# MATH 137 Quiz 2 – Solutions

Wednesday, January 22, 2024. Duration: 35 minutes.

**Notes:**

1. Answer all questions in the space provided.
2. For multiple choice and true/false questions, answer by filling in the bubble on the last page of the quiz.
3. Your grade will be influenced by how clearly you express your ideas and how well you organize your solutions. Show all details to get full marks. Numerical answers should be in exact values (no approximations).
4. There are a total of 20 possible points.
5. DO NOT write on the Crowdmark QR code at the top of the pages or your quiz will not be scanned (and will receive a grade of zero).
6. Use a dark pen or pencil.
7. No aids allowed.

(MC) Answer the following multiple choice questions on the last page of the quiz. Bubble (a),
[6] (b), (c), or (d). Note there is only one correct answer for each question.

1. Given $|x-4|<7$, which of the following is true of $|3x+1|$
   (a) $|3x+1|<8$
   (b) $|3x+1|>24$
   (c) $|3x+1|>8$
   (d) $|3x+1|<34$

   **Correct Answer:** (d) $|3x+1|<34$

2. Let $a_n$ be a sequence with $\lim_{n\to\infty} a_n = \infty$. Which of the following statements is FALSE about $a_n$?
   (a) For every real number $M>0$, there exists $N \in \mathbb{R}$ such that $a_n > M$ whenever $n > N$.
   (b) For any real number $M>0$, the terms of the sequence $a_n$ eventually exceeds $M$.
   (c) Given a real number $M>0$, there are only finitely many terms of the sequence $a_n$ such that $a_n \geq M$.
   (d) There exists $N \in \mathbb{R}$ such that $a_n > 0$ whenever $n > N$.

   **Correct Answer:** (c)

3. Consider the sequence $a_n = \sin\left(\frac{n\pi}{4}\right)$. Which of the following is NOT a subsequence of $a_n$?
   (a) $b_n = \left(\frac{1}{2}\right)^n$
   (b) $b_n = \frac{1}{\sqrt{2}}$
   (c) $b_n = (-1)^n$
   (d) $b_n = 0$

   **Correct Answer:** (a) $b_n = \left(\frac{1}{2}\right)^n$

(TF) True/False, answer on the last page of the quiz. Bubble (a) for True, (b) for False.

[1] 4. $\boxed{\text{TRUE}}$ or FALSE: A tail of a sequence is also a subsequence of the same sequence.

[1] 5. $\boxed{\text{TRUE}}$ or FALSE: The sequence $a_n = \begin{cases} 0 & \text{if} \quad n \leq 100 \\ 2^n & \text{if} \quad n > 100 \end{cases}$ diverges.

(SA) Short answer questions, marks only awarded for a correct final answer, you do not need to show any work.

[2] 1. Write down two equivalent definitions for $\lim_{n\to\infty} a_n = L$ where $L \in \mathbb{R}$.

**Solution:** The following are equivalent:

i. $\lim_{n\to\infty} a_n = L$.
ii. For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that if $n > N$, the $|a_n - L| < \varepsilon$.
iii. For every $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains a tail of $\{a_n\}$.
iv. For every $\varepsilon > 0$, the number of elements of $\{a_n\}$ that do not lie in the interval $(L - \varepsilon, L + \varepsilon)$ is finite.
v. Every interval $(a, b)$ containing $L$ contains a tail of $\{a_n\}$.
vi. Given any interval $(a, b)$ containing $L$, the number of elements of $\{a_n\}$ that do not lie in $(a, b)$ is finite.

[2] 2. Give examples of two sequences $a_n$ and $b_n$ such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$ but $\lim_{n\to\infty} (a_n - b_n) \neq 0$.

**Solution:** One possible answer is $a_n = n$ and $b_n = 2n$

(LA) The remaining questions are long answer questions, please show all of your work.

[3] 1. Show that if $|x - 2| < 1$ then $|2x^2 - 3x - 2| < 7$.

**Solution:** Our assumption $|x - 2| < 1$ implies $1 < x < 3$.Factoring we get:

$$\begin{aligned} 2x^2 - 3x - 2 &= 2x^2 - 4x + x - 2 \\ &= 2x(x - 2) + 1(x - 2) \\ &= (2x + 1)(x - 2). \end{aligned}$$

Since we have $1 < x < 3$, we use the bounds to find $|2x+1| < 7$ and $|x-2| < 1$. Therefore, we have $|2x^2 - 3x - 2| < 7$

[5] 2. Use the formal ( $\varepsilon - N$ ) definition of limits to prove $\lim_{n\to\infty} \frac{6n-3n^2-2}{(n-1)^2} = -3$

**Solution:** Let $\epsilon > 0$ be given. Pick $N > \frac{1}{\sqrt{\epsilon}} + 1$, $N \in \mathbb{R}$. Then, if $n > N$ we get

$$
\begin{aligned}
|a_n - L| &= \left|\frac{6n-3n^2-2}{(n-1)^2} + 3\right| \\
&= \left|\frac{6n-3n^2-2+3n^2-6n+3}{(n-1)^2}\right| \\
&= \left|\frac{1}{(n-1)^2}\right| \\
&= \frac{1}{(n-1)^2} \\
&< \frac{1}{(N-1)^2} \\
&< \frac{1}{\left(\frac{1}{\sqrt{\epsilon}}+1-1\right)^2} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon
\end{aligned}
$$

as desired.

This page is meant for rough work. Clearly indicate in the original question if part of your solution is here.