MATH 137 Lecture Notes - ~~Student~~ Instructor Copy

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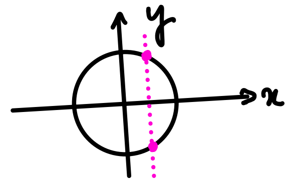
MathMatize

Pre-Calculus

Elementary Functions

- STOP and THINK:
  What is a function? A relation, a mapping:
  Give an example. $f(x) = x^2$
  Give an example of a relation that is not a function.
  Unit circle : $x^2 + y^2 = 1$
  One input $x = 0$
  Two outputs $y = \mp 1$



Fails the vertical line test.

- YOUR TURN: Evaluate the following:

  1. $f(x) = x^3 + 1,\ f(-1) = 0$
     $(-1)^3 + 1 = -1 + 1$
  2. $g(t) = \dfrac{1}{|x-1|},\ g(-2) = \dfrac{1}{3}$
     $\dfrac{1}{|-2-1|} = \dfrac{1}{|-3|}$
  3. $h(x) = |2 - 4x|,\ h(1) = 2$
     $|2 - 4(1)| = |-2|$
  4. $s(t) = \sqrt{2 - t^2},\ s(2) =$ Not defined for $\mathbb{R}$.
     $\sqrt{2 - (2)^2} = \sqrt{-2}$

- Find the domain and the range for the functions listed below. (DESMOS)

  1. $f(x) = x^3 + 1$
     DOMAIN: $\mathbb{R}$ or $(-\infty, \infty)$

     RANGE: $\mathbb{R}$ or $(-\infty, \infty)$
  2. $g(t) = \dfrac{1}{|x-1|}$
     DOMAIN: $(-\infty, 1) \cup (1, \infty)$

     RANGE: $(0, \infty)$
  3. $h(x) = |2 - 4x|$
     DOMAIN: $\mathbb{R}$ or $(-\infty, \infty)$

     RANGE: $[0, \infty)$
  4. $s(t) = \sqrt{2 - t^2}$
     DOMAIN: $[-\sqrt{2}, \sqrt{2}]$

     RANGE: $[0, \sqrt{2}]$

Definition: Function
Let X and Y be sets. A **function** f is

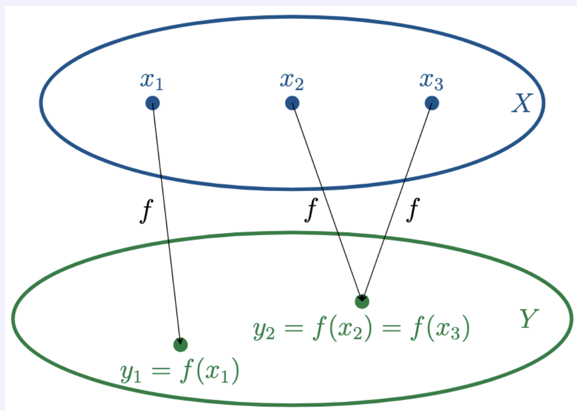
in

a mapping that assigns to each $x \in X$

exactly one $y = f(x) \in Y$

We use the notation

$$f : X \longrightarrow Y \quad \text{or} \quad x \longmapsto f(x)$$



Definition: Domain of a Function
Let $f : X \to Y$ be a function. We call the set of numbers for which the function f is defined the **domain** of f. More formally,

subset

$$D = \operatorname{dom}(f) = \{x : f(x) \text{ is well-defined}\} \subseteq X$$

Definition: Range of a Function
The **range** of a function $f : D \to \mathbb{R}$ is the set

$$\operatorname{range}(f) = \{ f(x) : x \in D \}$$

Definition: Odd and Even Function
A function f is called **even** if

$$f(x) = f(-x) \quad \text{for all } x \in D$$

A function g is called **odd** if

$$g(x) = -g(-x) \quad \text{for all } x \in D.$$

EXAMPLE: For each of the functions below, give its domain and range. Also, determine whether they are even or odd.

