MATH 137 Lecture Notes - Student Copy

Burcu Tuncer Karabina

Divergence to Infinity

Consider the sequence $\{a_n\} = n^2$.
GeoGebra Activity: Limit of a sequence 2
According to our observation,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \underline{\hspace{3cm}}$$

Definition: Divergence to ∞
We say that a sequence $\{a_n\}$ **diverges** to ∞, or $\lim_{n\to\infty} a_n = \infty$,

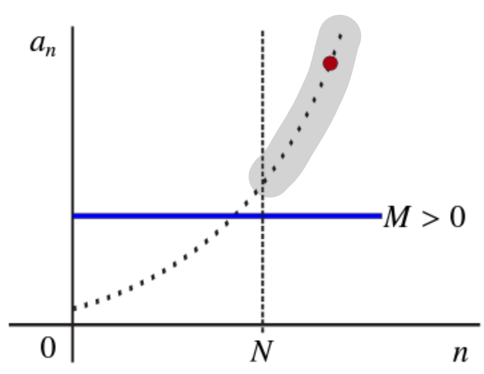
$\forall\, m > 0 \quad \exists N \in \mathbb{R}$ s.t.

$a_n > m \quad \forall n > N$

Equivalently,

any interval (m, ∞) contains a tail of $\{a_n\}$

The graphical representation will look like the figure below:



Consider the sequence $\{a_n\} = -n^2$.
GeoGebra Activity: Limit of a sequence 3
According to our observation,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -n^2 = \ldots\ldots\ldots\ldots\ldots\ldots$$

Definition: Divergence to $-\infty$
We say that a sequence $\{a_n\}$ **diverges** to $-\infty$ or $\lim_{n\to\infty} a_n = -\infty$ if,

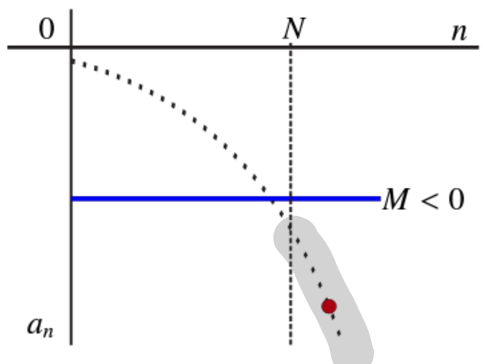
$\forall m < 0 \quad \exists N \in \mathbb{R}$ s.t.

$a_m < m \quad \forall n > N$

Equivalently,

any interval $(-\infty, m)$ contains a tail of $\{a_n\}$.

The graphical representation will look like the figure below:



QUIZ 2 cut off.

$\forall\, m<0 \quad \exists N\in\mathbb{R}$ s.t. $a_n < m \quad \forall n>N$.

EXAMPLE: Show that $\lim\limits_{n\to\infty}(1-n) = -\infty$

Let $m<0$. Let $N \boxed{> 1-m}$

If $n>N$, then

$$a_n = 1-n < 1-N < 1-(1-m) = m$$

since
$n>N$
$-n<-N$
$1-n<1-N$

ASIDE

$1-N<m$

$1-m<N$

USEFULL to know:

$$\lim_{n\to\infty} n^\alpha \begin{cases} \alpha>0, & \lim\limits_{n\to\infty} n^\alpha = \infty \\ \alpha<0, & \lim\limits_{n\to\infty} n^\alpha = 0 \end{cases}$$

Ex: $\lim\limits_{n\to\infty} n^3 = \infty$ and $\lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$

Arithmetic Rules for Limits

- STOP and THINK:
  Let $\{a_n\}$ and $\{b_n\}$ be two sequences converging to 3 and 4 consecutively. Are the following statements true?
  - $\{2a_n\}$ converges to 6.
  - $\{a_n + b_n\}$ converges to 7.
  - $\{a_nb_n\}$ converges to 12.

