Ch. 2: Getting Started

About this lecture

- Study a few simple algorithms for sorting
  - Insertion Sort
  - Selection Sort, Bubble Sort (Exercises)
  - Merge Sort
- Show why these algorithms are correct
- Try to analyze the efficiency of these algorithms (how fast they run)

The Sorting Problem

Input: A list of *n* numbers

Output: Arrange the numbers in increasing order

Remark: Sorting has many applications.

If the list is already sorted, we can search a number in the list faster.

Insertion Sort

- A good algorithm for sorting a small number of elements
- It works the way you might sort a hand of playing cards:
  - Start with an empty left hand and the cards face down on the table
  - Then remove one card at a time from the table, and insert it into the correct position in the left hand
  - To find the correct position for a card, compare it with each of the cards already in the hand, from right to left
  - Finally, the cards held in the left hand are sorted

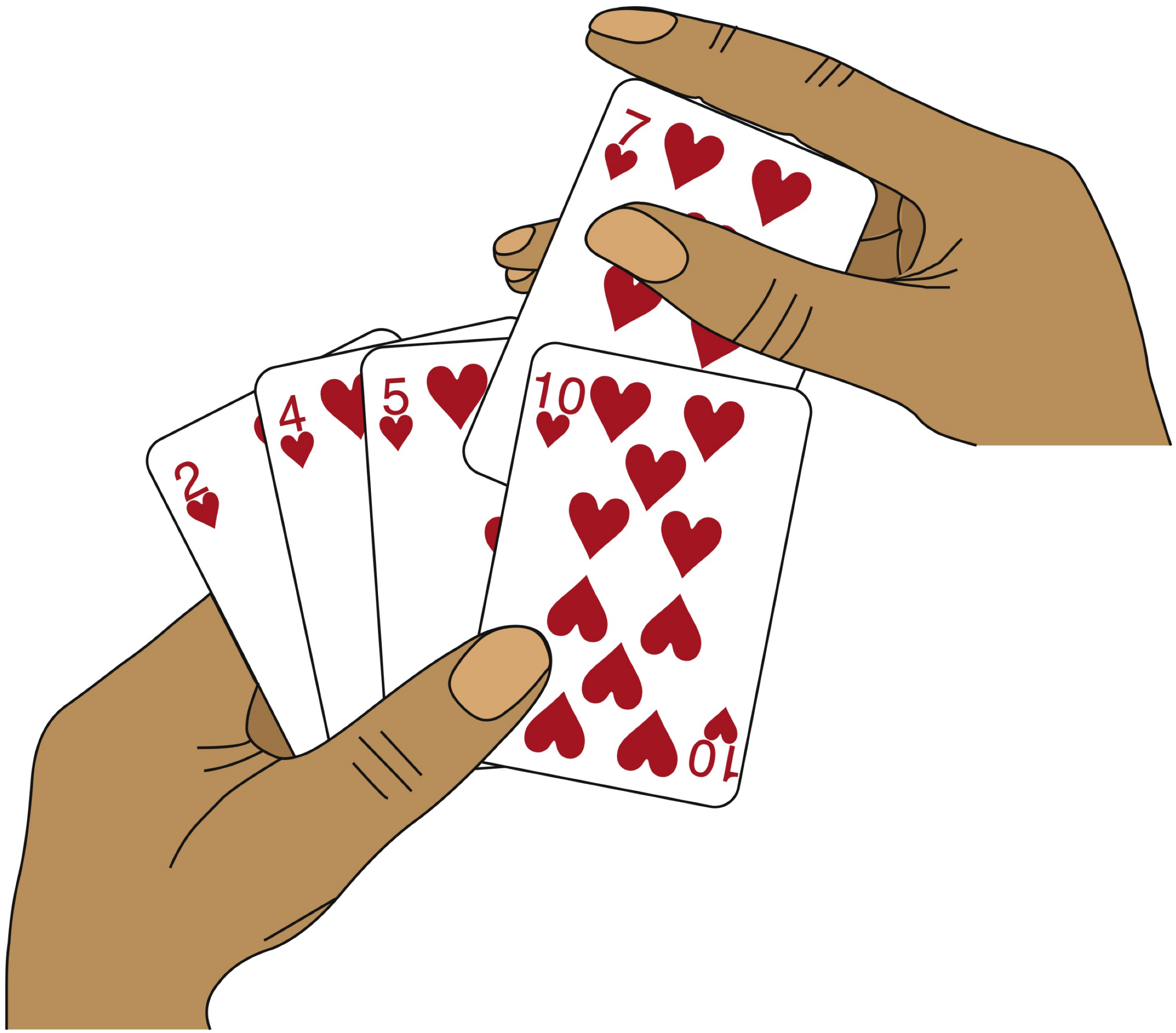
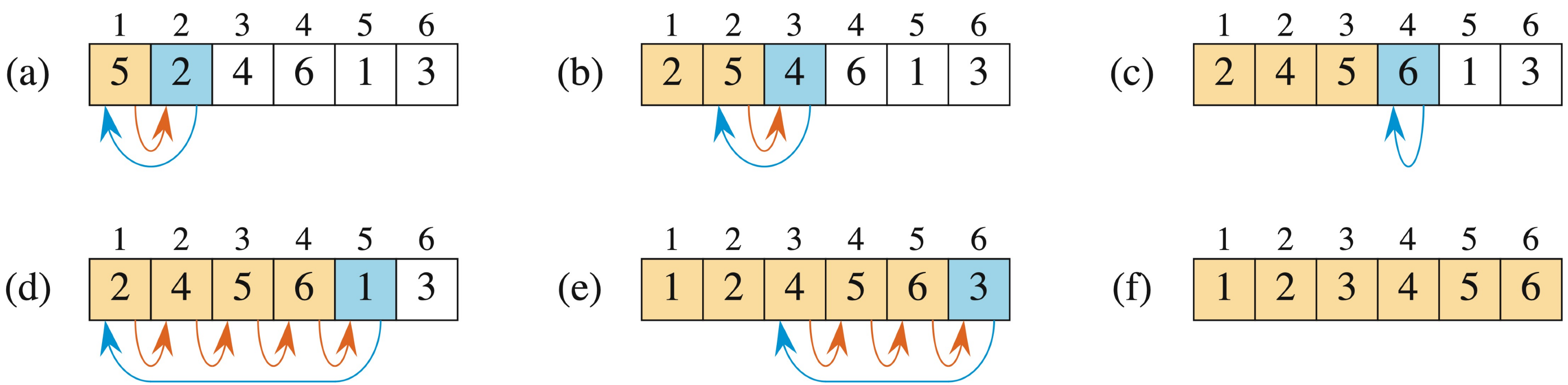


