MATH 137 Lecture Notes - Student Copy

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Theorem: Triangle Inequality II
Let $x, y \in \mathbb{R}$. Then

$$|x+y| \leq |x| + |y|$$

In your own words:

Proof: Let $x, y \in \mathbb{R}$.

$$|x+y| = |x-(-y)|$$
$$\leq |x-0| + |0-(-y)|$$
$$= |x| + |y|$$

RECALL

Relabel Δ-inequality

$$|a-b| \leq |a-c| + |c-b|$$

Choose $a = x$, $b = -y$, $c = 0$

$$|x-(-y)| \leq |x-0| + |0-(-y)|$$

Common Tricks: $|x-y| \leq |x-z| + |z-y|$

choose $z=0$ then $|x-y| \leq |x| + |y|$ $\circledast$ $(|-y| = |y|)$

$$|x| = |x+y-y| \leq |x+y| + |-y| = |x+y| + |y|$$

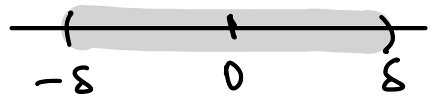
$$|x| \leq |x+y| + |y| \Rightarrow |x| - |y| \leq |x+y|.$$

$$|y| = |y+x-x| \leq |y+x| + |-x| = |x+y| + |x|$$

$$|y| \leq |x+y| + |x| \Rightarrow |y| - |x| \leq |x+y|$$

Inequalities of the form $|x| < \delta$ where $\delta \in \mathbb{R}$:

$\delta > 0$



if $x \geq 0$ then $|x| = x$

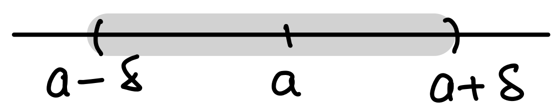
$$|x| < \delta \Rightarrow x < \delta$$

if $x < 0$ then $|x| = -x$

$$|x| < \delta \Rightarrow -x < \delta \Rightarrow x > -\delta$$

Therefore, $-\delta < x < \delta$ or $x \in (-\delta, \delta)$

Inequalities of the form $|x - a| < \delta$ where $\delta, a \in \mathbb{R}$:



$|x-a| < \delta \Rightarrow -\delta < x - a < \delta$ from previous part.

$\Rightarrow a - \delta < x < a + \delta$

OR

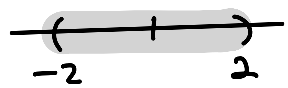
$x \in (a-\delta, a+\delta)$

$|x-a| \leq \delta \Rightarrow x \in [a-\delta, a+\delta]$

$0 < |x-a| < \delta \Rightarrow x \in (a-\delta, a+\delta) \setminus \{a\}$ since $|x-a|$ can't be zero

or $x \in (a-\delta, a) \cup (a, a+\delta)$

Example: Solve $|x| < 2$.



$$|x| < 2 \Rightarrow -2 < x < 2$$

Example: Solve $|x| \leq -2$.

No solution.

Example: Solve $|2x-1| < 5$. $\Rightarrow x > 3$ T/F?

$$|2x-1| < 5 \Rightarrow -5 < 2x-1 < 5$$

$$-4 < 2x < 6$$

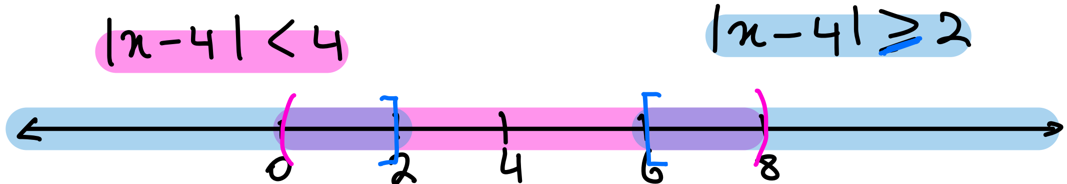
$$-2 < x < 3$$

Example: Solve $|1-2x| < 5$.

$|2x-1| = |1-2x|$ so the solution is the same as above.

Example: Solve $|2x-1| < -5$.

No solution.



