac{x}{4} - 1 = 0 \quad \text{or} & \cos\dfrac{x}{4} + 2 = 0 \\
2\cos\dfrac{x}{4} = 1 & \cos\dfrac{x}{4} = -2 \\
\cos\dfrac{x}{4} = \dfrac{1}{2} & \\
\dfrac{x}{4} = \dfrac{\pi}{3} \quad \text{or} \quad \dfrac{5\pi}{3} & \\
x = \dfrac{4\pi}{3} \quad \text{or} \quad \dfrac{20\pi}{3} &
\end{array}$$

$\frac{20\pi}{3}$ is eliminated as a solution because it is outside of the range and $\cos\frac{x}{4} = -2$ will not generate any solutions because -2 is outside of the range of cosine. Therefore, the only solution is $\frac{4\pi}{3}$.

Concept Problem Solution

The equation you were given is

$2\sin x \cos x = \cos x$

To solve this:

$2\sin x \cos x = \cos x$

Subtract $\cos x$ from both sides and factor it out of the equation:

$$2\sin x \cos x - \cos x = 0$$
$$\cos x(2\sin x - 1) = 0$$

Now set each factor equal to zero and solve. The first is $\cos x$:

$$\cos x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

And now for the other term:

$$2\sin x - 1 = 0$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Practice

Solve each equation for x over the interval $[0, 2\pi)$.

1. $\cos^2(x) + 2\cos(x) + 1 = 0$
2. $1 - 2\sin(x) + \sin^2(x) = 0$
3. $2\cos(x)\sin(x) - \cos(x) = 0$
4. $\sin(x)\tan^2(x) - \sin(x) = 0$
5. $\sec^2(x) = 4$
6. $\sin^2(x) - 2\sin(x) = 0$
7. $3\sin(x) = 2\cos^2(x)$
8. $2\sin^2(x) + 3\sin(x) = 2$
9. $\tan(x)\sin^2(x) = \tan(x)$
10. $2\sin^2(x) + \sin(x) = 1$
11. $2\cos(x)\tan(x) - \tan(x) = 0$
12. $\sin^2(x) + \sin(x) = 2$
13. $\tan(x)(2\cos^2(x) + 3\cos(x) - 2) = 0$

14. $\sin^2(x) + 1 = 2\sin(x)$
15. $2\cos^2(x) - 3\cos(x) = 2$

3.5 Trigonometric Equations Using the Quadratic Formula

Here you'll learn how to use the quadratic equation to find solutions of trig functions.

Solving equations is a fundamental part of mathematics. Being able to find which values of a variable fit an equation allows us to determine all sorts of interesting behavior, both in math and in the sciences. Solving trig equations for angles that satisfy the equation is one application of mathematical methods for solving equations. Suppose someone gave you the following equation:

$$3\sin^2\theta + 8\sin\theta - 3 = 0$$

Can you solve it? You might think it looks familiar.. almost like a quadratic equation, except the "x" has been replaced with a trig function? As it turns out, you're right on track. Read this Concept, and by the end, you'll be able to use the quadratic equation to solve for values of theta that satisfy the equation shown above.

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Solving Trigonometric Equations Using the Quadratic Formula

Guidance

When solving quadratic equations that do not factor, the quadratic formula is often used.

Remember that the quadratic equation is:

$ax^2 + bx + c = 0$ (where a, b, and c are constants)

In this situation, you can use the quadratic formula to find out what values of "x" satisfy the equation.

The same method can be applied when solving trigonometric equations that do not factor. The values for a is the numerical coefficient of the function's squared term, b is the numerical coefficient of the function term that is to the first power and c is a constant. The formula will result in two answers and both will have to be evaluated within the designated interval.

Example A

Solve $3\cot^2 x - 3\cot x = 1$ for exact values of x over the interval $[0, 2\pi]$.

Solution:

$$3\cot^2 x - 3\cot x = 1$$
$$3\cot^2 x - 3\cot x - 1 = 0$$

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The equation will not factor. Use the quadratic formula for $\cot x$, $a = 3, b = -3, c = -1$.

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$

$$\cot x = \frac{3 \pm \sqrt{9 + 12}}{6}$$

$$\cot x = \frac{3 + \sqrt{21}}{6} \qquad \text{or} \qquad \cot x = \frac{3 - \sqrt{21}}{6}$$

$$\cot x = \frac{3 + 4.5826}{6} \qquad\qquad\qquad \cot x = \frac{3 - 4.5826}{6}$$

$$\cot x = 1.2638 \qquad\qquad\qquad\qquad \cot x = -0.2638$$

$$\tan x = \frac{1}{1.2638} \qquad\qquad\qquad\qquad \tan x = \frac{1}{-0.2638}$$

$$x = 0.6694, 3.81099 \qquad\qquad\qquad x = 1.8287, 4.9703$$

Example B

Solve $-5\cos^2 x + 9\sin x + 3 = 0$ for values of x over the interval $[0, 2\pi]$.

Solution: Change $\cos^2 x$ to $1 - \sin^2 x$ from the Pythagorean Identity.

$$-5\cos^2 x + 9\sin x + 3 = 0$$
$$-5(1 - \sin^2 x) + 9\sin x + 3 = 0$$
$$-5 + 5\sin^2 x + 9\sin x + 3 = 0$$
$$5\sin^2 x + 9\sin x - 2 = 0$$

$$\sin x = \frac{-9 \pm \sqrt{9^2 - 4(5)(-2)}}{2(5)}$$

$$\sin x = \frac{-9 \pm \sqrt{81 + 40}}{10}$$

$$\sin x = \frac{-9 \pm \sqrt{121}}{10}$$

$$\sin x = \frac{-9 + 11}{10} \quad \text{and} \quad \sin x = \frac{-9 - 11}{10}$$

$$\sin x = \frac{1}{5} \quad \text{and} \quad -2$$

$$\sin^{-1}(0.2) \quad \text{and} \quad \sin^{-1}(-2)$$

$x \approx .201 \; rad$ and $\pi - .201 \approx 2.941$

This is the only solutions for x since -2 is not in the range of values.

Example C

Solve $3\sin^2 x - 6\sin x - 2 = 0$ for values of x over the interval $[0, 2\pi]$.

Solution:

$$3\sin^2 x - 6\sin x - 2 = 0$$

$$\sin x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)}$$

$$\sin x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$\sin x = \frac{6 \pm \sqrt{12}}{6}$$

$$\sin x = \frac{6 + 3.46}{10} \text{ and } \sin x = \frac{6 - 3.46}{10}$$

$$\sin x = .946 \text{ and } .254$$

$$\sin^{-1}(0.946) \text{ and } \sin^{-1}(0.254)$$

$x \approx 71.08 \ deg$ and $\approx 14.71 \ deg$

Vocabulary

Quadratic Equation: A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real constants.

Guided Practice

1. Solve $\sin^2 x - 2\sin x - 3 = 0$ for x over $[0, \pi]$.

2. Solve $\tan^2 x + \tan x - 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3. Solve the trigonometric equation such that $5\cos^2 \theta - 6\sin \theta = 0$ over the interval $[0, 2\pi]$.

Solutions:

1. You can factor this one like a quadratic.

$$\sin^2 x - 2\sin x - 3 = 0$$
$$(\sin x - 3)(\sin x + 1) = 0$$

$$\sin x - 3 = 0 \qquad\qquad\qquad\qquad \sin x + 1 = 0$$
$$\sin x = 3 \qquad \text{or} \qquad \sin x = -1$$
$$x = \sin^{-1}(3) \qquad\qquad\qquad\qquad x = \frac{3\pi}{2}$$

For this problem the only solution is $\frac{3\pi}{2}$ because sine cannot be 3 (it is not in the range).

2. $\tan^2 x + \tan x - 2 = 0$

$$\frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2} = \tan x$$
$$\frac{-1 \pm \sqrt{1+8}}{2} = \tan x$$
$$\frac{-1 \pm 3}{2} = \tan x$$
$$\tan x = -2 \quad \text{or} \quad 1$$

$\tan x = 1$ when $x = \frac{\pi}{4}$, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\tan x = -2$ when $x = -1.107 \ rad$

3. $5 \cos^2 \theta - 6 \sin \theta = 0$ over the interval $[0, 2\pi]$.

$$5 \left(1 - \sin^2 x\right) - 6 \sin x = 0$$
$$-5 \sin^2 x - 6 \sin x + 5 = 0$$
$$5 \sin^2 x + 6 \sin x - 5 = 0$$
$$\frac{-6 \pm \sqrt{6^2 - 4(5)(-5)}}{2(5)} = \sin x$$
$$\frac{-6 \pm \sqrt{36 + 100}}{10} = \sin x$$
$$\frac{-6 \pm \sqrt{136}}{10} = \sin x$$
$$\frac{-6 \pm 2\sqrt{34}}{10} = \sin x$$
$$\frac{-3 \pm \sqrt{34}}{5} = \sin x$$

$x = \sin^{-1}\left(\frac{-3+\sqrt{34}}{5}\right)$ or $\sin^{-1}\left(\frac{-3-\sqrt{34}}{5}\right) x = 0.6018 \ rad$ or $2.5398 \ rad$ from the first expression, the second expression will not yield any answers because it is out the the range of sine.

Concept Problem Solution

The original equation to solve was:

$3 \sin^2 \theta + 8 \sin \theta - 3 = 0$

Using the quadratic formula, with $a = 3, b = 8, c = -3$, we get:

$\sin \theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - (4)(3)(-3)}}{6} = \frac{-8 \pm \sqrt{100}}{6} = \frac{-8 \pm 10}{6} = \frac{1}{3} or - 3$

The solution of -3 is ignored because sine can't take that value, however:

$\sin^{-1} \frac{1}{3} = 19.471°$

Practice

Solve each equation using the quadratic formula.

1. $3x^2 + 10x + 2 = 0$
2. $5x^2 + 10x + 2 = 0$
3. $2x^2 + 6x - 5 = 0$

Use the quadratic formula to solve each quadratic equation over the interval $[0, 2\pi)$.

4. $3\cos^2(x) + 10\cos(x) + 2 = 0$
5. $5\sin^2(x) + 10\sin(x) + 2 = 0$
6. $2\sin^2(x) + 6\sin(x) - 5 = 0$
7. $6\cos^2(x) - 5\cos(x) - 21 = 0$
8. $9\tan^2(x) - 42\tan(x) + 49 = 0$
9. $\sin^2(x) + 3\sin(x) = 5$
10. $3\cos^2(x) - 4\sin(x) = 0$
11. $-2\cos^2(x) + 4\sin(x) = 0$
12. $\tan^2(x) + \tan(x) = 3$
13. $\cot^2(x) + 5\tan(x) + 14 = 0$
14. $\sin^2(x) + \sin(x) = 1$
15. What type of sine or cosine equations have no solution?

3.6 Cosine Sum and Difference Formulas

Here you'll learn to rewrite cosine functions with addition or subtraction in their arguments in a more easily solvable form.

While playing a board game with friends, you are using a spinner like this one:

When you tap the spinner with your hand, it rotates $110°$. However, at that moment, someone taps the game board and the spinner moves back a little to $80°$. One of your friends, who is a grade above you in math, starts talking to you about trig functions.

"Do you think you can calculate the cosine of the difference between those angles?" he asks.

"Hmm," you reply. "Sure. I think it's just $\cos(110° - 80°) = \cos 30°$."

Your friend smiles. "Are you sure?" he asks.

You realize you aren't sure at all. Can you solve this problem? Read this Concept, and by the end you'll be able to calculate the cosine of the difference of the angles.

Watch This

James Sousa:SumandDifference Identities for Cosine

Guidance

When thinking about how to calculate values for trig functions, it is natural to consider what the value is for the trig function of a difference of two angles. For example, is $\cos 15° = \cos(45° - 30°)$? Upon appearance, yes, it is. This section explores how to find an expression that would equal $\cos(45° - 30°)$. To simplify this, let the two given angles be a and b where $0 < b < a < 2\pi$.

Begin with the unit circle and place the angles a and b in standard position as shown in Figure A. Point Pt1 lies on the terminal side of b, so its coordinates are $(\cos b, \sin b)$ and Point Pt2 lies on the terminal side of a so its coordinates are $(\cos a, \sin a)$. Place the $a - b$ in standard position, as shown in Figure B. The point A has coordinates $(1, 0)$ and the Pt3 is on the terminal side of the angle $a - b$, so its coordinates are $(\cos[a - b], \sin[a - b])$.

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, $a - b$, are equal). Therefore the unknown side of each triangle must also be equal. That is: $d(A, P_3) = d(P_1, P_2)$

Applying the distance formula to the triangles in Figures A and B and setting them equal to each other:

$$\sqrt{[\cos(a - b) - 1]^2 + [\sin(a - b) - 0]^2} = \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2}$$

Square both sides to eliminate the square root.

$$[\cos(a-b)-1]^2 + [\sin(a-b)-0]^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$$

FOIL all four squared expressions and simplify.

$$\cos^2(a-b) - 2\cos(a-b) + 1 + \sin^2(a-b) = \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a - 2\sin a \sin b + \sin^2 b$$

$$\underbrace{\sin^2(a-b) + \cos^2(a-b)} - 2\cos(a-b) + 1 = \underbrace{\sin^2 a + \cos^2 a} - 2\cos a \cos b + \underbrace{\sin^2 b + \cos^2 b} - 2\sin a \sin b$$

$$1 - 2\cos(a-b) + 1 = 1 - 2\cos a \cos b + 1 - 2\sin a \sin b$$

$$2 - 2\cos(a-b) = 2 - 2\cos a \cos b - 2\sin a \sin b$$

$$-2\cos(a-b) = -2\cos a \cos b - 2\sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

In $\cos(a-b) = \cos a \cos b + \sin a \sin b$, the *difference* formula for cosine, you can substitute $a - (-b) = a + b$ to obtain: $\cos(a+b) = \cos[a-(-b)]$ or $\cos a \cos(-b) + \sin a \sin(-b)$. since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$, then $\cos(a+b) = \cos a \cos b - \sin a \sin b$, which is the *sum* formula for cosine.

The sum/difference formulas for cosine can be used to establish other identities:

Example A

Find an equivalent form of $\cos\left(\frac{\pi}{2} - \theta\right)$ using the cosine difference formula.

Solution:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = 0 \times \cos\theta + 1 \times \sin\theta, \text{ substitute } \cos\frac{\pi}{2} = 0 \text{ and } \sin\frac{\pi}{2} = 1$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

We know that is a true identity because of our understanding of the sine and cosine curves, which are a phase shift of $\frac{\pi}{2}$ off from each other.

The cosine formulas can also be used to find exact values of cosine that we weren't able to find before, such as $15° = (45° - 30°), 75° = (45° + 30°)$, among others.

Example B

Find the exact value of $\cos 15°$

Solution: Use the difference formula where $a = 45°$ and $b = 30°$.

$$\cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$\cos 15° = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$

$$\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example C

Find the exact value of $\cos \frac{5\pi}{12}$, in radians.

Solution: $\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$, notice that $\frac{\pi}{4} = \frac{3\pi}{12}$ and $\frac{\pi}{6} = \frac{2\pi}{12}$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Vocabulary

Cosine Sum Formula: The **cosine sum formula** relates the cosine of a sum of two arguments to a set of sine and cosines functions, each containing one argument.

Cosine Difference Formula: The **cosine difference formula** relates the cosine of a difference of two arguments to a set of sine and cosines functions, each containing one argument.

Guided Practice

1. Find the exact value for $\cos \frac{5\pi}{12}$

2. Find the exact value for $\cos \frac{7\pi}{12}$

3. Find the exact value for $\cos 345°$

Solutions:

1.

$$\cos\frac{5\pi}{12} = \cos\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

2.

$$\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

3.

$$\cos 345° = \cos(315° + 30°) = \cos 315° \cos 30° - \sin 315° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Concept Problem Solution

Prior to this Concept, it would seem that your friend was having some fun with you, since he figured you didn't know the cosine difference formula. But now, with this formula in hand, you can readily solve for the difference of the two angles:

$$\cos(110° - 80°)$$
$$= (\cos 110°)(\cos 80°) + (\sin 110°)(\sin 80°)$$
$$= (-.342)(.174) + (.9397)(.9848)$$
$$= -.0595 + .9254 = .8659$$

Therefore,

$\cos(110° - 80°) = .8659$

Practice

Find the exact value for each cosine expression.

1. $\cos 75°$
2. $\cos 105°$
3. $\cos 165°$
4. $\cos 255°$
5. $\cos -15°$

Write each expression as the cosine of an angle.

6. $\cos 96° \cos 20° + \sin 96° \sin 20°$
7. $\cos 4x \cos 3x - \sin 4x \sin 3x$
8. $\cos 37° \cos 12° + \sin 37° \sin 12°$
9. $\cos 59° \cos 10° - \sin 59° \sin 10°$
10. $\cos 5y \cos 2y + \sin 5y \sin 2y$
11. Prove that $\cos(x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\cos(x) + \sin(x))$
12. If $\cos(x)\cos(y) = \sin(x)\sin(y)$, then what does $\cos(x+y)$ equal?
13. Prove that $\cos(x - \frac{\pi}{2}) = \sin(x)$
14. Use the fact that $\cos(\frac{\pi}{2} - x) = \sin(x)$ (shown in examples), to show that $\sin(\frac{\pi}{2} - x) = \cos(x)$.
15. Prove that $\cos(x - y) + \cos(x + y) = 2\cos(x)\cos(y)$.

3.7 Sine Sum and Difference Formulas

Here you'll learn to rewrite sine functions with addition or subtraction in their arguments in a more easily solvable form.

You've gotten quite good at knowing the values of trig functions. So much so that you and your friends play a game before class everyday to see who can get the most trig functions of different angles correct. However, your friend Jane keeps getting the trig functions of more angles right. You're amazed by her memory, until she smiles one day and tells you that she's been fooling you all this time.

"What you do you mean?" you say.

"I have a trick that lets me calculate more functions in my mind by breaking them down into sums of angles." she replies.

You're really surprised by this. And all this time you thought she just had an amazing memory!

"Here, let me show you," she says. She takes a piece of paper out and writes down:

$\sin \frac{7\pi}{12}$

"This looks like an unusual value to remember for a trig function. So I have a special rule that helps me to evaluate it by breaking it into a sum of different numbers."

By the end of this Concept, you'll be able to calculate the above function using a special rule, just like Jane does.

Watch This

MEDIA
Click image to the left for more content.

James Sousa:SumandDifference Identities for Sine

Guidance

Our goal here is to figure out a formula that lets you break down a the sine of a sum of two angles (or a difference of two angles) into a simpler formula that lets you use the sine of only one argument in each term.

To find $\sin(a+b)$:

$$
\begin{aligned}
\sin(a+b) &= \cos\left[\frac{\pi}{2} - (a+b)\right] && \text{Set } \theta = a+b \\
&= \cos\left[\left(\frac{\pi}{2} - a\right) - b\right] && \text{Distribute the negative} \\
&= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b && \text{Difference Formula for cosines} \\
&= \sin a \cos b + \cos a \sin b && \text{Co-function Identities}
\end{aligned}
$$

In conclusion, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a-b)$:

$$
\begin{aligned}
\sin(a-b) &= \sin[a+(-b)] \\
&= \sin a \cos(-b) + \cos a \sin(-b) &&\text{Use the sine sum formula} \\
\sin(a-b) &= \sin a \cos b - \cos a \sin b &&\text{Use } \cos(-b) = \cos b, \text{ and } \sin(-b) = -\sin b
\end{aligned}
$$

In conclusion, $\sin(a-b) = \sin a \cos b - \cos a \sin b$, so, this is the *difference* formula for sine.

Example A

Find the exact value of $\sin \frac{5\pi}{12}$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$
\begin{aligned}
\sin \frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) \\
&= \sin \frac{3\pi}{12} \cos \frac{2\pi}{12} + \cos \frac{3\pi}{12} \sin \frac{2\pi}{12} \\
\sin \frac{5\pi}{12} &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}
$$

Example B

Given $\sin \alpha = \frac{12}{13}$, where α is in Quadrant II, and $\sin \beta = \frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha + \beta)$.

Solution: To find the exact value of $\sin(\alpha + \beta)$, here we use $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$. The values of $\sin \alpha$ and $\sin \beta$ are known, however the values of $\cos \alpha$ and $\cos \beta$ need to be found.

Use $\sin^2 \alpha + \cos^2 \alpha = 1$, to find the values of each of the missing cosine values.

For $\cos a : \sin^2 \alpha + \cos^2 \alpha = 1$, substituting $\sin \alpha = \frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2 + \cos^2 \alpha = \frac{144}{169} + \cos^2 \alpha = 1$ or $\cos^2 \alpha = \frac{25}{169} \cos \alpha = \pm \frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos \alpha = -\frac{5}{13}$.

For $\cos \beta$ use $\sin^2 \beta + \cos^2 \beta = 1$ and substitute $\sin \beta = \frac{3}{5}, \left(\frac{3}{5}\right)^2 + \cos^2 \beta = \frac{9}{25} + \cos^2 \beta = 1$ or $\cos^2 \beta = \frac{16}{25}$ and $\cos \beta = \pm \frac{4}{5}$ and since β is in Quadrant I, $\cos \beta = \frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$
\begin{aligned}
\sin(\alpha + \beta) &= \frac{12}{13} \times \frac{4}{5} + \left(-\frac{5}{13}\right) \times \frac{3}{5} \text{ or } \frac{48}{65} - \frac{15}{65} \\
\sin(\alpha + \beta) &= \frac{33}{65}
\end{aligned}
$$

Example C

Find the exact value of $\sin 15°$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$
\begin{aligned}
\sin 15° &= \sin (45° - 30°) \\
&= \sin 45° \cos 30° + \cos 45° \sin 30° \\
\sin 15° &= (.707) \times (.866) + (.707) \times (.5) \\
&= (.612262) \times (.3535) \\
&= .2164
\end{aligned}
$$

Vocabulary

Sine Sum Formula: The **sine sum formula** relates the sine of a sum of two arguments to a set of sine and cosines functions, each containing one argument.

Sine Difference Formula: The **sine difference formula** relates the sine of a difference of two arguments to a set of sine and cosines functions, each containing one argument.

Guided Practice

1. Find the exact value for $\sin 345°$

2. Find the exact value for $\sin \frac{17\pi}{12}$

3. If $\sin y = -\frac{5}{13}$, y is in quad III, and $\sin z = \frac{4}{5}$, z is in quad II find $\sin(y+z)$

Solutions:

1.

$$
\sin 345° = \sin(300° + 45°) = \sin 300° \cos 45° + \cos 300° \sin 45°
$$
$$
= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}
$$

2.

$$
\sin \frac{17\pi}{12} = \sin \left(\frac{9\pi}{12} + \frac{8\pi}{12} \right) = \sin \left(\frac{3\pi}{4} + \frac{2\pi}{3} \right) = \sin \frac{3\pi}{4} \cos \frac{2\pi}{3} + \cos \frac{3\pi}{4} \sin \frac{2\pi}{3}
$$
$$
= \frac{\sqrt{2}}{2} \cdot (-\frac{1}{2}) + -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{-\sqrt{2} - \sqrt{6}}{4}
$$

3.

If $\sin y = -\frac{5}{13}$ and in Quadrant III, then cosine is also negative. By the Pythagorean Theorem, the second leg is $12(5^2 + b^2 = 13^2)$, so $\cos y = -\frac{12}{13}$. If the $\sin z = \frac{4}{5}$ and in Quadrant II, then the cosine is also negative. By the Pythagorean Theorem, the second leg is $3(4^2 + b^2 = 5^2)$, so $\cos = -\frac{3}{5}$. To find $\sin(y+z)$, plug this information into the sine sum formula.

$$
\sin(y+z) = \sin y \cos z + \cos y \sin z
$$
$$
= -\frac{5}{13} \cdot -\frac{3}{5} + -\frac{12}{13} \cdot \frac{4}{5} = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}
$$

}}

Concept Problem Solution

With the sine sum formula, you can break the sine into easier to calculate quantities:

$$\sin\frac{7\pi}{12} = \sin\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$$
$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) + cos(\frac{\pi}{3})sin(\frac{\pi}{4})$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Practice

Find the exact value for each sine expression.

1. $\sin 75°$
2. $\sin 105°$
3. $\sin 165°$
4. $\sin 255°$
5. $\sin -15°$

Write each expression as the sine of an angle.

6. $\sin 46° \cos 20° + \cos 46° \sin 20°$
7. $\sin 3x \cos 2x - \cos 3x \sin 2x$
8. $\sin 54° \cos 12° + \cos 54° \sin 12°$
9. $\sin 29° \cos 10° - \cos 29° \sin 10°$
10. $\sin 4y \cos 3y + \cos 4y \sin 2y$
11. Prove that $\sin(x - \frac{\pi}{2}) = -\cos(x)$
12. Suppose that x, y, and z are the three angles of a triangle. Prove that $\sin(x+y) = \sin(z)$
13. Prove that $\sin(\frac{\pi}{2} - x) = \cos(x)$
14. Prove that $\sin(x+\pi) = -\sin(x)$
15. Prove that $\sin(x-y) + \sin(x+y) = 2\sin(x)\cos(y)$

3.8 Tangent Sum and Difference Formulas

Here you'll learn to rewrite tangent functions with addition or subtraction in their arguments in a more easily solvable form.

Suppose you were given two angles and asked to find the tangent of the difference of them. For example, can you compute:

$$\tan(120° - 40°)$$

Would you just subtract the angles and then take the tangent of the result? Or is something more complicated required to solve this problem? Keep reading, and by the end of this Concept, you'll be able to calculate trig functions like the one above.

Watch This

MEDIA
Click image to the left for more content.

James Sousa:SumandDifference Identities for Tangent

Guidance

In this Concept, we want to find a formula that will make computing the tangent of a sum of arguments or a difference of arguments easier. As first, it may seem that you should just add (or subtract) the arguments and take the tangent of the result. However, it's not quite that easy.

To find the sum formula for tangent:

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} \qquad \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b} \qquad \text{Substituting the sum formulas for sine and cosine}$$

$$= \frac{\frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b}}{\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b}} \qquad \text{Divide both the numerator and the denominator by } \cos a \cos b$$

$$= \frac{\frac{\sin a \cos b}{\cos a \cos b} + \frac{\sin b \cos a}{\cos a \cos b}}{\frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b}} \qquad \text{Reduce each of the fractions}$$

$$= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a \sin b}{\cos a \cos b}} \qquad \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \qquad \text{Sum formula for tangent}$$

In conclusion, $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Example A

Find the exact value of $\tan 285°$.

Solution: Use the difference formula for tangent, with $285° = 330° - 45°$

$$
\begin{aligned}
\tan(330° - 45°) &= \frac{\tan 330° - \tan 45°}{1 + \tan 330° \tan 45°} \\
&= \frac{-\frac{\sqrt{3}}{3} - 1}{1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \\
&= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\
&= \frac{-9 - 6\sqrt{3} - 3}{9 - 3} \\
&= \frac{-12 - 6\sqrt{3}}{6} \\
&= -2 - \sqrt{3}
\end{aligned}
$$

To verify this on the calculator, $\tan 285° = -3.732$ and $-2 - \sqrt{3} = -3.732$.

Example B

Verify the tangent difference formula by finding $\tan \frac{6\pi}{6}$, since this should be equal to $\tan \pi = 0$.

Solution: Use the difference formula for tangent, with $\tan \frac{6\pi}{6} = \tan(\frac{7\pi}{6} - \frac{\pi}{6})$

$$
\begin{aligned}
\tan(\frac{7\pi}{6} - \frac{\pi}{6}) &= \frac{\tan \frac{7\pi}{6} - \tan \frac{\pi}{6}}{1 + \tan \frac{7\pi}{6} \tan \frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6}}{1 - \frac{\sqrt{2}}{6} \cdot \frac{\sqrt{2}}{6}} = \frac{0}{1 - \frac{2}{36}} \\
&= \frac{0}{\frac{34}{36}} \\
&= 0
\end{aligned}
$$

Example C

Find the exact value of $\tan 165°$.

Solution: Use the difference formula for tangent, with $165° = 225° - 60°$

$$\tan(225° - 60°) = \frac{\tan 225° - \tan 60°}{1 + \tan 225° \tan 60°}$$
$$= \frac{1 - \sqrt{3}}{1 - 1 \cdot \sqrt{3}} = 1$$

Vocabulary

Tangent Sum Formula: The **tangent sum formula** relates the tangent of a sum of two arguments to a set of tangent functions, each containing one argument.

Tangent Difference Formula: The **tangent difference formula** relates the tangent of a difference of two arguments to a set of tangent functions, each containing one argument.

Guided Practice

1. Find the exact value for $\tan 75°$

2. Simplify $\tan(\pi + \theta)$

3. Find the exact value for $\tan 15°$

Solutions:

1.

$$\tan 75°$$
$$= \tan(45° + 30°)$$
$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$
$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3 + \sqrt{3}}{3}}{\frac{3 - \sqrt{3}}{3}}$$
$$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}}$$
$$= \frac{9 + 6\sqrt{3} + 3}{9 - 3} = \frac{12 + 6\sqrt{3}}{6}$$
$$= 2 + \sqrt{3}$$

2. $\tan(\pi + \theta) = \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta} = \frac{\tan \theta}{1} = \tan \theta$

3.

$$\tan 15°$$
$$= \tan(45° - 30°)$$
$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$
$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3 - \sqrt{3}}{3}}{\frac{3 + \sqrt{3}}{3}}$$
$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$
$$= \frac{9 + 6\sqrt{3} + 3}{9 - 3} = \frac{12 + 6\sqrt{3}}{6}$$
$$= 2 + \sqrt{3}$$

Concept Problem Solution

The Concept Problem asks you to find:

$\tan(120° - 40°)$

You can use the tangent difference formula:

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

to help solve this. Substituting in known quantities:

$$\tan(120° - 40°) = \frac{\tan 120° - \tan 40°}{1 + (\tan 120°)(\tan 40°)} = \frac{-1.732 - .839}{1 + (-1.732)(.839)} = \frac{-2.571}{-.453148} = 5.674$$

Practice

Find the exact value for each tangent expression.

1. $\tan \frac{5\pi}{12}$
2. $\tan \frac{11\pi}{12}$
3. $\tan -165°$
4. $\tan 255°$
5. $\tan -15°$

Write each expression as the tangent of an angle.

6. $\frac{\tan 15° + \tan 42°}{1 - \tan 15° \tan 42°}$
7. $\frac{\tan 65° - \tan 12°}{1 + \tan 65° \tan 12°}$
8. $\frac{\tan 10° + \tan 50°}{1 - \tan 10° \tan 50°}$
9. $\frac{\tan 2y + \tan 4}{1 - \tan 2 \tan 4y}$
10. $\frac{\tan x - \tan 3x}{1 + \tan x \tan 3x}$
11. $\frac{\tan 2x - \tan y}{1 + \tan 2x \tan y}$

12. Prove that $\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan(x)}{1 - \tan(x)}$
13. Prove that $\tan\left(x - \frac{\pi}{2}\right) = -\cot(x)$
14. Prove that $\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$
15. Prove that $\tan(x + y)\tan(x - y) = \frac{\tan^2(x) - \tan^2(y)}{1 - \tan^2(x)\tan^2(y)}$

3.9 Applications of Sum and Difference Formulas

Here you'll learn to solve problems using the sum and difference formulas for trig functions, including the sine sum/difference formulas, cosine sum/difference formulas, and tangent sum/difference formulas.

You are quite likely familiar with the values of trig functions for a variety of angles. Angles such as $30°$, $60°$, and $90°$ are common. However, if you were asked to find the value of a trig function for a more rarely used angle, could you do so? Or what if you were asked to find the value of a trig function for a sum of angles? For example, if you were asked to find $\sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$ could you?

Read on, and in this section, you'll get practice with simplifying trig functions of angles using the sum and difference formulas.

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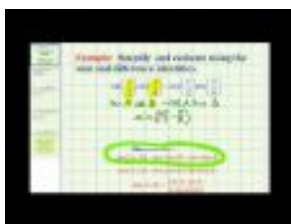
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James Sousa Example:Simplify a TrigExpression Using the Sum and DifferenceIdentities

Guidance

Quite frequently one of the main obstacles to solving a problem in trigonometry is the inability to transform the problem into a form that makes it easier to solve. Sum and difference formulas can be very valuable in helping with this.

Here we'll get some extra practice putting the sum and difference formulas to good use. If you haven't gone through them yet, you might want to review the Concepts on the Sum and Difference Formulas for sine, cosine, and tangent.

Example A

Verify the identity $\frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

$$
\begin{aligned}
\cot x \cot y + 1 &= \frac{\cos(x-y)}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} \qquad \text{Expand using the cosine difference formula.} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + 1 \\
\cot x \cot y + 1 &= \cot x \cot y + 1 \qquad\qquad\qquad \text{cotangent equals cosine over sine}
\end{aligned}
$$

Example B

Solve $3\sin(x - \pi) = 3$ in the interval $[0, 2\pi)$.

Solution: First, get $\sin(x - \pi)$ by itself, by dividing both sides by 3.

$$\frac{3\sin(x - \pi)}{3} = \frac{3}{3}$$
$$\sin(x - \pi) = 1$$

Now, expand the left side using the sine difference formula.

$$\sin x \cos \pi - \cos x \sin \pi = 1$$
$$\sin x(-1) - \cos x(0) = 1$$
$$-\sin x = 1$$
$$\sin x = -1$$

The $\sin x = -1$ when x is $\frac{3\pi}{2}$.

Example C

Find all the solutions for $2\cos^2\left(x + \frac{\pi}{2}\right) = 1$ in the interval $[0, 2\pi)$.

Solution: Get the $\cos^2\left(x + \frac{\pi}{2}\right)$ by itself and then take the square root.

$$2\cos^2\left(x + \frac{\pi}{2}\right) = 1$$
$$\cos^2\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$$
$$\cos\left(x + \frac{\pi}{2}\right) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Now, use the cosine sum formula to expand and solve.

$$\cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2} = \frac{\sqrt{2}}{2}$$
$$\cos x(0) - \sin x(1) = \frac{\sqrt{2}}{2}$$
$$-\sin x = \frac{\sqrt{2}}{2}$$
$$\sin x = -\frac{\sqrt{2}}{2}$$

The $\sin x = -\frac{\sqrt{2}}{2}$ is in Quadrants III and IV, so $x = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

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Vocabulary

Difference Formula: A **difference formula** is a formula to help simplify a trigonometric function of the difference of two angles, such as $\sin(a-b)$.

Sum Formula: A **sum formula** is a formula to help simplify a trigonometric function of the sum of two angles, such as $\sin(a+b)$.

Guided Practice

1. Find all solutions to $2\cos^2\left(x+\frac{\pi}{2}\right)=1$, when x is between $[0,2\pi)$.

2. Solve for all values of x between $[0,2\pi)$ for $2\tan^2\left(x+\frac{\pi}{6}\right)+1=7$.

3. Find all solutions to $\sin\left(x+\frac{\pi}{6}\right)=\sin\left(x-\frac{\pi}{4}\right)$, when x is between $[0,2\pi)$.

Solutions:

1. To find all the solutions, between $[0,2\pi)$, we need to expand using the sum formula and isolate the $\cos x$.

$$2\cos^2\left(x+\frac{\pi}{2}\right)=1$$
$$\cos^2\left(x+\frac{\pi}{2}\right)=\frac{1}{2}$$
$$\cos\left(x+\frac{\pi}{2}\right)=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$$
$$\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}=\pm\frac{\sqrt{2}}{2}$$
$$\cos x\cdot 0-\sin x\cdot 1=\pm\frac{\sqrt{2}}{2}$$
$$-\sin x=\pm\frac{\sqrt{2}}{2}$$
$$\sin x=\pm\frac{\sqrt{2}}{2}$$

This is true when $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}$, or $\frac{7\pi}{4}$

2. First, solve for $\tan(x+\frac{\pi}{6})$.

$$2\tan^2\left(x+\frac{\pi}{6}\right)+1=7$$
$$2\tan^2\left(x+\frac{\pi}{6}\right)=6$$
$$\tan^2\left(x+\frac{\pi}{6}\right)=3$$
$$\tan\left(x+\frac{\pi}{6}\right)=\pm\sqrt{3}$$

Now, use the tangent sum formula to expand for when $\tan(x+\frac{\pi}{6})=\sqrt{3}$.

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$$\frac{\tan x + \tan \frac{\pi}{6}}{1 - \tan x \tan \frac{\pi}{6}} = \sqrt{3}$$

$$\tan x + \tan \frac{\pi}{6} = \sqrt{3}\left(1 - \tan x \tan \frac{\pi}{6}\right)$$

$$\tan x + \frac{\sqrt{3}}{3} = \sqrt{3} - \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3}$$

$$\tan x + \frac{\sqrt{3}}{3} = \sqrt{3} - \tan x$$

$$2\tan x = \frac{2\sqrt{3}}{3}$$

$$\tan x = \frac{\sqrt{3}}{3}$$

This is true when $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

If the tangent sum formula to expand for when $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$, we get no solution as shown.

$$\frac{\tan x + \tan \frac{\pi}{6}}{1 - \tan x \tan \frac{\pi}{6}} = -\sqrt{3}$$

$$\tan x + \tan \frac{\pi}{6} = -\sqrt{3}\left(1 - \tan x \tan \frac{\pi}{6}\right)$$

$$\tan x + \frac{\sqrt{3}}{3} = -\sqrt{3} + \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3}$$

$$\tan x + \frac{\sqrt{3}}{3} = -\sqrt{3} + \tan x$$

$$\frac{\sqrt{3}}{3} = -\sqrt{3}$$

Therefore, the tangent sum formula cannot be used in this case. However, since we know that $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$ when $x + \frac{\pi}{6} = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$, we can solve for x as follows.

$$x + \frac{\pi}{6} = \frac{5\pi}{6}$$

$$x = \frac{4\pi}{6}$$

$$x = \frac{2\pi}{3}$$

$$x + \frac{\pi}{6} = \frac{11\pi}{6}$$

$$x = \frac{10\pi}{6}$$

$$x = \frac{5\pi}{3}$$

Therefore, all of the solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

3. To solve, expand each side:

$$\sin\left(x+\frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

$$\sin\left(x-\frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$$

Set the two sides equal to each other:

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$$

$$\sqrt{3}\sin x + \cos x = \sqrt{2}\sin x - \sqrt{2}\cos x$$

$$\sqrt{3}\sin x - \sqrt{2}\sin x = -\cos x - \sqrt{2}\cos x$$

$$\sin x\left(\sqrt{3} - \sqrt{2}\right) = \cos x\left(-1 - \sqrt{2}\right)$$

$$\frac{\sin x}{\cos x} = \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}}$$

$$\tan x = \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$$

$$= \frac{-\sqrt{3} - \sqrt{2} + \sqrt{6} - 2}{3 - 2}$$

$$= -2 + \sqrt{6} - \sqrt{3} - \sqrt{2}$$

As a decimal, this is -2.69677, so $\tan^{-1}(-2.69677) = x, x = 290.35°$ and $110.35°$.

Concept Problem Solution

To find $\sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$, use the sine sum formula:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{2}\right) \times \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{2}\right) \times \sin\left(\frac{\pi}{4}\right)$$

$$= (-1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{2}}{2}$$

Practice

Prove each identity.

1. $\cos(3x) + \cos(x) = 2\cos(2x)\cos(x)$
2. $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$

3. $\sin(3x) = 3\cos^2(x)\sin(x) - \sin^3(x)$
4. $\sin(4x) + \sin(2x) = 2\sin(3x)\cos(x)$
5. $\tan(5x)\tan(3x) = \frac{\tan^2(4x) - \tan^2(x)}{1 - \tan^2(4x)\tan^2(x)}$
6. $\cos((\frac{\pi}{2} - x) - y) = \sin(x + y)$

Use sum and difference formulas to help you graph each function.

7. $y = \cos(3)\cos(x) + \sin(3)\sin(x)$
8. $y = \cos(x)\cos(\frac{\pi}{2}) + \sin(x)\sin(\frac{\pi}{2})$
9. $y = \sin(x)\cos(\frac{\pi}{2}) + \cos(x)\sin(\frac{\pi}{2})$
10. $y = \sin(x)\cos(\frac{3\pi}{2}) - \cos(3)\sin(\frac{\pi}{2})$
11. $y = \cos(4x)\cos(2x) - \sin(4x)\sin(2x)$
12. $y = \cos(x)\cos(x) - \sin(x)\sin(x)$

Solve each equation on the interval $[0, 2\pi)$.

13. $2\sin(x - \frac{\pi}{2}) = 1$
14. $4\cos(x - \pi) = 4$
15. $2\sin(x - \pi) = \sqrt{2}$

3.10 Double Angle Identities

Here you'll learn the double angle identities and how to use them to rewrite trig equations into a more easily solvable form.

Finding the values for trig functions is pretty familiar to you by now. The trig functions of some particular angles may even seem obvious, since you've worked with them so many times. In some cases, you might be able to use this knowledge to your benefit to make calculating the values of some trig equations easier. For example, if someone asked you to evaluate

$\cos 120°$

without consulting a table of trig values, could you do it?

You might notice right away that this is equal to four times $30°$. Can this help you? Read this Concept, and at its conclusion you'll know how to use certain formulas to simplify multiples of familiar angles to solve problems.

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James Sousa:Double Angle Identities

Guidance

Here we'll start with the sum and difference formulas for sine, cosine, and tangent. We can use these identities to help derive a new formula for when we are given a trig function that has twice a given angle as the argument. For example, $\sin(2\theta)$. This way, if we are given θ and are asked to find $\sin(2\theta)$, we can use our new double angle identity to help simplify the problem. Let's start with the derivation of the double angle identities.

One of the formulas for calculating the sum of two angles is:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

This is the double angle formula for the sine function. The same procedure can be used in the sum formula for cosine, start with the sum angle formula:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

This is one of the double angle formulas for the cosine function. Two more formulas can be derived by using the Pythagorean Identity, $\sin^2\alpha + \cos^2\alpha = 1$.

$\sin^2\alpha = 1 - \cos^2\alpha$ and likewise $\cos^2\alpha = 1 - \sin^2\alpha$

Using $\sin^2\alpha = 1 - \cos^2\alpha$:
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \cos^2\alpha - (1 - \cos^2\alpha)$$
$$= \cos^2\alpha - 1 + \cos^2\alpha$$
$$= 2\cos^2\alpha - 1$$

Using $\cos^2\alpha = 1 - \sin^2\alpha$:
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= (1 - \sin^2\alpha) - \sin^2\alpha$$
$$= 1 - \sin^2\alpha - \sin^2\alpha$$
$$= 1 - 2\sin^2\alpha$$

Therefore, the double angle formulas for $\cos 2\alpha$ are:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

Finally, we can calculate the double angle formula for tangent, using the tangent sum formula:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

If α and β are both the same angle in the above formula, then

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

We can use these formulas to help simplify calculations of trig functions of certain arguments.

Example A

If $\sin a = \frac{5}{13}$ and a is in Quadrant II, find $\sin 2a$, $\cos 2a$, and $\tan 2a$.

Solution: To use $\sin 2a = 2\sin a\cos a$, the value of $\cos a$ must be found first.

$$= \cos^2 a + \sin^2 a = 1$$
$$= \cos^2 a + \left(\frac{5}{13}\right)^2 = 1$$
$$= \cos^2 a + \frac{25}{169} = 1$$
$$= \cos^2 a = \frac{144}{169}, \cos a = \pm\frac{12}{13}$$

.

However since a is in Quadrant II, $\cos a$ is negative or $\cos a = -\frac{12}{13}$.

$$\sin 2a = 2\sin a\cos a = 2\left(\frac{5}{13}\right) \times \left(-\frac{12}{13}\right) = \sin 2a = -\frac{120}{169}$$

For $\cos 2a$, use $\cos(2a) = \cos^2 a - \sin^2 a$

$$\cos(2a) = \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \text{ or } \frac{144 - 25}{169}$$
$$\cos(2a) = \frac{119}{169}$$

For $\tan 2a$, use $\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$. From above, $\tan a = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$.

$$\tan(2a) = \frac{2 \cdot \frac{-5}{12}}{1 - \left(\frac{-5}{12}\right)^2} = \frac{\frac{-5}{6}}{1 - \frac{25}{144}} = \frac{\frac{-5}{6}}{\frac{119}{144}} = -\frac{5}{6} \cdot \frac{144}{119} = -\frac{120}{119}$$

Example B

Find $\cos 4\theta$.

Solution: Think of $\cos 4\theta$ as $\cos(2\theta + 2\theta)$.

$$\cos 4\theta = \cos(2\theta + 2\theta) = \cos 2\theta\cos 2\theta - \sin 2\theta\sin 2\theta = \cos^2 2\theta - \sin^2 2\theta$$

Now, use the double angle formulas for both sine and cosine. For cosine, you can pick which formula you would like to use. In general, because we are proving a cosine identity, stay with cosine.

$$= (2\cos^2 \theta - 1)^2 - (2\sin\theta\cos\theta)^2$$
$$= 4\cos^4 \theta - 4\cos^2 \theta + 1 - 4\sin^2 \theta\cos^2 \theta$$
$$= 4\cos^4 \theta - 4\cos^2 \theta + 1 - 4(1 - \cos^2 \theta)\cos^2 \theta$$
$$= 4\cos^4 \theta - 4\cos^2 \theta + 1 - 4\cos^2 \theta + 4\cos^4 \theta$$
$$= 8\cos^4 \theta - 8\cos^2 \theta + 1$$

Example C

Solve the trigonometric equation $\sin 2x = \sin x$ such that $(-\pi \leq x < \pi)$

Solution: Using the sine double angle formula:

$$\sin 2x = \sin x$$
$$2\sin x \cos x = \sin x$$
$$2\sin x \cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$

$$2\cos x - 1 = 0$$
$$2\cos x = 1$$

$$\sin x = 0$$

$$x = 0, -\pi \qquad \cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

Vocabulary

Double Angle Identity: A **double angle identity** relates the a trigonometric function of two times an argument to a set of trigonometric functions, each containing the original argument.

Guided Practice

1. If $\sin x = \frac{4}{5}$ and x is in Quad II, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$

2. Find the exact value of $\cos^2 15° - \sin^2 15°$

3. Verify the identity: $\cos 3\theta = 4\cos^3 \theta - 3\cos\theta$

Solutions:

1. If $\sin x = \frac{4}{5}$ and in Quadrant II, then cosine and tangent are negative. Also, by the Pythagorean Theorem, the third side is $3(b = \sqrt{5^2 - 4^2})$. So, $\cos x = -\frac{3}{5}$ and $\tan x = -\frac{4}{3}$. Using this, we can find $\sin 2x, \cos 2x$, and $\tan 2x$.

$$\cos 2x = 1 - \sin^2 x \qquad\qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
$$= 1 - 2 \cdot \left(\frac{4}{5}\right)^2 \qquad\qquad = \frac{2 \cdot -\frac{4}{3}}{1 - \left(-\frac{4}{3}\right)^2}$$
$$\sin 2x = 2\sin x \cos x \qquad\qquad = 1 - 2 \cdot \frac{16}{25} \qquad\qquad = \frac{-\frac{8}{3}}{1 - \frac{16}{9}} = -\frac{8}{3} \div -\frac{7}{9}$$
$$= 2 \cdot \frac{4}{5} \cdot -\frac{3}{5} \qquad\qquad = 1 - \frac{32}{25} \qquad\qquad = -\frac{8}{3} \cdot -\frac{9}{7}$$
$$= -\frac{24}{25} \qquad\qquad = -\frac{7}{25} \qquad\qquad = \frac{24}{7}$$

2. This is one of the forms for $\cos 2x$.

$$\begin{aligned}
\cos^2 15° - \sin^2 15° &= \cos(15° \cdot 2) \\
&= \cos 30° \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

3. Step 1: Use the cosine sum formula

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$
$$\cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

Step 2: Use double angle formulas for $\cos 2\theta$ and $\sin 2\theta$

$$= (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta)\sin \theta$$

Step 3: Distribute and simplify.

$$\begin{aligned}
&= 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta \\
&= -\cos \theta(-2\cos^2 \theta + 2\sin^2 \theta + 1) \\
&= -\cos \theta[-2\cos^2 \theta + 2(1 - \cos^2 \theta) + 1] \qquad \rightarrow \text{Substitute } 1 - \cos^2 \theta \text{ for } \sin^2 \theta \\
&= -\cos \theta[-2\cos^2 \theta + 2 - 2\cos^2 \theta + 1] \\
&= -\cos \theta(-4\cos^2 \theta + 3) \\
&= 4\cos^3 \theta - 3\cos \theta
\end{aligned}$$

Concept Problem Solution

Since the problem wants you to find:

$\cos 120°$

You can simplify this into a familiar angle:

$\cos(2 \times 60°)$

And then apply the double angle identity:

$$\begin{aligned}
\cos(2 \times 60°) &= 2\cos^2 60° - 1 \\
&= (2)(\cos 60°)(\cos 60°) - 1 \\
&= (2)(\frac{1}{2})(\frac{1}{2}) - 1 \\
&= -\frac{1}{2}
\end{aligned}$$

Practice

Simplify each expression so that it is in terms of $\sin(x)$ and $\cos(x)$.

1. $\sin 2x + \cos x$
2. $\sin 2x + \cos 2x$
3. $\sin 3x + \cos 2x$
4. $\sin 2x + \cos 3x$

Solve each equation on the interval $[0, 2\pi)$.

