Development of Quantum Mechanics

14.1 INTRODUCTION

Newton's laws describe the motion of particles in classical mechanics and Maxwell's equations describe the electromagnetic fields in classical electromagnetism. The classical mechanics correctly explains the motion of celestial bodies like planets, stars, macroscopic and microscopic terrestrial bodies moving with non-relativistic speeds. However, classical theory does not hold in the region of atomic dimensions, i.e. it cannot explain the non-relativistic motion of electrons, protons etc. Classical theory could not explain the stability of atoms, spectral distribution of blackbody radiation, the origin of discrete spectra of atoms, etc. Also, classical mechanics could not explain a large number of observed phenomena like photoelectric effect, Compton effect, Raman effect, etc. So, the insufficiency of classical mechanics led to the development of quantum mechanics. Quantum mechanics is the description of motion and interaction of particles at the small scales where the discrete nature of the physical world becomes important. The quantum mechanics for the atomic system led to the explanation of discrete energy levels as well as the postulation of different quantum numbers. *Niels Bohr* had a large influence on the development of quantum mechanics through his so called Copenhagen Interpretation, a philosophical construct that was formulated to provide a fundamental framework for understanding the implicit assumptions, limitations, and applicability of the theory of quantum mechanics.

The development of quantum mechanics took place in two stages. The first stage began with *Max Planck's hypothesis* according to which the radiation is emitted or absorbed by matter in discrete packets or quanta of energy. This energy is equal to $h\nu$, where h is Planck's constant and ν is the frequency of radiation. This hypothesis led to a theory which was not completely satisfactory being a mixture of classical and non-classical concepts. The second stage of quantum mechanics began in 1925 along with two points of views. For example, *matrix mechanics* was introduced by *Heisenberg*, in which only observed quantities like frequencies and intensities of spectral lines are taken into account and unobserved quantities like positions, velocities, etc. in electronic orbits are omitted. Another form of quantum mechanics is called *wave mechanics*, whose theory was developed by *Schroedinger* in 1926. In this mechanics, concepts of classical wave theory and deBroglie's wave particle relationship are combined with each other. With the application of quantum mechanics, several problems of atomic physics have been solved. However, this mechanics also has certain limitations. Therefore, a more complete theory of particles called quantum field theory has been accepted since 1947. In order to understand the development of wave mechanics, we begin with the blackbody radiation.

14.2 BLACKBODY RADIATION: SPECTRAL DISTRIBUTION

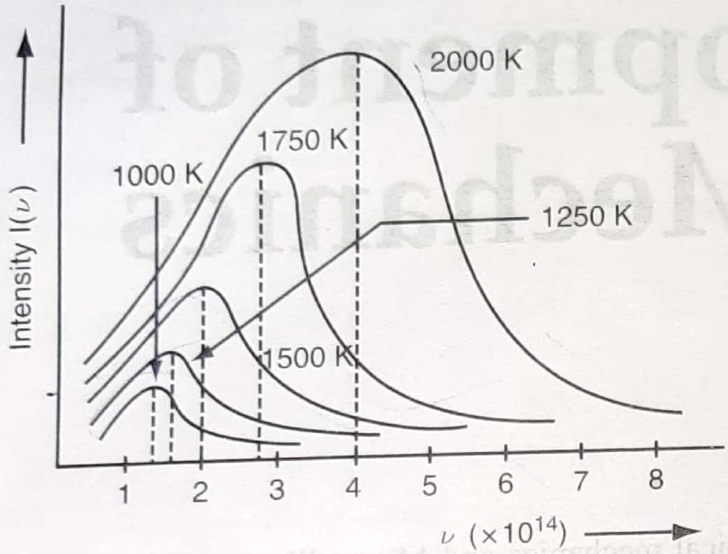

Fig. 14.1

A body that completely absorbs radiations of all wavelengths incident on it is referred to as a blackbody. When such a body is heated, it emits radiations which we call as blackbody radiations. A cavity made out of a hollow container of any material (iron or copper) with a narrow opening and painted with lampblack in the inside portion gives a close approximation to a perfectly blackbody. When any radiation falls on this hole, it enters the cavity, gets reflected by the wall of the cavity and eventually gets absorbed. Now if we heat the container at various temperatures, it will emit radiations of all the frequencies (or wavelengths). So, the emitted radiation from a blackbody is a continuous spectrum, i.e. it contains radiation of all the frequencies.

Let the intensity of emitted radiation be $I(\nu)\,d\nu$ between the frequencies ν and $\nu+d\nu$. The experimental measurements of intensity $I(\nu)$ with different ν is shown in Fig. 14.1 for different values of temperature T. These plots show that

(i) The distribution of frequencies is a function of temperature of the blackbody.

(ii) With the increase in temperature, the total amount of emitted radiation $I(\nu)\,d\nu$ increases.

(iii) The position of the maximum peak shifts toward higher frequencies with increasing equilibrium temperature.

The classical electromagnetic theory or wave theory together with classical thermodynamics does not explain the characteristics of the blackbody spectrum. However, Planck's hypothesis can explain these characteristics together with the use of classical thermodynamics.

Classical wave theory says that the electromagnetic radiation inside the cavity of the blackbody at an equilibrium temperature T forms the standing waves and the number of standing waves (possible modes) that can fit in the cavity depends on the wavelength. The number of possible modes in the cavity is large if the wavelength is small. However, for large wavelengths the number of possible modes is small. According to Rayleigh and Jeans, this increase in the number of modes is proportional to $1/\lambda^2$ or ν^2 and also each of the standing waves must be assigned an average kinetic energy kT, where k is the Boltzmann constant. This leads to the following Rayleigh-Jeans law (details discussed later)

$$I(\nu)d\nu = \frac{8\pi\nu^2}{c^3}kT d\nu$$

This relation shows that $I(\nu)$ is proportional to the square of ν. The corresponding plot is shown in Fig. 14.2. It is clear from the figure that the experimental data does not agree with the theory; the agreement is good only for smaller values of ν. The disagreement at high frequencies, i.e. in the UV region, is called ultraviolet catastrophe. Thus, the spectral distribution of a blackbody could not be explained on the basis of classical theory. This difficulty was resolved by Planck in 1900, when he stated that by assuming electromagnetic radiation to be emitted or absorbed in bundles of size $h\nu$, one could correctly predict the spectrum of blackbody radiation. As mentioned earlier, this bundle of energy is called a quantum. The quanta of high frequencies have high energies and those of low frequencies have low energies. Thus, the atoms and molecules in the cavity will emit radiation only if they have energy in the excess of $h\nu$. For low frequencies ν, there will be a large number of atoms and molecules that might have this excess energy. Since the bundles become quite bigger for higher frequencies ν, the number of atoms or molecules having energies in the excess of $h\nu$ decreases. It means for large ν, the intensity $I(\nu)$ does not increase rather decreases.

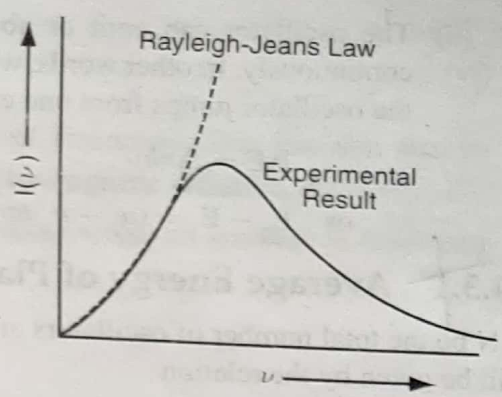

Fig. 14.2

For the explanation of blackbody radiation, Planck made a use of the Maxwell-Boltzmann distribution. According to this distribution, the number of molecules N_n with energy E is given by

$$N_n = N_0 e^{-E/kT}$$

In the above expression, N_0 refers to the number of molecules in a system in equilibrium at temperature T. Planck combined the expression of N_n with his quantum hypothesis $E = nh\nu$ and calculated the mean energy. Finally he arrived at the following expression for the distribution of the maximum intensity of radiation in the spectrum of blackbody.

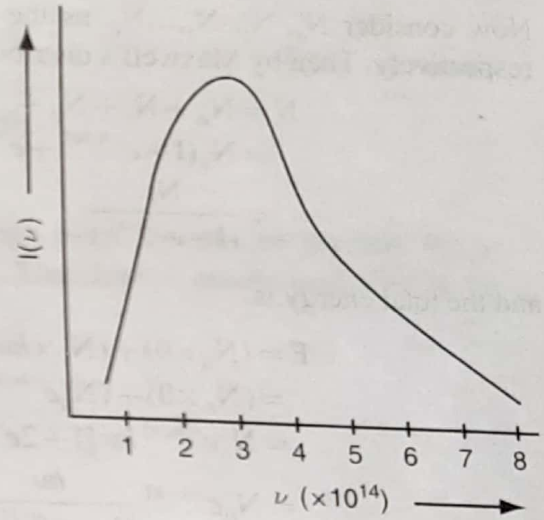

Fig. 14.3

$$I(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{(h\nu/kT)} - 1}$$

This expression is referred to as Planck's radiation law. This theoretical formula fits very well with the experimental data for the entire wavelength, as shown in Fig. 14.3. Thus Planck's quantum theory was able to interpret fully different characteristics of blackbody radiation which classical theory could not.

14.3 PLANCK'S QUANTUM HYPOTHESIS

Max Planck in 1900 introduced the quantum theory of radiation to explain the distribution of energy in the spectrum of *blackbody* radiation. He assumed that the atoms of the walls of a blackbody behave as oscillators and each has a characteristic frequency of oscillation. He made the following two revolutionary assumptions about the atomic oscillator.

(i) An oscillator cannot have any arbitrary value of energy but can have only discrete energies as per the following relation

$$E = nh\nu$$

where $n = 0,1,2,\ldots$, ν and h are known as frequency of oscillation and Planck's constant ($=6.62 \times 10^{-34}$ J sec), respectively. This relation shows that the energy of the oscillation is quantised.

(ii) The oscillator can emit or absorb energy only in the form of packets of energy ($h\nu$) but not continuously. In other words, we can say that the emission or absorption of energy occurs only when the oscillator jumps from one energy state to another along with the energy difference given by

$$\Delta E = \Delta n h\nu$$

or $$E_2 - E_1 = (n_2 - n_1)h\nu$$

14.3.1 Average Energy of Planck's Oscillators

If N be the total number of oscillators and E as the total energy of these oscillators, then the average energy will be given by the relation

$$\bar{E} = \frac{E}{N} \qquad \text{(i)}$$

Now consider N_0, N_1, N_2,...N_n, as the number of oscillators having energy values 0, $h\nu$, $2h\nu$, ..., $nh\nu$, respectively. Then by Maxwell's distribution formula $N_n = N_0 e^{-nh\nu/kT}$, we have

$$\begin{aligned} N &= N_0 + N_1 + N_2 + \dots \\ &= N_0(1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + \dots) \\ &= \frac{N_0}{(1 - e^{-h\nu/kT})} \end{aligned} \qquad \text{(ii)}$$

and the total energy is

$$\begin{aligned} E &= (N_0 \times 0) + (N_1 \times h\nu) + (N_2 \times 2h\nu) + \dots \\ &= (N_0 \times 0) + (N_0 e^{-h\nu/kT} \times h\nu) + (N_0 e^{-2h\nu/kT} \times 2h\nu) + \dots \\ &= N_0 e^{-h\nu/kT} h\nu\, [1 + 2e^{-h\nu/kT} + 3e^{-2h\nu/kT} + \dots] \\ &= N_0 e^{-h\nu/kT} \frac{h\nu}{(1 - e^{-h\nu/kT})^2} \end{aligned} \qquad \text{(iii)}$$

Putting the values of N and E from above relation in Eq. (i), we get

$$\bar{E} = \frac{E}{N} = \frac{h\nu e^{-h\nu/kT}}{1 - e^{-h\nu/kT}} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$\bar{E} = \frac{h\nu}{e^{h\nu/kT} - 1} \qquad \text{(iv)}$$

This is the expression for average energy of a Planck's oscillator.

14.3.2 Planck's Radiation Formula

The energy density of radiation (u_ν) in the frequency range ν and $\nu + d\nu$ depending upon the average energy of an oscillator is given by

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \times \bar{E} \qquad \text{(v)}$$

Putting the value of $\bar{E}$ from Eq. (iv) in Eq. (v) gives

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} d\nu$$

$$\text{or}\quad u_\nu\,d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/kT}-1} \qquad \text{(vi)}$$

The above relation is known as Planck's radiation formula in terms of frequency. This law can also be expressed in terms of wavelength λ of the radiation. Since $\nu = \frac{c}{\lambda}$ for electromagnetic radiation, $d\nu = -\frac{c}{\lambda^2}d\lambda$. Further, we know that the frequency is reciprocal of wavelength or in other words an increase in frequency corresponds to a decrease in wavelength. Therefore

$$u_\lambda d\lambda = -u_\nu d\nu$$

$$\text{or}\quad u_\lambda d\lambda = -\frac{8\pi h}{c^3}\frac{\left(\frac{c}{\lambda}\right)^3\left(-\frac{c}{\lambda^2}d\lambda\right)}{e^{hc/\lambda kT}-1}$$

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda \qquad \text{(vii)}$$

The above relation is known as Planck's formula in terms of wavelength.