Example: Solve $2 \leq |x-4| < 4$

Geometrically, $x \in (0,2] \cup [6,8)$

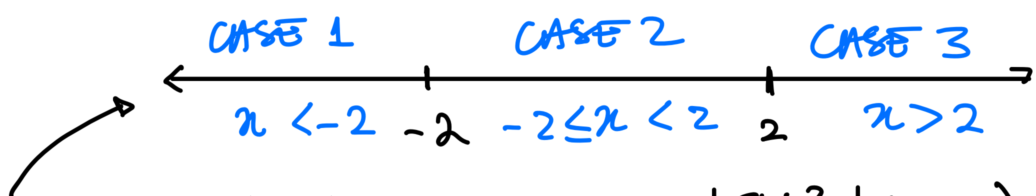
Algebraically, $|x-4| < 4 \Rightarrow -4 < x-4 < 4$

$0 < x < 8$

$|x-4| \geq 2 \Rightarrow x-4 \geq 2$ or $x-4 \leq -2$

$x \geq 6$ or $x \leq 2$

Altogether, we get $0 < x \leq 2$ or $6 \leq x < 8$.



Example: Solve $\dfrac{|x+2|}{|x-2|} > 5$ (same $\left|\dfrac{x+2}{x-2}\right| > 5$)

CASE 1 : $x < -2$

$|x+2| = -(x+2) = -x-2$

$|x-2| = -(x-2) = -x+2$

$\dfrac{|x+2|}{|x-2|} > 5 \Rightarrow \dfrac{-x-2}{-x+2} > 5 \Rightarrow -x-2 > -5x+10$

$\Rightarrow x > 3$

BUT, $x < -2$ so no sol^n. (*)

why?

CASE 2: $-2 \leq x < 2$

$|x+2| = x+2$

$|x-2| = -(x-2) = -x+2$

$\dfrac{|x+2|}{|x-2|} > 5 \Rightarrow \dfrac{x+2}{-x+2} > 5 \Rightarrow x+2 > -5x+10$

$\Rightarrow x > 4/3$

BUT $-2 \leq x < 2$, $4/3 < x < 2$ (**)

CASE 3: $x > 2$

$|x+2| = x+2$

$|x-2| = x-2$

$\dfrac{|x+2|}{|x-2|} > 5 \Rightarrow \dfrac{x+2}{x-2} > 5 \Rightarrow x+2 > 5x-10 \Rightarrow x < 3$

BUT $x > 2$, so $2 < x < 3$ (***)

*, **, and *** implies $x \in (4/3, 2) \cup (2, 3)$

THINK-PAIR-SHARE:

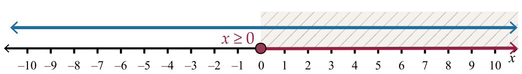
THINK: Take a minute to think about what you have learned today.
PAIR-SHARE: Discuss the question and the solution provided with one of the students sitting next to you. Are there any errors? If we run out of time in class for this activity, you can pair and share with a friend later.
Exercise: Solve $|x| > x - 2$.
Solution:

Case 1: $x \geq 0$

If $x \geq 0$, then $|x| = x$.
Therefore, we solve

$$\begin{aligned} x &> x - 2 \\ 0x &> -2 \qquad \text{Notice that } 0x > -2 \text{ is true for all values of } x. \\ 0 &> -2 \end{aligned}$$

Graphing the solution sets $x \geq 0$ and the set of all real numbers and identifying the region of overlap, we have the following:



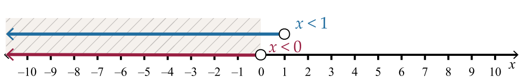
Therefore, $x \geq 0$.

Case 2: $x < 0$

If $x < 0$, then $|x| = -x$.
Therefore, we solve

$$\begin{aligned} -x &> x - 2 \\ -2x &> -2 \\ x &< 1 \end{aligned}$$

Graphing the solution sets $x < 0$ and $x < 1$ and identifying the region of overlap, we have the following:



Therefore, $x < 0$.

Putting It Together

Combining the cases, we see that the solution set consists of all real numbers. This is denoted by writing "The solution set is $\mathbb{R}$" or by writing $\{x \mid x \in \mathbb{R}\}$.

Sequences and Their Limits

A central object of interest in calculus is a sequence, an ordered list of numbers. Sequences can be used to approximate continuous processes through discrete data points, which naturally leads to the following definition.

Infinite Sequence
An **infinite sequence** of real numbers is a list of numbers in a definite order

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

where $a_i \in \mathbb{R}$ for $i \in \mathbb{N}$. We use the following notations:

$$\{a_1, \ldots, a_n, \ldots\} \text{ or } \{a_n\}_{n=1}^{\infty} \text{ or } \{a_n\}$$

In this course, we understand $\mathbb{N}$ as the set of the natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$ and note that in other courses or texts, the convention might be to include 0 in this set. We also note that a sequence can start at any integer value n, though in most cases in MATH 137, the sequences will start at $n = 1$.