5. $\sin(2x) = 2\sin(x)$
6. $\cos(2x) = \sin(x)$
7. $\sin(2x) - \tan(x) = 0$
8. $\cos^2(x) + \cos(x) = \cos(2x)$
9. $\cos(2x) = \cos(x)$

Simplify each expression so that only one calculation would be needed in order to evaluate.

10. $2\cos^2(15°) - 1$
11. $2\sin(25°)\cos(25°)$
12. $1 - 2\sin^2(35°)$
13. $\cos^2(60°) - \sin^2(60°)$
14. $2\sin(125°)\cos(125°)$
15. $1 - 2\sin^2(32°)$

3.11 Half Angle Formulas

Here you'll learn what the half angle formulas are and how to derive them.

After all of your experience with trig functions, you are feeling pretty good. You know the values of trig functions for a lot of common angles, such as $30°, 60°$ etc. And for other angles, you regularly use your calculator. Suppose someone gave you an equation like this:

$\cos 75°$

Could you solve it without the calculator? You might notice that this is half of $150°$. This might give you a hint!

When you've completed this Concept, you'll know how to solve this problem and others like it where the angle is equal to half of some other angle that you're already familiar with.

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James Sousa:Half Angle Identities

Guidance

Here we'll attempt to derive and use formulas for trig functions of angles that are half of some particular value. To do this, we'll start with the double angle formula for cosine: $\cos 2\theta = 1 - 2\sin^2 \theta$. Set $\theta = \frac{\alpha}{2}$, so the equation above becomes $\cos 2\frac{\alpha}{2} = 1 - 2\sin^2 \frac{\alpha}{2}$.

Solving this for $\sin \frac{\alpha}{2}$, we get:

$$\cos 2\frac{\alpha}{2} = 1 - 2\sin^2 \frac{\alpha}{2}$$
$$\cos \alpha = 1 - 2\sin^2 \frac{\alpha}{2}$$
$$2\sin^2 \frac{\alpha}{2} = 1 - \cos \alpha$$
$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$$
$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin\frac{\alpha}{2} = -\sqrt{\dfrac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

This formula shows how to find the sine of half of some particular angle.

One of the other formulas that was derived for the cosine of a double angle is:

$\cos 2\theta = 2\cos^2\theta - 1$. Set $\theta = \frac{\alpha}{2}$, so the equation becomes $\cos 2\frac{\alpha}{2} = -1 + 2\cos^2\frac{\alpha}{2}$. Solving this for $\cos\frac{\alpha}{2}$, we get:

$$\cos 2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1$$
$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$$
$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$\cos\frac{\alpha}{2} = \sqrt{\dfrac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos\frac{\alpha}{2} = -\sqrt{\dfrac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

This formula shows how to find the cosine of half of some particular angle.

Let's see some examples of these two formulas (sine and cosine of half angles) in action.

Example A

Determine the exact value of $\sin 15°$.

Solution: Using the half angle identity, $\alpha = 30°$, and $15°$ is located in the first quadrant. Therefore, $\sin\frac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}}$.

$$\sin 15° = \sqrt{\frac{1-\cos 30°}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$$

Plugging this into a calculator, $\sqrt{\dfrac{2-\sqrt{3}}{4}} \approx 0.2588$. Using the sine function on your calculator will validate that this answer is correct.

Example B

Use the half angle identity to find exact value of $\sin 112.5°$

Solution: since $\sin\frac{225°}{2} = \sin 112.5°$, use the half angle formula for sine, where $\alpha = 225°$. In this example, the angle $112.5°$ is a second quadrant angle, and the sin of a second quadrant angle is positive.

$$\sin 112.5° = \sin \frac{225°}{2}$$

$$= \pm \sqrt{\frac{1 - \cos 225°}{2}}$$

$$= + \sqrt{\frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{2}}$$

$$= \sqrt{\frac{\frac{2}{2} + \frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2 + \sqrt{2}}{4}}$$

Example C

Use the half angle formula for the cosine function to prove that the following expression is an identity: $2\cos^2 \frac{x}{2} - \cos x = 1$

Solution: Use the formula $\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$ and substitute it on the left-hand side of the expression.

$$2 \left(\sqrt{\frac{1 + \cos \theta}{2}} \right)^2 - \cos \theta = 1$$

$$2 \left(\frac{1 + \cos \theta}{2} \right) - \cos \theta = 1$$

$$1 + \cos \theta - \cos \theta = 1$$

$$1 = 1$$

Vocabulary

Half Angle Identity: A **half angle identity** relates the a trigonometric function of one half of an argument to a set of trigonometric functions, each containing the original argument.

Guided Practice

1. Prove the identity: $\tan \frac{b}{2} = \frac{\sec b}{\sec b \csc b + \csc b}$

2. Verify the identity: $\cot \frac{c}{2} = \frac{\sin c}{1 - \cos c}$

3. Prove that $\sin x \tan \frac{x}{2} + 2\cos x = 2\cos^2 \frac{x}{2}$

Solutions:

1.

Step 1: Change right side into sine and cosine.

$$\tan\frac{b}{2} = \frac{\sec b}{\sec b\csc b + \csc b}$$

$$= \frac{1}{\cos b} \div \csc b(\sec b + 1)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1}{\cos b} + 1\right)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1 + \cos b}{\cos b}\right)$$

$$= \frac{1}{\cos b} \div \frac{1 + \cos b}{\sin b\cos b}$$

$$= \frac{1}{\cos b} \cdot \frac{\sin b\cos b}{1 + \cos b}$$

$$= \frac{\sin b}{1 + \cos b}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\sqrt{\frac{1 - \cos b}{1 + \cos b}} = \frac{\sin b}{1 + \cos b}$$

$$\frac{1 - \cos b}{1 + \cos b} = \frac{\sin^2 b}{(1 + \cos b)^2}$$

$$(1 - \cos b)(1 + \cos b)^2 = \sin^2 b(1 + \cos b)$$

$$(1 - \cos b)(1 + \cos b) = \sin^2 b$$

$$1 - \cos^2 b = \sin^2 b$$

2.

Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\cot\frac{c}{2} = \frac{\sin c}{1 - \cos c}$$

$$= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1 + \cos c}{1 - \cos c}}$$

$$\sqrt{\frac{1 + \cos c}{1 - \cos c}} = \frac{\sin c}{1 - \cos c}$$

$$\frac{1 + \cos c}{1 - \cos c} = \frac{\sin^2 c}{(1 - \cos c)^2}$$

$$(1 + \cos c)(1 - \cos c)^2 = \sin^2 c(1 - \cos c)$$

$$(1 + \cos c)(1 - \cos c) = \sin^2 c$$

$$1 - \cos^2 c = \sin^2 c$$

3.

$$\sin x \tan \frac{x}{2} + 2\cos x = \sin x \left(\frac{1 - \cos x}{\sin x} \right) + 2\cos x$$

$$\sin x \tan \frac{x}{2} + 2\cos x = 1 - \cos x + 2\cos x$$

$$\sin x \tan \frac{x}{2} + 2\cos x = 1 + \cos x$$

$$\sin x \tan \frac{x}{2} + 2\cos x = 2\cos^2 \frac{x}{2}$$

Concept Problem Solution

The original question asked you to find $\cos 75°$. If you use the half angle formula, then $\alpha = 150°$

Substituting this into the half angle formula:

$$\sin \frac{150°}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - \cos 150°}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Practice

Use half angle identities to find the exact values of each expression.

1. $\sin 22.5°$
2. $\sin 75°$
3. $\sin 67.5°$
4. $\sin 157.5°$
5. $\cos 22.5°$
6. $\cos 75°$
7. $\cos 157.5°$
8. $\cos 67.5°$
9. Use the two half angle identities presented in this concept to prove that $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$.
10. Use the result of the previous problem to show that $\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{\sin x}$.
11. Use the result of the previous problem to show that $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos x}$.

Use half angle identities to help you find all solutions to the following equations in the interval $[0, 2\pi)$.

12. $\sin^2 x = \cos^2\left(\frac{x}{2}\right)$
13. $\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$
14. $\cos^2 x = \sin^2\left(\frac{x}{2}\right)$
15. $\sin^2\left(\frac{x}{2}\right) = 2\cos^2 x - 1$

3.12 **Trigonometric Equations Using Half Angle Formulas**

Here you'll learn how to solve trig equations using the half angle formulas.

As you've seen many times, the ability to find the values of trig functions for a variety of angles is a critical component to a course in Trigonometry. If you were given an angle as the argument of a trig function that was half of an angle you were familiar with, could you solve the trig function?

For example, if you were asked to find

$\sin 22.5°$

would you be able to do it? Keep reading, and in this Concept you'll learn how to do this.

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James Sousa Example:Determine aSine Function Usinga Half Angle Identity

Guidance

It is easy to remember the values of trigonometric functions for certain common values of θ. However, sometimes there will be fractional values of known trig functions, such as wanting to know the sine of half of the angle that you are familiar with. In situations like that, a half angle identity can prove valuable to help compute the value of the trig function.

In addition, half angle identities can be used to simplify problems to solve for certain angles that satisfy an expression. To do this, first remember the half angle identities for sine and cosine:

$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin \frac{\alpha}{2} = -\sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

$\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 + \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

When attempting to solve equations using a half angle identity, look for a place to substitute using one of the above identities. This can help simplify the equation to be solved.

Example A

Solve the trigonometric equation $\sin^2\theta = 2\sin^2\frac{\theta}{2}$ over the interval $[0, 2\pi)$.

Solution:

$$\sin^2\theta = 2\sin^2\frac{\theta}{2}$$
$$\sin^2\theta = 2\left(\frac{1-\cos\theta}{2}\right) \qquad \text{Half angle identity}$$
$$1-\cos^2\theta = 1-\cos\theta \qquad \text{Pythagorean identity}$$
$$\cos\theta - \cos^2\theta = 0$$
$$\cos\theta(1-\cos\theta) = 0$$

Then $\cos\theta = 0$ or $1-\cos\theta = 0$, which is $\cos\theta = 1$.

$\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, or 2π.

Example B

Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$

Solution:

To solve $2\cos^2\frac{x}{2} = 1$, first we need to isolate cosine, then use the half angle formula.

$$2\cos^2\frac{x}{2} = 1$$
$$\cos^2\frac{x}{2} = \frac{1}{2}$$
$$\frac{1+\cos x}{2} = \frac{1}{2}$$
$$1+\cos x = 1$$
$$\cos x = 0$$

$\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Example C

Solve $\tan\frac{a}{2} = 4$ for $0° \le a < 360°$

Solution:

To solve $\tan\frac{a}{2} = 4$, first isolate tangent, then use the half angle formula.

$$\tan\frac{a}{2} = 4$$

$$\sqrt{\frac{1-\cos a}{1+\cos a}} = 4$$

$$\frac{1-\cos a}{1+\cos a} = 16$$

$$16 + 16\cos a = 1 - \cos a$$

$$17\cos a = -15$$

$$\cos a = -\frac{15}{17}$$

Using your graphing calculator, $\cos a = -\frac{15}{17}$ when $a = 152°, 208°$

Vocabulary

Half Angle Identity: A **half angle identity** relates the a trigonometric function of one half of an argument to a set of trigonometric functions, each containing the original argument.

Guided Practice

1. Find the exact value of $\cos 112.5°$

2. Find the exact value of $\sin 105°$

3. Find the exact value of $\tan\frac{7\pi}{8}$

Solutions:

1.

$$\cos 112.5°$$

$$= \cos\frac{225°}{2}$$

$$= -\sqrt{\frac{1+\cos 225°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2-\sqrt{2}}}{2}$$

2.

$$\sin 105°$$
$$= \sin \frac{210°}{2}$$
$$= \sqrt{\frac{1 - \cos 210°}{2}}$$
$$= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$$
$$= \sqrt{\frac{\frac{2 - \sqrt{3}}{2}}{2}}$$
$$= \sqrt{\frac{2 - \sqrt{3}}{4}}$$
$$= \frac{\sqrt{2 - \sqrt{3}}}{2}$$

3.

$$\tan \frac{7\pi}{8}$$
$$= \tan \frac{1}{2} \cdot \frac{7\pi}{4}$$
$$= \frac{1 - \cos \frac{7\pi}{4}}{\sin \frac{7\pi}{4}}$$
$$= \frac{1 - \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$
$$= \frac{\frac{2 - \sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$
$$= -\frac{2 - \sqrt{2}}{\sqrt{2}}$$
$$= \frac{-2\sqrt{2} + 2}{2}$$
$$= -\sqrt{2} + 1$$

Concept Problem Solution

Knowing the half angle formulas, you can compute $\sin 22.5°$ easily:

$$\sin 22.5° = \sin\left(\frac{45°}{2}\right)$$

$$= \sqrt{\frac{1-\cos 45°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

Practice

Use half angle identities to find the exact value of each expression.

1. $\tan 15°$
2. $\tan 22.5°$
3. $\cot 75°$
4. $\tan 67.5°$
5. $\tan 157.5°$
6. $\tan 112.5°$
7. $\cos 105°$
8. $\sin 112.5°$
9. $\sec 15°$
10. $\csc 22.5°$
11. $\csc 75°$
12. $\sec 67.5°$
13. $\cot 157.5°$

Use half angle identities to help solve each of the following equations on the interval $[0, 2\pi)$.

14. $3\cos^2\left(\frac{x}{2}\right) = 3$
15. $4\sin^2 x = 8\sin^2\left(\frac{x}{2}\right)$

3.13 Sum to Product Formulas for Sine and Cosine

Here you'll learn how to use the sum to product formulas to rewrite equations involving the sum of sine and cosine functions into equations involving the products of sine and cosine functions.

Can you solve problems that involve the sum of sines or cosines? For example, consider the equation:

$\cos 10t + \cos 3t$

You could just compute each expression separately and add their values at the end. However, there is an easier way to do this. You can simplify the equation first, and then solve.

Read this Concept, and at the end of it, you'll be able to simplify this equation and transform it into a product of trig functions instead of a sum!

Watch This

In the first portion of this video, you'll learn about the Sum to Product formulas.

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James Sousa:Sumto Product and Product to Sum Identities

Guidance

In some problems, the product of two trigonometric functions is more conveniently found by the sum of two trigonometric functions by use of identities such as this one:

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \times \cos \frac{\alpha - \beta}{2}$$

This can be verified by using the sum and difference formulas:

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\left[\sin\left(\tfrac{\alpha}{2}+\tfrac{\beta}{2}\right)\cos\left(\tfrac{\alpha}{2}-\tfrac{\beta}{2}\right)\right]$$
$$= 2\left[\left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\left(\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\right]$$
$$= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\cos^2\tfrac{\beta}{2}+\sin^2\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\beta}{2}\cos^2\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin^2\tfrac{\beta}{2}\cos\tfrac{\alpha}{2}\right]$$
$$= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\left(\sin^2\tfrac{\beta}{2}+\cos^2\tfrac{\beta}{2}\right)+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\left(\sin^2\tfrac{\alpha}{2}+\cos^2\tfrac{\alpha}{2}\right)\right]$$
$$= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\right]$$
$$= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin\frac{\beta}{2}\cos\frac{\beta}{2}$$
$$= \sin\left(2\cdot\frac{\alpha}{2}\right)+\sin\left(2\cdot\frac{\beta}{2}\right)$$
$$= \sin\alpha+\sin\beta$$

The following variations can be derived similarly:

$$\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\times\cos\frac{\alpha+\beta}{2}$$
$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\times\sin\frac{\alpha-\beta}{2}$$

Here are some examples of this type of transformation from a sum of terms to a product of terms.

Example A

Change $\sin 5x-\sin 9x$ into a product.

Solution: Use the formula $\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\times\cos\frac{\alpha+\beta}{2}$.

$$\sin 5x-\sin 9x = 2\sin\frac{5x-9x}{2}\cos\frac{5x+9x}{2}$$
$$= 2\sin(-2x)\cos 7x$$
$$= -2\sin 2x\cos 7x$$

Example B

Change $\cos(-3x)+\cos 8x$ into a product.

Solution: Use the formula $\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$

$$\cos(-3x)+\cos(8x) = 2\cos\frac{-3x+8x}{2}\cos\frac{-3x-8x}{2}$$
$$= 2\cos(2.5x)\cos(-5.5x)$$
$$= 2\cos(2.5x)\cos(5.5x)$$

Example C

Change $2\sin 7x \cos 4x$ to a sum.

Solution: This is the reverse of what was done in the previous two examples. Looking at the four formulas above, take the one that has sine and cosine as a product, $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$. Therefore, $7x = \frac{\alpha+\beta}{2}$ and $4x = \frac{\alpha-\beta}{2}$.

$$7x = \frac{\alpha+\beta}{2} \qquad\qquad 4x = \frac{\alpha-\beta}{2}$$

and

$$14x = \alpha+\beta \qquad\qquad 8x = \alpha-\beta$$
$$\alpha = 14x-\beta \qquad\qquad 8x = [14x-\beta]-\beta$$

so

$$-6x = -2\beta$$
$$3x = \beta$$

$$\alpha = 14x-3x$$
$$\alpha = 11x$$

So, this translates to $\sin(11x) + \sin(3x)$. A shortcut for this problem, would be to notice that the sum of $7x$ and $4x$ is $11x$ and the difference is $3x$.

Vocabulary

Sum to Product Formula: A **sum to product formula** relates the sum or difference of two trigonometric functions to the product of two trigonometric functions.

Guided Practice

1. Express the sum as a product: $\sin 9x + \sin 5x$

2. Express the difference as a product: $\cos 4y - \cos 3y$

3. Verify the identity (using sum-to-product formula): $\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$

Solutions:

1. Using the sum-to-product formula:

$$\sin 9x + \sin 5x$$
$$\frac{1}{2}\left(\sin\left(\frac{9x+5x}{2}\right)\cos\left(\frac{9x-5x}{2}\right)\right)$$
$$\frac{1}{2}\sin 7x \cos 2x$$

2.

Using the difference-to-product formula:

$$\cos 4y - \cos 3y$$

$$-2\sin\left(\frac{4y+3y}{2}\right)\sin\left(\frac{4y-3y}{2}\right)$$

$$-2\sin\frac{7y}{2}\sin\frac{y}{2}$$

3.

Using the difference-to-product formulas:

$$\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$$

$$\frac{-2\sin\left(\frac{3a+5a}{2}\right)\sin\left(\frac{3a-5a}{2}\right)}{2\sin\left(\frac{3a-5a}{2}\right)\cos\left(\frac{3a+5a}{2}\right)}$$

$$-\frac{\sin 4a}{\cos 4a}$$

$$-\tan 4a$$

Concept Problem Solution

Prior to learning the sum to product formulas for sine and cosine, evaluating a sum of trig functions, such as

$$\cos 10t + \cos 3t$$

might have been considered difficult. But you can easily transform this equation into a product of two trig functions using:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$

Substituting the known quantities:

$$\cos 10t + \cos 3t = 2\cos\frac{13t}{2} \times \cos\frac{7t}{2} = 2\cos(6.5t)\cos(3.5t)$$

Practice

Change each sum or difference into a product.

1. $\sin 3x + \sin 2x$
2. $\cos 2x + \cos 5x$
3. $\sin(-x) - \sin 4x$
4. $\cos 12x + \cos 3x$
5. $\sin 8x - \sin 4x$
6. $\sin x + \sin\frac{1}{2}x$
7. $\cos 3x - \cos(-3x)$

Change each product into a sum or difference.

8. $-2\sin 3.5x\sin 2.5x$
9. $2\cos 3.5x\sin 0.5x$
10. $2\cos 3.5x\cos 5.5x$

11. $2\sin 6x \cos 2x$
12. $-2\sin 3x \sin x$
13. $2\sin 4x \cos x$
14. Show that $\cos\frac{A+B}{2}\cos\frac{A-B}{2} = \frac{1}{2}(\cos A + \cos B)$.
15. Let $u = \frac{A+B}{2}$ and $v = \frac{A-B}{2}$. Show that $\cos u \cos v = \frac{1}{2}(\cos(u+v) + \cos(u-v))$.

Here you'll learn how to convert equations involving the product of sine and cosine functions into equations involving the sum of sine and cosine functions.

Let's say you are in class one day, working on calculating the values of trig functions, when your instructor gives you an equation like this:

$\sin 75° \sin 15°$

Can you solve this sort of equation? You might want to just calculate each term separately and then compute the result. However, there is another way. You can transform this product of trig functions into a sum of trig functions.

Read on, and by the end of this Concept, you'll know how to solve this problem by changing it into a sum of trig functions.

Watch This

In the second portion of this video you'll learn about Product to Sum formulas.

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James Sousa:Sum to Product and Product to Sum Identities

Guidance

Here we'll begin by deriving formulas for how to convert the product of two trig functions into a sum or difference of trig functions.

There are two formulas for transforming a product of sine or cosine into a sum or difference. First, let's look at the product of the sine of two angles. To do this, we need to start with the cosine of the difference of two angles.

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \text{ and } \cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) - \cos(a+b) = \cos a \cos b + \sin a \sin b - (\cos a \cos b - \sin a \sin b)$$
$$\cos(a-b) - \cos(a+b) = \cos a \cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b$$
$$\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$$
$$\frac{1}{2}[\cos(a-b) - \cos(a+b)] = \sin a \sin b$$

The following product to sum formulas can be derived using the same method:

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

Armed with these four formulas, we can work some examples.

Example A

Change $\cos 2x \cos 5y$ to a sum.

Solution: Use the formula $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$. Set $\alpha = 2x$ and $\beta = 5y$.

$$\cos 2x \cos 5y = \frac{1}{2}\left[\cos(2x-5y)+\cos(2x+5y)\right]$$

Example B

Change $\frac{\sin 11z + \sin z}{2}$ to a product.

Solution: Use the formula $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$. Therefore, $\alpha+\beta = 11z$ and $\alpha-\beta = z$. Solve the second equation for α and plug that into the first.

$$\alpha = z+\beta \rightarrow (z+\beta)+\beta = 11z \qquad\qquad \text{and} \quad \alpha = z+5z = 6z$$
$$z+2\beta = 11z$$
$$2\beta = 10z$$
$$\beta = 5z$$

$\frac{\sin 11z + \sin z}{2} = \sin 6z \cos 5z$. Again, the sum of $6z$ and $5z$ is $11z$ and the difference is z.

Example C

Solve $\cos 5x + \cos x = \cos 2x$.

Solution: Use the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$.

$$\cos 5x + \cos x = \cos 2x$$
$$2\cos 3x \cos 2x = \cos 2x$$
$$2\cos 3x \cos 2x - \cos 2x = 0$$
$$\cos 2x(2\cos 3x - 1) = 0$$

$$\swarrow \qquad\qquad \searrow$$

$$\cos 2x = 0 \qquad\qquad 2\cos 3x - 1 = 0$$
$$2\cos 3x = 1$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad \cos 3x = \frac{1}{2}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad\qquad 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$
$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$$

Vocabulary

Product to Sum Formula: A **product to sum formula** relates the product of two trigonometric functions to the sum of two trigonometric functions.

Guided Practice

1. Express the product as a sum: $\sin(6\theta)\sin(4\theta)$

2. Express the product as a sum: $\sin(5\theta)\cos(2\theta)$

3. Express the product as a sum: $\cos(10\theta)\sin(3\theta)$

Solutions:

1. Using the product-to-sum formula:

$$\sin 6\theta \sin 4\theta$$
$$\frac{1}{2}\left(\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)\right)$$
$$\frac{1}{2}(\cos 2\theta - \cos 10\theta)$$

2. Using the product-to-sum formula:

$$\sin 5\theta \cos 2\theta$$
$$\frac{1}{2}\left(\sin(5\theta + 2\theta) - \sin(5\theta - 2\theta)\right)$$
$$\frac{1}{2}(\sin 7\theta - \sin 3\theta)$$

3. Using the product-to-sum formula:

$$\cos 10\theta \sin 3\theta$$

$$\frac{1}{2}\left(\sin(10\theta + 3\theta) - \sin(10\theta - 3\theta)\right)$$

$$\frac{1}{2}(\sin 13\theta - \sin 7\theta)$$

Concept Problem Solution

Changing $\sin 75° \sin 15°$ to a product of trig functions can be accomplished using

$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

Substituting in known values gives:

$\sin 75° \sin 15° = \frac{1}{2}\left[\cos(60°) - \cos(90°)\right] = \frac{1}{2}[\frac{1}{2} - 0] = \frac{1}{4}$

Practice

Express each product as a sum or difference.

1. $\sin(5\theta)\sin(3\theta)$
2. $\sin(6\theta)\cos(\theta)$
3. $\cos(4\theta)\sin(3\theta)$
4. $\cos(\theta)\cos(4\theta)$
5. $\sin(2\theta)\sin(2\theta)$
6. $\cos(6\theta)\sin(8\theta)$
7. $\sin(7\theta)\cos(4\theta)$
8. $\cos(11\theta)\cos(2\theta)$

Express each sum or difference as a product.

9. $\frac{\sin 8\theta + \sin 6\theta}{2}$
10. $\frac{\sin 6\theta - \sin 2\theta}{2}$
11. $\frac{\cos 12\theta + \cos 6\theta}{2}$
12. $\frac{\cos 12\theta - \cos 4\theta}{2}$
13. $\frac{\sin 10\theta + \sin 4\theta}{2}$
14. $\frac{\sin 8\theta - \sin 2\theta}{2}$
15. $\frac{\cos 8\theta - \cos 4\theta}{2}$

3.15 Triple-Angle Formulas and Linear Combinations

Here you'll learn to derive equations for formulas with triple angles using existing trig identities, as well as to construct linear combinations of trig functions.

In other Concepts you've dealt with double angle formulas. This was useful for finding the value of an angle that was double your well known value. Now consider the idea of a "triple angle formula". If someone gave you a problem like this:

$\sin 135°$

Could you compute its value?

Keep reading, and at the end of this Concept you'll know how to simplify equations such as this using the triple angle formula.

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Derivinga Triple Angle Formula

Guidance

Double angle formulas are great for computing the value of a trig function in certain cases. However, sometimes different multiples than two times and angle are desired. For example, it might be desirable to have three times the value of an angle to use as the argument of a trig function.

By combining the sum formula and the double angle formula, formulas for triple angles and more can be found.

Here, we take an equation which takes a linear combination of sine and cosine and converts it into a simpler cosine function.

$A\cos x + B\sin x = C\cos(x - D)$, where $C = \sqrt{A^2 + B^2}$, $\cos D = \frac{A}{C}$ and $\sin D = \frac{B}{C}$.

You can also use the TI-83 to solve trigonometric equations. It is sometimes easier than solving the equation algebraically. Just be careful with the directions and make sure your final answer is in the form that is called for. You calculator cannot put radians in terms of π.

Example A

Find the formula for $\sin 3x$

Solution: Use both the double angle formula and the sum formula.

$$\begin{aligned}
\sin 3x &= \sin(2x + x)\\
&= \sin(2x)\cos x + \cos(2x)\sin x\\
&= (2\sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x\\
&= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x\\
&= 3\sin x \cos^2 x - \sin^3 x\\
&= 3\sin x(1 - \sin^2 x) - \sin^3 x\\
&= 3\sin x - 4\sin^3 x
\end{aligned}$$

Example B

Transform $3\cos 2x - 4\sin 2x$ into the form $C\cos(2x - D)$

Solution: $A = 3$ and $B = -4$, so $C = \sqrt{3^2 + (-4)^2} = 5$. Therefore $\cos D = \frac{3}{5}$ and $\sin D = -\frac{4}{5}$ which makes the reference angle is $-53.1°$ or -0.927 radians. since cosine is positive and sine is negative, the angle must be a fourth quadrant angle. D must therefore be $306.9°$ or 5.36 radians.The final answer is $3\cos 2x - 4\sin 2x = 5\cos(2x - 5.36)$.

Example C

Solve $\sin x = 2\cos x$ such that $0 \le x \le 2\pi$ using a graphing calculator.

Solution: In $y =$, graph $y1 = \sin x$ and $y2 = 2\cos x$.

Next, use **CALC** to find the intersection points of the graphs.

Vocabulary

Linear Combination: A **linear combination** is a set of terms that are added or subtracted from each other with a multiplicative constant in front of each term.

Triple Angle Identity: A **triple angle identity** relates the a trigonometric function of three times an argument to a set of trigonometric functions, each containing the original argument.

Guided Practice

1. Transform $5\cos x - 5\sin x$ to the form $C\cos(x - D)$

2. Transform $-15\cos 3x - 8\sin 3x$ to the form $C\cos(x - D)$

3. Derive a formula for $\tan 4x$.

Solutions:

1. If $5\cos x - 5\sin x$, then $A = 5$ and $B = -5$. By the Pythagorean Theorem, $C = 5\sqrt{2}$ and $\cos D = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, because B is negative, D is in Quadrant IV. Therefore, $D = \frac{7\pi}{4}$. Our final answer is $5\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$.

2. If $-15\cos 3x - 8\sin 3x$, then $A = -15$ and $B = -8$. By the Pythagorean Theorem, $C = 17$. Because A and B are both negative, D is in Quadrant III, which means $D = \cos^{-1}\left(\frac{15}{17}\right) = 0.49 + \pi = 3.63 \; rad$. Our final answer is $17\cos 3(x - 3.63)$.

3.

$$\begin{aligned}
\tan 4x &= \tan(2x + 2x) \\
&= \frac{\tan 2x + \tan 2x}{1 - \tan 2x \tan 2x} \\
&= \frac{2\tan 2x}{1 - \tan^2 2x} \\
&= \frac{2 \cdot \frac{2\tan x}{1 - \tan^2 x}}{1 - \left(\frac{2\tan x}{1 - \tan^2 x}\right)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \div \frac{(1 - \tan^2 x)^2 - 4\tan^2 x}{(1 - \tan^2 x)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \div \frac{1 - 2\tan^2 x + \tan^4 x - 4\tan^2 x}{(1 - \tan^2 x)^2} \\
&= \frac{4\tan x}{1 - \tan^2 x} \cdot \frac{(1 - \tan^2 x)^2}{1 - 6\tan^2 x + \tan^4 x} \\
&= \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}
\end{aligned}$$

Concept Problem Solution

Using the triple angle formula we learned in this Concept for the sine function, we can break the angle down into three times a well known angle:

$$\sin 3x = 3\sin x - 4\sin^3 x$$

we can solve this problem.

$$\begin{aligned}
\sin(3 \times 45°) &= 3\sin 45° - 4\sin^3 45° \\
&= 3\frac{\sqrt{2}}{2} - 4\left(\frac{\sqrt{2}}{2}\right)^3 \\
&= 3\frac{\sqrt{2}}{2} - \left(\frac{4(2)^{2/3}}{8}\right) \\
&= \frac{3\sqrt{2} - 2\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Practice

Transform each expression to the form $C\cos(x - D)$.

1. $3\cos x - 2\sin x$
2. $2\cos x - \sin x$
3. $-4\cos x + 5\sin x$
4. $7\cos x - 6\sin x$

5. $11\cos x + 9\sin x$
6. $14\cos x + 2\sin x$
7. $-2\cos x - 4\sin x$

Derive a formula for each expression.

8. $\sin 4x$
9. $\cos 6x$
10. $\cos 4x$
11. $\csc 2x$
12. $\cot 2x$

Find all solutions to each equation in the interval $[0, 2\pi)$.

13. $\cos x + \cos 3x = 0$
14. $\sin 2x = \cos 3x$
15. $\cos 2x + \cos 4x = 0$

Summary

In this chapter identities and equations were presented to make computation of certain types of trigonometric equations simpler. These identities and equations began with trigonometric functions of the complement of an angle and ways to identify certain functions as being "even" or "odd".

After these topics, methods to solve trigonmetric equations by factoring and/or using the quadratic formula were presented.

This was followed up with by ways to simplify computation of certain types of trig functions and combinations of trig functions. Formulas and identities were presented for sums and differences of trig functions, products and quotients of trig functions, and how to compute a trig function for half of a given angle or twice a given angle.

CHAPTER **4**

Inverse Trigonometric Functions

Chapter Outline

Introduction

In this Chapter you'll learn how to find the inverse of trigonometric functions. An inverse function is a function that "undoes" another function. For example, the inverse of multiplying by three is dividing by three.

Trigonometric functions deal with relationships between sides of a triangle. And since they are functions of a angle, applying the inverse trigonometric function will give back the original angle under consideration.

Here you'll learn to find inverse functions, graph them, and apply them.

4.1 Definition of the Inverse of Trigonometric Ratios

Here you'll learn how to define the inverse of trigonometric ratios and apply them.

You are in band practice one day when something catches your eye. One of your fellow students has a musical instrument that you haven't seen played before - a triangle. This is an instrument that is a metal triangle that the musician plays by striking it. Walking over to your classmate, you ask him about the instrument. During a conversation about the triangle, you wonder if it would be possible to make different types of triangle instruments to make different sounds.

You ask your science teacher about this, and she is excited that you have taken an interest in the topic. Together you decide to devise some new instruments based on different length sides of triangles, and then try them out to see how they sound. To begin, your teacher asks you to create a list of a few different sides of triangle lengths, and then list the interior angles of the triangles you come up with.

You immediately get to work and generate a list. The first triangle has sides of 12 cm, 35 cm, and 37 cm, and is a right triangle. However, you realize that you aren't sure of the interior angles of the triangles. You are about to get out a paper and pencil to start plotting the triangles when you start to wonder if there might be an easier way to use math in finding the measurement of the angles instead of plotting them and measuring by hand.

As it turns out, there is a way to do this. Read this Concept on inverse trig functions, and then you'll know how to find the measures of the interior angles, as well as find the interior angles of the first triangle you devised for your list.

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James Sousa:Introductioto Inverse Trigonometric Functions

Guidance

In this Concept, we'll discuss and apply examples of the inverse functions for trig ratios.

Recall from Chapter 1, the ratios of the six trig functions and their inverses, with regard to the unit circle.

$$\sin\theta = \frac{y}{r} \rightarrow \sin^{-1}\frac{y}{r} = \theta \qquad\qquad \cos\theta = \frac{x}{r} \rightarrow \cos^{-1}\frac{x}{r} = \theta$$

$$\tan\theta = \frac{y}{x} \rightarrow \tan^{-1}\frac{y}{x} = \theta \qquad\qquad \cot\theta = \frac{x}{y} \rightarrow \cot^{-1}\frac{x}{y} = \theta$$

$$\csc\theta = \frac{r}{y} \rightarrow \csc^{-1}\frac{r}{y} = \theta \qquad\qquad \sec\theta = \frac{r}{x} \rightarrow \sec^{-1}\frac{r}{x} = \theta$$

These ratios can be used to find any θ in standard position or in a triangle.

In a sense, this is a way of "undoing" a trig function. Before, to find a trig function, you would use the ratio of two sides. Now, by using the inverse trig ratio, you can find angles when you need them.

Let's investigate this by doing a few examples.

Example A

Find the measure of the angles below.

a.

b.

Solution: For part a, you need to use the sine function and part b utilizes the tangent function. Because both problems require you to solve for an angle, the inverse of each must be used.

a. $\sin x = \frac{7}{25} \rightarrow \sin^{-1} \frac{7}{25} = x \rightarrow x = 16.26°$

b. $\tan x = \frac{40}{9} \rightarrow \tan^{-1} \frac{40}{9} = x \rightarrow x = 77.32°$

The trigonometric value $\tan\theta = \frac{40}{9}$ of the angle is known, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. The inverse of the tangent function is read "tangent inverse" and is also called the arctangent relation. The inverse of the cosine function is read "cosine inverse" and is also called the arccosine relation. The inverse of the sine function is read "sine inverse" and is also called the arcsine relation.

Example B

Find the angle, θ, in standard position.

Solution: The $\tan\theta = \frac{y}{x}$ or, in this case, $\tan\theta = \frac{8}{-11}$. Using the inverse tangent, you get $\tan^{-1} -\frac{8}{11} = -36.03°$. This means that the reference angle is $36.03°$. This value of $36.03°$ is the angle you also see if you move counterclockwise from the -x axis. To find the corresponding angle in the second quadrant (which is the same as though you started at the +x axis and moved counterclockwise), subtract $36.03°$ from $180°$, yielding $143.97°$.

Recall that inverse trigonometric functions are also used to find the angle of depression or elevation.

Example C

A new outdoor skating rink has just been installed outside a local community center. A light is mounted on a pole 25 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the skating rink. If the end of the rink is 60 feet from the pole, at what angle of depression should the light be installed?

Solution: In this diagram, the angle of depression, which is located outside of the triangle, is not known. Recall, the angle of depression equals the angle of elevation. For the angle of elevation, the pole where the light is located is the opposite and is 25 feet high. The length of the rink is the adjacent side and is 60 feet in length. To calculate the measure of the angle of elevation the trigonometric ratio for tangent can be applied.

$$\tan\theta = \frac{25}{60}$$
$$\tan\theta = 0.4166$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(0.4166)$$
$$\theta = 22.6°$$

The angle of depression at which the light must be placed to light the rink is $22.6°$

Vocabulary

Trigonometric Inverse: The **trigonometric inverse** is a function that undoes a trig function to give the original argument of the function. It can also be used to find an angle from the ratio of two sides.

Guided Practice

1. Find the value of the missing angle.

2. Find the value of the missing angle.

3. What is the value of the angle with its terminal side passing through (-14, -23)?

Solutions:

1. $\cos\theta = \frac{12}{17} \to \cos^{-1}\frac{12}{17} = 45.1°$

2. $\sin\theta = \frac{25}{36} \to \sin^{-1}\frac{31}{36} = 59.44°$

3. This problem uses tangent inverse. $\tan x = \frac{-23}{-14} \to x = \tan^{-1}\frac{23}{14} = 58.67°$ (value graphing calculator will produce). However, this is the reference angle. Our angle is in the third quadrant because both the x and y values are negative. The angle is $180° + 58.67° = 238.67°$.

Concept Problem Solution

Since you now know about inverse trigonometric ratios, you know that you can apply the inverse of a trig function to help solve this problem. For example, it is straightforward to apply the tangent function:

$$\tan\theta = \frac{35}{12}$$
$$\theta = \tan^{-1}\frac{35}{12}$$
$$\theta = 71.08°$$

You can find the other angle the in a similar manner, this time using the sine function:

$$\sin\theta = \frac{12}{37}$$
$$\theta = \sin^{-1}\frac{12}{37}$$
$$\theta = 18.92°$$

Practice

Find the measure of angle A in each triangle below.

1.
2.
3.

4.

5.

Use inverse tangent to find the value of the angle with its terminal side passing through each of the given points.

6. (-3,4)
7. (12,13)
8. (-4, -7)
9. (5, -4)
10. (-6, 10)
11. (-3, -10)
12. (6, 8)

Use inverse trigonometry to solve each problem.

13. A 30 foot building casts a 60 foot shadow. What is the angle that the sun hits the building?
14. Over 3 miles (horizontal), a road rises 100 feet (vertical). What is the angle of elevation?
15. An 80 foot building casts a 20 foot shadow. What is the angle that the sun hits the building?

4.2 Exact Values for Inverse Sine, Cosine, and Tangent

Here you'll learn to find the angle for common values of inverse trigonometric functions.

You are working with a triangular brace in shop class. The brace is a right triangle, and the length of one side of the bracket is $\sqrt{3} \approx 1.732$ and it is connected to the other side at a right angle. The length of the other side is 1. You need to find the angle that the third piece makes with the first piece, labelled below as "C":

Can you find the angle between the legs of the brace?

By the time you finish reading this Concept, you'll be able to answer this question.
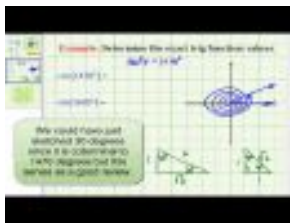
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James Sousa Example:Determine Trig FunctionValues Using Reference Triangles

Guidance

Inverse trig functions can be useful in a variety of math problems for finding angles that you need to know. In many cases, such as angles involving multiples of $30°, 60°$ and $90°$, the values of trig functions are often memorized, since they are used so often.

Recall the unit circle and the critical values. With the inverse trigonometric functions, you can find the angle value (in either radians or degrees) when given the ratio and function. Make sure that you find all solutions within the given interval.

Example A

Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

Solution: This is a value from the special right triangles and the unit circle.

Recall that $-\frac{\sqrt{3}}{2}$ is from the $30 - 60 - 90$ triangle. The reference angle for sin and $\frac{\sqrt{3}}{2}$ would be $60°$. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

Example B

Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

Solution: This is a value from the special right triangles and the unit circle.

$-\frac{\sqrt{2}}{2}$ is from an isosceles right triangle. The reference angle is then $45°$. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

Example C

Find the exact value of the expression without a calculator, in $[0, 2\pi)$.

$\tan^{-1}\sqrt{3}$

Solution: This is a value from the special right triangles and the unit circle.

$\sqrt{3}$ is also from a $30-60-90$ triangle. Tangent is $\sqrt{3}$ for the reference angle $60°$. Tangent is positive in the first and third quadrants, so the answer would be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.

Vocabulary

Trigonometric Inverse: The **trigonometric inverse** is a function that undoes a trig function to give the original argument of the function. It can also be used to find an angle from the ratio of two sides.

Guided Practice

1. Find the exact value of the inverse function of $\cos^{-1}(0)$, without a calculator in $[0, 2\pi)$

2. Find the exact value of the inverse function of $\tan^{-1}\left(-\sqrt{3}\right)$, without a calculator in $[0, 2\pi)$

3. Find the exact value of the inverse function of $\sin^{-1}\left(-\frac{1}{2}\right)$, without a calculator in $[0, 2\pi)$

Solutions:

1. $\frac{\pi}{2}, \frac{3\pi}{2}$
2. $\frac{2\pi}{3}, \frac{5\pi}{3}$
3. $\frac{11\pi}{6}, \frac{7\pi}{6}$

Concept Problem Solution

Using your knowledge of the values of trig functions for angles, you can work backward to find the angle that the brace makes:

$$\tan C = \frac{1}{\sqrt{3}}$$
$$\tan^{-1} C = \tan^{-1}\frac{1}{\sqrt{3}}$$
$$C = 60°$$

Practice

Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

1. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
2. $\cos^{-1}\left(\frac{1}{2}\right)$
3. $\sin^{-1}(1)$
4. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
5. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
6. $\tan^{-1}(-1)$
7. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
8. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
9. $\csc^{-1}\left(\sqrt{2}\right)$
10. $\sec^{-1}(-2)$
11. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$
12. $\sec^{-1}\left(\frac{2\sqrt{3}}{2}\right)$
13. $\csc^{-1}\left(-\frac{2\sqrt{3}}{2}\right)$
14. $\cot^{-1}\left(-\sqrt{3}\right)$
15. $\cot^{-1}(-1)$

4.3 Inverse of Functions through Algebraic Manipulation

Here you'll learn how to find the inverse of functions through algebraic manipulation.

If you were given a function, such as $f(x) = \frac{2x}{x+7}$, can you tell if the function has an inverse? Is there a way that you could find its inverse through algebraic manipulation?

Read on, and you'll find out how to do just that in this Concept.

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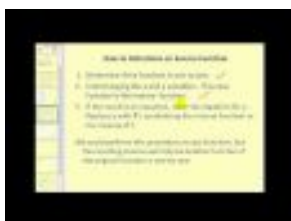
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James Sousa:InverseFunctions

Guidance

An "inverse" is something that undoes a function, giving back the original argument. For example, a function such as $y = \frac{1}{3}x$ has an inverse function of $y = 3x$, since any value placed into the first function will be returned as what it originally was if it is input into the second function. In this case, it is easy to see that to "undo" multiplication by $\frac{1}{3}$, you should multiply by 3. However, in many cases it may not be easy to infer by examination what the inverse of a function is.

To start, let's examine what is required for a function to have an inverse. It is important to remember that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function is one-to-one when its graph passes both the vertical and the horizontal line test. This means that every vertical and horizontal line will intersect the graph in exactly one place.

This is the graph of $f(x) = \frac{x}{x+1}$. The graph suggests that f is one-to-one because it passes both the vertical and the horizontal line tests. To find the inverse of f, switch **the x and y and solve for** y.

First, switch x and y.

$$x = \frac{y}{y+1}$$

Next, multiply both sides by $(y+1)$.

$$(y+1)x = \frac{y}{y+1}(y+1)$$
$$x(y+1) = y$$

Then, apply the distributive property and put all the y terms on one side so you can pull out the y.

$$xy + x = y$$
$$xy - y = -x$$
$$y(x - 1) = -x$$

Divide by $(x - 1)$ to get y by itself.

$$y = \frac{-x}{x - 1}$$

Finally, multiply the right side by $\frac{-1}{-1}$.

$$y = \frac{x}{1 - x}$$

Therefore the inverse of f is $f^{-1}(x) = \frac{x}{1-x}$.

The symbol f^{-1} is read "f inverse" and is not the reciprocal of f.

Example A

Find the inverse of $f(x) = \frac{1}{x-5}$ algebraically.

Solution: To find the inverse algebraically, switch $f(x)$ to y and then switch x and y.

$$y = \frac{1}{x - 5}$$
$$x = \frac{1}{y - 5}$$
$$x(y - 5) = 1$$
$$xy - 5x = 1$$
$$xy = 5x + 1$$
$$y = \frac{5x + 1}{x}$$

Example B

Find the inverse of $f(x) = 5 \sin^{-1}\left(\frac{2}{x-3}\right)$

Solution:

a.

$$f(x) = 5\sin^{-1}\left(\frac{2}{x-3}\right)$$
$$x = 5\sin^{-1}\left(\frac{2}{y-3}\right)$$
$$\frac{x}{5} = \sin^{-1}\left(\frac{2}{y-3}\right)$$
$$\sin\frac{x}{5} = \left(\frac{2}{y-3}\right)$$
$$(y-3)\sin\frac{x}{5} = 2$$
$$(y-3) = \frac{2}{\sin\frac{x}{5}}$$
$$y = \frac{2}{\sin\frac{x}{5}} + 3$$

Example C

Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x+4)$

Solution:

$$x = 4\tan^{-1}(3y+4)$$
$$\frac{x}{4} = \tan^{-1}(3y+4)$$
$$\tan\frac{x}{4} = 3y+4$$
$$\tan\frac{x}{4} - 4 = 3y$$
$$\frac{\tan\frac{x}{4} - 4}{3} = y$$
$$f^{-1}(x) = \frac{\tan\frac{x}{4} - 4}{3}$$

Vocabulary

Horizontal Line Test: The **horizontal line test** is a test applied to a function to see how many times the graph of a function intersects an arbitrary horizontal line drawn across the coordinate grid. A function passes this test if it intersects the horizontal line in only one place, no matter where the horizontal line is drawn.

One to One Function: A **one to one function** is a function that passes both the horizontal and vertical line tests.

Vertical Line Test: The **vertical line test** is a test applied to a function to see how many times the graph of a function intersects an arbitrary vertical line drawn across the coordinate grid. A function passes this test if it intersects the vertical line in only one place, no matter where the vertical line is drawn.

Guided Practice

1. Find the inverse of $f(x) = 2x^3 - 5$

2. Find the inverse of $y = \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right)$

3. Find the inverse of $g(x) = 2\sin(x-1) + 4$

Solutions:

1.

$$
\begin{aligned}
f(x) &= 2x^3 - 5 \\
y &= 2x^3 - 5 \\
x &= 2y^3 - 5 \\
x + 5 &= 2y^3 \\
\frac{x+5}{2} &= y^3 \\
\sqrt[3]{\frac{x+5}{2}} &= y
\end{aligned}
$$

2.

$$
\begin{aligned}
y &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right) \\
x &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}y - 5\right) \\
3x &= \tan^{-1}\left(\frac{3}{4}y - 5\right) \\
\tan(3x) &= \frac{3}{4}y - 5 \\
\tan(3x) + 5 &= \frac{3}{4}y \\
\frac{4(\tan(3x) + 5)}{3} &= y
\end{aligned}
$$

3.

$$
\begin{aligned}
g(x) &= 2\sin(x-1) + 4 \\
y &= 2\sin(x-1) + 4 \\
x &= 2\sin(y-1) + 4 \\
x - 4 &= 2\sin(y-1) \\
\frac{x-4}{2} &= \sin(y-1) \\
\sin^{-1}\left(\frac{x-4}{2}\right) &= y - 1 \\
1 + \sin^{-1}\left(\frac{x-4}{2}\right) &= y
\end{aligned}
$$

Concept Problem Solution

Since the original function is:

$f(x) = y = \frac{2x}{x+7}$

You can first switch all of the "x" and "y" values:

$x = \frac{2y}{y+7}$

You can then rearrange the equation and isolate "y":

$$x(y+7) = 2y$$
$$xy + 7x = 2y$$
$$xy - 2y = -7x$$
$$y(x-2) = -7x$$
$$y = \frac{-7x}{x-2}$$

The inverse function is written as $f^{-1}(x) = \frac{-7x}{x-2}$

Practice

Find the inverse of each function.