Figure 2.1 Sorting a hand of cards using insertion sort.

Insertion Sort

- Operates in *n* rounds.
- At each round, Swap towards the left side ; Stop until seeing an item with a smaller value.



Question: Why is this algorithm correct?

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i – 1].
    j = i – 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j – 1
    A[j + 1] = key
```

Correctness of Insertion Sort

- **Three properties for Loop Invariant:**
  - **Initialization:** It is true prior to the first iteration of the loop
  - **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration
  - **Termination:** When the loop terminates, array is sorted.
- **Loop invariant for Insertion sort:** At the start of each iteration of the for loop of lines 1-8, the subarray A[1..j-1] consists of the elements originally subarray A[1..j-1], but in sorted order. After the lines 4-8, the subarray A[1..j] consists of the elements originally subarray A[1..j] in sorted order.

Analyzing the Running Time

- Which of candidate algorithms is the best?
- Compare their running time on a computer
  - But there are many kinds of computers !!!

Standard assumption: Our computer is a RAM (Random Access Machine), so that

- each arithmetic (such as $+$, $-$, $\times$, $\div$), memory access, and control (such as conditional jump, subroutine call, return) takes constant amount of time

Analyzing the Running Time

- Suppose that our algorithms are now described in terms of RAM operations

  ➔ we can count # of each operation used

  ➔ we can measure the running time!

- Running time is usually measured as a function of the input size
  – E.g., *n* in our sorting problem

Insertion Sort (Running Time)

The following is a pseudo-code for Insertion Sort.

Each line requires constant RAM operations.

INSERTION-SORT(A)	*cost*	*times*	
for $j \leftarrow 2$ **to** *length*$[A]$	c_1	n	**Why ?**
do *key* $\leftarrow A[j]$	c_2	$n-1$	
$\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$	
$i \leftarrow j-1$	c_4	$n-1$	
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$	
do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$	
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$	
$A[i+1] \leftarrow key$	c_8	$n-1$	

t_j = # of times *key* is compared at round j

Insertion Sort (Running Time)

- Let $T(n)$ denote the running time of insertion sort, on an input of size n
- By combining terms, we have

$$T(n) = c_1 n + (c_2+c_4+c_8)(n-1) + c_5 \sum t_j + (c_6+c_7) \sum (t_j - 1)$$

- The values of t_j are dependent on the input