1. The sequence $\{\frac{1}{n+1}\}_{n=1}^{\infty}$ is $\{\underset{a_1}{\frac{1}{2}}, \underset{a_2}{\frac{1}{3}}, \underset{a_3}{\frac{1}{4}}, \ldots, \underset{a_n}{\frac{1}{n+1}}, \ldots\}$
2. The sequence $\{(-1)^n\}_{n=1}^{\infty}$ $\{-1, 1, -1, 1, -1, 1, \ldots\}$

Alternatively, we might define the sequence **recursively**, that is, a_n is defined in terms of one or more of the previous terms $a_{n-1}, a_{n-2}, \ldots$. In this case, we also need to define one or more starting values, depending on the depth of the recursion, which is the number of previous terms a_n depends on.
For example, consider the sequence defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{1 + a_n}$$

Then we have

$a_1 = 1$

- $a_2 = 1/(1+a_1) = 1/(1+1) = 1/2$
- $a_3 = 1/(1+a_2) = 1/(1+1/2) = 2/3$
- $a_4 = 1/(1+a_3) = 1/(1+2/3) = 3/5$
- $a_5 =$ $\vdots$
- $a_6 =$

$\vdots$

- $a_{2756} =$ Computer or find a closed formula (general term)

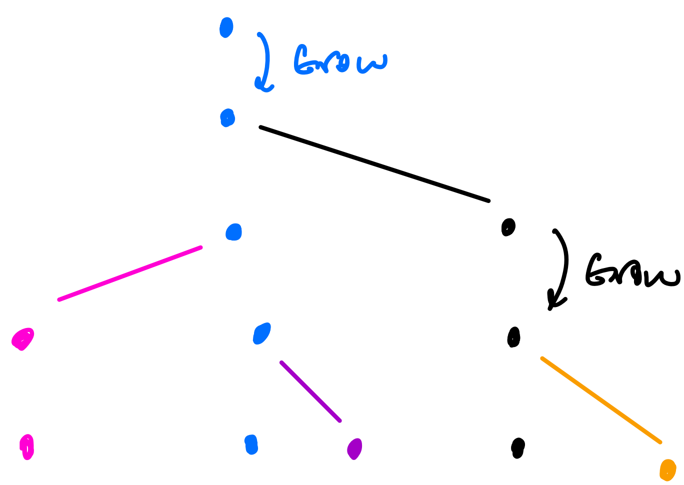
EXAMPLE (Fibonacci sequence):
The Fibonacci sequence, named after a famous Italian mathematician, is defined recursively as follows:

$$F_1 = 1 \; , \; F_2 = 1 \; , \; F_{n+2} = F_{n+1} + F_n \quad \text{for } n > 2$$

That is, the next term is the sum of the two previous terms, which makes this a recursive sequence with depth 2. From the expression, we can see it takes the following values:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

let F_n be the # of pairs of rabbits.



$F_1 = 1$

$F_2 = 1$

$F_3 = 2$

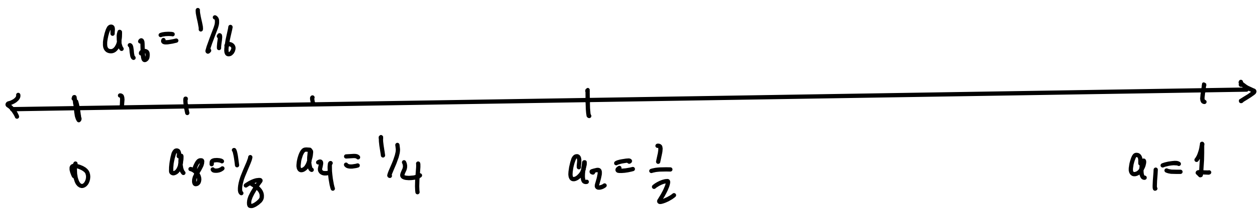
$F_4 = 3$

$F_5 = 5$

Message: Visualization might be helpful !?

It can be helpful to visualize sequences on a real line of numbers, which we illustrate in the following two examples.