1. $f(x) = 3x + 5$
2. $g(x) = 0.2x - 7$
3. $h(x) = 0.1x^2$
4. $k(x) = 5x + 6$
5. $f(x) = \sqrt{x-4}$
6. $g(x) = (x)^{\frac{1}{3}} + 1$
7. $h(x) = (x+1)^3$
8. $k(x) = \frac{x^2}{3}$
9. $f(x) = -2 + 4\sin^{-1}(x+7)$
10. $g(x) = 1 + 3\tan^{-1}(2x+1)$
11. $h(x) = 4\cos^{-1}(3x)$
12. $k(x) = -1\tan^{-1}(6x)$
13. $j(x) = 5 + 2\sin^{-1}(x+5)$
14. $m(x) = -2\tan(3x+1)$
15. $p(x) = 5 - 6\sin(\frac{x}{2})$

4.4 Inverses by Mapping

Here you'll learn to find the inverse of a function through a process of plotting points of the function.

Your instructor gives you a function, $f(x) = (x-1)^2 + 3$, where $x \geq 1$, and asks you to find the inverse. You are all set to start manipulating the equation, when your Instructor specifies that you should find the inverse by graphing instead of by algebraic manipulation.

Are you able to do this?

Keep reading, and by the end of this Concept, you'll be able to find the inverse of the function through graphing instead of algebra.

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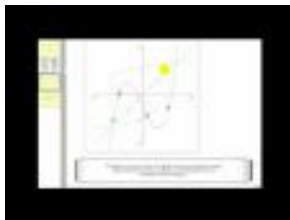
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James Sousa:Animation Inverse Function

Guidance

Determining an inverse function algebraically can be both involved and difficult, so it is useful to know how to map f to f^{-1}. The graph of f can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a,b) and (b,a) in the coordinate plane are symmetric with respect to the line $y = x$.

The points (a,b) and (b,a) are reflections of each other across the line $y = x$.

Example A

Find the inverse of $f(x) = \frac{1}{x-5}$ by mapping.

Solution: From the last section, we know that the inverse of this function is $y = \frac{5x+1}{x}$. To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.

A: (4, -1)

B: (4.8, -5)

C: (2, -0.3)

D: (0, -0.2)

E: (5.3, 3.3)

F: (6, 1)

G: (8, 0.3)

H: (11, 0.2)

Now, take these eight points, switch the x and y and plot (y,x). Connect them to make the inverse function.

$A^{-1}: (-1,4)$

$B^{-1}: (-5,4.8)$

$C^{-1}: (-0.3,2)$

$D^{-1}: (-0.2,0)$

$E^{-1}: (3.3,5.3)$

$F^{-1}: (1,6)$

$G^{-1}: (0.3,8)$

$H^{-1}: (0.2,11)$

Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Example B

Find the inverse of $f(x) = x^2 - 4$.

Solution: Let's use the graphic approach for this one. The function is graphed in blue and its inverse is red.

Clearly, the inverse relation is not a function because it does not pass the vertical line test. This is because all parabolas fail the horizontal line test. To "make" the inverse a function, we restrict the domain of the original function. For parabolas, this is fairly simple. To find the inverse of this function algebraically, we get $f^{-1}(x) = \sqrt{x+4}$. Technically, however, the inverse is $\pm\sqrt{x+4}$ because the square root of any number could be positive or negative. So, the inverse of $f(x) = x^2 - 4$ is both parts of the square root equation, $\sqrt{x+4}$ and $-\sqrt{x+4}$. $\sqrt{x+4}$ will yield the top portion of the horizontal parabola and $-\sqrt{x+4}$ will yield the bottom half. Be careful, because if you just graph $f^{-1}(x) = \sqrt{x+4}$ in your graphing calculator, it will only graph the top portion of the inverse.

This technique of sectioning the inverse is applied to finding the inverse of trigonometric functions because it is periodic.

Example C

Find the inverse of $f(x) = \frac{x-1}{3x+2}$ mapping.

Solution: To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.

A: (0, -.5)

B: (-1, 2)

C: (1, 0)

D: (-2, .75)

E: (2, .125)

F: (-3, .57)

G: (3, .18)

Now, take these seven points, switch the x and y and plot (y,x). Connect them to make the inverse function.

$A^{-1} : (-.5, 0)$

$B^{-1} : (2, -1)$

$C^{-1} : (0, 1)$

$D^{-1} : (.75, -2)$

$E^{-1} : (.125, 2)$

$F^{-1} : (.57, -3)$

$G^{-1} : (.18, 3)$

Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Vocabulary

Inverse Function: An **inverse function** is a function that undoes another function.

Mapping: Mapping is a procedure involving the plotting of points on a coordinate grid to see the behavior of a function.

Guided Practice

1.Study the following graph and answer these questions:

(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

2. Find the inverse of $f(x) = x^2 + 2x - 15$ using the mapping principle.

3. Find the inverse of $y = 1 + 2\sin x$ using the mapping principle.

Solutions:

1. The graph represents a one-to-one function. It passes both a vertical and a horizontal line test. The inverse would be a function.

2. By selecting 4-5 points and switching the x and y values, you will get the red graph below.

3. By selecting 4-5 points and switching the x and y values, you will get the red graph below.

Concept Problem Solution

The original equation is $f(x) = (x-1)^2 + 3$.

Here is a plot of the function:

Notice that the domain of the function under examination has to be restricted to 1 or greater, so that the function will pass the horizontal line test. Some points that are on this graph are:

A: (1, 3)

B: (2, 4)

C: (3, 7)

To map the inverse function, first take each point and switch the "x" and "y" values:

$A^{-1} : (3, 1)$

$B^{-1} : (4,2)$

$C^{-1} : (7,3)$

Then connect these dots, and you can see the graph of the inverse function. The inverse function graph looks like this:

In this case, the range of the function has to be restricted to be 1 or greater, so that the inverse function will pass the vertical line test.

Practice

For each of the following graphs answer these questions:

(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

1.
2.
3.
4.
5.
6.

Find the inverse of each function using the mapping principle.

7. $y = x^2 + x - 2$
8. $y = x^3$
9. $y = \sin(x - \frac{\pi}{2})$
10. $y = \cos(2x)$
11. $y = \frac{1}{x}$
12. $y = x^2 - 9$
13. $y = -2 + \sin(\frac{1}{2}x)$
14. What type of points will be in the same place in both a function and its inverse?
15. When you graph a function and its inverse on the same set of axes, where is the line of reflection? Why?

4.5 Inverses of Trigonometric Functions

Here you'll learn to find the domain and range of inverse trigonometric functions by utilizing the reflection principle.

Your instructor gives you a trigonometric function, $f(x) = 3\sin(x) + 5$, and asks you to find the inverse. You are all set to start manipulating the equation, when you realize that you don't know just how to do this. Your instructor suggests that you try finding the inverse through graphing instead.

Are you able to do this?

Keep reading, and by the end of this Concept, you'll be able to find the inverse of trig function and others through graphing instead of algebra.

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James Sousa:Introduction toInverseSine, Inverse Cosine, and Inverse Tangent

Guidance

In other Concepts, two different ways to find the inverse of a function were discussed: graphing and algebra. However, when finding the inverse of trig functions, it is easy to find the inverse of a trig function through graphing. Consider the graph of a sine function shown here:

In order to consider the inverse of this function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of f is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ a new function $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ takes on all values once and only once.

The inverse of $f(x)$ is represented by the symbol $f^{-1}(x)$, and $y = f^{-1}(x) \Leftrightarrow f(y) = x$. The inverse of $\sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ will be written as $\sin^{-1} x$. or $\arcsin x$.

$$\left.\begin{cases} y = \sin^{-1} x \\ \qquad or \\ y = \arcsin x \end{cases}\right\} \Leftrightarrow \sin y = x$$

In this Concept we will use both $\sin^{-1} x$ and $\arcsin x$ and both are read as "the inverse sine of x" or "the number between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ whose sine is x."

The graph of $y = \sin^{-1} x$ is obtained by applying the inverse reflection principle and reflecting the graph of $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ in the line $y = x$. The domain of $y = \sin x$ becomes the range of $y = \sin^{-1} x$, and hence the range of $y = \sin x$ becomes the domain of $y = \sin^{-1} x$.

Another way to view these graphs is to construct them on separate grids. If the domain of $y = \sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse sine function $y = \sin^{-1} x$ is the inverse of the restricted section of the sine function.

The domain of $y = \sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range is [-1, 1].

The restriction of $y = \sin x$ is a one-to-one function and it has an inverse that is shown below.

The domain of $y = \sin^{-1}$ is [-1, 1] and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse functions for cosine and tangent are defined by following the same process as was applied for the inverse sine function. However, in order to create one-to-one functions, different intervals are used. The cosine function is restricted to the interval $0 \leq x \leq \pi$ and the new function becomes $y = \cos x, 0 \leq x \leq \pi$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$ The result is the graph of $y = \cos^{-1} x$ (also expressed as $y = \arccos x$).

Again, construct these graphs on separate grids to determine the domain and range. If the domain of $y = \cos x$ is restricted to the interval $[0, \pi]$, the result is a restricted one-to one function. The inverse cosine function $y = \cos^{-1} x$ is the inverse of the restricted section of the cosine function.

The domain of $y = \cos x$ is $[0, \pi]$ and the range is [-1, 1].

The restriction of $y = \cos x$ is a one-to-one function and it has an inverse that is shown below.

The statements $y = \cos x$ and $x = \cos y$ are equivalent for $y-$values in the restricted domain $[0, \pi]$ and $x-$values between -1 and 1.

The domain of $y = \cos^{-1} x$ is [-1, 1] and the range is $[0, \pi]$.

The tangent function is restricted to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the new function becomes $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$. The result is the graph of $y = \tan^{-1} x$ (also expressed as $y = \arctan x$).

Graphing the two functions separately will help us to determine the domain and range. If the domain of $y = \tan x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse tangent function $y = \tan^{-1} x$ is the inverse of the restricted section of the tangent function.

The domain of $y = \tan x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range is $[-\infty, \infty]$.

The restriction of $y = \tan x$ is a one-to-one function and it has an inverse that is shown below.

The statements $y = \tan x$ and $x = \tan y$ are equivalent for $y-$values in the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x-$values between -4 and +4.

The domain of $y = \tan^{-1} x$ is $[-\infty, \infty]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The above information can be readily used to evaluate inverse trigonometric functions without the use of a calculator. These calculations are done by applying the restricted domain functions to the unit circle. To summarize:

TABLE 4.1:

Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
$y = \sin x$		$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	[-1, 1]	1 AND 4
	$y = \arcsin x$ $y = \sin^{-1} x$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
$y = \cos x$		$[0, \pi]$	[-1, 1]	1 AND 2
	$y = \arccos x$ $y = \cos^{-1} x$	[-1, 1]	$[0, \pi]$	
$y = \tan x$		$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$	1 AND 4

TABLE 4.1: (continued)

Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
	$y = \arctan x$ $y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	

Now that the three trigonometric functions and their inverses have been summarized, let's take a look at the graphs of these inverse trigonometric functions.

Example A

Establish an alternative domain that makes $y = sin(x)$ a one to one function.

Solution: Any number of possible solutions can be given, but the important point is that the function must pass the "horizontal line test" and the "vertical line test". This means that a horizontal line drawn through the graph will intersect the function in only one place, and a vertical line drawn through the graph will intersect the function in only one place.

For the sine curve, this means that the function can't "turn over" or "go in the other direction", since then it couldn't pass the horizontal line test. So any part of the function that starts at the bottom of the "y" values and stops at the top of the "y" values will work. (Any value that starts at the top of the "y" values and stops at the bottom of the "y" values will work as well.

In this example, you can see that the function starts at $\frac{\pi}{2}$ and stops at $\frac{3\pi}{2}$.

Example B

Find the range of the function given in Example A

Solution: You can see that the function still has the same "y" range of values, since the function $y = \sin x$ moves up and down between -1 and 1. Therefore, the range is $-1 \leq y \leq 1$.

Example C

Find the domain and range of the inverse of the function given in Example A.

Solution: Since the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function, you only have to take the "x" and "y" values of the original function and reverse them to get the domain and range range of the inverse function.

Therefore, the domain of $y = \sin^{-1} x$ as described in Example A is $-1 \leq x \leq 1$ and the range is $\frac{\pi}{2} \leq y \leq \frac{3\pi}{2}$.

Vocabulary

Inverse Function: An **inverse function** is a function that undoes another function.

Guided Practice

1. Sketch a graph of $y = \frac{1}{2} \cos^{-1}(3x + 1)$. Sketch $y = \cos^{-1} x$ on the same set of axes and compare how the two differ.

2. Sketch a graph of $y = 3 - \tan^{-1}(x-2)$. Sketch $y = \tan^{-1} x$ on the same set of axes and compare how the two differ.

3. Graph $y = 2\sin^{-1}(2x)$

Solutions:

1.

$y = \frac{1}{2}\cos^{-1}(3x+1)$ is in blue and $y = \cos^{-1}(x)$ is in red. Notice that $y = \frac{1}{2}\cos^{-1}(3x+1)$ has half the amplitude and is shifted over -1. The 3 seems to narrow the graph.

2.

$y = 3 - \tan^{-1}(x-2)$ is in blue and $y = \tan^{-1} x$ is in red. $y = 3 - \tan^{-1}(x-2)$ is shifted up 3 and to the right 2 (as indicated by point C, the "center") and is flipped because of the $-\tan^{-1}$.

3.

Concept Problem Solution

To find the inverse of this function through graphing, first restrict the domain of the function so that it is one to one. A graph of $f(x) = 3\sin(x) + 5$, restricted so that the domain is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ looks like this:

If you apply the inverse reflection principle, you can see that the inverse of this function looks like this:

Practice

1. Why does the domain of a trigonometric function have to be restricted in order to find its inverse function?
2. If the domain of $f(x) = \cos(x)$ is $[0, \pi]$, what is the domain and range of $f^{-1}(x)$?.
3. If the domain of $g(x) = \sin(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, what is the domain and range of $g^{-1}(x)$?.
4. Establish an alternative domain that makes $y = \cos(x)$ a function.
5. What is the domain and range of the inverse of the function from the previous problem.
6. Establish an alternative domain that makes $y = \tan(x)$ a function.
7. What is the domain and range of the inverse of the function from the previous problem.

Sketch a graph of each function. Use the domains presented in this concept.

8. $y = 2\sin^{-1}(3x-1)$
9. $y = -3 + \cos^{-1}(2x)$
10. $y = 1 + 2\tan^{-1}(x+2)$
11. $y = 4\sin^{-1}(x-4)$
12. $y = 2 + \cos^{-1}(x+3)$
13. $y = 1 + \cos^{-1}(2x-3)$
14. $y = -3 + \tan^{-1}(3x+1)$
15. $y = -1 + 2\sin^{-1}(x+5)$

4.6 Composition of Trig Functions and Their Inverses

Here you'll learn what compositions of functions are and how to compute them.

You've considered trigonometric functions, and you've considered inverse functions, and now it's time consider how to compose trig functions and their inverses. If someone were to ask you to apply the inverse of a trig function to a different trig function, would you be able to do this? For example, can you find $\sin^{-1}(\cos(\frac{3\pi}{2}))$?

When you complete this Concept, you'll be able to solve this problem.

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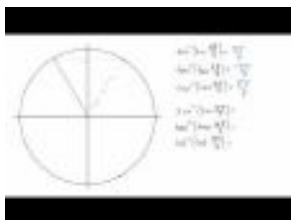
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Inverse trigfunctions: Composition

Guidance

In other Concepts, you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad\qquad \cos(\cos^{-1}(x)) = x \qquad\qquad \tan(\tan^{-1}(x)) = x$$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig. function of a trig. function, will only be true for values of x within the restricted domains.

$$\sin^{-1}(\sin(x)) = x \qquad\qquad \cos^{-1}(\cos(x)) = x \qquad\qquad \tan^{-1}(\tan(x)) = x$$

These equations are better known as composite functions. However, it is not necessary to only have a function and its inverse acting on each other. In fact, is is possible to have composite function that are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.

When solving these types of problems, start with the function that is composed inside of the other and work your way out. Use the following examples as a guideline.

Example A

Find $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$.

Solution: We know that $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, within the defined restricted domain. Then, we need to find $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$. So, the above properties allow for a short cut. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$, think of it like the sine and sine inverse cancel each other out and all that is left is the $\frac{\sqrt{2}}{2}$.

Example B

Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution: For all of these types of problems, the answer is restricted to the inverse functions' ranges.

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$: First find $\tan^{-1}\sqrt{3}$, which is $\frac{\pi}{3}$. Then find $\cos\frac{\pi}{3}$. Your final answer is $\frac{1}{2}$. Therefore, $\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$.

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

Example C

Without using technology, find the exact value of each of the following:

a. $\cos(\tan^{-1}(-1))$

b. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

Solution: For all of these types of problems, the answer is restricted to the inverse functions' ranges.

a. $\cos(\tan^{-1}(-1)) = \cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

b. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Vocabulary

Composite Function: A **composite function** is a set of two different trigonometric functions applied to an argument in conjunction with one another.

Guided Practice

1. Find the exact value of $\cos^{-1}\frac{\sqrt{3}}{2}$, without a calculator, over its restricted domain.

2. Evaluate: $\sin\left(\cos^{-1}\frac{5}{13}\right)$

3. Evaluate: $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right)$

Solutions:

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1. $\frac{\pi}{6}$

2.

$$\cos\theta = \frac{5}{13}$$
$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta$$
$$\sin\theta = \frac{12}{13}$$

3. $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right) \rightarrow \sin\theta = -\frac{6}{11}$.

The third side is $b = \sqrt{121 - 36} = \sqrt{85}$.

$\tan\theta = -\frac{6}{\sqrt{85}} = -\frac{6\sqrt{85}}{85}$

Concept Problem Solution

To solve this problem: $\sin^{-1}(\cos(\frac{3\pi}{2}))$, you can work outward.

First find:

$\cos(\frac{3\pi}{2}) = 0$

Then find:

$\sin^{-1} 0 = 0$

or

$\sin^{-1} 0 = \pi$

Practice

Without using technology, find the exact value of each of the following.

1. $\sin\left(\sin^{-1}\frac{1}{2}\right)$
2. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$
3. $\tan\left(\tan^{-1}\sqrt{3}\right)$
4. $\cos\left(\sin^{-1}\frac{1}{2}\right)$
5. $\tan\left(\cos^{-1}1\right)$
6. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$
7. $\sin^{-1}\left(\sin\frac{\pi}{2}\right)$
8. $\cos^{-1}\left(\tan\frac{\pi}{4}\right)$
9. $\tan^{-1}\left(\sin\pi\right)$
10. $\sin^{-1}\left(\cos\frac{\pi}{3}\right)$
11. $\cos^{-1}\left(\sin-\frac{\pi}{4}\right)$
12. $\tan\left(\sin^{-1}0\right)$
13. $\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

14. $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$
15. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

4.7 Definition of Inverse Reciprocal Trig Functions

Here you'll learn inverse relationships for the secant, cosecant, and cotangent functions.

So far you've had to deal with trig functions, reciprocal functions, and inverse functions. Now you'll start to see inverse reciprocal functions. For example, can you compute

$$\sec^{-1} \frac{2}{\sqrt{3}}$$

As it turns out, this can be readily computed.

At the end of this Concept, you'll know how to compute this and other inverse reciprocal functions.

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Reciprocal Trig Functions

Guidance

We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$
\begin{aligned}
y &= \sin^{-1} x \\
x &= \sin y \\
\frac{1}{x} &= \csc y \\
\csc^{-1} \frac{1}{x} &= y \\
\csc^{-1} \frac{1}{x} &= \sin^{-1} x
\end{aligned}
$$

Because cosecant and secant are inverses, $\sin^{-1} \frac{1}{x} = \csc^{-1} x$ is also true.

The inverse reciprocal identity for cosine and secant can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid.

$$\sec^{-1} \frac{1}{x} = \cos^{-1} x \leftrightarrow \cos^{-1} \frac{1}{x} = \sec^{-1} x$$

Tangent and cotangent have a slightly different relationship. Recall that the graph of cotangent differs from tangent by a reflection over the $y-$axis and a shift of $\frac{\pi}{2}$. As an equation, this can be written as $\cot x = -\tan\left(x - \frac{\pi}{2}\right)$. Taking the inverse of this function will show the inverse reciprocal relationship between arccotangent and arctangent.

$$y = \cot^{-1} x$$
$$y = -\tan^{-1}\left(x - \frac{\pi}{2}\right)$$
$$x = -\tan\left(y - \frac{\pi}{2}\right)$$
$$-x = \tan\left(y - \frac{\pi}{2}\right)$$
$$\tan^{-1}(-x) = y - \frac{\pi}{2}$$
$$\frac{\pi}{2} + \tan^{-1}(-x) = y$$
$$\frac{\pi}{2} - \tan^{-1} x = y$$

Remember that tangent is an odd function, so that $\tan(-x) = -\tan(x)$. Because tangent is odd, its inverse is also odd. So, this tells us that $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$. To graph arcsecant, arccosecant, and arccotangent in your calculator you will use these conversion identities: $\sec^{-1} x = \cos^{-1}\frac{1}{x}, \csc^{-1} x = \sin^{-1}\frac{1}{x}, \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$. Note: It is also true that $\cot^{-1} x = \tan^{-1}\frac{1}{x}$.

Now, let's apply these identities to some problems that will give us an insight into how they work.

Example A

Evaluate $\sec^{-1}\sqrt{2}$

Solution: Use the inverse reciprocal property. $\sec^{-1} x = \cos^{-1}\frac{1}{x} \rightarrow \sec^{-1}\sqrt{2} = \cos^{-1}\frac{1}{\sqrt{2}}$. Recall that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, $\sec^{-1}\sqrt{2} = \cos^{-1}\frac{\sqrt{2}}{2}$, and we know that $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$.

Example B

Find the exact value of each expression within the restricted domain, without a calculator.

a. $\sec^{-1}\sqrt{2}$

b. $\cot^{-1}\left(-\sqrt{3}\right)$

c. $\csc^{-1}\frac{2\sqrt{3}}{3}$

Solution: For each of these problems, first find the reciprocal and then determine the angle from that.

a. $\sec^{-1}\sqrt{2} = \cos^{-1}\frac{\sqrt{2}}{2}$ From the unit circle, we know that the answer is $\frac{\pi}{4}$.

b. $\cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\sqrt{3}\right)$ From the unit circle, the answer is $\frac{5\pi}{6}$.

c. $\csc^{-1}\frac{2\sqrt{3}}{3} = \sin^{-1}\frac{\sqrt{3}}{2}$ Within our interval, there are is one answer, $\frac{\pi}{3}$.

Example C

Using technology, find the value in radian measure, of each of the following:

a. $\arcsin 0.6384$

b. $\arccos(-0.8126)$

c. $\arctan(-1.9249)$

Solution:

a.

b.

c.

Make sure that your calculator's MODE is RAD (radians).

Vocabulary

Inverse Function: An **inverse function** is a function that undoes another function.

Reciprocal Function: A **reciprocal function** is a function that when multiplied by the original function gives the number 1 as a result.

Guided Practice

1. $\sec^{-1}(-2)$

2. $\cot^{-1}(-1)$

3. $\csc^{-1}\left(\sqrt{2}\right)$

Solutions:

1. $\frac{2\pi}{3}$

2. $-\frac{\pi}{4}$

3. $\frac{\pi}{4}$

Concept Problem Solution

The original goal was to evaluate $\sec^{-1}\frac{2}{\sqrt{3}}$

You can start with the inverse reciprocal property:

$\sec^{-1}x = \cos^{-1}\frac{1}{x}$

Substituting in values for "x" gives:

$\sec^{-1}\frac{2}{\sqrt{3}} = \cos^{-1}\frac{1}{\frac{2}{\sqrt{3}}}$

This can be rewritten as:

$\cos^{-1}\frac{\sqrt{3}}{2}$

And

$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

Therefore,

$\sec^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{6}$

Practice

Using technology, find the value in radian measure, of each of the following.

1. $\sin^{-1}(.345)$
2. $\cos^{-1}(.87)$
3. $\csc^{-1}(4)$
4. $\sec^{-1}(2.32)$
5. $\cot^{-1}(5.2)$

Find the exact value of each expression within the restricted domain, without a calculator.

6. $\sec^{-1}(\frac{2\sqrt{3}}{3})$
7. $\csc^{-1}(1)$
8. $\cot^{-1}(\sqrt{3})$
9. $\csc^{-1}(2)$
10. $\sec^{-1}(\sqrt{2})$
11. $\cot^{-1}(1)$
12. $\cos^{-1}(\frac{1}{2})$
13. $\sec^{-1}(2)$
14. $\cot^{-1}(\frac{\sqrt{3}}{3})$
15. $\sin^{-1}(\frac{\sqrt{3}}{2})$

4.8 Composition of Inverse Reciprocal Trig Functions

Here you'll learn how to create compositions of the inverses of secant, cosecant, and cotangent functions.

Composing functions involves applying one function and then applying another function afterward. In the case of inverse reciprocal functions, you could create compositions of functions such as sec^{-1}, csc^{-1}, and cot^{-1}.

Consider the following problem:

$$\csc(\cot^{-1} \sqrt{3})$$

Can you solve this problem?

Keep reading, and at the conclusion of this Concept, you'll be able to do so.

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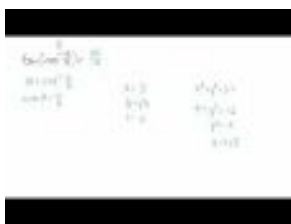
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HardTrig Inverse Composition Problems

Guidance

Just as you can apply one function and then another whenever you'd like, you can do the same with inverse reciprocal trig functions. This process is called composition. Here we'll explore some examples of composition for these inverse reciprocal trig functions by doing some problems.

Example A

Without a calculator, find $\cos\left(\cot^{-1} \sqrt{3}\right)$.

Solution:

First, find $\cot^{-1} \sqrt{3}$, which is also $\tan^{-1} \frac{\sqrt{3}}{3}$. This is $\frac{\pi}{6}$. Now, find $\cos \frac{\pi}{6}$, which is $\frac{\sqrt{3}}{2}$. So, our answer is $\frac{\sqrt{3}}{2}$.

Example B

Without a calculator, find $\sec^{-1} \left(\csc \frac{\pi}{3}\right)$.

Solution: First, $\csc \frac{\pi}{3} = \frac{1}{\sin \frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Then $\sec^{-1} \frac{2\sqrt{3}}{3} = \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

Example C

Evaluate $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Solution: Even though this problem is not a critical value, it can still be done without a calculator. Recall that sine is the opposite side over the hypotenuse of a triangle. So, 3 is the opposite side and 5 is the hypotenuse. This is a Pythagorean Triple, and thus, the adjacent side is 4. To continue, let $\theta = \sin^{-1}\frac{3}{5}$ or $\sin\theta = \frac{3}{5}$, which means θ is in the Quadrant 1 (from our restricted domain, it cannot also be in Quadrant II). Substituting in θ we get $\cos\left(\sin^{-1}\frac{3}{5}\right) = \cos\theta$ and $\cos\theta = \frac{4}{5}$.

Vocabulary

Inverse Function: An **inverse function** is a function that undoes another function.

Reciprocal Function: A **reciprocal function** is a function that when multiplied by the original function gives the number 1 as a result.

Guided Practice

1. Find the exact value of $\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ without a calculator, over its restricted domains.

2. Find the exact value of $\sec^{-1}(\tan(\cot^{-1}1))$ without a calculator, over its restricted domains.

3. Find the exact value of $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$ without a calculator, over its restricted domains.

Solutions:

1. $\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \csc\frac{\pi}{6} = 2$

2. $\sec^{-1}(\tan(\cot^{-1}1)) = \sec^{-1}\left(\tan\frac{\pi}{4}\right) = \sec^{-1}1 = 0$

3. $\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1}0 = 0$

Concept Problem Solution

The first step in this problem is to ask yourself "What angle would produce a cotangent of $\sqrt{2}$?"

Since values for "x" and "y" around the unit circle are all fractions, and cotangent is equal to $\frac{x}{y}$, you need to find a pair of equations on the unit circle which, when divided by each other, give $\sqrt{2}$ as the answer.

When looking around the unit circle, you can see that $\cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

Therefore, $\cot^{-1}\sqrt{3} = 30°$

Then you can apply the next function:

$\csc 30° = \frac{hypotenuse}{opposite} = \frac{1}{\frac{1}{2}} = 2$

And so

$\csc(\cot^{-1}\sqrt{3}) = 2$

Practice

Without using technology, find the exact value of each of the following. Use the restricted domain for each function.

1. $\sin\left(\sec^{-1}\sqrt{2}\right)$
2. $\cos\left(\csc^{-1}1\right)$
3. $\tan\left(\cot^{-1}\sqrt{3}\right)$
4. $\cos\left(\csc^{-1}2\right)$
5. $\cot\left(\cos^{-1}1\right)$
6. $\csc\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$
7. $\sec^{-1}\left(\cos\pi\right)$
8. $\cot^{-1}\left(\tan\frac{\pi}{4}\right)$
9. $\sec^{-1}\left(\csc\frac{\pi}{4}\right)$
10. $\csc^{-1}\left(\sec\frac{\pi}{3}\right)$
11. $\cos^{-1}\left(\cot-\frac{\pi}{4}\right)$
12. $\tan\left(\cot^{-1}0\right)$
13. $\sin\left(\csc^{-1}\frac{2\sqrt{3}}{3}\right)$
14. $\cot^{-1}\left(\sin\frac{\pi}{2}\right)$
15. $\cos\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right)$

4.9 Trigonometry in Terms of Algebra

Here you'll learn how to write trigonometric relationships and their compositions in terms of lengths of triangle sides.

You are babysitting your little cousin while doing your homework. While working on your trig functions, your cousin asks you what you are doing. While trying to explain sine, cosine, and tangent, your cousin is very confused. She doesn't understand what you mean by those words, but really wants to understand what the functions mean. Can you define the trig functions in terms of the relationships of sides for your little cousin?

By the end of this Concept, you'll understand how to do this.

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UnitCircle Definition of Trig Functions

Guidance

All of the trigonometric functions can be rewritten in terms of only x, when using one of the inverse trigonometric functions. Starting with tangent, we draw a triangle where the opposite side (from θ) is defined as x and the adjacent side is 1. The hypotenuse, from the Pythagorean Theorem would be $\sqrt{x^2 + 1}$. Substituting $\tan^{-1} x$ for θ, we get:

$$\tan\theta = \frac{x}{1}$$

$$\tan\theta = x \qquad\qquad hypotenuse = \sqrt{x^2 + 1}$$

$$\theta = \tan^{-1} x$$

$$\sin(\tan^{-1} x) = \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad\qquad \csc(\tan^{-1} x) = \csc\theta = \frac{\sqrt{x^2+1}}{x}$$

$$\cos(\tan^{-1} x) = \cos\theta = \frac{1}{\sqrt{x^2+1}} \qquad\qquad \sec(\tan^{-1} x) = \sec\theta = \sqrt{x^2+1}$$

$$\tan(\tan^{-1} x) = \tan\theta = x \qquad\qquad \cot(\tan^{-1} x) = \cot\theta = \frac{1}{x}$$

Example A

Find $\sin(\tan^{-1} 3x)$.

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Solution: Instead of using x in the ratios above, use $3x$.

$$\sin(\tan^{-1} 3x) = \sin\theta = \frac{3x}{\sqrt{(3x)^2 + 1}} = \frac{3x}{\sqrt{9x^2 + 1}}$$

Example B

Find $\sec^2(\tan^{-1} x)$.

Solution: This problem might be better written as $[\sec(\tan^{-1} x)]^2$. Therefore, all you need to do is square the ratio above.

$$[\sec(\tan^{-1} x)]^2 = \left(\sqrt{x^2 + 1} \right)^2 = x^2 + 1$$

You can also write the all of the trig functions in terms of arcsine and arccosine. However, for each inverse function, there is a different triangle. You will derive these formulas in the exercise for this section.

Example C

Find $\csc^3(\tan^{-1} 4x)$.

Solution: This example is similar to both of the examples above. First, use $4x$ instead of x in the ratios above. Second, the $\csc^3$ is the same as taking the csc function and cubing it.

$$[\csc(\tan^{-1} 4x)]^3 = \left(\frac{\sqrt{(4x)^2 + 1}}{x} \right)^3 = \frac{(16x^2 + 1)^{3/2}}{x^3}$$

Vocabulary

Trigonometric Function: A **trigonometric function** is a function describing a relationship between two sides of a triangle.

Guided Practice

1. Express $\cos^2(\tan^{-1} x)$ as an algebraic expression involving no trigonometric functions.

2. Express $\cot(\tan^{-1} x^2)$ as an algebraic expression involving no trigonometric functions.

3. To find trigonometric functions in terms of sine inverse, use the following triangle.

Determine the sine, cosine and tangent in terms of arcsine. Find $\tan(\sin^{-1} 2x^3)$.

Solutions:

1. $\frac{1}{x^2 + 1}$

2. $\frac{1}{x^2}$

3. The adjacent side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

$$\sin(\sin^{-1}x) = \sin\theta = x$$
$$\cos(\sin^{-1}x) = \cos\theta = \sqrt{1-x^2}$$
$$\tan(\sin^{-1}x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\sin^{-1}(2x^3)) = \frac{2x^3}{\sqrt{1-(2x^3)^2}} = \frac{2x^3}{\sqrt{1-4x^6}}$$

Concept Problem Solution

As it turns out, it's very easy to explain trig functions in terms of ratios. If you look at the unit circle

you can see that each trig function can be represented as a ratio of two sides. The value of any trig function can be represented as the length of one of the sides of the triangle (shown with two red sides and the black hypotenuse) divided by the length of one of the other sides. In fact, you should explain to your cousin, the words like "sine", "cosine", and "tangent" are just conveniences in this case to describe relationships that keep coming up over and over again. It would be possible to just describe the trig functions in terms of relationships of one side to another, if you'd like.

Using the sides of a triangle made on the unit circle, if the side opposite the angle is called "x":

$$\sin = \frac{opposite}{hypotenuse} = \frac{x}{\sqrt{x^2+1}}$$
$$\cos = \frac{adjacent}{hypotenuse} = \frac{1}{\sqrt{x^2+1}}$$
$$\tan = \frac{opposite}{adjacent} = \frac{x}{1}$$

So as you can see, since trig functions are really just relationships between sides, it is possible to work with them in whatever form you want; either in terms of the usual "sine", "cosine" and "tangent", or in terms of algebra.

Practice

Rewrite each expression as an algebraic expression involving no trigonometric functions.

1. $\sin(\tan^{-1}5x)$
2. $\cos(\tan^{-1}2x^2)$
3. $\cot(\tan^{-1}3x^2)$
4. $\sin(\cos^{-1}x)$
5. $\sin(\cos^{-1}3x)$
6. $\cos(\sin^{-1}2x^2)$
7. $\csc(\cos^{-1}x)$
8. $\sec(\sin^{-1}x)$
9. $\cos^2(\tan^{-1}3x^2)$
10. $\sin(\sec^{-1}x)$
11. $\cos(\csc^{-1}x)$

12. $\sin(\tan^{-1} 2x^3)$
13. $\cos(\sin^{-1} 3x)$
14. $\sin(\sec^{-1} x)$
15. $\cos(\cot^{-1} x)$

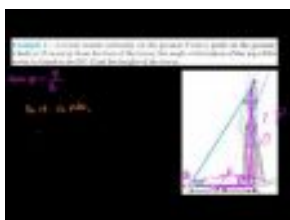# 4.10 Applications of Inverse Trigonometric Functions

Here you'll learn how to apply inverse trig functions to solve problems you would see in real life situations.

You are mapping out a hiking trip to a new spot in the state park near your home. On your map, you know that you are supposed to go on a course which ends with you having moved 2.5 miles East and 3 miles South. Your map, with the start and end points, is shown here:

Now you need to calculate the angle that you need to walk with respect to due East. Can you find a way to calculate this angle using inverse trig functions?

Read on, and when you complete this Concept, you'll be able to determine the answer.

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Height and Distance WordProblemApplicationof Trigonometry

Guidance

The following problems are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements.

Example A

On a cold winter day the sun streams through your living room window and causes a warm, toasty atmosphere. This is due to the angle of inclination of the sun which directly affects the heating and the cooling of buildings. Noon is when the sun is at its maximum height in the sky and at this time, the angle is greater in the summer than in the winter. Because of this, buildings are constructed such that the overhang of the roof can act as an awning to shade the windows for cooling in the summer and yet allow the sun's rays to provide heat in the winter. In addition to the construction of the building, the angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90° -$ latitude $+ -23.5° \cdot \cos\left[(N+10)\frac{360}{365}\right]$ where N represents the number of the day of the year that corresponds to the date of the year. Note: This formula is accurate to $\pm\frac{1°}{2}$

a. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $42°$, March 10th, the 69th day of the year.

Solution:

$$\text{Angle of sun} = 90° - 42° + -23.5° \cdot \cos\left[(69+10)\frac{360}{365}\right]$$

$$\text{Angle of sun} = 48° + -23.5°(0.2093)$$

$$\text{Angle of sun} = 48° - 4.92°$$

$$\text{Angle of sun} = 43.08°$$

b. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $20°$, September 21st.

Solution:

$$\text{Angle of sun} = 90° - 20° + -23.5° \cdot \cos\left[(264+10)\frac{360}{365}\right]$$

$$\text{Angle of sun} = 70° + -23.5°(0.0043)$$

$$\text{Angle of sun} = 70.10°$$

Example B

A tower, 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower. What angle will the guy wire make with the ground?

Solution: Draw a picture.

$$\tan\theta = \frac{opp.}{adj.}$$

$$\tan\theta = \frac{28.4}{5}$$

$$\tan\theta = 5.68$$

$$\tan^{-1}(\tan\theta) = \tan^{-1}(5.68)$$

$$\theta = 80.02°$$

The following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$. In addition, technology will also be used to complete the solution.

Example C

In the main concourse of the local arena, there are several viewing screens that are available to watch so that you do not miss any of the action on the ice. The bottom of one screen is 3 feet above eye level and the screen itself is 7 feet high. The angle of vision (inclination) is formed by looking at both the bottom and top of the screen.

a. Sketch a picture to represent this problem.

b. Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does the maximum value for the angle of vision occur?

Solution:

a.

b.

$$\theta_2 = \tan\theta - \tan\theta_1$$

$$\tan\theta = \frac{10}{x} \text{ and } \tan\theta_1 = \frac{3}{x}$$

$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.

From the graph, it can be seen that the maximum occurs when $x \approx 5.59$ $ft.$ and $\theta \approx 32.57°$.

Vocabulary

Inverse Trigonometric Function: An **inverse trigonometric function** is a function that undoes a trigonometric function to return the original angle used as the argument.

Guided Practice

1. The intensity of a certain type of polarized light is given by the equation $I = I_0 \sin 2\theta \cos 2\theta$. Solve for θ.

2. The following diagram represents the ends of a water trough. The ends are actually isosceles trapezoids, and the length of the trough from end-to-end is ten feet. Determine the maximum volume of the trough and the value of θ that maximizes that volume.

3. A boat is docked at the end of a 10 foot pier. The boat leaves the pier and drops anchor 230 feet away 3 feet straight out from shore (which is perpendicular to the pier). What was the bearing of the boat from a line drawn from the end of the pier through the foot of the pier?

Solutions:

1.

$$I = I_0 \sin 2\theta \cos 2\theta$$

$$\frac{I}{I_0} = \frac{I_0}{I_0} \sin 2\theta \cos 2\theta$$

$$\frac{I}{I_0} = \sin 2\theta \cos 2\theta$$

$$\frac{2I}{I_0} = 2\sin 2\theta \cos 2\theta$$

$$\frac{2I}{I_0} = \sin 4\theta$$

$$\sin^{-1}\frac{2I}{I_0} = 4\theta$$

$$\frac{1}{4}\sin^{-1}\frac{2I}{I_0} = \theta$$

2. The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin\theta)(\cos\theta)$ and that of the rectangle is $(1)(\cos\theta)$. This means that

the volume can be determined by the function $V(\theta) = 10(\cos\theta + \sin\theta\cos\theta)$, and this function can be graphed as follows to find the maximum volume and the angle θ where it occurs.

Therefore, the maximum volume is approximately 13 cubic feet and occurs when θ is about $30°$.

3.

$$\cos x = \frac{7}{230} \to x = \cos^{-1}\frac{7}{230}$$
$$x = 88.26°$$

Concept Problem Solution

You can set up a triangle that matches the physical situation of this problem. Here is what it should look like:

Using the tangent function, you can solve for the angle you need to find:

$$\theta = \tan^{-1}\left(\frac{3}{-2.5}\right)$$
$$\theta = -50.19$$

Practice

1. The distance from a boat to a lighthouse is 100 feet and the lighthouse is 120 feet tall. What is the angle of depression from the top of the Lighthouse to the boat.
2. You are standing 100 feet from an arch that is 68 feet tall. At what angle do you have to look up to see the top of the arch? Assume you are 5 feet tall.
3. The angle of elevation of the top of a church to a point 100 feet away from the base is $60°$. Find the height of the church.

You are standing looking at a large painting on the wall. The bottom of the painting is 1 foot above your eye level. The painting is 10 feet tall. Assume you are standing x feet from the painting and that angle θ is formed by the lines of vision to the bottom and to the top of the painting.

4. Draw a picture to represent this situation.
5. Solve for θ in terms of x.
6. If you are standing 10 feet from the painting, what is θ?
7. If $\theta = 30°$, how far are you standing from the wall (to the nearest foot)?

You are watching a hot-air balloon that was 300 feet from you when it started rising from the ground. Assume the height of the balloon is x and θ is the angle of elevation from the ground where you are standing up to the balloon.

8. Solve for x in terms of θ.
9. Solve for θ in terms of x.
10. What is the angle of elevation when the hot-air balloon is 500 feet above the ground?
11. How high above the ground is the balloon when the angle of elevation is $80°$?

Recall that if the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90° -$ latitude $+ -23.5° \cdot \cos\left[(N+10)\frac{360}{365}\right]$ where N represents the number of the day of the year that corresponds to the date of the year.

12. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $30°$, April 12th, the 102th day of the year.
13. Determine the measurement of the sun's angle of inclination for a building located at a latitude of $50°$, August 14th, the 226th day of the year.

14. A tower, 50 feet high, is secured with a guy wire anchored 8 feet from the base of the tower. What angle will the guy wire make with the ground?
15. A 30 foot tall flagpole casts a 12 foot shadow. What is the angle that the sun hits the flagpole?

Summary

This Chapter discussed inverse trigonometric functions, including how to find them through graphing and algebraic means. Once the inverse functions were found, information about how to find compositions of inverse trig functions and inverse reciprocal functions were discussed. These topics were then presented in terms of Algebra.

The Chapter concluded with applications of inverse trig functions to real life situations.

CHAPTER **5**   **Triangles and Vectors**

Chapter Outline

Introduction

Finding unknown quantities in triangles, such as the lengths of sides and the measure of angles, is a critical portion of the study of trigonometry. This Chapter deals with how to find such unknown quantities in different cases where certain quantities are known.

Also introduced in this Chapter are vectors. Consider for a moment the idea that all things cannot be represented with only a number. For example, while you can count the number of pencils in your desk with a number, you cannnot completely describe the force you apply when pushing on your desk with just a number. This is because, in addition to the number describing the strength of the force, you need something describing the direction of the force (in this case, pushing down). Vectors are a way to describe these kinds of quantities; they have both a magnitude and a direction.

5.1 Sides of an Oblique Triangle

Here you'll learn to find the unknown side of a triangle using the Law of Cosines when two sides of the triangle and the angle between them are known.

You are playing a game called "Over the Line", where you stand at one corner of a triangle and hit a ball. The field looks like this:

Points are scored by hitting the ball so that it lands beyond the first line in the triangle, but before the second line.

Given that the angle on the left hand side of the triangle is $15°$, and the length of the sides of the triangle going to the first scoring line are 30 yards, can you calculate the length of the line you have to hit the ball past to score?

Watch This

James Sousa Example:Applicationof the Law of Cosines

Guidance

This Concept takes ideas that have only been applied to right triangles and interprets them so that they can be used for any type of triangle. First, the laws of sines and cosines take the Pythagorean Theorem and ratios and apply them to any triangle.

The Law of Cosines is a generalization of the Pythagorean Theorem, where the angle C is the angle between the two given sides of a triangle:

$$c^2 = a^2 + b^2 - 2(a)(b)\cos C$$

You'll notice that if this were a right triangle, $\cos C = \cos 90° = 0$, and so the third term would disappear, leaving the familiar Pythagorean Theorem.

One case where we can use the Law of Cosines is when we know two sides and the included angle in a triangle (SAS) and want to find the third side.

Example A

Using $\triangle DEF, \angle E = 12°, d = 18$, and $f = 16.8$. Find e.

Solution: Since $\triangle DEF$ isn't a right triangle, we cannot use the Pythagorean Theorem or trigonometry functions to find the third side. However, we can use our newly derived Law of Cosines.

$$e^2 = 18^2 + 16.8^2 - 2(18)(16.8)\cos 12 \qquad \text{Law of Cosines}$$
$$e^2 = 324 + 282.24 - 2(18)(16.8)\cos 12 \qquad \text{Simplify squares}$$
$$e^2 = 324 + 282.24 - 591.5836689 \qquad \text{Multiply}$$
$$e^2 = 14.6563311 \qquad \text{Add and subtract from left to right}$$
$$e \approx 3.8 \qquad \text{Square root}$$

*Note that the negative answer is thrown out as having no geometric meaning in this case.

Example B

An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order for a kitchen to be utilized effectively, these three amenities must form a triangle with each other. This is known as the "work triangle." By design, the three parts of the work triangle must be no less than 3 feet apart and no more than 7 feet apart103° angle with the stove and the refrigerator, will the distance between the stove and the refrigerator remain within the confines of the work triangle?

Solution: In order to find the distance from the sink to the refrigerator, we need to find side x. To find side x, we will use the Law of Cosines because we are dealing with an obtuse triangle (and thus have no right angles to work with). We know the length two sides: the sink to the stove and the sink to the refrigerator. We also know the included angle (the angle between the two known lengths) is $103°$. This means we have the SAS case and can apply the Law of Cosines.

$$x^2 = 3.6^2 + 5.7^2 - 2(3.6)(5.7)\cos 103 \qquad \text{Law of Cosines}$$
$$x^2 = 12.96 + 32.49 - 2(3.6)(5.7)\cos 103 \qquad \text{Simplify squares}$$
$$x^2 = 12.96 + 32.49 + 9.23199127 \qquad \text{Multiply}$$
$$x^2 = 54.68199127 \qquad \text{Evaluate}$$
$$x \approx 7.4 \qquad \text{Square root}$$

No, this triangle does not conform to the definition of a work triangle. The sink and the refrigerator are too far apart by 0.4 feet.

Example C

Using $\triangle JKL$, $\angle J = 2°$, $l = 25$, and $k = 27$. Find j.

Solution: Since $\triangle JKL$ isn't a right triangle, we cannot use the Pythagorean Theorem or trigonometry functions to find the third side. However, we can use our newly derived Law of Cosines.

$$j^2 = 25^2 + 27^2 - 2(25)(27)\cos 2 \qquad \text{Law of Cosines}$$
$$j^2 = 625 + 729 - 2(25)(27)\cos 2 \qquad \text{Simplify squares}$$
$$j^2 = 625 + 729 - 1349.18 \qquad \text{Multiply}$$
$$j^2 = 4.82 \qquad \text{Add and subtract from left to right}$$
$$j \approx 2.20 \qquad \text{Square root}$$

*Note that the negative answer is thrown out as having no geometric meaning in this case.

Vocabulary

Included Angle: The **included angle** in a triangle is the angle between two known sides.

Law of Cosines: The **law of cosines** is a rule involving the sides of an oblique triangle stating that the square of a side of the triangle is equal to the sum of the squares of the other two sides plus two times the lengths of the other two sides times the cosine of the angle opposite the side being computed.

Oblique Triangle: An **oblique triangle** is a triangle without a right angle as one of its internal angles.

Side Angle Side Triangle: A **side angle side triangle** is a triangle where two of the sides and the angle between them are known quantities.

Guided Practice

1. Find side "a" in this triangle, where $\angle A = 50°, b = 8, c = 11$

2. Find side "l" in this triangle where $\angle L = 79.5°, m = 22.4, p = 13.17$

3. Find side "b" in this triangle where $\angle B = 67.2°, d = 43, e = 39$

Solutions:

1. $a^2 = 8^2 + 11^2 - 2 \cdot 8 \cdot 11 \cdot \cos 50°, a \approx 8.5$

2. $l^2 = 22.4^2 + 13.17^2 - 2 \cdot 22.4 \cdot 13.17 \cdot \cos 79.5°, l \approx 23.8$

3. $b^2 = 39^2 + 43^2 - 2 \cdot 39 \cdot 43 \cdot \cos 67.2°, b \approx 45.5$

Concept Problem Solution

Since you know that the length of each of the other 2 sides is 30 yards, and the angle is $15°$, you can use the Law of Cosines to find the length of the third side.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
$$c^2 = 30^2 + 30^2 - (2)(30)(30)\cos 15°$$
$$c^2 = 900 + 900 - 1738.67$$
$$c^2 = 1800 - 1738.67$$
$$c^2 = 61.33$$
$$c \approx 7.83$$

Practice

1. State the Law of Cosines.

For each triangle below, state the values of a, b, and C.

2.
3.
4.
5.

6.

7.

8.

Now, for each triangle, solve for the missing side using the Law of Cosines.

9.

10.

11.

12.

13.

14.

15.