Theorem: Arithmetic Rules for Limits of Sequences
Let $\{a_n\}$ and $\{b_n\}$ be sequences. Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, where $a, b \in \mathbb{R}$.
Then the following rules hold: meaning these limits exist!

1. For any $c \in \mathbb{R}$, if $a_n = c$ $\forall n$, then $\lim_{n\to\infty} a_n = c = a$
2. For any $c \in \mathbb{R}$, $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = ca$
3. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a+b$
4. $\lim_{n\to\infty} a_nb_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = ab$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n = a/b$ if $b \neq 0$
6. If $\lim_{n\to\infty} b_n = \pm\infty$ and $\lim_{n\to\infty} a_n = k$ for some $k \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
7. If $a_n \geqslant 0$ for all n and $\alpha > 0$, then $\lim_{n\to\infty} a_n^{\alpha} = \left[\lim_{n\to\infty} a_n\right]^{\alpha} = a^{\alpha}$
8. For any $\underline{k \in \mathbb{N}}$, $\lim_{n\to\infty} a_{n+k} = a$
9. If $\alpha > 0$, then $\lim_{n\to\infty} n^{\alpha} = \infty$
10. If $\alpha < 0$ then $\lim_{n\to\infty} n^{\alpha} = 0$

Proof: (3) let $\varepsilon > 0$.

since $\lim_{n\to\infty} a_n = a$, $\exists N_1 \in \mathbb{R}$ s.t. if $n > N_1$ then $|a_n - a| < \varepsilon/2$

since $\lim_{n\to\infty} b_n = b$, $\exists N_2 \in \mathbb{R}$ s.t. if $n > N_2$ then $|b_n - b| < \varepsilon/2$

let $N \boxed{= \max\{N_1, N_2\}}$. If $n > N$, then

$|(a_n+b_n) - (a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

Example: $\lim_{n\to\infty} \frac{3n+7}{n+4}$ $= \lim_{n\to\infty} \frac{\frac{1}{n}(3n+7)}{\frac{1}{n}(n+4)} = \lim_{n\to\infty} \frac{3+\frac{7}{n}}{1+\frac{4}{n}}$

$= \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{7}{n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{4}{n}}$ this step can be skipped.

$= \frac{3+0}{1+0}$ by arithmetic rules.

Example: $\lim_{n\to\infty} \frac{n^3+n^2+1}{2n^3+7n^2-1}$ $= \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^3+n^2+1)}{\frac{1}{n^3}(2n^3+7n^2-1)}$

$= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^3}}{2+\frac{7}{n}-\frac{1}{n^3}}$

$= \frac{1+0+0}{2+0-0} = \frac{1}{2}$ by arithmetic rules.

Example: $\lim_{n\to\infty} \frac{n+1}{n^2+1}$ $= \lim_{n\to\infty} \frac{\frac{1}{n^2}(n+1)}{\frac{1}{n^2}(n^2+1)}$

Also note that n^2+1 grows faster than $n+1$. This should give you intuition about what the limit should be.

$= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} = \frac{0+0}{1+0} = 0$

ALWAYS SHOW SOME WORK!

Revisit #5, what if $b=0$?

Some interesting examples: need to handle these individually!

- $\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$
- $\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} n = \infty$
- $\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$

Theorem:
Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $\lim_{n\to\infty} b_n = 0$. Also assume that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists. Then

$$\lim_{n\to\infty} a_n = 0$$

Proof: $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot b_n = L \cdot 0 = 0$

since both limits exist.

Corollary:
Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} a_n \neq 0$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} \text{ DNE.}$$

Example: $\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^3+3n)}{\frac{1}{n^3}(n^2+1)}$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{\frac{1}{n}+\frac{1}{n^3}} = \infty$$

conv. to 1

conv. to 0

! Before we apply a theorem, we always check to see if the conditions of the theorem are satisfied!

