Chapter 8-2: Sorting in Linear Time

About this lecture

- Sorting algorithms we studied so far
  - Insertion, Merge, Heapsort, Quicksort

  ➔ determine sorted order by comparison

- We will look at 3 new sorting algorithms
  - Counting Sort, Radix Sort, Bucket Sort

  ➔ assume some properties on the input, and determine the sorted order by counting

Counting Sort

extra info on values

- Input: Array A[1..n] of n integers, each has value from [0,k]
- Output: Sorted array of the n integers
- Idea 1: Create B[1..n] to store the output
- Idea 2: Process A[1..n] from right to left
  - ✓ Use k + 1 counters: c[0], c[1], …, c[k]
    - ➢ c[i] is the # of elements with value i

Counting Sort (Step 1)

1. Initialize c[0], c[1], …, c[k] to 0
2. /* First, set c[j] = # elements with value j */

   (1.1) For x = 1,2,…,n, increase c[A[x]] by 1
3. /* Set c[j] = the number of elements less than or equal to value j (iteratively) */

   (1.2) For y = 1,2,…,k, c[y] = c[y-1] + c[y]

Time for Step 1 = O(n + k)

Counting Sort (Step 2)

```
/*  Process A from right to left */
    For x = n, n-1,…,2, 1
    {   /* Process next element */
        B[c[A[x]]] = A[x];
        /* Update counter */
        Decrease c[A[x]] by 1;
    }
```

Time for Step 2 = O(n)

Counting Sort (Details)

Before Running

A

2	1	2	5	3	3	1	2

next element

k+1 counters

c[0] =0 , c[1] = 0, c[2] = 0,

c[3] = 0, c[4] = 0, c[5] = 0

B

Counting Sort (Details)

Step 1.1: For x = 1,2,…,n, increase c[A[x]] by 1

A

2	1	2	5	3	3	1	2

next element

k+1 counters

c[0] =0 , c[1] = 0, c[2] = 0,

c[3] = 0, c[4] = 0, c[5] = 0

B

Counting Sort (Details)

Step 1.1: For x = 1,2,…,n, increase c[A[x]] by 1

A

2	1	2	5	3	3	1	2

next element

k+1 counters

c[0] =0 , c[1] = 0, c[2] = 1,

c[3] = 0, c[4] = 0, c[5] = 0

B

Counting Sort (Details)

Step 1.1: For x = 1,2,…,n, increase c[A[x]] by 1

A

2	1	2	5	3	3	1	2

next element

k+1 counters

c[0] =0 , c[1] = 1, c[2] = 1,

c[3] = 0, c[4] = 0, c[5] = 0

B

Counting Sort (Details)