Insertion Sort (Running Time)

- Best Case:

  The input list is sorted, so that all $t_j = 1$

  Then, $T(n) = c_1 n + (c_2+c_4+c_5+c_8)(n-1)$

  $= Kn + c$ ➔ linear function of n

- Worst Case:

  The input list is sorted in decreasing order, so that all $t_j = j-1$

  Then, $T(n) = K_1 n^2 + K_2 n + K_3$

  ➔ quadratic function of n

Worst-Case Running Time

- In our course (and in most CS research), we concentrate on worst-case time

- Some reasons for this:
  1. Gives an upper bound of running time
  2. Worst case occurs fairly often

- Remark: Some people also study average-case running time (they assume input is drawn randomly)

Divide and Conquer

➔ Divide a big problem into smaller subproblems

➔ Solve (Conquer) smaller subproblems recursively

➔ Combine the results to solve the original one

- The above idea is called Divide-and-Conquer
- Smart idea to solve complex problems
- Can we apply this idea for sorting ?

Divide-and-Conquer for Sorting

- What is a smaller subproblem?
  - ➔ e.g., sorting fewer numbers
  - ➔ Let's divide the list into two shorter lists

- Next, solve smaller subproblems (how?)

- Finally, combine the results
  - ➔ "Merging" two sorted lists into a single sorted list (how?)

Merge Sort