14.3.3 Wien's Law and Rayleigh-Jeans Law

With the help of Planck's radiation formula Wien's law and Rayleigh-Jeans law can be derived. When the wavelength λ and temperature T are very small, then $e^{hc/\lambda kT} >> 1$. Therefore, 1 can be neglected in the denominator of Eq. (vii).

Thus

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}e^{-hc/\lambda kT}d\lambda$$

By substituting $8\pi hc = \text{A}$ and $\frac{hc}{k} = \text{B}$, we get

$$u_\lambda d\lambda = \frac{\text{A}}{\lambda^5}e^{-\text{B}/\lambda\text{T}}d\lambda \qquad \text{(viii)}$$

This is known as *Wien's law*, which is valid at low temperature T and small wavelength λ.

For high temperature T and large wavelength λ, $e^{hc/\lambda kT}$ can be approximated to $1+\frac{hc}{\lambda kT}$. Then we have from Eq. (vii)

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5\left(1+\frac{hc}{\lambda kT}-1\right)}d\lambda$$

$$u_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda \qquad \text{(ix)}$$

This is known as *Rayleigh-Jeans Law*.

14.4 SIMPLE CONCEPT OF QUANTUM THEORY

As discussed earlier, Planck's hypothesis says that the radiation does not emit in continuous fashion rather it gets emitted in discrete packets of energy equal to $h\nu$. These packets are referred to as quanta or photons.

Therefore, it can be said that the exchange of energy between the radiation and the matter takes place in discrete set of values. In view of the application of quantum theory, it is necessary to be aware of the photon.

14.4.1 Photon: Mass, Energy and Momentum

Photon is an elementary particle that is massless and has no charge. It is a bundle of energy or packet of energy emitted by a source of radiation. It moves with velocity of light. It can carry energy and momentum. We know that the mass m of the particle moving with v, comparable with the velocity of light c, is given by as per the special theory of relativity.

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{i}$$

where m is the relativistic mass of the particle and m_0 is its rest mass. Since the photon is moving with the velocity of the light, we substitute $v = c$ in Eq. (i). With this the moving mass m of the photon becomes $m = \infty$, which is not possible. So, if the photon moves with the velocity of light then the zero in the numerator balances the zero in the denominator, i.e. $m = 0/0$; this is an indeterminate quantity. It means if we take rest mass of the photon to be zero, this value should not particularly disturb us due the fact that the photons are never at rest and always keep moving with the velocity of light.

The energy of a photon is given below as

$$\mathrm{E} = h\nu \tag{ii}$$

If m is the moving mass of the photon, then according to special theory of relativity, the following relation gives the energy

$$\mathrm{E} = mc^2 \tag{iii}$$

so $\quad mc^2 = h\nu$

$$\text{or} \quad m = \frac{h\nu}{c^2} \tag{iv}$$

Now the energy relation

$$\mathrm{E}^2 = p^2c^2 + m_0^2 c^4 \tag{v}$$

Since $m_0 = 0$, $\mathrm{E} = pc$ and the momentum of the photon is given by

$$p = \frac{\mathrm{E}}{c} = \frac{mc^2}{c} = mc \tag{vi}$$

$$\text{or} \quad p = \frac{\mathrm{E}}{c} = \frac{h\nu}{c} \tag{vii}$$

Thus, if a photon of frequency ν is to be treated as a particle, then the characteristics of the photon are given as

$$m_0 = 0,\ \mathrm{E} = h\nu,\ m = h\nu/c^2 \text{ and } p = h\nu/c \tag{viii}$$

These characteristics of the photons are useful in the discussion of Compton effect, which establishes the photon hypothesis.

14.5 WAVE PARTICLE DUALITY

The phenomena of interference, diffraction and polarisation can be explained on the basis of wave theory of light. However, the wave nature of light fails to explain the phenomena of Compton effect, photoelectric effect, the continuous X-ray spectrum and the blackbody radiation. In the light of these facts, physicists assumed the particle nature of electromagnetic radiation (light). These entire phenomena can be explained on the basis of quantum hypothesis, according to which electromagnetic radiation is propagated in small packets or bundles. These packets are called photons. It means that light or electromagnetic radiation exhibits wave and particle properties both. Hence, light or electromagnetic radiation has dual nature, i.e. it behaves like a particle as well as a wave. This dual characteristic property of radiation is called dual nature of light or wave particle duality.

14.6 PHOTOELECTRIC EFFECT

The photoelectric effect refers to the emission or ejection of electrons from the surface of a metal (generally) in response to incident light. Energy contained within the incident light is absorbed by the electrons within the metal, gaining sufficient energy to be 'knocked' out of, i.e. emitted from, the surface of the metal. According to the classical Maxwell wave theory of light, the more intense incident light should eject the electrons from the metal with their greater energy. It means the average energy carried by an ejected (photoelectric) electron should increase with the intensity of the incident light. In fact, *Lenard* found that this was not so. Rather, he observed the energies of the emitted electrons to be independent of the intensity of the incident radiation. In 1905, *Einstein* resolved this paradox successfully by proposing that the incident light consists of individual quanta, called photons, that interact with the electrons in the metal like discrete particles, rather than as continuous waves. He adopted the Planck's quantum hypothesis and applied it to the electromagnetic radiation. For a given frequency ν or colour (λ) of the incident radiation, each photon carries the energy $E = h\nu$. According to Einstein's model, increase in the intensity of the light corresponds to the enhancement in the number of incident photons per unit time (flux), while the energy of each photon remains the same as long as the frequency of the radiation is kept constant. It means, increasing the intensity of the incident radiation would cause greater numbers of electrons to be ejected and each electron would carry the same average energy because each incident photon carries the same energy. Likewise, in Einstein's model, increasing the frequency ν rather than the intensity of the incident radiation would increase the average energy of the emitted electrons. Both of these predictions were confirmed experimentally. It is interesting to note that the rate of increase of the energy of the ejected electrons with increasing frequency enables us to determine the value of Planck's constant h, as the frequency can be measured.

14.6.1 Theoretical Explanation

In photoemission, one quantum is absorbed by one electron. If the electron is some distance into material of the cathode, some energy will be lost as it moves towards the surface. There will always be some electrostatic cost as the electron leaves the surface. This is known as the work function ϕ_0. The electrons those are very close to the surface will be the most energetic, and they will leave the cathode with kinetic energy given by

$$E_K = h\nu - \phi_0$$

or $$E_K = h\nu - h\nu_0$$

where $h\nu_0 = \phi_0$

Therefore, it is clear that there is a minimum light frequency called *threshold frequency* ν_0 for a given metal for which the quantum of energy is equal to the work function. Light below that frequency, no matter how bright, will not cause photoemission.

14.6.2 Experiment

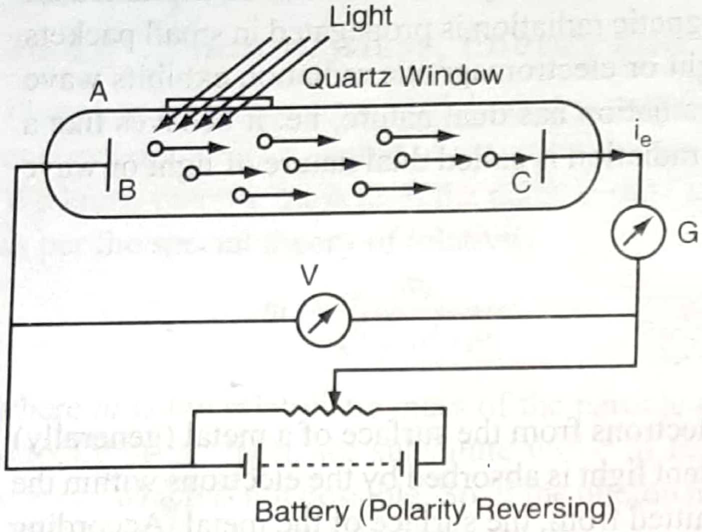

Fig. 14.4

An experimental arrangement to the photoelectric effect is shown in Fig. 14.4. It consists of a vacuum tube A, which contains a metallic plate B and a charge collecting plate C. When light is incident on the plate B through the quartz window, electrons are ejected from the metallic surface. The collector is kept at positive potential V with respect to the metallic plate, which is at zero potential. So, due to this positive potential the collector C collects these ejected electrons. Therefore, a current i_e is produced, which can be measured by the galvanometer G. We can increase the current i_e by increasing the potential V until i_e reaches a constant value, i.e. it approaches a saturation.

By using the reversing switch, we apply the negative potential to the collecting plate C. Under this situation, the electrons are repelled by C and only those electrons whose energy is greater than the potential energy eV will be able to reach the collector C. So we get some current in the galvanometer G. The applied potential for which the current i_e becomes zero, i.e. $i_e = 0$, is called stopping potential V_0. The relation between the maximum kinetic energy of the electrons E_K and stopping potential V_0 is given as below.

$$E_K = \frac{1}{2}mv^2_{\max} = e|V_0|$$

We can obtain the following results by performing detailed experiment under various conditions.

(1) The photoelectric current i_e increases with the increasing intensity I of the incident radiation, if the frequency is kept constant.

(2) There is no time lag between illumination of the metal surface and the emission of electrons.

(3) If the frequency of the incident radiation is greater than the threshold frequency ν_0 (certain minimum frequency), only then the emission of electrons takes place.

(4) The maximum kinetic energy E_K of the photoelectrons is independent of the intensity I of the incident light. This is shown in Fig. 14.5 in which we observe that the stopping potential is same for the light of three different intensities having same frequency.

(5) The maximum kinetic energy of the photoelectrons depends on the frequency of the incident radiation. From Fig. 14.6, we observe that at different frequencies, stopping potential is also different but the saturation current remains the same.

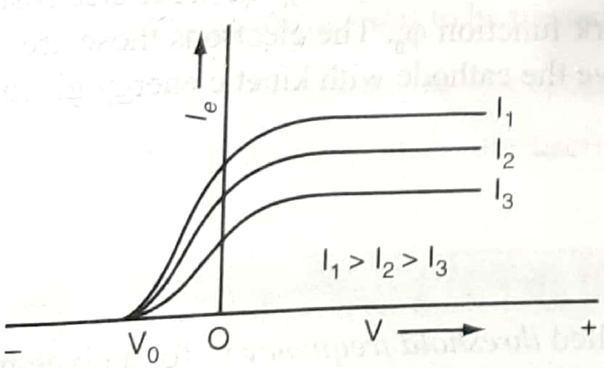

Fig. 14.5

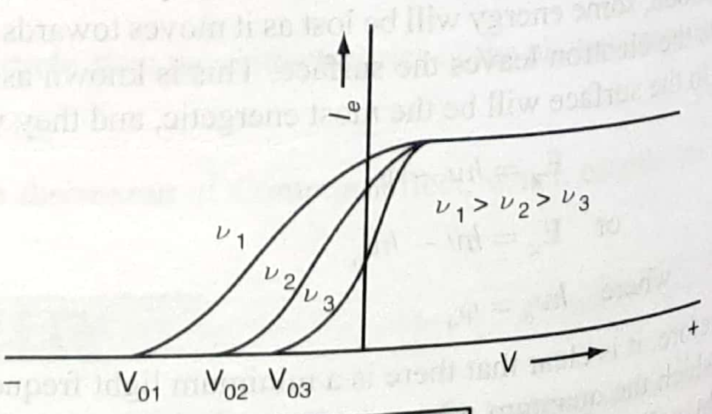

Fig. 14.6

(6) There is a linear relation between maximum kinetic energy and frequency. We show this in Fig. 14.7 for three different metals cesium, potassium and tungsten, which satisfy the following relation

$$E_K = a_1\nu + a_2$$

Here a_1 is the slope of the straight line and a_2 is the intercept. From the figure it is clear that though a_1 remains the same for all surfaces, a_2 is different for different metals.

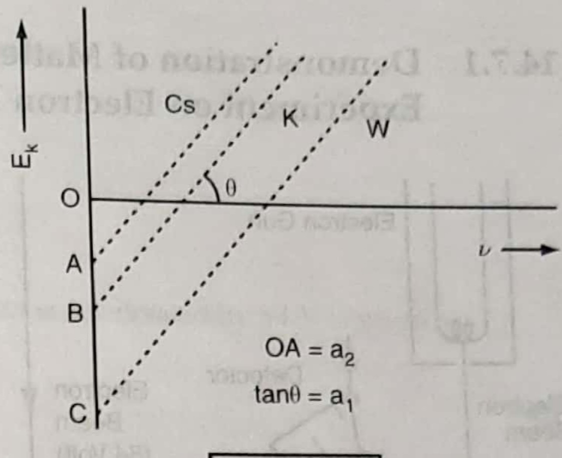

Fig. 14.7

The photoelectric effect is perhaps the most direct and convincing evidence of the existence of photons and the *'corpuscular'* nature of light and electromagnetic radiation. That is, it provides undeniable evidence of the quantisation of the electromagnetic field and the limitations of the classical field equations of Maxwell. *Albert Einstein* received the Nobel Prize in Physics in 1921 for explaining the photoelectric effect and for his contributions to the theoretical physics.