Step 1.1: For x = 1,2,…,n, increase c[A[x]] by 1

A

2	1	2	5	3	3	1	2

next element

k+1 counters

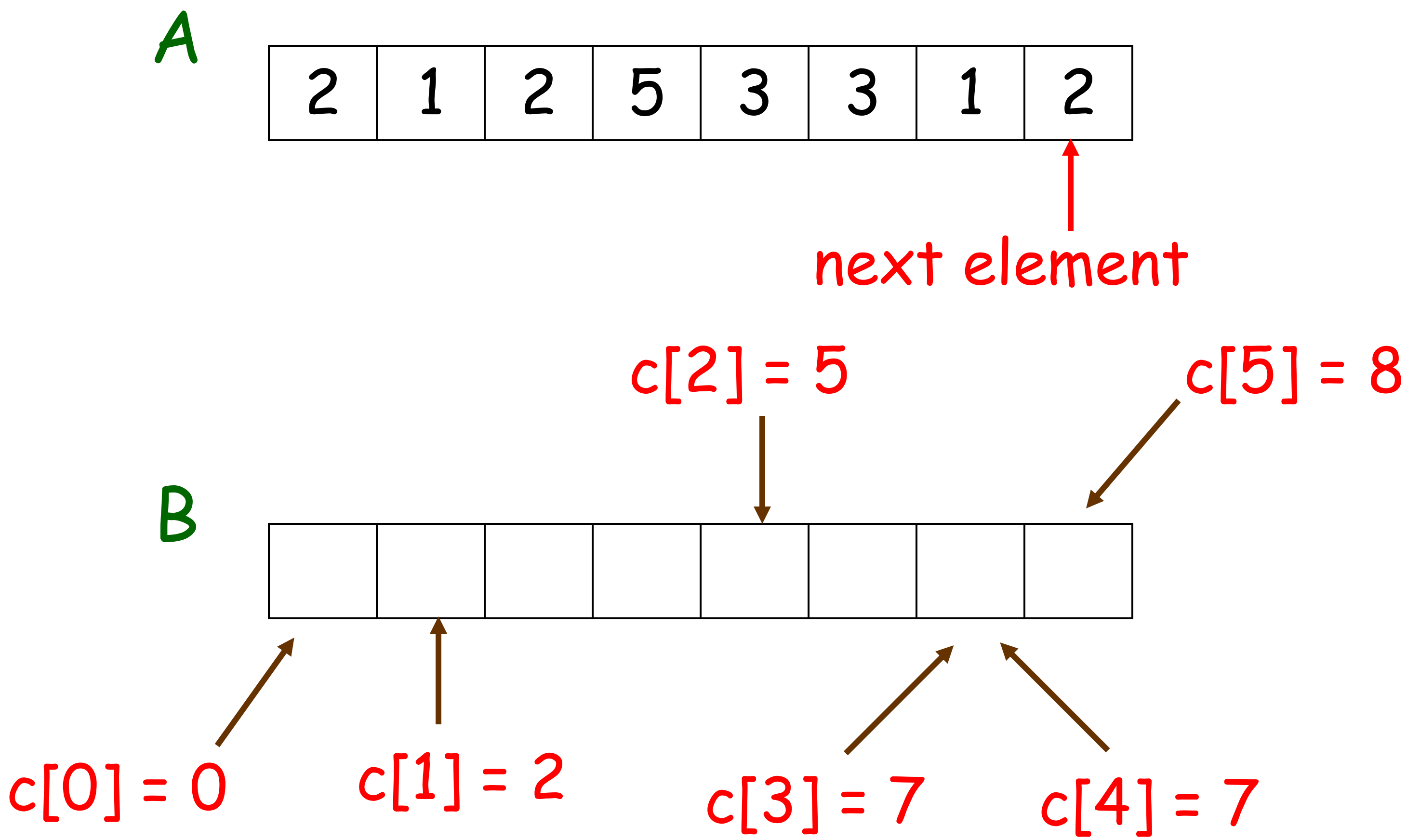
c[0] =0 , c[1] = 2, c[2] = 3,

c[3] = 2, c[4] = 0, c[5] = 1

B

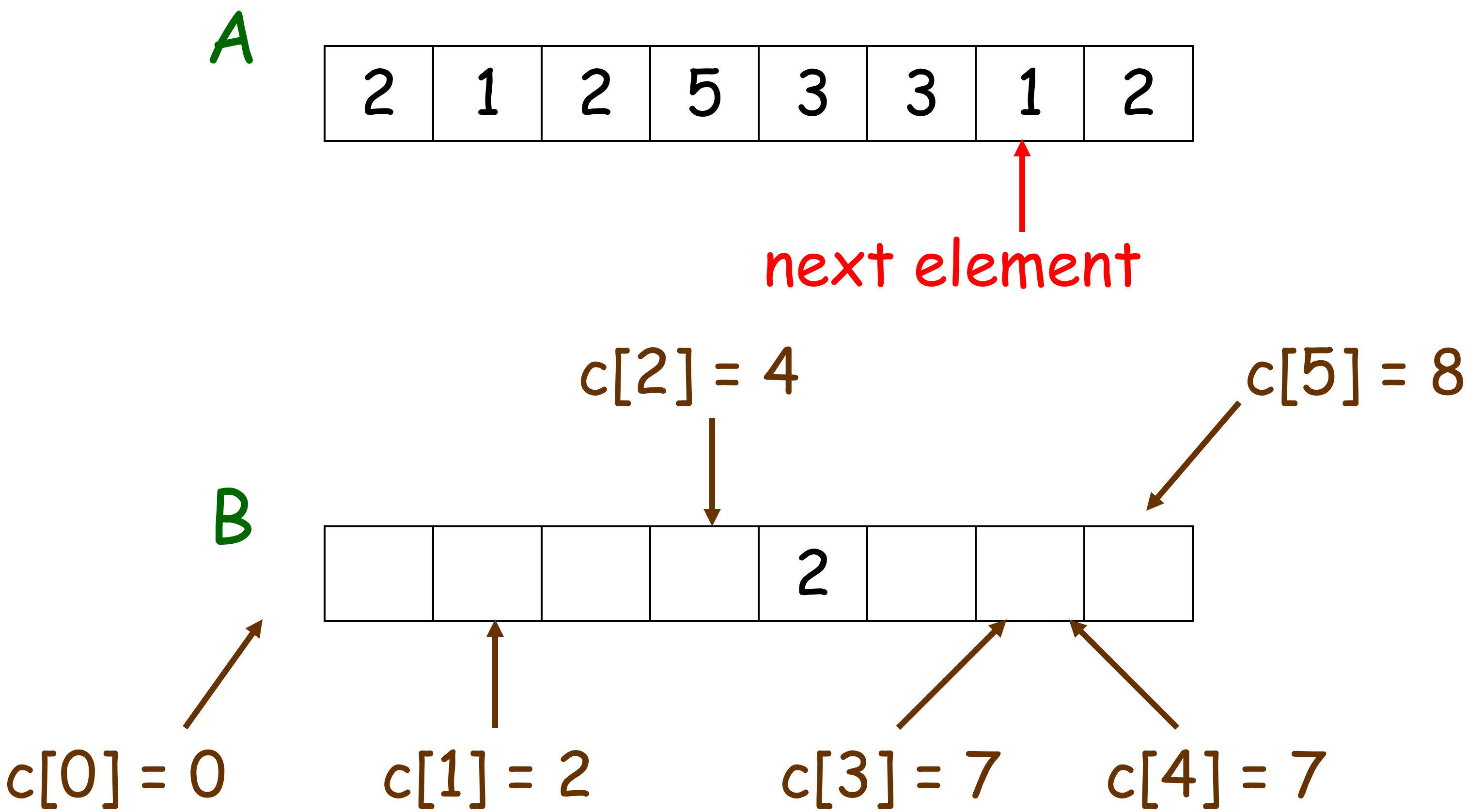
Counting Sort (Details)

Step 1.2: For y = 1,2,…,k, c[y] = c[y-1] + c[y]



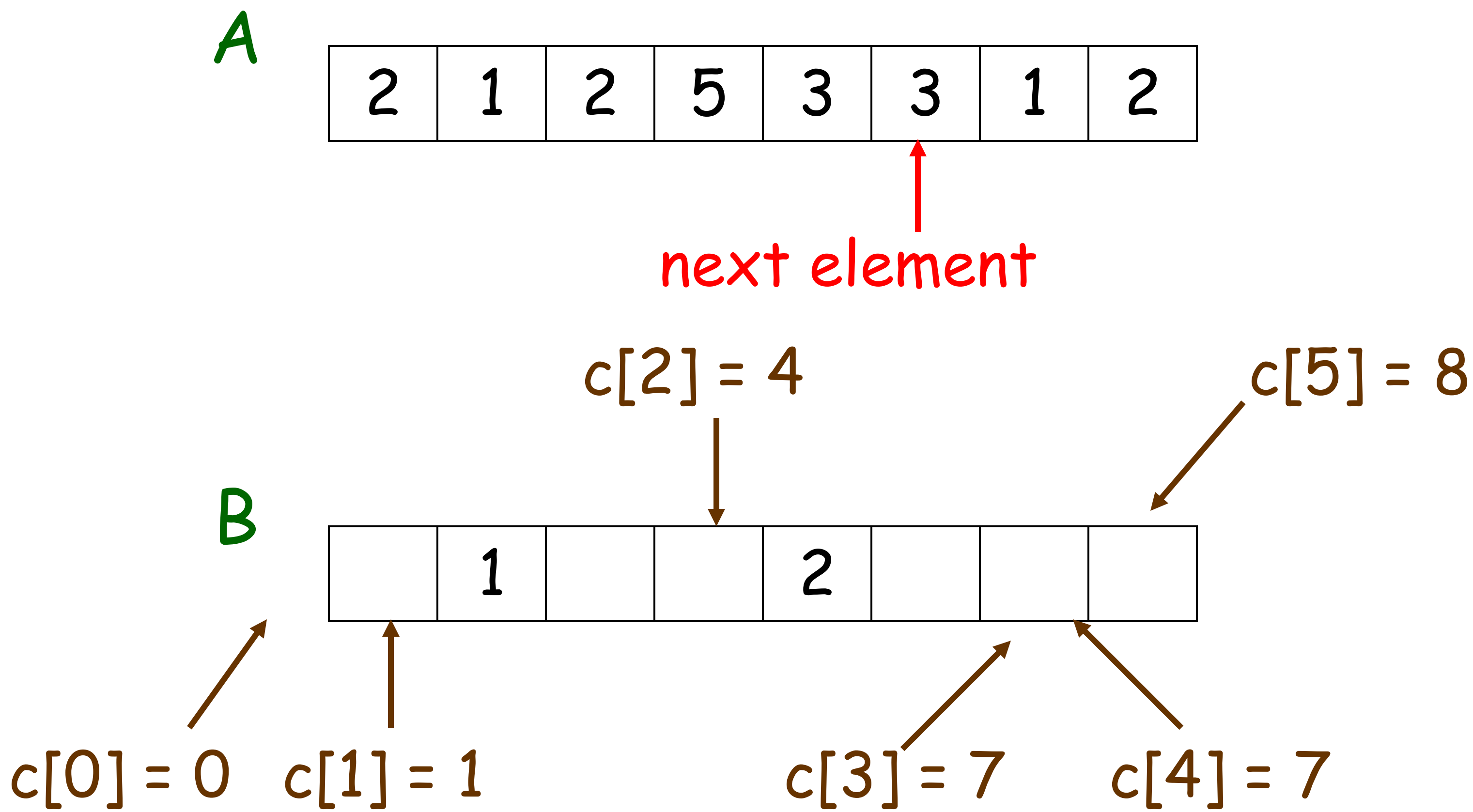
Counting Sort (Details)