- The previous algorithm, using divide-and-conquer approach, is called Merge Sort
- The key steps are summarized as follows:

  Step 1. Divide list to two halves, A and B

  Step 2. Sort A using Merge Sort

  Step 3. Sort B using Merge Sort

  Step 4. Merge sorted lists of A and B

Question: Why is this algorithm correct?

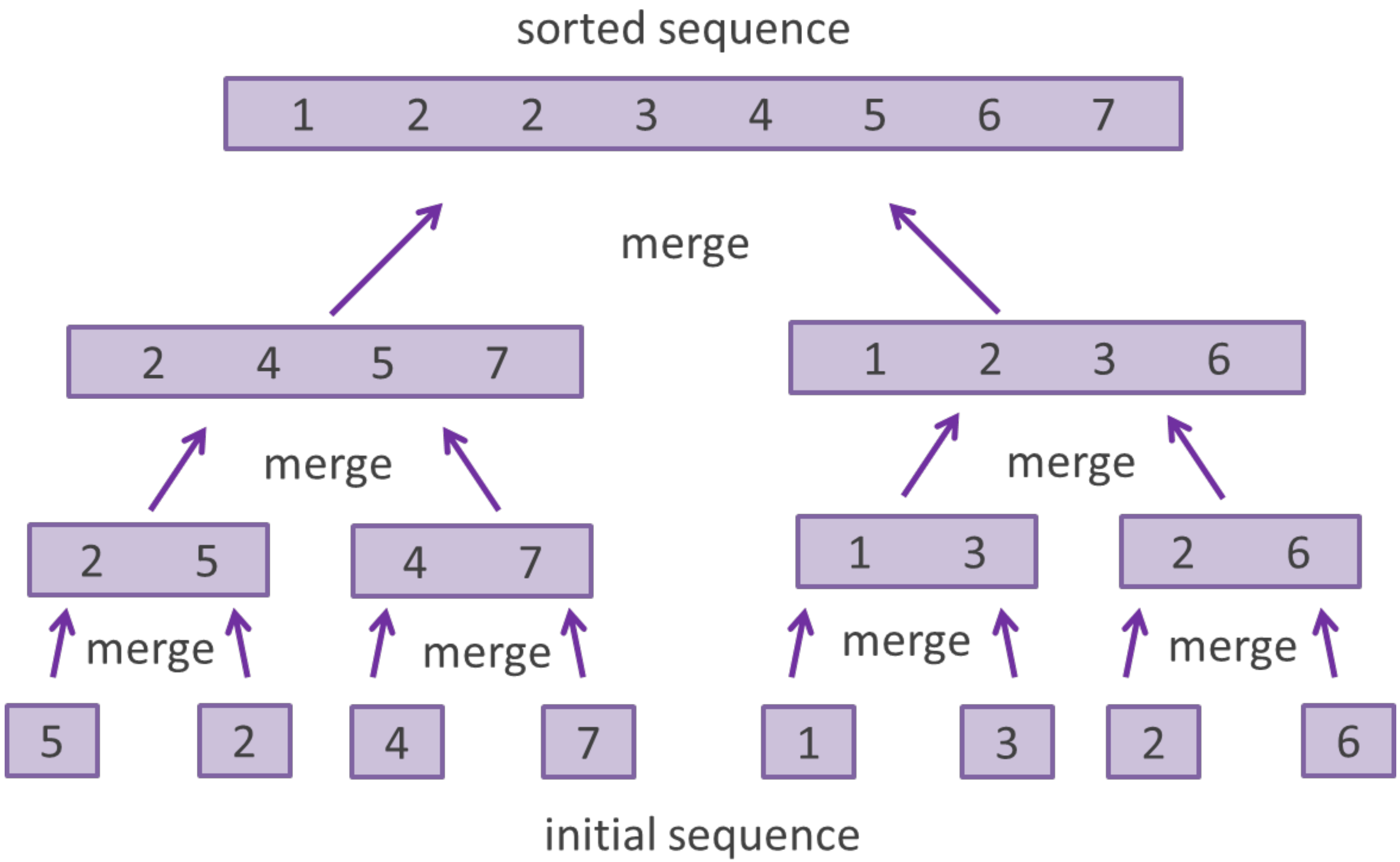


Figure 2.4 The operation of merge sort on the array $A = \langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$. The lengths of the sorted sequences being merged increase as the algorithm progresses from bottom to top.

Merge Sort (Running Time)

The following is a partial pseudo-code for Merge Sort.

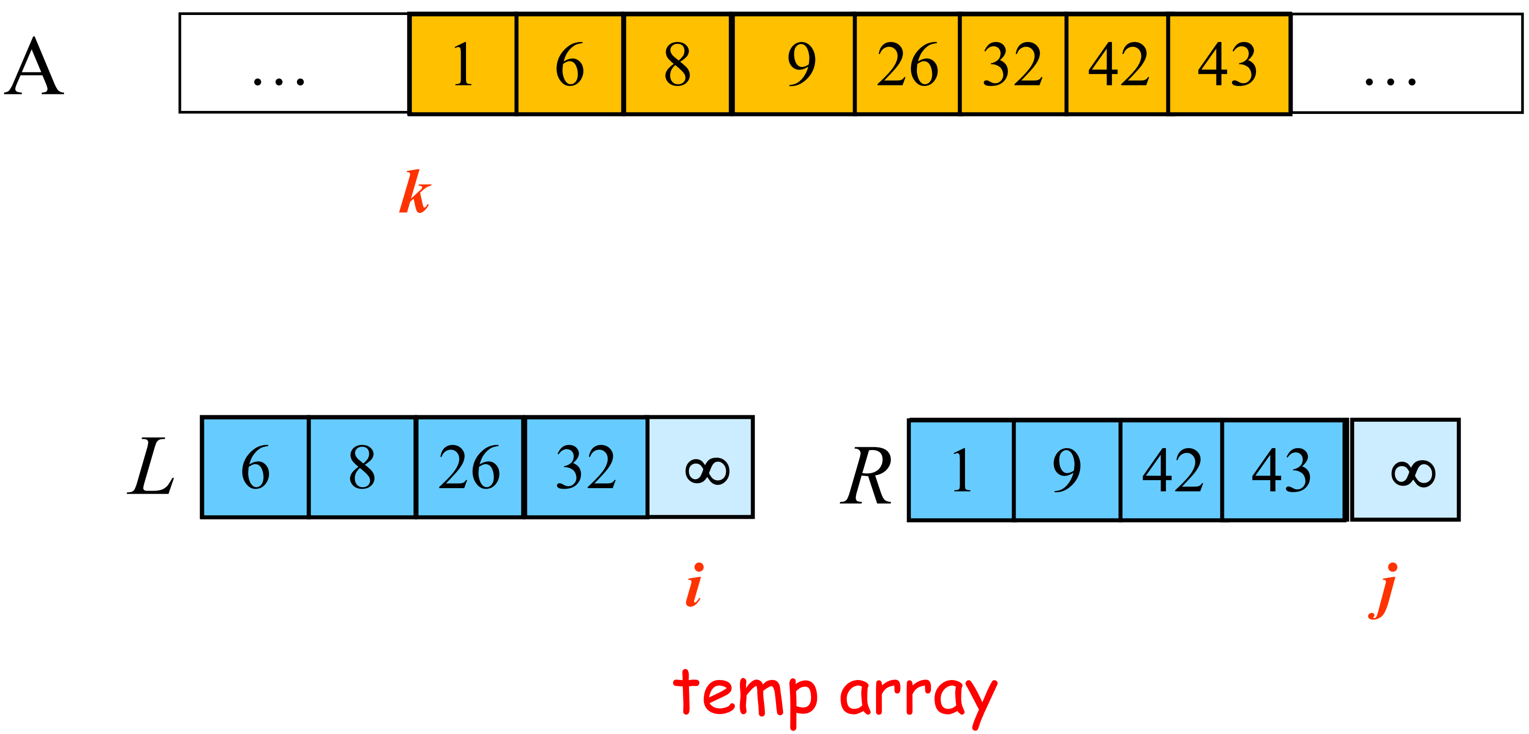
```
MERGE-SORT(A, p, r)
if p ≥ r                           // zero or one element?
    return
q = ⌊(p + r)/2⌋                    // midpoint of A[p : r]
MERGE-SORT(A, p, q)                // recursively sort A[p : q]
MERGE-SORT(A, q + 1, r)            // recursively sort A[q + 1 : r]
// Merge A[p : q] and A[q + 1 : r] into A[p : r].
MERGE(A, p, q, r)
```

The subroutine MERGE(A,p,q,r) is missing.

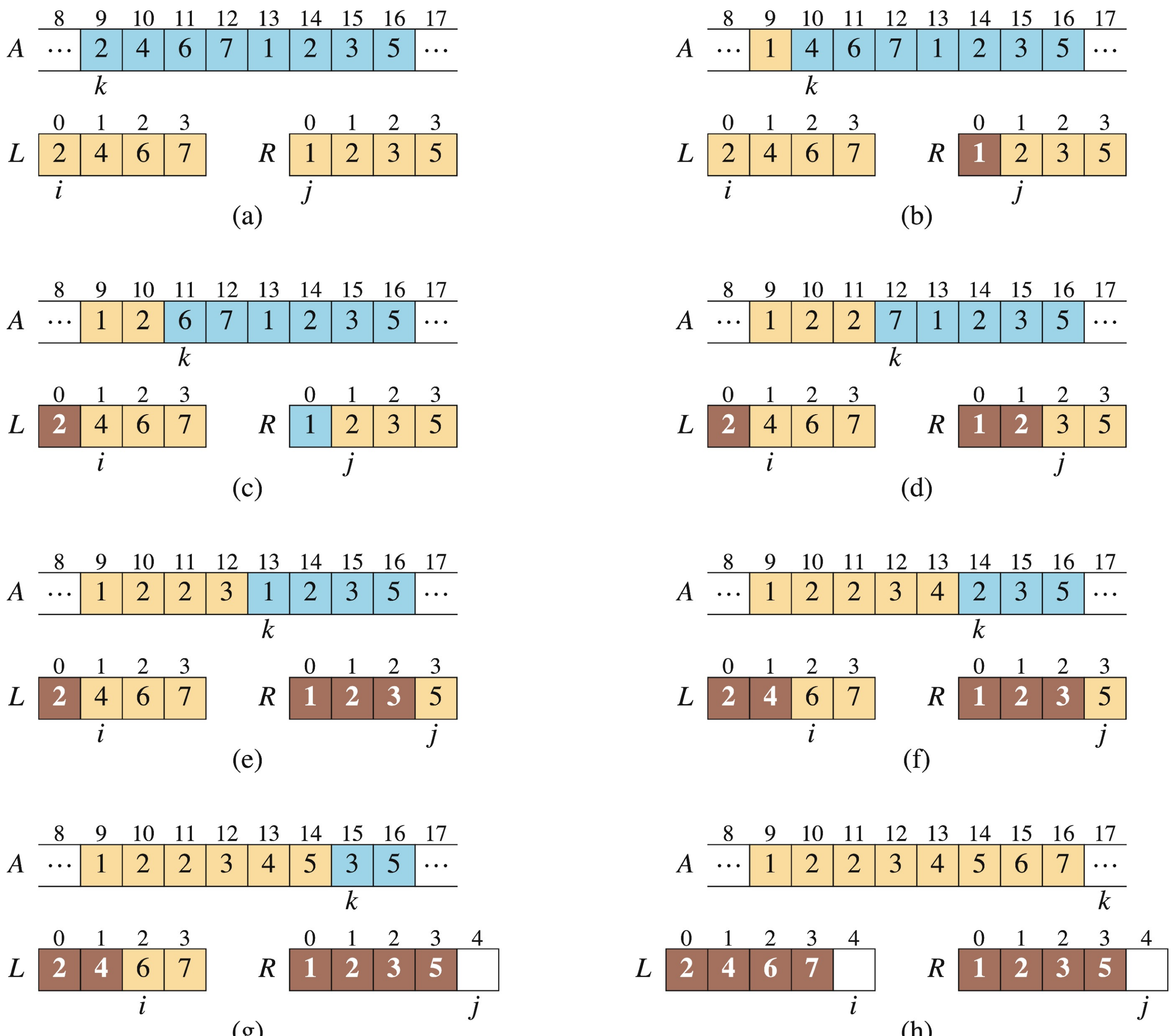
Can you complete it?

Hint: Create two temp arrays for merging

Merge – Example



Merge – Example



Procedure Merge