14.7 deBROGLIE WAVES: MATTER WAVES

In 1924, Louis deBroglie proposed in his doctoral dissertation that there was a fundamental relation between waves and particles. Therefore, the energy of the photon according to special theory of relativity is given by

$$E = h\nu \tag{i}$$

and momentum p is

$$p = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{ii}$$

Here, it can be noted that E and p are the characteristics of the partricles, and ν and λ are the characteristics of the waves. From above relations, we see that these sets of quantities are related to each other by the Planck's constant h. deBroglie also suggested that the dual nature of electromagnetic radiation may be extended to material particles such as electrons, protons, neutrons etc. It means that a moving particle, whatever its nature be, has wave properties associated with it. The waves associated with these particles are known as matter waves or deBroglie waves. The difference between the electromagnetic radiation and elementary particles is that in the case of photons, $m_0 = 0$ and $v = c$ but in the case of material particles $m_0 \neq 0$ and $v < c$. deBroglie gave the following hypothesis, which is applicable to all matters, radiation and particles.

(i) If there is a particle of momentum p, its motion is associated with a wave of wavelength

$$\lambda = \frac{h}{p} \tag{iii}$$

(ii) If there is a wave of wavelength λ, the square of the amplitude of the wave at any point in space is proportional to the probability of observing, at that point in space, a particle of momentum

$$p = \frac{h}{\lambda} \tag{iv}$$

The dual nature of matter can be proved if we could show that a beam of particles also exhibits the phenomenon of diffraction pattern just like the electromagnetic waves show the phenomena of diffraction and interference.

14.7.1 Demonstration of Matter Waves: Davisson-Germer Experiment on Electron Diffraction

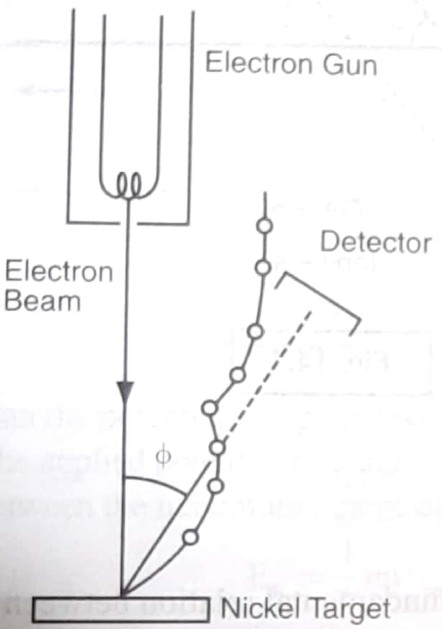

Fig. 14.8(a)

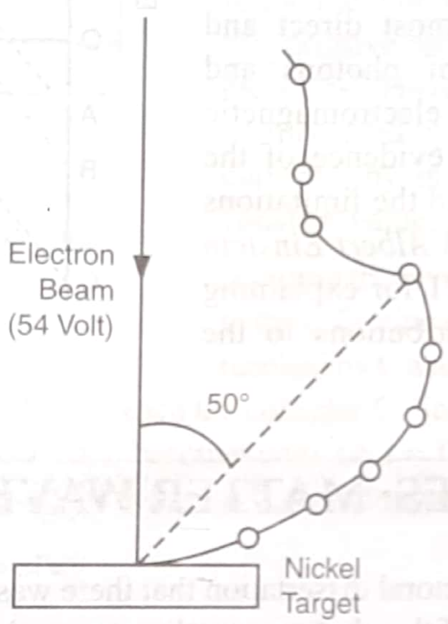

Fig. 14.8(b)

The Davisson-Germer experiment was conducted in 1927 which confirmed the deBroglie hypothesis, according to which particles of matter (such as electrons) have wave properties. This demonstration of the wave particle duality was important historically in the establishment of quantum mechanics and of the Schroedinger equation. The experimental setup is shown in Fig. 14.8a. Here the electrons from a heated filament or electron gun were accelerated by a voltage V and allowed to fall on the surface of nickel target. *Davisson* and *Germer* measured the intensity of the scattered electrons as a function of the angle ϕ and plotted it in the form of polar diagram. Fig. 14.8b shows the results from the accelerating voltage of 54 V. For this case, there is an intense scattering or a pronounced peak at an angle of $\phi = 50°$. Such deflection can be explained by assuming that the electron beam has a wave associated with it. This situation is similar to the Bragg deflection. So the waves associated with the electron beam were satisfying Bragg's law, which caused a diffraction peak. In order to prove this, consider Fig. 14.9 that shows atomic planes in Ni crystal. Here $\theta = 50°, \phi = (180 - 50)/2 = 65°$, $d = 0.91$ Å. Hence, for $n = 1$ from Bragg's law $n\lambda = 2d \sin\theta$ gives

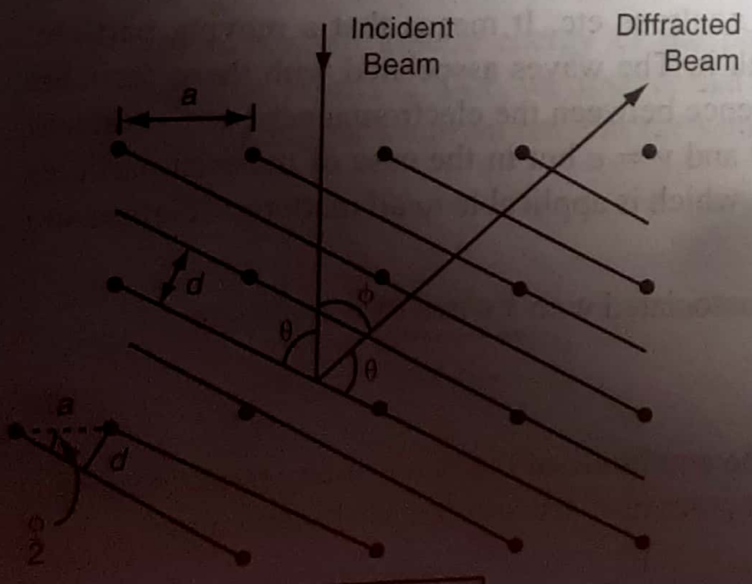

Fig. 14.9

$$\lambda = 2d \sin\theta = 2 \times 0.91 \times \sin 65° = 1.65 \text{ Å} \quad \text{(i)}$$

Since Bragg's law basically talks about the diffraction of X rays, this experiment enables us to treat the electrons as waves and the wavelength associated with the electrons should be 1.65 Å, if they are scattered at $\phi = 50°$.

Now we apply deBroglie's hypothesis. Since the electron of mass m gains the velocity v when it gets accelerated through a potential difference of V, we write the following relation for the energy for the nonrelativistic motion of the electron

$$\frac{1}{2}mv^2 = eV$$

So the deBroglie wavelength associated with the electron is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2meV}}$$

$$\text{or } \lambda = \sqrt{\frac{150}{V}}\ \text{Å}$$

Therefore, deBroglie wavelength associated with the electron that is accelerated by 54 V is given as

$$\lambda = \sqrt{\frac{150}{V}}\ \text{Å} = \sqrt{\frac{150}{54}}\ \text{Å} = 1.67\ \text{Å} \tag{ii}$$

A comparision of Eq. (i) with Eq. (ii) shows that the value of the wavelength λ is the same in both the cases. It means there is a wave called deBroglie wave associated with the electrons. Therefore, this confirms the deBroglie hypothesis.

14.8 COMPTON EFFECT: COMPTON SCATTERING

As per classical electromagnetic theory, when an electromagnetic radiation (frequency ν) is incident on free charges (say, electrons), the free charges absorb this radiation and start oscillating at frequency ν. Then these oscillating charges radiate electromagnetic waves of the same frequency ν. This type of scattering where the change in frequency or wavelength does not take place is called *coherent scattering*. This coherent scattering has been observed with the radiation in visible range and also at longer wavelengths. However, this predication of classical theory fails in the case of scattering of radiation of very short wavelengths like X-rays. Here the scattered X-rays are found to consist of two frequencies: ν and ν_1. The wavelength λ corresponding to the frequency ν is called unmodified wavelength or unmodified radiation, whereas the wavelength λ_1 corresponding to the frequency ν_1 is called modified wavelength or modified radiation. This type of scattering is known as *incoherent scattering*.

The Compton effect or Compton scattering is related to the scattering of X-rays (electromagnetic waves of very short wavelength) by free electrons. A. H. Compton found that when X-rays are scattered by a solid material (say carbon in which the loosely bound electrons are assumed to be almost free) the scattered X-ray radiations carry the longer wavelength. This phenomenon of increase in the wavelength (or decrease in frequency) of X-ray radiations by scattering is called the Compton effect. This effect was explained by using the quantum theory of radiations. On the basis of this theory, these radiations are made up of photons of energy $h\nu$. These photons in the incident X-rays collide with the free electrons of the target (Fig. 14.10). If the collision is elastic, then the energy or wavelength of the scattered photons remains the same. If the collision is inelastic, then the incident photon transfers some of the energy to the electron. Thus, as a result, the energy of the scattered X-ray photon decreases (or wavelength increases).

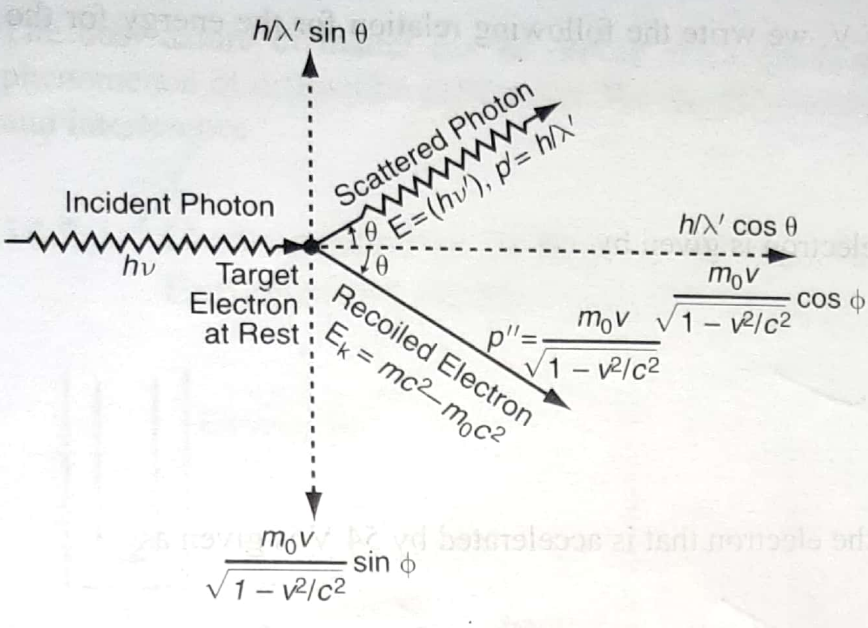

Fig. 14.10

Let us assume,

λ = wavelength of incident X-rays

λ' = wavelength of scattered X-rays

$\Delta\lambda = \lambda' - \lambda$ = Compton shift

Energy of incident X-ray photon $= h\nu = hc/\lambda$

Momentum of incident X-ray photon $= h/\lambda$

Energy of the scattered X-ray photon $h\nu' = hc/\lambda'$

Momentum of the scattered X-ray photon $= h/\lambda'$

Kinetic energy of the recoiled electron

$$= (m - m_0)c^2 = \left[\frac{m_0}{\sqrt{1-v^2/c^2}} - m_0\right]c^2$$

Momentum of the recoiled electron

$$= mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

m = moving mass or relativistic mass of the electron and m_0 is the rest mass.