Step 2: Process next element of A and update corresponding counter



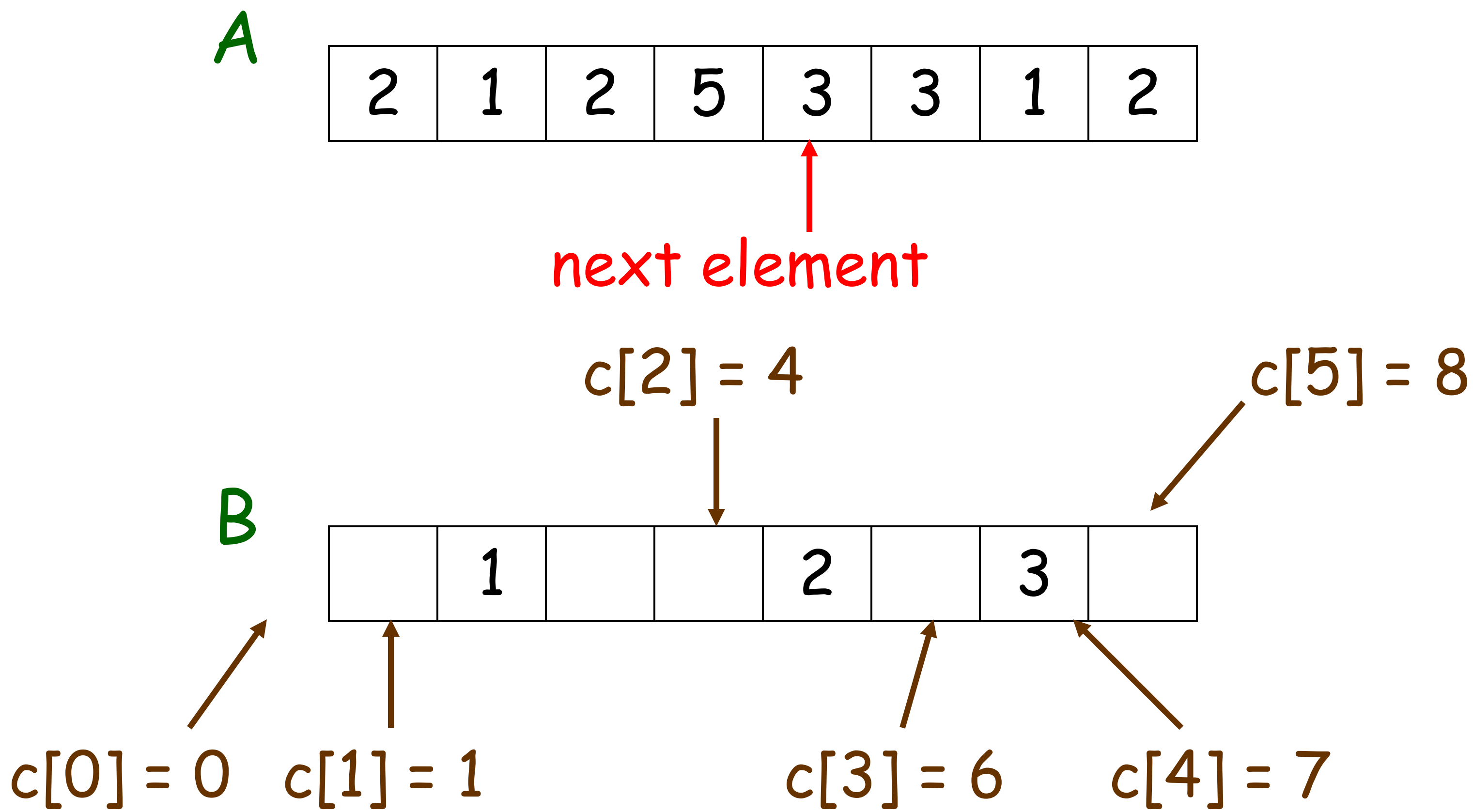
Counting Sort (Details)

Step 2: Process next element of A and update corresponding counter



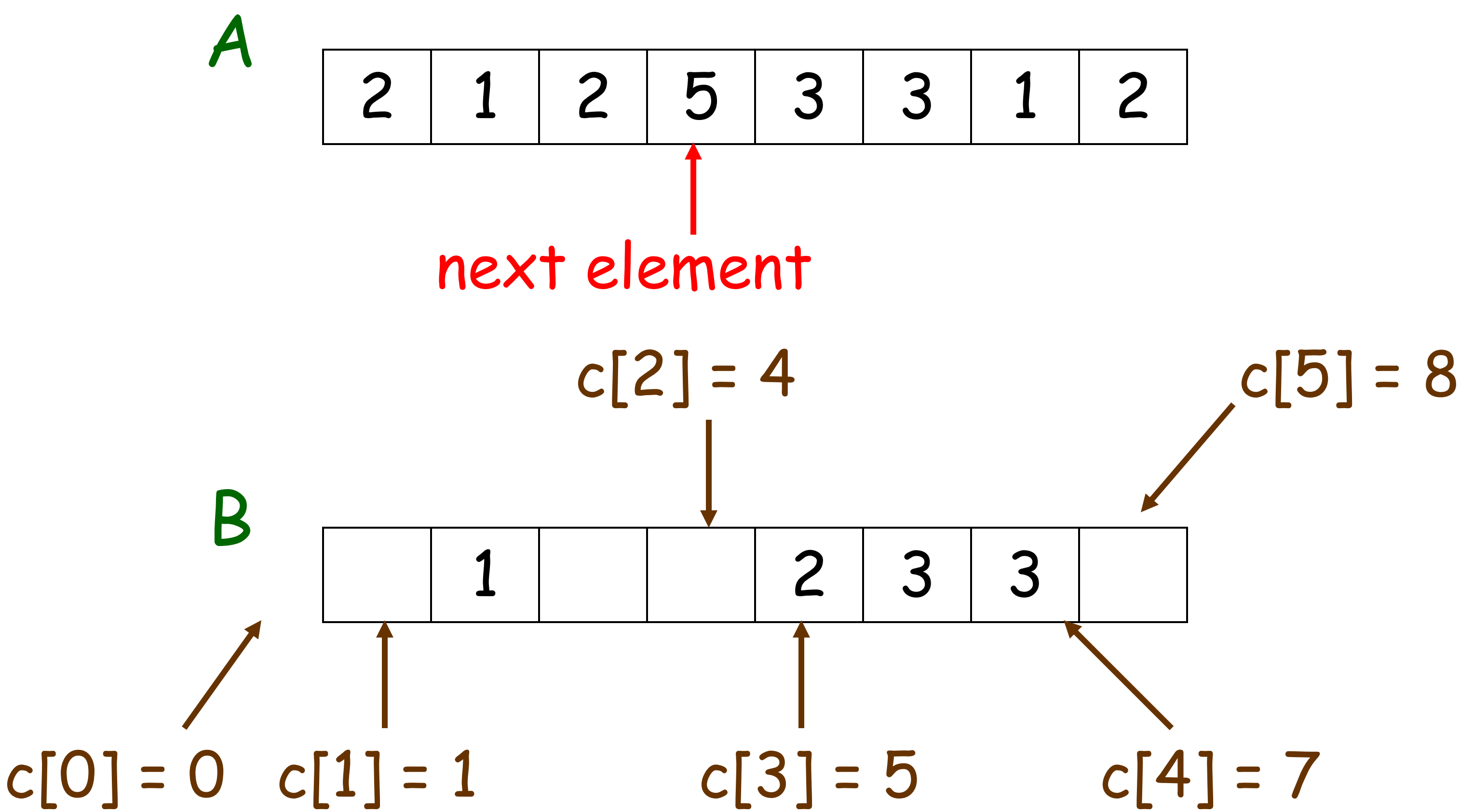
Counting Sort (Details)