```
Merge(A, p, q, r)
n1 ← q − p + 1
n2 ← r − q
      for i ← 1 to n1
          do L[i] ← A[p + i − 1]
      for j ← 1 to n2
          do R[j] ← A[q + j]
      L[n1+1] ← ∞
      R[n2+1] ← ∞
      i ← 1
      j ← 1
      for k ←p to r
          do if L[i] ≤ R[j]
              then A[k] ← L[i]
                      i ← i + 1
              else A[k] ← R[j]
                      j ← j + 1
```

Input: Array containing sorted subarrays $A[p..q]$ and $A[q+1..r]$.

Output: Merged sorted subarray in $A[p..r]$.

Sentinels, to avoid having to check if either subarray is fully copied at each step.

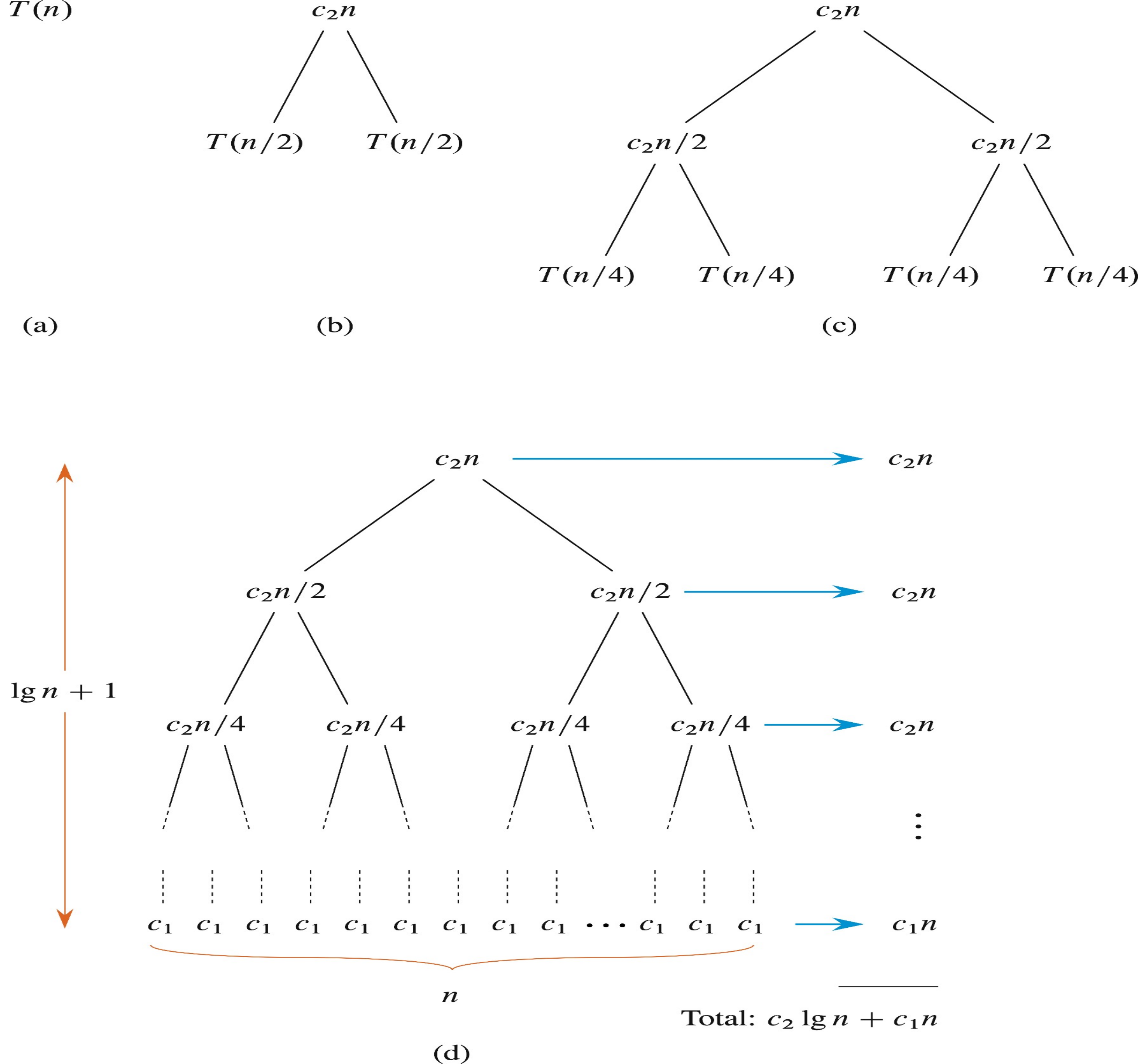
Merge Sort (Running Time)

- Let $T(n)$ denote the running time of merge sort, on an input of size n
- Suppose we know that Merge( ) of two lists of total size n runs in $c_1 n$ time
- Then, we can write T(n) as:

$$T(n) = 2T(n/2) + c_1 n \qquad \text{when } n > 1$$

$$T(n) = c_2 \qquad \text{when } n = 1$$

- Solving the recurrence, we have
- $T(n) = c_1 n \log n + c_2 n$


T(n)
c2n
T(n/2)
T(n/2)
c2n/2
T(n/4)
(a)
(b)
(c)
c1