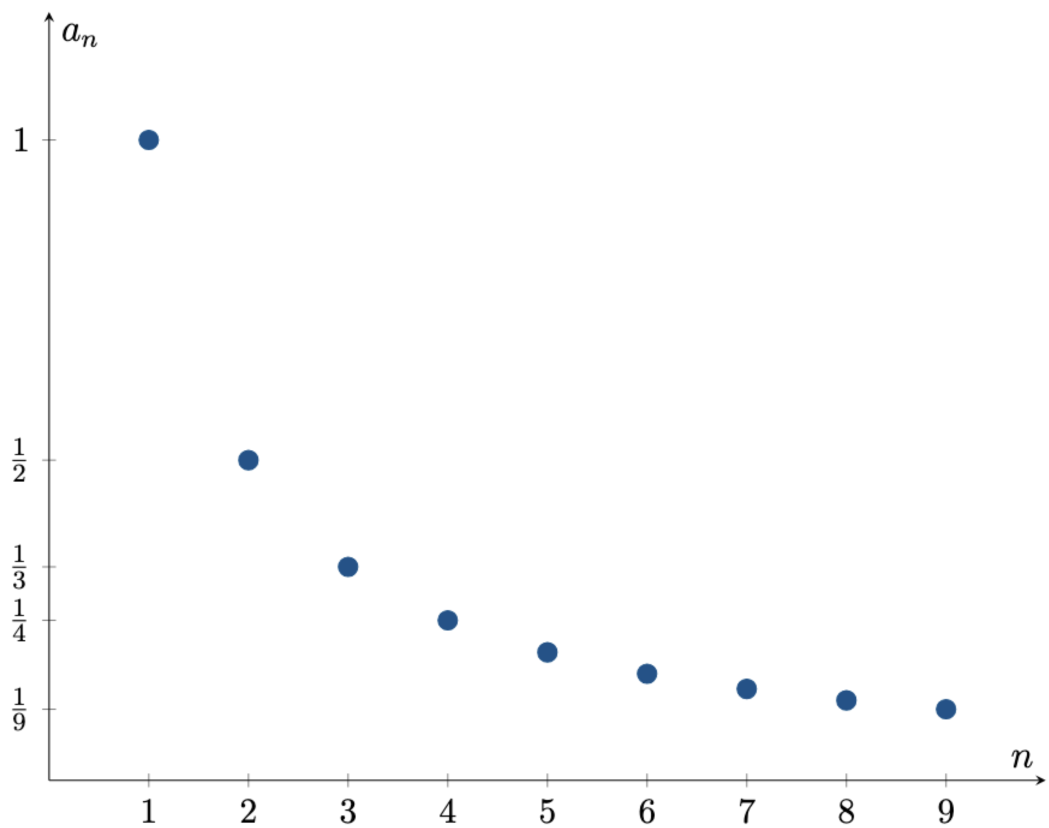
EXAMPLE: Consider the sequence $\{a_n\}$ with $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$.



An alternative way to think of a sequence is as a function mapping from the natural numbers to the real numbers, i.e., as a function f with

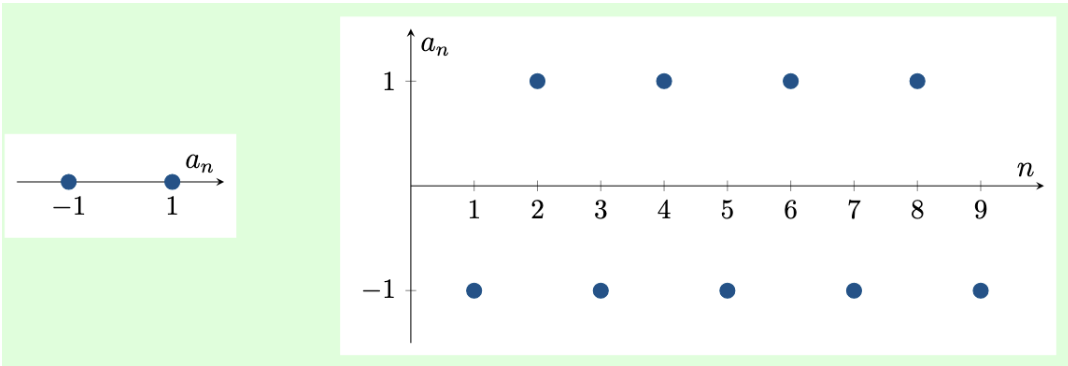
$$f : \mathbb{N} \to \mathbb{R} \quad \text{with} \quad f(n) = a_n.$$

EXAMPLE: For the sequence $\{1/n\}_{n=1}^{\infty}$, we have $a_n = 1/n$, or $f(n) = 1/n$. We can plot the function f in a 2-dimensional plot, displaying the pairs $(n, f(n))$, as depicted below.



The 2-dimensional plot is typically more informative, as we can see how the values of the sequence evolve for different n, rather than just seeing what values the sequence takes. This is particularly well-illustrated by the sequence in the next example.

