Implementing Numerical Methods to Explore and Evaluate ODE and Integral Equations with Known Exact Solutions

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Abstract

In this project, I have implemented numerical methods to solve an Ordinary Differential Equation (ODE) and an integral equations which have exact solution.

1 Introduction and Background

For ODE, I solved the differential equation first for a LRC circuit (which has exact solution and behaves like a harmonic oscillator when R=0) by implementing our own 2nd order differential equation solver using the well-established Runge-Kutta 4th Order Method and then comparing it to the solution given by the SciPy's integrate.solve ivp function. Then I made the plots for the time evolution of current and charge for the LRC circuit for different condition in order to mimic underdamped, critically damped and overdamped situation.

The time evolution of current in the LRC circuit is given by in general,

$$V_R = IR \tag{1}$$

$$V_L = L\frac{dI}{dt} \tag{2}$$

$$V_C = \frac{Q}{C} \tag{3}$$

$$V_S = V_R + V_L + V_C \tag{4}$$

where VS is the source voltage.

Again, for the RLC circuit with $V = 0$ (no external driving voltage) and $R = 0$ (no resistance), the circuit becomes a simple LC circuit. The circuit is described by a charge q on the capacitor and a current i through the inductor.

The governing differential equation in this case, derived from Kirchhoff's voltage law, is:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \tag{5}$$

This equation represents a simple harmonic oscillator with no damping or unforced damped harmonic oscillator which is not similar to an RLC circuit with constant voltage.

To solve this equation, we use the ansatz

$$q(t) = A\cos(\omega t + \phi)$$

where A is the amplitude, ω is the angular frequency, and ϕ is the phase constant.

Substituting this into the differential equation:

$$-LA\omega^2\cos(\omega t + \phi) + \frac{A}{C}\cos(\omega t + \phi) = 0 \tag{6}$$

This reduces to:

$$-LA\omega^2 + \frac{A}{C} = 0 \tag{7}$$

From which we can derive:

$$\omega = \frac{1}{\sqrt{LC}} \tag{8}$$

Thus, the solution is:

$$q(t) = A\cos\left(\frac{t}{\sqrt{LC}} + \phi\right) \tag{9}$$

The constants A and ϕ can be determined from initial conditions. For example, given an initial charge q_0 and an initial current i_0, you can determine A and ϕ.

Given the relation

$$i(t) = \frac{dq(t)}{dt}$$

the current $i(t)$ is:

$$i(t) = -A\frac{1}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}} + \phi\right) \tag{10}$$

This represents the simple harmonic motion of the charge and current in the LC circuit. Without resistance, the oscillations will continue indefinitely because there's no mechanism to dissipate energy.

This second-order differential equation has three types of solutions, depending on the discriminant, $R^2 - 4L/C$:

1. **Underdamped** ($R^2 < 4L/C$): The system oscillates with an exponentially decaying envelope.
2. **Critically Damped** ($R^2 = 4L/C$): The system returns to equilibrium in the shortest time without oscillating.
3. **Overdamped** ($R^2 > 4L/C$): The system returns to equilibrium without oscillating.

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For the second part of the project, which is an integral problem, I picked up Coulomb's law to find the electric field for symmetric one-dimensional charge distribution. I employed integrator using the riemann sum, the trapezoidal rule, and simpson's rule and then compared the result with SciPy implementation.

The electric field due to a charged distribution is given by Coulomb's law. For a point charge, the electric field is given by:

$$E = \frac{kQ}{r^2} \tag{11}$$

Where:

- E is the electric field
- k is the Coulomb constant, $9 \times 10^9\,\mathrm{N.m^2/C^2}$
- Q is the charge
- r is the distance from the point charge.

When considering extended charge distributions, the electric field at a point is the superposition of the electric fields due to infinitesimal charge elements in the distribution. For a one-dimensional charge distribution described by a linear charge density $\lambda(x)$, the electric field at a distance r from a point x on the distribution is given by:

$$dE = \frac{k\lambda(x)dx}{r^2} \tag{12}$$

The total electric field at a given location due to the entire distribution is found by integrating this expression over the distribution.

2 Method and Implementation

Libraries such as `numpy` for numerical operations, `matplotlib` for plotting, and `scipy.integrate` for numerical integration were imported.