According to the law of conservation of energy

Energy of the incident photon = Energy of the scattered photon + Energy of the recoiled electron.

$$E = E' + E_K$$

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + (m - m_0)c^2$$

or $$\frac{h}{\lambda} - \frac{h}{\lambda'} = \frac{m_0 c}{\sqrt{1-v^2/c^2}} - m_0 c$$

No need of derivation

or $$\frac{h}{\lambda} - \frac{h}{\lambda'} + m_0 c = \frac{m_0 c}{\sqrt{1-v^2/c^2}} \quad \text{(i)}$$

According to the law of conservation of momentum,

$$\frac{h}{\lambda'}\sin\theta = \frac{m_0 v}{\sqrt{1-v^2/c^2}}\sin\phi \quad \text{(ii)}$$

and $$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + \frac{m_0 v}{\sqrt{1-v^2/c^2}}\cos\phi$$

or $$\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta = \frac{m_0 v}{\sqrt{1-v^2/c^2}}\cos\phi \quad \text{(iii)}$$

Squaring and adding Eqs. (ii) and (iii), we get

$$\frac{h^2}{\lambda'^2} + \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta = \frac{m_0^2 v^2}{1-v^2/c^2}$$

or $$\frac{h^2}{\lambda'^2} + \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta = \frac{m_0^2 v^2 c^2}{c^2 - v^2} \quad \text{(iv)}$$

On squaring Eq. (i), we get

$$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}+m_0{}^2c^2-\frac{2h^2}{\lambda\lambda'}-\frac{2hm_0c}{\lambda'}+\frac{2hm_0c}{\lambda}=\frac{m_0{}^2c^2}{1-v^2/c^2}=\frac{m_0{}^2c^4}{c^2-v^2}$$

or $$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}+2hm_0c\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\frac{m_0{}^2c^4}{c^2-v^2}-m_0{}^2c^2$$

or $$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}+2hm_0c\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)=\frac{m_0{}^2c^4-m_0{}^2c^4+m_0{}^2v^2c^2}{c^2-v^2}$$

$$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}+2hm_0c\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)=\frac{m_0{}^2v^2c^2}{c^2-v^2} \quad \text{(v)}$$

On comparing Eqs. (iv) and (v), we get

$$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}+2hm_0c\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)=\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}\cos\theta$$

or $$2hm_0c\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)=\frac{2h^2}{\lambda\lambda'}(1-\cos\theta)$$

or $$(\lambda'-\lambda)=\Delta\lambda=\frac{h}{m_0c}(1-\cos\theta)$$

or $$\boxed{\Delta\lambda=\frac{h}{m_0c}(1-\cos\theta)} \quad \text{(vi)}$$

only CM

where, h = Planck's constant. m_0 = rest mass of electron, c = velocity of light and θ = angle of scattering of the photon. This is just to emphasize the RHS contains the angle of scattering of photon not of the electron.

14.8.1 Verification of Compton Effect

A beam of monochromatic X-ray of known wavelength is made incident on a graphite target. The intensity distribution with wavelength of monochromatic X-ray scattered at different angles is measured by Bragg's X-ray spectrometer. The intensity distribution with wavelength for different angles is shown in Fig. 14.11. It may be noted that diffraction patterns have two diffraction peaks-one corresponding to modified radiation and the other corresponding to unmodified radiation. The difference between two peaks on the wavelength axis provides the Compton shift. It can be concluded from the diffraction patterns that greater scattering angle yields greater Compton shift. For example, $\Delta\lambda=\lambda'-\lambda=h/m_0c\,(1-\cos\theta)$ at $\theta=90^\circ$ gives $\Delta\lambda=0.024$ Å.

Hence, Compton effect is experimentally verified.

14.8.2 Why Compton Effect is not observable with Visible Light?

As mentioned, the Compton effect is observed significantly with the X-rays which are very short wavelength radiations. This can be confirmed, if we use the visible light (λ = 4000 Å – 7000 Å) in place of X-rays and calculate the Compton shift. For this we use

$$\Delta\lambda=\frac{h}{m_0c}(1-\cos\theta)=0.0242(1-\cos\theta)\text{Å}$$

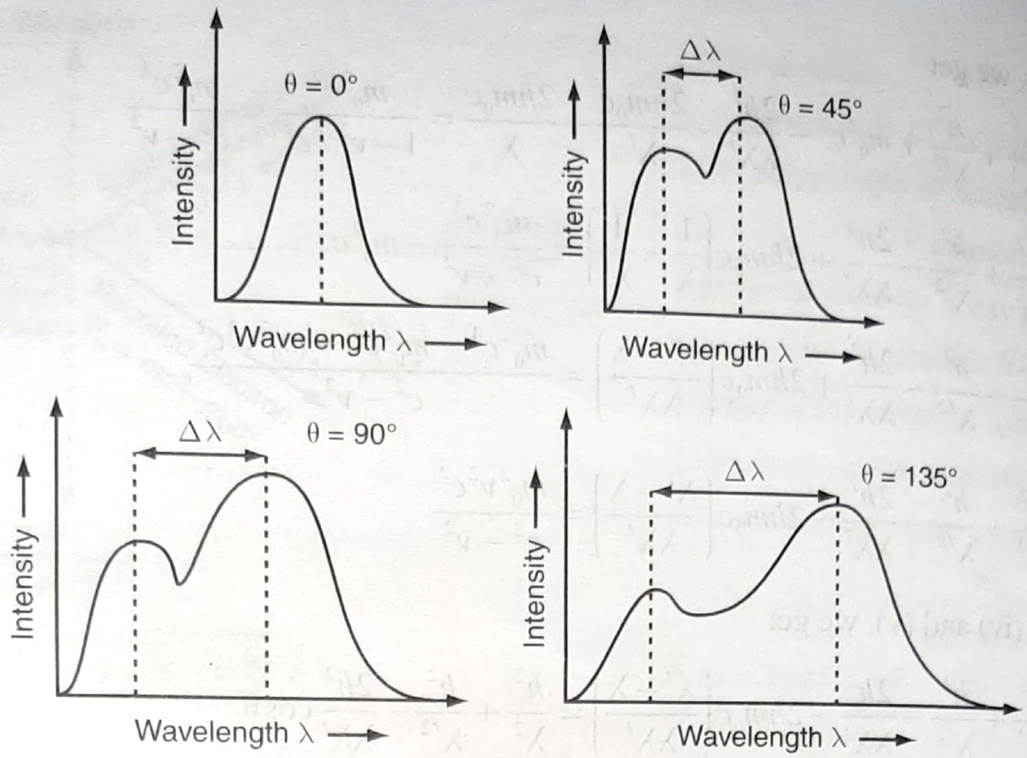

Fig. 14.11

For maximum shift, θ = 180°. Hence

$$(\Delta\lambda)_{max} = 0.0484 \text{ Å}$$

The percentage Compton shift for λ = 4000 Å

$$= \frac{(\Delta\lambda)_{max}}{\lambda} \times 100 \simeq 0.001\%$$

Similarly, the percentage Compton shift for larger wavelength of visible light, i.e. for λ = 7000 Å, would be 0.0007%. So, you can see that the Compton shift for the case of visible light is not significant. For this reason, the X-rays are appropriate for realizing the Compton effect or Compton scattering.

14.9 PHASE AND GROUP VELOCITIES: deBROGLIE WAVES

Phase Velocity

Waves have already been discussed in Chapter 1. However, here we will discuss phase and group velocities in the context of deBroglie waves. We can write the deBroglie wave travelling along the $+x$ direction as

$$y = a \sin(\omega t - kx) \quad (i)$$

where a is the amplitude, ω (=2πν) is the angular frequency and k (=2π/λ) is the propagation constant of the wave. By the definition, the ratio of angular frequency ω to the propagation constant k is the phase (or wave) velocity. If we represent the phase velocity by u, then

$$\boxed{u = \frac{\omega}{k}}$$

$(\omega t - kx)$ is called the phase of the wave motion. It means the particle of the constant phase travels such that $\omega t - kx =$ constant.

$$\text{or} \quad \frac{d}{dt}(\omega t - kx) = 0$$

$$\omega - k\frac{dx}{dt} = 0$$

$$\text{or} \quad \frac{dx}{dt} = u = \frac{\omega}{k} \tag{ii}$$

where $u = \dfrac{dx}{dt}$ is the phase (or wave) velocity. Thus the wave velocity is the velocity of planes of constant phase which advances through the medium. We can write the phase velocity $u = \nu\lambda$ and for an electromagnetic wave $E = h\nu$, or $\nu = E/h$

According to deBroglie $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$

$$u = \nu\lambda = \frac{E}{h} \times \frac{h}{mv} = \frac{mc^2}{mv} = \frac{c^2}{v}$$

$$u = \frac{c^2}{v} \tag{iii}$$

Since $c >> v$, Eq. (iii) implies that the phase velocity of deBroglie wave associated with the particle moving with velocity v is greater than c, the velocity of light.

Group Velocity

As we have seen, the phase velocity of a wave associated with a particle comes out to be greater than the velocity of light. This difficulty can be overcome by assuming that each moving particle is associated with a group of waves or a wave packet rather than a single wave. In this context, deBroglie waves are represented by a wave packet and hence we have *group velocity* associated with them. In order to realize the concept of group velocity, we consider the combination of two waves, resultant of which is shown in Fig. 14.12. The two waves are represented by the following relations (i)

$$y_1 = a \sin(\omega_1 t - k_1 x) \tag{ii}$$

$$\text{and} \quad y_2 = a \sin(\omega_2 t - k_2 x)$$

Their superposition gives

$$y = y_1 + y_2 = a\,[\sin(\omega_1 t - k_1 x) + \sin(\omega_2 t - k_2 x)]$$

$$\text{or} \quad y = 2a \sin\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \tag{iii}$$

$$\therefore \quad y = 2a \cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \sin(\omega t - kx)$$

No derivative required

where $\omega = \dfrac{\omega_1 + \omega_2}{2}$, $k = \dfrac{k_1 + k_2}{2}$

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Eq. (iii) can be re-written as

$$y = 2a\cos\left[\frac{(\Delta\omega)t}{2} - \frac{(\Delta k)x}{2}\right]\sin(\omega t - kx) \qquad \text{(iv)}$$

where $\Delta\omega = \omega_1 - \omega_2$ and $\Delta k = k_1 - k_2$.

The resultant wave Eq. (iv) has two parts.

(i) A wave of frequency ω, propagation constant k and the velocity u, given by

$$u = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda$$

which is the phase velocity or wave velocity.

(ii) Another wave of frequency $\Delta\omega/2$, propagation constant $\Delta k/2$ and the velocity $G = \dfrac{\Delta\omega}{\Delta k}$. This velocity is the velocity of envelope of the group of waves, i.e. it is the velocity of the wave packet (shown by dotted lines) and is known as group velocity.

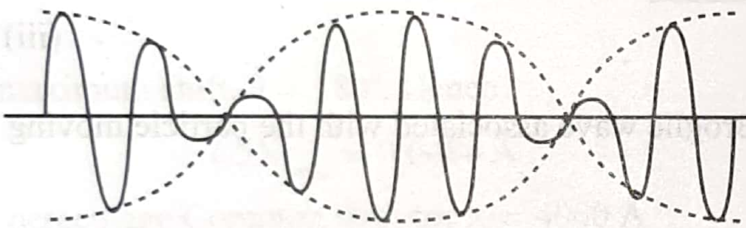

Fig. 14.12

For the waves having small difference in their frequencies and wave numbers, we can write

$$G = \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} = \frac{\partial(2\pi\nu)}{\partial(2\pi/\lambda)} = \frac{\partial\nu}{\partial(1/\lambda)} = -\lambda^2\frac{\partial\nu}{\partial\lambda}$$

$$G = -\frac{\lambda^2}{2\pi}\frac{\partial\omega}{\partial\lambda}$$

This is the expression for the group velocity.

14.9.1 Relation between Group Velocity and Phase Velocity

If u be the phase (wave) velocity, then the group velocity can be written as

$$G = \frac{d\omega}{dk} = \frac{d}{dk}(uk) \qquad \left[u = \frac{\omega}{k}\right]$$

$$\text{or} \quad G = u + k\frac{du}{dk}$$

$$\text{But} \quad k = \frac{2\pi}{\lambda} \Rightarrow dk = -\frac{2\pi}{\lambda^2}d\lambda \quad \text{and} \quad \frac{k}{dk} = -\frac{\lambda}{d\lambda}$$

Therefore, the group velocity is given by

$$G = u + \left(-\frac{\lambda}{d\lambda}\right)du$$

$$\text{or} \quad G = u - \lambda\frac{du}{d\lambda}$$

This relation shows that the group velocity G is less than the phase velocity u in a dispersive medium where u is a function of k or λ. However, in a non-dispersive medium, the velocity u is independent of k, i.e. the wave of all wavelength travel with the same speed, i.e. $\frac{du}{d\lambda} = 0$. Then $\mathbf{G} = \boldsymbol{u}$. This is true for electromagnetic waves in vacuum and the elastic waves in homogenous medium.

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Example 1 Calculate the frequency and wavelength of a photon whose energy is 75 eV.

Solution Given energy $E = 75\ eV = 75 \times 1.6 \times 10^{-19}$ J.

Formula used is

$$E = h\nu = \frac{hc}{\lambda}$$

$$\text{Frequency } (\nu) = \frac{E}{h} = \frac{75 \times 1.6 \times 10^{-19}}{6.62 \times 10^{-34}}$$

$$= 18.13 \times 10^{15}\ \text{Hz}$$

and wavelength $\lambda = \dfrac{c}{\nu} = \dfrac{3 \times 10^{8}}{18.13 \times 10^{15}}$

$$= 1.655 \times 10^{-8}\ \text{m}$$

$$= 165.5 \times 10^{-10}\ \text{m}$$

or $\lambda = \mathbf{165.5\ Å}$

Example 2 Find the number of quanta of energy emitted per second if a radio station operates at a frequency of 98 MHz and radiates power of 2×10^5 W.