Generalization:

$$\lim_{n\to\infty} \frac{b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \ldots + b_j n^j}{c_0 + c_1 n + c_2 n^2 + c_3 n^3 + \ldots + c_k n^k} = \begin{cases} b_j/c_k & \text{if } j = k \\ 0 & \text{if } j < k \\ \infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} > 0 \\ -\infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} < 0 \end{cases}$$

Examples:

- $\lim_{n\to\infty} \frac{3n+1}{2n-2} = \frac{3}{2}$
- $\lim_{n\to\infty} \frac{4n^2+5n}{n^3-1} = 0$
- $\lim_{n\to\infty} \frac{7-n^4}{1+n^3} = -\infty$

Example (Another technique): $\lim_{n\to\infty} \sqrt{n^2+4} - n$

[Conjugate method]

$$\lim_{n\to\infty} (\sqrt{n^2+4} - n) \cdot \frac{\sqrt{n^2+4} + n}{\sqrt{n^2+4} + n}$$

Recall:
$(a+b)(a-b) = a^2 - b^2$

$$= \lim_{n\to\infty} \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} = \lim_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot 4}{\frac{1}{n}\left(\sqrt{n^2+4}+n\right)} = \lim_{n\to\infty} \frac{\cancel{\frac{4}{n}} \to 0}{\sqrt{1+\cancel{\frac{4}{n^2}}}+1}$$

goes in as $\frac{1}{n^2}$ since $n>0$

0

$$= \frac{0}{1+1} = \frac{0}{2} = 0$$

Example (Recursively defined sequence): $a_1 = 2$ and $a_{n+1} = \dfrac{5 + a_n}{2}$

$$\lim_{n\to\infty} a_n = L$$

(Note that this is a very strong assumption we also have to prove! But we don't have enough tools yet!

By rule #8, $\lim_{n\to\infty} a_{n+1} = L$ [choose $k=1$]

$$\underbrace{\lim_{n\to\infty} a_n}_{L} = \underbrace{\lim_{n\to\infty} a_{n+1}}_{L} = \underbrace{\lim_{n\to\infty} \frac{5+a_n}{2}}_{\frac{5+L}{2}}$$

by arithmetic rules of limits

$$L = \frac{5+L}{2}$$

$$2L = 5 + L$$

$$L = 5 \checkmark$$

Squeeze Theorem (for Sequences)

Is the answer correct? MAYBE!

- STOP and THINK:
  Consider the sequence $a_n = \dfrac{\sin n}{n}$. Is the following solution true? NOT

$$\lim_{n\to\infty} \frac{\sin n}{n} = \underbrace{\left(\lim_{n\to\infty} \sin n\right)}_{\text{DNE}} \underbrace{\left(\lim_{n\to\infty} \frac{1}{n}\right)}_{0} = \left(\lim_{n\to\infty} \sin n\right)(0) = 0$$

DNE

So we can't apply arithmetic rule #4.

$$-1 \leq \sin n \leq 1$$

$$\underset{\to 0}{-\frac{1}{n}} \leq \frac{\sin n}{n} \leq \underset{\to 0}{\frac{1}{n}} \qquad \text{since } n>0$$

Can I claim $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$?

GeoGebra Activity: Squeeze Theorem for Sequences useful for $\lim$ with $(-1)^n$, sin/cos.

Theorem: Squeeze Theorem
If $a_n \le b_n \le c_n$ for all $n > M$ (for some $M \in \mathbb{N}$) and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then

$$\lim_{n\to\infty} b_n = L$$

Proof: Let $\varepsilon > 0$. Since $a_n \to L$ and $c_n \to L$, we can find $N \in \mathbb{R}$ s.t. $n > N$ $a_n \in (L-\varepsilon, L+\varepsilon)$ and $c_n \in (L-\varepsilon, L+\varepsilon)$

Then we have

$$L-\varepsilon < a_n \le b_n \le c_n < L+\varepsilon$$

So $b_n \in (L-\varepsilon, L+\varepsilon)$. Therefore, $b_n \to L$.