EXAMPLE: Consider the sequence $\{(-1)^n\}$, which alternates between -1 and $+1$. The sequence is displayed below: The 1-dimensional plot on the left just shows two values, while the 2-dimensional plot on the right shows how the sequence is oscillating.



Definition: Subsequence
Let $\{a_n\}$ be a sequence of real numbers and $\{n_1, n_2, n_3, \ldots\}$ be a sequence of natural numbers, where $n_1 < n_2 < n_3 < \cdots$. Then the sequence

$$\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_k}, \ldots\}$$

is a **subsequence** of $\{a_n\}$.

EXAMPLE: Consider $\{a_n\}$ with $a_n = (-1)^n$. List two subsequences for $\{a_n\}$.
Solution:

$\{1, 1, 1, \ldots\}$

$\{-1, -1, -1, \ldots\}$

$\{1, -1, -1, 1, -1, 1, -1, \ldots\}$

Example: Consider $\{a_n\} = \dfrac{1}{n}$. List two subsequences for $\{a_n\}$.
Solution:

$\{1/2, 1/3, 1/4, \ldots\}$

$\{1/5, 1/7, 1/9, \ldots\}$

$\{1/4, 1/8, 1/16, \ldots\}$

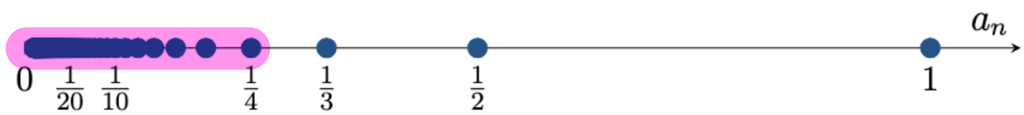
How many more?

$$\underbrace{a_1, a_2, a_3}_{\text{delete}}, a_4, a_5, \ldots\ldots$$

Definition: Tail of a Sequence Let $\{a_n\}$ be a given sequence and $k \in \mathbb{N}$. The subsequence

$$\{a_k, a_{k+1}, a_{k+2}, \ldots, a_n, \ldots\}$$

is called the **tail** of $\{a_n\}$ **with cutoff** k, and is the sequence $\{a_n\}$ with the first $k-1$ points deleted.



Limits of Sequences

Definition: Limit of a Sequence
Let $\{a_n\}$ be a sequence and $L \in \mathbb{R}$. We say that L is the **limit** of $\{a_n\}$ if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{R}$ s.t. if $n > N$
then $|a_n - L| < \varepsilon$

If such an L exists, we say that $\{a_n\}$ **converges to** L and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

If no such L exists then we say that $\{a_n\}$ **diverges**.

GeoGebra Activity: Explore Formal Definition Limit of a sequence 1

ε : Tolerance
N : Cutoff

For every tolerance $\varepsilon > 0$, there exists a cutoff N such that after the cutoff N, the tail of the sequence $\{a_n\}$ are in $(L-\varepsilon, L+\varepsilon)$

because $|a_n - L| < \varepsilon \iff L - \varepsilon < a_n < L + \varepsilon$

EXAMPLE: Consider $\{a_n\}$ with $a_n = \frac{1}{n^2}$. Intuitively, we guess the limit is $L = 0$, but how do we show this more formally?

Solution:

$\varepsilon = \frac{1}{100}$. Find N such that $|a_n - L| < \varepsilon$

$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$

$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100} \Rightarrow \frac{1}{n^2} < \frac{1}{100} \Rightarrow \frac{1}{n} < \frac{1}{10} \Rightarrow n > 10.$

If we set $N = 10$, $\forall n > N$, $|a_n - 0| < \frac{1}{100}$.

This is <u>NOT</u> a proof that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$.

Because our choice of $\varepsilon > 0$ should be arbitrary.

The general procedure of proving that L is the limit of a given sequence $\{a_n\}$:

1. Fix some given $\varepsilon > 0$.
2. In a side computation, find the bound N so that $|a_n - L| < \varepsilon$ holds for all $n > N$.
3. Then, show that for all $n > N$, the inequality $|a_n - L| < \varepsilon$ indeed holds.

EXAMPLE: Consider the sequence $\{a_n\} = \dfrac{1}{n^2}$.