16. Prove that the Law of Cosines is equivalent to the Pythagorean Theorem for all right triangles.

5.2 Determination of Unknown Angles Using Law of Cosines

Here you'll learn to find an unknown angle in a triangle where the lengths of all three sides are known using the Law of Cosines.

You and a group of your friends are out in the country playing paintball. You have a playing field that is a triangle with sides of 50 meters, 50 meters, and 80 meters. You're trying to figure out what the angle is between the side that has a length of 50 meters and the other side that has a length of 50 meters. Is there a way to do this?

By the end of this Concept, you'll be able to calculate the unknown angle of a triangle using information about the sides.

Watch This
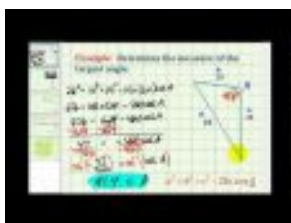
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James Sousa Example:Determine the Measure of anAngle of a TriangleGiven the Length ofThree Sides

Guidance

The Law of Cosines is a natural extension of the Pythagorean Theorem that allows us to perform calculations to find the sides of triangles that are oblique.

Remember that the Law of Cosines is a generalization of the Pythagorean Theorem, where the angle C is the angle between the two given sides of a triangle:

$$c^2 = a^2 + b^2 - 2(a)(b)\cos C$$

You'll notice that if this were a right triangle, $\cos C = \cos 90° = 0$, and so the third term would disappear, leaving the familiar Pythagorean Theorem.

Another situation where we can apply the Law of Cosines is when we know all three sides in a triangle (SSS) and we need to find one of the angles. The Law of Cosines allows us to find any of the three angles in the triangle. First, we will look at how to apply the Law of Cosines in this case, and then we will look at the real-world application given in the Concept Problem above.

Example A

An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order for a kitchen to be utilized effectively, these three amenities must form a triangle with each other. This is known as the "work triangle." By design, the three parts of the work triangle must be no less than 3 feet apart and no more than 7 feet apart103° angle with the stove and the refrigerator. If the architect moves the stove so that it is 4.2 feet from the sink and makes the fridge 6.8 feet from the stove, how does this affect the angle the sink forms with the stove and the refrigerator?

Solution: In order to find how the angle is affected, we will again need to use the Law of Cosines, but because we do not know the measures of any of the angles, we solve for Y.

$$
\begin{aligned}
6.8^2 &= 4.2^2 + 5.7^2 - 2(4.2)(5.7)\cos Y && \text{Law of Cosines} \\
46.24 &= 17.64 + 32.49 - 2(4.2)(5.7)\cos Y && \text{Simplify squares} \\
46.24 &= 17.64 + 32.49 - 47.88\cos Y && \text{Multiply} \\
46.24 &= 50.13 - 47.88\cos Y && \text{Add} \\
-3.89 &= -47.88\cos Y && \text{Subtract} \\
0.0812447786 &= \cos Y && \text{Divide} \\
85.3^\circ &\approx Y && \cos^{-1}(0.081244786)
\end{aligned}
$$

The new angle would be 85.3°, which means it would be 17.7° less than the original angle.

Example B

In oblique $\triangle MNO, m = 45, n = 28$, and $o = 49$. Find $\angle M$.

Solution: Since we know all three sides of the triangle, we can use the Law of Cosines to find $\angle M$.

$$
\begin{aligned}
45^2 &= 28^2 + 49^2 - 2(28)(49)\cos M && \text{Law of Cosines} \\
2025 &= 784 + 2401 - 2(28)(49)\cos M && \text{Simplify squares} \\
2025 &= 784 + 2401 - 2744\cos M && \text{Multiply} \\
2025 &= 3185 - 2744\cos M && \text{Add} \\
-1160 &= -2744\cos M && \text{Subtract } 3185 \\
0.422740525 &= \cos M && \text{Divide by } -2744 \\
65^\circ &\approx M && \cos^{-1}(0.422740525)
\end{aligned}
$$

It is important to note that we could use the Law of Cosines to find $\angle N$ or $\angle O$ also.

Example C

Sam is building a retaining wall for a garden that he plans on putting in the back corner of his yard. Due to the placement of some trees, the dimensions of his wall need to be as follows: side $1 = 12ft$, side $2 = 18ft$, and side $3 = 22ft$. At what angle do side 1 and side 2 need to be? Side 2 and side 3? Side 1 and side 3?

Solution: Since we know the measures of all three sides of the retaining wall, we can use the Law of Cosines to find the measures of the angles formed by adjacent walls. We will refer to the angle formed by side 1 and side 2 as $\angle A$, the angle formed by side 2 and side 3 as $\angle B$, and the angle formed by side 1 and side 3 as $\angle C$. First, we will find $\angle A$.

$$22^2 = 12^2 + 18^2 - 2(12)(18)\cos A \qquad \text{Law of Cosines}$$
$$484 = 144 + 324 - 2(12)(18)\cos A \qquad \text{Simplify squares}$$
$$484 = 144 + 324 - 432\cos A \qquad \text{Multiply}$$
$$484 = 468 - 432\cos A \qquad \text{Add}$$
$$16 = -432\cos A \qquad \text{Subtract 468}$$
$$-0.037037037 \approx \cos A \qquad \text{Divide by } -432$$
$$92.1° \approx A \qquad \cos^{-1}(-0.037037037)$$

Next we will find the measure of $\angle B$ also by using the Law of Cosines.

$$18^2 = 12^2 + 22^2 - 2(12)(22)\cos B \qquad \text{Law of Cosines}$$
$$324 = 144 + 484 - 2(12)(22)\cos B \qquad \text{Simplify squares}$$
$$324 = 144 + 484 - 528\cos B \qquad \text{Multiply}$$
$$324 = 628 - 528\cos B \qquad \text{Add}$$
$$-304 = -528\cos B \qquad \text{Subtract 628}$$
$$0.575757576 = \cos B \qquad \text{Divide by } -528$$
$$54.8° \approx B \qquad \cos^{-1}(0.575757576)$$

Now that we know two of the angles, we can find the third angle using the Triangle Sum Theorem, $\angle C = 180 - (92.1 + 54.8) = 33.1°$.

Vocabulary

Side Side Side Triangle: A **side side side triangle** is a triangle where the lengths of all three sides are known quantities.

Guided Practice

1. Find the largest angle in the triangle below, where $t = 6, r = 7, i = 11$

2. Find the smallest angle in the triangle below, where $q = 17, d = 12.8, r = 18.6, \angle Q = 62.4°$

3. Find the second largest angle in the triangle below, where $c = 9, d = 11, m = 13$

Solutions:

1. $11^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cdot \cos I, \angle I \approx 115.4°$

2. $12.8^2 = 17^2 + 18.6^2 - 2 \cdot 17 \cdot 18.6 \cdot \cos D, \angle D \approx 41.8°$

3. $11^2 = 9^2 + 13^3 - 2 \cdot 9 \cdot 13 \cdot \cos D, \angle D \approx 56.5°$

Concept Problem Solution

You can use the Law of Cosines to solve this problem:

$$c^2 = a^2 + b^2 + 2ab\cos\theta$$
$$80^2 = 50^2 + 50^2 + (2)(50)(50)\cos\theta$$
$$80^2 - 50^2 - 50^2 = (2)(50)(50)\cos\theta$$
$$6400 - 2500 - 2500 = (2)(50)(50)\cos\theta$$
$$1400 = (2)(50)(50)\cos\theta$$
$$\cos\theta = \frac{1400}{5000}$$
$$\theta = \cos^{-1}(.28)$$
$$\theta = 73.74°$$

The angle in your paintball course is rather large, measuring $73.74°$

Practice

1. If you know the lengths of all three sides of a triangle, how can you identify the smallest angle of the triangle? The largest angle?
2. If you know the measures of two angles of a triangle, how can you find the measure of the third angle?

Use the triangle below to answer questions 3-5.

3. What is the measure of the smallest angle of the triangle?
4. What is the measure of the largest angle of the triangle?
5. What is the measure of the third angle of the triangle?

Use the triangle below to answer questions 6-8.

6. What is the measure of the smallest angle of the triangle?
7. What is the measure of the largest angle of the triangle?
8. What is the measure of the third angle of the triangle?

Use the triangle below to answer questions 9-11.

9. What is the measure of the smallest angle of the triangle?
10. What is the measure of the largest angle of the triangle?
11. What is the measure of the third angle of the triangle?

Use the triangle below to answer questions 12-14.

12. What is the measure of the smallest angle of the triangle?
13. What is the measure of the largest angle of the triangle?
14. What is the measure of the third angle of the triangle?

Use the triangle below to answer questions 15-17.

15. What is the measure of the smallest angle of the triangle?
16. What is the measure of the largest angle of the triangle?
17. What is the measure of the third angle of the triangle?

Use the triangle below to answer questions 18-20.

18. What is the measure of the smallest angle of the triangle?
19. What is the measure of the largest angle of the triangle?
20. What is the measure of the third angle of the triangle?

5.3 Identify Accurate Drawings of Triangles

Here you'll learn to determine if drawings and diagrams of triangles are rendered correctly using your knowledge of trigonometry.

Your friend is creating a new board game that involves several different triangle shaped pieces. However, the game requires accurate measurements of several different pieces that all have to fit together. She brings some of the pieces to you and asks if you can verify that her measurements of the pieces' side lengths and angles are correct.

You take out the first piece. According to your friend, the piece has sides of length 4 in, 5 in and 7 in, and the angle between the side of the length 4 and the side of length 5 is 78°. She's very confident in the lengths of the sides, but not quite sure if she measured the angle correctly. Is there a way to determine if your friend's game piece has the correct measurements, or did she make a mistake?

It is indeed possible to determine if your friend's measurements are correct or not. At the end of this Concept, you'll be able to tell your friend if her measurements were accurate.

Watch This

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Click image to the left for more content.

James Sousa:TheLawof Cosines:Applications

Guidance

Our extension of the analysis of triangles draws us naturally to oblique triangles. The Law of Cosines can be used to verify that drawings of oblique triangles are accurate. In a right triangle, we might use the Pythagorean Theorem to verify that all three sides are the correct length, or we might use trigonometric ratios to verify an angle measurement. However, when dealing with an obtuse or acute triangle, we must rely on the Law of Cosines.

Example A

In $\triangle ABC$ at the right, $a = 32, b = 20$, And $c = 16$. Is the drawing accurate if it labels $\angle C$ as $35.2°$? If not, what should $\angle C$ measure?

Solution: We will use the Law of Cosines to check whether or not $\angle C$ is $35.2°$.

$$16^2 = 20^2 + 32^2 - 2(20)(32)\cos 35.2 \qquad \text{Law of Cosines}$$
$$256 = 400 + 1024 - 2(20)(32)\cos 35.2 \qquad \text{Simply squares}$$
$$256 = 400 + 1024 - 1045.94547 \qquad \text{Multiply}$$
$$256 \neq 378.05453 \qquad \text{Add and subtract}$$

Since $256 \neq 378.05453$, we know that $\angle C$ is not $35.2°$. Using the Law of Cosines, we can figure out the correct measurement of $\angle C$.

$$
\begin{aligned}
16^2 &= 20^2 + 32^2 - 2(20)(32)\cos C & &\text{Law of Cosines} \\
256 &= 400 + 1024 - 2(20)(32)\cos C & &\text{Simplify Squares} \\
256 &= 400 + 1024 - 1280\cos C & &\text{Multiply} \\
256 &= 1424 - 1280\cos C & &\text{Add} \\
-1168 &= -1280\cos C & &\text{Subtract 1424} \\
0.9125 &= \cos C & &\text{Divide} \\
24.1° &\approx \angle C & &\cos^{-1}(0.9125)
\end{aligned}
$$

For some situations, it will be necessary to utilize not only the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios to verify that a triangle or quadrilateral has been drawn accurately.

Example B

A builder received plans for the construction of a second-story addition on a house. The diagram shows how the architect wants the roof framed, while the length of the house is 20 ft. The builder decides to add a perpendicular support beam from the peak of the roof to the base. He estimates that new beam should be 8.3 feet high, but he wants to double-check before he begins construction. Is the builder's estimate of 8.3 feet for the new beam correct? If not, how far off is he?

Solution: If we knew either $\angle A$ or $\angle C$, we could use trigonometric ratios to find the height of the support beam. However, neither of these angle measures are given to us. Since we know all three sides of $\triangle ABC$, we can use the Law of Cosines to find one of these angles. We will find $\angle A$.

$$
\begin{aligned}
14^2 &= 12^2 + 20^2 - 2(12)(20)\cos A & &\text{Law of Cosines} \\
196 &= 144 + 400 - 480\cos A & &\text{Simplify} \\
196 &= 544 - 480\cos A & &\text{Add} \\
-348 &= -480\cos A & &\text{Subtract} \\
0.725 &= \cos A & &\text{Divide} \\
43.5° &\approx \angle A & &\cos^{-1}(0.725)
\end{aligned}
$$

Now that we know $\angle A$, we can use it to find the length of BD.

$$
\begin{aligned}
\sin 43.5 &= \frac{x}{12} \\
12\sin 43.5 &= x \\
8.3 &\approx x
\end{aligned}
$$

Yes, the builder's estimate of 8.3 feet for the support beam is accurate.

Example C

In $\triangle CIR, c = 63, i = 52$, and $r = 41.9$. Find the measure of all three angles.

Solution:

$$63^2 = 52^2 + 41.9^2 - 2 \cdot 52 \cdot 41.9 \cdot \cos C$$
$$52^2 = 63^2 + 41.9^2 - 2 \cdot 63 \cdot 41.9 \cdot \cos I$$
$$180° - 83.5° - 55.1° = 41.4°$$
$$\angle C \approx 83.5°$$
$$\angle I \approx 55.1°$$
$$\angle R \approx 41.4°$$

Vocabulary

Law of Cosines: The **law of cosines** relates the lengths of the sides of a triangle that is not a right triangle.

Guided Practice

1. Find AD using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three.

2. Find HK using the Pythagorean Theorem, Law of Cosines, trig functions, or any combination of the three if $JK = 3.6, KI = 5.2, JI = 1.9, HI = 6.7$, and $\angle KJI = 96.3°$.

3. Use the Law of Cosines to determine whether or not the following triangle is drawn accurately. If not, determine how far the measurement of side "d" is from the correct value.

Solutions:

1. First, find AB. $AB^2 = 14.2^2 + 15^2 - 2 \cdot 14.2 \cdot 15 \cdot \cos 37.4°, AB = 9.4. \sin 23.3° = \frac{AD}{9.4}, AD = 3.7$.

2. $\angle HJI = 180° - 96.3° = 83.7°$ (these two angles are a linear pair). $6.7^2 = HJ^2 + 1.9^2 - 2 \cdot HJ \cdot 1.9 \cdot \cos 83.7°$. This simplifies to the quadratic equation $HJ^2 - 0.417HJ - 41.28$. Using the quadratic formula, we can determine that $HJ \approx 6.64$. So, since $HJ + JK = HK, 6.64 + 3.6 \approx HK \approx 10.24$.

3. To determine this, use the Law of Cosines and solve for d to determine if the picture is accurate. $d^2 = 12^2 + 24^2 - 2 \cdot 12 \cdot 24 \cdot \cos 30°, d = 14.9$, which means d in the picture is off by 1.9.

Concept Problem Solution

Since your friend is certain of the lengths of the sides of the triangle, you should use those as the known quantities in the Law of Cosines and solve for the angle:

$$7^2 = 5^2 + 4^2 + (2)(5)(4) \cos \theta$$
$$49 = 25 + 16 + 40 \cos \theta$$
$$49 - 25 - 16 = 40 \cos \theta$$
$$\frac{8}{40} = \cos \theta$$
$$cos^{-1} \frac{8}{40} = \theta$$
$$\theta = 78.46$$

So as it turns out, your friend is rather close. Her measurements were probably slight inaccurate because of her round off from the protractor.

Practice

1. If you know the lengths of all three sides of a triangle and the measure of one angle, how can you determine if the triangle is drawn accurately?

Determine whether or not each triangle is labelled correctly.

2.
3.
4.
5.
6.
7.

Determine whether or not each described triangle is possible. Assume angles have been rounded to the nearest degree.

8. In $\triangle BCD$, b=4, c=4, d=5, and $m\angle B = 51°$.
9. In $\triangle ABC$, a=7, b=4, c=9, and $m\angle B = 34°$.
10. In $\triangle BCD$, b=3, c=2, d=7, and $m\angle D = 138°$.
11. In $\triangle ABC$, a=8, b=6, c=13.97, and $m\angle C = 172°$.
12. In $\triangle ABC$, a=4, b=4, c=9, and $m\angle B = 170°$.
13. In $\triangle BCD$, b=3, c=5, d=4, and $m\angle C = 90°$.
14. In $\triangle ABC$, a=8, b=3, c=6, and $m\angle A = 122°$.
15. If you use the Law of Cosines to solve for $m\angle C$ in $\triangle ABC$ where a=3, b=7, and c=12, you will an error. Explain why.

5.4 Derivation of the Triangle Area Formula

Here you'll learn how to derive and apply a formula for the area of a triangle that involves the sine function.

While in the lunch room with your friends one day, you're discussing different ways you can use the things you've learned in math class. You tell your friends that you've been learning a lot about triangles, such as how to find their area. One of your friends looks down at your plate and starts to smile.

"Alright," he says. "If you're so good at things involving triangles, I dare you to find something simple. Tell me the area of your slice of pizza." He points down at the pizza on your plate.

The pizza is shaped like a triangle. But unfortunately its not a right triangle. The outer edge is 5 inches long, and the long sides are 7 inches long. The angle between the edge and the long side of the slice is $69°$. Is there any way to tell the area of your pizza slice?

Read through this Concept, and at its end, you'll be able to answer your friend's challenge.

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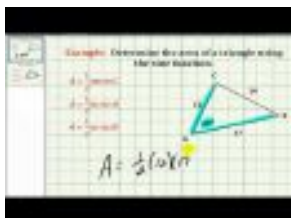
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James Sousa Example:Determine the Area of aTriangle Using the SineFunction

Guidance

We can use the area formula from Geometry, $A = \frac{1}{2}bh$, as well as the sine function, to derive a new formula that can be used when the height, or altitude, of a triangle is unknown.

In $\triangle ABC$ below, BD is altitude from B to AC. We will refer to the length of BD as h since it also represents the height of the triangle. Also, we will refer to the area of the triangle as K to avoid confusing the area with $\angle A$.

$$k = \frac{1}{2}bh \qquad\qquad\qquad \text{Area of a triangle}$$

$$k = \frac{1}{2}b(c\sin A) \qquad\qquad \sin A = \frac{h}{c} \text{ therefore } c\sin A = h$$

$$k = \frac{1}{2}bc\sin A \qquad\qquad \text{Simplify}$$

We can use a similar method to derive all three forms of the area formula, regardless of the angle:

$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}ac\sin B$$
$$K = \frac{1}{2}ab\sin C$$

The formula $K = \frac{1}{2}bc\sin A$ requires us to know two sides and the included angle (SAS) in a triangle. Once we know these three things, we can easily calculate the area of an oblique triangle.

Example A

In $\triangle ABC, \angle C = 62°, b = 23.9$, and $a = 31.6$. Find the area of the triangle.

Solution: Using our new formula, $K = \frac{1}{2}ab\sin C$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(31.6)(23.9)\sin 62$$
$$K \approx 333.4$$

Example B

The Pyramid Hotel recently installed a triangular pool. One side of the pool is 24 feet, another side is 26 feet, and the angle in between the two sides is $87°$. If the hotel manager needs to order a cover for the pool, and the cost is $35 per square foot, how much can he expect to spend?

Solution: In order to find the cost of the cover, we first need to know the area of the cover. Once we know how many square feet the cover is, we can calculate the cost. In the illustration above, you can see that we know two of the sides and the included angle. This means we can use the formula $K = \frac{1}{2}bc\sin A$.

$$K = \frac{1}{2}(24)(26)\sin 87$$
$$K \approx 311.6$$
$$311.6 \, sq.ft. \times \$35/sq.ft. = \$10,905.03$$

The cost of the cover will be $10,905.03.

Example C

In $\triangle GHI, \angle I = 15°, g = 14.2$, and $h = 7.9$. Find the area of the triangle.

Solution: Using our new formula, $K = \frac{1}{2}ab\sin C$, which is the same as $K = \frac{1}{2}gh\sin I$, plug in what is known and solve for the area.

$$K = \frac{1}{2}(14.2)(7.9)\sin 15$$
$$K \approx 14.52$$

Vocabulary

Oblique Triangle: An **oblique triangle** is a triangle that does not have $90°$ as one of its interior angles.

SAS Triangle: An **SAS triangle** is a triangle where two sides and the angle in between them are known quantities.

Guided Practice

1. A farmer needs to replant a triangular section of crops that died unexpectedly. One side of the triangle measures 186 yards, another measures 205 yards, and the angle formed by these two sides is $148°$.

What is the area of the section of crops that needs to be replanted?

2. The farmer goes out a few days later to discover that more crops have died. The side that used to measure 205 yards now measures 288 yards. How much has the area that needs to be replanted increased by?

3. Find the perimeter of the quadrilateral at the left If the area of $\triangle DEG = 56.5$ and the area of $\triangle EGF = 84.7$.

Solutions:

1. Use $K = \frac{1}{2}bc\sin A, K = \frac{1}{2}(186)(205)\sin 148°$. So, the area that needs to be replaced is 10102.9 square yards.

2. $K = \frac{1}{2}(186)(288)\sin 148° = 14193.4$, the area has increased by 4090.5 yards.

3. You need to use the $K = \frac{1}{2}bc\sin A$ formula to find DE and GF.

$$56.5 = \frac{1}{2}(13.6)DE\sin 39° \rightarrow DE = 13.2 \qquad 84.7 = \frac{1}{2}(13.6)EF\sin 60° \rightarrow EF = 14.4$$

Second, you need to find sides DG and GF using the Law of Cosines.

$$DG^2 = 13.2^2 + 13.6^2 - 2 \cdot 13.2 \cdot 13.6 \cdot \cos 39° \rightarrow DG = 8.95$$
$$GF^2 = 14.4^2 + 13.6^2 - 2 \cdot 14.4 \cdot 13.6 \cdot \cos 60° \rightarrow GF = 14.0$$

The perimeter of the quadrilateral is 50.55.

Concept Problem Solution

$K = \frac{1}{2}bc\sin A$

where in this case, one of the sides is equal to 5, the other is equal to 7, and the angle is $69°$.

$K = \frac{1}{2}(5)(7)\sin 69° = 16.34 in^2$

Practice

Find the area of each triangle.

1. $\triangle ABC$ if a=13, b=15, and $m\angle C = 71°$.
2. $\triangle ABC$ if b=8, c=4, and $m\angle A = 67°$.
3. $\triangle ABC$ if b=34, c=29, and $m\angle A = 138°$.
4. $\triangle ABC$ if a=3, b=7, and $m\angle C = 80°$.
5. $\triangle ABC$ if a=4.8, c=3.7, and $m\angle B = 43°$.
6. $\triangle ABC$ if a=12, b=5, and $m\angle C = 20°$.

7. $\triangle ABC$ if a=3, b=10, and $m\angle C = 50°$.
8. $\triangle ABC$ if a=5, b=9, and $m\angle C = 14°$.
9. $\triangle ABC$ if a=5, b=7, and c=11.
10. $\triangle ABC$ if a=7, b=8, and c=9.
11. $\triangle ABC$ if a=12, b=14, and c=4.
12. A farmer measures the three sides of a triangular field and gets 114, 165, and 257 feet. What is the measure of the largest angle of the triangle?
13. Using the information from the previous problem, what is the area of the field?

Another field is a quadrilateral where three sides measure 30, 50, and 60 yards, and two angles measure $130°$ and $140°$, as shown below.

14. Find the area of the quadrilateral. Hint: divide the quadrilateral into two triangles and find the area of each.
15. Find the length of the fourth side.
16. Find the measures of the other two angles.

5.5 Heron's Formula

Here you'll learn to apply Heron's formula for finding the area of a triangle when the lengths of all three sides are known.

You are in History class learning about different artifacts from other cultures, when the subject of pyramids is presented by your teacher. He informs the class that pyramids come in a variety of sizes, designs, and styles, and are found not only in Egypt, but in many other countries around the world. He tells everyone that a typical pyramid might be approximately 200 meters long at the base and 175 meters up each of the diagonal sides.

Your mind wanders back to your math class from that morning, and you find yourself wondering if there is a straightforward way to determine the area of one of the faces of the pyramid from the information you have been given.

Do you think there is a way to do this?

As it turns out, there is a very straightforward way to determine the area of a triangle when you know the lengths of all three sides. At the end of this Concept, you'll be able to calculate the area of one of the pyramid's sides.

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James Sousa:Heron'sFormula

Guidance

One way to find the area of an oblique triangle when we know two sides and the included angle is by using the formula $K = \frac{1}{2} bc \sin A$. We could also find the area of a triangle in which we know all three sides by first using the Law of Cosines to find one of the angles and then using the formula $K = \frac{1}{2} bc \sin A$. While this process works, it is time-consuming and requires a lot of calculation. Fortunately, we have another formula, called Heron's Formula, which allows us to calculate the area of a triangle when we know all three sides. It is derived from $K = \frac{1}{2} bc \sin A$, the Law of Cosines and the Pythagorean Identity.

$K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$ or half of the perimeter of the triangle.

Example A

In $\triangle ABC, a = 23, b = 46$, and $c = 41$. Find the area of the triangle.

Solution: First, you need to find s: $s = \frac{1}{2}(23 + 41 + 46) = 55$. Now, plug s and the three sides into Heron's Formula and simplify.

$$K = \sqrt{55(55-23)(55-46)(55-41)}$$
$$K = \sqrt{55(32)(9)(14)}$$
$$k = \sqrt{221760}$$
$$K \approx 470.9$$

Example B

A handyman is installing a tile floor in a kitchen. Since the corners of the kitchen are not exactly square, he needs to have special triangular shaped tile made for the corners. One side of the tile needs to be 11.3", the second side needs to be 11.9",and the third side is 13.6". If the tile costs $4.89 per square foot, and he needs four of them, how much will it cost to have the tiles made?

Solution: In order to find the cost of the tiles, we first need to find the area of one tile. Since we know the measurements of all three sides, we can use Heron's Formula to calculate the area.

$$s = \frac{1}{2}(11.3 + 11.9 + 13.6) = 18.4$$
$$K = \sqrt{18.4(18.4 - 11.3)(18.4 - 11.9)(18.4 - 13.6)}$$
$$K = \sqrt{18.4(7.1)(6.5)(4.8)}$$
$$K = \sqrt{4075.968}$$
$$K \approx 63.843 \ in^2$$

The area of one tile is 63.843 square inches. The cost of the tile is given to us in square feet, while the area of the tile is in square inches. In order to find the cost of one tile, we must first convert the area of the tile into square feet.

$$1 \text{ square foot} = 12in \times 12in = 144in^2$$
$$\frac{63.843}{144} = 0.443 \ ft^2 \qquad\qquad \text{Covert square inches into square feet}$$
$$0.443 \ ft^2 \times 4.89 = 2.17 \qquad\qquad \text{Multiply by the cost of the tile.}$$
$$2.17 \times 4 = 8.68$$

The cost for four tiles would be $8.68.

Example C

In $\triangle GHI, g = 11, h = 24$, and $i = 18$. Find the area of the triangle.

Solution: First, you need to find s: $s = \frac{1}{2}(11 + 24 + 18) = 26.5$. Now, plug s and the three sides into Heron's Formula and simplify.

$$K = \sqrt{26.5(26.5 - 11)(26.5 - 24)(26.5 - 18)}$$
$$K = \sqrt{26.5(15.5)(2.5)(8.5)}$$
$$k = \sqrt{8728.44}$$
$$K \approx 93.43$$

Vocabulary

Heron's Formula: Heron's formula is a formula to calculate the area of a triangle when the lengths of all three sides are known.

Guided Practice

1. Use Heron's formula to find the area of a triangle with the following sides: $HC = 4.1, CE = 7.4$, and $HE = 9.6$

2. The Pyramid Hotel is planning on repainting the exterior of the building. The building has four sides that are isosceles triangles with bases measuring 590 ft and legs measuring 375 ft.

a. What is the total area that needs to be painted? b. If one gallon of paint covers 25 square feet, how many gallons of paint are needed?

3. A contractor needs to replace a triangular section of roof on the front of a house. The sides of the triangle are 8.2 feet, 14.6 feet, and 16.3 feet. If one bundle of shingles covers $33\frac{1}{3}$ square feet and costs $15.45, how many bundles does he need to purchase? How much will the shingles cost him? How much of the bundle will go to waste?

Solutions:

1. $A = 14.3$

2. a. Use Heron's Formula, then multiply your answer by 4, for the 4 sides.

$s = \frac{1}{2}(375 + 375 + 590) = 670$

$A = \sqrt{670(670 - 375)(670 - 375)(670 - 590)} = 68,297.4$

The area multiplied by 4: $68,297.4 \cdot 4 = 273,189.8$ total square feet.

b. $\frac{273,189.8}{25} \approx 10,928$ gallons of paint are needed.

3. Using Heron's Formula, s and the area are: $s = \frac{1}{2}(8.2 + 14.6 + 16.3) = 19.55$ and $A = \sqrt{19.55(19.55 - 8.2)(19.55 - 14.6)(19.5}$ $59.75\ sq.ft$. He will need 2 bundles $\left(\frac{59.75}{33.3} = 1.8\right)$. The shingles will cost him $2 \cdot \$15.45 = \$30.90$ and 6.92 square feet will go to waste $(66.67 - 59.75 = 6.92)$.

Concept Problem Solution

You can use Heron's Formula to find the area of one of the faces of the pyramid.

The equation for the area of the triangle is: $K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$ or half of the perimeter of the triangle.

so, in this case,

$s = \frac{1}{2}(200 + 175 + 175) = \frac{550}{2} = 225$

And

$K = \sqrt{225(225 - 200)(225 - 175)(225 - 175)} = 3,750$ square meters.

Practice

Find the area of each triangle with the three given side lengths.

1. 2, 14, 15
2. 6, 8, 9
3. 10, 14, 20

4. 11, 15, 6
5. 4, 4, 4
6. 4, 5, 3
7. 32, 40, 50
8. 20, 18, 22
9. 20, 20, 20
10. 18, 17, 12
11. 9, 12, 10
12. 11, 18, 8
13. Describe when it makes the most sense to use Heron's formula to find the area of a triangle.
14. A tiling is made of 30 congruent triangles. The lengths of the sides of each triangle are 3 inches, 5 inches, and 7 inches. What is the area of the tiling?
15. What type of triangle with have the maximum area for a given perimeter? Show or explain your reasoning.

5.6 Determination of Unknown Triangle Measures Given Area

Here you'll learn how to apply rules for finding unknown triangle quantities when the area is known.

You are working on creating a mobile for your art class. A mobile is a piece of art that has a rod with different shapes hanging from it, so they can spin.

To create your project, you need to cut a set of triangles that have a variety of sizes. You are about to start cutting triangles, when your friend, who is helping you with the project, comes over. She tells you that each piece needs to have a rod through the side of it to balance the shape in a certain way. She wants you to make a piece that looks like this:

You have already cut a triangle by cutting a piece out of construction paper. You know that one side of your triangle is 6 inches long, but you don't know the length of the other two sides. Can you use the information you have to find the length of b in the mobile piece above? (The area of the triangle is $25in^2$, and the interior angle between the six inch side and the side you want to know is $35°$).

By the end of this Concept, you'll be able to solve this problem.

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How do you find the height of a triangleif you knowthearea and the base

Guidance

In this section, we will look at situations where we know the area but need to find another part of the triangle, as well as an application involving a quadrilateral. All of this will involve the use of the Law of Cosines, Law of Sines, and the Alternate Formula for the Area of a Triangle.

Example A

The jib sail on a sailboat came untied and the rope securing it was lost. If the area of the jib sail is 56.1 square feet, use the figure and information below to find the length of the rope.

Solution: Since we know the area, one of the sides, and one angle of the jib sail, we can use the formula $K = \frac{1}{2} bc \sin A$ to find the side of the jib sail that is attached to the mast. We will call this side y.

$$56.1 = \frac{1}{2} \, 28(y) \sin 11$$

$$56.1 = 2.671325935 \, y$$

$$21.0 = y$$

Now that we know side y, we know two sides and the included angle in the triangle formed by the mast, the rope, and the jib sail. We can now use the Law of Cosines to calculate the length of the rope.

$$x^2 = 21^2 + 27^2 - 2(21)(27) \cos 18$$

$$x^2 = 91.50191052$$

$$x \approx 9.6 \; ft$$

The length of the rope is approximately 9.6 feet.

Example B

In quadrilateral $QUAD$ below, the area of $\triangle QUD = 5.64$, the area of $\triangle UAD = 6.39$, $\angle QUD = 31°$, $\angle DUA = 40°$, and $UD = 7.8$. Find the perimeter of $QUAD$.

Solution: In order to find the perimeter of $QUAD$, we need to know sides QU, QD, UA, and AD. Since we know the area, one side, and one angle in each of the triangles, we can use $K = \frac{1}{2} \, bc \sin A$ to figure out QU and UA.

$$5.64 = \frac{1}{2}(7.8)(QU) \sin 31 \qquad\qquad 6.39 = \frac{1}{2}(7.8)UA \sin 40$$

$$2.8 \approx QU \qquad\qquad\qquad\qquad\qquad 2.5 \approx UA$$

Now that we know QU and UA, we know two sides and the included angle in each triangle (SAS). This means that we can use the Law of Cosines to find the other two sides, QD and DA. First we will find QD and DA.

$$QD^2 = 2.8^2 + 7.8^2 - 2(2.8)(7.8) \cos 31 \qquad\qquad DA^2 = 2.5^2 + 7.8^2 - 2(2.5)(7.8) \cos 40$$

$$QD^2 = 31.23893231 \qquad\qquad\qquad\qquad\qquad DA^2 = 37.21426672$$

$$QD \approx 5.6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad DA \approx 6.1$$

Finally, we can calculate the perimeter since we have found all four sides of the quadrilateral.

$$pQUAD = 2.8 + 5.6 + 6.1 + 2.5 = 17$$

Example C

In $\triangle ABC, BD$ is an altitude from B to AC. The area of $\triangle ABC = 232.96, AB = 16.2$, and $AD = 14.4$. Find DC.

Solution:

First, find BD by using the Pythagorean Theorem. $BD = \sqrt{16.2^2 - 14.4^2} = 7.42$. Then, using the area and formula $(A = \frac{1}{2}bh)$, you can find AC. $232.96 = \frac{1}{2}(7.42)AC \rightarrow AC = 62.78$. $DC = 62.78 - 14.4 = 48.38$.

Vocabulary

Law of Cosines: The **law of cosines** is an equation relating the length of one side of a triangle to the lengths of the other two sides and the sine of the angle included between the other two sides.

Law of Sines: The **law of sines** is an equation relating the sine of an interior angle of a triangle divided by the side opposite that angle to a different interior angle of the same triangle divided by the side opposite that second angle.

Guided Practice

1. Find "h" in the triangle below: Area $= 1618.98, b = 36.3$

2. Find $\angle A$ in the triangle below: Area $= 387.6, b = 25.6, c = 32.9$

3. Find the area of $\triangle ABC$ below: Area $\triangle ABD = 16.96, AD = 3.2, \angle DBC = 49.6°$

Solutions:

1. Since we know the area, one of the sides (18.15), and one angle of the triangle (45°), we can use the formula $K = \frac{1}{2} bc \sin A$ to find the other side of the triangle. We can then use the Pythagorean Theorem to find the height of the triangle.

This gives a result of:

$h = 89.2$

2. Since we know the area and the lengths of two of the sides of the triangle, we can use the formula $K = \frac{1}{2} bc \sin A$ to solve for the included angle, which gives:

$\angle A = 67°$

3. Area of $\triangle ABC = 83.0$

Concept Problem Solution

Since you know that the mobile piece is six inches on one side, and that the area of the triangle is $25 in^2$, you can use the formula $K = \frac{1}{2} ab \sin C$ to find the length of the other side:

$$K = \frac{1}{2} ab \sin C$$
$$25 = \frac{1}{2}(6)(b) \sin 35°$$
$$25 = 1.72b$$
$$b = \frac{25}{1.72}$$
$$b = 14.53 in$$

Practice

1. The area of the triangle below is $138 in^2$. Solve for x, the height.
2.
3. The area of the triangle below is $250 cm^2$. A height is given on the diagram. Solve for x.
4.

Use the triangle below for questions 3-5. The area of the large triangle is $65 cm^2$.

3. Solve for x.
4. Find the perimeter of the large triangle.
5. Find the measure of all three angles of the large triangle.

Use the triangle below for questions 6-8. The area of the triangle is $244cm^2$.

6. Solve for θ.
7. Solve for x.
8. Find the perimeter of the triangle.

Use the triangle below for questions 9-11. The area of the triangle is $299.8in^2$.

9. Solve for x.
10. Solve for y.
11. Find the measure of the other two angles of the triangle.

Use the triangle below for questions 12-15. The area of the large triangle is $84in^2$.

12. Solve for x.
13. Solve for y.
14. Solve for z.
15. Solve for θ.

5.7 Angle-Angle-Side Triangles

Here you'll learn to use the Law of Sines to find the length of an unknown side of a triangle when two angles and the length of one of the other sides are known.

You and a friend decide to go fly kites on a breezy Saturday afternoon. While sitting down to make your kites, you are working on make the best shape possible to catch the breeze. While your friend decides to go with a diamond shaped kite, you try out making a triangle shaped one. While trying to glue the kite together, you make the first and second piece lock together with a 70° angle. The angle between the first and third pieces is 40°. Finally, you also have measured the length of the second piece and found that it is 22 inches long.

Your kite looks like this:

Is there a way to find out, using math, what the length of the third side will be?

Keep reading, and you'll be able to answer this question at the end of this Concept.

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James Sousa Example:Solvinga Triangle Using the Law ofSines GivenTwoAngles and One Side

Guidance

The Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. This is a ratio between the sine of an angle in a triangle and the length of the side opposite that angle to the sine of a different angle in that triangle and the length of the side opposing that second angle.

The Law of Sines allows us to find many quantities of interest in triangles by comparing sides and interior angles as a ratio. One case where we can to use the Law of Sines is when we know two of the angles in a triangle and a non-included side (AAS).

Example A

Using $\triangle GMN$, $\angle G = 42°$, $\angle N = 73°$ and $g = 12$. Find n.

Since we know two angles and one non-included side (g), we can find the other non-included side (n).

$$\frac{\sin 73°}{n} = \frac{\sin 42°}{12}$$
$$n \sin 42° = 12 \sin 73°$$
$$n = \frac{12 \sin 73°}{\sin 42°}$$
$$n \approx 17.15$$

Example B

Continuing on from Example A, find $\angle M$ and m.

Solution: $\angle M$ is simply $180° - 42° - 73° = 65°$. To find side m, you can now use either the Law of Sines or Law of Cosines. Considering that the Law of Sines is a bit simpler and new, let's use it. It does not matter which side and opposite angle you use in the ratio with $\angle M$ and m.

Option 1: $\angle G$ and g

$$\frac{\sin 65°}{m} = \frac{\sin 42°}{12}$$
$$m \sin 42° = 12 \sin 65°$$
$$m = \frac{12 \sin 65°}{\sin 42°}$$
$$m \approx 16.25$$

Option 2: $\angle N$ and n

$$\frac{\sin 65°}{m} = \frac{\sin 73°}{17.15}$$
$$m \sin 73° = 17.15 \sin 65°$$
$$m = \frac{17.15 \sin 65°}{\sin 73°}$$
$$m \approx 16.25$$

Example C

A business group wants to build a golf course on a plot of land that was once a farm. The deed to the land is old and information about the land is incomplete. If AB is 5382 feet, BC is 3862 feet, $\angle AEB$ is 101°, $\angle BDC$ is 74°, $\angle EAB$ is 41° and $\angle DCB$ is 32°, what are the lengths of the sides of each triangular piece of land? What is the total area of the land?

Solution: Before we can figure out the area of the land, we need to figure out the length of each side. In $\triangle ABE$, we know two angles and a non-included side. This is the AAS case. First, we will find the third angle in $\triangle ABE$ by using the Triangle Sum Theorem. Then, we can use the Law of Sines to find both AE and EB.

$$\angle ABE = 180 - (41 + 101) = 38°$$

$$\frac{\sin 101}{5382} = \frac{\sin 38}{AE} \qquad\qquad \frac{\sin 101}{5382} = \frac{\sin 41}{EB}$$

$$AE(\sin 101) = 5382(\sin 38) \qquad\qquad EB(\sin 101) = 5382(\sin 41)$$

$$AE = \frac{5382(\sin 38)}{\sin 101} \qquad\qquad EB = \frac{5382(\sin 41)}{\sin 101}$$

$$AE = 3375.5\ feet \qquad\qquad EB \approx 3597.0\ feet$$

Next, we need to find the missing side lengths in $\triangle DCB$. In this triangle, we again know two angles and a non-included side (AAS), which means we can use the Law of Sines. First, let's find $\angle DBC = 180 - (74 + 32) = 74°$. Since both $\angle BDC$ and $\angle DBC$ measure $74°$, $\triangle DCB$ is an isosceles triangle. This means that since BC is 3862 feet, DC is also 3862 feet. All we have left to find now is DB.

$$\frac{\sin 74}{3862} = \frac{\sin 32}{DB}$$

$$DB(\sin 74) = 3862(\sin 32)$$

$$DB = \frac{3862(\sin 32)}{\sin 74}$$

$$DB \approx 2129.0\ feet$$

Finally, we need to calculate the area of each triangle and then add the two areas together to get the total area. From the last section, we learned two area formulas, $K = \frac{1}{2}\ bc \sin A$ and Heron's Formula. In this case, since we have enough information to use either formula, we will use $K = \frac{1}{2}\ bc \sin A$ since it is less computationally intense.

First, we will find the area of $\triangle ABE$.

$\triangle ABE$:

$$K = \frac{1}{2}(3375.5)(5382)\sin 41$$

$$K = 5,959,292.8\ ft^2$$

$\triangle DBC$:

$$K = \frac{1}{2}(3862)(3862)\sin 32$$

$$K = 3,951,884.6\ ft^2$$

The total area is $5,959,292.8 + 3,951,884.6 = 9,911,177.4\ ft^2$.

Vocabulary

Angle Angle Side Triangle: An **angle angle side triangle** is a triangle where two of the angles and the non-included side are known quantities.

Guided Practice

1. Find side "d" in the triangle below with the following information: $e = 214.9, D = 39.7°, E = 41.3°$

2. Find side "o" in the triangle below with the following information: $M = 31°, O = 9°, m = 15$

3. Find side "q" in the triangle below with the following information: $Q = 127°, R = 21.8°, r = 3.62$

Solutions:

1. $\frac{\sin 41.3°}{214.9} = \frac{\sin 39.7°}{d}, d = 208.0$

2. $\frac{\sin 9°}{o} = \frac{\sin 31°}{15}, o = 4.6$

3. $\frac{\sin 127°}{q} = \frac{\sin 21.8°}{3.62}, q = 7.8$

Concept Problem Solution

Since you know two angles and one non-included side of the kite, you can find the other non-included side using the Law of Sines. Set up a ratio using the angles and side you know and the side you don't know.

$$\frac{\sin 70°}{x} = \frac{\sin 40°}{22}$$
$$x = \frac{22 \sin 70°}{\sin 40°}$$
$$x \approx 32.146$$

The length of the dowel rod on the unknown side will be approximately 32 inches.

Practice

In $\triangle ABC$, $m\angle A = 50°$, $m\angle B = 34°$, and a=6.

1. Find the length of b.
2. Find the length of c.

In $\triangle KMS$, $m\angle K = 42°$, $m\angle M = 26°$, and k=14.

3. Find the length of m.
4. Find the length of s.

In $\triangle DEF$, $m\angle D = 52°$, $m\angle E = 78°$, and d=23.

5. Find the length of e.
6. Find the length of f.

In $\triangle PQR$, $m\angle P = 2°$, $m\angle Q = 79°$, and p=20.

7. Find the length of q.
8. Find the length of r.

In $\triangle DOG$, $m\angle D = 50°$, $m\angle G = 59°$, and o=12.

9. Find the length of d.
10. Find the length of g.

In $\triangle CAT$, $m\angle C = 82°$, $m\angle T = 4°$, and a=8.

11. Find the length of c.
12. Find the length of t.

In $\triangle YOS$, $m\angle Y = 65°$, $m\angle O = 72°$, and s=15.

13. Find the length of o.
14. Find the length of y.

In $\triangle HCO$, $m\angle H = 87°$, $m\angle C = 14°$, and o=19.

15. Find the length of h.
16. Find the length of c.

5.8 Angle-Side-Angle Triangles

Here you'll learn to find an unknown side of a triangle using the Law of Sines when two angles and the length of the side between them are known.

You're eating lunch in the cafeteria one afternoon while working on your math homework. Lately you seem to notice the triangular shapes in everything. At home, at school, with your friends. It seems like triangles are everywhere. And you find yourself trying to apply what you are learning in math class to all of the triangles around you. And today is no exception. As you start to take a bite of your chip, you suddenly recognize that familiar shape - the triangle.

You estimate the length of one of the sides of the chip to be 3 cm. You also can tell that the angle adjacent to the 3 cm side is $50°$ and the angle adjacent on the other side of the 3 cm edge is $60°$. Can you find the lengths of the other two sides using techniques from your math class?

Read on, and at the end of this Concept, you'll be able to solve this problem.

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Law of Sines- Solving ASA Triangle

Guidance

The Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. This is a ratio between the sine of an angle in a triangle and the length of the side opposite that angle to the sine of a different angle in that triangle and the length of the side opposing that second angle.

One case where we use the Law of Sines is when we know two angles in a triangle and the **included** side (ASA). For instance, in $\triangle TRI$:

$\angle T, \angle R$, and i are known

$\angle T, \angle I,$ and r are known

$\angle R, \angle I,$ and t are known

In this case, the Law of Sines allows us to find either of the non-included sides.

Example A

In the triangle above, $\triangle TRI, \angle T = 83°, \angle R = 24°$, and $i = 18.5$. Find the measure of t.

Solution: Since we know two angles and the included side, we can find either of the non-included sides using the Law of Sines. Since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal $180°$.

$$\angle I = 180 - (83 + 24)$$
$$\angle I = 180 - 107$$
$$\angle I = 73°$$

Now that we know $\angle I = 73°$, we can use the Law of Sines to find t.

$$\frac{\sin 73}{18.5} = \frac{\sin 83}{t}$$
$$t(\sin 73) = 18.5(\sin 83)$$
$$t = \frac{18.5(\sin 83)}{\sin 73}$$
$$t \approx 19.2$$

Notice how we wait until the last step to input the values into the calculator. This is so our answer is as accurate as possible.

Example B

In order to avoid a large and dangerous snowstorm on a flight from Chicago to Buffalo, pilot John starts out $27°$ off of the normal flight path. After flying 412 miles in this direction, he turns the plane toward Buffalo. The angle formed by the first flight course and the second flight course is $88°$. For the pilot, two issues are pressing:

1. What is the total distance of the modified flight path?
2. How much further did he travel than if he had stayed on course?

Solution, Part 1: In order to find the total distance of the modified flight path, we need to know side x. To find side x, we will need to use the Law of Sines. Since we know two angles and the included side, this is an ASA case. Remember that in the ASA case, we need to first find the third angle in the triangle.

$$MissingAngle = 180 - (27 + 88) = 65° \qquad \text{The sum of angles in a triangle is 180}$$
$$\frac{\sin 65}{412} = \frac{\sin 27}{x} \qquad\qquad\qquad \text{Law of Sines}$$
$$x(\sin 65) = 412(\sin 27) \qquad\qquad \text{Cross multiply}$$
$$x = \frac{412(\sin 27)}{\sin 65} \qquad\qquad\qquad \text{Divide by sin 65}$$
$$x \approx 206.4 \; miles$$

The total distance of the modified flight path is $412 + 206.4 = 618.4 \; miles$.

Solution, Part 2: To find how much farther John had to travel, we need to know the distance of the original flight path, y. We can use the Law of Sines again to find y.

$$\frac{\sin 65}{412} = \frac{\sin 88}{y} \qquad \text{Law of Sines}$$

$$y(\sin 65) = 412(\sin 88) \qquad \text{Cross multiply}$$

$$y = \frac{412(\sin 88)}{\sin 65} \qquad \text{Divide by } \sin 65$$

$$y \approx 454.3 \; miles$$

John had to travel $618.4 - 454.3 = 164.1 \; miles$ farther.

Example C

In the triangle shown here:

The sides given are $\angle A = 14°$, $\angle B = 18°$, and c=11. Find the length of side "a".

Solution: Since we know two angles and the included side, we can find either of the non-included sides using the Law of Sines. Since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal $180°$.

$$\angle C = 180 - (18 + 14)$$
$$\angle C = 180 - 32$$
$$\angle C = 148°$$

Use the Law of Sines to find the length of side "a":

$$\frac{\sin 14}{a} = \frac{\sin 148}{11}$$
$$11(\sin 14) = a(\sin 148)$$
$$a = \frac{11(\sin 14)}{\sin 148}$$
$$a \approx 5.02$$

Vocabulary

Angle Side Angle Triangle: An **angle side angle triangle** is a triangle where two angles and the length of the side in between them are known quantities.