Solution Given $\nu = 98 \times 10^6$ cycles / sec and Power (P) $= 2 \times 10^5\ W = 2 \times 10^5$ J/sec.

Energy of each quanta is

$$E = h\nu$$

$$\therefore E = 6.62 \times 10^{-34} \times 98 \times 10^{6}$$

$$= 6.4876 \times 10^{-26}\ \text{J/quanta}$$

$$= 6.5 \times 10^{-26}\ \text{J/quanta}$$

Number of quanta emitted per second

$$= \frac{\text{Power}}{\text{quantum energy}}$$

$$= \frac{2 \times 10^{5}\ (\text{J / sec})}{6.5 \times 10^{-26}\ (\text{J / quanta})}$$

$$= \mathbf{3.08 \times 10^{30}\ quanta/sec}$$

Example 3 A certain spectral line has wavelength 4000 Å. Calculate the energy of the photon.

Solution Given $\lambda = 4.0 \times 10^{-7}$ m.

Formula used is

$$E_k = h\nu = \frac{hc}{\lambda}$$
$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{4\times10^{-7}\,m}$$
$$= \mathbf{4.965 \quad 10^{-19}\,J}$$

Example 4 Calculate the number of photons of green light of wavelength 5000 Å require to make one erg of energy.

Solution Given $\lambda = 5\times10^{-7}$ m.

Formula used is

$$E = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^{8}}{5\times10^{-7}}$$

$= 3.972\times10^{-19}$ J

$= 3.972\times10^{-12}$ erg

Number of photons of green light emitted (per energy)

$$= \frac{1.0}{3.972\times10^{-12}}$$
$$= 252\times10^{9}$$

Example 5 Calculate the wavelength of a photon of energy 5×10^{-19} J.

Solution Given $E = 5\times10^{-19}$ J

Formula used is

$$E = \frac{hc}{\lambda}$$

$$\text{or} \quad \lambda = \frac{hc}{E} = \frac{6.62\times10^{-34}\times3\times10^{8}}{5\times10^{-19}}$$
$$= 3.972\times10^{-7}\,m$$
$$= \mathbf{4000\ Å}$$

Example 6 Calculate the energy of an electron of wavelength 4.35×10^{-7} m.

Solution Given $E = 4.35\times10^{-7}$ m.

Formula used is

$$E = h\nu = \frac{hc}{\lambda}.$$
$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{4.35\times10^{-7}}$$
$$= \mathbf{4.566\times10^{-19}\,J}$$

Formula used is

$$E_k = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{4 \times 10^{-7}\,m}$$

$$= \mathbf{4.965 \quad 10^{-19}\,J}$$

Example 4 Calculate the number of photons of green light of wavelength 5000 Å require to make one of energy.

Solution Given $\lambda = 5 \times 10^{-7}$ m.

Formula used is

$$E = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{5 \times 10^{-7}}$$

$$= 3.972 \times 10^{-19}\,J$$

$$= 3.972 \times 10^{-12}\text{ erg}$$

Number of photons of green light emitted (per energy)

$$= \frac{1.0}{3.972 \times 10^{-12}}$$

$$= \mathbf{252 \times 10^9}$$

Example 5 Calculate the wavelength of a photon of energy 5×10^{-19} J.

Solution Given $E = 5 \times 10^{-19}$ J

Formula used is

$$E = \frac{hc}{\lambda}$$

or $$\lambda = \frac{hc}{E} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{5 \times 10^{-19}}$$

$$= 3.972 \times 10^{-7}\,m$$

$$= \mathbf{4000\ Å}$$

Example 6 Calculate the energy of an electron of wavelength 4.35×10^{-7} m.

Solution Given $E = 4.35 \times 10^{-7}$ m.

Formula used is

$$E = h\nu = \frac{hc}{\lambda}.$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{4.35 \times 10^{-7}}$$

$$= \mathbf{4.566 \times 10^{-19}\,J}$$

Example 7 How many watts of power at the threshold is received by the eye, if it receives 120 photons per second of the visible light of wavelength = 5600 Å.

Solution Given $\lambda = 5.6 \times 10^{-7}$ m and number of photons = 120.

Energy of a photon $E = h\nu = \dfrac{hc}{\lambda}$

$$\text{or} \quad E = \frac{6.62\times10^{-34}\times3\times10^{8}}{5.6\times10^{-7}} = 3.55\,\text{J}$$

The energy received by the eye per second = 3.55 × 120 J/sec

= **425.57 W**

Example 8 How many photons of yellow light of wavelength 5500 Å constitute 1.5 J of energy.

Solution Given $\lambda = 5.5 \times 10^{-7}$ m and energy of n photons = 1.5 J

Formula used is $\quad E = h\nu = \dfrac{hc}{\lambda}$

Energy of a photon of yellow light, i.e.

$$E = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^{8}}{5.5\times10^{-7}}$$
$$= 3.61\times10^{-19}\,\text{J}$$

Given

$n\times$ energy of one photon = 1.5 J

$$\text{or} \quad n = \frac{1.5}{3.61\times10^{-19}}$$
$$= \mathbf{4.155\times10^{18}}$$

Example 9 Calculate the work function, stopping potential and maximum velocity of photoelectrons for a light of wavelength 4350 Å when it incidents on sodium surface. Consider the threshold wavelength of photoelectrons to be 5420 Å.

Solution Given $\lambda_0 = 5.42 \times 10^{-7}$ m and $\lambda = 4.35 \times 10^{-7}$ m.

Formulae used are

$$\phi_0 = \frac{hc}{\lambda_0} = h\nu_0$$

$$\frac{1}{2}mv_{max}^2 = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right] \quad \text{and}$$

$$eV = h\nu - h\nu_0 = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right]$$

$$\text{or} \quad eV = \frac{1}{2}mv_{max}^2 = (E_k)_{max}$$

$$\phi_0 = \frac{hc}{\lambda_0} = \frac{6.62\times10^{-34}\times3\times10^{8}}{5.42\times10^{-7}} = 3.664\times10^{-19}\,\text{J}$$

$$\phi_0 = \frac{hc}{\lambda_0} = \frac{6.62\times10^{-34}\times3\times10^8}{5.42\times10^{-7}} = 3.664\times10^{-19}\,\text{J}$$

$$\frac{1}{2}mv_{max}^2 = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right]$$

$$v_{max}^2 = \frac{2hc}{m}\left[\frac{\lambda_0 - \lambda}{\lambda_0\lambda}\right]$$

$$= \frac{2\times6.62\times10^{-34}\times3\times10^8}{9.1\times10^{-31}}\left[\frac{(5.42-4.35)\times10^{-7}}{5.42\times4.35\times10^{-14}}\right] = 0.1981\times10^{1}$$

$$\therefore v_{max} = 0.445\times10^6\,\text{m/sec}$$

$$= 4.45\times10^5\,\text{m/sec}$$

$$\text{eV} = \frac{1}{2}mv_{max}^2$$

The stopping potential

$$V = \frac{mv_{max}^2}{2e} = \frac{9.1\times10^{-31}\times(4.45\times10^5)^2}{2\times1.6\times10^{-19}}$$

$$= \mathbf{0.56\ volts}$$

Example 10 The threshold frequency for photoelectric emission in copper is 1.1×10^{15} Hz. Find th maximum energy in eV when light of frequency 1.2×10^{15} Hz is directed on the copper surfac

Solution Given $\nu_0 = 1.1 \times 10^{15}$ Hz and $\nu = 1.2 \times 10^{15}$ Hz.

Formula used is

$$\frac{1}{2}mv_{max}^2 = h\nu - h\nu_0 = h(\nu - \nu_0)$$

$$= 6.62\times10^{-34}[1.2-1.1]\times10^{15}$$

$$= 0.662\times10^{-19}\,\text{J}$$

$$= \mathbf{0.414\,eV}$$

Example 11 Calculate the work function in electron volts of a metal, given that photoelectric threshold (i) 6200 Å (ii) 5000 Å.

Solution Given (i) $\lambda_0 = 6.2 \times 10^{-7}$ m (ii) $\lambda_0 = 5.0 \times 10^{-7}$ m.

$$\phi_0 = h\nu_0 = \frac{hc}{\lambda_0}$$

(i) $$\phi_0 = \frac{6.62\times10^{-34}\times3\times10^8}{6.2\times10^{-7}\times1.6\times10^{-19}}\,\text{eV}$$

$$= \mathbf{2.0\ eV}$$

(ii) $\lambda_0 = 5.0\times10^{-7}\,\text{m}$

$$\phi_0 = \frac{6.62\times10^{-34}\times3\times10^8}{5\times10^{-7}\times1.6\times10^{-19}}$$

$$= 2.483\,\text{eV}$$

$$= \mathbf{2.48\ eV}$$

Example 12 Find out the maximum energy of the photoelectron, work function and threshold frequency when a light of wavelength 3132 Å is incident on a surface of cesium and the stopping potential for the photo electron is 1.98 volt.

Solution Given V= 1.98 volts and λ=3.132x10^{-7}m.

Formulae used are

$$E_k = \frac{1}{2}mv^2_{max} = eV_0, \quad V_0 = \text{stopping potential}$$

$$\text{and } E_k = h(\nu - \nu_0) = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

Then maximum energy of the photoelectron (E_{max})

$$= eV_0 = 1.6\times10^{-19}\times1.98 \text{ J}$$

$$E_k = 3.168\times10^{-19} \text{ J}$$

$$E_k = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$$3.168\times10^{-19} = 6.62\times10^{-34}\times3\times10^8\left[\frac{1}{3.132\times10^{-7}} - \frac{1}{\lambda_0}\right]$$

$$\frac{1}{6.2689\times10^{-7}} = \frac{1}{3.132\times10^{-7}} - \frac{1}{\lambda_0}$$

$$\text{or} \quad \frac{1}{\lambda_0} = 3.193\times10^6 - 1.595\times10^6 = 1.598\times10^6$$

$$\lambda_0 = \frac{1}{1.598\times10^6} = 6258 \text{ Å}$$

$$\text{Work function}(\phi_0) = \frac{hc}{\lambda_0} = \frac{6.62\times10^{-34}\times3\times10^8}{6.258\times10^{-7}}$$

$$= \mathbf{3.174\times10^{-19}\ J}$$

Example 13 Is it possible to liberate an electron from a metal surface having work function 4.8 eV with an incident radiation of wavelength (i) 5000 Å and (ii) 2000 Å.

Solution Given ϕ_0 = 4.8 eV.

Formula used is $E_k = \frac{hc}{\lambda}$.

$$\text{(i) Energy}(E_k) = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^8}{5\times10^{-7}} \text{ J}$$

$$= \frac{6.62\times10^{-34}\times3\times10^8}{5\times10^{-7}\times1.6\times10^{-19}} \text{ eV}$$

$$= \mathbf{2.48\ eV}$$

From the above it is clear that the energy corresponding to wavelength 5000 Å is found to be less than the work function i.e. 4.8 eV. So it will not be able to liberate an electron.

(ii) $E_k = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{2.0 \times 10^{-7}} = 9.93 \times 10^{-19}$ J

or $E_k = \frac{9.93 \times 10^{-19}}{1.6 \times 10^{-19}}$ eV $= 6.206$ eV

$E_k = \mathbf{6.21\ eV}$

As the energy corresponding to wavelength 2000 Å is greater than the work function. So it is sufficient to liberate electrons.

Example 14 Find the maximum energy of the photoelectron, the work function and threshold frequency, if the potassium surface is illuminated by a light of wavelength 5893 Å. The stopping potential for the emitted electron be 0.36V.

Solution Given stopping potential $V_0 = 0.36$V and $\lambda = 5893$ Å.

Formula used is

$$E_k = eV = h\nu - \phi_0.$$
$$E_k = eV = 0.36 eV$$

Work function

$$(\phi_0) = h\nu - eV = \frac{hc}{\lambda} - eV$$
$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{5.893 \times 10^{-7} \times 1.6 \times 10^{-19}} - 0.36\ \text{eV}$$
$$= 2.11 - 0.36 = 1.75 eV$$

Thus the work function is 1.75 eV.

Threshold frequency

$$\nu_0 = \frac{\phi}{h}$$
$$= \frac{1.75 \times 1.6 \times 10^{-19}}{6.62 \times 10^{-34}}$$
$$= \mathbf{4.23 \times 10^{14}\ cycles/sec}$$

Example 15 Find the maximum kinetic energy of the emitted electrons and the stopping potential if the light of wavelength 5890 Å is incident on the surface for which threshold ~~frequency~~ is 7320 Å. wavelength.

Solution Given $\lambda = 5.89 \times 10^{-7}$ m and $\lambda_0 = 7.32 \times 10^{-7}$ m.