Step 2: Process next element of A and update corresponding counter



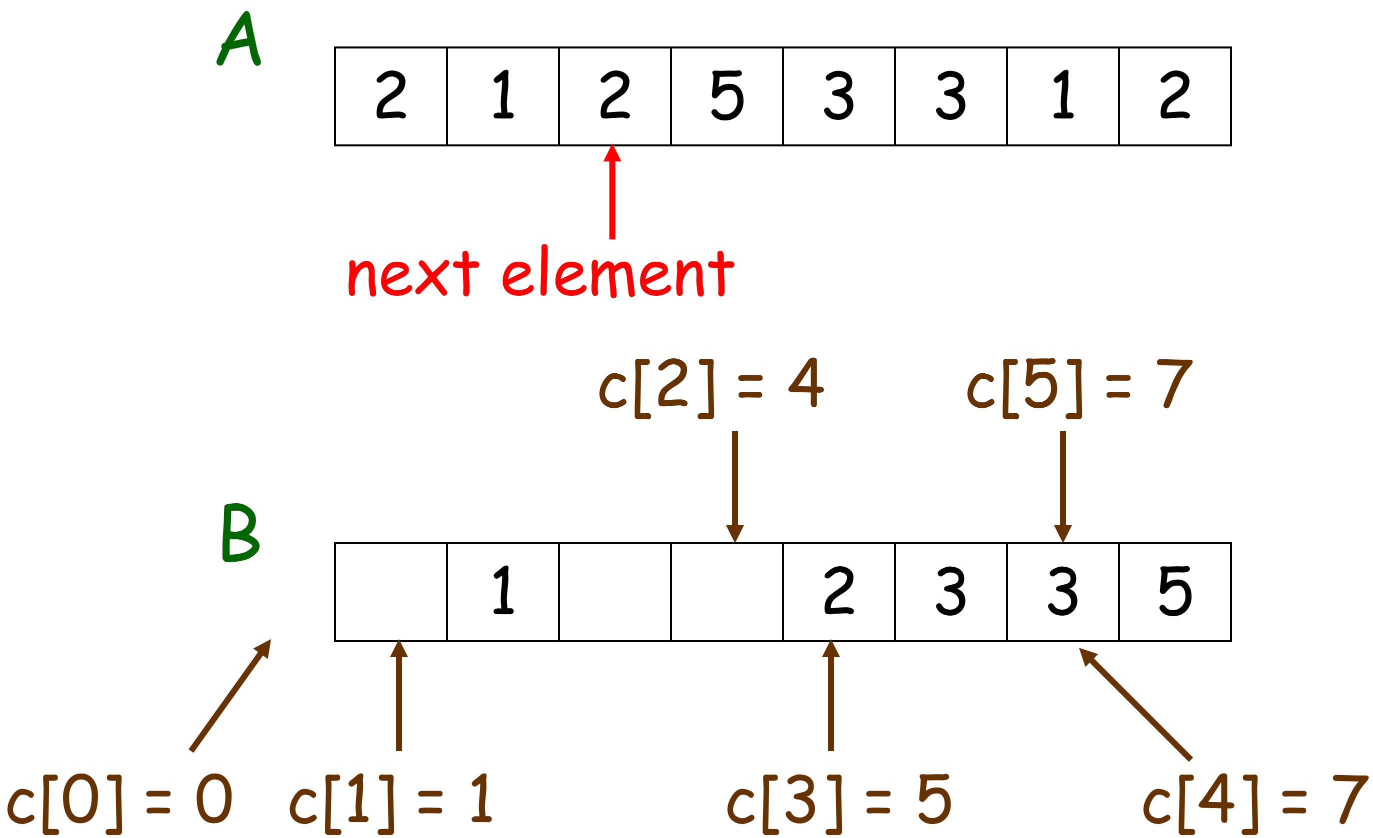
Counting Sort (Details)

Step 2: Process next element of A and update corresponding counter



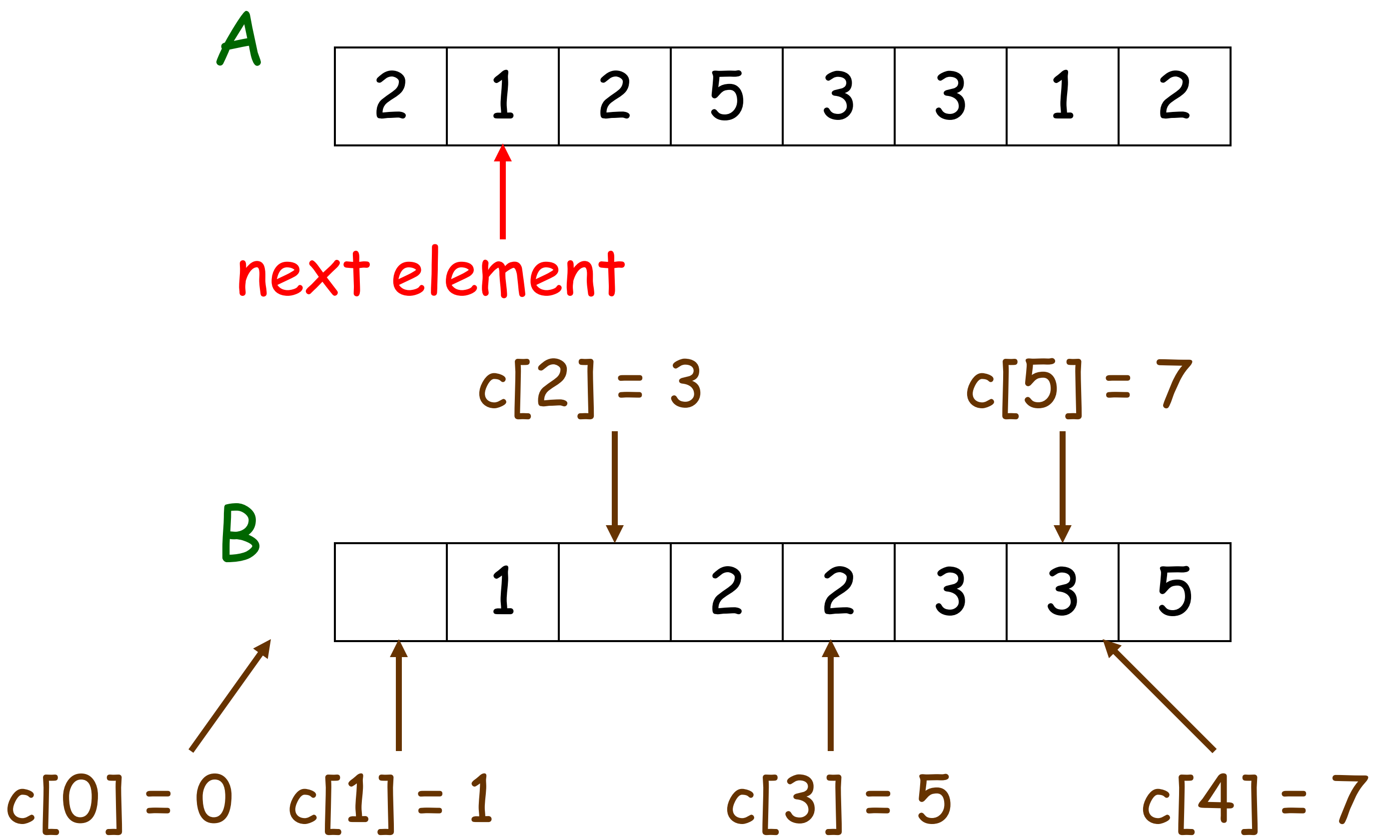
Counting Sort (Details)