1. $f(x) = x^4$,
   (a) Domain: $(-\infty, \infty)$
   (b) Range: $[0, \infty)$
   (c) Odd, even, neither:

   $f(-x) = (-x)^4 = x^4 = f(x)$

   f is even.

2. $g(x) = -\frac{1}{x}$,
   (a) Domain: $(-\infty, 0) \cup (0, \infty)$
   (b) Range: $(-\infty, 0) \cup (0, \infty)$
   (c) Odd, even, neither:

   $g(-x) = -\frac{1}{-x} = \frac{1}{x} = -g(x)$

   g is odd.

3. $h(x) = \frac{1}{\sqrt{x}}$.
   (a) Domain: $(0, \infty)$
   (b) Range: $(0, \infty)$
   (c) Odd, even, neither:

   $h(-x) = \frac{1}{\sqrt{-x}}$ neither odd nor even.

YOUR TURN:

T **or F:** Any even power function of the form $f(x) = x^{2k}$ for some $k \in \mathbb{N}$ is even.

$$f(-x) = (-x)^{2k} = [(-x)^2]^k = (x^2)^k = x^{2k} = f(x)$$

T **or F:** Any odd power function of the form $g(x) = x^{2k-1}$ for some $k \in \mathbb{N}$ is odd.

$$g(-x) = (-x)^{2k-1} = (-1)^{2k-1}(x)^{2k-1} = (-1)\,x^{2k-1} = -x^{2k-1} = -g(x)$$

QUESTION: Is it possible to find a function that is both even and odd? Yes $f(x)=0$.

Definition: Root of a function
Suppose f is a function, and suppose there is $x \in D$ so that $f(x) = 0$. Then, we call x a **root** of f.

$f(x)$

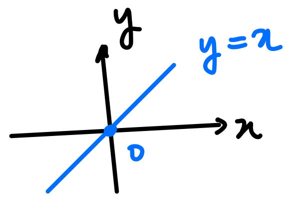
- no roots : $f(x) = e^x$ has no roots
- one root : $p(x) = x - 1$ has one root, $x=1$
- multiple roots : $q(x) = (x-2)(x-1)$ has two roots $x=1$ and $x=2$.

For polynomials of the form $p(x) = ax^2 + bx + c$, we can use the quadratic formula $\Delta = b^2 - 4ac$

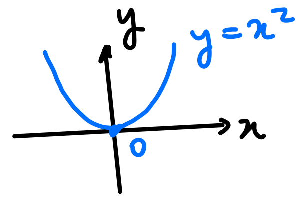
- if $\Delta < 0$, $p(x)$ has no roots
- if $\Delta = 0$, $p(x)$ has one (repeated) root.
- if $\Delta > 0$, $p(x)$ has two distinct roots.

Parent Functions:

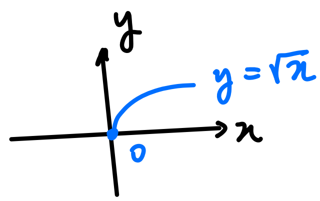
- $f(x) = x$
  - DOMAIN: $\mathbb{R}$
  - RANGE: $\mathbb{R}$



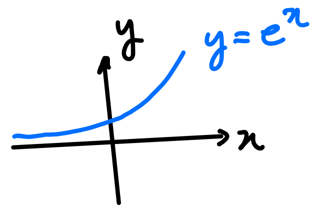
- $f(x) = x^2$
  - DOMAIN: $\mathbb{R}$
  - RANGE: $[0,\infty)$



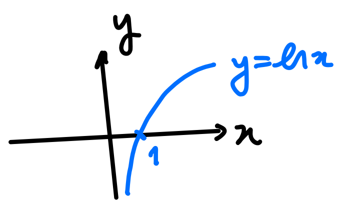
- $f(x) = \sqrt{x}$
  - DOMAIN: $[0,\infty)$
  - RANGE: $[0,\infty)$



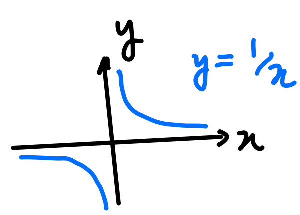
- $f(x) = e^x$
  - DOMAIN: $\mathbb{R}$
  - RANGE: $(0,\infty)$



