Revision

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Inner product space

Ex $f(t) = 3t - 5, \quad g(t) = t^2$

$$\langle f\ g\rangle = \int_0^1 f(t)\,g(t)\,dt \quad \longleftarrow$$

(a) $\langle f\ g\rangle$

(b) $\|f\|$

(c) $\|g\|$

(d) angle between f & g.

Sol (a) $\langle f\ g\rangle = \int_0^1 f(t)\,g(t)\,dt = \int_0^1 (3t-5)\,t^2\,dt$

$$= \int_0^1 3t^3 - 5t^2\,dt = \left[\frac{3t^4}{4} - \frac{5t^3}{3}\right]_0^1$$

$$= \left(\frac{3}{4} - \frac{5}{3}\right) - (0) = \frac{-11}{12} \text{ Ans}$$

(b) $\|f\| = \sqrt{\langle f\ f\rangle}$

$$\|f\|^2 = \langle f\ f\rangle = \int_0^1 f(t)\,f(t)\,dt = \int_0^1 (3t-5)(3t-5)\,dt$$

$$= \int_0^1 9t^2 + 25 - 30t\,dt = 13$$

$\|f\| = \sqrt{13}$

(c) $\|g\| = \sqrt{\langle g\ g\rangle}$

$$\|g\|^2 = \langle g\ g\rangle = \int_0^1 g(t)\,g(t)\,dt = \int_0^1 t^2 \cdot t^2\,dt$$

$$= \int_0^1 t^4\,dt$$

$$= \left[\frac{t^5}{5}\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}$$

$$\|g\| = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

(d) $\cos\theta = \dfrac{\langle f \; g \rangle}{\|f\| \; \|g\|}$

$$\boxed{\begin{array}{l} \cos\theta = \dfrac{u \cdot v}{|u|\,|v|} \\ \cos\theta = \dfrac{\langle u \; v \rangle}{\|u\| \; \|v\|} \end{array}}$$

$$= \frac{-11/12}{\sqrt{13}\;\sqrt{5}/5} = \frac{-55}{12\sqrt{65}} \quad \text{Ans}$$

Ex $u = (2, 3, 5)$, $v = (1, -4, 3)$

find the angle 'θ' between u & v.

Sol $\cos\theta = \dfrac{\langle u \quad v \rangle}{\|u\| \; \|v\|}$

$$\langle u \quad v \rangle = (2)(1) + (3)(-4) + 5(3)$$
$$= 2 - 12 + 15 = 5$$

$$\|u\| = \sqrt{(2)^2 + (3)^2 + (5)^2} = \sqrt{38}$$

$$\|v\| = \sqrt{(1)^2 + (-4)^2 + 3^2} = \sqrt{26}$$

$$\cos\theta = \frac{5}{\sqrt{38}\;\sqrt{26}} \Rightarrow \theta = \cos^{-1}\frac{5}{\sqrt{38}\;\sqrt{26}} \quad \text{Ans}$$

Type 2 : Transformation of Matrices

a) find Associated Matrix

$T : V \longrightarrow W$

1) write standard basis of domain (V)

2) find Image of St. basis

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3) write coeff. Matrix $= A$

4) Associated Matrix $= A' = [T]$

Ex:- $T : \underbrace{R^2} \to R^3$

$T(x, y) = (x+y, 2x-y, y)$

$B = \{e_1, e_2\}$ be a basis of R^2

where $e_1 = (1, 0)$, e_2 $(0, 1)$

$T(e_1) = T(1, 0) = (1, 2, 0)$

$T(e_2) = T(0, 1) = (1, -1, 1)$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Associated Matrix $= [T] = A' = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$ Ans

Ex:- $T : R^3 \to R^2$

$T(x, y, z) = (x+z, y-z)$

$B = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

find Transformation matrix of B

Sol $[T] = [T : B] = A'$

$T(e_1) = T(1, 0, 0) = (1, 0)$

$T(e_2) = T(0, 1, 0) = (0, 1)$

$T(e_3) = T(0, 0, 1) = (1, -1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$[T] = [T : B] = A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Note: Order of transformation matrix = (dim of Codomain) x (dim of domain)