Guided Practice

1. Find side "a" in the triangle below using the following information: $b = 16, A = 11.7°, C = 23.8°$

2. Find side "a" in the triangle below using the following information: $k = 6.3, J = 16.2°, L = 40.3°$

3. Even though ASA and AAS triangles represent two different cases of the Law of Sines, what do they both have in common?

Solutions:

1. $\frac{\sin 11.7°}{a} = \frac{\sin 144.5°}{16}, a = 5.6$

2. $\frac{\sin 40.3°}{l} = \frac{\sin 123.5°}{6.3}, l = 4.9$

3. Student answers will vary but they should notice that in both cases you know or can find an angle and the side across from it.

Concept Problem Solution

You can use the Law of Sines to find the length of either of the other 2 sides. However, first it is good to note that since the sum of the interior angles of a triangle must equal $180°$, the third angle in the triangle must measure $180° - 50° - 60° = 70°$.

Now to set up the ratios:

$$\frac{\sin 60}{a} = \frac{\sin 70}{3}$$
$$3(\sin 60) = a(\sin 70)$$
$$a = \frac{3(\sin 60)}{\sin 70}$$
$$a \approx 2.7647$$

The length of one of the other 2 sides is approximately 2.7647 centimeters.

To find the length of the last side:

$$\frac{\sin 50}{b} = \frac{\sin 70}{3}$$
$$3(\sin 50) = b(\sin 70)$$
$$b = \frac{3(\sin 50)}{\sin 70}$$
$$b \approx 2.46$$

The length of the other one of the 2 unknown sides is approximately 2.46 centimeters.

Practice

In $\triangle ABC$, $m\angle A = 40°$, $m\angle B = 67°$, and c=6.

1. Find $m\angle C$.
2. Find the length of a.
3. Find the length of b.

In $\triangle DEF$, $m\angle D = 36°$, $m\angle E = 101°$, and f=11.

4. Find $m\angle F$.
5. Find the length of d.
6. Find the length of e.

In $\triangle BIG$, $m\angle B = 56°$, $m\angle I = 71°$, and g=23.

7. Find $m\angle G$.
8. Find the length of b.
9. Find the length of i.

In $\triangle APL$, $m\angle A = 79°$, $m\angle P = 40°$, and l=15.

10. Find $m\angle L$.
11. Find the length of a.
12. Find the length of p.

In $\triangle SAU$, $m\angle S = 5°$, $m\angle A = 99°$, and u=21.

13. Find $m\angle U$.
14. Find the length of s.
15. Find the length of a.

5.9 Possible Triangles with Side-Side-Angle

Here you'll learn how to determine the number of solutions for triangles where two sides and the non-included angle are known.

Your team has just won the flag in a flag football tournament at your school. As a reward, you get to take home the flag and keep it until the next game, when the other team will try to win it back. The flag looks like this:

It makes an isosceles triangle. You start to wonder how many different possible triangles there are for different lengths of sides. For example, if you make an oblique triangle that has a given angle greater than ninety degrees, how many ways are there to do this? Can you determine how many different possible triangles there are if the triangle is an isosceles triangle?

By the end of this Concept, you'll be able to determine the answer to this question, as well as for a variety of other triangle leg lengths.

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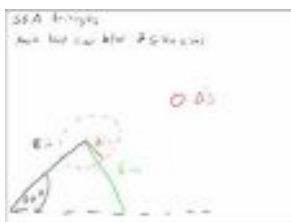
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Determining the Amount of Triangles Resulting from an SSA Triangle

Guidance

In Geometry, you learned that two sides and a non-included angle do not necessarily define a unique triangle. Consider the following cases given a, b, and $\angle A$:

Case 1:$(a < b)$

In this case $a < b$ and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2:$(a < b)$

In this case, $a < b$ and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3:$(a < b)$

In this case, $a < b$ and side a meets the base at exactly two points. Since two triangles exist, there are <u>two</u> solutions.

Case 4:$(a = b)$

In this case $a = b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5:$(a > b)$

In this case, $a > b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous Case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b. If you are faced with the first situation, where $a < b$, we can still tell how many solutions there will be by using a and $b \sin A$.

TABLE 5.1:

	If:	Then:
a.	$a < b$	No solution, one solution, two solutions
i.	$a < b \sin A$	No solution
ii.	$a = b \sin A$	One solution
iii.	$a > b \sin A$	Two solutions
b.	$a = b$	One solution
c.	$a > b$	One solution

Example A

Determine if the sides and angle given determine no, one or two triangles. The set contains an angle, its opposite side and the side between them.

$a = 5, b = 8, A = 62.19°$

Solution:

$5 < 8, 8 \sin 62.19° = 7.076$. So $5 < 7.076$, which means there is no solution.

Example B

Determine if the sides and angle given determine no, one or two triangles. The set contains an angle, its opposite side and the side between them.

$c = 14, b = 10, B = 15.45°$

Solution: Even though a, b and $\angle A$ is not used in this example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

$10 < 14, 14 \sin 15.45° = 3.73$. So $10 > 3.73$, which means there are two solutions.

Example C

Determine if the sides and angle given determine no, one or two triangles. The set contains an angle, its opposite side and the side between them.

$d = 16, g = 11, D = 44.94°$

Solution: Even though a, b and $\angle A$ is not used in this example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

$16 > 11$, there is one solution.

Vocabulary

Side Side Angle Triangle: A **side side angle triangle** is a triangle where the length of two sides and one of the angles that is not between the two sides are known quantities.

Guided Practice

1.Determine how many solutions there would be for a triangle based on the given information and by calculating $b\sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

$A = 32.5°, a = 26, b = 37$

2. Determine how many solutions there would be for a triangle based on the given information and by calculating $b\sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

$A = 42.3°, a = 16, b = 26$

3. Determine how many solutions there would be for a triangle based on the given information and by calculating $b\sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

$A = 47.8°, a = 13.48, b = 18.2$

Solutions:

1. $A = 32.5°, a = 26, b = 3726 > 19.9$ 2 solutions

2. $A = 42.3°, a = 16, b = 2616 < 17.5$ 0 solutions

3. $A = 47.8°, a = 13.48, b = 18.213.48 = 13.48$ 1 solution

Concept Problem Solution

As you now know, when two sides of a triangle with an included side are known, and the lengths of the two sides are equal, there is one possible solution. Since an isosceles triangle meets these criteria, there is only one possible solution.

Practice

Determine if the sides and angle given determine no, one or two triangles. The set contains an angle, its opposite side and another side of the triangle.

1. $a = 6, b = 6, A = 45°$
2. $a = 4, b = 7, A = 115°$
3. $a = 5, b = 2, A = 68°$
4. $a = 7, b = 6, A = 34°$
5. $a = 5, b = 3, A = 89°$
6. $a = 4, b = 4, A = 123°$
7. $a = 6, b = 8, A = 57°$
8. $a = 4, b = 9, A = 24°$
9. $a = 12, b = 11, A = 42°$
10. $a = 15, b = 17, A = 96°$
11. $a = 9, b = 10, A = 22°$
12. In $\triangle ABC$, a=4, b=5, and $m\angle A = 32°$. Find the possible value(s) of c.
13. In $\triangle DEF$, d=7, e=5, and $m\angle D = 67°$. Find the possible value(s) of f.
14. In $\triangle KQD$, $m\angle K = 20°$, k=24, and d=31. Find $m\angle D$.
15. In $\triangle MRS$, $m\angle M = 70°$, m=44, and r=25. Find $m\angle R$.

5.10 Law of Sines

Here you'll learn how to apply the Law of Sines when different types of triangles are presented.

While working in art class you are trying to design pieces of glass that you will eventually fit together into a sculpture. You are drawing out what you think will be a diagram of one of the pieces. You have a side of length 14 inches, and side of length 17 inches, and an angle next to the 17 in side of 35° (not the angle between the 14 in and 17 in pieces, but the one at the other end of the 17 in piece). Your diagram looks like this:

It occurs to you that you could use your knowledge of math to find out if you are going to be able to finish the drawing and make a piece that could actually be built.

Can you figure out how to do this?

By the end of this Concept, you'll know how to apply the Law of Sines to determine the number of possible solutions for a triangle.

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James Sousa:TheLawof Sines: The AmbiguousCase

Guidance

In $\triangle ABC$ below, we know two sides and a non-included angle. Remember that the Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. Since we know a, b, and $\angle A$, we can use the Law of Sines to find $\angle B$. However, since this is the SSA case, we have to watch out for the Ambiguous case. Since $a < b$, we could be faced with situations where either no possible triangles exist, one possible triangle exists, or two possible triangles exist.

To find out how many solutions there are in an ambiguous case, compare the length of a to $b\sin A$. If $a < b\sin A$, then there are no solutions. If $a = b\sin A$, then there is one solution. If $a > b\sin A$, then there are two solutions.

Example A

Find $\angle B$.

Solution: Use the Law of Sines to determine the angle.

$$\frac{\sin 41}{12} = \frac{\sin B}{23}$$
$$23\sin 41 = 12\sin B$$
$$\frac{23\sin 41}{12} = \sin B$$
$$1.257446472 = \sin B$$

Since no angle exists with a sine greater than 1, there is no solution to this problem.

We also could have compared a and $b\sin A$ beforehand to see how many solutions there were to this triangle.

$a = 12, b\sin A = 15.1$: since $12 < 15.1, a < b\sin A$ which tells us there are no solutions.

Example B

In $\triangle ABC, a = 15, b = 20$, and $\angle A = 30°$. Find $\angle B$.

Solution: Again in this case, $a < b$ and we know two sides and a non-included angle. By comparing a and $b\sin A$, we find that $a = 15, b\sin A = 10$. Since $15 > 10$ we know that there will be two solutions to this problem.

$$\frac{\sin 30}{15} = \frac{\sin B}{20}$$
$$20\sin 30 = 15\sin B$$
$$\frac{20\sin 30}{15} = \sin B$$
$$0.6666667 = \sin B$$
$$\angle B = 41.8°$$

There are two angles less than $180°$ with a sine of 0.6666667, however. We found the first one, $41.8°$, by using the inverse sine function. To find the second one, we will subtract $41.8°$ from $180°, \angle B = 180° - 41.8° = 138.2°$.

To check to make sure $138.2°$ is a solution, we will use the Triangle Sum Theorem to find the third angle. Remember that all three angles must add up to $180°$.

$$180° - (30° + 41.8°) = 108.2° \qquad \text{or} \qquad 180° - (30° + 138.2°) = 11.8°$$

This problem yields two solutions. Either $\angle B = 41.8°$ or $138.2°$.

Example C

A boat leaves lighthouse A and travels 63km. It is spotted from lighthouse B, which is 82km away from lighthouse A. The boat forms an angle of $65.1°$ with both lighthouses. How far is the boat from lighthouse B?

Solution: In this problem, we again have the SSA angle case. In order to find the distance from the boat to the lighthouse (a) we will first need to find the measure of $\angle A$. In order to find $\angle A$, we must first use the Law of Sines to find $\angle B$. Since $c > b$, this situation will yield exactly one answer for the measure of $\angle B$.

$$\frac{\sin 65.1°}{82} = \frac{\sin B}{63}$$
$$\frac{63 \sin 65.1°}{82} = \sin B$$
$$0.6969 \approx \sin B$$
$$\angle B = 44.2°$$

Now that we know the measure of $\angle B$, we can find the measure of angle A, $\angle A = 180° - 65.1° - 44.2° = 70.7°$. Finally, we can use $\angle A$ to find side a.

$$\frac{\sin 65.1°}{82} = \frac{\sin 70.7°}{a}$$
$$\frac{82 \sin 70.7°}{\sin 65.1°} = a$$
$$a = 85.3$$

The boat is approximately 85.3 km away from lighthouse B.

Vocabulary

Law of Cosines: The **law of cosines** is an equation relating the length of one side of a triangle to the lengths of the other two sides and the sine of the angle included between the other two sides.

Law of Sines: The **law of sines** is a rule applied to triangles stating that the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle in the triangle to the side opposite that angle.

Guided Practice

1. Prove using the Law of Sines: $\frac{a-c}{c} = \frac{\sin A - \sin C}{\sin C}$

2. Find all possible measures of angle B if any exist for the following triangle values: $A = 32.5°, a = 26, b = 37$

3. Find all possible measures of angle B if any exist for the following triangle values: $A = 42.3°, a = 16, b = 26$

Solutions:

1.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$c \sin A = a \sin C$$
$$c \sin A - c \sin C = a \sin C - c \sin C$$
$$c(\sin A - \sin C) = \sin C(a - c)$$
$$\frac{\sin A - \sin C}{\sin C} = \frac{a - c}{c}$$

2. $\frac{\sin 32.5°}{26} = \frac{\sin B}{37} \rightarrow B = 49.9°$ or $180° - 49.9° = 130.1°$

3. no solution

Concept Problem Solution

A drawing of this situation looks like this:

You can start by using the Law of Sines:

$\frac{\sin A}{a} = \frac{\sin B}{b}$

and substitute known values:

$\frac{\sin 35}{14} = \frac{\sin B}{17}$

Then solving for $\sin B$:

$\sin B = \frac{17 \sin 35°}{14}$

And so

$B \approx 44.15°$

Since the interior angles of any triangle add up to $180°$, we can find $\angle C$:

$$C = 180° - 35° - 44.15°$$
$$C = 100.85$$

This information can be used again in the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 35}{14} = \frac{\sin 100.86}{c}$$
$$c = \frac{14 \sin 100.86}{\sin 35}$$
$$c = \frac{13.75}{.5735}$$
$$c = 23.976$$

Practice

Find all possible measures of angle B if any exist for each of the following triangle values.

1. $A = 30°, a = 13, b = 15$
2. $A = 42°, a = 21, b = 12$
3. $A = 22°, a = 36, b = 37$
4. $A = 87°, a = 14, b = 12$
5. $A = 31°, a = 25, b = 44$
6. $A = 59°, a = 37, b = 41$
7. $A = 81°, a = 22, b = 20$
8. $A = 95°, a = 31, b = 34$
9. $A = 112°, a = 12, b = 15$
10. $A = 78°, a = 20, b = 16$
11. In $\triangle ABC$, a=10 and $m\angle B = 39°$. What's a possible value for b that would produce two triangles?

12. In $\triangle ABC$, a=15 and $m\angle B = 67°$. What's a possible value for b that would produce no triangles?
13. In $\triangle ABC$, a=21 and $m\angle B = 99°$. What's a possible value for b that would produce one triangle?
14. Bill and Connie are each leaving for school. Connie's house is 4 miles due east of Bill's house. Bill can see the school in the direction 40° east of north. Connie can see the school on a line 51° west of north. What is the straight line distance of each person from the school?
15. Rochelle and Rose are each looking at a hot air balloon. They are standing 2 miles apart. The angle of elevation for Rochelle is 30° and the angle of elevation for Rose is 34°. How high off the ground is the balloon?
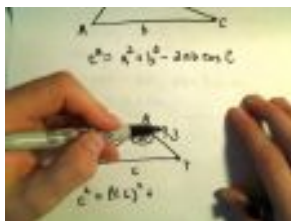
5.11 Law of Cosines

Here you'll learn to apply the Law of Cosines in different situations involving triangles.

While helping your mom bake one day, the two of you get an unusual idea. You want to cut the cake into pieces, and then frost over the surface of each piece. You start by cutting out a slice of the cake, but you don't quite cut the slice correctly. It ends up being an oblique triangle, with a 5 inch side, a 6 inch side, and an angle of $70°$ between the sides you measured. Can you help your mom determine the length of the third side, so she can figure out how much frosting to put out?

By the end of this Concept, you'll know how to find the length of the third side of the triangle in cases like this by using the Law of Cosines.

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The Law of Cosines

Guidance

The Law of Cosines is a fantastic extension of the Pythagorean Theorem to oblique triangles. In this Concept, we show some interesting ways to utilize this formula to analyze real world situations.

Example A

In a game of pool, a player must put the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a $168°$ angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?

Note: This is actually a trick shot performed by spinning the eight ball, and the eight ball will not actually travel in straight-line trajectories. However, to simplify the problem, assume that it travels in straight lines.

Solution: In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X).

$$y^2 = 6.8^2 + 2.1^2 - 2(6.8)(2.1)\cos 168°$$
$$y^2 = 78.59$$
$$y = 8.86 \; feet$$

The distance from the spot on the bumper to the pocket is 8.86 feet. We can now use this distance and the Law of Sines to find angle X. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since we know all three sides this problem will yield only one solution.

$$\frac{\sin 168°}{8.86} = \frac{\sin X}{6.8}$$
$$\frac{6.8\sin 168°}{8.86} = \sin X$$
$$0.1596 \approx \sin B$$
$$\angle B = 8.77°$$

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios. We will also look at another real-world application involving the SSA case.

Example B

Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain. The mountains forms a 103° angle with Person 1 and Person 3, while Person 2 forms a 92.7° angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.

Solution: In the triangle formed by the three people, we know two sides and the included angle (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

To find x:

$$x^2 = 131.5^2 + 67.8^2 - 2(131.5)(67.8)\cos 92.7$$
$$x^2 = 22729.06397$$
$$x = 150.8 \; yds$$

Now that we know $x = 150.8$, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since $150.8 > 72.6$, we know that we will have only one solution to this problem.

$$\frac{\sin 103}{150.8} = \frac{\sin Y}{72.6}$$
$$\frac{72.6\sin 103}{150.8} = \sin Y$$
$$0.4690932805 = \sin Y$$
$$28.0 \approx \angle Y$$

Example C

Katie is constructing a kite shaped like a triangle.

She knows that the lengths of the sides are a = 13 inches, b = 20 inches, and c = 19 inches. What is the measure of the angle between sides "a" and "b"?

Solution: Since she knows the length of each of the sides of the triangle, she can use the Law of Cosines to find the angle desired:

$$c^2 = a^2 + b^2 - 2(a)(b)\cos C$$
$$19^2 = 13^2 + 20^2 - (2)(13)(20)\cos C$$
$$361 = 169 + 400 - 520\cos C$$
$$-208 = -520\cos C$$
$$\cos C = 0.4$$
$$C \approx 66.42°$$

Vocabulary

Law of Cosines: The **law of cosines** is a rule involving the sides of an oblique triangle stating that the square of a side of the triangle is equal to the sum of the squares of the other two sides plus two times the lengths of the other two sides times the cosine of the angle opposite the side being computed.

Guided Practice

1. You are cutting a triangle out for school that looks like this:

Find side c (which is the side opposite the 14° angle) and $\angle B$ (which is the angle opposite the side that has a length of 14).

2. While hiking one day you walk for 2 miles in one direction. You then turn 110° to the left and walk for 3 more miles. Your path looks like this:

When you turn to the left again to complete the triangle that is your hiking path for the day, how far will you have to walk to complete the third side? What angle should you turn before you start walking back home?

3. A support at a construction site is being used to hold up a board so that it makes a triangle, like this:

If the angle between the support and the ground is 17°, the length of the support is 2.5 meters, and the distance between where the board touches the ground and the bottom of the support is 3 meters, how far along the board is the support touching? What is the angle between the board and the ground?

Solutions:

1. You know that two of the sides have lengths of 11 and 14 inches, and that the angle between them is 14°. You can use this to find the length of the third side:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
$$c^2 = 121 + 196 - (2)(11)(14)(.97)$$
$$c^2 = 121 + 196 - 307.384$$
$$c^2 = 9.16$$
$$c = 3.03$$

And with this you can use the Law of Sines to solve for the unknown angle:

$$\frac{\sin 14°}{3.03} = \frac{\sin B}{11}$$
$$\sin B = \frac{11\sin 14°}{3.03}$$
$$\sin B = .878$$
$$B = \sin^{-1}(.0307) = 61.43°$$

2. Since you know the lengths of two of the legs of the triangle, along with the angle between them, you can use the Law of Cosines to find out how far you'll have to walk along the third leg:

$$c^2 = a^2 + b^2 + 2ab\cos 70°$$
$$c^2 = 4 + 1 + (2)(2)(1)(.342)$$
$$c^2 = 6.368$$
$$c = \sqrt{6.368} \approx 2.52$$

Now you have enough information to solve for the interior angle of the triangle that is supplementary to the angle you need to turn:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 70°}{2.52} = \frac{\sin B}{2}$$
$$\sin B = \frac{2\sin 70°}{2.52} = \frac{1.879}{2.52} = .746$$
$$B = \sin^{-1}(.746) = 48.25°$$

The angle $48.25°$ is the interior angle of the triangle. So you should turn $90° + (90° - 48.25°) = 90° + 41.75° = 131.75°$ to the left before starting home.

3. You should use the Law of Cosines first to solve for the distance from the ground to where the support meets the board:

$$c^2 = a^2 + b^2 + 2ab\cos 17°$$
$$c^2 = 6.25 + 9 + (2)(2.5)(3)\cos 17°$$
$$c^2 = 6.25 + 9 + (2)(2.5)(3)(.956)$$
$$c^2 = 26.722$$
$$c \approx 5.17$$

And now you can use the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 17°}{5.17} = \frac{\sin B}{2.5}$$
$$\sin B = \frac{2.5 \sin 17°}{5.17} = .1414$$
$$B = \sin^{-1}(.1414) = 8.129°$$

Concept Problem Solution

You can use the Law of Cosines to help your mom find out the length of the third side on the piece of cake:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 5^2 + 6^2 + (2)(5)(6)\cos 70°$$
$$c^2 = 25 + 36 + 60(.342)$$
$$c^2 = 81.52$$
$$c \approx 9.03$$

The piece of cake is just a little over 9 inches long.

Practice

In $\triangle ABC$, a=12, b=15, and c=20.

1. Find $m\angle A$.
2. Find $m\angle B$.
3. Find $m\angle C$.

In $\triangle DEF$, d=25, e=13, and f=16.

4. Find $m\angle D$.
5. Find $m\angle E$.

6. Find $m\angle F$.

In $\triangle KBP$, k=19, $\angle B = 61°$, and p=12.

7. Find the length of b.
8. Find $m\angle K$.
9. Find $m\angle P$.
10. While hiking one day you walk for 5 miles due east, then turn to the left and walk 3 more miles 30° west of north. At this point you want to return home. How far are you from home if you were to walk in a straight line?
11. A parallelogram has sides of 20 and 31 ft, and an angle of 46°. Find the length of the longer diagonal of the parallelogram.
12. Dirk wants to find the length of a long building from one side (point A) to the other (point B). He stands outside of the building (at point C), where he is 500 ft from point A and 220 ft from point B. The angle at C is 94°. Find the length of the building.

Determine whether or not each triangle is possible.

13. a=12, b=15, c=10
14. a=1, b=5, c=4
15. $\angle A = 32°$, a=8, b=10

5.12 General Solutions of Triangles

In this Concept we'll use the Pythagorean Theorem, trigonometry functions, the Law of Sines, and the Law of Cosines to solve various triangles. Our focus will be on understanding when it is appropriate to use each method as well as how to apply the methods above in real-world and applied problems.

While talking with your little sister one day, the conversation turns to shapes. Your sister is only in junior high school, so while she knows some things about right triangles, such as the Pythagorean Theorem, she doesn't know anything about other types of triangles. You show her an example of an oblique triangle by drawing this on a piece of paper:

Fascinated, she tells you that she knows how to calculate the area of a triangle using the familiar formula $\frac{1}{2}bh$ and the lengths of sides if the triangle is a right triangle, but that she can't use the formulas on the triangle you just drew.

"Do you know how to find the lengths of sides of the triangle and the area?" she asks.

Read on, and at the end of this Concept, you'll be able to answer your sister's question.

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James Sousa:TheLawof Sines: The Basics

Guidance

In the previous sections we have discussed a number of methods for finding a missing side or angle in a triangle. Previously, we only knew how to do this in right triangles, but now we know how to find missing sides and angles in oblique triangles as well. By combining all of the methods we've learned up until this point, it is possible for us to find all missing sides and angles in any triangle we are given.

Below is a chart summarizing the triangle techniques that we have learned up to this point. This chart describes the type of triangle (either right or oblique), the given information, the appropriate technique to use, and what we can find using each technique.

TABLE 5.2:

Type of Triangle:	Given Information:	Technique:	What we can find:
Right	Two sides	Pythagorean Theorem	Third side
Right	One angle and one side	Trigonometric ratios	Either of the other two sides
Right	Two sides	Trigonometric ratios	Either of the other two angles

TABLE 5.2: (continued)

Type of Triangle:	Given Information:	Technique:	What we can find:
Oblique	2 angles and a non-included side (AAS)	Law of Sines	The other non-included side
Oblique	2 angles and the included side (ASA)	Law of Sines	Either of the non-included sides
Oblique	2 sides and the angle opposite one of those sides (SSA) – Ambiguous case	Law of Sines	The angle opposite the other side (can yield no, one, or two solutions)
Oblique	2 sides and the included angle (SAS)	Law of Cosines	The third side
Oblique	3 sides	Law of Cosines	Any of the three angles

Example A

In $\triangle ABC, a = 12, b = 13, c = 8$. Solve the triangle.

Solution: Since we are given all three sides in the triangle, we can use the Law of Cosines. Before we can solve the triangle, it is important to know what information we are missing. In this case, we do not know any of the angles, so we are solving for $\angle A, \angle B$, and $\angle C$. We will begin by finding $\angle A$.

$$12^2 = 8^2 + 13^2 - 2(8)(13)\cos A$$
$$144 = 233 - 208\cos A$$
$$-89 = -208\cos A$$
$$0.4278846154 = \cos A$$
$$64.7 \approx \angle A$$

Now, we will find $\angle B$ by using the Law of Cosines. Keep in mind that you can now also use the Law of Sines to find $\angle B$. Use whatever method you feel more comfortable with.

$$13^2 = 8^2 + 12^2 - 2(8)(12)\cos B$$
$$169 = 208 - 192\cos B$$
$$-39 = -192\cos B$$
$$0.2031 = \cos B$$
$$78.3° \approx \angle B$$

We can now quickly find $\angle C$ by using the Triangle Sum Theorem, $180° - 64.7° - 78.3° = 37°$

Example B

In triangle $DEF, d = 43, e = 37$, and $\angle F = 124°$. Solve the triangle.

Solution: In this triangle, we have the SAS case because we know two sides and the included angle. This means that we can use the Law of Cosines to solve the triangle. In order to solve this triangle, we need to find side $f, \angle D$, and $\angle E$. First, we will need to find side f using the Law of Cosines.

$$f^2 = 43^2 + 37^2 - 2(43)(37)\cos 124$$
$$f^2 = 4997.351819$$
$$f \approx 70.7$$

Now that we know f, we know all three sides of the triangle. This means that we can use the Law of Cosines to find either $\angle D$ or $\angle E$. We will find $\angle D$ first.

$$43^2 = 70.7^2 + 37^2 - 2(70.7)(37)\cos D$$
$$1849 = 6367.49 - 5231.8\cos D$$
$$-4518.49 = -5231.8\cos D$$
$$0.863658779 = \cos D$$
$$30.3° \approx \angle D$$

To find $\angle E$, we need only to use the Triangle Sum Theorem, $\angle E = 180 - (124 + 30.3) = 25.7°$.

Example C

In triangle $ABC, A = 43°, B = 82°$, and $c = 10.3$. Solve the triangle.

Solution: This is an example of the ASA case, which means that we can use the Law of Sines to solve the triangle. In order to use the Law of Sines, we must first know $\angle C$, which we can find using the Triangle Sum Theorem, $\angle C = 180° - (43° + 82°) = 55°$.

Now that we know $\angle C$, we can use the Law of Sines to find either side a or side b.

$$\frac{\sin 55}{10.3} = \frac{\sin 43}{a} \qquad\qquad \frac{\sin 55}{10.3} = \frac{\sin 82}{b}$$
$$a = \frac{10.3\sin 43}{\sin 55} \qquad\qquad b = \frac{10.3\sin 82}{\sin 55}$$
$$a = 8.6 \qquad\qquad b = 12.5$$

Vocabulary

Law of Cosines: The **law of cosines** is an equation relating the length of one side of a triangle to the lengths of the other two sides and the sine of the angle included between the other two sides.

Law of Sines: The **law of sines** is an equation relating the sine of an interior angle of a triangle divided by the side opposite that angle to a different interior angle of the same triangle divided by the side opposite that second angle.

Guided Practice

1. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

$A = 69°, B = 12°, a = 22.3$, find b

2. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

$a = 1.4, b = 2.3, C = 58°$, find c.

3. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have. If a triangle has no solution or two solutions, explain why.

$a = 3.3, b = 6.1, c = 4.8$, find A.

Solutions:

1. AAS, Law of Sines, one solution

2. SAS, Law of Cosines, one solution

3. SSS, Law of Cosines, one solution

Concept Problem Solution

Since you know that two of the angles are $23°$ and $28°$, the third angle in the triangle must be $180° - 23° - 28° = 129°$. Using these angles and the knowledge that one of the sides has a length of 4, you can solve for the lengths of the other two sides using the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 23°}{a} = \frac{\sin 129°}{4}$$
$$a = \frac{4 \sin 23°}{\sin 129°} = \frac{1.56}{.777}$$
$$a \approx 2$$

And repeating the process for the third side:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 23°}{2} = \frac{\sin 28°}{c}$$
$$c = \frac{2 \sin 28°}{\sin 23°} = \frac{.939}{.781}$$
$$c \approx 1.2$$

Now you know all three angles and all three sides. You can use Heron's formula or the alternative formula for the area of a triangle to find the area:

$$K = \frac{1}{2} bc \sin A$$
$$K = \frac{1}{2}(4)(1.2) \sin 23°$$
$$K = \frac{1}{2}(4)(1.2)(.391)$$
$$K \approx .9384$$

Practice

Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of the triangle and label the given information. Also, state how many solutions (if any) the triangle would have.

1. $a = 3, b = 4, C = 71°$, find c.
2. $a = 8, b = 7, c = 9$, find A.
3. $A = 135°, B = 12°, c = 100$, find a.
4. $a = 12, b = 10, A = 80°$, find c.
5. $A = 50°, B = 87°, a = 13$, find b.
6. In $\triangle ABC$, $a = 15, b = 19, c = 20$. Solve the triangle.
7. In $\triangle DEF$, $d = 12, E = 39°, f = 17$. Solve the triangle.
8. In $\triangle PQR$, $P = 115°, Q = 30°, q = 10$. Solve the triangle.
9. In $\triangle MNL$, $m = 5, n = 9, L = 20°$. Solve the triangle.
10. In $\triangle SEV$, $S = 50°, E = 44°, s = 12$. Solve the triangle.
11. In $\triangle KTS$, $k = 6, t = 15, S = 68°$. Solve the triangle.
12. In $\triangle WRS$, $w = 3, r = 5, s = 6$. Solve the triangle.
13. In $\triangle DLP$, $D = 52°, L = 110°, p = 8$. Solve the triangle.
14. In $\triangle XYZ$, $x = 10, y = 12, z = 9$. Solve the triangle.
15. In $\triangle AMF$, $A = 99°, m = 15, f = 16$. Solve the triangle.

5.13 Directed Line Segments

Here you'll learn to define a vector mathematically and draw it.

While babysitting your nephew one day you have a discussion regarding math. As a second grader, he is learning to add and subtract quantities. He asks you about what you're doing in math class, and you explain to him that you have just been introduced to vectors. Once you explain to him that a vector is a mathematical quantity that has both magnitude and direction, his question is both simple and yet brilliant: "Why do you need vectors? Can't everything just be described with numbers that don't have direction?"

At the end of this Concept, you'll know how to answer your nephew.

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James Sousa:Introduction toVectors

Guidance

A vector is represented diagrammatically by a directed line segment or arrow. A **directed line segment** has both **magnitude** and **direction**. **Magnitude** refers to the length of the directed line segment and is usually based on a scale. The vector quantity represented, such as influence of the wind or water current may be completely invisible.

A 25 mph wind is blowing from the northwest. If $1 \ cm = 5 \ mph$, then the vector would look like this:

An object affected by this wind would travel in a southeast direction at 25 mph.

A vector is said to be in **standard position** if its **initial point** is at the origin. The initial point is where the vector begins and the **terminal point** is where it ends. The axes are arbitrary. They just give a place to draw the vector.

If we know the coordinates of a vector's initial point and terminal point, we can use these coordinates to find the magnitude and direction of the vector.

All vectors have **magnitude**. This measures the total distance moved, total velocity, force or acceleration. "Distance" here applies to the magnitude of the vector even though the vector is a measure of velocity, force, or acceleration. In order to find the magnitude of a vector, we use the distance formula. A vector can have a negative magnitude. A force acting on a block pushing it at 20 lbs north can be also written as vector acting on the block from the south with a magnitude of -20 lbs. Such negative magnitudes can be confusing; making a diagram helps. The -20 lbs south can be re-written as +20 lbs north without changing the vector. Magnitude is also called the **absolute value** of a vector.

Example A

If a vector starts at the origin and has a terminal point with coordinates (3,5), find the length of the vector.

If we know the coordinates of the initial point and the terminal point, we can find the magnitude by using the distance formula. Initial point (0,0) and terminal point (3,5).

Solution: $|\vec{v}| = \sqrt{(3-0)^2 + (5-0)^2} = \sqrt{9+25} = 5.8$ The magnitude of $\vec{v}$ is 5.8.

If we don't know the coordinates of the vector, we must use a ruler and the given scale to find the magnitude. Also notice the notation of a vector, which is usually a lower case letter (typically $u, v,$ or w) in italics, with an arrow over it, which indicates direction. If a vector is in standard position, we can use trigonometric ratios such as sine, cosine and tangent to find the **direction** of that vector.

Example B

If a vector is in standard position and its terminal point has coordinates of (12, 9) what is the direction?

Solution: The horizontal distance is 12 while the vertical distance is 9. We can use the tangent function since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{9}{12}$$
$$\tan^{-1}\frac{9}{12} = 36.9°$$

So, the direction of the vector is 36.9°.

If the vector isn't in standard position and we don't know the coordinates of the terminal point, we must a protractor to find the direction.

Two vectors are **equal** if they have the same magnitude and direction. Look at the figures below for a visual understanding of **equal vectors**.

Example C

Determine if the two vectors are equal.

$\vec{a}$ is in standard position with terminal point (-4, 12)

$\vec{b}$ has an initial point of (7, -6) and terminal point (3, 6)

Solution: You need to determine if both the magnitude and the direction are the same.

$$\begin{aligned}
\text{Magnitude}: \quad &|\vec{a}| = \sqrt{(0-(-4))^2 + (0-12)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
&|\vec{b}| = \sqrt{(7-3)^2 + (-6-6)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
\text{Direction}: \quad &\vec{a} \rightarrow \tan\theta = \frac{12}{-4} \rightarrow \theta = 108.43° \\
&\vec{b} \rightarrow \tan\theta = \frac{-6-6}{7-3} = \frac{-12}{4} \rightarrow \theta = 108.43°
\end{aligned}$$

Because the magnitude and the direction are the same, we can conclude that the two vectors are equal.

Vocabulary

Directed Line Segment: A **directed line segment** is a portion of a line that has both a magnitude and direction.

Magnitude: A **magnitude** is the length of a line segment or vector.

Vector: A **vector** is a type of mathematical quantity that has both a magnitude and a direction.

Guided Practice

1. Given the initial and terminal coordinates below, find the magnitude and direction of the vector that results.

initial (2, 4) terminal (8, 6)

2. Given the initial and terminal coordinates below, find the magnitude and direction of the vector that results.

initial (5, -2) terminal (3, 1)

3. Assume $\vec{a}$ is in standard position. For the terminal point (12, 18), find the magnitude and direction of the vector.

Solutions:

1. $|\vec{a}| = \sqrt{(2-8)^2 + (4-6)^2} = 6.3$, direction $= \tan^{-1}\left(\frac{4-6}{2-8}\right) = 18.4°$

2. $|\vec{a}| = \sqrt{(5-3)^2 + (-2-1)^2} = 3.6$, direction $= \tan^{-1}\left(\frac{-2-1}{5-3}\right) = 123.7°$. Note that when you use your calculator to solve for $\tan^{-1}\left(\frac{-2-1}{5-3}\right)$, you will get $-56.3°$. The calculator produces this answer because the range of the calculator's $y = \tan^{-1} x$ function is limited to $-90° < y < 90°$. You need to sketch a draft of the vector to see that its direction when placed in standard position is into the second quadrant (and not the fourth quadrant), and so the correct angle is calculated by moving the angle into the second quadrant through the equation $-56.3° + 180° = 123.7°$.

3. $|\vec{a}| = \sqrt{12^2 + 18^2} = 21.6$, direction $= \tan^{-1}\left(\frac{18}{12}\right) = 56.3°$

Concept Problem Solution

Your nephew's thinking is quite good. Many things in the world can be described by numbers, without the use of direction. However, math needs to "line up" with reality. If something doesn't work with just numbers, there needs to be a new type of mathematical quantity to describe the behavior completely. For example, consider two cars that are both moving at 25 miles per hour. Will they collide?

You can see that there isn't enough information to answer the question. You don't know which way the cars are going. If the two cars are going in the same direction, then they won't collide. If, however, they are going directly at each other, then they will certainly collide. In order to describe the behavior of the cars completely, a quantity is needed that is not just the magnitude of the car's motion, but also the direction - which is why vectors are needed.

And that is how you should answer your nephew.

Practice

1. What is the difference between the magnitude and direction of a vector?
2. How can you determine the magnitude of a vector if you know its initial point and terminal point?
3. How can you determine the direction of a vector if you know its initial point and terminal point?
4. How can you determine whether or not two vectors are equal?
5. If a vector starts at the origin and has a terminal point with coordinates (2, 7), find the magnitude of the vector.
6. If a vector is in standard position and its terminal point has coordinates of (3, 9), what is the direction of the vector?
7. If a vector has an initial point at (1, 6) and has a terminal point at (5, 9), find the magnitude of the vector.
8. If a vector has an initial point at (1, 4) and has a terminal point at (8, 7), what is the direction of the vector?

Given the initial and terminal coordinates below, find the magnitude and direction of the vector that results.

9. initial (4, -1); terminal (5, 3)
10. initial (2, -3); terminal (4, 5)
11. initial (3, 2); terminal (0, 3)
12. initial (-2, 5); terminal (2, 1)

Determine if the two vectors are equal.

13. $\vec{a}$ is in standard position with terminal point (1, 5) and $\vec{b}$ has an initial point (3, -2) and terminal point (4, 2).
14. $\vec{c}$ has an initial point (-3, 1) and terminal point (1, 2) and $\vec{d}$ has an initial point (3, 5) and terminal point (7, 6).
15. $\vec{e}$ is in standard position with terminal point (2, 3) and $\vec{f}$ has an initial point (1, -6) and terminal point (3, -9).

5.14 Vector Addition

Here you'll learn how to add two vectors with either the triangle method or the parallelogram method.

You and a friend are pulling a box across a floor. However, each of you is pulling at a different angle. A diagram of your efforts looks like this:

Each of these forces is a vector. Can you determine the net force you and your friend are applying to the box? To find the net result of the effort, you need to add the vectors for each of the forces. By the end of this Concept, you'll be able to accomplish this.

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Vector Addition:head-to-tail method

Guidance

The sum of two or more vectors is called the **resultant** of the vectors. There are two methods we can use to find the resultant: the parallelogram method and the triangle method.

The Parallelogram Method:*It is important to note that we cannot use the parallelogram method to find the sum of a vector and itself.*

To find the sum of the resultant vector, we would again use a ruler and a protractor to find the magnitude and direction.

If you look closely, you'll notice that the parallelogram method is really a version of the triangle or tip-to-tail method. If you look at the top portion of the figure above, you can see that one side of our parallelogram is really vector b translated.

The Triangle Method:*This method is also referred to as the tip-to-tail method.*

To find the sum of the resultant vector we would use a ruler and a protractor to find the magnitude and direction.

The resultant vector can be much longer than either $\vec{a}$ or $\vec{b}$, or it can be shorter. Below are some more examples of the triangle method.

Example A

Example B

Example C

Vocabulary

Parallelogram Method: The **parallelogram method** is a method of adding vectors by creating a parallelogram from the two vectors being added.

Resultant: A **resultant** is a vector representing the sum of two or more vectors.

Triangle Method: The **triangle method** is a method of adding vectors by connecting the tail of one vector to the head of another vector.

Guided Practice

1. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if $|\vec{m}| = 29.8 |\vec{n}| = 37.7$

2. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if $|\vec{m}| = 2.8 |\vec{n}| = 5.4$

3. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if $|\vec{m}| = 11.9 |\vec{n}| = 9.4$

Solutions:

1. For the problem, use the Pythagorean Theorem to find the magnitude and $\tan \theta = \frac{|\vec{n}|}{|\vec{m}|}$

magnitude $= 48.1$, direction $= 51.7°$

2. For the problem, use the Pythagorean Theorem to find the magnitude and $\tan \theta = \frac{|\vec{n}|}{|\vec{m}|}$

magnitude $= 6.1$, direction $= 62.6°$

3. For the problem, use the Pythagorean Theorem to find the magnitude and $\tan \theta = \frac{|\vec{n}|}{|\vec{m}|}$

magnitude $= 15.2$, direction $= 38.3°$

Concept Problem Solution

A triangle method diagram of the vectors being added looks like this:

As you can see, the resultant force has a magnitude of 100 Newtons at an angle of $45°$

Practice

$\vec{a}$ is in standard position with terminal point (1, 5) and $\vec{b}$ is in standard position with terminal point (4, 2).

1. Find the coordinates of the terminal point of the resultant vector.
2. What is the magnitude of the resultant vector?
3. What is the direction of the resultant vector?

$\vec{c}$ is in standard position with terminal point (4, 3) and $\vec{d}$ is in standard position with terminal point (2, 2).

4. Find the coordinates of the terminal point of the resultant vector.
5. What is the magnitude of the resultant vector?
6. What is the direction of the resultant vector?

$\vec{e}$ is in standard position with terminal point (3, 2) and $\vec{f}$ is in standard position with terminal point (-1, 2).

7. Find the coordinates of the terminal point of the resultant vector.
8. What is the magnitude of the resultant vector?
9. What is the direction of the resultant vector?

$\vec{g}$ is in standard position with terminal point (5, 5) and $\vec{h}$ is in standard position with terminal point (4, 2).

10. Find the coordinates of the terminal point of the resultant vector.
11. What is the magnitude of the resultant vector?
12. What is the direction of the resultant vector?

$\vec{i}$ is in standard position with terminal point (1, 5) and $\vec{j}$ is in standard position with terminal point (-3, 1).

13. Find the coordinates of the terminal point of the resultant vector.
14. What is the magnitude of the resultant vector?
15. What is the direction of the resultant vector?
16. Vectors $\vec{k}$ and $\vec{l}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{k}$ and $\vec{l}$) of their resultant if $|\vec{k}| = 42$ and $|\vec{l}| = 30$.
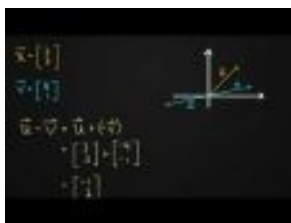
5.15 Vector Subtraction

Here you'll learn how to subtract two vectors using the triangle method.

You and a friend are trying to position a heavy sculpture out in front of your school. Fortunately, the sculpture is on rollers, so you can move it around easily and slide it into place. While you are applying force to the sculpture, it starts to move. The vectors you and your friend are applying look like this:

However, the sculpture starts to move too far and overshoots where it is supposed to be. You quickly tell your friend to pull instead of push, in effect subtracting her force vector, where before it was being added. Can you represent this graphically?

By the end of this Concept, you'll know how to represent the subtraction of vectors and answer this question.

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Vector Subtraction

Guidance

As you know from Algebra, $A - B = A + (-B)$. When we think of vector subtraction, we must think about it in terms of adding a negative vector. A **negative** vector is the same magnitude of the original vector, but its direction is opposite.

In order to subtract two vectors, we can use either the triangle method or the parallelogram method from above. The only difference is that instead of adding vectors A and B, we will be adding A and $-B$.

Example A

Using the triangle method for subtraction.

Example B

Example C

Vocabulary

Negative Vector: A **negative vector** is a vector that is the same in magnitude as the original vector, but opposite in direction.

Triangle Method: The **triangle method** is a method of adding vectors by connecting the tail of one vector to the head of another vector.

Guided Practice

1. For the vector subtraction below, make a diagram of the subtraction. $\vec{a} - \vec{d}$

2. For the vector subtraction below, make a diagram of the subtraction. $\vec{b} - \vec{a}$

3. For the vector subtraction below, make a diagram of the subtraction. $\vec{d} - \vec{c}$

Solutions:

1.

2.

3.

Concept Problem Solution

As you've seen in this Concept, subtracting a vector is the same as adding the negative of the original vector. This is exactly like the rule for adding a negative number to a positive number. Therefore, to change your friend's force vector to a subtraction instead of an addition, you need to change the direction by 180° while keeping the magnitude the same. The graph looks like this:

Practice

$\vec{a}$ is in standard position with terminal point (1, 5) and $\vec{b}$ is in standard position with terminal point (4, 2).

1. Find the coordinates of the terminal point of $\vec{a} - \vec{b}$.
2. What is the magnitude of $\vec{a} - \vec{b}$?
3. What is the direction of $\vec{a} - \vec{b}$?

$\vec{c}$ is in standard position with terminal point (4, 3) and $\vec{d}$ is in standard position with terminal point (2, 2).

4. Find the coordinates of the terminal point of $\vec{c} - \vec{d}$.
5. What is the magnitude of $\vec{c} - \vec{d}$?
6. What is the direction of $\vec{c} - \vec{d}$?

$\vec{e}$ is in standard position with terminal point (3, 2) and $\vec{f}$ is in standard position with terminal point (-1, 2).

7. Find the coordinates of the terminal point of $\vec{e} - \vec{f}$.
8. What is the magnitude of $\vec{e} - \vec{f}$?
9. What is the direction of $\vec{e} - \vec{f}$?

$\vec{g}$ is in standard position with terminal point (5, 5) and $\vec{h}$ is in standard position with terminal point (4, 2).

10. Find the coordinates of the terminal point of $\vec{g} - \vec{h}$.
11. What is the magnitude of $\vec{g} - \vec{h}$?
12. What is the direction of $\vec{g} - \vec{h}$?

$\vec{i}$ is in standard position with terminal point (1, 5) and $\vec{j}$ is in standard position with terminal point (-3, 1).

13. Find the coordinates of the terminal point of $\vec{i} - \vec{j}$.
14. What is the magnitude of $\vec{i} - \vec{j}$?
15. What is the direction of $\vec{i} - \vec{j}$?

5.16 Resultant of Two Displacements

In this Concept you will learn to interpret the resultant of vector addition as a physical situation involving the combination of two displacements.

You are outside playing frisbee on a warm afternoon with your friends. You are trying to throw the frisbee to your friend, but unfortunately, the wind is blowing and keeps pushing your frisbee away from your friend and off course. If you are throwing the frisbee at 20 miles per hour at your friend who is due North of you, and the wind is blowing toward the East at 5 miles per hour, what is the actual trajectory of the frisbee?

Keep reading, and by the end of this Concept, you'll understand how to apply vector addition to exactly this sort of problem.

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Adding Vectors Part 1 (ResultantVelocity)

Guidance

We can use vectors to find direction, velocity, and force of moving objects. In this section we will look at a few applications where we will use resultants of vectors to find speed, direction, and other quantities. A displacement is a distance considered as a vector. If one is 10 ft away from a point, then any point at a radius of 10 ft from that point satisfies the condition. If one is 28 degrees to the east of north, then only one point satisfies this.

Example A

A cruise ship is traveling south at 22 mph. A wind is also blowing the ship eastward at 7 mph. What speed is the ship traveling at and in what direction is it moving?

Solution: In order to find the direction and the speed the boat is traveling, we must find the resultant of the two vectors representing 22 mph south and 7 mph east. Since these two vectors form a right angle, we can use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the resultant vector.

First, we will find the speed.

$$22^2 + 7^2 = x^2$$
$$533 = x^2$$
$$23.1 = x$$

The ship is traveling at a speed of 23.1mph.

To find the direction, we will use tangent, since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{7}{22}$$

$$\tan^{-1}\frac{7}{22} = 17.7°$$

The ship's direction is $S17.7°E$.

Example B

A hot air balloon is rising at a rate of 13 ft/sec, while a wind is blowing at a rate of 22 ft/sec. Find the speed at which the balloon is traveling as well as its angle of elevation.

Solution: First, we will find the speed at which our balloon is rising. Since we have a right triangle, we can use the Pythagorean Theorem to find calculate the magnitude of the resultant.

$$x^2 = 13^2 + 22^2$$

$$x^2 = 653$$

$$x = 25.6 \ ft/\sec$$

The balloon is traveling at rate of 25.6 feet per second.

To find the angle of elevation of the balloon, we need to find the angle it makes with the horizontal. We will find the $\angle A$ in the triangle and then we will subtract it from $90°$.

$$\tan A = \frac{22}{13}$$

$$A = \tan^{-1}\frac{22}{13}$$

$$A = 59.4°$$

Angle with the horizontal $= 90 - 59.4 = 30.6°$.

The balloon has an angle of elevation of $30.6°$.

Example C

Continuing on with the previous example, find:

a. How far from the lift off point is the balloon in 2 hours? Assume constant rise and constant wind speed. (this is total displacement

b. How far must the support crew travel on the ground to get under the balloon? (horizontal displacement

c. If the balloon stops rising after 2 hours and floats for another 2 hours, how far from the initial point is it at the end of the 4 hours? How far away does the crew have to go to be under the balloon when it lands?