Formula used is

$$E_k = h\nu - h\nu_0 = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$
$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{10^{-7}}\left[\frac{1}{5.89} - \frac{1}{7.32}\right]$$
$$= 19.86 \times 10^{-19}\left[\frac{7.32 - 5.89}{5.89 \times 7.32}\right]$$
$$= \mathbf{6.587 \times 10^{-20}\ J}$$

$$Ve = E_k \quad \text{or} \quad V = \frac{E_k}{e}$$

$$\text{Stopping potential(V)} = \frac{6.587\times10^{-20}}{1.6\times10^{-19}}$$
$$= \mathbf{0.412\ V}$$

Example 16 The threshold wavelength for photoelectric emission in tungsten is 2300 Å. What wavelength of light must be used in order for electrons with a maximum energy of 1.5 eV to be ejected?

Solution Given $\lambda_0 = 2.3 \times 10^{-7}$ m and $E_k = 1.5$ eV.

Formula used is

$$E = h\nu - h\nu_0 = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$$\frac{1}{\lambda} - \frac{1}{\lambda_0} = \frac{E}{hc}$$

$$\text{or} \quad \frac{1}{\lambda} = \frac{1.5\times1.6\times10^{-19}}{6.62\times10^{-34}\times3\times10^{8}} + \frac{1}{2.3\times10^{-7}}$$

$$= 1.2085\times10^{6} + 4.3478\times10^{6}$$

$$\frac{1}{\lambda} = 5.556\times10^{6}$$

$$\text{or} \quad \lambda = 1.7998\times10^{-7}\ \text{m}$$

$$\lambda = \mathbf{1799.8\ Å}$$

Example 17 The work function of tungsten is 4.53 eV. If ultraviolet light of wavelength 1500 Å is incident on the surface, does it cause photoelectron emission? If so, what is the kinetic energy of the emitted electron?

Solution Given work function $\phi_0 = 4.53$ eV and $\lambda = 1.5 \times 10^{-7}$ m.

Formula used is $E_k = \dfrac{hc}{\lambda}$

Energy corresponding to incident photon of wavelength 1.5×10^{7} m

$$E_k = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.5\times10^{-7}}\ \text{J}$$

$$= 13.24\times10^{-19}\ \text{J}$$

$$E_k = 8.28\ \text{eV}$$

The kinetic energy of the electron

$$E_k = \frac{1}{2}mv^2_{max} = h\nu - h\nu_0 = h\nu - \phi_0$$

$$= 8.28 - 4.53$$

$$= \mathbf{3.75\ eV}$$

Example 18 The work function of sodium metal is 2.3 eV. What is the longest wavelength of light that cause photoelectric emission from sodium?

Solution Given $\phi_0 = 2.3 \text{ eV} = 2.3 \times 1.6 \times 10^{-19}$ J.

$$\phi_0 = h\nu_0 = \frac{hc}{\lambda_0}$$

Longest wavelength = Threshold wavelength

$$\lambda_0 = \frac{hc}{\phi_0} = \frac{6.62\times10^{-34}\times3\times10^{8}}{2.3\times1.6\times10^{-19}}$$

$$= \mathbf{5396.74\ Å}$$

Example 19 Evaluate the threshold wavelength of photoelectric material whose work function is 2.0 eV.

Solution Given $\phi_0 = 2.0 \text{ eV} = 2 \times 1.6 \times 10^{-19}$ J.

Formula used is

$$\lambda = \frac{hc}{\phi_0}$$

$$\text{or} \quad \lambda = \frac{6.62\times10^{-34}\times3\times10^{8}}{2\times1.6\times10^{-19}}$$

$$= \mathbf{6206\ Å}$$

Example 20 Calculate the threshold wavelength and the wavelength of incident electromagnetic radiation so that the photoelectrons emitted from potassium have a maximum kinetic energy of 4eV. Take the work function of potassium as 2.2 eV.

Solution Given $E_{max} = 4.0 \times 1.6 \times 10^{-19}$ J and $\phi_0 = 2.2 \times 1.6 \times 10^{-19}$ J.

Formulae used are

$$\phi_0 = h\nu_0 = \frac{hc}{\lambda_0} \quad \text{and} \quad E_k = h\nu - \phi_0 = h\nu - h\nu_0$$

$$\lambda_0 = \frac{hc}{\phi_0} = \frac{6.62\times10^{-34}\times3\times10^{8}}{2.2\times1.6\times10^{-19}}$$

$$\lambda_0 = 5642 \text{ Å (Threshold wavelength)}$$

$$E_k = 4\times1.6\times10^{-19} = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right]$$

$$\text{or} \quad \frac{1}{\lambda} = \frac{4\times1.6\times10^{-19}}{6.62\times10^{-34}\times3\times10^{8}} + \frac{1}{\lambda_0}$$

$$= 3.223\times10^{6} + 1.772\times10^{6}$$

$$\frac{1}{\lambda} = 4.995\times10^{6}$$

$$\lambda = \frac{10}{4.995\times10^{7}}$$

$$= \mathbf{2002\ Å}$$

Example 21 Ultraviolet light of wavelength 350 nm and intensity 1.0 watt/m² is directed at a potassium surface. (i) Find the maximum kinetic energy of photoelectron (ii) 0.5% of incident photons produce photoelectrons, how many photoelectrons are emitted per second if the surface of potassium is 1.0 cm². Work function of potassium is 2.1 eV.

Solution Given $\lambda = 3.5 \times 10^{-7}$ m and $\phi_0 = 2.1$ eV.

(i) Formula used is

$$E_k = \frac{1}{2}mv_{\max}^2 = h\nu - \phi_0.$$

$$= \frac{hc}{\lambda} - \phi_0 = \frac{6.62\times10^{-34}\times3\times10^8}{3.5\times10^{-7}\times1.6\times10^{-19}} - 2.1\text{ eV}$$

$$E_k = (3.546 - 2.1)\text{eV} = 1.45\text{ eV}$$

$$= 2.3136\times10^{-19}\text{ J}$$

$$= \mathbf{2.314\times10^{-19}\ J}$$

(ii) Energy incident per second on 1.0 cm² surface of potassium = 10^{-4} Joule

The energy which produces photoelectron per second = 0.5%.

Effective energy which will be used to produce photoelectrons = $\frac{0.5}{100}\times10^{-4}\text{ J} = 5\times10^{-7}\text{ J}$

Minimum energy required to eject one electron from the surface

$= 2.314 \times 10^{-19}$ J

So the number of electrons emitted per second from 1.0 cm² area of the surface of potassium will be $= \frac{5\times10^{-7}}{2.314\times10^{-19}}$

$= \mathbf{2.16\times10^{12}}$

Example 22 Calculate the value of Planck's constant from the following data, assuming that the electronic charge e has value of 1.6×10^{-19} Coulomb. A surface when irradiated with light of wavelength 5896 Å emits electrons for which the stopping potential is 0.12 volts. When the same surface is irradiated with light of wavelength 2830 Å, it emits electrons for which the stopping potential is 2.2 volts.

Solution If the radiation of wavelength is incident on the surface of the metal having work function ϕ_0 and stopping potential V_0 for the emitted electrons, then ϕ_0 and V_0 satisfy the following relation.

$$eV_0 = \frac{hc}{\lambda} - \phi_0 \qquad \text{(i)}$$

(i) Given $\lambda = 5.896 \times 10^{-7}$m and $V_0 = 0.12$ volts

$$\frac{hc}{\lambda} = eV_0 + \phi_0$$

$$\frac{h\times3\times10^8}{5.896\times10^{-7}} = 1.6\times10^{-19}\times0.12 + \phi_0 \qquad \text{(ii)}$$

(ii) Given $\lambda = 2.83 \times 10^{-7}$ m and $V_0 = 2.2$ volts, then

$$\frac{h\times3\times10^8}{2.83\times10^{-7}} = 1.6\times10^{-19}\times2.2 + \phi_0 \qquad \text{(iii)}$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$h\left[\frac{3\times10^{8}}{2.83\times10^{-7}}-\frac{3\times10^{8}}{5.896\times10^{-7}}\right]=\left[1.6\times10^{-19}\times2.2-1.6\times10^{-19}\times0.12\right]$$

$$h\times\frac{3\times10^{15}[5.896-2.83]}{2.83\times5.896}=1.6\times10^{-19}\times2.08$$

$$h=\mathbf{6.04\times10^{-34}\ Jsec}$$

Example 23 Calculate Compton shift if X-rays of wavelength 1.0 Å are scattered from a carbon block. The scattered radiation is viewed at 90° to the incident beam.

Solution Given $\lambda = 1.0$ Å $= 10^{-10}$m and $\phi = 90°$.

Formula used is

$$\Delta\lambda=\frac{h}{m_0c}(1-\cos\phi).$$

$$=\frac{6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos90^{\circ})$$

$$=0.242\times10^{-11}\text{m}$$

$$=0.024\times10^{-10}\text{m}$$

$$=\mathbf{0.0242\ Å}$$

Example 24 An X-ray photon is found to have doubled its wavelength on being scattered by 90°. Find the energy and wavelength of incident photon.

Solution Given $\phi = 90°$.

Formula used is

$$\Delta\lambda=\frac{h}{m_0c}(1-\cos\phi). \quad \text{(i)}$$

$$=\frac{6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos90^{\circ})$$

$$=\mathbf{0.242\times10^{-11}m=0.024\ Å}$$

As $\Delta\lambda = \lambda' - \lambda$, where λ is the wavelength of incident photon and λ' is the wavelength of scattered photon, then

$$\lambda' = \lambda + \Delta\lambda \quad \text{(ii)}$$

Given

$$\lambda' = 2\lambda \quad \text{(iii)}$$

From Eqs. (2) and (3), we get

$$2\lambda = \lambda + \Delta\lambda$$

or $\lambda = \Delta\lambda = 0.0242 \times 10^{-10}$ m $= 0.0242$ Å

Energy of the incident photon (E) $= h\nu = \dfrac{hc}{\lambda}$

$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{0.0242\times10^{-10}} = \mathbf{0.513\ MeV}$$

Example 25 Calculate the wavelength of incident X-ray photon which produces recoil electron of energy 4.0 KeV in Compton effect. The electron recoils in the direction incident photon and photon is scattered at an angle of 180º.

Solution $\phi = 180^\circ$ and energy of the recoiled electron = 4000 eV.

Let λ be the wavelength of incident X-ray photon and λ' be the scattered photon, then according to the law of conservation of energy.

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = \text{Kinetic energy of the recoiled electron}$$

$$= \frac{1}{2}mv^2 = 4\times10^3\ \text{eV} = 4\times10^3\times1.6\times10^{-19}\ \text{J}$$

$$\text{or} \quad \frac{hc}{\lambda} - \frac{hc}{\lambda'} = 6.4\times10^{-16}\ \text{J} \qquad \text{(i)}$$

According to the principal of conservation of linear momentum in the direction incident photon

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + mv\cos\theta \qquad \text{(ii)}$$

$$= \frac{h}{\lambda'}\cos180^\circ + mv\cos0^\circ = -\frac{h}{\lambda'} + mv$$

$$\frac{h}{\lambda} + \frac{h}{\lambda'} = mv \qquad \text{(iii)}$$

Momentum ($p = mv$) can be calculated as

$$\frac{1}{2}mv^2 = 4.0k\,\text{eV} = 4\times10^3\times1.6\times10^{-19}\ \text{J}$$

$$= 6.4\times10^{-16}\ \text{J}$$

$$mv^2\frac{m}{m} = 2\times6.4\times10^{-16}\ \text{J}$$

$$(mv)^2 = 2m\times6.4\times10^{-16}\ \text{J} = 2\times9.1\times10^{-31}\times6.4\times10^{-16}$$

$$= 1.1648\times10^{-45} = 11.648\times10^{-46}$$

$$mv = 34.13\times10^{-24}\ \text{kg m sec}^{-1} \qquad \text{(iv)}$$

By using Eqs. (iii) and (iv) then, we get

$$\frac{h}{\lambda} + \frac{h}{\lambda'} = 34.13\times10^{-24}$$

Multiplying by velocity of light

$$\frac{hc}{\lambda} + \frac{hc}{\lambda'} = 102.4\times10^{-16} \qquad \text{(v)}$$

By adding Eq. (v) with Eq. (i), we get

$$2\frac{hc}{\lambda} = (102.4 + 6.4) \times 10^{-16}$$

$$= 108.79 \times 10^{-16}$$

$$\lambda = \frac{2hc}{108.79 \times 10^{-16}} = 0.365 \times 10^{-10}\,\text{m}$$

$$\lambda = \mathbf{0.365Å}$$

Example 26 X-rays with $\lambda = 1$ Å are scattered from a carbon block. The scattered radiation is vie[wed at] 90^0 to the incident beam.

(i) What is Compton shift $\Delta\lambda$?

(ii) What kinetic energy is imparted to the recoil electron?