Use the formal definition of the limit of a sequence to show that $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$

Solution:

Let $\varepsilon > 0$. Let $N \boxed{> \frac{1}{\sqrt{\varepsilon}}}$

such that if $n > N$

$$|a_n - L| = \left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{N^2} \quad \text{since } n > N$$

$$< \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2}$$

$$= \frac{1}{1/\varepsilon}$$

$$= \varepsilon$$

ASIDE

WANT

$$\frac{1}{N^2} < \varepsilon$$

$$\frac{1}{\varepsilon} < N^2$$

$$\frac{1}{\sqrt{\varepsilon}} < N$$

$(\varepsilon, N > 0)$

$\square$

EXAMPLE: Use the formal definition of a limit to show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Solution:

Let $\varepsilon > 0$. Let $N \boxed{> \frac{1}{\varepsilon^2}}$

such that if $n > N$ then

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \quad \text{since } n > N$$

$$< \frac{1}{\sqrt{1/\varepsilon^2}}$$

$$= \varepsilon$$

ASIDE

WANT

$$\frac{1}{\sqrt{N}} < \varepsilon$$

$$\frac{1}{\varepsilon} < \sqrt{N}$$

$$\frac{1}{\varepsilon^2} < N$$

EXAMPLE: Use the formal definition of a limit to show that $\displaystyle\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

Solution:

Let $\varepsilon > 0$. Let $N > \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right)$ s.t.

If $n > N$ then

$$\left| \frac{n}{2n+3} - \frac{1}{2} \right| = \left| \frac{2n - 2n - 3}{4n+6} \right|$$

$$= \left| \frac{-3}{4n+6} \right|$$

$$= \frac{3}{4n+6}$$

$$< \frac{3}{4N+6} \quad \text{since } n > N$$

$$< \frac{3}{4\left(\frac{1}{4}\left(\frac{3}{6} - 6\right)\right) + 6}$$

$$= \varepsilon$$

ASIDE

WANT

$$\frac{3}{4N+6} < \varepsilon$$

$$\frac{3}{\varepsilon} < 4N + 6$$

$$\frac{3}{\varepsilon} - 6 < 4N$$

$$\frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right) < N$$

$\square$

EXAMPLE: Use the formal definition of a limit to show that $\lim_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$.

Solution:

let $\varepsilon > 0$. let $N \boxed{> 7/9\varepsilon}$ s.t.

if $n > N$ then

$$\left| \frac{n^2}{3n^2+7n} - \frac{1}{3} \right| = \left| \frac{\cancel{3n^2} - \cancel{3n^2} - 7n}{9n^2+21n} \right|$$

$$= \left| \frac{-7n}{9n^2+21n} \right|$$

$$= \frac{7n}{9n^2+21n}$$

$$< \frac{7n}{9n^2}$$

$$= \frac{7}{9n}$$

$$< \frac{7}{9N} \quad \text{since } n > N$$

$$< \frac{7}{9\left(\frac{7}{9\varepsilon}\right)}$$

$$= \varepsilon$$

ASIDE

WANT

$$\frac{7}{9N} < \varepsilon$$

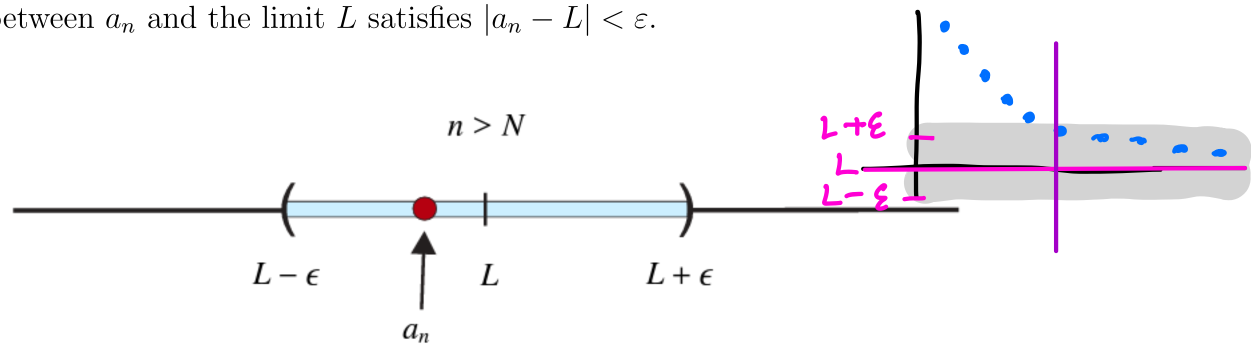
$$\frac{7}{9\varepsilon} < N$$

$\square$

Common mistakes:

- ε should be arbitrary, so your proof should always start with "Let $\varepsilon > 0$". We never fix ε! From here, your aim is to find a fixed N in terms of ε.
- Never assume $|a_n - L| < \varepsilon$. This is ultimately what we want to conclude!
- In practice, we barely use this formal definition. Do not use this definition unless you are asked to! Later, we will develop better methods for finding these limits.