Solution:

a. After two hours, the balloon will be 184,320 feet from the lift off point (25.6 ft/sec multiplied by 7200 seconds in two hours).

b. After two hours, the horizontal displacement will be 158,400 feet (22ft/sec multiplied by 7200 seconds in two hours).

c. After two hours, the balloon will have risen 93,600 feet. After an additional two hours of floating (horizontally only) in the 22ft/sec wind, the balloon will have traveled 316,800 feet horizontally (22ft/second times 14,400 seconds in four hours).

We must recalculate our resultant vector using Pythagorean Theorem.

$$x = \sqrt{93600^2 + 316800^2} = 330338 \; ft.$$

The balloon is 330,338 feet from its initial point. The crew will have to travel 316,800 feet or 90 miles (horizontal displacement) to be under the balloon when it lands.

Vocabulary

Displacement: A **displacement** is a distance considered as a vector.

Guided Practice

1. Does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$? Explain your answer.

2. A plane is traveling north at a speed of 225 mph while an easterly wind is blowing the plane west at 18 mph. What is the direction and the speed of the plane?

3. Two workers are pulling on ropes attached to a tree stump. One worker is pulling the stump east with 330 Newtons of forces while the second working is pulling the stump north with 410 Newtons of force. Find the magnitude and direction of the resultant force on the tree stump.

Solutions:

1. When two vectors are summed, the magnitude of the resulting vector is almost always different than the sum of the magnitudes of the two initial vectors. The only times that $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ would be true is when 1) the magnitude of at least one of the two vectors to be added is zero, or 2) both of the vectors to be added have the same direction.

2. Speed (magnitude): $\sqrt{18^2 + 225^2} = 225.7$ and its direction is $\tan\theta = \frac{18}{225} = N4.6°W$.

3. The magnitude is $\sqrt{330^2 + 410^2} = 526.3$ Newtons and the direction is $\tan^{-1}\left(\frac{410}{330}\right) = E51.2°N$.

Concept Problem Solution

When you draw each of the vectors, one for your throw of the frisbee and one for the wind, they look like this:

And combining them, as we saw in this Concept, results in this:

The diagonal line in above diagram is the result of both of the vector displacements, and is therefore the actual path of the frisbee.

Practice

A ship is traveling north at 30 mph. A constant 5 mph wind is coming from the east (blowing west).

1. What is the ship's actual speed?
2. In what direction is the ship moving?
3. How far does the ship travel in 2 hours?
4. If the ship travels for 2 hours and then just floats for another two hours, how far has the trip traveled?
5. If the ship travels for 2 hours and then just floats for another two hours, how far from its starting point is the ship?

A balloon is rising at a rate of 20 cm/s, while a wind is blowing at a constant rate of 15 cm/s.

6. What is the balloon's actual speed?
7. What is the balloon's angle of elevation?
8. At what rate would the wind have to be blowing for the balloon to be traveling at 50 cm/s?
9. After 3 hours, how far must the support crew travel on the ground to get under the balloon?

A plane is traveling west at a speed of 500 mph while a northerly wind is blowing the plane south at 30 mph.

10. What is the actual speed of the plane?
11. What is the actual direction of the plane?
12. How far will the plane travel in 4 hours?
13. How long will it take for the plane to travel exactly 3000 miles?

A plane is takes off at a rate of 230 mph with an angle of 30° to the ground.

14. What is the rate at which the plane is receding from the ground?
15. What is its ground speed?

5.17 Vector Multiplied by a Scalar

Here you'll learn how to multiply a vector by a scalar, what the resulting quantity is, and how to represent it.

While at summer camp, you are enjoying a tug of war with your friends. You are on one side of the rope, pulling with a force of 200 N. The vector for this force can be represented like this:

You decide to really go for the win and pull as hard as you possibly can. As it turns out, you are pulling with twice the force you were before. Do you know how you can represent the vector for this?

At the conclusion of this Concept you'll understand exactly how to understand and represent the results of multiplying a vector by a scalar.

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Multiplying a Vectorby a Scalar

Guidance

In working with vectors there are two kinds of quantities employed. The first is the vector, a quantity that has both magnitude and direction. The second quantity is a scalar. Scalars are just numbers. The magnitude of a vector is a scalar quantity. A vector can be multiplied by a real number. This real number is called a **scalar**. The product of a vector $\vec{a}$ and a scalar k is a vector, written $\vec{ka}$. It has the same direction as $\vec{a}$ with a magnitude of $k|\vec{a}|$ if $k > 0$. If $k < 0$, the vector has the opposite direction of $\vec{a}$ and a magnitude of $k|\vec{a}|$.

Example A

The speed of the wind before a hurricane arrived was 20 mph from the SSE ($N22.5°W$). It quadrupled when the hurricane arrived. What is the current vector for wind velocity?

Solution: The wind is coming now at 80 mph from the same direction.

Example B

A sailboat was traveling at 15 knots due north. After realizing he had overshot his destination, the captain turned the boat around and began traveling twice as fast due south. What is the current velocity vector of the ship?

Solution: The ship is traveling at 30 knots in the opposite direction.

If the vector is expressed in coordinates with the starting end of the vector at the origin, this is called standard form. To perform a scalar multiplication, we multiply our scalar by both the coordinates of our vector. The word

scalar comes from "scale." Multiplying by a scalar just makes the vectors longer or shorter, but doesn't change their direction.

Example C

Consider the vector from the origin to (4, 6). What would the representation of a vector that had three times the magnitude be?

Solution: Here $k = 3$ and $\vec{v}$ is the directed segment from (0,0) to (4, 6).

Multiply each of the components in the vector by 3.

$$\vec{kv} = (0,0) \; to \; (12,18)$$

The new coordinates of the directed segment are (0, 0), (12, 18).

Vocabulary

Magnitude: A **magnitude** is the length of a line segment or vector.

Scalar: A **scalar** is a number.

Vector: A **vector** is a type of mathematical quantity that has both a magnitude and a direction.

Guided Practice

1. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (0,0) \; to \; (5,4)$ and $\vec{a} = 2\vec{b}$.

2. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (0,0) \; to \; (5,4)$ and $\vec{a} = -\frac{1}{2}\vec{b}$.

3. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (0,0) \; to \; (5,4)$ and $\vec{a} = 0.6\vec{b}$.

Solutions:

1. $2\vec{b} = 2\langle 5,4\rangle = \langle 10,8\rangle = 10\hat{i} + 8\hat{j}$

2. $-\frac{1}{2}\vec{c} = -\frac{1}{2}\langle -3,7\rangle = \langle 1.5,-3.5\rangle = 1.5\hat{i} - 3.5\hat{j}$

3. $0.6\vec{b} = 0.6\langle 5,4\rangle = \langle 3,2.4\rangle = 3\hat{i} + 2.4\hat{j}$

Concept Problem Solution

Since the original vector looked like this:

The new vector is equal to 2 times the old vector. As we found in this Concept, multiplication of a vector by a scalar doesn't change the direction of the vector, only its magnitude. Therefore, the new vector looks like this:

Its length is twice the length of the original vector, and its direction is unchanged.

Practice

$\vec{a}$ is in standard position with terminal point (1, 5) and $\vec{b}$ is in standard position with terminal point (4, 2).

1. Find the coordinates of the terminal point of $2\vec{a}$.

2. Find the coordinates of the terminal point of $\frac{1}{2}\vec{b}$.
3. Find the coordinates of the terminal point of $6\vec{a}$.

$\vec{c}$ is in standard position with terminal point (4, 3) and $\vec{d}$ is in standard position with terminal point (2, 2).

4. Find the coordinates of the terminal point of $3\vec{c} + 2\vec{d}$.
5. Find the coordinates of the terminal point of $4\vec{c} - 0.3\vec{d}$.
6. Find the coordinates of the terminal point of $\vec{c} - 3\vec{d}$.

$\vec{e}$ is in standard position with terminal point (3, 2) and $\vec{f}$ is in standard position with terminal point (-1, 2).

7. Find the coordinates of the terminal point of $4\vec{e} + 5\vec{f}$.
8. Find the coordinates of the terminal point of $3\vec{e} - 3\vec{f}$.
9. Find the coordinates of the terminal point of $5\vec{e} + \frac{3}{4}\vec{f}$.

$\vec{g}$ is in standard position with terminal point (5, 5) and $\vec{h}$ is in standard position with terminal point (4, 2).

10. Find the coordinates of the terminal point of $\vec{g} + 4\vec{h}$.
11. Find the coordinates of the terminal point of $5\vec{g} - 2\vec{h}$.
12. Find the coordinates of the terminal point of $2\vec{g} - 3\vec{h}$.

$\vec{i}$ is in standard position with terminal point (1, 5) and $\vec{j}$ is in standard position with terminal point (-3, 1).

13. Find the coordinates of the terminal point of $3\vec{i} - \vec{j}$.
14. Find the coordinates of the terminal point of $0.5\vec{i} - 0.6\vec{j}$.
15. Find the coordinates of the terminal point of $6\vec{i} + 1.2\vec{j}$.

5.18 Translation of Vectors and Slope

Here you'll learn what vector translation is and how to perform it.

You and your friends are at a ski weekend for your school. While skiing, you go down a hill that is rather steep. You decide to use a vector to represent your motion from the top of the hill going down. Counting the top of the hill as the origin, you ski down a slope and measure how far your "x" and "y" positions have changed. As it turns out, you can represent this displacement with the vector $(12, -256)$. Can you calculate the incline (slope) of the hill you came down?

By the end of this Concept, you'll understand how to determine the slope of vectors and answer this question.

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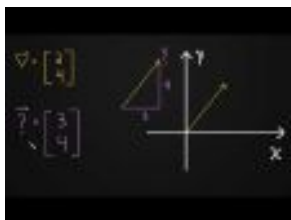
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Translational Property of Vectors

Guidance

Vectors with the same magnitude and direction are equal. This means that the same ordered pair could represent many different vectors. For instance, the ordered pair (4, 8) can represent a vector in standard position where the initial point is at the origin and the terminal point is at (4, 8). This vector could be thought of as the resultant of a horizontal vector with a magnitude or 4 units and a vertical vector with a magnitude of 8 units. Therefore, any vector with a horizontal component of 4 and vertical component of 8 could also be represented by the ordered pair (4, 8).

If you think back to Algebra, you know that the slope of a line is the change in y over the change in x, or the vertical change over the horizontal change.

Example A

Consider the vector from (4, 7) to (12, 11). What would the representation of a vector that had 2.5 times the magnitude be?

Solution: Here, $k = 2.5$ and $\vec{v}$ = the directed segment from (4, 7) to (12, 11).

Mathematically, two vectors are equal if their direction and magnitude are the same. The positions of the vectors do not matter. This means that if we have a vector that is not in standard position, we can translate it to the origin. The initial point of $\vec{v}$ is (4, 7). In order to **translate** this to the origin, we would need to add (-4, -7) to both the initial and terminal points of the vector.

Initial point: $(4,7) + (-4,-7) = (0,0)$

Terminal point: $(12,11) + (-4,-7) = (8,4)$

Now, to calculate $\vec{kv}$:

$$\vec{kv} = (2.5(8), 2.5(4))$$
$$\vec{kv} = (20, 10)$$

The new coordinates of the directed segment are (0, 0) and (20, 10). To translate this back to our original terminal point:

Initial point: $(0,0) + (4,7) = (4,7)$

Terminal point: $(20, 10) + (4,7) = (24, 17)$

The new coordinates of the directed segment are (4, 7) and (24, 17).

Example B

What is the slope of a vector starting from the origin with terminal coordinates (5 , 7)?

Solution: Since the slope is defined as the change in "y" divided by the change in "x", we can find the slope of this vector:

$$\frac{\Delta y}{\Delta x} = \frac{7-0}{5-0} = \frac{7}{5} = 1.4$$

Example C

A vector starts at the origin and has terminal coordinates (11 , 17). What would the new coordinates of the tail and tip of the vector be if the vector were shifted 15 units along the "x" axis?

Solution: The vector maintains the same orientation in space, it is just moved down the "x" axis. Therefore, only the "x" coordinates the of the vector's tail and tip change.

So the new coordinates of the tail of the vector are:

$$(0 + 15, 0) = (15, 0)$$

And the new coordinates of the tip are:

$$(11 + 15, 17) = (26, 17)$$

Vocabulary

Translate: To move a vector on a coordinate system without changing its length or orientation is to **translate** it.

Vector: A **vector** is a type of mathematical quantity that has both a magnitude and a direction.

Guided Practice

1. Find the magnitude of the horizontal and vertical components of the following vector given the following coordinates of their initial and terminal points.

initial $= (-3, 8)$ terminal $= (2, -1)$

2. Find the magnitude of the horizontal and vertical components of the following vector given the following coordinates of their initial and terminal points.

initial $= (7, 13)$ terminal $= (11, 19)$

3. Find the magnitude of the horizontal and vertical components of the following vector given the following coordinates of their initial and terminal points.

initial $= (4.2, -6.8)$ terminal $= (-1.3, -9.4)$

Solutions:

1. The vector needs to be translated to (0,0). Also, recall that magnitudes are always positive.

$(-3, 8) + (3, -8) = (0, 0)$ $(2, -1) + (3, -8) = (5, -9)$

horizontal $= 5$, vertical $= 9$

2. The vector needs to be translated to (0,0). Also, recall that magnitudes are always positive.

$(7, 13) + (-7, -13) = (0, 0)$ $(11, 19) + (-7, -13) = (4, 6)$

horizontal $= 4$, vertical $= 6$

3. The vector needs to be translated to (0,0). Also, recall that magnitudes are always positive.

$(4.2, -6.8) + (-4.2, 6.8) = (0, 0)$ $(-1.3, -9.4) + (-4.2, 6.8) = (-5.5, -2.6)$

horizontal $= 5.5$, vertical $= 2.6$

Concept Problem Solution

Since the slope is defined as the change in "y" divided by the change in "x", we can find the slope of the vector representing your trip down the hill:

$$\frac{\Delta y}{\Delta x} = \frac{-256 - 0}{12 - 0} = \frac{-256}{12} \approx -21.33$$

This means that for every foot the hill changed in the "x" direction, it went down 21.33 feet in the "y" direction. That's a steep hill indeed!

Practice

In each question below, the initial and terminal coordinates for a vector are given. If the vector is translated so that it is in standard position (with the initial point at the origin), what are the new terminal coordinates?

1. initial (2, 5) and terminal (7, -1)
2. initial (4, 3) and terminal (3, -5)
3. initial (8, 1) and terminal (-4, 7)
4. initial (-2, 7) and terminal (3, 5)
5. initial (4, -3) and terminal (4, 3)
6. initial (0, 2) and terminal (6, -4)

Find the slope of each vector below with the given terminal coordinates. Assume the vector is in standard position.

7. terminal (6, 7)

8. terminal (3, 6)
9. terminal (-2, 4)
10. terminal (5, 8)
11. terminal (1, 3)

Find the magnitude of the horizontal and vertical components of each vector given the coordinates of their initial and terminal points.

12. initial (1, 5) and terminal (1, -3)
13. initial (4, 5) and terminal (6, -5)
14. initial (6, 1) and terminal (-4, 4)
15. initial (-2, 3) and terminal (2, 5)

5.19 Unit Vectors and Components

Here you'll learn how to break down a vector into component vectors and unit vectors.

While working in your math class at school, the instructor passes everyone a map of Yourtown. She asks you to find your house and place a red dot on it, and then find the school and place a blue dot there. Your map looks like this:

She then asks you to break down the trip to your school in terms of component vectors and unit vectors. Are you able to do this?

Keep reading, and by the end of this Concept, you'll understand how to break a vector into its components and graph it using unit vector notation.

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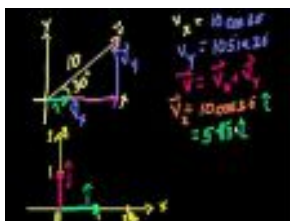
James Sousa:Unit VectorNotation

Guidance

A **unit vector** is a vector that has a magnitude of one unit and can have any direction. Traditionally $\hat{i}$ (read "i hat") is the unit vector in the x direction and $\hat{j}$ (read "j hat") is the unit vector in the y direction. $|\hat{i}| = 1$ and $|\hat{j}| = 1$. Unit vectors on perpendicular axes can be used to express all vectors in that plane. Vectors are used to express position and motion in three dimensions with $\hat{k}$ ("k hat") as the unit vector in the z direction. We are not studying 3D space in this course. The unit vector notation may seem burdensome but one must distinguish between a vector and the components of that vector in the direction of the $x-$ or $y-$axis. The unit vectors carry the meaning for the direction of the vector in each of the coordinate directions. The number in front of the unit vector shows its magnitude or length. Unit vectors are convenient if one wishes to express a 2D or 3D vector as a sum of two or three orthogonal components, such as $x-$ and $y-$axes, or the $z-$axis. (Orthogonal components are those that intersect at right angles.)

Component vectors of a given vector are two or more vectors whose sum is the given vector. The sum is viewed as equivalent to the original vector. Since component vectors can have any direction, it is useful to have them perpendicular to one another. Commonly one chooses the x and y axis as the basis for the unit vectors. Component vectors do not have to be orthogonal.

A vector from the origin (0, 0) to the point (8, 0) is written as $8\hat{i}$. A vector from the origin to the point (0, 6) is written as $6\hat{j}$.

The reason for having the component vectors perpendicular to one another is that this condition allows us to use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the components. One can solve vector problems without use of unit vectors if specific information about orientation or direction in space such as N, E, S or W is part of the problem.

Example A

What are the component vectors of the vector shown here?

Solution: Since the length of the vector is 5, and the angle the vector makes with the x axis is $53.13°$, the "x" component of the vector is:

$$|V_x| = |\vec{V}|\cos 53.13°$$
$$|V_x| = (5)(.6) = 3$$

And the "y" component is:

$$|V_y| = |\vec{V}|\sin 53.13°$$
$$|V_y| = (5)(.8) = 4$$

And we have the familiar 3, 4, 5 triangle, where the vector is the hypotenuse.

Example B

Why are unit vectors required when dealing with vector addition?

Solution:

Unit vectors are required because it is necessary to have like quantities for addition. If there are two numbers, they can be added. If there are two vectors, they can be added. But if you have a number and a vector, they can't be added. Having unit vectors along with a magnitude makes a quantity a vector.

Example C

What are the unit vectors and the lengths of the component vectors when

$\vec{V} = 7\hat{i} + 9\hat{j}$

Solution:

The unit vectors in this case are $\hat{i}$ and $\hat{j}$. In some courses and books you might see the notation for unit vectors written instead as $\hat{x}$ and $\hat{y}$.

The length of the component vector in the $\hat{i}$ direction is 7, and the component vector in the $\hat{j}$ direction is 9.

Vocabulary

Component Vectors: The **component vectors** of a given vector are two or more vectors whose sum is the given vector.

Unit Vector: A **unit vector** is a vector having a length of one.

Guided Practice

1. An inclined ramp is 12 feet long and forms an angle of 28.2° with the ground. Find the horizontal and vertical components of the ramp.

2. A wind vector has a magnitude of 25 miles per hour with an angle of 20° with respect to the east. Determine how much the wind is blowing to the north and how much it is blowing to the east.

3. A vector $|\vec{V}|$ has a magnitude of 25 inches, and is at an angle of 80° with respect to the positive "x" axis. Write the vector in component and unit vector notation.

Solutions:

1. $y = 12\sin 28.2° = 5.7, x = 12\cos 28.2° = 10.6$

2. Since the vector has an angle of 20° with respect to the horizontal, the component to the east is $25\cos 20° = 23.49$ miles per hour. In the same way, the component to the north is $25\sin 20° = 8.55$ miles per hour.

3. The "x" component is $25\cos 80° = 4.34$. The "y" component is $25\sin 80° = 24.62$. Therefore, the vector can be written as $|\vec{V}| = 4.34\hat{i} + 24.62\hat{j}$.

Concept Problem Solution

In this Concept you learned that breaking a vector down into its components involves adding the portion of the vector along the "y" axis to the portion of the vector along the "x" axis. To accomplish this in the case of the map, you only need to write down the length the vector has in the "x" direction (along with an "x" unit vector) and then add to it the length the vector has in the "y" direction (along with a "y" unit vector). Your map should look like this:

Practice

1. Describe how to find the vertical and horizontal components of a vector when given the magnitude and direction of the vector.
2. $\vec{a}$ has a magnitude of 6 and a direction of 100°. Find the components of the vector.
3. $\vec{b}$ has a magnitude of 3 and a direction of 60°. Find the components of the vector.
4. $\vec{c}$ has a magnitude of 2 and a direction of 84°. Find the components of the vector.
5. $\vec{d}$ has a magnitude of 5 and a direction of 32°. Find the components of the vector.
6. $\vec{e}$ has a magnitude of 2 and a direction of 45°. Find the components of the vector.
7. $\vec{f}$ has a magnitude of 7 and a direction of 70°. Find the components of the vector.
8. A plane is flying on a bearing of 50° at 450 mph. Find the component form of the velocity of the plane. What does the component form tell you?
9. A baseball is thrown at a 20° angle with the horizontal with an initial speed of 30 mph. Find the component form of the initial velocity.
10. A plane is flying on a bearing of 300° at 500 mph. Find the component form of the velocity of the plane.
11. A plane is flying on a bearing of 150° at 470 mph. At the same time, there is a wind blowing at a bearing of 200° at 60 mph. What is the component form of the velocity of the plane?
12. Using the information from the previous problem, find the actual ground speed of the plane.
13. Wind is blowing at a magnitude of 50 mph with an angle of 25° with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?
14. Find a unit vector in the direction of $\vec{a}$, a vector in standard position with terminal point (-4, 3).
15. Find a unit vector in the direction of $\vec{b}$, a vector in standard position with terminal point (5, 1).

5.20 Resultant as the Sum of Two Components

Here you'll learn how to express a vector as the sum of two component vectors.

You are working in science class on a "weather unit". As part of this class, you are tasked with going out and checking the wind speed each day at a meter behind your school. The wind speed you record for the day is 20 mph at a $E50°N$ trajectory. This means that the wind is blowing at an angle $50°$ taken from the direction that would be due East. At the conclusion of each day, you are supposed to break the wind speed (which is a vector) into two components: the portion that is in a North/South direction and the portion that is in an East/West direction. Can you figure out how to do this?

By the end of this Concept, you'll be able to break vectors into their individual components using trig relationships.

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How to Add Vectors UsingComponents

Guidance

We can look at any vector as the resultant of two perpendicular components. If we generalize some vector $\vec{q}$ into perpendicular components, $|\vec{r}|\hat{i}$ is the horizontal component of a vector $\vec{q}$ and $|\vec{s}|\hat{j}$ is the vertical component of $\vec{q}$. Therefore $\vec{r}$ is a magnitude, $|\vec{r}|$, times the unit vector in the x direction and $\vec{s}$ is its magnitude, $|\vec{s}|$, times the unit vector in the y direction. The sum of $\vec{r}$ plus $\vec{s}$ is: $\vec{r} + \vec{s} = \vec{q}$. This addition can also be written as $|\vec{r}|\hat{i} + |\vec{s}|\hat{j} = \vec{q}$.

If we are given the vector $\vec{q}$, we can find the components of $\vec{q}, \vec{r}$, and $\vec{s}$ using trigonometric ratios if we know the magnitude and direction of $\vec{q}$.

This is accomplished by taking the magnitude of the vector times the cosine of the vector's angle to find the horizontal component, and the magnitude of the vector times the sine of the vector's angle to find the vertical component.

Example A

If $|\vec{q}| = 19.6$ and its direction is $73°$, find the horizontal and vertical components.

Solution: If we know an angle and a side of a right triangle, we can find the other remaining sides using trigonometric ratios. In this case, $\vec{q}$ is the hypotenuse of our triangle, $\vec{r}$ is the side adjacent to our $73°$ angle, $\vec{s}$ is the side opposite our $73°$ angle, and $\vec{r}$ is directed along the x−axis.

To find $\vec{r}$, we will use cosine and to find $\vec{s}$ we will use sine. Notice this is a scalar equation so all quantities are just numbers. It is written as the quotient of the magnitudes, not the vectors.

$$\cos 73 = \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}$$

$$\cos 73 = \frac{r}{19.6}$$

$$r = 19.6 \cos 73$$

$$r = 5.7$$

$$\sin 73 = \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}$$

$$\sin 73 = \frac{s}{19.6}$$

$$s = 19.6 \sin 73$$

$$s = 18.7$$

The horizontal component is 5.7 and the vertical component is 18.7. One can rewrite this in vector notation as $5.7\hat{i} + 18.7\hat{j} = \vec{q}$. The components can also be written $\vec{q} = \langle 5.7, 18.7 \rangle$, with the horizontal component first, followed by the vertical component. Be careful not to confuse this with the notation for plotted points.

Example B

If $|\vec{m}| = 12.1$ and its direction is $31°$, find the horizontal and vertical components.

To find $\vec{r}$, we will use cosine and to find $\vec{s}$ we will use sine. Notice this is a scalar equation so all quantities are just numbers. It is written as the quotient of the magnitudes, not the vectors.

$$\cos 31 = \frac{|\vec{r}|}{|\vec{m}|} = \frac{r}{m}$$

$$\cos 31 = \frac{r}{12.1}$$

$$r = 12.1 \cos 31$$

$$r = 10.37$$

$$\sin 31 = \frac{|\vec{s}|}{|\vec{m}|} = \frac{s}{m}$$

$$\sin 31 = \frac{s}{12.1}$$

$$s = 12.1 \sin 31$$

$$s = 6.23$$

Example C

If $|\vec{r}| = 15$ and $|\vec{s}| = 11$, find the resultant vector length and angle.

Solution: We can view each of these vectors on the coordinate system here:

Each of these vectors then serves as sides in a right triangle. So we can use the Pythagorean Theorem to find the length of the resultant:

$$c^2 = a^2 + b^2$$

$$c^2 = 15^2 + 11^2$$

$$c^2 = 225 + 121$$

$$c^2 = 346$$

$$c = \sqrt{346} \approx 18.60$$

The angle of rotation that the vector makes with the "x" axis can be found using the tangent function:

$$\tan\theta = \frac{opposite}{adjacent}$$
$$\tan\theta = \frac{11}{15}$$
$$\tan\theta = .73$$
$$\theta = \tan^{-1}.73$$
$$\theta \approx 36.13$$

Vocabulary

Component Vectors: The **component vectors** of a given vector are two or more vectors whose sum is the given vector.

Resultant: The **resultant** is a vector that is the sum of two or more vectors.

Guided Practice

1. Find the magnitude of the horizontal and vertical components of the following vector if the resultant vector's magnitude and direction are given as magnitude $= 75$ \qquad direction $= 35°$.

2. Find the magnitude of the horizontal and vertical components of the following vector if the resultant vector's magnitude and direction are given as magnitude $= 3.4$ \qquad direction $= 162°$.

3. Find the magnitude of the horizontal and vertical components of the following vector if the resultant vector's magnitude and direction are given as magnitude $= 15.9$ \qquad direction $= 12°$.

Solutions:

1. $\cos 35° = \frac{x}{75}, \sin 35° = \frac{y}{75}, x = 61.4, y = 43$

2. $\cos 162° = \frac{x}{3.4}, \sin 162° = \frac{y}{3.4}, x = 3.2, y = 1.1$

3. $\cos 12° = \frac{x}{15.9}, \sin 12° = \frac{y}{15.9}, x = 15.6, y = 3.3$

Concept Problem Solution

From this Concept you've learned how to take a vector and break it into components using trig functions. If you draw the wind speed you recorded as a vector:

You can find the "x" and "y" components. These are the same as the part of the wind that is blowing to the East and the part of the wind that is blowing to the North.

East component: $\cos 50° = \frac{x}{20}$

$20\cos 50° = x$

x = 12.86 mph

North component: $\sin 50° = \frac{y}{20}$

$20\sin 50° = y$

y = 15.32 mph

Practice

Find the horizontal and vertical components of the following vectors given the resultant vector's magnitude and direction.

1. magnitude $= 65$ direction $= 22°$.
2. magnitude $= 34$ direction $= 15°$.
3. magnitude $= 29$ direction $= 160°$.
4. magnitude $= 100$ direction $= 320°$.
5. magnitude $= 320$ direction $= 200°$.
6. magnitude $= 15$ direction $= 110°$.
7. magnitude $= 10$ direction $= 80°$.
8. magnitude $= 90$ direction $= 290°$.
9. magnitude $= 87$ direction $= 10°$.
10. magnitude $= 42$ direction $= 150°$.

11. If $|\vec{r}| = 12$ and $|\vec{s}| = 8$, find the resultant vector magnitude and angle.
12. If $|\vec{r}| = 14$ and $|\vec{s}| = 6$, find the resultant vector magnitude and angle.
13. If $|\vec{r}| = 9$ and $|\vec{s}| = 24$, find the resultant vector magnitude and angle.
14. Will cosine always be used to find the horizontal component of a vector?
15. If you know the component form of a vector, how can you find its magnitude and direction?

5.21 Resultant as Magnitude and Direction

Here you'll learn how to express a vector as a combination of the vector's magnitude (length) and its direction (which way it is pointing).

You are out playing soccer with friends after school one day when you and a friend kick the soccer ball at the same time. However, you kick the ball with 55 N of force with a vector like this:

and your friend kicks the ball with 70 N of force with a vector like this:

The angle between the two vectors is 74°, and combined the graph of both vectors looks like this:

Can you represent the net force on the ball? What will the resultant vector be like?

Keep reading, and by the time you finish this Concept, you'll be able to make this calculation and draw the resultant vector.

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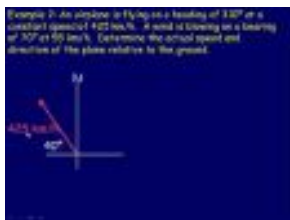
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Applicationsof Vector Addition

Guidance

If we don't have two perpendicular vectors, we can still find the magnitude and direction of the resultant without a graphic estimate with a construction using a compass and ruler. This can be accomplished using both the Law of Sines and the Law of Cosines.

Let's investigate some examples of this.

Example A

$\vec{A}$ makes a 54° angle with $\vec{B}$. The magnitude of $\vec{A}$ is 13.2. The magnitude of $\vec{B}$ is 16.7. Find the magnitude and direction the resultant makes with the smaller vector.

There is no preferred orientation such as a compass direction or any necessary use of x and y coordinates. The problem can be solved without the use of unit vectors.

Solution: In order to solve this problem, we will need to use the parallelogram method. Since vectors only have magnitude and direction, one can move them on the plane to any position one wishes, as long as the magnitude and direction remain the same. First, we will complete the parallelogram: Label the vectors. Move $\vec{b}$ so its tail is on the tip of $\vec{a}$. Move $\vec{a}$ so its tail is on the tip of $\vec{b}$. This makes a parallelogram because the angles did not change during the translation. Put in labels for the vertices of the parallelogram.

Since opposite angles in a parallelogram are congruent, we can find $\angle A$.

$$\angle CBD + \angle CAD + \angle ACB + \angle BDA = 360$$
$$2\angle CBD + 2\angle ACB = 360$$
$$\angle ACB = 54°$$
$$2\angle CBD = 360 - 2(54)$$
$$\angle CBD = \frac{360 - 2(54)}{2} = 126$$

Now, we know two sides and the included angle in an oblique triangle. This means we can use the Law of Cosines to find the magnitude of our resultant.

$$x^2 = 13.2^2 + 16.7^2 - 2(13.2)(16.7)\cos 126$$
$$x^2 = 712.272762$$
$$x = 26.7$$

To find the direction, we can use the Law of Sines since we now know an angle and a side across from it. We choose the Law of Sines because it is a proportion and less computationally intense than the Law of Cosines.

$$\frac{\sin \theta}{16.7} = \frac{\sin 126}{26.7}$$
$$\sin \theta = \frac{16.7\sin 126}{26.7}$$
$$\sin \theta = 0.5060143748$$
$$\theta = \sin^{-1} 0.5060 = 30.4°$$

The magnitude of the resultant is 26.7 and the direction it makes with the smaller vector is 30.4° counterclockwise.

We can use a similar method to add three or more vectors.

Example B

Vector A makes a 45° angle with the horizontal and has a magnitude of 3. Vector B makes a 25° angle with the horizontal and has a magnitude of 5. Vector C makes a 65° angle with the horizontal and has a magnitude of 2. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.

Solution: To begin this problem, we will find the resultant using Vector A and Vector B. We will do this using the parallelogram method like we did above.

Since Vector A makes a 45° angle with the horizontal and Vector B makes a 25° angle with the horizontal, we know that the angle between the two ($\angle ADB$) is 20°.

To find $\angle DBE$:

$$2\angle ADB + 2\angle DBE = 360$$
$$\angle ADB = 20°$$
$$2\angle DBE = 360 - 2(20)$$
$$\angle DBE = \frac{360 - 2(20)}{2} = 160$$

Now, we will use the Law of Cosines to find the magnitude of *DE*.

$$DE^2 = 3^2 + 5^2 - 2(3)(5)\cos 160$$
$$DE^2 = 62$$
$$DE = 7.9$$

Next, we will use the Law of Sines to find the measure of $\angle EDB$.

$$\frac{\sin 160}{7.9} = \frac{\sin \angle EDB}{3}$$
$$\sin \angle EDB = \frac{3 \sin 160}{7.9}$$
$$\sin \angle EDB = .1299$$
$$\angle EDB = \sin^{-1} 0.1299 = 7.46°$$

We know that Vector *B* forms a 25° angle with the horizontal so we add that value to the measure of $\angle EDB$ to find the angle *DE* makes with the horizontal. Therefore, *DE* makes a 32.46° angle with the horizontal.

Next, we will take *DE*, and we will find the resultant vector of *DE* and Vector *C* from above. We will repeat the same process we used above.

Vector *C* makes a 65° angle with the horizontal and *DE* makes a 32° angle with the horizontal. This means that the angle between the two ($\angle CDE$) is 33°. We will use this information to find the measure of $\angle DEF$.

$$2\angle CDE + 2\angle DEF = 360$$
$$\angle CDE = 33°$$
$$2\angle DEF = 360 - 2(33)$$
$$\angle DEF = \frac{360 - 2(33)}{2} = 147$$

Now we will use the Law of Cosines to find the magnitude of *DF*.

$$DF^2 = 7.9^2 + 2^2 - 2(7.9)(2)\cos 147$$
$$DF^2 = 92.9$$
$$DF = 9.6$$

Next, we will use the Law of Sines to find $\angle FDE$.

$$\frac{\sin 147}{9.6} = \frac{\sin \angle FDE}{2}$$
$$\sin \angle FDE = \frac{2 \sin 147}{9.6}$$
$$\sin \angle FDE = .1135$$
$$\angle FDE = \sin^{-1} 0.1135 = 6.5° = 7°$$

Finally, we will take the measure of $\angle FDE$ and add it to the 32° angle that *DE* forms with the horizontal. Therefore, DF forms a 39° angle with the horizontal.

Example C

Two forces of 310 lbs and 460 lbs are acting on an object. The angle between the two forces is $61.3°$. What is the magnitude of the resultant? What angle does the resultant make with the smaller force?

Solution: We do not need unit vectors here as there is no preferred direction like a compass direction or a specific axis. First, to find the magnitude we will need to figure out the other angle in our parallelogram.

$$2\angle ACB + 2\angle CAD = 360$$
$$\angle ACB = 61.3°$$
$$2\angle CAD = 360 - 2(61.3)$$
$$\angle CAD = \frac{360 - 2(61.3)}{2} = 118.7$$

Now that we know the other angle, we can find the magnitude using the Law of Cosines.

$$x^2 = 460^2 + 310^2 - 2(460)(310)\cos 118.7°$$
$$x^2 = 444659.7415$$
$$x = 667$$

To find the angle the resultant makes with the smaller force, we will use the Law of Sines.

$$\frac{\sin\theta}{460} = \frac{\sin 118.7}{666.8}$$
$$\sin\theta = \frac{460\sin 118.7}{666.8}$$
$$\sin\theta = .6049283888$$
$$\theta = \sin^{-1} 0.6049 = 37.2°$$

Vocabulary

Resultant: The **resultant** is a vector that is the sum of two or more vectors.

Guided Practice

1. Forces of 140 Newtons and 186 Newtons act on an object. The angle between the forces is $43°$. Find the magnitude of the resultant and the angle it makes with the larger force.

2. An airplane is traveling at a speed of 155 km/h. It's heading is set at $83°$ while there is a 42.0 km/h wind from $305°$. What is the airplane's actual heading?

3. If $\overrightarrow{AB}$ is any vector, what is $\overrightarrow{AB} + \overrightarrow{BA}$?

Solutions:

1. magnitude $= 304, 18.3°$ between resultant and larger force

2. Recall that headings and angles in triangles are complementary. So, an $83°$ heading translates to $7°$ from the horizontal. Adding that to $35°$ ($270°$ from $305°$) we get $42°$ for two of the angles in the parallelogram. So, the

other angles in the parallelogram measure 138° each, $\frac{360-2(42)}{2}$. Using 138° in the Law of Cosines, we can find the diagonal or resultant, $x^2 = 42^2 + 155^2 - 2(42)(155)\cos 138$, so $x = 188.3$. We then need to find the angle between the resultant and the speed using the Law of Sines. $\frac{\sin a}{42} = \frac{\sin 138}{188.3}$, so $a = 8.6°$. To find the actual heading, this number needs to be added to 83°, getting 91.6°.

3. *BA* is the same vector as *AB*, but because it starts with *B* it is in the opposite direction. Therefore, when you add the two together, you will get (0,0).

Concept Problem Solution

As you've seen in this Concept, you can represent the vector resulting from both of your forces as the resultant of vector addition. Since vectors only have magnitude and direction, one can move them on the plane to any position one wishes, as long as the magnitude and direction remain the same. First, we will complete the parallelogram: Label the vectors. Move $\vec{b}$ so its tail is on the tip of $\vec{a}$. Move $\vec{a}$ so its tail is on the tip of $\vec{b}$. This makes a parallelogram because the angles did not change during the translation. Put in labels for the vertices of the parallelogram.

Since opposite angles in a parallelogram are congruent, we can find angle A.

$$\angle CBD + \angle CAD + \angle ACB + \angle BDA = 360$$
$$2\angle CBD + 2\angle ACB = 360$$
$$\angle ACB = 72°$$
$$2\angle CBD = 360 - 2(72)$$
$$\angle CBD = \frac{360 - 2(72)}{2} = 108$$

Now, we know two sides and the included angle in an oblique triangle. This means we can use the Law of Cosines to find the magnitude of our resultant.

$$x^2 = 70^2 + 55^2 - 2(70)(55)\cos 72$$
$$x^2 = 5545.569$$
$$x = 74.47$$

To find the direction, we can use the Law of Sines since we now know an angle and a side across from it. We choose the Law of Sines because it is a proportion and less computationally intense than the Law of Cosines.

$$\frac{\sin \theta}{70} = \frac{\sin 108}{74.47}$$
$$\sin \theta = \frac{70 \sin 108}{74.47}$$
$$\sin \theta = 0.89397$$
$$\theta = \sin^{-1} 0.89397 = 63.68°$$

The combined force of your kick along with your friend's is 74.47 Newtons and the direction it makes with your kick is 63.68° counterclockwise.

Practice

$\vec{a}$ makes a 42° angle with $\vec{b}$. The magnitude of $\vec{a}$ is 15. The magnitude of $\vec{b}$ is 22.

1. Find the magnitude of the resultant.
2. Find the angle of the resultant makes with the smaller vector.

$\vec{c}$ makes a 80° angle with $\vec{d}$. The magnitude of $\vec{c}$ is 70. The magnitude of $\vec{d}$ is 45.

3. Find the magnitude of the resultant.
4. Find the angle of the resultant makes with the smaller vector.

$\vec{e}$ makes a 50° angle with $\vec{f}$. The magnitude of $\vec{e}$ is 32. The magnitude of $\vec{f}$ is 10.

5. Find the magnitude of the resultant.
6. Find the angle of the resultant makes with the smaller vector.

$\vec{g}$ makes a 100° angle with $\vec{h}$. The magnitude of $\vec{g}$ is 50. The magnitude of $\vec{h}$ is 35.

7. Find the magnitude of the resultant.
8. Find the angle of the resultant makes with the smaller vector.
9. Two forces of 100 lbs and 120 lbs are acting on an object. The angle between the two forces is 50°. What is the magnitude of the resultant?
10. Using the information from the previous problem, what angle does the resultant make with the larger force?
11. A force of 50 lbs acts on an object at an angle of 30°. A second force of 75 lbs acts on the object at an angle of −10°. What is the magnitude of the resultant force?
12. Using the information from the previous problem, what is the direction of the resultant force?
13. A plane is flying on a bearing of 30° at a speed of 450 mph. A wind is blowing with the bearing 200° at 50 mph. What is the plane's actual direction?
14. Vector A makes a 30° angle with the horizontal and has a magnitude of 4. Vector B makes a 55° angle with the horizontal and has a magnitude of 6. Vector C makes a 75° angle with the horizontal and has a magnitude of 3. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.
15. Vector A makes a 12° angle with the horizontal and has a magnitude of 5. Vector B makes a 25° angle with the horizontal and has a magnitude of 2. Vector C makes a 60° angle with the horizontal and has a magnitude of 7. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.

Summary

This Chapter introduced ways to find unknown quantities in triangles when other quantities are known. Included were derivations and applications of the Law of Sines and the Law of Cosines. This was followed by alternate formulas for finding the area of a triangle, including Heron's Formula.

Cases where quantities were known, such as two angles and the not included side, or two angles and the included side, were presented, along with methods to find the other unknown quantities. This led to the "ambiguous case", where triangles with two sides and the not included angle as known quantities were addressed.

The Chapter then turned to vectors, including their definition, translation, addition, subtraction, multiplication by a scalar, and decomposition into combinations of unit vectors.

CHAPTER **6**

Polar System

Chapter Outline

Introduction

When graphing numbers, you are accustomed to plotting points on a "rectangular coordinate system", involving "x" and "y" coordinates. However, there is another way to plot numbers, called a "polar coordinate system". This Chapter will introduce you to how to plot numbers on this coordinate system, as well as how to translate a plot from rectangular coordinates to polar coordinates, and vice versa.

After this, you'll be introduced to how to integrate your knowledge of complex numbers and their functions into this newly described method of plotting.

6.1 Plots of Polar Coordinates

Here you'll learn how to express a point in a polar coordinate system as a distance from an origin and the angle with respect to an axis.

While playing a game of darts with your friend, you decide to see if you can plot the coordinates of where your darts land. The dartboard looks like this

While trying to set up a rectangular coordinate system, your friend tells you that it would be easier to plot the positions of your darts using a "polar coordinate system". Can you do this?

At the end of this Concept, you'll be able to accomplish this plotting task successfully.

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James Sousa:Introduction toPolar Coordinates

Guidance

The graph paper that you have used for plotting points and sketching graphs has been rectangular grid paper. All points were plotted in a rectangular form (x, y) by referring to a set of perpendicular $x-$ and $y-$ axes. In this section you will discover an alternative to graphing on rectangular grid paper – graphing on circular grid paper.

Look at the two options below:

You are all familiar with the rectangular grid paper shown above. However, the circular paper lends itself to new discoveries. The paper consists of a series of concentric circles-circles that share a common center. The common center O, is known as the pole or origin and the polar axis is the horizontal line r that is drawn from the pole in a positive direction. The point P that is plotted is described as a directed distance r from the pole and by the angle that $\overline{OP}$ makes with the polar axis. The coordinates of P are (r, θ).

These coordinates are the result of assuming that the angle is rotated counterclockwise. If the angle were rotated clockwise then the coordinates of P would be $(r, -\theta)$. These values for P are called polar coordinates and are of the form $P(r, \theta)$ where r is the absolute value of the distance from the pole to P and θ is the angle formed by the polar axis and the terminal arm $\overline{OP}$.

Example A

Plot the point $A(5, -255°)$ and the point $B(3, 60°)$

Solution, A: To plot A, move from the pole to the circle that has $r = 5$ and then rotate $255°$**clockwise** from the polar axis and plot the point on the circle. Label it A.

Solution, B: To plot B, move from the pole to the circle that has $r = 3$ and then rotate $60°$**counter clockwise** from the polar axis and plot the point on the circle. Label it B.

Example B

Determine four pairs of polar coordinates that represent the following point $P(r, \theta)$ such that $-360° \leq \theta \leq 360°$.

Solution: Pair $1 \rightarrow (4, 120°)$. Pair $2 \rightarrow (4, -240°)$ comes from using $k = -1$ and $(r, \theta + 360°k), (4, 120° + 360(-1))$. Pair $3 \rightarrow (-4, 300°)$ comes from using $k = 0$ and $(-r, \theta + [2k+1]180°), (-4, 120° + [2(0)+1]180°)$. Pair $4 \rightarrow (-4, -60°)$ comes from using $k = -1$ and $(-r, \theta + [2k+1]180°), (-4, 120° + [2(-1)+1]180°)$.

Example C

Plot the following coordinates in polar form and give their description in polar terms: (1,0), (0,1), (-1,0), (-1,1)

Solution: The points plotted are shown above. Since each point is 1 unit away from the origin, we know that the radius of each point in polar form will be equal to 1.

The first point lies on the positive 'x' axis, so the angle in polar coordinates is $0°$. The second point lies on the positive 'y' axis, so the angle in polar coordinates is $90°$. The third point lies on the negative 'x' axis, so the angle in polar coordinates is $180°$. The fourth point lies on the negative 'y' axis, so the angle in polar coordinates is $270°$.

Vocabulary

Polar Coordinates: A set of **polar coordinates** are a set of coordinates plotted on a system that uses the distance from the origin and angle from an axis to describe location.

Guided Practice

1. Plot the point: $M(2.5, 210°)$
2. $S\left(-3.5, \frac{5\pi}{6}\right)$
3. $A\left(1, \frac{3\pi}{4}\right)$

Solutions:

1.

2.

3.

Concept Problem Solution

Since you have the positions of the darts on the board with both the distance from the origin and the angle they make with the horizontal, you can describe them using polar coordinates.

As you can see, the positions of the darts are:

$(3, 45°), (6, 90°)$

and

$(4, 0°)$

Practice

Plot the following points on a polar coordinate grid.

1. $(3, 150°)$
2. $(2, 90°)$
3. $(5, 60°)$
4. $(4, 120°)$
5. $(3, 210°)$
6. $(-2, 120°)$
7. $(4, -90°)$
8. $(-5, -30°)$
9. $(2, -150°)$
10. $(-3, 300°)$

Give three alternate sets of coordinates for the given point within the range $-360° \leq \theta \leq 360°$.

11. $(3, 60°)$
12. $(2, 210°)$
13. $(4, 330°)$
14. Find the length of the arc between the points $(2, 30°)$ and $(2, 90°)$.
15. Find the area of the sector created by the origin and the points $(4, 30°)$ and $(4, 90°)$.
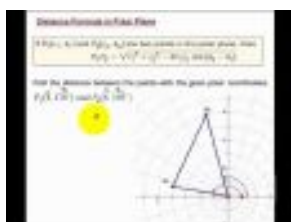
6.2 Distance Between Two Polar Coordinates

Here you'll learn how to find the distance between two points that are plotted in a polar coordinate system.

When playing a game of darts with your friend, the darts you throw land in a pattern like this

You and your friend decide to find out how far it is between the two darts you threw. If you know the positions of each of the darts in polar coordinates, can you somehow find a formula to let you determine the distance between the two darts?

At the end of this Concept, you'll be able to answer this question.

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DistanceFormulain Polar Plane

Guidance

Just like the Distance Formula for x and y coordinates, there is a way to find the distance between two polar coordinates. One way that we know how to find distance, or length, is the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$ or $a = \sqrt{b^2 + c^2 - 2bc\cos A}$. If we have two points (r_1, θ_1) and (r_2, θ_2), we can easily substitute r_1 for b and r_2 for c. As for A, it needs to be the angle between the two radii, or $(\theta_2 - \theta_1)$. Finally, a is now distance and you have $d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$.

Example A

Find the distance between $(3, 60°)$ and $(5, 145°)$.

Solution: After graphing these two points, we have a triangle. Using the new Polar Distance Formula, we have $d = \sqrt{3^2 + 5^2 - 2(3)(5)\cos 85°} \approx 5.6$.

Example B

Find the distance between $(9, -45°)$ and $(-4, 70°)$.

Solution: This one is a little trickier than the last example because we have negatives. The first point would be plotted in the fourth quadrant and is equivalent to $(9, 315°)$. The second point would be $(4, 70°)$ reflected across the pole, or $(4, 250°)$. Use these two values of θ for the formula. Also, the radii should always be positive when put into the formula. That being said, the distance is $d = \sqrt{9^2 + 4^2 - 2(9)(4)\cos(315 - 250)°} \approx 8.16$.

Example C

Find the distance between $(2, 10°)$ and $(7, 10°)$.

Solution: This problem is straightforward from looking at the relationship between the points. The two points lie at the same angle, so the straight line distance between them is $7 - 2 = 5$. However, we can confirm this using the distance formula:

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos 0°} = \sqrt{4 + 49 - 28} = \sqrt{25} = 5.$$

Vocabulary

Polar Coordinates: A set of **polar coordinates** are a set of coordinates plotted on a system that uses the distance from the origin and angle from an axis to describe location.