Step 2: Process next element of *A* and update corresponding counter



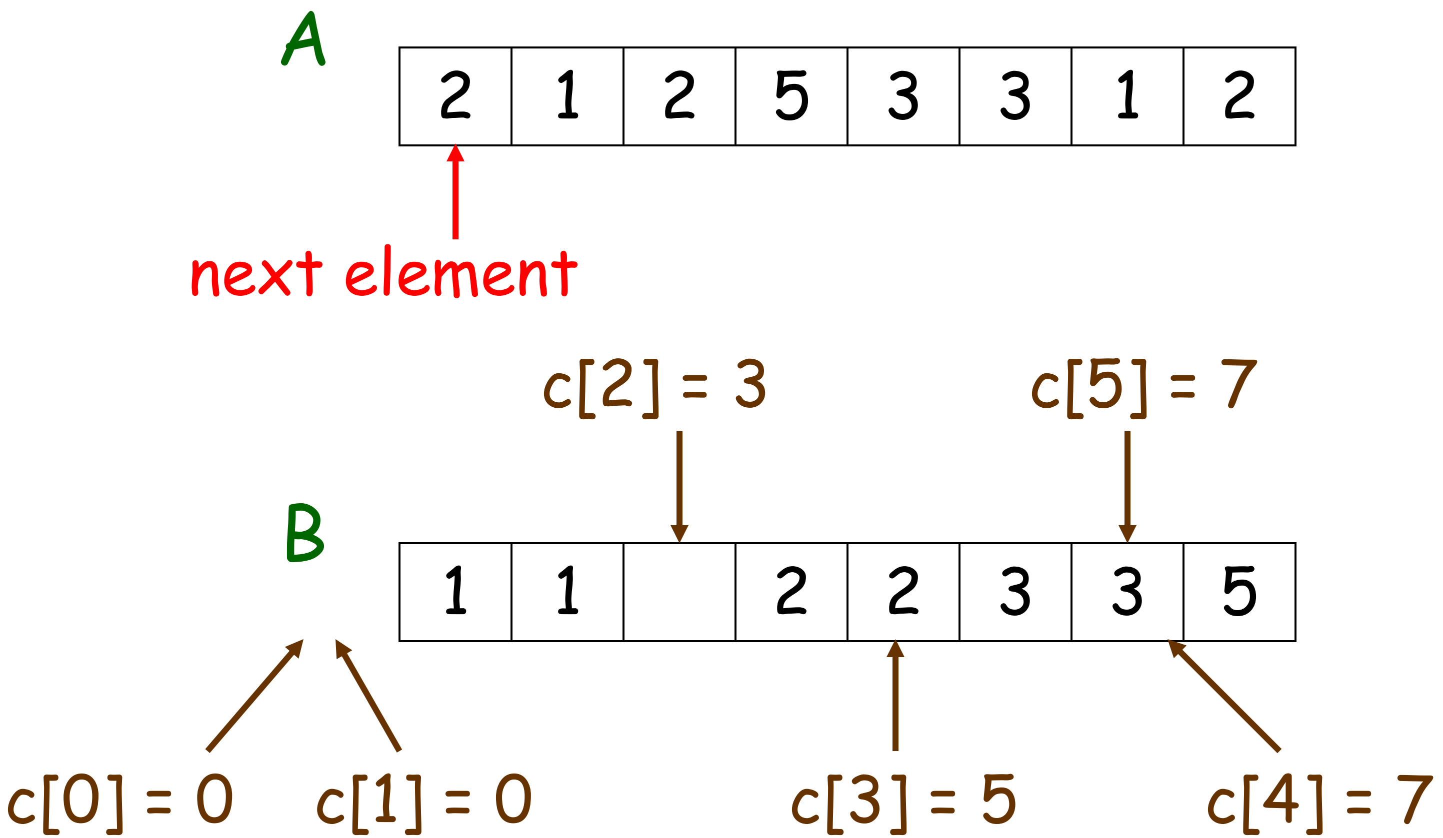
Counting Sort (Details)

Step 2: Process next element of A and update corresponding counter



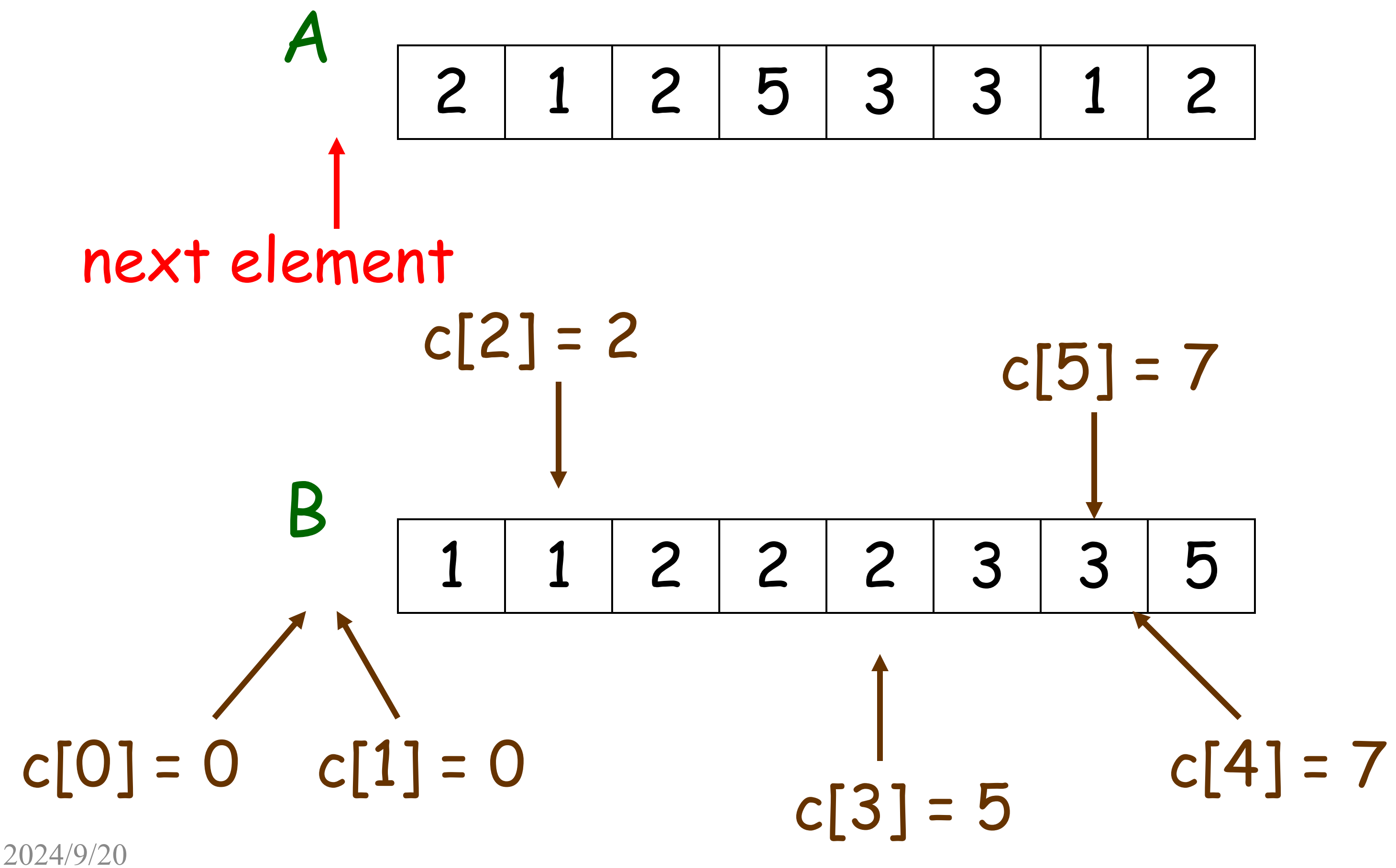
Counting Sort (Details)