Solution Given $\lambda = 1 \times 10^{-10}$ m.

Formula used for Compton shift is

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi).$$

$$\phi = 90°$$

$$\Delta\lambda = \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^8}(1 - \cos 90°)$$

$$= 2.425 \times 10^{-12}\,\text{m}$$

Let λ be the wavelength of incident X-ray photon and λ' be the scattered photon, then according to the law of conservat[ion] of energy

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_k = \frac{hc}{\lambda + \Delta\lambda} + E_k$$

$$E_k = \frac{hc}{\lambda} - \frac{hc}{\lambda + \Delta\lambda} = \frac{hc\Delta\lambda}{\lambda(\lambda + \Delta\lambda)}$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8 \times 2.425 \times 10^{-12}}{1 \times 10^{-10} \times (1 + 0.02425) \times 10^{-10}}$$

$$= 47.02 \times 10^{-18}\,\text{J}$$

$$= \mathbf{294\ eV}$$

Example 27 X-ray of wavelength 0.144 Å are scattered from a carbon target. Find maximum shift i[n] wavelength and maximum energy of recoil electron.

Solution Given $\lambda = 0.144 \times 10^{-10}$ m.

Formula used for Compton shift

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi).$$

The Compton shift will be maximum if $\phi = 180°$

$$[\Delta\lambda]_{max} = \frac{6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos180°)$$
$$= \frac{2\times6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}$$
$$= 0.485\times10^{-11}\text{m}$$
$$= 0.0485\text{Å}$$

The kinetic energy of the recoil electron is given by the relation

$$E_k = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$
$\lambda' = \lambda + \Delta\lambda$, then
$$E_k = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda+\Delta\lambda}\right]$$
$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{10^{-10}}\left[\frac{1}{0.144} - \frac{1}{0.144+0.0485}\right]$$
$$= 34.75\times10^{-16}\text{J}$$
$$= \mathbf{21.72\,keV}$$

Example 28 X- rays of wavelength 0.2 Å are scattered from a target. Calculate the wavelength of X-ray scattered through 45°. Also find the maximum kinetic energy of the recoil electron.

Solution Given λ = 0.2 Å = 0.2 × 10^{-10} m and ϕ = 45°.

$$\Delta\lambda = \frac{h}{mc}(1-\cos45°) = \frac{6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}[1-0.7071] = 0.0071\text{ Å}$$

Therefore, wavelength of scattered X-rays

$$\lambda' = \lambda + \Delta\lambda = 0.2 + 0.0071 = 0.2071\text{ Å}$$
$$E_k = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

Thus kinetic energy is maximum if λ' is maximum. The maximum value of λ' can be obtained by the relation $\lambda' = \lambda + \Delta\lambda_m$. Maximum value of $\Delta\lambda$ is obtained at ϕ = 180°.

$$\Delta\lambda_m = \frac{h}{m_0c}(1-\cos\phi) = \frac{6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos180°)$$
$$= \frac{2\times6.62\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}} = 0.0485\text{ Å}$$
$$\lambda' = 0.2 + 0.0485$$
$$= 0.2485\text{ Å}$$

Hence, maximum kinetic energy i.e

$$E_k = \frac{6.62\times10^{-34}\times3\times10^{8}}{10^{-10}}\left[\frac{1}{0.2} - \frac{1}{0.2485}\right]$$
$$= \mathbf{19.38\times10^{-16}\,J}$$

Example 29 Calculate the deBroglie wavelength associated with the automobile of mass 2×10^3 kg which is moving with a speed 96 km/hr.

Solution Given $m = 2\times10^3\,\text{kg},\ v = \dfrac{96\times10^3}{60\times60}\,\text{m/sec} = 26.67\,\text{m/sec.}$

deBroglie wavelength is given as

$$\lambda = \frac{h}{mv} = \frac{6.62\times10^{-34}}{2\times10^3\times26.67}$$
$$= 0.124\times10^{-37}\,\text{m}$$
$$= \mathbf{1.24\times10^{-38}\,m}$$

Example 30 A particle of charge q and mass m is accelerated through a potential difference V. Find its deBroglie wavelength. Calculate the wavelength (λ), if the particle is an electron and V = 50 volts.

Solution When a particle of charge q and mass m is accelerated through a potential V, then deBroglie wavelength is given by

$$\lambda = \frac{h}{mv} \tag{i}$$

and $$E_k = \frac{1}{2}mv^2 = qV \quad \text{or} \quad m^2v^2 = 2mqV$$

or $$mv = \sqrt{2mqV} \tag{ii}$$

By using Eqs. (i) and (ii), we obtain

$$\lambda = \frac{h}{\sqrt{2mqV}}$$

Given $q = 1.6 \times 10^{-19}$ C and V = 50 volts, then

$$\lambda = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times50}}$$
$$= \mathbf{1.74\ \text{Å}}$$

Example 31 Calculate the wavelength of thermal neutrons at 27°C, given mass of neutron = 1.67×10^{-27}kg, Planck's constant $h = 6.6 \times 10^{-34}$ J·s and Boltzmann's constant $k = 1.37\times10^{-23}\text{JK}^{-1}$.

Solution Given $T = 27°C = 27 + 273 = 300K$, $m = 1.67\times10^{-27}$kg, $h = 6.6\times10^{-34}$ Jsec and $k = 1.376\times10^{-23}\text{JK}^{-1}$.

deBroglie wavelength is given by

$$\lambda = \frac{h}{mv} \tag{i}$$

$$E_k = \frac{1}{2}mv^2 = \frac{3}{2}kT \quad \text{or} \quad (mv)^2 = 3mkT \tag{ii}$$

or $$mv = \sqrt{3mkT}$$

Then, $\lambda = \dfrac{h}{\sqrt{3mkT}} = \dfrac{6.6\times10^{-34}}{\sqrt{3\times1.67\times10^{-27}\times1.376\times10^{-23}\times300}}$

$= 1.452\times10^{-10}\,\text{m}$

or $\lambda = \mathbf{1.452\,\text{Å}}$

Example 32 A proton is moving with a speed 2×10^8 m/sec. Find the wavelength of matter wave associated with it.

Solution Given $v = 2\times10^8$ m/sec.

Formula used for deBroglie wavelength is

$$\lambda = \frac{h}{mv} = \frac{6.62\times10^{-34}}{1.67\times10^{-27}\times2\times10^8}$$

$$= \mathbf{1.98\times10^{-15}\,m}$$

Example 33 The deBroglie wavelength associated with an electron is 0.1Å. Find the potential difference by which the electron is accelerated.

Solution Given $\lambda = 0.1\times10^{-10}$ m.

deBroglie wavelength in terms of potential difference is given by

$$\lambda = \frac{h}{\sqrt{2mqV}}$$

or $2mqV = \dfrac{h^2}{\lambda^2}$

or $V = \dfrac{h^2}{2mq\lambda^2} = \dfrac{(6.62\times10^{-34})^2}{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times(10^{-11})^2}$

$= \mathbf{15.05\,kV}$

Example 34 Calculate the deBroglie wavelength of an α-particle accelerated through a potential difference of 200 volts.

Solution Given V = 200 volts, $q = q_\alpha = 2e = 3.2\times10^{-19}$ C and $m = m_\alpha = 4m_p$.

deBroglie wavelength in terms of potential difference

$$\lambda = \frac{h}{\sqrt{2m_\alpha qV}} = \frac{6.62\times10^{-34}}{\sqrt{2\times4\times1.67\times10^{27}\times2\times1.6\times10^{-19}\times200}}$$

$$= \frac{6.62\times10^{-34}}{92.468\times10^{-23}} = 0.07159\times10^{-11}$$

$$\lambda = \mathbf{7.16\times10^{-13}\,m}$$

Example 35 Calculate the deBroglie wavelength of an average Helium atom in furnace of 400 K. Given $k = 1.38\times10^{-23}$ J/K

Solution Given T = 400 K, $k = 1.38 \times 10^{-23}$ J/K and mass of Helium atom $= 4m_p = 4 \times 1.67 \times 10^{-27}$ kg.

deBroglie wavelength in terms of temperature i.e.

$$\lambda = \frac{h}{\sqrt{3mkT}} = \frac{6.62\times10^{-34}}{\sqrt{3\times4\times1.67\times10^{-27}\times1.38\times10^{-23}\times400}}$$

$$= \frac{6.62\times10^{-34}}{105.176\times10^{-25}} = 0.6294\text{Å}$$

$$\lambda = \mathbf{0.6294Å}$$

Example 36 Calculate the deBroglie wavelength associated with a neutron moving with a velocity of 2000 m/sec.

Solution Given v = 2000 m/sec and $m = 1.67 \times 10^{-27}$ kg.

deBroglie wavelength

$$\lambda = \frac{h}{mv} = \frac{6.62\times10^{-34}}{1.67\times10^{-27}\times2000}$$

$$= 1.98\times10^{-10}\,\text{m}$$

$$= \mathbf{1.98Å}$$

Example 37 Calculate the energy in eV corresponding to a wavelength of 1.0 Å for electron and neutron. Given $h = 6.6 \times 10^{-34}$ J sec, mass of electron $= 9.1 \times 10^{-31}$ kg and mass of the neutron $= 1.7 \times 10^{-27}$ kg.

Solution

Formula used is

$$\lambda = \frac{h}{mv} \quad \text{or} \quad v = \frac{h}{\lambda m}$$

$$\text{or} \quad v = \frac{6.6\times10^{-34}}{1.0\times10^{-10}\times1.7\times10^{-27}}$$

$$= \mathbf{3.88\times10^3\,m/sec}$$

If the velocity is much less than the velocity of light, it can be considered as non-relativistic case and hence deBroglie wavelength can be obtained by the relation.

$$\lambda = \frac{h}{\sqrt{2mE}} \quad \text{or} \quad \lambda^2 = \frac{h^2}{2mE}$$

For Electron

$$\text{or} \quad E = \frac{h^2}{2m\lambda^2} = \frac{(6.62\times10^{-34})^2}{2\times9.1\times10^{-31}\times(10^{-10})^2}$$

$$= \frac{43.8244\times10^{-68}}{18.2\times10^{-51}} = 2.41\times10^{-17}\,\text{J}$$

$$= 1.51\times100 = 151\text{eV}$$

$$\mathbf{E = 151eV}$$

For neutron

$$E = \frac{(6.62 \times 10^{-34})^2}{2 \times 1.7 \times 10^{-27} \times (10^{-10})^2} = \frac{43.8244 \times 10^{-68}}{3.4 \times 10^{-47}}$$
$$= 12.89 \times 10^{-21} \text{J}$$
$$= \mathbf{0.081\ eV}$$

Example 38 Calculate deBroglie wavelength of an electron whose kinetic energy is (i) 500 eV, (ii) 50 eV and (iii) 1.0 eV.

Solution Formula used is

$$\lambda = \frac{h}{\sqrt{2mE}}$$

(i) $E = 500\,\text{eV} = 500 \times 1.6 \times 10^{-19} = 8.0 \times 10^{-17}\,\text{J}$

$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 8.0 \times 10^{-17}}} = 5.486 \times 10^{-11}\,\text{m}$$
$$= \mathbf{0.5486\text{Å}}$$

(ii) $E = 50\,\text{eV} = 50 \times 1.6 \times 10^{-19} = 8.0 \times 10^{-18}$

$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 8 \times 10^{-18}}}$$
$$\lambda = 1.735 \times 10^{-10}\,\text{m}$$

or $\lambda = \mathbf{1.735\text{Å}}$

(iii) $E = 1.0\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$

$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}}}$$
$$\lambda = \mathbf{12.267\text{Å}}$$

Example 39 Calculate the ratio of deBroglie wavelengths associated with the neutrons with kinetic energies of 1.0 eV and 510 eV.

Solution Formula used is

$$\lambda = \frac{h}{\sqrt{2mE}}.$$

For $E = 1.0$ eV $= 1.6 \times 10^{-19}$ J and $m_n = 1.7 \times 10^{-27}$ kg

$$\lambda_1 = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 1.7 \times 10^{-27} \times 1.6 \times 10^{-19}}}$$
$$= 2.838 \times 10^{-11}$$
$$\lambda_1 = \mathbf{0.284\text{Å}}$$

For $E = 510\,eV = 510 \times 1.6 \times 10^{-19} = 816 \times 10^{-19}\,J$

$$\lambda_2 = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 1.7 \times 10^{-27} \times 816 \times 10^{-19}}}$$

$$\lambda_2 = 0.01257\text{Å}$$

$$= \mathbf{0.0126\ \text{Å}}$$

and ratio of deBroglie wavelength is

$$\frac{\lambda_1}{\lambda_2} = \frac{0.284}{0.0126} = \mathbf{22.54 : 1}$$

Example 40 Calculate the ratio of deBroglie waves associated with a proton and an electron each having the kinetic energy as 20 M eV [$m_p = 1.67 \times 10^{-27}$ kg and $m_e = 9.1 \times 10^{-31}$ kg].