Guided Practice

1. Given P_1 and P_2, calculate the distance between the points.

$P_1(1, 30°)$ and $P_2(6, 135°)$

2. Given P_1 and P_2, calculate the distance between the points.

$P_1(2, -65°)$ and $P_2(9, 85°)$

3. Given P_1 and P_2, calculate the distance between the points.

$P_1(-3, 142°)$ and $P_2(10, -88°)$

Solutions:

1. Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$.

$$P_1P_2 = \sqrt{1^2 + 6^2 - 2(1)(6)\cos(135° - 30°)}$$
$$P_1P_2 \approx 6.33 \ units$$

2. Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$.

$$P_1P_2 = \sqrt{2^2 + 9^2 - 2(2)(9)\cos 150°}$$
$$= 10.78$$

3. Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$.

$$P_1P_2 = \sqrt{3^2 + 10^2 - 2(3)(10)\cos(322 - 272)°}$$
$$= 8.39$$

Concept Problem Solution

Using the Distance Formula for points in a polar plot, it is possible to determine the distance between the 2 darts:

$$d = \sqrt{3^2 + 6^2 - 2(4)(6)\cos 45°}$$
$$d = \sqrt{9 + 36 - 48(.707)}$$
$$d = \sqrt{45 - 33.936}$$
$$d \approx 11.064$$

Practice

Find the distance between each set of points.

1. $(1, 150°)$ and $(2, 130°)$
2. $(4, 90°)$ and $(5, 210°)$
3. $(6, 60°)$ and $(2, 90°)$
4. $(2, 120°)$ and $(1, 150°)$
5. $(7, 210°)$ and $(4, 300°)$
6. $(-4, 120°)$ and $(2, 100°)$
7. $(3, -90°)$ and $(5, 150°)$
8. $(-4, -30°)$ and $(3, 250°)$
9. $(7, -150°)$ and $(4, 130°)$
10. $(-2, 300°)$ and $(2, 10°)$
11. Find the length of the arc between the points $(3, 40°)$ and $(3, 150°)$.
12. Find the length of the arc between the points $(1, 10°)$ and $(1, 70°)$.
13. Find the area of the sector created by the origin and the points $(4, 20°)$ and $(4, 110°)$.
14. Find the area of the sector created by the origin and the points $(2, 100°)$ and $(2, 180°)$.
15. Find the area of the sector created by the origin and the points $(5, 110°)$ and $(5, 160°)$.

6.3 Transformations of Polar Graphs

Here you'll learn how to alter graphs expressed in polar coordinates by changing the constants and/or functions used to describe the graph.

While playing around with your calculator one day, you create a polar plot that looks like this

Your teacher tells you that this is a polar plot with an equation $2 + 2\sin\theta$. You decide you'd like to rotate the graph, so that it is actually on its side, like this

Can you find a way to do this by changing the equation you plot? Read on, and by the end of this Concept, you'll be able to do exactly that.

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Example of Graphof Polar Equation

Guidance

Just as in graphing on a rectangular grid, you can also graph polar equations on a polar grid. These equations may be simple or complex. To begin, you should try something simple like $r = k$ or $\theta = k$ where k is a constant. The solution for $r = 1.5$ is simply all ordered pairs such that $r = 1.5$ and θ is any real number. The same is true for the solution of $\theta = 30°$. The ordered pairs will be any real number for r and θ will equal $30°$. Here are the graphs for each of these polar equations.

Example A

On a polar plane, graph the equation $r = 1.5$

Solution: The solution is all ordered pairs of (r, θ) such that r is always 1.5. This means that it doesn't matter what θ is, so the graph is a circle with radius 1.5 and centered at the origin.

Example B

On a polar plane, graph the equation $\theta = 30°$

Solution: For this example, the r value, or radius, is arbitrary. θ must equal $30°$, so the result is a straight line, with an angle of elevation of $30°$.

To begin graphing more complicated polar equations, we will make a table of values for $y = \sin\theta$ or in this case $r = \sin\theta$. When the table has been completed, the graph will be drawn on a polar plane by using the coordinates (r, θ).

Example C

Graph the following polar equations on the same polar grid and compare the graphs.

$$r = 5 + 5\sin\theta \qquad\qquad\qquad r = 5 - 5\sin\theta$$
$$r = 5(1 + \sin\theta) \qquad\qquad\qquad r = 5(1 - \sin\theta)$$

Solution:

The cardioid is symmetrical about the positive $y-$axis and the point of indentation is at the pole. The result of changing $+$ to $-$ is a reflection in the $x-$axis. The cardioid is symmetrical about the negative $y-$axis and the point of indentation is at the pole.

Changing the value of a to a negative did not change the graph of the cardioid.

It is also possible to create a sinusoidal curve called a limaçon. It has $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ as its polar equation. Not all limaçons have the inner loop as a part of the shape. Some may curve to a point, have a simple indentation (known as a dimple) or curve outward. The shape of the limaçon depends upon the ratio of $\frac{a}{b}$ where a is a constant and b is the coefficient of the trigonometric function.

As we've seen with cardioids, it is possible to create transformations of graphs of limaçons by changing values of constants in the equation of the shape.

Vocabulary

Cardioid: A **carioid** is a graph of two heart shaped loops reflected across the "x" axis.

Limacon: A **limacon** is a graph with a sinusoidal curve looping around the origin.

Transformation: A **transformation** is a change performed on a graph by changing the constants and/or the functions of the polar equations.

Guided Practice

1. Graph the curve $r = -3 - 3\cos\theta$

2. Graph the curve $r = 2 + 4\sin\theta$

3. Graph the curve $r = 4$

Solutions:

1. $r = -3 - 3\cos\theta$

2. $r = 2 + 4\sin\theta$

3. $r = 4$

Concept Problem Solution

As you've seen in this section, transformations to the graph of a cardioid can be accomplished by 2 different ways. In this case, you want to rotate the graph so that it is around the "x" axis instead of the "y" axis. To accomplish this,

you change the function from a sine function to a cosine function:

$r = 2 + 2\cos\theta$

Practice

Graph each equation.

1. $r = 4$
2. $\theta = 60°$
3. $r = 2$
4. $\theta = 110°$

Graph each function using your calculator and sketch on your paper.

5. $r = 3 + 3\sin(\theta)$
6. $r = 2 + 4\sin(\theta)$
7. $r = 1 - 5\sin(\theta)$
8. $r = 2 - 2\sin(\theta)$
9. $r = 3 + 6\sin(\theta)$
10. $r = -3 + 6\sin(\theta)$
11. Analyze the connections between the equations and their graphs above. Make a hypothesis about how to graph $r = a + b\sin(\theta)$ for positive or negative values of a and b where $b \geq a$.

Graph each function using your calculator and sketch on your paper.

12. $r = 3 + 3\cos(\theta)$
13. $r = 2 + 4\cos(\theta)$
14. $r = 1 - 5\cos(\theta)$
15. $r = 2 - 2\cos(\theta)$
16. $r = 3 + 6\cos(\theta)$
17. $r = -3 + 6\cos(\theta)$
18. Analyze the connections between the equations and their graphs above. Make a hypothesis about how to graph $r = a + b\cos(\theta)$ for positive or negative values of a and b where $b \geq a$.
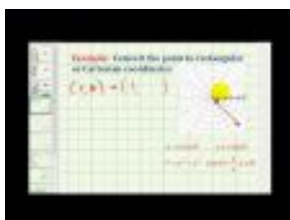
6.4 Polar to Rectangular Conversions

Here you'll learn how to convert a position described in polar coordinates to the equivalent position in rectangular coordinates.

You are hiking one day with friends. When you stop to examine your map, you mark your position on a polar plot with your campsite at the origin, like this

You decide to plot your position on a different map, which has a rectangular grid on it instead of a polar plot. Can you convert your coordinates from the polar representation to the rectangular one?

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James Sousa Example:Converta pointin polar coordinates torectangularcoordinates

Guidance

Just as x and y are usually used to designate the rectangular coordinates of a point, r and θ are usually used to designate the polar coordinates of the point. r is the distance of the point to the origin. θ is the angle that the line from the origin to the point makes with the positive $x-$axis. The diagram below shows both polar and Cartesian coordinates applied to a point P. By applying trigonometry, we can obtain equations that will show the relationship between polar coordinates (r, θ) and the rectangular coordinates (x, y)

The point P has the polar coordinates (r, θ) and the rectangular coordinates (x, y).

Therefore

$$x = r\cos\theta \qquad\qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad\qquad \tan\theta = \frac{y}{x}$$

These equations, also known as coordinate conversion equations, will enable you to convert from polar to rectangular form.

Example A

Given the following polar coordinates, find the corresponding rectangular coordinates of the points: $W(4, -200°), H\left(4, \frac{\pi}{3}\right)$

Solution:

a) For $W(4, -200°), r = 4$ and $\theta = -200°$

$$x = r\cos\theta \qquad\qquad\qquad\qquad y = r\sin\theta$$
$$x = 4\cos(-200^\circ) \qquad\qquad\qquad y = 4\sin(-200^\circ)$$
$$x = 4(-.9396) \qquad\qquad\qquad\qquad y = 4(.3420)$$
$$x \approx -3.76 \qquad\qquad\qquad\qquad\quad y \approx 1.37$$

The rectangular coordinates of W are approximately $(-3.76, 1.37)$.

b) For $H\left(4, \frac{\pi}{3}\right), r = 4$ and $\theta = \frac{\pi}{3}$

$$x = r\cos\theta \qquad\qquad\qquad\qquad y = r\sin\theta$$
$$x = 4\cos\frac{\pi}{3} \qquad\qquad\qquad\qquad y = 4\sin\frac{\pi}{3}$$
$$x = 4\left(\frac{1}{2}\right) \qquad\qquad\qquad\qquad y = 4\left(\frac{\sqrt{3}}{2}\right)$$
$$x = 2 \qquad\qquad\qquad\qquad\qquad y = 2\sqrt{3}$$

The rectangular coordinates of H are $(2, 2\sqrt{3})$ or approximately $(2, 3.46)$.

In addition to writing polar coordinates in rectangular form, the coordinate conversion equations can also be used to write polar equations in rectangular form.

Example B

Write the polar equation $r = 4\cos\theta$ in rectangular form.

Solution:

$$r = 4\cos\theta$$
$$r^2 = 4r\cos\theta \qquad\qquad\qquad \textit{Multiply both sides by r.}$$
$$x^2 + y^2 = 4x \qquad\qquad\qquad r^2 = x^2 + y^2 \textit{ and } x = r\cos\theta$$

The equation is now in rectangular form. The r^2 and θ have been replaced. However, the equation, as it appears, does not model any shape with which we are familiar. Therefore, we must continue with the conversion.

$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \qquad\qquad \textit{Complete the square for } x^2 - 4x.$$
$$(x - 2)^2 + y^2 = 4 \qquad\qquad\qquad \textit{Factor } x^2 - 4x + 4.$$

The rectangular form of the polar equation represents a circle with its centre at $(2, 0)$ and a radius of 2 units.

This is the graph represented by the polar equation $r = 4\cos\theta$ for $0 \le \theta \le 2\pi$ or the rectangular form $(x-2)^2 + y^2 = 4$.

Example C

Write the polar equation $r = 3\csc\theta$ in rectangular form.

Solution:

$$r = 3\csc\theta$$
$$\frac{r}{\csc\theta} = 3 \qquad\qquad divide\ by\csc\theta$$
$$r \cdot \frac{1}{\csc\theta} = 3$$
$$r\sin\theta = 3 \qquad\qquad \sin\theta = \frac{1}{\csc\theta}$$
$$y = 3 \qquad\qquad y = r\sin\theta$$

Vocabulary

Polar Coordinates: A set of **polar coordinates** are a set of coordinates plotted on a system that uses the distance from the origin and angle from an axis to describe location.

Rectangular Coordinates: A set of **rectangular coordinates** are a set of coordinates plotted on a system using basis axes at right angles to each other.

Guided Practice

1. Write the polar equation $r = 6\cos\theta$ in rectangular form.

2. Write the polar equation $r\sin\theta = -3$ in rectangular form.

3. Write the polar equation $r = 2\sin\theta$ in rectangular form.

Solutions:

1.

$$r = 6\cos\theta$$
$$r^2 = 6r\cos\theta$$
$$x^2 + y^2 = 6x$$
$$x^2 - 6x + y^2 = 0$$
$$x^2 - 6x + 9 + y^2 = 9$$
$$(x - 3)^2 + y^2 = 9$$

2.

$$r\sin\theta = -3$$
$$y = -3$$

3.

$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$
$$y^2 - 2y = -x^2$$
$$y^2 - 2y + 1 = -x^2 + 1$$
$$(y-1)^2 = -x^2 + 1$$
$$x^2 + (y-1)^2 = 1$$

Concept Problem Solution

You can see from the map that your position is represented in polar coordinates as $(3, 70°)$. Therefore, the radius is equal to 3 and the angle is equal to $70°$. The rectangular coordinates of this point can be found as follows:

$$x = r\cos\theta \qquad\qquad\qquad y = r\sin\theta$$
$$x = 3\cos(70°) \qquad\qquad\qquad y = 3\sin(70°)$$
$$x = 3(.342) \qquad\qquad\qquad y = 3(.94)$$
$$x \approx 1.026 \qquad\qquad\qquad y \approx 2.82$$

Practice

Given the following polar coordinates, find the corresponding rectangular coordinates of the points.

1. $(2, \frac{\pi}{6})$
2. $(4, \frac{2\pi}{3})$
3. $(5, \frac{\pi}{3})$
4. $(3, \frac{\pi}{4})$
5. $(6, \frac{3\pi}{4})$

Write each polar equation in rectangular form.

6. $r = 3\sin\theta$
7. $r = 2\cos\theta$
8. $r = 5\csc\theta$
9. $r = 3\sec\theta$
10. $r = 6\cos\theta$
11. $r = 8\sin\theta$
12. $r = 2\csc\theta$
13. $r = 4\sec\theta$
14. $r = 3\cos\theta$
15. $r = 5\sin\theta$

6.5 Rectangular to Polar Conversions

Here you'll learn how to convert a position described in a rectangular coordinate system to the equivalent position described in a polar coordinate system.

You are looking at a map of your state with a rectangular coordinate grid. It looks like this

You realize that if you convert the the coordinates of Yourtown (labelled with a "YT") to polar coordinates, you can more easily see the distance between the Capitol at the origin and Yourtown. Can you make this conversion from rectangular to polar coordinates? Keep reading, and at the end of this Concept, you'll be able to perform the conversion.

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James Sousa Example:Converta Pointin Rectangular Coordinates to PolarCoordinatesUsing Radians

Guidance

When converting rectangular coordinates to polar coordinates, we must remember that there are many possible polar coordinates. We will agree that when converting from rectangular coordinates to polar coordinates, one set of polar coordinates will be sufficient for each set of rectangular coordinates. Most graphing calculators are programmed to complete the conversions and they too provide one set of coordinates for each conversion. The conversion of rectangular coordinates to polar coordinates is done using the Pythagorean Theorem and the Arctangent function. The Arctangent function only calculates angles in the first and fourth quadrants so π radians must be added to the value of θ for all points with rectangular coordinates in the second and third quadrants.

In addition to these formulas, $r = \sqrt{x^2 + y^2}$ is also used in converting rectangular coordinates to polar form.

Example A

Convert the following rectangular coordinates to polar form.

$P(3, -5)$

Solution: For $P(3, -5)$ $x = 3$ and $y = -5$. The point is located in the fourth quadrant and $x > 0$.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan\frac{y}{x}$$
$$r = \sqrt{(3)^2 + (-5)^2} \qquad\qquad \theta = \tan^{-1}\left(-\frac{5}{3}\right)$$
$$r = \sqrt{34} \qquad\qquad \theta \approx -1.03$$
$$r \approx 5.83$$

The polar coordinates of $P(3, -5)$ are $P(5.83, -1.03)$.

Example B

Convert the following rectangular coordinates to polar form.

$Q(-9, -12)$

Solution: For $Q(-9, -12)$ $x = -9$ and $y = -5$. The point is located in the third quadrant and $x < 0$.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan\frac{y}{x} + \pi$$
$$r = \sqrt{(-9)^2 + (-12)^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{-12}{-9}\right) + \pi$$
$$r = \sqrt{225} \qquad\qquad \theta \approx 4.07$$
$$r = 15$$

The polar coordinates of $Q(-9, -12)$ are $Q(15, 4.07)$

Example C

Convert the following rectangular coordinates to polar form.

$Q(2, 7)$

Solution: For $Q(2, 7)$ $x = 2$ and $y = 7$. The point is located in the first quadrant and $x > 0$.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan\frac{y}{x} + \pi$$
$$r = \sqrt{(2)^2 + (7)^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{7}{2}\right) + \pi$$
$$r = \sqrt{54} \qquad\qquad \theta \approx 74.05$$
$$r \approx 7.35$$

The polar coordinates of $Q(2, 7)$ are $Q(7.35, 74.05)$

Vocabulary

Polar Coordinates: A set of **polar coordinates** are a set of coordinates plotted on a system that uses the distance from the origin and angle from an axis to describe location.

Rectangular Coordinates: A set of **rectangular coordinates** are a set of coordinates plotted on a system using basis axes at right angles to each other.

Guided Practice

1. Write the following rectangular point in polar form: $A(-2, 5)$ using radians

2. Write the following rectangular point in polar form: $B(5, -4)$ using radians

3. Write the following rectangular point in polar form: $C(1, 9)$ using degrees

Solutions:

1. For $A(-2, 5)$ $x = -2$ and $y = 5$. The point is located in the second quadrant and $x < 0$.

$$r = \sqrt{(-2)^2 + (5)^2} = \sqrt{29} \approx 5.39, \ \theta = \arctan\frac{5}{-2} + \pi = 1.95.$$

The polar coordinates for the rectangular coordinates $A(-2, 5)$ are $A(5.39, 1.95)$

2. For $B(5, -4)$ $x = 5$ and $y = -4$. The point is located in the fourth quadrant and $x > 0$.

$$r = \sqrt{(5)^2 + (-4)^2} = \sqrt{41} \approx 6.4, \ \theta = \tan^{-1}\left(\frac{-4}{5}\right) \approx -0.67$$

The polar coordinates for the rectangular coordinates $B(5, -4)$ are $A(6.40, -0.67)$

3. $C(1, 9)$ is located in the first quadrant.

$$r = \sqrt{1^2 + 9^2} = \sqrt{82} \approx 9.06, \ \theta = \tan^{-1}\frac{9}{1} \approx 83.66.$$

Concept Problem Solution

To convert these rectangular coordinates into polar coordinates, first use the Pythagorean Theorem:

$$r = \sqrt{(4)^2 + (2)^2} = \sqrt{20} \approx 4.47$$

and then use the tangent function to find the angle:

$$\theta = \arctan\frac{4}{2} = 63.43°$$

The polar coordinates for Yourtown are $YT(4.47, 63.43°)$

Practice

Write the following points, given in rectangular form, in polar form using radians where $0 \leq \theta \leq 2\pi$.

1. (1,3)
2. (2,5)
3. (-2,3)
4. (2,-1)
5. (3,2)
6. (4,5)
7. (-1,2)
8. (-3,3)

9. (-2,5)
10. (1,-4)
11. (5,2)
12. (1,6)

For each equation, convert the rectangular equation to polar form.

13. $x = 5$
14. $2x - 3y = 5$
15. $2x + 4y = 2$
16. $(x-1)^2 + y^2 = 1$
17. $(x+3)^2 + (y+3)^2 = 18$
18. $y = 7$

Rectangular to Polar Form for Equations

Here you'll learn to convert equations expressed in rectangular coordinates to equations expressed in polar coordinates through substitution.

You are working diligently in your math class when your teacher gives you an equation to graph:

$(x+1)^2 - (y+2)^2 = 7$

As you start to consider how to rearrange this equation, you are told that the goal of the class is to convert the equation to polar form instead of rectangular form.

Can you find a way to do this?

By the end of this Concept, you'll be able to convert this equation to polar form.

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James Sousa Example:Find the Polar Equationfora Line

Guidance

Interestingly, a rectangular coordinate system isn't the only way to plot values. A polar system can be useful. However, it will often be the case that there are one or more equations that need to be converted from rectangular to polar form. To write a rectangular equation in polar form, the conversion equations of $x = r\cos\theta$ and $y = r\sin\theta$ are used.

If the graph of the polar equation is the same as the graph of the rectangular equation, then the conversion has been determined correctly.

$$(x-2)^2 + y^2 = 4$$

The rectangular equation $(x-2)^2 + y^2 = 4$ represents a circle with center (2, 0) and a radius of 2 units. The polar equation $r = 4\cos\theta$ is a circle with center (2, 0) and a radius of 2 units.

Example A

Write the rectangular equation $x^2 + y^2 = 2x$ in polar form.

Solution: Remember $r = \sqrt{x^2+y^2}, r^2 = x^2+y^2$ and $x = r\cos\theta$.

$$x^2 + y^2 = 2x$$
$$r^2 = 2(r\cos\theta) \qquad\qquad \textit{Pythagorean Theorem and } x = r\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$r = 2\cos\theta \qquad\qquad \textit{Divide each side by } r$$

Example B

Write $(x-2)^2 + y^2 = 4$ in polar form.

Remember $x = r\cos\theta$ and $y = r\sin\theta$.

$$(x-2)^2 + y^2 = 4$$
$$(r\cos\theta - 2)^2 + (r\sin\theta)^2 = 4 \qquad\qquad x = r\cos\theta \textit{ and } y = r\sin\theta$$
$$amp; r^2\cos^2\theta - 4r\cos\theta + 4 + r^2\sin^2\theta = 4 \qquad\qquad \textit{expand the terms}$$
$$amp; r^2\cos^2\theta - 4r\cos\theta + r^2\sin^2\theta = 0 \qquad\qquad \textit{subtract } 4 \textit{ from each side}$$
$$amp; r^2\cos^2\theta + r^2\sin^2\theta = 4r\cos\theta \qquad\qquad \textit{isolate the squared terms}$$
$$amp; r^2(\cos^2\theta + \sin^2\theta) = 4r\cos\theta \qquad\qquad \textit{factor } r^2 - \textit{a common factor}$$
$$amp; r^2 = 4r\cos\theta \qquad\qquad \textit{Pythagorean Identity}$$
$$amp; r = 4\cos\theta \qquad\qquad \textit{Divide each side by } r$$

Example C

Write the rectangular equation $(x+4)^2 + (y-1)^2 = 17$ in polar form.

$$(x+4)^2 + (y-1)^2 = 17$$
$$(r\cos\theta + 4)^2 + (r\sin\theta - 1)^2 = 17 \qquad\qquad x = r\cos\theta \textit{ and } y = r\sin\theta$$
$$amp; r^2\cos^2\theta + 8r\cos\theta + 16 + r^2\sin^2\theta - 2r\sin\theta + 1 = 17 \qquad\qquad \textit{expand the terms}$$
$$amp; r^2\cos^2\theta + 8r\cos\theta - 2r\sin\theta + r^2\sin^2\theta = 0 \qquad\qquad \textit{subtract } 17 \textit{ from each side}$$
$$amp; r^2\cos^2\theta + r^2\sin^2\theta = -8r\cos\theta + 2r\sin\theta \qquad\qquad \textit{isolate the squared terms}$$
$$amp; r^2(\cos^2\theta + \sin^2\theta) = -2r(4\cos\theta - \sin\theta) \qquad\qquad \textit{factor } r^2 - \textit{a common factor}$$
$$amp; r^2 = -2r(4\cos\theta - \sin\theta) \qquad\qquad \textit{Pythagorean Identity}$$
$$amp; r = -2(4\cos\theta - \sin\theta) \qquad\qquad \textit{Divide each side by } r$$

Vocabulary

Polar Coordinates: A set of **polar coordinates** are a set of coordinates plotted on a system that uses the distance from the origin and angle from an axis to describe location.

Rectangular Coordinates: A set of **rectangular coordinates** are a set of coordinates plotted on a system using basis axes at right angles to each other.

Guided Practice

1. Write the rectangular equation $(x-4)^2 + (y-3)^2 = 25$ in polar form.

2. Write the rectangular equation $3x - 2y = 1$ in polar form.

3. Write the rectangular equation $x^2 + y^2 - 4x + 2y = 0$ in polar form.

Solutions:

1.

$$(x-4)^2 + (y-3)^2 = 25$$
$$x^2 - 8x + 16 + y^2 - 6y + 9 = 25$$
$$x^2 - 8x + y^2 - 6y + 25 = 25$$
$$x^2 - 8x + y^2 - 6y = 0$$
$$x^2 + y^2 - 8x - 6y = 0$$
$$r^2 - 8(r\cos\theta) - 6(r\sin\theta) = 0$$
$$r^2 - 8r\cos\theta - 6r\sin\theta = 0$$
$$r(r - 8\cos\theta - 6\sin\theta) = 0$$
$$r = 0 \text{ or } r - 8\cos\theta - 6\sin\theta = 0$$
$$r = 0 \text{ or } r = 8\cos\theta + 6\sin\theta$$

From graphing $r - 8\cos\theta - 6\sin\theta = 0$, we see that the additional solutions are 0 and 8.

2.

$$3x - 2y = 1$$
$$3r\cos\theta - 2r\sin\theta = 1$$
$$r(3\cos\theta - 2\sin\theta) = 1$$
$$r = \frac{1}{3\cos\theta - 2\sin\theta}$$

3.

$$x^2 + y^2 - 4x + 2y = 0$$
$$r^2\cos^2\theta + r^2\sin^2\theta - 4r\cos\theta + 2r\sin\theta = 0$$
$$r^2(\sin^2\theta + \cos^2\theta) - 4r\cos\theta + 2r\sin\theta = 0$$
$$r(r - 4\cos\theta + 2\sin\theta) = 0$$
$$r = 0 \text{ or } r - 4\cos\theta + 2\sin\theta = 0$$
$$r = 0 \text{ or } r = 4\cos\theta - 2\sin\theta$$

Concept Problem Solution

The original equation to convert is:

$(x+1)^2 - (y+2)^2 = 7$

You can substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the equation, and then simplify:

$$(r\cos\theta + 1)^2 - (r\sin\theta + 2)^2 = 7$$
$$(r^2\cos^2\theta + 2r\cos\theta + 1) - (r^2\sin^2\theta + 2r\cos\theta + 4) = 7$$
$$r^2(\cos^2\theta + \sin^2\theta) + 4r\cos\theta + 5 = 7$$
$$r^2 + 4r\cos\theta = 2$$

Practice

Write each rectangular equation in polar form.

1. $x = 3$
2. $y = 4$
3. $x^2 + y^2 = 4$
4. $x^2 + y^2 = 9$
5. $(x-1)^2 + y^2 = 1$
6. $(x-2)^2 + (y-3)^2 = 13$
7. $(x-1)^2 + (y-3)^2 = 10$
8. $(x+2)^2 + (y+2)^2 = 8$
9. $(x+5)^2 + (y-1)^2 = 26$
10. $x^2 + (y-6)^2 = 36$
11. $x^2 + (y+2)^2 = 4$
12. $2x + 5y = 11$
13. $4x - 7y = 10$
14. $x + 5y = 8$
15. $3x - 4y = 15$

6.7 Intersections of Polar Curves

Here you'll learn to algebraically find the intersection of two curves written in polar coordinates.

You are working in Art Class one afternoon and decide to draw the "Olympic Rings". These are a set of rings that lock together and are the symbol of the Olympics.

You do this on the computer by generating a circle for each of the rings using an equation in polar coordinates. The equations you use for the first two rings are $r = 2\sin\theta$ and $r = 2\cos\theta$. Is it possible for your to find the angles where these first two rings intersect each other?

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James Sousa:Graphing Polar Equations ontheTI84 Graphing Calculator

Guidance

When you worked with a system of linear equations with two unknowns, finding the point of intersection of the equations meant finding the coordinates of the point that satisfied both equations. If the equations are rectangular equations for curves, determining the point(s) of intersection of the curves involves solving the equations algebraically since each point will have one ordered pair of coordinates associated with it.

Example A

Solve the following system of equations algebraically:

$$x^2 + 4y^2 - 36 = 0$$
$$x^2 + y = 3$$

Solution: Before solving the system, graph the equations to determine the number of points of intersection.

The graph of $x^2 + 4y^2 - 36 = 0$ is an ellipse and the graph represented by $x^2 + y = 3$ is a parabola. There are three points of intersection. To determine the exact values of these points, algebra must be used.

$$amp; x^2 + 4y^2 - 36 = 0 \rightarrow x^2 + 4y^2 = 36 \qquad x^2 + 4y^2 + 0y = 36 \qquad x^2 + 4y^2 + 0y = 36$$
$$amp; x^2 + y = 3 \rightarrow x^2 + 0y^2 + y = 3 \qquad -1(x^2 + 0y^2 + y = 3) \qquad -x^2 - 0y^2 - y = -3$$

$$\overline{\qquad\qquad\qquad\qquad}$$
$$4y^2 - y = 33$$

Using the quadratic formula, $a = 4\ b = -1\ c = -33$

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-33)}}{2(4)}$$

$$y = \frac{1 + 23}{8} = 3 \qquad y = \frac{1 - 23}{8} = -2.75$$

These values must be substituted into one of the original equations.

$$
\begin{array}{ll}
x^2 + y = 3 & \qquad\qquad x^2 + y = 3 \\
x^2 + 3 = 3 & \qquad\qquad x^2 + (-2.75) = 3 \\
x^2 = 0 & \qquad\qquad x^2 = 5.75 \\
x = 0 & \qquad\qquad x = \pm\sqrt{5.75} \approx 2.4
\end{array}
$$

The three points of intersection as determined algebraically in Cartesian representation are $A(0,3), B(2.4, -2.75)$ and $C(2.4, 2.75)$.

If we are working with polar equations to determine the polar coordinates of a point of intersection, we must remember that there are many polar coordinates that represent the same point. Remember that switching to polar form changes a great deal more than the notation. Unlike the Cartesian system which has one name for each point, the polar system has an infinite number of names for each point. One option would be to convert the polar coordinates to rectangular form and then to convert the coordinates for the intersection points back to polar form. Perhaps the best option would be to explore some examples. As these examples are presented, be sure to use your graphing calculator to create your own visual representations of the equations presented.

Example B

Determine the polar coordinates for the intersection point(s) of the following polar equations: $r = 1$ and $r = 2\cos\theta$.

Solution: Begin with the graph. Using the process described in the technology section in this chapter; create the graph of these polar equations on your graphing calculator. Once the graphs are on the screen, use the **trace** function and the arrow keys to move the cursor around each graph. As the cursor is moved, you will notice that the equation of the curve is shown in the upper left corner and the values of θ, x, y are shown (in decimal form) at the bottom of the screen. The values change as the cursor is moved.

$$
\begin{array}{ll}
r = 1 & \qquad\qquad 2\cos\theta = 1 \\
r = 2\cos\theta & \qquad\qquad \cos\theta = \frac{1}{2} \\
& \qquad\qquad \cos^{-1}(\cos\theta) = \cos^{-1}\frac{1}{2}
\end{array}
$$

$\theta = \frac{\pi}{3}$ in the first quadrant and $\theta = \frac{5\pi}{3}$ in the fourth quadrant.

The points of intersection are $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right)$. However, these two solutions only cover the possible values $0 \le \theta \le 2\pi$. If you consider that $\cos\theta = \frac{1}{2}$ is true for an infinite number of theta these solutions must be extended to include $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right) + 2\pi k, k\varepsilon Z$. Now the solutions include all possible rotations.

Example C

Find the intersection of the graphs of $r = \sin\theta$ and $r = 1 - \sin\theta$

Solution: Begin with the graph. You can create these graphs using your graphing calculator.

$$r = \sin\theta \qquad\qquad \sin\theta = 1 - \sin\theta$$
$$r = 1 - \sin\theta \qquad\qquad 2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$amp; r = \sin\theta \qquad\qquad \theta = \frac{\pi}{6} \text{ in the first quadrant and } \theta = \frac{5\pi}{6} \text{ in the second quadrant.}$$
$$amp; r = \sin\frac{\pi}{6} \qquad\qquad \text{The intersection points are } \left(\frac{1}{2}, \frac{\pi}{6}\right) \text{ and } \left(\frac{1}{2}, \frac{5\pi}{6}\right)$$
$$amp; r = \frac{1}{2} \qquad\qquad \text{Another intersection point seems to be the origin } (0,0).$$

If you consider that $\sin\theta = \frac{1}{2}$ is true for an infinite number of theta as was $\cos\theta = \frac{1}{2}$ in the previous example, the same consideration must be applied to include all possible solutions. To prove if the origin is indeed an intersection point, we must determine whether or not both curves pass through $(0, 0)$.

$$r = \sin\theta \qquad\qquad\qquad r = 1 - \sin\theta$$
$$0 = \sin\theta \qquad\qquad\qquad 0 = 1 - \sin\theta$$
$$r = 0 \qquad\qquad\qquad 1 = \sin\theta$$
$$\frac{\pi}{2} = 0$$

From this investigation, the point $(0, 0)$ was on the curve $r = \sin\theta$ and the point $\left(0, \frac{\pi}{2}\right)$ was on the curve $r = 1 - \sin\theta$. Because the second coordinates are different, it seems that they are two different points. However, the coordinates represent the same point $(0,0)$. The intersection points are $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and $(0,0)$.

Sometimes it is helpful to convert the equations to rectangular form, solve the system and then convert the polar coordinates back to polar form.

Vocabulary

Intersection: The **intersection** is the place where the graphs of two equations meet each other.

Guided Practice

1. Find the intersection of the graphs of $r = \sin 3\theta$ and $r = 3\sin\theta$.

2. Find the intersection of the graphs of $r = 2 + 2\sin\theta$ and $r = 2 - 2\cos\theta$

3. Determine two polar curves that will never intersect.

Solutions:

1.

There appears to be one point of intersection.

$$r = \sin 3\,\theta \qquad\qquad\qquad\qquad r = 3\sin\theta$$
$$0 = \sin 3\,\theta \qquad\qquad\qquad\qquad 0 = 3\sin\theta$$
$$0 = \theta \qquad\qquad\qquad\qquad\qquad 0 = \sin\theta$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 = \theta$$

The point of intersection is $(0, 0)$

2.

$$r = 2 + 2\sin\theta \qquad\qquad\qquad r = 2 + 2\sin\theta$$
$$r = 2 + 2\sin\left(\frac{3\pi}{4}\right) \qquad\qquad r = 2 + 2\sin\frac{7\pi}{4}$$
$$r \approx 3.4 \qquad\qquad\qquad\qquad r \approx 0.59$$
$$r = 2 + 2\sin\theta \qquad\qquad\qquad r = 2 - 2\cos\theta$$
$$0 = 2 + 2\sin\theta \qquad\qquad\qquad 0 = 2 - 2\cos\theta$$
$$-1 = \sin\theta \qquad\qquad\qquad\qquad 1 = \cos\theta$$
$$\theta = \frac{3\pi}{2} \qquad\qquad\qquad\qquad \theta = 0$$

Since both equations have a solution at $r = 0$, that is $(0, \frac{3\pi}{2})$ and $(0, 0)$, respectively, and these two points are equivalent, the two equations will intersect at $(0, 0)$.

$$r = 2 + 2\sin\theta$$
$$r = 2 - 2\cos\theta$$
$$2 + 2\sin\theta = 2 - 2\cos\theta$$
$$2\sin\theta = -2\cos\theta$$
$$\frac{2\sin\theta}{2\cos\theta} = -\frac{2\cos\theta}{2\cos\theta}$$
$$\frac{\sin\theta}{\cos\theta} = -1$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4}$$

The points of intersection are $\left(3.4, \frac{3\pi}{4}\right)$, $\left(0.59, \frac{7\pi}{4}\right)$ and $(0,0)$.

3. There are several answers here. The most obvious are any two pairs of circles, for example $r = 3$ and $r = 9$.

Concept Problem Solution

The goal is the find the place where the two equations meet. Therefore, you want the point where they are equal. Mathematically, this is:

$$2\cos\theta = 2\sin\theta$$
$$\cos\theta = \sin\theta$$
$$\frac{\sin\theta}{\cos\theta} = 1$$
$$\tan\theta = 1$$
$$\theta = 45°, 225°$$

Practice

Find all points of intersection for each of the following pairs of graphs. Answers should be in polar coordinates with $0 \le \theta < 2\pi$.

1. $r = 2$ and $r = 2\cos\theta$
2. $r = 3$ and $r = 3\sin\theta$
3. $r = 1$ and $r = 2\sin\theta$
4. $r = \sin\theta$ and $r = 1 + \sin\theta$
5. $r = \sin 2\theta$ and $r = 2\sin\theta$
6. $r = 3 + 3\sin\theta$ and $r = 3 - 3\cos\theta$
7. $r = \cos 3\theta$ and $r = \sin 3\theta$
8. $r = \sin 2\theta$ and $r = \sin 3\theta$
9. $r = 3\cos\theta$ and $r = 2 - \cos\theta$
10. $r = \cos\theta$ and $r = 1 - \cos\theta$
11. $r = 3\sin\theta$ and $r = 2 - \sin\theta$
12. $r^2 = \sin\theta$ and $r^2 = \cos\theta$
13. $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$
14. Explain why one point of intersection of polar graphs can be represented by an infinite number of polar coordinate pairs.
15. Explain why the point (0,0) in polar coordinates is not always found as a point of intersection when solving for points of intersection algebraically.

6.8 Equivalent Polar Curves

Here you'll learn to determine if two polar equations are equivalent by inspection of their respective graphs.

While working on a problem in math class, you get a solution with a certain equation. In this case, your solution is $3 + 2\cos(\theta)$. Your friend comes over and tells you that he thinks he has solved the problem. However, when he shows you his paper, his equation looks different from yours. His solution is $-3 + 2\cos(-\theta)$. Is there a way you can determine if the two equations are equivalent?

At the conclusion of this Concept, you'll be able to determine if the solutions of you and your friend are equivalent.

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Familiesof Polar Curves: Circles, Cardiods,andLimacons

Guidance

The expression "same only different" comes into play in this Concept. We will graph two distinct polar equations that will produce two equivalent graphs. Use your graphing calculator and create these curves as the equations are presented.

In some other Concepts, graphs were generated of a limaçon, a dimpled limaçon, a looped limaçon and a cardioid. All of these were of the form $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The easiest way to see what polar equations produce equivalent curves is to use either a graphing calculator or a software program to generate the graphs of various polar equations.

Example A

Plot the following polar equations and compare the graphs.

a)

$$r = 1 + 2\sin\theta$$
$$r = -1 + 2\sin\theta$$

b)

$$r = 4\cos\theta$$
$$r = 4\cos(-\theta)$$

Solution: By looking at the graphs, the result is the same. So, even though a is different in both, they have the same graph. We can assume that the sign of a does not matter.

b) These functions also result in the same graph. Here, θ differed by a negative. So we can assume that the sign of θ does not change the appearance of the graph.

Example B

Graph the equations $x^2 + y^2 = 16$ and $r = 4$. Describe the graphs.

Solution:

Both equations, one in rectangular form and one in polar form, are circles with a radius of 4 and center at the origin.

Example C

Graph the equations $(x-2)^2 + (y+2)^2 = 8$ and $r = 4\cos\theta - 4\sin\theta$. Describe the graphs.

Solution: There is not a visual representation shown here, but on your calculator you should see that the graphs are circles centered at (2, -2) with a radius $2\sqrt{2} \approx 2.8$.

Vocabulary

Equivalent Polar Curves: A set of **equivalent polar curves** are two equations that are different in appearance but that produce identical graphs.

Guided Practice

1. Write the rectangular equation $x^2 + y^2 = 6x$ in polar form and graph both equations. Should they be equivalent?

2. Determine if $r = -2 + \sin\theta$ and $r = 2 - \sin\theta$ are equivalent *without* graphing.

3. Determine if $r = -3 + 4\cos(-\pi)$ and $r = 3 + 4\cos\pi$ are equivalent *without* graphing.

Solutions:

1.

$$
\begin{aligned}
x^2 + y^2 &= 6x \\
r^2 &= 6(r\cos\theta) \qquad\qquad\qquad r^2 = x^2 + y^2 \qquad \text{and} \qquad x = y\cos\theta \\
r &= 6\cos\theta \qquad\qquad\qquad\quad \text{divide by } r
\end{aligned}
$$

Both equations produced a circle with center $(3,0)$ and a radius of 3.

2. $r = -2 + \sin\theta$ and $r = 2 - \sin\theta$ are not equivalent because the sine has the opposite sign. $r = -2 + \sin\theta$ will be primarily above the horizontal axis and $r = 2 - \sin\theta$ will be mostly below. However, the two do have the same pole axis intercepts.

3. $r = -3 + 4\cos(-\pi)$ and $r = 3 + 4\cos\pi$ are equivalent because the sign of "a" does not matter, nor does the sign of θ.

Concept Problem Solution

As you learned in this Concept, we can compare graphs of equations to see if the equations are the same or not.

A graph of $3 + 2\cos(\theta)$ looks like this:

And a graph of $-3 + 2\cos(-\theta)$ looks like this:

As you can see from the plots, your friend is correct. Your graph and his are the same, therefore the equations are equivalent.

Practice

For each equation in rectangular form given below, write the equivalent equation in polar form.

1. $x^2 + y^2 = 4$
2. $x^2 + y^2 = 6y$
3. $(x-1)^2 + y^2 = 1$
4. $(x-4)^2 + (y-1)^2 = 17$
5. $x^2 + y^2 = 9$

For each equation below in polar form, write another equation in polar form that will produce the same graph.

6. $r = 4 + 3\sin\theta$
7. $r = 2 - \sin\theta$
8. $r = 2 + 2\cos\theta$
9. $r = 3 - \cos\theta$
10. $r = 2 + \sin\theta$

Determine whether each of the following sets of equations produce equivalent graphs without

11. $r = 3 - \sin\theta$ and $r = 3 + \sin\theta$
12. $r = 1 + 2\sin\theta$ and $r = -1 + 2\sin\theta$
13. $r = 3\sin\theta$ and $r = 3\sin(-\theta)$
14. $r = 2\cos\theta$ and $r = 2\cos(-\theta)$
15. $r = 1 + 3\cos\theta$ and $r = 1 - 3\cos\theta$

6.9 Trigonometric Form of Complex Numbers

Here you'll learn how to express a complex number in trigonometric form by understanding the relationship between the rectangular form of complex numbers and their corresponding polar form.

You have begun working with complex numbers in your math class. While describing numbers in the complex plane, you realize that the plotting of a complex number is a lot like plotting a set of points on a rectangular coordinate system.

You also learned in math class that you can convert coordinates from a rectangular system into a polar system. As you are considering this, you plot the the complex number $2 + 3i$. Can you somehow convert this into a type of polar plot that you've done before?

Read on, and at the conclusion of this Concept, you'll be able to convert this number into a polar form.

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James Sousa:ComplexNumbersin Trigonometric Form

Guidance

A number in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, or $\sqrt{-1}$, is called a complex number. Despite their names, complex numbers and imaginary numbers have very real and significant applications in both mathematics and in the real world. Complex numbers are useful in pure mathematics, providing a more consistent and flexible number system that helps solve algebra and calculus problems. We will see some of these applications in the examples throughout this Concept.

The following diagram will introduce you to the relationship between complex numbers and polar coordinates.

In the figure above, the point that represents the number $x + yi$ was plotted and a vector was drawn from the origin to this point. As a result, an angle in standard position, θ, has been formed. In addition to this, the point that represents $x + yi$ is r units from the origin. Therefore, any point in the complex plane can be found if the angle θ and the $r-$ value are known. The following equations relate x, y, r and θ.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x}$$

If we apply the first two equations to the point $x + yi$ the result would be:

$$x + yi = r\cos\theta + ri\sin\theta \rightarrow r(\cos\theta + i\sin\theta)$$

The right side of this equation $r(\cos\theta + i\sin\theta)$ is called the **polar** or **trigonometric** form of a complex number. A shortened version of this polar form is written as r *cis* θ. The length r is called the **absolute value** or the **modulus**, and the angle θ is called the **argument** of the complex number. Therefore, the following equations define the polar form of a complex number:

$$r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x} \qquad\qquad x + yi = r(\cos\theta + i\sin\theta)$$

It is now time to implement these equations perform the operation of converting complex numbers in standard form to complex numbers in polar form. You will use the above equations to do this.

Example A

Represent the complex number $5 + 7i$ graphically and express it in its polar form.

Solution: Here is the graph of $5 + 7i$.

Converting to polar from rectangular, $x = 5$ and $y = 7$.

$$r = \sqrt{5^2 + 7^2} = 8.6 \qquad\qquad\qquad \tan\theta = \frac{7}{5}$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}\frac{7}{5}$$
$$\theta = 54.5°$$

So, the polar form is $8.6(\cos 54.5° + i\sin 54.5°)$.

Another widely used notation for the polar form of a complex number is $r\angle\theta = r(\cos\theta + i\sin\theta)$. Finally, there is a third way to write a complex number, in the form of r*cis*θ, where "r" is the length of the vector in polar form, and θ is the angle the vector makes with the positive "x" axis. This makes a total of three ways to write the polar form of a complex number.

$$x + yi = r(\cos\theta + i\sin\theta) \qquad\qquad x + yi = r\,cis\theta \qquad\qquad x + yi = r\angle\theta$$

Example B

Express the following polar form of each complex number using the shorthand representations.

a) $4.92(\cos 214.6° + i\sin 214.6°)$

b) $15.6(\cos 37° + i\sin 37°)$

Solution:

a) $4.92\angle 214.6°$

4.92 *cis* $214.6°$

b) $15.6\angle 37°$

15.6 *cis* $37°$

Example C

Represent the complex number $-3.12 - 4.64i$ graphically and give two notations of its polar form.

Solution: From the rectangular form of $-3.12 - 4.64i$ $x = -3.12$ and $y = -4.64$

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-3.12)^2 + (-4.64)^2}$$
$$r = 5.59$$

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{-4.64}{-3.12}$$
$$\theta = 56.1°$$

This is the reference angle so now we must determine the measure of the angle in the third quadrant. $56.1° + 180° = 236.1°$

One polar notation of the point $-3.12 - 4.64i$ is $5.59(\cos 236.1° + i\sin 236.1°)$. Another polar notation of the point is $5.59\angle 236.1°$

So far we have expressed all values of theta in degrees. Polar form of a complex number can also have theta expressed in radian measure. This would be beneficial when plotting the polar form of complex numbers in the polar plane.

The answer to the above example $-3.12 - 4.64i$ with theta expressed in radian measure would be:

$$\tan\theta = \frac{-4.64}{-3.12}$$

$$\tan\theta = .9788 \text{(reference angle)}$$

$$0.9788 + 3.14 = 4.12 \text{ rad.}$$

$$5.59(\cos 4.12 + i\sin 4.12)$$

Now that we have explored the polar form of complex numbers and the steps for performing these conversions, we will look at an example in circuit analysis that requires a complex number given in polar form to be expressed in standard form.

Vocabulary

Complex Number: A **complex number** is a number having both real and imaginary components.

Guided Practice

1. The impedance Z, in ohms, in an alternating circuit is given by $Z = 4650\angle -35.2°$. Express the value for Z in standard form. (In electricity, negative angles are often used.)

2. Express the following complex numbers in their polar form.

    1. $4 + 3i$

2. $-2+9i$
3. $7-i$
4. $-5-2i$

3. Express the complex number $6-8i$ graphically and write it in its polar form.

Solutions:

1. The value for Z is given in polar form. From this notation, we know that $r = 4650$ and $\theta = -35.2°$ Using these values, we can write:

$$Z = 4650(\cos(-35.2°)+i\sin(-35.2°))$$
$$x = 4650\cos(-35.2°) \to 3800$$
$$y = 4650\sin(-35.2°) \to -2680$$

Therefore the standard form is $Z = 3800 - 2680i$ ohms.

2. #$4+3i \to x = 4, y = 3$

$$r = \sqrt{4^2+3^2} = 5, \tan\theta = \frac{3}{4} \to \theta = 36.87° \to 5(\cos 36.87° + i\sin 36.87°)$$

1. $-2+9i \to x = -2, y = 9$

$$r = \sqrt{(-2)^2+9^2} = \sqrt{85} \approx 9.22, \tan\theta = -\frac{9}{2} \to \theta = 102.53° \to 9.22(\cos 102.53° + i\sin 102.53°)$$

2. $7-i \to x = 7, y = -1$

$$r = \sqrt{7^2+1^2} = \sqrt{50} \approx 7.07, \tan\theta = -\frac{1}{7} \to \theta = 351.87° \to 7.07(\cos 351.87° + i\sin 351.87°)$$

3. $-5-2i \to x = -5, y = -2$

$$r = \sqrt{(-5)^2+(-2)^2} = \sqrt{29} \approx 5.39, \tan\theta = \frac{2}{5} \to \theta = 201.8° \to 5.39(\cos 201.8° + i\sin 201.8°)$$

3. $6-8i$

$$6-8i \qquad\qquad\qquad \tan\theta = \frac{y}{x}$$
$$x = 6 \text{ and } y = -8 \qquad\qquad \tan\theta = \frac{-8}{6}$$
$$r = \sqrt{x^2+y^2} \qquad\qquad\qquad \theta = -53.1°$$
$$r = \sqrt{(6)^2+(-8)^2}$$
$$r = 10$$

Since θ is in the fourth quadrant then $\theta = -53.1° + 360° = 306.9°$ Expressed in polar form $6-8i$ is $10(\cos 306.9° + i\sin 306.9°)$ or $10\angle 306.9°$

Concept Problem Solution

You can now convert $2 + 3i$ into polar form by using the equations giving the radius and angle of the number's position in the complex plane:

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(2)^2 + (3)^2}$$
$$r = \sqrt{13}$$

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{3}{2}$$
$$\theta = 56.31°$$

Therefore, the polar form of $2 + 3i$ is $\sqrt{13}(\cos 56.31° + i\sin 56.31°)$.