Step 2: Process next element of A and
update corresponding counter



Counting Sort (Details)

Step 2: Done when all elements of A are processed



Counting Sort (Running Time)

Conclusion:

- Running time = $O(n + k)$

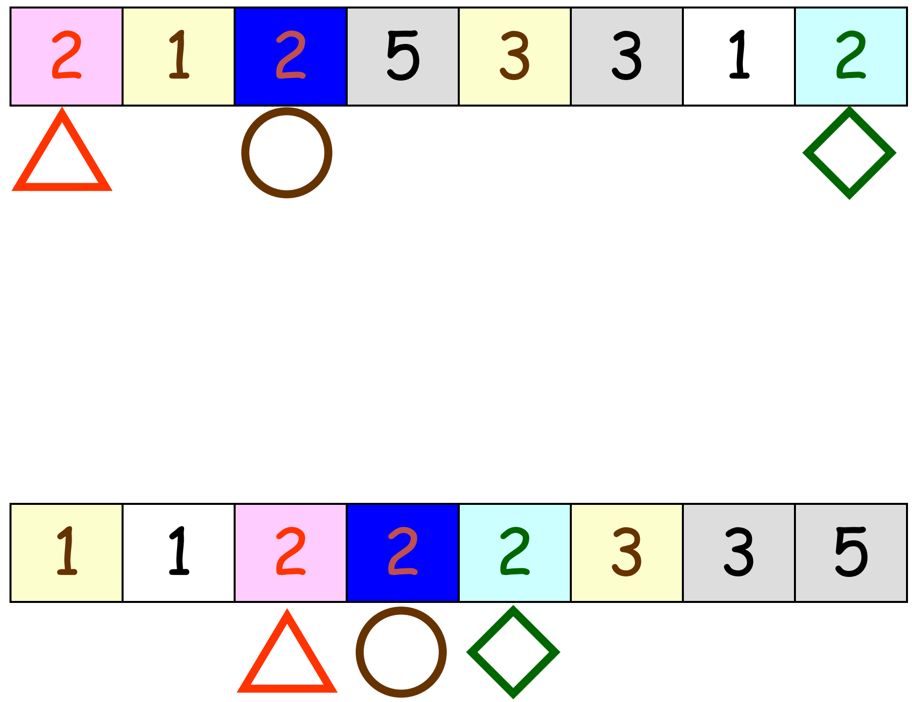
  ➔ if $k = O(n)$, time is (asymptotically) optimal

- Counting sort is also stable :
  - elements with same value appear in same order in before and after sorting

Stable Sort

Before Sorting

After Sorting



Radix Sort

extra info on values

- Input: Array A[1..n] of n integers,
  each has d digits, and
  each digit has value from [0,k]
- Output: Sorted array of the n integers
- Idea: Sort in d rounds
  - At Round j, stable sort A on digit j (where rightmost digit = digit 1)

Radix Sort (Example Run)

Before Running

1 9 0 4
2 5 7 9
1 8 7 4
6 3 5 5
4 4 3 2
8 3 1 8
1 3 0 4

4 digits

Radix Sort (Example Run)

Round 1: Stable sort digit 1

1 9 0 4	4 4 3 2
2 5 7 9	1 9 0 4
1 8 7 4	1 8 7 4
6 3 5 5	1 3 0 4
4 4 3 2	6 3 5 5
8 3 1 8	8 3 1 8
1 3 0 4	2 5 7 9

Radix Sort (Example Run)

Round 2: Stable sort digit 2

4 4 3 2		1 9 0 4
1 9 0 4		1 3 0 4
1 8 7 4		8 3 1 8
1 3 0 4	→	4 4 3 2
6 3 5 5		6 3 5 5
8 3 1 8		1 8 7 4
2 5 7 9		2 5 7 9

After Round 2, last 2 digits are sorted (why?)

Radix Sort (Example Run)

Round 3: Stable sort digit 3

1 9 0 4	1 3 0 4
1 3 0 4	8 3 1 8
8 3 1 8	6 3 5 5
4 4 3 2	4 4 3 2
6 3 5 5	2 5 7 9
1 8 7 4	1 8 7 4
2 5 7 9	1 9 0 4

After Round 3, last 3 digits are sorted (why?)

Radix Sort (Example Run)

Round 4: Stable sort digit 4

1 3 0 4			1 3 0 4
8 3 1 8			1 8 7 4
6 3 5 5			1 9 0 4
4 4 3 2		⇒	2 5 7 9
2 5 7 9			4 4 3 2
1 8 7 4			6 3 5 5
1 9 0 4			8 3 1 8

After Round 4, last 4 digits are sorted (why?)

Radix Sort (Example Run)

Done when all digits are processed

1 3 0 4

1 8 7 4

1 9 0 4

2 5 7 9

4 4 3 2

6 3 5 5

8 3 1 8

The array is sorted (why?)

Radix Sort (Correctness)

Question:

"After r rounds, last r digits are sorted"

Why ??

Answer:

This can be proved by induction :

The statement is true for $r = 1$

Assume the statement is true for $r = k$

Then …

Radix Sort (Correctness)

At Round k+1,

- if two numbers differ in digit "k+1", their relative order [based on last k+1 digits] will be correct after sorting digit "k+1"
- if two numbers match in digit "k+1", their relative order [based on last k+1 digits] will be correct after stable sorting digit "k+1" (why?)

➔ Last "k+1" digits sorted after Round "k+1"

Radix Sort (Summary)

Conclusion:

- After d rounds, last d digits are sorted, so that the numbers in $A[1..n]$ are sorted

- There are d rounds of stable sort, each can be done in $O(n + k)$ time

  ➔ Running time = $O(d(n + k))$

  - if $d=O(1)$ and $k=O(n)$, asymptotically optimal

Bucket Sort

extra info on values

- Input: Array A[1..n] of n elements, each is drawn uniformly at random from the interval [0,1)
- Output: Sorted array of the n elements
- Idea:

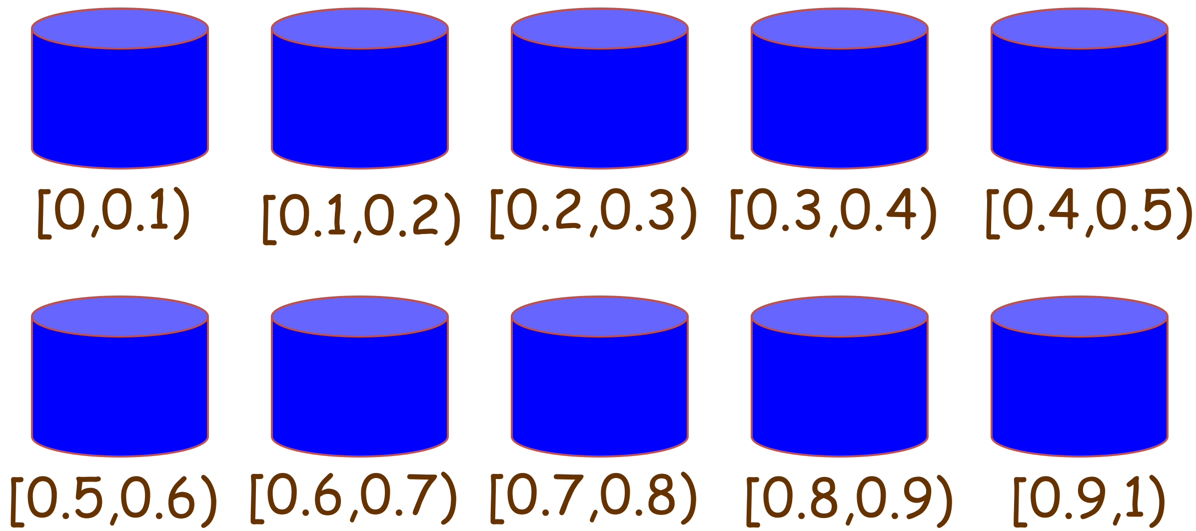
  Distribute elements into n buckets, so that each bucket is likely to have fewer elements → easier to sort

