Week 3:

Practice Problems

1. Prove (using the formal definition of a limit) the following

   (a) $\lim_{n\to\infty} \ln(n+10) = \infty$

   (b) $\lim_{n\to\infty} 1 - 2^n = -\infty$

2. Determine if the following statements are true or false. If true, argue your case mathematically, if false, provide a counterexample.

   (a) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty}(a_n + b_n) = \infty$

   (b) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty}(a_n - b_n) = 0$

   (c) If $a_n \le b_n \le c_n$ for all n, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = M$, then $\lim_{n\to\infty} b_n = K$ with $L \le K \le M$.

3. Prove (using the formal definition of a limit) that if $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. [Hint: Consider the cases $L = 0$ and $L \neq 0$ separately.]

4. Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = \infty$. Which of the following must be true?

   (a) Every term of $\{a_n\}$ is positive.

   (b) No subsequence of $\{a_n\}$ converges.

   (c) $\lim_{n\to\infty} 3n^2 a_n = \infty$

   (d) $\lim_{n\to\infty} \frac{a_n}{n} = \infty$

   (e) $\lim_{n\to\infty} -a_n = -\infty$

   (f) None of the above

5. Which cutoff values of N will guarantee that $a_n = 2 + n^2$ is greater than M, assuming $M > 5$ (select all that apply):

   (a) $N = \sqrt{M}$

   (b) $N = \sqrt{M - 5}$

   (c) $N = \frac{1}{M}$

   (d) $N = M$

   (e) $N = M^2 + 2$

   (f) None of the above

6. Assuming $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ where $a_n \geq 0$ and $b_n \geq 0$, determine

$$\lim_{n\to\infty} \left(a_n \sin(n) + b_n \cos(n)\right).$$

7. Let's examine how absolute values and limits interact.

   (a) The statement
   $$\text{If } \lim_{n\to\infty} |a_n| = |L| \text{ then } \lim_{n\to\infty} a_n = L$$
   is false in general. Provide a counter-example.

   (b) The statement
   $$\text{If } \lim_{n\to\infty} a_n = L \text{ then } \lim_{n\to\infty} |a_n| = |L|$$
   is true. Show this using the definition of limits. Hint: $||a| - |b|| \leq |a - b|$.

   (Even though it is not necessary for this question, you should be able to show that the hint is true.)

   (c) Is the statement
   $$\text{If } \lim_{n\to\infty} |a_n| = 0 \text{ then } \lim_{n\to\infty} a_n = 0$$
   true? If so, argue why, if not, provide a counterexample.

8. Compute the following limits using any method.

   (a) $\lim_{n\to\infty} \dfrac{\sin(n^2)}{n^2}$

   (b) $\lim_{n\to\infty} \dfrac{3n - (-1)^n}{n}$

   (c) $\lim_{n\to\infty} \dfrac{n!}{n^n}$ $\left[\text{Hint: Write } n! = 1 \cdot 2 \cdot 3 \ldots n \text{ and } n^n = n \cdot n \cdot n \ldots n \text{ and use the fact that } 0 < \dfrac{n!}{n^n}.\right]$

   (d) $\lim_{n\to\infty} \dfrac{3n^3 + 2n^2 - n - 1}{n^3 + n + 3}$

   (e) $\lim_{n\to\infty} \dfrac{n^2 + 2n - 6}{n + 1}$

9. Consider the sequence $\{a_n\}$ defined by $a_n = 3 - \dfrac{2}{n}$ when $n \geq 1$. Which of the following are true?

   (a) 4 is an upper bound of $\{a_n\}$.

   (b) 2.99 is an upper bound of $\{a_n\}$.

   (c) $\{a_n\}$ is a decreasing sequence.

   (d) 1 is a lower bound of $\{a_n\}$.

   (e) 1 is the greatest lower bound of $\{a_n\}$.

   (f) $\dfrac{14}{5}$ is the greatest lower bound for some tail of $\{a_n\}$.

   (g) None of the above.

10. Which of the following sequences are monotone?

(a) $\{(-1)^n + 3^n\}$

(b) $\{n^2 - 4n\}$

(c) $\left\{\sin(\frac{\pi}{4n})\right\}$

(d) $\left\{\frac{1}{n} + \frac{1}{2n+2}\right\}$

(e) None of the above.

11. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \dfrac{7 + a_n}{6}$ for $n \geq 1$.

(a) By induction, show that $\{a_n\}$ is an increasing sequence that is bounded above by 2.

(b) Prove that this sequence is convergent and find $\lim_{n\to\infty} a_n$.

12. Define a sequence $\{a_n\}$ by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) By induction, show that $\{a_n\}$ is an increasing sequence that is bounded above by 3.

(b) Prove that this sequence is convergent and find $\lim_{n\to\infty} a_n$.