System Parameters

L = 1 Henry
C = 1 Farad

Tested resistance values were: R = 0.5, 1 and 2 Ohm. and a special case of R = 0 ohm representing a simple harmonic oscillator (SHO). Based on Kirchhoff's voltage law, a system of first-order ODEs describing the RLC circuit was established.

$$\frac{dq}{dt} = i \tag{13}$$

$$\frac{di}{dt} = \frac{-Ri - \frac{q}{C}}{L} \tag{14}$$

Well-known solutions for overdamped, critically damped, and underdamped cases were derived. A special solution for the SHO was also included.

Numerical Methods Implemented

- Forward Euler
- Runge-Kutta (4th Order)
- SciPy's solve_ivp

For each resistance value, charge and current against time were plotted for all solutions.

For the second part, in order to compute the electric field due to the charge distributions, we used numerical integration methods. Specifically, we implemented: Riemann Sum, Trapezoidal rule and Simpson's rule. For each method, we compared our custom implementations to established implementations in the SciPy library to validate their accuracy.

For the uniform charge distribution, we assumed that the charge density λ is constant over the interval $[a, b]$. For the linear charge distribution, we assumed that the charge density λ varies linearly over the interval $[a, b]$. The plots generated for both distributions showcase the electric field's magnitude as a function of distance from the distribution.

3 Result and Discussion

With the absence of resistance, the system showed pure sinusoidal oscillations, as shown in Fig.1, the amplitude remained constant over time. This validated our implementation. For underdamped (R=0.5 Ohm), The circuit exhibited oscillations that gradually decayed over time, that is, amplitude diminishing over time. Most methods employed showed almost similar behavior. For Critically Damped (R =1.0 ohm) the system quickly returned to equilibrium without oscillating (fig.3). For overdamped, The plots showed a slower return (Fig.4) to equilibrium compared to the critically damped case, with no oscillations. It is evident from Fig.3 and 4 that SciPy showed significantly different behavior than the other approach. It might be because of the adaptive method it employs, refining the interval limit.

For part-2, the electric field decreases as the distance from the charge distribution increases, which is consistent with the inverse-square nature of Coulomb's law (Fig.5). omparing our custom integration methods with the SciPy methods, we found that they align closely, validating our implementations.

Conclusion

In most scenarios, while the Runge-Kutta method and solve_ivp closely matched the analytical solutions, the Forward Euler method showed significant deviations, especially in oscillatory cases. This

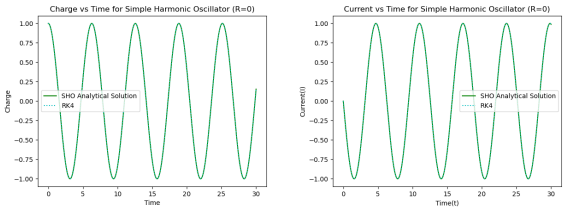
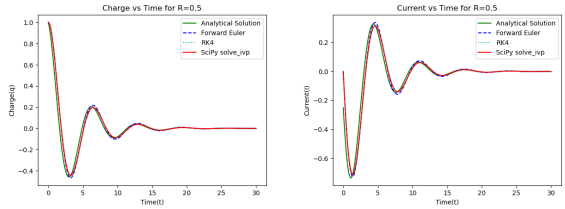


Figure 1: Enter Caption



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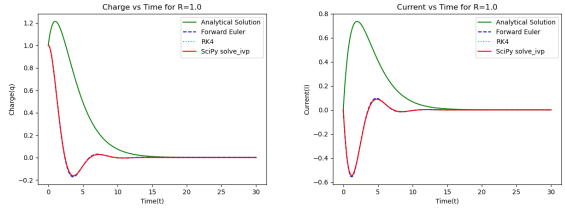

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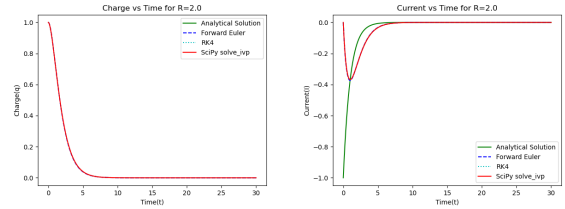

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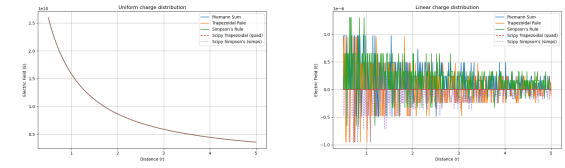

Figure 5: Enter Caption

disparity accentuates the importance of choosing an appropriate numerical method for simulating dynamic systems.The electric field due to one-dimensional charge distributions demonstrates the effectiveness of methods like Riemann sums, the trapezoidal rule, and Simpson's rule.