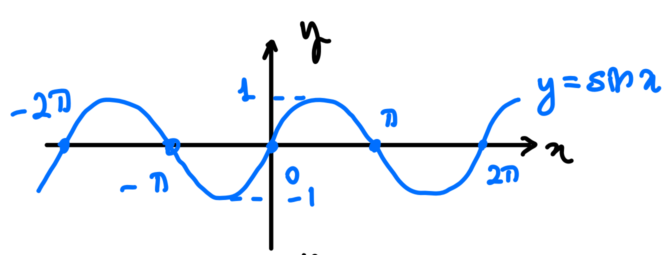
- $f(x) = \ln(x)$
  - DOMAIN: $(0,\infty)$
  - RANGE: $\mathbb{R}$



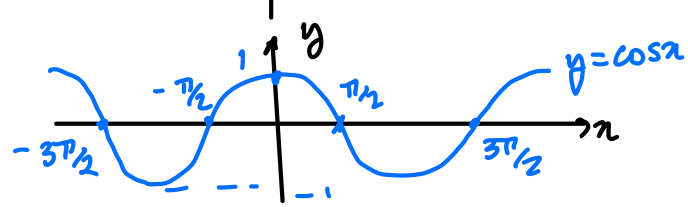
- $f(x) = \dfrac{1}{x}$
  - DOMAIN: $(-\infty,0)\cup(0,\infty)$
  - RANGE: $(-\infty,0)\cup(0,\infty)$



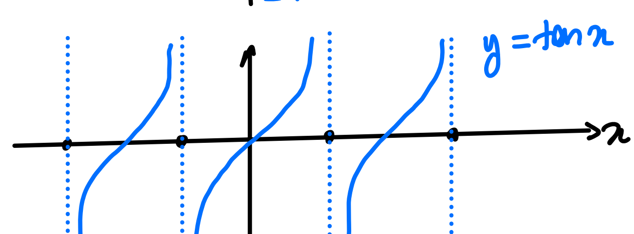
- $f(x) = \sin(x)$
  - DOMAIN: $\mathbb{R}$
  - RANGE: $[-1, 1]$



- $f(x) = \cos(x)$
  - DOMAIN: $\mathbb{R}$
  - RANGE: $[-1, 1]$



- $f(x) = \tan(x)$
  - DOMAIN: $\mathbb{R} \setminus (2k+1)\pi$
  - RANGE: $\mathbb{R}$



"0" where $\sin x = 0$

VA where $\cos x = 0$

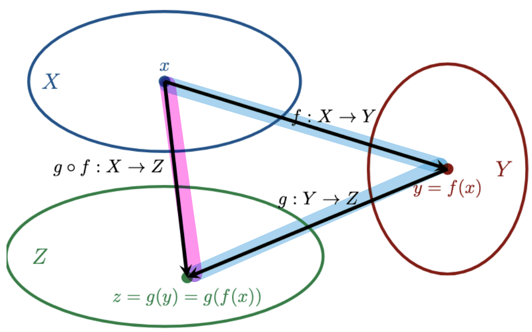
END OF L1

Compositions

A given function may have no root, one root, or multiple roots: We will see more examples throughout this review chapter.
We can build more complicated functions by summing, multiplying, or dividing other functions. For instance, the function $(f+g)$ is given by $f(x)+g(x)$ (eg, if $f(x)=x^2$ and $g(x)=x$, then $(f+g)(x)=f(x)+g(x)=x^2+x$). Another method to combine functions to create new functions is to "plug them into each other", which we define as follows.

Definition: Function Composition
Let $f: X \to Y$ and $g: Y \to Z$ be functions. The composition of f and g, denoted by $g \circ f$, is the function

$$g \circ f : X \to Z \quad \text{with} \quad (g \circ f)(x) = \underbrace{g(f(x))}_{g \text{ evaluated at } f}$$



The order of the composition matters: $(g \circ f)(x)$ is in general not $(f \circ g)(x)$; the two functions may not even have the same domain.