Bucket Sort (Details)

Before Running

0.78, 0.17, 0.39, 0.26, 0.72,
0.94, 0.21, 0.12, 0.23, 0.68

Step 1: Create n buckets

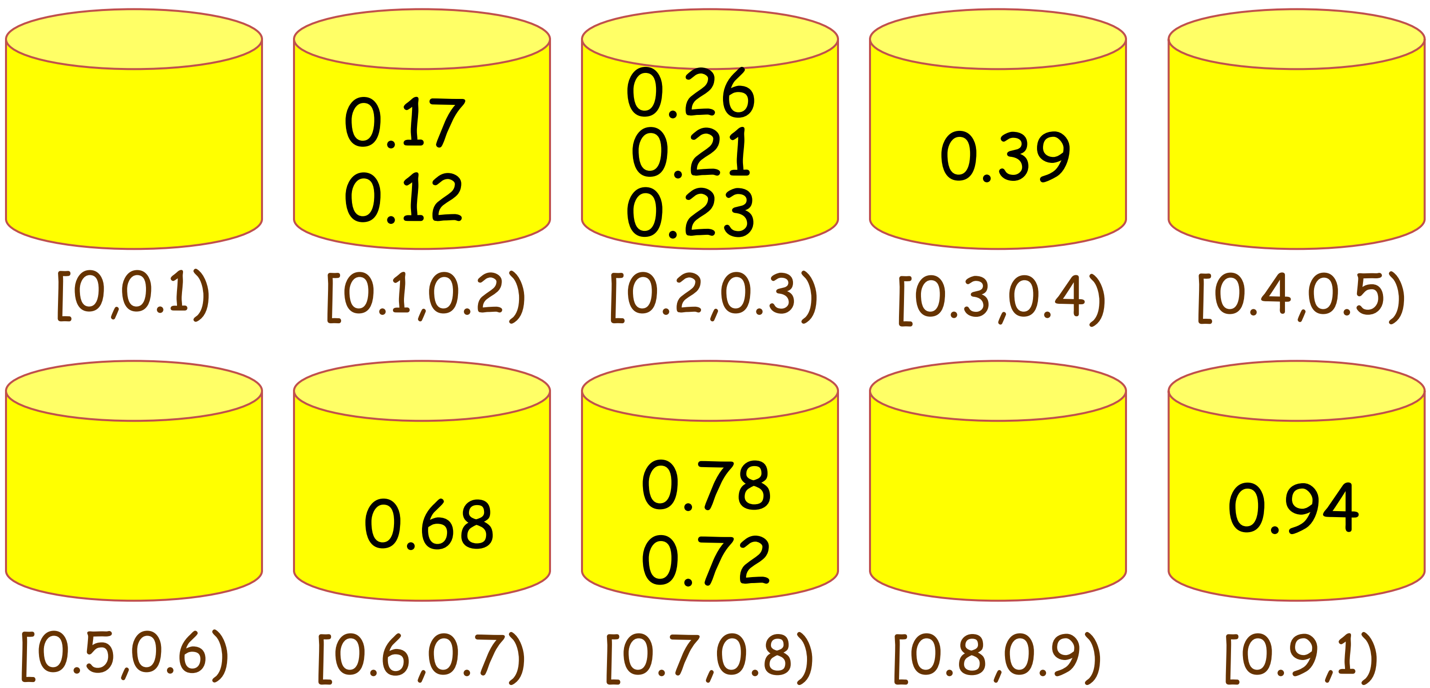


n = #buckets
= #elements

each bucket represents a subinterval of size 1/n

Bucket Sort (Details)

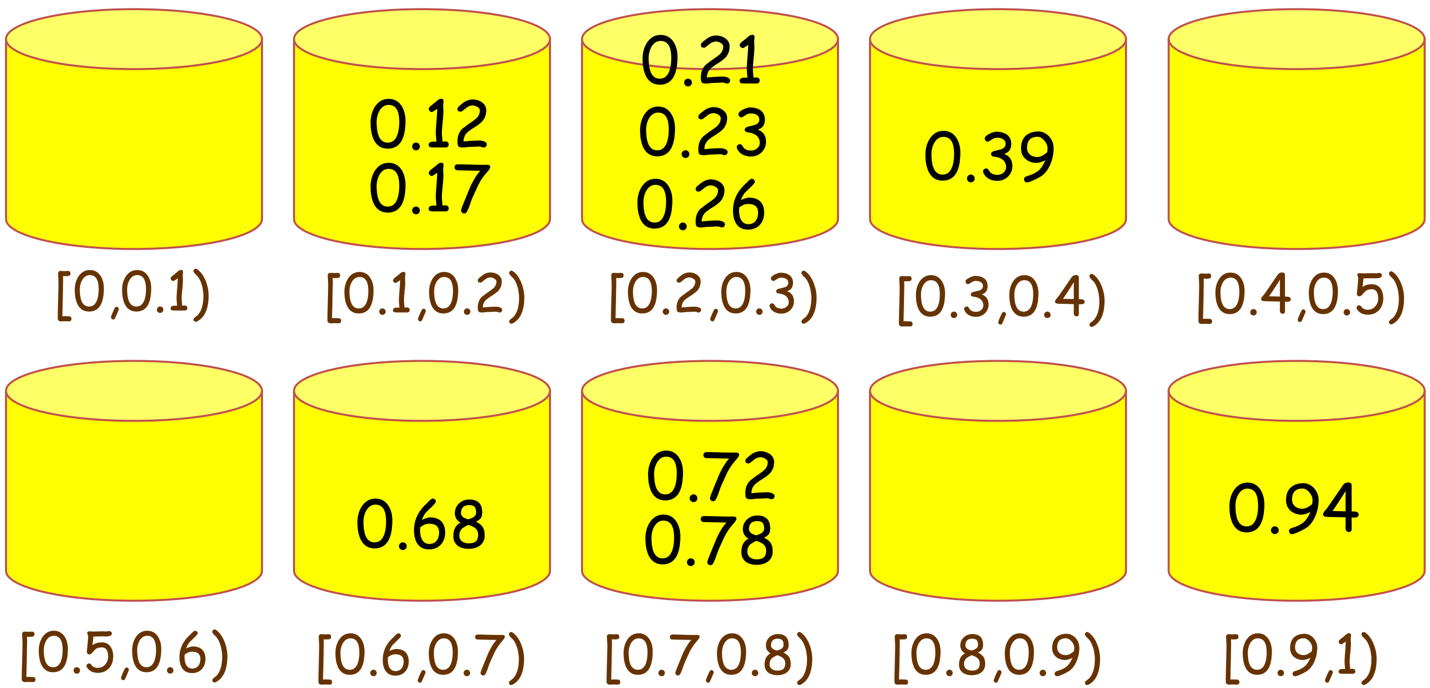
Step 2: Distribute each element to correct bucket