Example: $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$

$$-1 \le (-1)^n \le 1$$

$$-\frac{1}{n^2+1} \le \frac{(-1)^n}{n^2+1} \le \frac{1}{n^2+1} \qquad n^2+1>0$$

Since $\lim_{n\to\infty} \frac{1}{n^2+1} = \lim_{n\to\infty} \frac{-1}{n^2+1} = 0$, by the Squeeze Thrm

we find $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$

Example: $\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n}$

$$-1 \le \cos(n^2+7) \le 1$$

$$6 \le \cos(n^2+7)+7 \le 8$$

$$\frac{6}{n} \le \frac{\cos(n^2+7)+7}{n} \le \frac{8}{n} \quad , n>0$$

Since $\lim_{n\to\infty} \frac{6}{n} = \lim_{n\to\infty} \frac{8}{n} = 0$, by the squeeze Thrm, we find

$$\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n} = 0.$$

THINK-PAIR-SHARE:

THINK: Take a minute to think about what you have learned today.
PAIR-SHARE: Discuss the question and the solution provided with one of the students sitting next to you. If we run out of time in class for this activity, you can pair and share with a friend later.

Exercise: Prove (using the definition) that if $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

GO OVER THIS !

Proof: Case 1: $L = 0$: Suppose $\lim_{n\to\infty} a_n = 0$.

Let $\varepsilon > 0$.
Since $a_n \longrightarrow 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, we have $|a_n - 0| < \varepsilon^2$.
Hence, for $n > N$, we get

$$|a_n| < \varepsilon^2 \implies a_n < \varepsilon^2 \implies \sqrt{a_n} < \sqrt{\varepsilon^2} \implies \sqrt{a_n} < \varepsilon$$

as desired. Therefore, $\sqrt{a_n} \longrightarrow \sqrt{0} = 0$.
Case 2: $L \neq 0$: Suppose $\lim_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$.
Since $a_n \longrightarrow L$, we can find $N \in \mathbb{N}$ such that if $n > N$, we have $|a_n - L| < \sqrt{L}\varepsilon$.
Also, note that $a_n - L = (\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})$.
So, for $n > N$, we have

$$\begin{aligned} |\sqrt{a_n} - \sqrt{L}| &= \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \\ &< \frac{\sqrt{L}\varepsilon}{\sqrt{a_n} + \sqrt{L}} \\ &< \frac{\sqrt{L}\varepsilon}{\sqrt{L}} \\ &= \varepsilon \end{aligned}$$

as desired.
Therefore, $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Here is a GeoGebra activity to explore the formal definition of limit of a sequence: Explore Formal Definition of Limit of a Sequence

Recursive Sequences

We have encountered recursive sequences, such as the Fibonacci Sequence which clearly diverges to ∞. We've also learned how we can use limit rules to determine the limit of a recursive sequence, if that limit exists. This raises the question: How do we determine if a recursive sequence converges?
Before we describe how to determine if a recursive sequence converges, we first introduce some terminology.

Definition:Upper and Lower Bound
Let $S \subseteq \mathbb{R}$.

- α is an **upper bound** of S if

  $$x \leq \alpha \qquad \forall x \in S$$

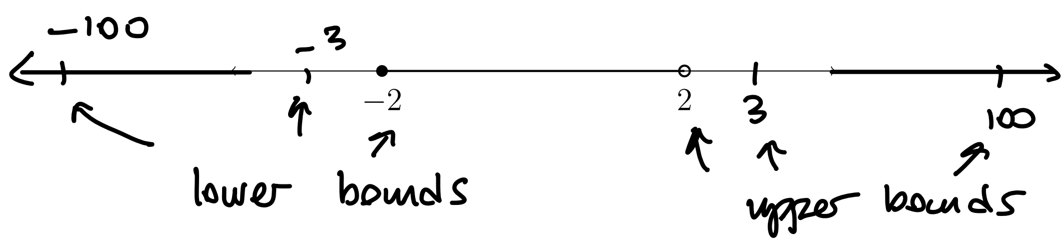
  and such a set S is called **bounded above**.
- β is a **lower bound** of S if

  $$\beta \leq x \qquad \forall x \in S$$

  and such a set S is called **bounded below**.
- We call S **bounded** if it is

  both bounded below and above.