EXAMPLE: Let $f(x) = x^2$ and $g(x) = x + 1$. Write down the functions $g \circ f$ and $f \circ g$ along with their domains and ranges.

Solution:

$$(f \circ g)(x) = f(g(x)) = (x+1)^2 = (x+1)^2$$

Domain: $f(g(x))$
↳ goes in g first
so domain is $\mathbb{R}$.

Range: outputs are $(x+1)^2$.
So range is $[0, \infty)$

$$(g \circ f)(x) = g(f(x)) = [x^2] + 1 = x^2 + 1$$

Domain: $g(f(x))$
↳ goes in f first
so domain is $\mathbb{R}$.

Range: outputs are x^2+1
so range is $[1, \infty)$

Next, we define the inverse function as the function f^{-1} that "undoes" what f did.

Definition: Inverse Function
Let $f : X \to Y$ be a function with domain X and range Y. Then f is **invertible** if

there exists a function $f^{-1} : Y \to X$ so that

$f(f^{-1}(x)) = x \;\; \forall x \in Y$ and $f^{-1}(f(x)) = x \;\; \forall x \in X$

If the inverse exists, we can find it as follows:

1. Write $y = f(x)$
2. Solve for x
3. Swap x and y

$$f^{-1} \neq \frac{1}{f}$$

EXAMPLE: Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 4$. Compute the inverse, $f^{-1}(x)$, for $x \in \mathbb{R}$.
Solution:

f

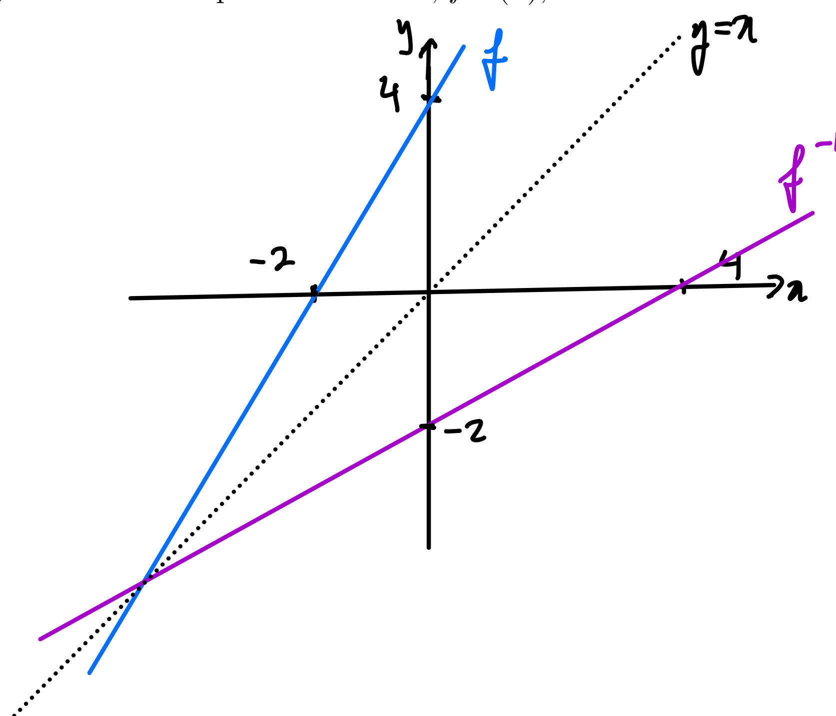
$$y = 2x + 4$$

$$y - 4 = 2x$$

$$x = \frac{y-4}{2}$$

f^{-1}

$$y = \frac{x-4}{2}$$



REMARK: ① Graphs of f and f^{-1} are reflections of each other with respect to $y = x$.

