

Chapter 16 Amortized Analysis I

About this lecture

- Given a data structure, **amortized analysis** studies in a sequence of operations, the **average time** to perform an operation
- Introduce **amortized cost** of an operation
- Three Methods for the Same Purpose
 - (1) **Aggregate Method**
 - (2) **Accounting Method**
 - (3) **Potential Method**



This Lecture

Amortized Cost

- In general, we can say something like:
 - OP_1 runs in amortized $O(x)$ time
 - OP_2 runs in amortized $O(y)$ time
 - OP_3 runs in amortized $O(z)$ time

Meaning:

For **any** sequence of operations with

$$\#OP_1 = n_1, \#OP_2 = n_2, \#OP_3 = n_3,$$

$$\text{worst-case total time} = O(n_1x + n_2y + n_3z)$$

Super Stack

- Your friend has created a **super stack**, which, apart from **PUSH/POP**, supports:
SUPER-POP(k): pop top k items
- Suppose **SUPER-POP** never pops more items than current stack size
- The time for **SUPER-POP** is $O(k)$
- The time for **PUSH/POP** is $O(1)$

Super Stack

- Suppose we start with an empty stack, and we have performed n operations
 - But we don't know the order

Questions:

- Worst-case time of a **SUPER-POP** ?

Ans. $O(n)$ time [why?]

- Total time of n operations in worst case ?

Ans. $O(n^2)$ time [correct, but not tight]

Super Stack

- Though we don't know the order of the operations, we still know that:
 - ✓ There are at most n PUSH/POP
 - Time spent on PUSH/POP = $O(n)$
 - ✓ # items popped by all SUPER-POP cannot exceed total # items ever pushed into stack (Why?)
 - Time spent on SUPER-POP = $O(n)$
 - ✓ So, total time of n operations = $O(n)$!!!

Amortized Cost

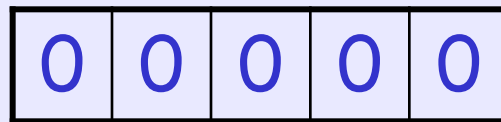
- So far, there are no assumptions on n and the order of operations. Thus, we have:

For **any** n and **any** sequence of n operations,
worst-case total time = $O(n)$

- We can think of each operation performs in average $O(n) / n = O(1)$ time
- We say **amortized cost** = $O(1)$ per operation
(or, each runs in **amortized** $O(1)$ time)

Binary Counter

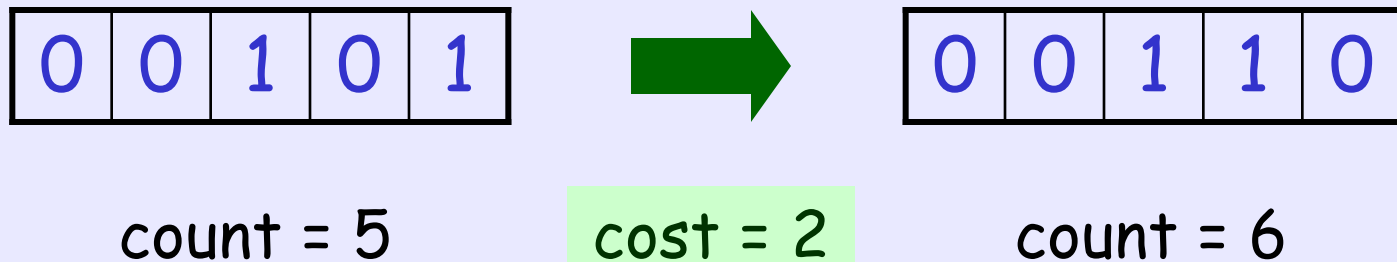
- Let us see another example of implementing a k -bit binary counter
- At the beginning, count is 0, and the counter will be like (assume $k=5$):



which is the binary representation of the count

Binary Counter

- When the counter is incremented, the content will change
- Example: content of counter when:

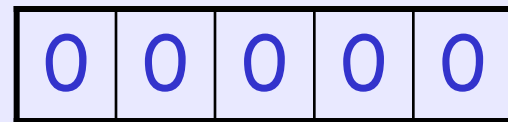
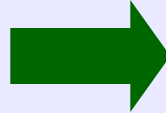
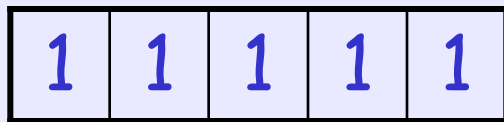


- The **cost** of the increment is equal to the number of bits flipped

Binary Counter

Special case:

When all bits in the counter are 1, an increment resets all bits to 0



count = MAX

cost = $k = 5$

count = 0

- The cost of the corresponding increment is equal to k , the number of bits flipped

Binary Counter

- Suppose we have performed n increments

Questions:

- Worst-case time of an increment ?

Ans. $O(k)$ time

- Total time of n operations in worst case ?

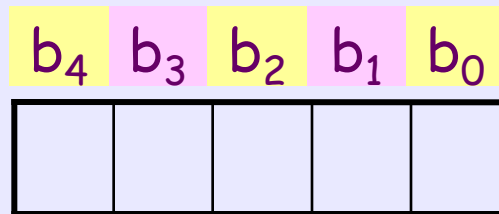
Ans. $O(nk)$ time [correct, but not tight]

Binary Counter

Let us denote the bits in the counter by

$b_0, b_1, b_2, \dots, b_{k-1},$

starting from the right



Observation:

b_i is flipped only once in every 2^i increments

Precisely, b_i is flipped at x^{th} increment $\Leftrightarrow x$ is divisible by 2^i

Amortized Cost

- So, for n increments, the total cost is:

$$\sum_{i=0 \text{ to } k} \lfloor n / 2^i \rfloor$$

$$\leq \sum_{i=0 \text{ to } k} (n / 2^i) < 2n$$

- By dividing total cost with #increments,
→ amortized cost of increment = $O(1)$

Aggregate Method

- The computation of **amortized cost** of an operation in **super stack** or **binary counter** follows similar steps:
 1. Find **total** cost (thus, an "aggregation")
 2. Divide **total** cost by **#operations**

This method is called **Aggregate Method**

Remarks

- In amortized analysis, the amortized cost to perform an operation is computed by the **average cost over all performed operations** in the worst case
- There is a different topic called **average-case analysis**, which studies average performance over all inputs
- Both are useful, but they just study different things

Example: Average-Case Analysis

- Consider sorting n numbers with quick sort
 - Find the number of comparisons on the quick sort
- Suppose each of the $n!$ possible orders is equally likely to be input
- Then, we may be able to compute the average # comparisons of the $n!$ inputs

Example: Average-Case Analysis

- In fact, we can show that

$$\text{average case} = \Theta(n \log n)$$

- So, we can say

$$\text{average sorting time} = \Theta(n \log n)$$

- However, we cannot say

$$\text{amortized sorting time} = \Theta(n \log n) \dots$$

why?

Homework

- Exercises: 16.1-1, 16.1-2, 16.1-3