EXAMPLE: Consider $S = [-2, 2)$. Find an upper and lower bound for S. Are they unique?



Definition: Least Upper Bound
Let $S \subset \mathbb{R}$.
Then α is called the **least upper bound** of S if

1. α is an upper bound for S
2. α is the smallest upper bound for S.

ie if α' is another upper bound for S, then $\alpha \le \alpha'$

We write

$lub(S)$ or $sup(S)$

The least upper bound is often called the **supremum** of S and is denoted by $sup(S)$.

Definition: Greatest Lower Bound
Let $S \subset \mathbb{R}$.
Then β is called the **greatest lower bound** of S if

1. β is a lower bound for S
2. β is the largest lower bound for S

ie if β' is another lower bound S, then $\beta' \le \beta$.

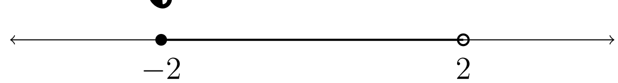
We write

$glb(S)$ or $inf(S)$

The greatest lower bound is often called the **infimum** of S and is denoted by $inf(S)$.

Example: Consider $S = [-2, 2)$. Find the supremum and infimum for S. Are they unique? Do they always belong to S? Do they always exist?

$glb(S) = -2 \in S$ $\qquad$ $lub(S) = 2 \notin S$



Consider $[1, \infty)$ has no upper bound.

Similarly, $(-\infty, 5)$ has no lower bound.

$\mathbb{R}$ has neither.

Definition: Monotonic Sequences
A sequence $\{a_n\}$ is called

- **incresing** if $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$
- strictly increasing ~~**non-decreasing**~~ if $a_n < a_{n+1} \quad \forall n \in \mathbb{N}$
- **decresing** if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$
- strictly decreasing ~~**non-increasing**~~ if $a_n > a_{n+1} \quad \forall n \in \mathbb{N}$
- **monotonic** if $\{a_n\}$ is either decreasing or increasing.

EXAMPLES:

Increasing Sequence

A sequence is increasing if each term is greater than or equal to the previous term.

Example: $\{1, 1, 2, 3, 5, 8, \ldots\}$

Decreasing Sequence

A sequence is decreasing if each term is less than or equal to the previous term.

Example: $\{-1, -1, -2, -3, -8, \ldots\}$

Strictly Inc.
~~**Non-Decreasing**~~ **Sequence**

A sequence is ~~non-decreasing~~ if each term is greater than the previous term.

Example: $\{1, 4, 9, 16, 25, \ldots\}$

Strictly Dec.
~~**Non-Increasing**~~ **Sequence**

A sequence is ~~non-increasing~~ if each term is less than the previous term.

Example: $\{1/2, 1/3, 1/4, \ldots\}$

YOUR TURN: Can a sequence $\{a_n\}$ be increasing and decreasing at the same time?

YES. constant sequences!

$a_n \leq a_{n+1}$ and $a_{n+1} \leq a_n \quad \forall n$

$\Rightarrow a_n = a_{n+1} \quad \forall n$

$\Rightarrow a_n$ is a constant sequence.

Remarks: Convergence and Boundedness

T / F : $\{a_n\}$ bounded $\Rightarrow$ $\{a_n\}$ converge

No: $\{\sin(n)\}$, $\{(-1)^n\}$

$\{a_n\}$ bounded + $\underbrace{?}_{\text{monotonic}}$ $\Rightarrow$ $\{a_n\}$ converge

Axiom:
If $S \subseteq \mathbb{R}$ is nonempty and bounded above, then S has a least upper bound. Similarly, if S is bounded below, then S has a greatest lower bound.

The following theorem allows us to conclude whether monotonic sequences converge and is an important tool we will use for recursive sequences.