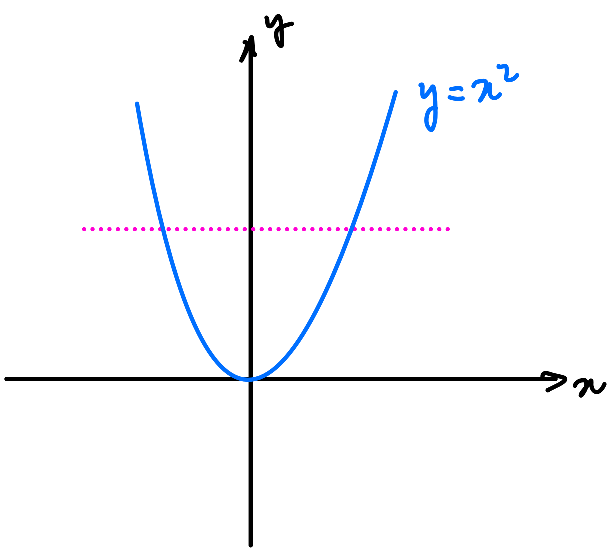
② If $f(a) = b$ then $f^{-1}(b) = a$.

If the inverse does not exist, we would be unable to complete Step 3. It's worth noting that Step 3 is not always easy!
Geometrically, the inverse function is the mirror image at the diagonal $y = x$, as will be illustrated in the next example.
EXAMPLE: For each of the following functions, determine the inverse function, if it exists.

$$f_1 : \mathbb{R} \to [0, \infty), \quad f_1(x) = x^2;$$
$$f_2 : [0, \infty) \to [0, \infty), \quad f_2(x) = x^2.$$

Solution:

$f_1(x) = x^2$



$y = x^2 \Rightarrow \quad x = \mp\sqrt{y}$

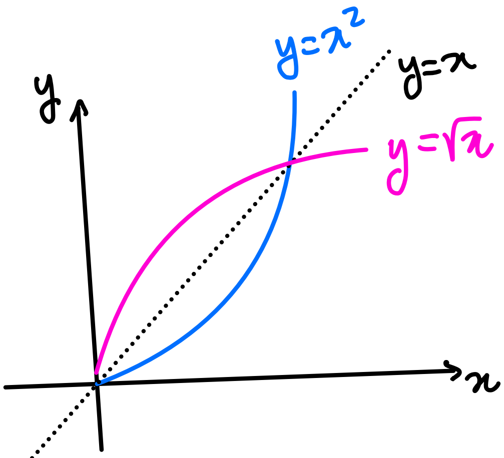
which one?

Fails horizontal line test.

Not invertible.

Solution continued:

$f_2 : [0, \infty) \to [0, \infty), \quad f_2(x) = x^2.$ This function is f_1 with "restricted domain"



f

$y = x^2 \Rightarrow x = \mp\sqrt{y}$

which one?

since $x \geq 0$

$\Rightarrow x = \sqrt{y}$

$\Rightarrow y = \sqrt{x}$ $\quad f^{-1}$

The previous example illustrated different situations that can arise when solving $f(x) = y$ for a function $f : X \to Y$ for some $x \in X$ and $y \in Y$.

- There is no solution $x \in X$ satisfying $f(x) = y$. In this case, $f^{-1}(y)$ does not exist.
- There are at least two solutions, say $x_1, x_2 \in X$, satisfying $f(x_1) = f(x_2) = y$. In this case, $f^{-1}(y)$ does not exist.
- There is exactly one solution $x \in X$ satisfying $f(x) = y$. In this case, $f^{-1}(y)$ exists and $x = f^{-1}(y)$.

Definition: Injective, Surjective, Bijective (Optional Material)
Let $f : X \to Y$ a function. We say that f is

- **surjective**, if

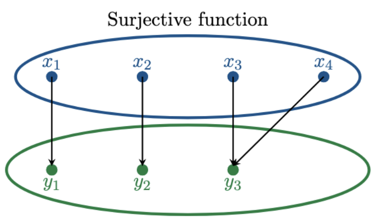
$$\forall y \in Y, \ \exists \text{ at least one } x \in X \text{ s.t. } f(x) = y$$

- **injective**, if

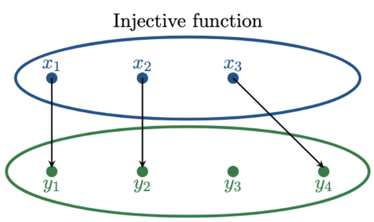
$$\forall y \in Y, \ \exists \text{ at most one } x \in X \text{ s.t. } f(x) = y$$