When proving $\lim_{n\to\infty} a_n = L$, we are given $\varepsilon > 0$ and we try to find N so that if $n > N$, the distance between a_n and the limit L satisfies $|a_n - L| < \varepsilon$.

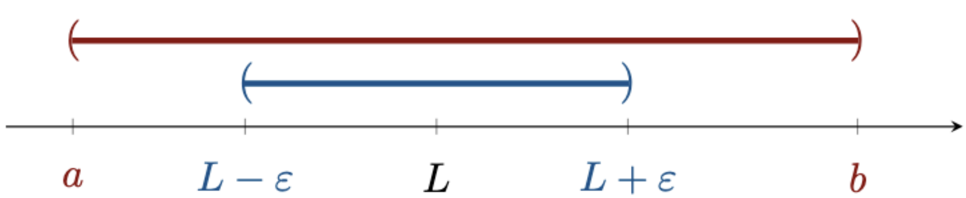


Notice that this inequality is equivalent to $a_n \in (L-\varepsilon, L+\varepsilon)$ for $n > N$. Also, the collection of $\{a_n\}$ for which $n > N$ is the tail of the sequence with cutoff N! Motivated by this, we can give an equivalent definition:

Definition: Limit of a Sequence II
Let $\{a_n\}$ be a sequence and $L \in \mathbb{R}$. Then, $\lim_{n\to\infty} a_n = L$ if

$\forall \varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$
contains a tail of $\{a_n\}$

We can generalize this even further! Since the above is true for every $\varepsilon > 0$, if we pick any open interval (a, b) containing L, then we can find a small enough ε so that $(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$. This is illustrated in the figure below. Therefore, any open interval containing L also contains a tail of $\{a_n\}$.



Let's collect all of these alternate (but equivalent) definitions together in the following theorem:

Definition: Equivalent Definitions of the Limit of a Sequence
The following are equivalent: (TFAE)

1. $\lim_{n \to \infty} a_n = L$
2. $\forall \varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$
3. $\forall \varepsilon > 0$, the number of elements of $\{a_n\}$ that don't lie in $(L-\varepsilon, L+\varepsilon)$ is finite.
4. Every interval (a,b) containing L contains a tail of $\{a_n\}$.
5. Given every interval (a,b) containing L, the number of terms of $\{a_n\}$ that don't lie in (a,b) is finite.

6. $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t. if $n > N$ then $|a_n - L| < \varepsilon$.

Uniqueness of Limits

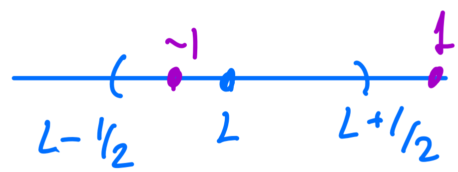
EXAMPLE: Consider $\{(-1)^n\} = \{-1, 1, -1, 1, \cdots\}$. It takes the values $+1$ and -1 infinitely often. Is it possible that both $+1$ and -1 are limits of this sequence?

Assume -1 is a limit.
Consider the interval $(-2, 0)$. This contains -1.
But it misses all the 1's (∞-many 1's) from this sequence. This is a contradiction.

This raises the question: Does the sequence $\{(-1)^n\}$ have a limit at all? We can prove it does not!

PROOF:

Suppose that $\lim_{n\to\infty} (-1)^n = L \in \mathbb{R}$.
Then $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t. if $n > N$ then
$|(-1)^n - L| < \varepsilon$.
Choose $\varepsilon = \frac{1}{2}$. Consider the interval $(L - \frac{1}{2}, L + \frac{1}{2})$.
This interval has length 1. But it must contain ∞-many 1's and -1's by def. But -1 and 1 are 2 units apart.
And that is a contradiction.



We can use a similar argument that we used in this example to prove formally that the limit of a sequence, if it exists, is unique.

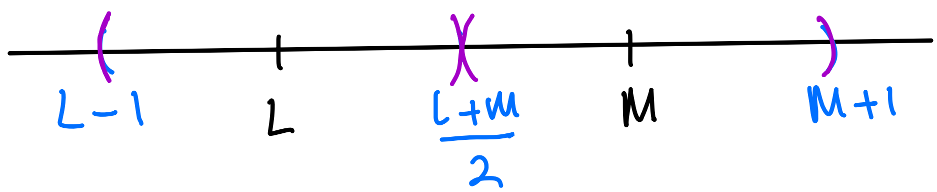
Theorem: Uniqueness of Sequence Limit
Let $\{a_n\}$ be a sequence.