Practice

Plot each of the following points in the complex plane.

1. $1 + i$
2. $2 - 3i$
3. $-2 - i$
4. i
5. $4 - i$

Find the trigonometric form of the complex numbers where $0 \leq \theta < 2\pi$.

6. $8 - 6i$
7. $5 + 12i$
8. $2 - 2i$
9. $3 + 3i$
10. $2 + 3i$
11. $5 - 6i$

Write each complex number in standard form.

12. $4(\cos 30° + i\sin 30°)$
13. $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
14. $2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$
15. $2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

6.10 Product Theorem

Here you'll learn to derive and apply the Product Theorem, which simplifies the multiplication of complex numbers.

What if you were given two complex numbers in polar form, such as $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right), 7\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$ and asked to multiply them? Would you be able to do this? How long would it take you?

After completing this Concept, you'll know the Product Theorem, which will make it easier to multiply complex numbers.

Watch This

In the first part of this video you'll learn about the product of complex numbers in trigonometric form.
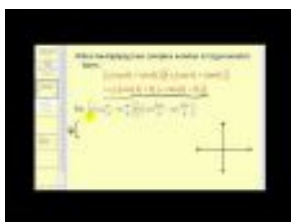
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James Sousa:TheProductandQuotient ofComplexNumbersin Trigonometric Form

Guidance

Multiplication of complex numbers in polar form is similar to the multiplication of complex numbers in standard form. However, to determine a general rule for multiplication, the trigonometric functions will be simplified by applying the sum/difference identities for cosine and sine. To obtain a general rule for the multiplication of complex numbers in polar from, let the first number be $r_1(\cos\theta_1+i\sin\theta_1)$ and the second number be $r_2(\cos\theta_2+i\sin\theta_2)$. The product can then be simplified by use of three facts: the definition $i^2=-1$, the sum identity $\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos(\alpha+\beta)$, and the sum identity $\sin\alpha\cos\beta+\cos\alpha\sin\beta=\sin(\alpha+\beta)$.

Now that the numbers have been designated, proceed with the multiplication of these binomials.

$$r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)$$
$$r_1r_2(\cos\theta_1\cos\theta_2+i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2+i^2\sin\theta_1\sin\theta_2)$$
$$r_1r_2[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)]$$
$$r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

Therefore:

$$r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

We can use this general formula for the product of complex numbers to perform computations.

Example A

Find the product of the complex numbers $3.61(\cos 56.3° + i\sin 56.3°)$ and $1.41(\cos 315° + i\sin 315°)$

Solution: Use the Product Theorem, $r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

$$3.61(\cos 56.3° + i\sin 56.3°) \cdot 1.41(\cos 315° + i\sin 315°)$$
$$= (3.61)(1.41)[\cos(56.3° + 315°) + i\sin(56.3° + 315°)]$$
$$= 5.09(\cos 371.3° + i\sin 371.3°)$$
$$= 5.09(\cos 11.3° + i\sin 11.3°)$$

*Note: Angles are expressed $0° \le \theta \le 360°$ unless otherwise stated.

Example B

Find the product of $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \cdot \sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Solution: First, calculate $r_1 r_2 = 5 \cdot \sqrt{3} = 5\sqrt{3}$ and $\theta = \theta_1 + \theta_2 = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$

$$5\sqrt{3}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

Example C

Find the product of the numbers $r_1 = 1 + i$ and $r_2 = \sqrt{3} - i$ by first converting them to trigonometric form.

Solution:

First, convert $1 + i$ to polar form:

$$r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\theta_1 = \arctan\frac{1}{1} = \frac{\pi}{4}$$

And now do the same with $\sqrt{3} - i$:

$$r_2 = \sqrt{\sqrt{3}^2 + (-1)^2} = 2$$
$$\theta_2 = \arctan-\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$$

And now substituting these values into the product theorem:

$$r_1 \times r_2 =$$

$$(2)(\sqrt{2})\left(\cos\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)\right)$$

$$(2)(\sqrt{2})\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right)$$

Vocabulary

Product Theorem: The **product theorem** is a theorem showing a simplified way to multiply complex numbers.

Guided Practice

1. Multiply together the following complex numbers. If they are not in polar form, change them before multiplying.

$2\angle 56°, 7\angle 113°$

2. Multiply together the following complex numbers. If they are not in polar form, change them before multiplying.

$3(\cos\pi + i\sin\pi), 10\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

3. Multiply together the following complex numbers. If they are not in polar form, change them before multiplying.

$2+3i, -5+11i$

Solutions:

1. $2\angle 56°, 7\angle 113° = (2)(7)\angle(56°+113°) = 14\angle 169°$

2. $3(\cos\pi+i\sin\pi), 10\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right) = (3)(10)cis\left(\pi+\frac{5\pi}{3}\right) = 30cis\frac{8\pi}{3} = 30cis\frac{2\pi}{3}$

3. $2+3i, -5+11i \rightarrow$ change to polar

$$x = 2, y = 3 \qquad\qquad\qquad\qquad x = -5, y = 11$$

$$r = \sqrt{2^2+3^2} = \sqrt{13} \approx 3.61 \qquad\qquad r = \sqrt{(-5)^2+11^2} = \sqrt{146} \approx 12.08$$

$$\tan\theta = \frac{3}{2} \rightarrow \theta = 56.31° \qquad\qquad \tan\theta = -\frac{11}{5} \rightarrow \theta = 114.44°$$

$$(3.61)(12.08)\angle(56.31°+114.44°) = 43.61\angle 170.75°$$

Concept Problem Solution

Since you want to multiply

$2(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}), 7\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

where $r_1 = 2, r_2 = 7, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{3\pi}{2}$,

you can use the equation

$r_1 r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$

and calculate:

$(2)(7)[\cos(\frac{\pi}{2}+\frac{3\pi}{2})+i\sin(\frac{\pi}{2}+\frac{3\pi}{2})]$

This simplifies to:

$$14[\cos(2\pi) + i\sin(2\pi)]$$
$$= 14[1 + i0]$$
$$= 14$$

Practice

Multiply each pair of complex numbers. If they are not in trigonometric form, change them before multiplying.

1. $3(\cos 32° + i\sin 32°) \cdot 2(\cos 15° + i\sin 15°)$
2. $2(\cos 10° + i\sin 10°) \cdot 10(\cos 12° + i\sin 12°)$
3. $4(\cos 45° + i\sin 45°) \cdot 8(\cos 62° + i\sin 62°)$
4. $2(\cos 60° + i\sin 60°) \cdot \frac{1}{2}(\cos 34° + i\sin 34°)$
5. $5(\cos 25° + i\sin 25°) \cdot 2(\cos 115° + i\sin 115°)$
6. $-3(\cos 70° + i\sin 70°) \cdot 3(\cos 85° + i\sin 85°)$
7. $7(\cos 85° + i\sin 85°) \cdot \sqrt{2}(\cos 40° + i\sin 40°)$
8. $(3 - 2i) \cdot (1 + i)$
9. $(1 - i) \cdot (1 + i)$
10. $(4 - i) \cdot (3 + 2i)$
11. $(1 + i) \cdot (1 + 4i)$
12. $(2 + 2i) \cdot (3 + i)$
13. $(1 - 3i) \cdot (2 + i)$
14. $(1 - i) \cdot (1 - i)$
15. Can you multiply a pair of complex numbers in standard form without converting to trigonometric form?

6.11 Quotient Theorem

Here you'll learn to derive and apply the Quotient Theorem to simplify the division of complex numbers.

Suppose you are given two complex numbers in polar form, such as $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $3(\cos\pi+i\sin\pi)$ and asked to divide them. Can you do this? How long will it take you?

By the end of this Concept, you'll know how to divide complex numbers using the Quotient Theorem.

Watch This

In the second part of this video you'll learn about the quotient of complex numbers in trigonometric form.
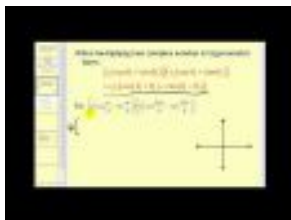
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James Sousa:TheProductandQuotient ofComplexNumbersin Trigonometric Form

Guidance

Division of complex numbers in polar form is similar to the division of complex numbers in standard form. However, to determine a general rule for division, the denominator must be rationalized by multiplying the fraction by the complex conjugate of the denominator. In addition, the trigonometric functions must be simplified by applying the sum/difference identities for cosine and sine as well as one of the Pythagorean identities. To obtain a general rule for the division of complex numbers in polar from, let the first number be $r_1(\cos\theta_1+i\sin\theta_1)$ and the second number be $r_2(\cos\theta_2+i\sin\theta_2)$. The product can then be simplified by use of five facts: the definition $i^2=-1$, the difference identity $\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)$, the difference identity $\sin\alpha\cos\beta-\cos\alpha\sin\beta=\sin(\alpha-\beta)$, the Pythagorean identity, and the fact that the **conjugate** of $\cos\theta_2+i\sin\theta_2$**is** $\cos\theta_2-i\sin\theta_2$.

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$$

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}\cdot\frac{(\cos\theta_2-i\sin\theta_2)}{(\cos\theta_2-i\sin\theta_2)}$$

$$\frac{r_1}{r_2}\cdot\frac{\cos\theta_1\cos\theta_2-i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2-i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2-i^2\sin^2\theta_2}$$

$$\frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta_2+\sin^2\theta_2}$$

$$\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

In general:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

We can use this rule for the computation of two complex numbers divided by one another.

Example A

Find the quotient of $(\sqrt{3} - i) \div (2 - i2\sqrt{3})$

Solution: Express each number in polar form.

$$\begin{array}{ll}
\sqrt{3} - i & 2 - i2\sqrt{3} \\
r_1 = \sqrt{x^2 + y^2} & r_2 = \sqrt{x^2 + y^2} \\
r_1 = \sqrt{(\sqrt{3})^2 + (-1)^2} & r_2 = \sqrt{(2)^2 + (-2\sqrt{3})^2} \\
r_1 = \sqrt{4} = 2 & r_2 = \sqrt{16} = 4
\end{array}$$

$$\frac{r_1}{r_2} = .5$$

$$\begin{array}{lll}
\theta_1 = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) & \theta_2 = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) & \theta = \theta_1 - \theta_2 \\
\theta_1 = 5.75959 \; rad. & \theta_2 = 5.23599 \; rad. & \theta = 5.75959 - 5.23599 \\
& & \theta = 0.5236
\end{array}$$

Now, plug in what we found to the Quotient Theorem.

$$\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = .5(\cos 0.5236 + i\sin 0.5236)$$

Example B

Find the quotient of the two complex numbers $28\angle 35°$ and $14\angle 24°$

Solution:

$$\begin{array}{lll}
\text{For } 28 \angle 35° & \text{For } 14 \angle 24° & \frac{r_1}{r_2} = \frac{28}{14} = 2 \\
r_1 = 28 & r_2 = 14 & \theta = \theta_1 - \theta_2 \\
\theta_1 = 35° & \theta_2 = 24° & \theta = 35° - 24° = 11°
\end{array}$$

$$\frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$
$$= 2\angle 11°$$

Example C

Using the Quotient Theorem determine $\frac{1}{4cis\frac{\pi}{6}}$

Solution:

Even though 1 is not a complex number, we can still change it to polar form.

$$1 \rightarrow x = 1, y = 0$$

$$r = \sqrt{1^2 + 0^2} = 1$$
$$\tan\theta = \frac{0}{1} = 0 \rightarrow \theta = 0°$$

$$\text{So, } \frac{1}{4cis\frac{\pi}{6}} = \frac{1cis0}{4cis\frac{\pi}{6}} = \frac{1}{4}cis\left(0 - \frac{\pi}{6}\right) = \frac{1}{4}cis\left(-\frac{\pi}{6}\right).$$

Vocabulary

Quotient Theorem: The **quotient theorem** is a theorem showing a simplified way to divide complex numbers.

Guided Practice

1. Divide the following complex numbers. If they are not in polar form, change them before dividing.

$\frac{2\angle 56°}{7\angle 113°}$

2. Divide the following complex numbers. If they are not in polar form, change them before dividing.

$\frac{10\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)}{5(\cos\pi + i\sin\pi)}$

3. Divide the following complex numbers. If they are not in polar form, change them before dividing.

$\frac{2+3i}{-5+11i}$

Solutions:

1. $\frac{2\angle 56°}{7\angle 113°} = \frac{2}{7}\angle(56° - 113°) = \frac{2}{7}\angle -57°$

2.

$$\frac{10\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)}{5(\cos\pi + i\sin\pi)} = 2\left(\cos\left(\frac{5\pi}{3} - \pi\right) + i\sin\left(\frac{5\pi}{3} - \pi\right)\right)$$
$$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

3. $\frac{2+3i}{-5+11i} \rightarrow$ change each to polar.

$x = 2, y = 3$ $\qquad\qquad$ $x = -5, y = 11$

$r = \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.61$ $\qquad$ $r = \sqrt{(-5)^2 + 11^2} = \sqrt{146} \approx 12.08$

$\tan\theta = \frac{3}{2} \rightarrow \theta = 56.31°$ $\qquad\qquad$ $\tan\theta = -\frac{11}{5} \rightarrow \theta = 114.44°$

$\frac{3.61}{12.08}\angle(56.31° - 114.44°) = 0.30\angle -58.13°$

Concept Problem Solution

You know that the 2 numbers to divide are $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $3(\cos\pi + i\sin\pi)$.

If you consider $r_1 = 2, r_2 = 3, \theta_1 = \frac{\pi}{3}, \theta_2 = \pi$, you can use the formula:

$\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Substituting values into this equation gives:

$$\frac{2}{3}[\cos(\frac{\pi}{3} - \pi) + i\sin(\frac{\pi}{3} - \pi)]$$
$$= \frac{2}{3}[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)]$$
$$= \frac{2}{3}\left(\left(-\frac{1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$
$$= -\frac{1}{3} - i\frac{\sqrt{3}}{3}$$

Practice

Divide each pair of complex numbers. If they are not in trigonometric form, change them before dividing.

1. $\frac{3(\cos 32° + i\sin 32°)}{2(\cos 15° + i\sin 15°)}$
2. $\frac{2(\cos 10° + i\sin 10°)}{10(\cos 12° + i\sin 12°)}$
3. $\frac{4(\cos 45° + i\sin 45°)}{8(\cos 62° + i\sin 62°)}$
4. $\frac{2(\cos 60° + i\sin 60°)}{\frac{1}{2}(\cos 34° + i\sin 34°)}$
5. $\frac{5(\cos 25° + i\sin 25°)}{2(\cos 115° + i\sin 115°)}$
6. $\frac{-3(\cos 70° + i\sin 70°)}{3(\cos 85° + i\sin 85°)}$
7. $\frac{7(\cos 85° + i\sin 85°)}{\sqrt{2}(\cos 40° + i\sin 40°)}$
8. $\frac{(3-2i)}{(1+i)}$
9. $\frac{(1-i)}{(1+i)}$
10. $\frac{(4-i)}{(3+2i)}$
11. $\frac{(1+i)}{(1+4i)}$
12. $\frac{(2+2i)}{(3+i)}$
13. $\frac{(1-3i)}{(2+i)}$
14. $\frac{(1-i)}{(1-i)}$
15. Can you divide a pair of complex numbers in standard form without converting to trigonometric form? How?

6.12 DeMoivre's Theorem

Here you'll learn how to simplify a complex number raised to a power using DeMoivre's Theorem

Imagine that you are in math class one day, and you are given the following number:

$[5(\cos 135° + i \sin 135°)]^4$

Your instructor wants you to find this number. Can you do it? How long will it take you? Probably a long time, if you want to take a number to the fourth power, you'd have to multiply the number by itself, over and over again.

However, there is a shortcut. Read on, and by the end of this Concept, you'll be able to use DeMoivre's Theorem to simplify the calculation of powers of complex numbers.

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James Sousa:DeMoivre's Theorem

Guidance

The basic operations of addition, subtraction, multiplication and division of complex numbers can all be carried out, albeit with some changes in form from what you may have seen with numbers having only real components. The addition and subtraction of complex numbers lent themselves best to numbers expressed in standard form. However multiplication and division were easily performed when the complex numbers were in polar form. Another operation that is performed using the polar form of complex numbers is the process of raising a complex number to a power.

The polar form of a complex number is $r(\cos\theta + i\sin\theta)$. If we allow z to equal the polar form of a complex number, it is very easy to see the development of a pattern when raising a complex number in polar form to a power. To discover this pattern, it is necessary to perform some basic multiplication of complex numbers in polar form.

If $z = r(\cos\theta + i\sin\theta)$ and $z^2 = z \cdot z$ then:

$$z^2 = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta)$$
$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]$$
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

Likewise, if $z = r(\cos\theta + i\sin\theta)$ and $z^3 = z^2 \cdot z$ then:

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta) \cdot r(\cos \theta + i\sin \theta)$$
$$z^3 = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

Again, if $z = r(\cos\theta + i\sin\theta)$ and $z^4 = z^3 \cdot z$ then

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

These examples suggest a general rule valid for all powers of z, or n. We offer this rule and assume its validity for all n without formal proof, leaving that for later studies. The general rule for raising a complex number in polar form to a power is called De Moivre's Theorem, and has important applications in engineering, particularly circuit analysis. The rule is as follows:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Where $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer.

Notice what this rule looks like geometrically. A complex number taken to the nth power has two motions: First, its distance from the origin is taken to the nth power; second, its angle is multiplied by n. Conversely, the roots of a number have angles that are evenly spaced about the origin.

Example A

Find $[2(\cos 120° + i\sin 120°)]^5$

Solution: $\theta = 120° = \frac{2\pi}{3}$ rad, using De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$(\cos 120° + i\sin 120°)^5 = 2^5\left[\cos 5\frac{2\pi}{3} + i\sin 5\frac{2\pi}{3}\right]$

$$= 32\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$
$$= 32\left(-\frac{1}{2} + -\frac{i\sqrt{3}}{2}\right)$$
$$= -16 - 16i\sqrt{3}$$

Example B

Find $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{10}$

Solution: Change into polar form.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2} \cdot -\frac{2}{1} \right) = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

$$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = \sqrt{1} = 1$$

The polar form of $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$ is $1\left(\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3} \right)$

Now use De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^{10} = 1^{10}\left[\cos 10\left(\frac{2\pi}{3} \right) + i\sin 10\left(\frac{2\pi}{3} \right) \right]$$

$$\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^{10} = 1\left(\cos \frac{20\pi}{3} + i\sin \frac{20\pi}{3} \right)$$

$$\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^{10} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \rightarrow \text{Standard Form}$$

Example C

Find $[3(\cos 45° + i\sin 45°)]^3$

Solution: $\theta = 45° = \frac{\pi}{4}$ rad, using De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$(\cos 45° + i\sin 45°)^3 = 3^3\left[\cos 3\frac{\pi}{4} + i\sin 3\frac{\pi}{4} \right]$

$$= 27\left(\cos \frac{3\pi}{4} + i\sin \frac{3\pi}{4} \right)$$

$$= 27\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \right)$$

$$= -13.5 + 13.5i\sqrt{2}$$

Vocabulary

DeMoivres Theorem: DeMoivres theorem relates a complex number raised to a power to a set of trigonometric functions by stating that the complex number raised to a power is equal to the trigonometric representation of the number with the power times the angle under consideration as the argument for the trigonometric form.

Guided Practice

1. Evaluate: $\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8$

2. Evaluate: $\left[3\left(\sqrt{3}-i\sqrt{3}\right)\right]^4$

3. Evaluate: $(\sqrt{5}-i)^7$

Solutions:

1. $\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8 = \left(\frac{\sqrt{2}}{2}\right)^8\left(\cos\frac{8\pi}{4}+i\sin\frac{8\pi}{4}\right) = \frac{1}{16}\cos 2\pi+\frac{i}{16}\sin 2\pi = \frac{1}{16}$

2.

$$\left[3(\sqrt{3}-i\sqrt{3})\right]^4 = (3\sqrt{3}-3i\sqrt{3})^4$$

$$r = \sqrt{(3\sqrt{3})^2+(3\sqrt{3})^2} = 3\sqrt{6}, \tan\theta = \frac{3\sqrt{3}}{3\sqrt{3}} = 1 \rightarrow 45°$$

$$= \left(3\sqrt{6}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^4 = (3\sqrt{6})^4\left(\cos\frac{4\pi}{4}+i\sin\frac{4\pi}{4}\right)$$

$$= 81(36)[-1+i(0)] = -2936$$

3.

$$(\sqrt{5}-i)^7 \rightarrow r = \sqrt{(\sqrt{5})^2+(-1)^2} = \sqrt{6}, \tan\theta = -\frac{1}{\sqrt{5}} \rightarrow \theta = 335.9°$$

$$\sqrt{6}(\cos 335.9°+i\sin 335.9°)^7 = (\sqrt{6})^7(\cos(7\cdot 335.9°)+i\sin(7\cdot 335.9°))$$

$$= 216\sqrt{6}(\cos 2351.3°+i\sin 2351.3°)$$

$$= 216\sqrt{6}(-0.981-0.196i)$$

$$= -519.04-103.7i$$

Concept Problem Solution

The original problem was:

$[5(\cos 135°+i\sin 135°)]^4$

In this problem, $r = 5, \theta = \frac{3\pi}{4}$

Using

$$z^n = [r(\cos\theta+i\sin\theta)]^n = r^n(\cos n\theta+i\sin n\theta)$$

, you can substitute to get:

$$[5(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})]^4$$
$$= 5^4(\cos 4\frac{3\pi}{4} + i\sin 4\frac{3\pi}{4})$$
$$= 625(\cos(3\pi) + i\sin(3\pi))$$
$$= 625(-1 + i0)$$
$$= -625$$

Practice

Use DeMoivre's Theorem to evaluate each expression. Write your answer in standard form.

1. $(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^3$
2. $[2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})]^2$
3. $[3(\cos\frac{3\pi}{2} + i\sin\frac{\pi}{2})]^5$
4. $(1 + i)^5$
5. $(1 - \sqrt{3}i)^3$
6. $(1 + 2i)^6$
7. $(\sqrt{3} - i)^5$
8. $(\frac{1}{2} + \frac{i\sqrt{3}}{2})^4$
9. $[3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^5$
10. $(2 - \sqrt{5}i)^5$
11. $(\sqrt{2} + \sqrt{2}i)^4$
12. $[2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})]^8$
13. $(-1 + \sqrt{2}i)^6$
14. $[5(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})]^3$
15. $(3 - 4i)^6$

6.13 DeMoivre's Theorem and nth Roots

Here you'll learn to simplify complex numbers to some root using DeMoivre's Theorem.

You are in math class one day when your teacher asks you to find $\sqrt{3i}$. Are you able to find roots of complex numbers? By the end of this Concept, you'll be able to perform this calculation.

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James Sousa:DeterminingtheNthRoots of a Complex Number

Guidance

Other Concepts in this course have explored all of the basic operations of arithmetic as they apply to complex numbers in standard form and in polar form. The last discovery is that of taking roots of complex numbers in polar form. Using De Moivre's Theorem we can develop another general rule – one for finding the nth root of a complex number written in polar form.

As before, let $z = r(\cos\theta + i\sin\theta)$ and let the nth root of z be $v = s(\cos\alpha + i\sin\alpha)$. So, in general, $\sqrt[n]{z} = v$ and $v^n = z$.

$$\sqrt[n]{z} = v$$
$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = s(\cos\alpha + i\sin\alpha)$$

$r(\cos\theta + i\sin\theta)^{\frac{1}{n}} = s(\cos\alpha + i\sin\alpha)$

$$r^{\frac{1}{n}}\left(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta\right) = s(\cos\alpha + i\sin\alpha)$$
$$r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) = s(\cos\alpha + i\sin\alpha)$$

From this derivation, we can conclude that $r^{\frac{1}{n}} = s$ or $s^n = r$ and $\alpha = \frac{\theta}{n}$. Therefore, for any integer $k(0, 1, 2, \ldots n-1)$, v is an nth root of z if $s = \sqrt[n]{r}$ and $\alpha = \frac{\theta + 2\pi k}{n}$. Therefore, the general rule for finding the nth roots of a complex number if $z = r(\cos\theta + i\sin\theta)$ is: $\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$. Let's begin with a simple example and we will leave θ in degrees.

Example A

Find the two square roots of $2i$.

Solution: Express $2i$ in polar form.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \cos\theta = 0$$
$$r = \sqrt{(0)^2 + (2)^2} \qquad\qquad \theta = 90°$$
$$r = \sqrt{4} = 2$$

$$(2i)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left(\cos\frac{90°}{2} + i\sin\frac{90°}{2}\right) = \sqrt{2}(\cos 45° + i\sin 45°) = 1 + i$$

To find the other root, add $360°$ to θ.

$$(2i)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left(\cos\frac{450°}{2} + i\sin\frac{450°}{2}\right) = \sqrt{2}(\cos 225° + i\sin 225°) = -1 - i$$

Example B

Find the three cube roots of $-2 - 2i\sqrt{3}$

Solution: Express $-2 - 2i\sqrt{3}$ in polar form:

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$$
$$r = \sqrt{16} = 4 \qquad\qquad \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \frac{4\pi}{3}$$

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

$$\sqrt[3]{-2 - 2i\sqrt{3}} = \sqrt[3]{4}\left(\cos\frac{\frac{4\pi}{3} + 2\pi k}{3} + i\sin\frac{\frac{4\pi}{3} + 2\pi k}{3}\right) \quad k = 0, 1, 2$$

$$z_1 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{0}{3}\right) + i\sin\left(\frac{4\pi}{9} + \frac{0}{3}\right)\right] \qquad\qquad k = 0$$
$$= \sqrt[3]{4}\left[\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right]$$
$$z_2 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right) + i\sin\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right)\right] \qquad\qquad k = 1$$
$$= \sqrt[3]{4}\left[\cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9}\right]$$
$$z_3 = \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right)\right] \qquad\qquad k = 2$$
$$= \sqrt[3]{4}\left[\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right]$$

In standard form: $z_1 = 0.276 + 1.563i, z_2 = -1.492 - 0.543i, z_3 = 1.216 - 1.02i$.

Example C

Calculate $\left(\cos \frac{\pi}{4} + i\sin \frac{\pi}{4}\right)^{1/3}$

Using the for of DeMoivres Theorem for fractional powers, we get:

$$\left(\cos \frac{\pi}{4} + i\sin \frac{\pi}{4}\right)^{1/3}$$
$$= \cos\left(\frac{1}{3} \times \frac{\pi}{4}\right) + i\sin\left(\frac{1}{3} \times \frac{\pi}{4}\right)$$
$$= \left(\cos \frac{\pi}{12} + i\sin \frac{\pi}{12}\right)$$

Vocabulary

DeMoivres Theorem: DeMoivres theorem relates a complex number raised to a power to a set of trigonometric functions.

Guided Practice

1. Find $\sqrt[3]{27i}$.

2. Find the principal root of $(1+i)^{\frac{1}{5}}$. Remember the principal root is the positive root i.e. $\sqrt{9} = \pm 3$ so the principal root is +3.

3. Find the fourth roots of $81i$.

Solutions:

1.

$$\sqrt[3]{27i} = (0+27i)^{\frac{1}{3}}$$

$$a = 0 \text{ and } b = 27$$

$$x = 0 \text{ and } y = 27$$

Polar Form

$$r = \sqrt{x^2 + y^2} \qquad \theta = \frac{\pi}{2}$$

$$r = \sqrt{(0)^2 + (27)^2}$$

$$r = 27$$

$$\sqrt[3]{27i} = \left[27\left(\cos(\frac{\pi}{2} + 2\pi k) + i\sin(\frac{\pi}{2} + 2\pi k)\right)\right]^{\frac{1}{3}} \text{ for } k = 0, 1, 2$$

$$\sqrt[3]{27i} = \sqrt[3]{27}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2} + 2\pi k\right) + i\sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2} + 2\pi k\right)\right] \text{ for } k = 0, 1, 2$$

$$\sqrt[3]{27i} = 3\left(\cos \frac{\pi}{6} + i\sin \frac{\pi}{6}\right) \text{ for } k = 0$$

$$\sqrt[3]{27i} = 3\left(\cos \frac{5\pi}{6} + i\sin \frac{5\pi}{6}\right) \text{ for } k = 1$$

$$\sqrt[3]{27i} = 3\left(\cos \frac{9\pi}{6} + i\sin \frac{9\pi}{6}\right) \text{ for } k = 2$$

$$\sqrt[3]{27i} = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), 3\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right), -3i$$

}}

2.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\sqrt{2}}{2} \qquad\qquad \text{Polar Form} = \sqrt{2}cis\frac{\pi}{4}$$

$$r = \sqrt{(1)^2 + (1)^2}$$

$$r = \sqrt{2}$$

$$(1+i)^{\frac{1}{5}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{\frac{1}{5}}$$

$$(1+i)^{\frac{1}{5}} = \sqrt{2}^{\frac{1}{5}}\left[\cos\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right) + i\sin\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right)\right]$$

$$(1+i)^{\frac{1}{5}} = \sqrt[10]{2}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)$$

In standard form $(1+i)^{\frac{1}{5}} = (1.06 + 1.06i)$ and this is the principal root of $(1+i)^{\frac{1}{5}}$.

3.

$81i$ in polar form is:

$$r = \sqrt{0^2 + 81^2} = 81, \tan\theta = \frac{81}{0} = und \rightarrow \theta = \frac{\pi}{2} \quad 81\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\left[81\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + i\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)\right]^{\frac{1}{4}}$$

$$3\left(\cos\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2\pi k}{4}\right)\right)$$

$$3\left(\cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)\right)$$

$$z_1 = 3\left(\cos\left(\frac{\pi}{8} + \frac{0\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{0\pi}{2}\right)\right) = 3\cos\frac{\pi}{8} + 3i\sin\frac{\pi}{8} = 2.77 + 1.15i$$

$$z_2 = 3\left(\cos\left(\frac{\pi}{8} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{\pi}{2}\right)\right) = 3\cos\frac{5\pi}{8} + 3i\sin\frac{5\pi}{8} = -1.15 + 2.77i$$

$$z_3 = 3\left(\cos\left(\frac{\pi}{8} + \frac{2\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{2\pi}{2}\right)\right) = 3\cos\frac{9\pi}{8} + 3i\sin\frac{9\pi}{8} = -2.77 - 1.15i$$

$$z_4 = 3\left(\cos\left(\frac{\pi}{8} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{3\pi}{2}\right)\right) = 3\cos\frac{13\pi}{8} + 3i\sin\frac{13\pi}{8} = 1.15 - 2.77i$$

Concept Problem Solution

Finding the two square roots of $3i$ involves first converting the number to polar form:

For the radius:

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(0)^2 + (3)^2}$$
$$r = \sqrt{9} = 3$$

And the angle:

$$\cos\theta = 0$$
$$\theta = 90°$$

$$(3i)^{\frac{1}{2}} = 3^{\frac{1}{2}}\left(\cos\frac{90°}{2} + i\sin\frac{90°}{2}\right) = \sqrt{3}(\cos 45° + i\sin 45°) = \frac{\sqrt{6}}{2}(1+i)$$

To find the other root, add 360° to θ.

$$(3i)^{\frac{1}{2}} = 3^{\frac{1}{2}}\left(\cos\frac{450°}{2} + i\sin\frac{450°}{2}\right) = \sqrt{3}(\cos 225° + i\sin 225°) = \frac{\sqrt{6}}{2}(-1-i)$$

Practice

Find the cube roots of each complex number. Write your answers in standard form.

1. $8(\cos 2\pi + i\sin 2\pi)$
2. $3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
3. $2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$
4. $(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
5. $(3 + 4i)$
6. $(2 + 2i)$

Find the principal fifth roots of each complex number. Write your answers in standard form.

7. $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$
8. $4(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$
9. $32(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
10. $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
11. $32i$
12. $(1 + \sqrt{5}i)$
13. Find the sixth roots of -64 and plot them on the complex plane.
14. How many solutions could the equation $x^6 + 64 = 0$ have? Explain.
15. Solve $x^6 + 64 = 0$. Use your answer to #13 to help you.

6.14 Equations Using DeMoivre's Theorem

Here you'll learn how to solve equations using DeMoivre's Theorem.

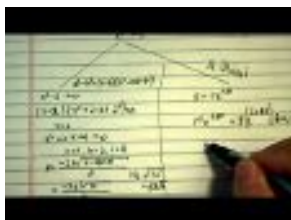You are given an equation in math class:

$x^4 = 16$

and asked to solve for "x". "Excellent!" you say. "This should be easy. The answer is 2."

"Not quite so fast," says your instructor.

"I want you to find the complex roots as well!"

Can you do this? Read on, and by the end of this Concept, you'll be able to solve equations to find complex roots.

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DeMoivres Theorem: Find all roots. Real and Imaginary.

Guidance

We've already seen equations that we would like to solve. However, up until now, these equations have involved solutions that were real numbers. However, there is no reason that solutions need to be limited to the real number line. In fact, some equations cannot be solved completely without the use of complex numbers. Here we'll explore a little more about complex numbers as solutions to equations.

The roots of a complex number are cyclic in nature. This means that when the roots are plotted on the complex plane, the nth roots are equally spaced on the circumference of a circle.

Since you began Algebra, solving equations has been an extensive topic. Now we will extend the rules to include complex numbers. The easiest way to explore the process is to actually solve an equation. The solution can be obtained by using De Moivre's Theorem.

Example A

Consider the equation $x^5 - 32 = 0$. The solution is the same as the solution of $x^5 = 32$. In other words, we must determine the fifth roots of 32.

Solution:

$$x^5 - 32 = 0 \text{ and } x^5 = 32.$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(32)^2 + (0)^2}$$
$$r = 32$$
$$\theta = \tan^{-1}\left(\frac{0}{32}\right) = 0$$

Write an expression for determining the fifth roots of $32 = 32 + 0i$

$$32^{\frac{1}{5}} = [32(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{\frac{1}{5}}$$
$$= 2\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right) k = 0, 1, 2, 3, 4$$

$$x_1 = 2\left(\cos\frac{0}{5} + i\sin\frac{0}{5}\right) \rightarrow 2(\cos 0 + i\sin 0) = 2 \qquad\qquad for\ k = 0$$

$$x_2 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) \approx 0.62 + 1.9i \qquad\qquad for\ k = 1$$

$$x_3 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right) \approx -1.62 + 1.18i \qquad\qquad for\ k = 2$$

$$x_4 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right) \approx -1.62 - 1.18i \qquad\qquad for\ k = 3$$

$$x_5 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right) \approx 0.62 - 1.9i \qquad\qquad for\ k = 4$$

Example B

Solve the equation $x^3 - 27 = 0$. This is the same as the equation $x^3 = 27$.

Solution:

$$x^3 = 27$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(27)^2 + (0)^2}$$
$$r = 27$$
$$\theta = \tan^{-1}\left(\frac{0}{27}\right) = 0$$

Write an expression for determining the cube roots of $27 = 27 + 0i$

$$27^{\frac{1}{3}} = [27(\cos(0+2\pi k)+i\sin(0+2\pi k)]^{\frac{1}{3}}$$

$$= 3\left(\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right) k = 0,1,2$$

$$x_1 = 3\left(\cos\frac{0}{3} + i\sin\frac{0}{3}\right) \rightarrow 3(\cos 0 + i\sin 0) = 3 \qquad\qquad for\ k = 0$$

$$x_2 = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \approx -1.5 + 2.6i \qquad\qquad for\ k = 1$$

$$x_3 = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \approx -1.5 - 2.6i \qquad\qquad for\ k = 2$$

Example C

Solve the equation $x^4 = 1$

Solution:

$$x^4 = 1$$
$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{(1)^2+(0)^2}$$
$$r = 1$$
$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

Write an expression for determining the cube roots of $1 = 1+0i$

$$1^{\frac{1}{4}} = [1(\cos(0+2\pi k)+i\sin(0+2\pi k)]^{\frac{1}{4}}$$

$$= 1\left(\cos\frac{2\pi k}{4} + i\sin\frac{2\pi k}{4}\right) k = 0,1,2,3$$

$$x_1 = 1\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) \rightarrow 3(\cos 0 + i\sin 0) = 1 \qquad\qquad for\ k = 0$$

$$x_2 = 1\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) = 0 + i = i \qquad\qquad for\ k = 1$$

$$x_3 = 1\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) = -1 - 0i = -1 \qquad\qquad for\ k = 2$$

$$x_4 = 1\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = 0 - i = -i \qquad\qquad for\ k = 3$$

Vocabulary

DeMoivres Theorem: DeMoivres theorem relates a complex number raised to a power to a set of trigonometric functions by stating that the complex number raised to a power is equal to the trigonometric representation of the number with the power times the angle under consideration as the argument for the trigonometric form.

Guided Practice

1. Rewrite the following in rectangular form: $[2(\cos 315° + i \sin 315°)]^3$

2. Solve the equation $x^4 + 1 = 0$. What shape do the roots make?

3. Solve the equation $x^3 - 64 = 0$. What shape do the roots make?

Solutions:

1.

$$r = 2 \text{ and } \theta = 315° \text{ or } \frac{7\pi}{4}.$$

$$z^n = [r(\cos\theta + i \sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

$$z^3 = 2^3 \left[(\cos 3 \left(\frac{7\pi}{4} \right) + i \sin 3 \left(\frac{7\pi}{4} \right) \right]$$

$$z^3 = 8 \left(\cos\frac{21\pi}{4} + i \sin\frac{21\pi}{4} \right)$$

$$z^3 = 8 \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right)$$

$$z^3 = -4\sqrt{2} - 4i\sqrt{2}$$

$\frac{21\pi}{4}$ is in the third quadrant so both are negative.

2.

$$x^4 + 1 = 0 \qquad\qquad\qquad r = \sqrt{x^2 + y^2}$$

$$x^4 = -1 \qquad\qquad\qquad r = \sqrt{(-1)^2 + (0)^2}$$

$$x^4 = -1 + 0i \qquad\qquad\qquad r = 1$$

$$\theta = \tan^{-1} \left(\frac{0}{-1} \right) + \pi = \pi$$

Write an expression for determining the fourth roots of $x^4 = -1 + 0i$

$$(-1 + 0i)^{\frac{1}{4}} = [1(\cos(\pi + 2\pi k) + i \sin(\pi + 2\pi k))]^{\frac{1}{4}}$$

$$(-1 + 0i)^{\frac{1}{4}} = 1^{\frac{1}{4}} \left(\cos\frac{\pi + 2\pi k}{4} + i \sin\frac{\pi + 2\pi k}{4} \right)$$

$$x_1 = 1 \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad\qquad \text{for } k = 0$$

$$x_2 = 1 \left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4} \right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad\qquad \text{for } k = 1$$

$$x_3 = 1 \left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4} \right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \qquad\qquad \text{for } k = 2$$

$$x_4 = 1 \left(\cos\frac{7\pi}{4} + i \sin\frac{7\pi}{4} \right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \qquad\qquad \text{for } k = 3$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the four roots will form the corners of a square.

3.

$x^3 - 64 = 0 \rightarrow x^3 = 64 + 0i$

$64 + 0i = 64(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k))$

$$x = (x^3)^{\frac{1}{3}} = (64 + 0i)^{\frac{1}{3}}$$
$$= \sqrt[3]{64}\left(\cos\left(\frac{0 + 2\pi k}{3}\right) + i\sin\left(\frac{0 + 2\pi k}{3}\right)\right)$$

$$z_1 = 4\left(\cos\left(\frac{0 + 2\pi 0}{3}\right) + i\sin\left(\frac{0 + 2\pi 0}{3}\right)\right)$$
$$= 4\cos 0 + 4i\sin 0$$
$$= 4 \text{ for } k = 0$$

$$z_2 = 4\left(\cos\left(\frac{0 + 2\pi}{3}\right) + i\sin\left(\frac{0 + 2\pi}{3}\right)\right)$$
$$= 4\cos\frac{2\pi}{3} + 4i\sin\frac{2\pi}{3} = -2 + 2i\sqrt{3} \text{ for } k = 1$$

$$z_3 = 4\left(\cos\left(\frac{0 + 4\pi}{3}\right) + i\sin\left(\frac{0 + 4\pi}{3}\right)\right)$$
$$= 4\cos\frac{4\pi}{3} + 4i\sin\frac{4\pi}{3}$$
$$= -2 - 2i\sqrt{3} \text{ for } k = 2$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the three roots will form the vertices of an equilateral triangle.

Concept Problem Solution

Since you want to find the fourth root of 16, there will be four solutions in all.

$$x^4 = 16.$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(16)^2 + (0)^2}$$
$$r = 16$$
$$\theta = \tan^{-1}\left(\frac{0}{16}\right) = 0$$

Write an expression for determining the fourth roots of $16 = 16 + 0i$

$$16^{\frac{1}{4}} = [16(\cos(0+2\pi k) + i\sin(0+2\pi k)]^{\frac{1}{4}}$$

$$= 2\left(\cos\frac{2\pi k}{4} + i\sin\frac{2\pi k}{4}\right) k = 0,1,2,3,4$$

$$x_1 = 2\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) \to 2(\cos 0 + i\sin 0) = 2 \qquad\qquad for\ k = 0$$

$$x_2 = 2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) = 2i \qquad\qquad for\ k = 1$$

$$x_3 = 2\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) = -2 \qquad\qquad for\ k = 2$$

$$x_4 = 2\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = -2i \qquad\qquad for\ k = 3$$

Therefore, the four roots of 16 are $2, -2, 2i, -2i$. Notice how you could find the two real roots if you seen complex numbers. The addition of the complex roots completes our search for the roots of equations.

Practice

Solve each equation.

1. $x^3 = 1$
2. $x^5 = 1$
3. $x^8 = 1$
4. $x^5 = -32$
5. $x^4 + 5 = 86$
6. $x^5 = -1$
7. $x^4 = -1$
8. $x^3 = 8$
9. $x^6 = -64$
10. $x^3 = -64$
11. $x^5 = 243$
12. $x^3 = 343$
13. $x^7 = -128$
14. $x^{12} = 1$
15. $x^6 = 1$

6.15 Geometry of Complex Roots

Here you'll learn how to plot the complex roots of equations in polar coordinates.

You've just finished a problem where you needed to solve the equation:

$x^4 = 16$

After solving for the roots, which were $2, -2, 2i, -2i$ your instructor asks you to plot them on the complex plane. Can you accomplish this?

Read on, and by the end of this Concept, you'll understand how to plot and interpret the geometry of complex roots.

Watch This

In the second part of this video you'll learn about the geometry of complex roots.

> MEDIA
> Click image to the left for more content.

James Sousa:DeterminingtheNthRoots of a Complex Number

Guidance

It's always good to get an intuitive feel for values by plotting them. This tendency extends into the complex numbers as well.

The five roots of the equation $x^5 - 32 = 0$ involve one real root and four complex ones. Let's take a look at a plot of these roots in the complex plane.

The nth roots of a complex number, when graphed on the complex plane, are equally spaced around a circle. So, instead of having all the roots, all that is necessary to graph the roots is one of them and the radius of the circle. For this particular example, the roots are $\frac{2\pi}{5}$ or $72°$ apart. This goes along with what we know about regular pentagons. The roots are $\frac{2\pi}{n}$ degrees apart.

Example A

Calculate the two roots for $x^2 = 1$ and represent them graphically.

Solution:

$$x^2 = 1$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (0)^2}$$
$$r = 1$$
$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

Write an expression for determining the two roots of $1 = 1 + 0i$

$$1^{\frac{1}{2}} = [1(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{\frac{1}{2}}$$
$$= 1\left(\cos\frac{2\pi k}{2} + i\sin\frac{2\pi k}{2}\right) k = 0, 1$$
$$x_1 = 1\left(\cos\frac{0}{2} + i\sin\frac{0}{2}\right) \rightarrow 1(\cos 0 + i\sin 0) = 1 \qquad\qquad for\ k = 0$$
$$x_2 = 1\left(\cos\frac{2\pi}{2} + i\sin\frac{2\pi}{2}\right) = -1 \qquad\qquad for\ k = 1$$

These roots are plotted here:

Example B

Calculate the three roots for $x^3 = 1$ and represent them graphically.

Solution: In standard form, $1 = 1 + 0i\ r = 1$ and $\theta = 0$. The polar form is $1 + 0i = 1[\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]$. The expression for determining the cube roots of $1 + 0i$ is:

$$(1 + 0i)^{\frac{1}{3}} = 1^{\frac{1}{3}}\left(\cos\frac{0 + 2\pi k}{3} + i\sin\frac{0 + 2\pi k}{3}\right)$$

When $k = 0, k = 1$ and $k = 2$ the three cube roots of 1 are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. When these three roots are represented graphically, the three points, on the circle with a radius of 1 (the cubed root of 1 is 1), form a triangle.

Example C

Calculate the four roots for $x^4 = 1$ and represent them graphically

Solution:

$$x^4 = 1$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (0)^2}$$
$$r = 1$$
$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

Write an expression for determining the cube roots of $1 = 1 + 0i$

$$1^{\frac{1}{4}} = [1(\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)]^{\frac{1}{4}}$$
$$= 1\left(\cos\frac{2\pi k}{4} + i\sin\frac{2\pi k}{4}\right) k = 0, 1, 2, 3$$
$$x_1 = 1\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right) \rightarrow 3(\cos 0 + i\sin 0) = 1 \qquad\qquad for\ k = 0$$
$$x_2 = 1\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) = 0 + i = i \qquad\qquad for\ k = 1$$
$$x_3 = 1\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) = -1 - 0i = -1 \qquad\qquad for\ k = 2$$
$$x_4 = 1\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = 0 - i = -i \qquad\qquad for\ k = 3$$

These roots are plotted here:

Vocabulary

DeMoivres Theorem: DeMoivres theorem relates a complex number raised to a power to a set of trigonometric functions by stating that the complex number raised to a power is equal to the trigonometric representation of the number with the power times the angle under consideration as the argument for the trigonometric form.

Guided Practice

1. In the examples above, you saw the complex roots determined for a number of different polynomial orders, such as x^2, x^3, x^4. What conclusion can you draw about the number of complex roots there are in relation to the order of the polynomial being solved?

2. What is the spacing in polar coordinates between the roots of the polynomial $x^6 = 12$?

3. Solve for the roots of the equation $x^2 - 3x + 5 = 0$ and plot them.

Solutions:

1. You can conclude that the total number of roots is the same as the order of the polynomial under consideration. For example, x^2 will have 2 roots, while x^3 will have 3 roots, etc.

2. Since there are six total roots, and all of the roots are equally spaced around a circle in the complex plane, there are $\frac{360°}{6} = 60°$ between roots.

3. This equation can be solved using the familiar quadratic formula. Notice that previously we used quadratics that gave real solutions. However, there is no reason to limit the solution set to the real numbers, now that you know how to utilize complex numbers as well.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - (4)(1)(5)}}{(2)(1)}$$
$$= \frac{3 \pm \sqrt{-11}}{2}$$
$$= \frac{3 \pm i\sqrt{11}}{2}$$
$$= \frac{3}{2} \pm \frac{i\sqrt{11}}{2}$$

A plot of these roots looks like this:

where the vertical axis is the imaginary number line.

Concept Problem Solution

You can see a plot of these roots here. Notice, as mentioned before, that the roots are placed equidistant around a circle that could be drawn in the plane with a radius of two.

Practice

1. How many roots does $x^5 = 1$ have?
2. Calculate the roots of $x^5 = 1$ and represent them graphically.
3. How many roots does $x^8 = 1$ have?
4. Calculate the roots of $x^8 = 1$ and represent them graphically.
5. How many roots does $x^{10} = 1$ have?
6. Calculate the roots of $x^{10} = 1$ and represent them graphically.
7. How many roots does $x^4 = 16$ have?
8. Calculate the roots of $x^4 = 16$ and represent them graphically.
9. How many roots does $x^3 = 27$ have?
10. Calculate the roots of $x^3 = 27$ and represent them graphically.
11. How do the solutions of the equation $x^3 = -1$ compare to the solutions of the equation $x^3 = 1$?
12. Describe how to represent the roots of $x^6 = 1$ graphically without first solving the equation.
13. Describe how to represent the roots of $x^{12} = 1$ graphically without first solving the equation.
14. Describe how to represent the roots of $x^4 = 81$ graphically without first solving the equation.
15. Describe how to represent the roots of $x^8 = 256$ graphically without first solving the equation.

Summary

This Chapter presented polar plots. Included were topics about how to plot values on a polar coordinate system, as well as how to translate between rectangular and polar coordinates. This was followed by how to describe complex numbers in trigonometric form, and theorems dealing with these relationships, including the Product Theorem, the

Quotient Theorem, and DeMoivre's Theorem. The Chapter concluded with sections on how to solve equations in complex numbers and the geometry of complex roots.