- **bijective**, if

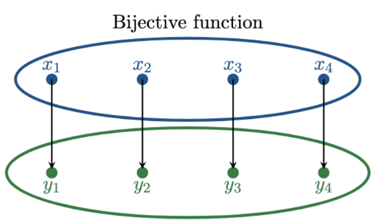
$$\forall y \in Y, \ \exists \text{ exactly one } x \in X \text{ s.t. } f(x) = y$$



Domain: $\{x_1, x_2, x_3, x_4\}$
Codomain: $\{y_1, y_2, y_3\}$
Range: $\{y_1, y_2, y_3\}$



Domain: $\{x_1, x_2, x_3\}$
Codomain: $\{y_1, y_2, y_3, y_4\}$
Range: $\{y_1, y_2, y_3\}$



Domain: $\{x_1, x_2, x_3, x_4\}$
Codomain: $\{y_1, y_2, y_3, y_4\}$
Range: $\{y_1, y_2, y_3, y_4\}$

REMARKS:

- By definition of the range of a function, if $Y = \text{range}(f)$, then $f : X \to Y$ is surjective.
- If the function f is injective, then $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- Bijective functions are also called **one-to-one** functions.
- For MATH 137, we are only interested in one-to-one (or bijective) functions as this relates to invertibility.

YOUR TURN:

EXAMPLE: Given $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x^2}$, find the domain and the range for $g(f(x))$.
Solution:

$$g(f(x)) = \frac{1}{(\sqrt{x})^2} = \frac{1}{x} \longrightarrow \text{Range } (0, \infty)$$

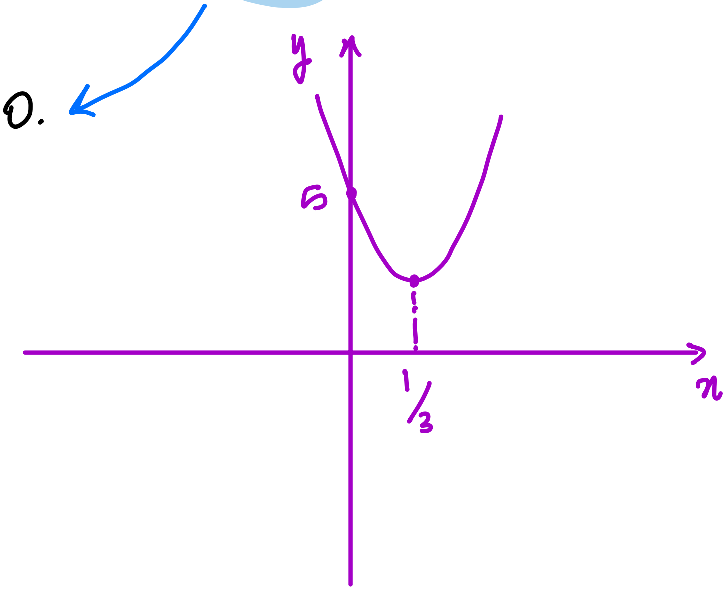
$\downarrow$

Domain : $[0, \infty)$

EXAMPLE: Given $f(x) = 3x^2 - 2x + 5$, find $f^{-1}(5)$. where $x \leq 1/3$.
Solution:

Note that $f(0) = 5$

Therefore, $f^{-1}(5) = 0$.