In above example $T: R^3 \to R^2$

$$O(T:B) = 2 \times 3$$

Type:- $T: V \to W$

$B_1 = \{v_1, v_2 \dots v_n\}$ is a basis of V

$B_2 = \{w_1, w_2 \dots w_m\}$ is a basis of W

$[T: B_1, B_2] = ?$

1) find $T(v_1), T(v_2) \dots T(v_n)$
2) write L.C of $w_1, w_2 \dots w_m$
3) find $T(v_1) = \alpha w_1 + \beta w_2 \dots \gamma w_m$

$T(v_2) = \quad ''$

$T(v_n) \dots \quad \dots$

A = coeff matrix

$[T: B_1, B_2] = A'$

Ex:- $T: R^2 \to R^2$

$T(x, y) = (2x - 3y, x + 4y)$

$B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,3), (2,5)\}$

find $[T: B_1, B_2]$

$B_1 = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}\}$

$B_2 = \{\underset{w_1}{(1,3)}, \underset{w_2}{(2,5)}\}$

Sol $T(v_1) = T(1,0) = (2, 1)$

$T(v_2) = T(0,1) = (-3, 4)$

$(x, y) = \alpha w_1 + \beta w_2 = \alpha(1,3) + \beta(2,5)$

$= (\alpha + 2\beta, 3\alpha + 5\beta)$

$\alpha + 2\beta = x$

$3\alpha + 6\beta = 3x$

$3\alpha + 5\beta = y$

$$\left.\begin{aligned} \alpha + 2\beta &= x \\ 3\alpha + 5\beta &= y \end{aligned}\right\}$$

$$\begin{aligned} 3\alpha + 6\beta &= 3x \\ 3\alpha + 5\beta &= y \\ \hline \beta &= 3x - y \end{aligned}$$

$$\alpha = x - 2\beta = x - 2(3x - y)$$
$$= -5x + 2y$$

$$(x, y) = \alpha w_1 + \beta w_2$$

$$(x, y) = (-5x + 2y) w_1 + (3x - y) w_2$$

$$T(v_1) = (2, 1) = -8 w_1 + 5 w_2$$
$$T(v_2) = (-3, 4) = 23 w_1 - 13 w_2$$

$$A = \begin{bmatrix} -8 & 5 \\ 23 & -13 \end{bmatrix}$$

$$[T : B_1, B_2] = A' = \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix} \quad \underline{\text{Ans}}$$

Note 1 : Let $T : V(F) \to V(F)$
and only B_1 is given
then B_2 is also same as B_1

$[T : B_1]$

Note 2 $[V : B] = [\alpha \quad \beta]^T = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

In the above example

$$\alpha = -5x + 2y$$
$$\beta = 3x - y$$

$$[V : B] = \begin{bmatrix} -5x + 2y \\ 3x - y \end{bmatrix}$$

Note 3. $[T(v) : B] = [T : B][v : B]$

$$= \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix} \begin{bmatrix} -5x+2y \\ 3x-y \end{bmatrix}$$

$$= \begin{bmatrix} 40x - 16y + 69x - 23y \\ -25x + 10y - 39x + 13y \end{bmatrix}$$

$$= \begin{bmatrix} 109x - 39y \\ -64x + 23y \end{bmatrix}$$

Ex:- $T : R^2 \to R^2$

$T(x, y) = (4x - 2y, 2x + y)$

$B = \{(1,1), (-1,0)\}$

$[T : B]$

Sol

$T(v_1) = T(1,1) = (2, 3)$

$T(v_2) = T(-1,0) = (-4, -2)$

$(x, y) = \alpha w_1 + \beta w_2 = \alpha(1,1) + \beta(-1,0)$

$(x, y) = (\alpha - \beta, \alpha)$

$\alpha = y$

$\alpha - \beta = x \Rightarrow \beta = y - x$

$(x, y) = \alpha w_1 + \beta w_2$

$= y w_1 + (y - x) w_2$

$T(v_1) = (2,3) = 3w_1 + (3-2)w_2 = 3w_1 + w_2$

$T(v_2) = (-4,-2) = -2w_1 + (-2+4)w_2 = -2w_1 + 2w_2$

$A = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ $[T : B] = A' = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$ Ans

$B_1 = \{(1,1), (-1,0)\}$
$v_1 \quad v_2$

$B_2 = \{(1,1), (-1,0)\}$
$w_1 \quad w_2$

Note

$[V : B] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y \\ y-x \end{bmatrix}$