Theorem: Monotone Convergence Theorem (MCT)
Let $\{a_n\}$ be an increasing sequence.

1. If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = \text{lub}(\{a_n\})$
2. If $\{a_n\}$ is NOT bounded above, then $\{a_n\}$ diverges to ∞

Similarly, let $\{b_n\}$ be a decreasing sequence.

1. If $\{b_n\}$ is bounded below, then $\{b_n\}$ converges to $L = \text{glb}(\{a_n\})$
2. If $\{b_n\}$ is NOT bounded below, then $\{b_n\}$ diverges to $-\infty$

PROOF:
We will prove the case where $\{a_n\}$ is increasing and bounded above. The other case is similar.

READ THE PROOF FROM INSTUCTOR'S NOTES!

#1) Suppose $\{a_n\}$ is bounded above and let $lub(\{a_n\}) = L$

Let $\varepsilon > 0$. Consider $L - \varepsilon$.

$L - \varepsilon < \overset{lub}{L}$ since $\varepsilon > 0$

$L - \varepsilon$ is not an upper bound since L is (ie $lub(\{a_n\}) = L$).

So we can find a term greater than $L-\varepsilon$ since it's not an upper bound. ie $\exists N \in \mathbb{N}$ s.t $a_N > L - \varepsilon$. (*)

But $\{a_n\}$ is a increasing sequence so if $n \geq N$ then $a_n \geq a_N$. (**)

By (*) and (**), we get $L - \varepsilon < a_N \leq a_n$

$L - \varepsilon < a_N \leq a_n < L < L + \varepsilon \Rightarrow |a_n - L| < \varepsilon$

since $L = lub$

Therefore, $\lim\limits_{n\to\infty} a_n = L$

#2) Suppose $\{a_n\}$ is not bounded above.

Let $M > 0$.

Since $\{a_n\}$ is not bounded above, we can find a term greater than M. ie $\exists N \in \mathbb{N}$ s.t. $a_N > M$. (*)

But $\{a_n\}$ is a increasing sequence so if $n \geq N$ then $a_n \geq a_N$. (**)

By (*) and (**), we get $a_n \geq a_N > M$.

Now try the decreasing proof yourself!

Mathematical Induction

Before we can use the MCT, we need to develop a proof technique called **Mathematical Induction**, a technique often used to show properties that need to hold for all natural numbers. Induction allows us to prove an infinite number of related statements! In fact, it helps us prove a sequence of statements P_n for $n \in \mathbb{N}$. For instance,

$$\text{Statement } P_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}$$

is a statement giving the formula for the sum of the first n natural numbers, for each $n \in \mathbb{N}$.

Method: Mathematical Induction Suppose we have a sequence of statements $P_1, P_2, P_3, \ldots, P_n, \ldots$, where $n \in \mathbb{N}$. If we can:

POMI

1. Prove P_1 is true, **(base case - BS)**
2. Prove: If P_k is true for some $k \in \mathbb{N}$ **(inductive hypothesis - IH)**, then P_{k+1} is true, **(inductive step - IS)**,

then we can conclude that P_n is true for all $n \in \mathbb{N}$.

You can think of mathematical induction as a sequence of dominoes falling: If the first one falls, and each domino can cause the next one to fall, then all dominoes will fall. Similarly, if the first statement P_1 is true, and any statement P_k being true causes P_{k+1} to be true, then all statements will be true. You will explore Mathematical Induction further in MATH 135. EXAMPLE: As an example, we show the formula for the sum of the first n integers:

$$P_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}.$$

- Base Case: $(n=1)$ $P_1 : 1 = \frac{1(1+1)}{2}$ is true.
  BC
- Inductive Hypothesis: Assume P_k is true $\forall k \in \mathbb{N}$
  IH $1+2+\cdots+k = \frac{k(k+1)}{2}$
- Inductive Step: Show P_{k+1} is true.
  IS

Don't skip.