LONG DIVISION:

EXAMPLE: Find $\dfrac{x^3+10x^2+13x-24}{x-1}$

- using synthetic division
- using long division

Synthetic division : Only for division by $x-a$.

$$\begin{array}{c|cccc} & 1 & 10 & 13 & -24 \\ 1 & \downarrow & 1 & 11 & 24 \\ \hline & \underbrace{1 \quad 11 \quad 24}_{} & & & \underbrace{0}_{R} \end{array}$$

$q(x) = x^2 + 11x + 2$

$$\frac{x^3+10x^2+13x-24}{x-1} = x^2 + 11x + 25 \quad \text{or}$$

Long division: $x^3 + 10x^2 + 13x - 24 = (x-1)(x^3 + 10x^2 + 13x - 24)$

$$\begin{array}{r|l} & x^2 + 11x + 24 \quad R \; 0 \\ \hline x-1 & \cancel{x^3} + 10x^2 + 13x - 24 \\ & -\cancel{x^3} + x^2 \quad \downarrow \quad \downarrow \\ \hline & \quad 11\cancel{x^2} + 13x \\ & -11\cancel{x^2} + 11x \\ \hline & \qquad 24\cancel{x} - 24 \\ & \quad -24\cancel{x} + 24 \\ \hline & \qquad\qquad 0 \end{array}$$

We have included the following example detailing each step for your notes:
Consider

$$(x^3 - 12x^2 + 38x - 17) : (x - 7)$$

1. Divide the leading term of the dividend x^3 by the leading term of the divisor x and put the result $\frac{x^3}{x} = x^2$ on top of the table.

$$\begin{array}{c|cccc} & x^2 & & & \\ \hline (x-7) & x^3 & -12x^2 & +38x & -17 \end{array}$$

Then, multiply this result x^2 with the divisor $(x - 7)$ and put the product $x^3 - 7x^2$ below the dividend.

$$\begin{array}{c|cccc} & x^2 & & & \\ \hline (x-7) & x^3 & -12x^2 & +38x & -17 \\ & x^3 & -7x^2 & & \end{array}$$

2. Next, subtract the dividend $x^3 - 12x^2 + 38x - 17$ from the obtained product $x^3 - 7x^2$ and write the difference $-5x^2 + 38x - 17$ at the bottom.

$$\begin{array}{c|cccc} & x^2 & & & \\ \hline (x-7) & x^3 & -12x^2 & +38x & -17 \\ & x^3 & -7x^2 & & \\ \hline & & -5x^2 & +38x & -17 \end{array}$$

3. We repeat Steps 1. and 2. with the new dividend $-5x^2 + 38x - 17$. That is, we divide its leading term $-5x^2$ by the leading term of the divisor x, and put the result $\frac{-5x^2}{x} = -5x$ on top of the table. Then, we subtract the product of the result and the divisor $(-5x) \cdot (x - 7) = -5x^2 + 35x$ from the new dividend $-5x^2 + 38x - 17$ and note the result $-5x^2 + 38x - 17 - (-5x^2 + 35x) = 3x - 17$ below.

$$\begin{array}{c|cccc} & x^2 & -5x & & \\ \hline (x-7) & x^3 & -12x^2 & +38x & -17 \\ & x^3 & -7x^2 & & \\ \hline & & -5x^2 & +38x & -17 \\ & & -5x^2 & +35x & \\ \hline & & & 3x & -17 \end{array}$$

4. Again, we repeat Steps 1. and 2. with the new dividend $3x - 17$. Its leading term $3x$ divided by the leading term of the divisor x is 3, which we put on top of the table. Then, subtract $3 \cdot (x - 7) = 3x - 21$ from $3x - 17$ to find the remainder $3x - 17 - 3x - 21 = 4$, which is noted at the bottom of the table.

	x^2	$-5x$	$+3$	
$(x-7)$	x^3	$-12x^2$	$+38x$	-17
	x^3	$-7x^2$		
		$-5x^2$	$+38x$	-17
		$-5x^2$	$+35x$	
			$3x$	-17
			$3x$	-21
				$+4$

5. Since the new dividend is a constant, its degree is 0, and this is smaller than the degree of the divisor (the divisor has degree 1). Hence, we stop and we can read off $q(x)$ on the top and $r(x)$ at the bottom of the table, which is repeated here:

	x^2	$-5x$	$+3$	
$(x-7)$	x^3	$-12x^2$	$+38x$	-17
	x^3	$-7x^2$		
		$-5x^2$	$+38x$	-17
		$-5x^2$	$+35x$	
			$3x$	-17
			$3x$	-21
				$+4$