If $\{a_n\}$ has a limit L,

then the limit L is unique.

PROOF:

Suppose $\{a_n\}$ has two limits L and M with $L \neq M$ and (wlog) $L < M$.



Since both L and M are limits, both intervals $(L-1, \frac{L+M}{2})$ and $(\frac{L+M}{2}, M+1)$ should contain a tail of $\{a_n\}$.

Observe that these two intervals are disjoint.

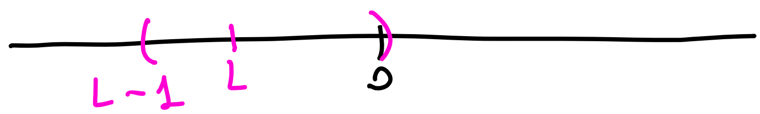
Since L is limit, $(L-1, \frac{L+M}{2})$ misses finitely many terms but M is a limit too and $(\frac{L+M}{2}, M+1)$ contains a tail too. This is a contradiction.

REMARKS

If we know that a sequence $\{a_n\}$ can only take certain values, we can infer something about the limit of the sequence, so it exists.

- If $a_n \geq 0$ for all n then $\{a_n\}$ cannot converge to a negative number! Why?

Assume that $L < 0$. Consider the interval $(L-1, 0)$ contains L but cannot contain any terms since $a_n \geq 0$



- More generally: if $\alpha \leq a_n \leq \beta$ $\forall n$, and $\lim\limits_{n\to\infty} a_n = L$ then $\alpha \leq L \leq \beta$.

- But can we also make the same statement using strict inequalities? That is, can we infer from $a_n > 0$ for all $n \in \mathbb{N}$ that then also $\lim\limits_{n\to\infty} a_n = L$ satisfies $L > 0$?

No! $a_n = \frac{1}{n} > 0$ $\forall n$ but $\lim\limits_{n\to\infty} \frac{1}{n} = 0 \not> 0$

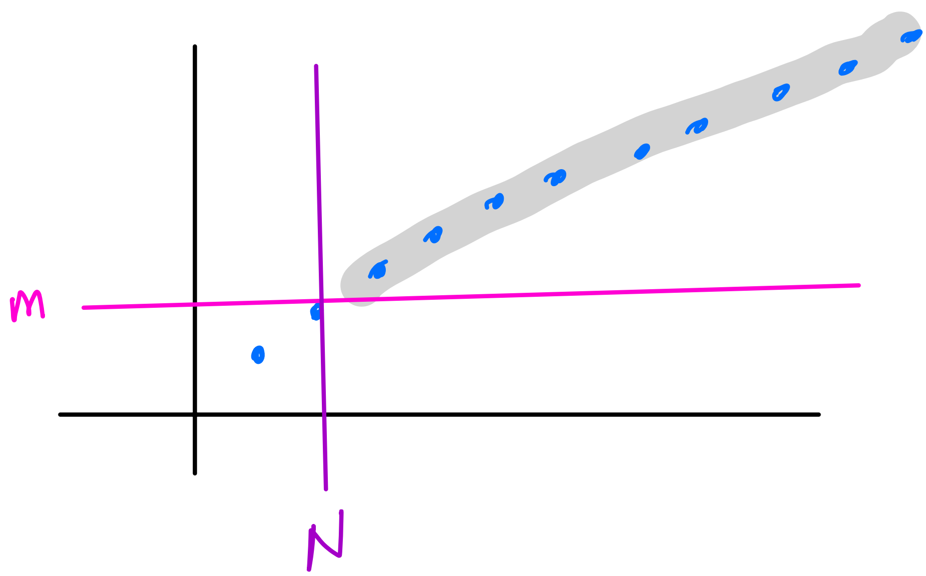
Divergence to ∞.

$\{a_n\}$ diverges to ∞, or $\lim_{n\to\infty} a_n = \infty$

if $\forall m > 0 \quad \exists N \in \mathbb{R}$ s.t.

$$a_n > m \quad \forall n > N$$

Equivalently, any interval of the form (m, ∞) contains a tail of $\{a_n\}$



Ex: $a_n = n$, $a_n = 2n$, $a_n = n^3$, $a_n = n+1$

T/F: $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$ then

$$\lim_{n\to\infty} (a_n - b_n) = 0$$

Counter ex: $a_n = 2n$, $b_n = n$

END OF WEEK 2 MATERIAL