$$\underbrace{1+2+\cdots+k}_{\text{by IH, this equals } \frac{k(k+1)}{2}} + (k+1) = \frac{k(k+1)}{2} + (k+1) \quad \text{by IH.}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}. \text{ So } P_{k+1} \text{ is true.}$$

By POMI, P_n is true $\forall n \in \mathbb{N}$.
↳ Don't skip

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Before we can use the MCT, we need to develop a proof technique called **Mathematical Induction**, a technique often used to show properties that need to hold for all natural numbers. Induction allows us to prove an infinite number of related statements! In fact, it helps us prove a sequence of statements P_n for $n \in \mathbb{N}$. For instance,

$$\text{Statement } P_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}$$

is a statement giving the formula for the sum of the first n natural numbers, for each $n \in \mathbb{N}$.

Method: Mathematical Induction Suppose we have a sequence of statements $P_1, P_2, P_3, \ldots, P_n, \ldots$, where $n \in \mathbb{N}$. If we can:

(POMI)

1. Prove P_1 is true, **(base case - BS)**
2. Prove: If P_k is true for some $k \in \mathbb{N}$ **(inductive hypothesis - IH)**, then P_{k+1} is true, **(inductive step - IS)**,

then we can conclude that P_n is true for all $n \in \mathbb{N}$.

You can think of mathematical induction as a sequence of dominoes falling: If the first one falls, and each domino can cause the next one to fall, then all dominoes will fall. Similarly, if the first statement P_1 is true, and any statement P_k being true causes P_{k+1} to be true, then all statements will be true. You will explore Mathematical Induction further in MATH 135. EXAMPLE: As an example, we show the formula for the sum of the first n integers:

$$P_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}.$$

- Base Case: $n = 1$ $\qquad 1 = \frac{1(2)}{2}$ is true.
  (BC)
- Inductive Hypothesis: Suppose P_k is true for some k.
  (IH)
  ie $\quad P_k : 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$
- Inductive Step: Show that P_{k+1} is true.
  (IS)

$$\underbrace{1 + 2 + \cdots + k}_{\text{equals } \frac{k(k+1)}{2} \text{ by induction hypothesis}} + k+1 = \frac{k(k+1)}{2} + (k+1) \quad \text{by IH.}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \quad \checkmark \quad \text{So, } P_{k+1} \text{ is true}$$

By the principles of mathematical induction, P_n is true for all $n \in \mathbb{N}$.

(or by POMI)

(or by PoMI)

In this course, we will use the MCT and induction to find limits of recursive sequences. To do this, we follow these steps:

Method: Recursive Sequence Limits via MCT

1. Prove that $\{a_n\}$ is monotonic.
2. Prove that $\{a_n\}$ is bounded above or below.
3. Apply MCT.
4. Find the limit

Sometimes we combine these 2 steps.

EXAMPLE: Consider the sequence $\{a_n\}$ recursively defined via

$$a_1 = 1, \quad a_{n+1} = \frac{3 + a_n}{2} \quad (n \geqslant 1).$$

Prove that this sequence converges and find its limit. **Solution:**

STEP 1) List first a few terms: $1, 2, 5/2, \dots$

Looks like this sequence is increasing.

CLAIM: $\{a_n\}$ is increasing.

WTS: $P_n: \ a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$

BC: $(n=1)$ $P_1: \ 1 \leq 2$ is true.

IH: P_k is true for some k. $a_k \leq a_{k+1}$

IS: By IH, we know that

$$a_k \leq a_{k+1}$$

$$a_k + 3 \leq a_{k+1} + 3$$

$$\frac{a_k + 3}{2} \leq \frac{a_{k+1} + 3}{2}$$

$$a_{k+1} \leq a_{k+2}$$

Therefore, P_{k+1} is true.

By the principles of mathematical induction, P_n is true for all $n \in \mathbb{N}$.
(or by PoMI)

Solution continued:

STEP 2: $\{a_n\}$ is bounded above.
Just make an educated guess. No need to guess lub.