That is,

$$q(x) = x^2 - 5x + 3, \quad r(x) = 4$$

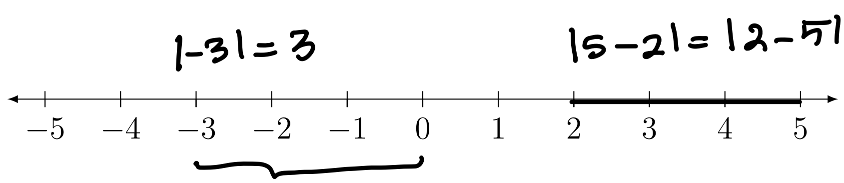
so that

$$\frac{x^3 - 12x^2 + 38x - 17}{x - 7} = x^2 - 5x + 3 + \frac{4}{x-7}.$$

Absolute Values

- STOP and THINK:
  Is the following statement true?
  The absolute value is a mechanism that simply drops negative signs.
  In other words, is $|-x| = x$ for all x?
- YOUR TURN: Evaluate the following:
  1. $|-3| =$ 3
  2. $|\pi| =$ π
  3. $|5000| =$ 5000

Absolute value as a distance function:



Remarks:

- $|x|$ measures the distance from x to 0.
- Distance btw a and b is same as distance btw b and a. Therefore,

$$|a-b| = |b-a| \quad \forall a,b \in \mathbb{R}$$

- $|-x| \neq x \quad \forall x.$

Let $x=-4$, $-x=4$

$$|-x| = |4| = 4 \neq x$$

Definition: Absolute Value
For each $x \in \mathbb{R}$, we define the **absolute value** of x by

$$|x| = \begin{cases} x & , \text{ if } x \geq 0 \quad \text{(keep sign)} \\ -x & , \text{ if } x < 0 \quad \text{(change sign)} \end{cases}$$

- STOP and THINK:
  Is the following statement true?

$$|x| = |-x|$$

if $x \geq 0$ then $-x \leq 0$, so $|x| = x$ and $|-x| = x$

$+$ keep $\quad$ $-$ change

In this case, $|x| = |-x|$.

if $x < 0$ then $-x > 0$, so $|x| = -x$ and $|-x| = -x$

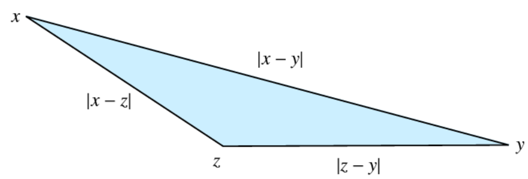
$-$ change $\quad$ $+$ keep

In this case, $|x| = |-x|$.

Therefore $|x| = |-x| \quad \forall x \in \mathbb{R}$.

Inequalities Involving Absolute Values

One of the most fundamental inequalities in all of mathematics is the Triangle Inequality.



Theorem: Triangle Inequality I
Let x,y, and z be any real numbers. Then

$$|x-y| \leq |x-z| + |z-y|$$

or

$$|x-z| + |z-y| \geq |x-y|$$

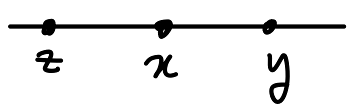
In your own words: WLOG, let $x \leq y$. Think of abs. val. as DISTANCE
Proof:

case 1
$z < x$

case 2
$x \leq z \leq y$

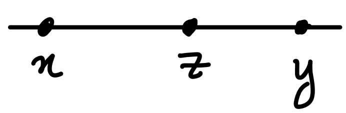
case 3
$y < z$

CASE 1: $z < x < y$



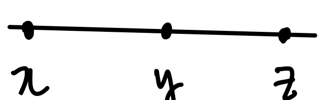
Clearly, $|x-y| < |z-y|$
So $|x-y| \leq |z-y| + |x-z|$

CASE 2: $x \leq y \leq z$



$|x-y| = |x-z| + |z-y|$

CASE 3: $x < y < z$



Clearly, $|x-y| < |x-z|$
So $|x-y| \leq |x-z| + |z-y|$

END OF WEEK 1 MATERIAL