Metropolis-Hastings algorithm

- generalisation of the Metropolis Algorithm
- For the Metropolis Algorithm, the proposal distribution must be symmetric:

$$g(x' \mid x_t) = g(x_t \mid x')$$

prob of proposing x' if we are at state x_t = prob of proposing x_t if we are at state x'

the two prob are equal because $g(x)$ is symmetric.

Metropolis-Hastings algorithm does not require the proposal distribution be symmetric.

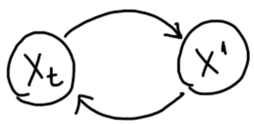
$$g(x' \mid x_t) \neq g(x_t \mid x')$$

M.H. Algorithm:

1. Pick initial state x_0. We have a target distribution $P(x)$.
2. iterate:
   - 2a. propose a candidate $x' \sim g(x' \mid x_t)$ based on current state x_t
   - 2b. Calculate acceptance prob. $p_{move} = \min\left(1, \frac{P(x')}{P(x_t)} \cdot \frac{g(x_t \mid x')}{g(x' \mid x_t)}\right)$
   - 2c. accept or reject x'. Generate $U \sim \text{Unif}(0,1)$
     - If $U \leq p_{move}$, then accept. $x_{t+1} = x'$
     - If $U > p_{move}$, then reject. $x_{t+1} = x_t$

(If $P(x)$ is not known, you can use $f(x) \propto P(x)$.)

M.H. algorithm works because there is detailed balance at the target distribution.



The amount flowing from X_t to X' is exactly equal to the amount flowing from X' to X_t.

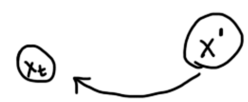
This will be true for any two states if all states follow the target distribution $P(X)$.

$P(X)$ = the probability of being in state X in the target distribution.

$P(X'|X_t)$ = Prob of transitioning from state X_t to state X'.

We want to show: $$P(X_t) \cdot P(X'|X_t) = P(X') \cdot P(X_t|X')$$

prob of starting at state X_t · prob of transition to X' from X_t = prob of starting at state X' · prob of transition to X_t from X'



A trivial case: If the proposed location is rejected then the next state $X' = X_t$

Both sides of the equation are equal.

We want to show: $P(x_t)\cdot P(x'|x_t) = P(x')P(x_t|x')$

$$= P(x_t)\cdot g(x'|x_t)\cdot \min\left(1, \frac{P(x')}{P(x_t)}\cdot\frac{g(x_t|x')}{g(x'|x_t)}\right)$$

I will distribute $g(x'|x_t)$ through the min function.

I will replace 1 with $\frac{P(x_t)}{P(x_t)}$

$$P(x_t)\cdot\min\left(\frac{P(x_t)}{P(x_t)}\cdot g(x'|x_t), \frac{P(x')}{P(x_t)}\cdot\frac{g(x_t|x')}{\cancel{g(x'|x_t)}}\cdot \cancel{g(x'|x_t)}\right)$$

I distribute $P(x_t)$ through the min function.

$$\min\left(\cancel{P(x_t)}\cdot\frac{P(x_t)}{\cancel{P(x_t)}}\cdot g(x'|x_t), \cancel{P(x_t)}\cdot\frac{P(x')}{\cancel{P(x_t)}}\cdot g(x_t|x')\right)$$

$$= \min\left(P(x_t)\cdot g(x'|x_t), P(x')g(x_t|x')\right)$$

$$= P(x')\cdot g(x_t|x')\cdot \min\left(1, \frac{P(x_t)}{P(x')}\cdot\frac{g(x'|x_t)}{g(x_t|x')}\right)$$

distribute through min function

replace with $\frac{P(x')}{P(x')}$

$$= P(x')\min\left(\frac{P(x')}{P(x')}\cdot g(x_t|x'), \frac{P(x_t)}{P(x')}\cdot\frac{g(x'|x_t)}{\cancel{g(x_t|x')}}\cdot \cancel{g(x_t|x')}\right)$$

distribute through min function.

$$= \min\left(\cancel{P(x')}\cdot\frac{P(x')}{\cancel{P(x')}}\cdot g(x_t|x'), \cancel{P(x')}\cdot\frac{P(x_t)}{\cancel{P(x')}}\cdot g(x'|x_t)\right)$$

$$= \min\left(P(x')g(x_t|x'), P(x_t)g(x'|x_t)\right)$$

they are equal