c2n
c1n
lg n + 1
n
Total: c2 lg n + c1n
(d)

Which Algorithm is Faster?

- Unfortunately, we still cannot tell
  - Since constants in running times are unknown
- But we do know that if *n* is VERY large, the worst-case time of Merge Sort must be smaller than that of Insertion Sort
- Merge Sort is **asymptotically** faster than Insertion Sort
- How to do the merge sort if n is not a power of 2?

天平與撞球

- 你有8顆撞球，其中一顆比較重，唯一的工具是一根天平，請問你最少要稱幾次，才能找出較重的那顆球?

天平與撞球

•先比(1+2+3) 與 (4+5+6)球 的重量，如一樣重則有瑕疵的球為7, 8其中之一。

•如不一樣重，則比較重的一邊任兩顆球，即可求得答案。

•延伸題：(1)若有9顆撞球，請問你最少要稱幾次，才能找出較重的那顆球? (2)若有*N*顆撞球呢?

天平與撞球

- 你有8顆撞球，其中一顆重量跟其他7顆不一樣重，唯一的工具是一根天平，請問你最少要稱幾次，才能找出不一樣重的那顆球?

Exercise

- Problem: 2-1, 2-4
- Exercises: 2.1-5, 2.2-4, 2.3-4, 2.3-5, 2.3-7, 2.3-8

Exercise

- Suppose we have *N* identical-looking balls numbered 1 through *N*, and only one of them is a counterfeit ball whose weight is different from the others. Suppose further that you have one balance scale. Develop a method for finding the counterfeit ball with a minimum number of weighing times in the worst case.