CLAIM: $a_n \leq 5 \quad \forall n \in \mathbb{N}$

BC: $(n=1)$ $\quad a_1 = 1 \leq 5$ is true.

IH: P_k is true for some k. $\quad a_k \leq 5$

IS: By P_k, we know that

$$a_k \leq 5$$

$$\frac{a_k+3}{2} \leq \frac{5+3}{2} = 4 \leq 5$$

$$a_{k+1} \leq 5$$

Therefore, P_{k+1} is true.

By the principles of mathematical induction, P_n is true for all $n \in \mathbb{N}$.

STEP 3: Since $\{a_n\}$ is increasing and bounded above, by MCT, $\{a_n\}$ conv. to its lub

STEP 4: Let $L = \text{lub}(\{a_n\})$. By step 4, we know that $a_n \to L$.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n+3}{2} = \frac{L+3}{2}$$

Solving $L = \frac{L+3}{2}$, we find $L = 3$.

Notice that both claims, $a_n \leq a_{n+1}$ and $a_n \leq 5$, were proven using the same steps. In fact, we can prove both increasing and bounded above in a single use of Mathematical Induction as seen in the following example.

EXAMPLE:
Consider the sequence $\{a_n\}$ recursively defined via

$$a_1 = 2, \quad a_{n+1} = \sqrt{7 + a_n} \quad (n \geqslant 1).$$

Prove that this sequence converges and find its limit.

① & ②

$a_1 = 2, \ a_2 = \sqrt{7+2} = 3, \ a_3 = \sqrt{7+3} = \sqrt{10}, \ \ldots$

CLAIM: $\{a_n\}$ is non-decreasing and bounded above by 10.
WTS: $a_n \le a_{n+1}$ and $a_n \le 10$ for all $n \in \mathbb{N}$.

Base case $(n=1)$: $a_1 = 2, \ a_2 = 3$

$a_1 \le a_2$ ✓ $\qquad a_1 = 2 \le 10$ ✓

IH: Suppose $a_k \le a_{k+1}$ and $a_k \le 10$ for some $k \ge 1$.

IS:

$a_k \le a_{k+1}$ by IH.
$a_k + 7 \le a_{k+1} + 7$
$\sqrt{a_k + 7} \le \sqrt{a_{k+1} + 7}$
$a_{k+1} \le a_{k+2}$ ✓✓

$a_k \le 10$ by IH.
$a_k + 7 \le 17$
$\sqrt{a_k + 7} \le \sqrt{17} \le 10$
$a_{k+1} \le 10$ ✓✓

Therefore, $\{a_n\}$ is non-decreasing and bounded above by 10 by POMI.

③ Thus, by MCT, $\{a_n\}$ converges.

④ Let $\lim_{n\to\infty} a_n = L$.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7 + a_n} = \sqrt{7 + L}$

$\Rightarrow L = \sqrt{L + 7}$

$\Rightarrow L + 7 = L^2$

$\Rightarrow L^2 - L - 7 = 0$

$\Rightarrow L_{1,2} = \dfrac{1 \mp \sqrt{29}}{2}$

$L = \cancel{\dfrac{1 - \sqrt{29}}{2}} < 2 = a_1$ not even an upper bound!

$\boxed{L = \dfrac{1 + \sqrt{29}}{2}}$

REMARK:
The order of the steps does matter! We cannot perform the last step, unless we know that $\lim_{n\to\infty} a_n$ exists.
For instance, consider the sequence $\{a_n\}$ with

$$a_1 = 1, \quad a_{n+1} = 2a_n + 1.$$

The first few terms are

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 7, \quad a_4 = 15, \quad \dots$$

Had we carelessly started with the last step, and argued $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$, we would have obtain the ridiculous result $L = 2 \cdot L + 1$ or $L = -1$. But the given sequence $\{a_n\}$ does not converge (and, in fact, is never negative). While the given sequence is monotonic, it is not bounded, and we cannot apply MCT.

END OF WEEK 3 MATERIAL