If Bucket j represents subinterval $[\, j/n, (j+1)/n\,)$,
element with value x should be in Bucket $\lfloor xn \rfloor$

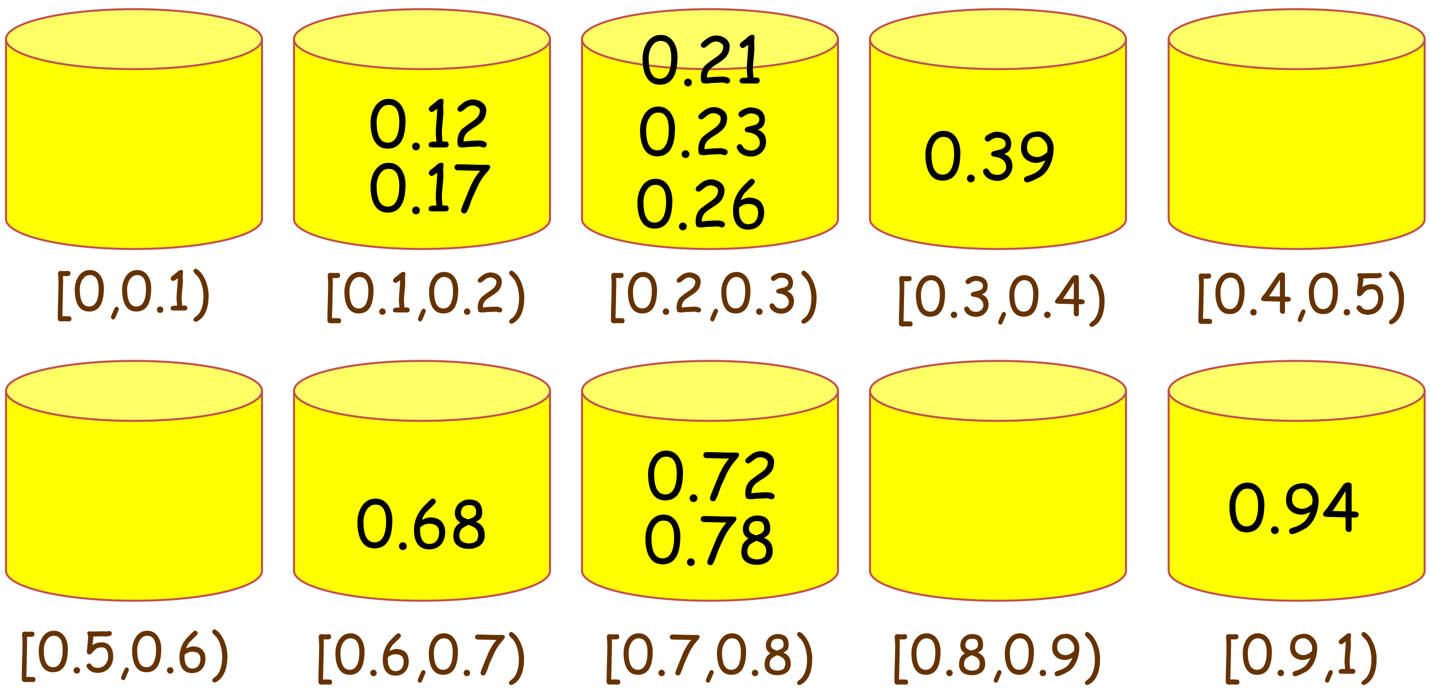
Bucket Sort (Details)

Step 3: Sort each bucket (by insertion sort)



Bucket Sort (Details)

Step 4: Collect elements from Bucket 0 to Bucket n-1



Sorted Output 0.12, 0.17, 0.21, 0.23, 0.26, 0.39, 0.68, 0.72, 0.78, 0.94

Bucket Sort (Running Time)

- Let X = # comparisons in all insertion sort

  Running time = $\Theta(n + X)$ varies on input

  ➔ worst-case running time = $\Theta(n^2)$

- How about average running time?

Finding average of X (i.e. #comparisons) gives average running time

Average Running Time

Let n_j = # elements in Bucket j

$$X \le c(n_0^2 + n_1^2 + \ldots + n_{n-1}^2)$$

varies on input

So, $E[X] \le E[c(n_0^2 + n_1^2 + \ldots + n_{n-1}^2)]$

$$= c\,E[n_0^2 + n_1^2 + \ldots + n_{n-1}^2]$$

$$= c\,(E[n_0^2] + E[n_1^2] + \ldots + E[n_{n-1}^2])$$

$$= cn\,E[n_0^2] \qquad \text{(by uniform distribution)}$$

Average Running Time

Textbook (pages 202-203) shows that

$E[n_0^2] = 2 - (1/n)$

➔ $E[X] \le cn\, E[n_0^2] = 2cn - c$

In other words, $E[X] = O(n)$

➔ Average running time = $\Theta(n)$

For Interested Classmates

The following is how we can show

$$E[n_0^2] = 2 - (1/n)$$

Recall that n_0 = # elements in Bucket 0

So, suppose we set

$A_k = 1$ if element k is in Bucket 0

$A_k = 0$ if element k not in Bucket 0

Then, $n_0 = A_1 + A_2 + \ldots + A_n$

For Interested Classmates

Then,

$$
\begin{aligned}
E[n_0^2] &= E[(A_1 + A_2 + \ldots + A_n)^2] \\
&= E[\, A_1^2 + A_2^2 + \ldots + A_n^2 \\
&\qquad + A_1A_2 + A_1A_3 + \ldots + A_1A_n \\
&\qquad + A_2A_1 + A_2A_3 + \ldots + A_2A_n \\
&\qquad + \ldots \\
&\qquad + A_nA_1 + A_nA_2 + \ldots + A_nA_{n-1} \,]
\end{aligned}
$$

$$= E[A_1^2] + E[A_2^2] + \ldots + E[A_n^2]$$
$$+ E[A_1A_2] + \ldots + E[A_nA_{n-1}]$$
$$= n\,E[A_1^2] + n(n-1)\,E[A_1A_2]$$

The value of A_1^2 is either 1 (when $A_1 = 1$),
or 0 (when $A_1 = 0$)

The first case happens with 1/n chance (when element 1 is in Bucket 0), so

$$E[A_1^2] = 1/n * 1 + (1- 1/n) * 0 = 1/n$$

For A_1A_2, it is either 1 (when A_1=1 and A_2=1),
or 0 (otherwise)

The first case happens with $1/n^2$ chance (when both element 1 and element 2 are in Bucket 0), so

$E[A_1A_2] = 1/n^2 * 1 + (1- 1/n^2) * 0 = 1/n^2$

Thus, $E[n_0^2] = n\, E[A_1^2] + n(n-1)\, E[A_1A_2]$

$= n\,(1/n) + n(n-1)\,(1/n^2)$

$= 2 - 1/n$

Practice at Home

- Exercises: 8.2-5, 8.2-6, 8.2-7, 8.3-2, 8.3-4, 8.3-5, 8.4-2, 8.4-4
- Problem 8-2, 8-3
- It is known that $\Omega(n \log n)$ is a lower bound for sorting problems. However, we have seen algorithms like counting sort or radix sort which can sort n items in $O(n)$ time. Is there a contradiction? If not, please explain. Can we conclude that the performance of counting sort or radix sort is better than merge sort or heapsort?