Solution Given energy of each proton and electron is $20 \times 10^6 \times 10^{-19}\,J = 3.2 \times 10^{-12}\,J$.

Formula used is

$$\lambda = \frac{h}{\sqrt{2mE}}$$

For proton

$$\lambda_p = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 1.67 \times 10^{-27} \times 3.2 \times 10^{-12}}}$$

$$= 6.4 \times 10^{-15}\,m$$

For electron

$$\lambda_e = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 3.2 \times 10^{-12}}}$$

$$= 2.74 \times 10^{-13}\,m$$

The ratio of λ_p to λ_e is

$$\lambda_p : \lambda_e = \mathbf{1{:}43}$$

Example 41 Calculate the deBroglie wavelength of 1.0 M eV proton. Do we require relativistic calculation?

Solution Given Energy $E = 1.0 \times 10^6 \times 1.6 \times 10^{-19}\,J = 1.6 \times 10^{-13}\,J$

Formula used for velocity of Proton

$$E = \frac{1}{2}mv^2 \quad \text{or} \quad v^2 = \frac{2E}{m}$$

$$\text{or} \quad v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-13}}{1.67 \times 10^{-27}}}$$

$$= \mathbf{1.38 \times 10^7\,m/sec}$$

From the above result it is clear that the velocity of proton is nearly one twentieth of the velocity of light. So the relativistic calculations are not required.

Example 42 Calculate the deBroglie wavelength associated with a proton moving with a velocity equal to $\frac{1}{20}^{th}$ of velocity of light.

Solution Given $v = \frac{c}{20} = \frac{3\times10^8}{20} = 1.5\times10^7$ m/sec and $m = 1.67\times10^{-27}$ kg.

Formula used is

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.62\times10^{-34}}{1.67\times10^{-27}\times1.5\times10^7}$$
$$= \mathbf{2.643\times10^{-14}\,m}$$

Example 43 Calculate the kinetic energy of a proton and an electron so that the deBroglie wavelengths associated with them is the same and equal to 5000 Å.

Solution Given wavelength of proton and electron = 5.0×10^{-7} m.

Formula used in

$$\lambda = \frac{h}{\sqrt{2mE}} \quad \text{or} \quad E = \frac{h^2}{2m\lambda^2}$$

For proton $m = m_p = 1.67 \times 10^{-27}$ kg and $\lambda = 5.0 \times 10^{-7}$ m

$$E = \frac{(6.62\times10^{-34})^2}{2\times1.67\times10^{-27}\times(5.0\times10^{-7})^2}$$
$$= \frac{43.8244\times10^{-68}}{83.5\times10^{-41}} = 0.5248\times10^{-27}\,\text{J}$$
$$= \mathbf{5.248\times10^{-28}\,J}$$

For electron $m = m_e = 9.1 \times 10^{-31}$

$$E = \frac{(6.62\times10^{-34})^2}{2\times9.1\times10^{-31}\times(5\times10^{-7})^2} = \frac{43.8244\times10^{-68}}{4.55\times10^{-43}}$$
$$E = \mathbf{9.63\times10^{-25}\,J}$$

Example 44 Find deBroglie wavelength of an electron in the first Bohr's orbit of hydrogen atom.

Solution Energy of an electron in the first Bohr's orbit of hydrogen atom can be obtained by using the relation

$$E_n = \frac{-13.6}{n^2}$$

$$E_1 = \frac{-13.6}{1^2} = -13.6\text{eV}$$
$$E_1 = -13.6\times1.6\times10^{-19}\,\text{J} = -2.176\times10^{-18}\,\text{J}$$

Magnitude of energy is $= 2.176\times10^{-18}$ J

$$\text{Wavelength } \lambda = \frac{h}{\sqrt{2mE}} = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times21.76\times10^{-19}}}$$
$$= 3.3\times10^{-10}\,\text{m}$$
$$= \mathbf{3.3\,Å}$$

Example 45 Calculate the ratio of deBroglie wavelengths of a hydrogen atom and helium atom at room temperature, when they move with thermal velocities. Given mass of hydrogen atom

$m_H = 1.67 \times 10^{-27}$ kg and mass of helium atom $m_{He} = 4 \times m_p = 4 \times 1.67 \times 10^{-27}$ kg at room temperature T = 27°C = 300 K and Boltzmann's constant k = 1.376×10^{-23} J/K.

Solution deBroglie wavelength can be calculated by the relation

$$\lambda = \frac{h}{\sqrt{3mkT}}$$

For Hydrogen atom

$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{3 \times 1.67 \times 10^{-27} \times 1.376 \times 10^{-23} \times 300}}$$
$$= 1.456 \times 10^{-10}\text{ m}$$
$$\lambda = \mathbf{1.456 Å}$$

For Helium atom

$$\lambda_{He} = \frac{6.62 \times 10^{-34}}{\sqrt{3 \times 4 \times 1.67 \times 10^{-27} \times 1.376 \times 10^{-23} \times 300}}$$
$$= 0.728 \times 10^{-10}\text{ m}$$
$$= \mathbf{0.728 Å}$$

The ratio of wavelengths i.e

$$\frac{\lambda_H}{\lambda_{He}} = \frac{1.456}{0.728} = \frac{2}{1}$$
$$\boldsymbol{\lambda_H : \lambda_{He} = 2 : 1}$$

Example 46 A proton and a deuteron have the same kinetic energy. Which has a longer wavelength?

Solution m_p = mass of porton, $m_d = 2m_p$ and v_p and v_d are the velocities of proton and deuteron.

Kinetic energy of proton is given by

$$E_p = \frac{1}{2} m_p v_p^2$$

and kinetic energy of deuteron is

$$E_d = \frac{1}{2} m_d v_d^2 = \frac{1}{2}(2m_p)v_d^2$$
$$E_d = m_p v_d^2$$

But $E_p = E_d$, then

$$m_p v_d^2 = \frac{1}{2} m_p v_p^2$$

or $$v_d = \frac{v_p}{\sqrt{2}}$$

deBroglie wavelength corresponding to moving proton and deuteron are

$$\lambda_p = \frac{h}{m_p v_p} \quad \text{and}$$

$$\lambda_d = \frac{h}{m_d v_d} = \frac{h}{2m_p \frac{v_p}{\sqrt{2}}} = \frac{h}{\sqrt{2} m_p v_p}$$

$$\frac{\lambda_d}{\lambda_p} = \frac{h}{\sqrt{2}\, m_p v_p} \times \frac{m_p v_p}{h} = \frac{1}{\sqrt{2}}$$

$$\lambda_p = \sqrt{2}\, \lambda_d$$

i.e. proton has a longer wavelength.

Example 47 Find the phase and group velocities of an electron whose deBroglie wavelength is 1.2Å.

Solution Formula used is

$$\lambda = \frac{h}{mv}.$$

v_g = Group velocity = Particle velocity = v

$$v = \frac{h}{m\lambda} = \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 1.2 \times 10^{-10}}$$

$$v = 6.06 \times 10^6 \text{ m/sec} = \text{ group velocity} = v_g$$

$$\text{or} \quad v_g = 6.06 \times 10^6 \text{ m/sec}$$

$$\text{Phase velocity } v_p = \frac{\omega}{k}$$

$$\text{Energy E} = h\nu$$

$$\text{or} \quad \text{E} = \frac{h}{2\pi} 2\pi\nu = \hbar\omega$$

$$\text{and momentum } p = \frac{h}{\lambda}$$

$$\text{or} \quad \text{P} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$$

$$\text{or} \quad v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{\text{E}}{p}$$

$$\text{and} \quad \text{E} = \frac{1}{2}mv^2 \text{ and } p = mv$$

$$\text{or} \quad \text{E} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

$$v_p = \frac{\text{E}}{p}$$

$$v_p = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{h/\lambda}{2m} = \frac{h}{2m\lambda}$$

$$= \frac{6.62 \times 10^{-34}}{2 \times 9.1 \times 10^{-31} \times 1.2 \times 10^{-10}}$$

$$v_p = \mathbf{3.03 \times 10^6 \, m/sec}$$

From the above result it is clear that the phase velocity is just half of group velocity.

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Quantum Mechanics

15.1 INTRODUCTION

The wave like and particle behaviour of electrons and photons have been discussed in the previous chapter. However, all the subatomic particles like protons, neutrons, α particles, etc. show their dual nature, i.e. sometimes they behave as particle and sometimes as wave. Various types of explanation to understand this wave particle duality led to the development of quantum mechanics. Quantum mechanics deals with the behaviour and characteristics of matter, in the subatomic level, and energy. With the development of quantum theory, queries like stability of electron orbits and blackbody radiation could be explained scientifically.

Basics of quantum theory were developed by Planck, Einstein, Schroedinger and Heisenberg. As discussed earlier, *Planck* in 1900 established that all forms of matter emit or absorb energy in units, called quanta. Prior to this theory, it was assumed that energy existed only in the form of electromagnetic waves. In 1905, *Einstein* stated that not only energy but also radiation is quantifiable. He came to the conclusion that the energy (E) of light depends on its frequency (ν) as per the relation $E = h\nu$. *deBroglie* in 1924 proposed the principle of wave particle duality according to which matter and energy are similar and there is no profound difference in terms of their composition and behaviour. According to him, both matter and energy can behave either as waves or particles depending upon the condition. Schroedinger discovered the wave equation and contributed to the development of quantum mechanics. In 1927 *Heisenberg* proposed the uncertainty principle according to which it is impossible to measure the precise values of momentum and position of a subatomic particle. This way the modern quantum theory was developed in the early 20th century. As we have already seen, quantum physics mainly deals with waves and the subatomic particles of matter. For this reason quantum theory is also referred to as quantum wave mechanics.

15.2 HEISENBERG UNCERTAINTY PRINCIPLE

Heisenberg uncertainty principle is perhaps the best known result of the wave particle duality, i.e. the concept of waves or wave packet associated with a moving particle. According to Heisenberg uncertainty principle it is impossible to determine simultaneously the exact position and momentum (or velocity) of a small moving particle like electron.

As discussed earlier, the quantity $|\psi(x, t)|^2 \Delta x$ represents the probability that the particle is within the region between x and $x+\Delta x$. It means that there is an uncertainty in the location of the position of the particle and Δx is a measure of the uncertainty. The uncertainty in the position would be less if Δx is smaller, i.e. if the wave packet is very narrow. The narrow wave packet means the range of wavelength $\Delta\lambda$ between λ and $\lambda+\Delta\lambda$ is smaller or the range of wave numbers Δk between k and $k+\Delta k$ is larger. So Δx is inversely proportional to Δk, i.e.

$$\Delta x \propto \frac{1}{\Delta k}$$

We may approximate this as $\Delta x \Delta k \approx 1$. Taking $\hbar = \frac{h}{2\pi}$, we get $p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$, $\Delta k = \frac{\Delta p}{\hbar}$. Therefore

$$\Delta x \Delta p \approx \hbar$$

The above relation represents the lowest limit of accuracy. Therefore, we can write more generally.

$$\boxed{\Delta x \Delta p \geq \hbar}$$

The principle of uncertainty can also be represented in terms of energy E and time t. Since $\frac{\Delta p}{\Delta t} = \Delta F$, we can write $\frac{\Delta(mv)}{\Delta t} = \Delta F$

$$\text{or} \quad \Delta p = \Delta F \times \Delta t$$

Putting this value of Δp in the expression $\Delta x \Delta p \geq \hbar$ we obtain

$$\Delta x \times (\Delta F \times \Delta t) \geq \hbar$$

$$\text{or} \quad [\Delta F \times \Delta x]\, \Delta t \geq \hbar \quad \text{or}$$

$$\boxed{\Delta E\, \Delta t \geq \hbar}$$

The principle of uncertainty can also be expressed in terms of angular momentum and angle. Suppose we have a particle at a particular angular position θ and its angular momentum is L_θ. Then the limits in the uncertainties $\Delta\theta$ and ΔL_θ are given by the relation $\Delta\theta\, \Delta L_\theta \geq \hbar$.

15.2.1 Mathematical Proof

Heisenberg's uncertainty principle can be proved on the basis of deBroglie's wave concept that a material particle in motion is equivalent to a group of waves or wave packet, the group velocity G being equal to the particle velocity v. Consider a simple case of wave packet which is formed by the superposition of two simple harmonic plane waves of equal amplitudes a and having nearly equal frequencies ω_1 and ω_2. The two waves can be represented by the equations.

$$y_1 = a \sin(\omega_1 t - k_1 x)$$

$$y_2 = a \sin(\omega_2 t - k_2 x)$$

where k_1 and k_2 are their propagation constants and $\frac{\omega_1}{k_1}$ and $\frac{\omega_2}{k_2}$ are their respective phase velocities. The resultant wave due to superposition of these wave is given by

$$y = y_1 + y_2$$

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$$\text{or} \qquad \frac{\Delta k}{2}\Delta x = \pi \qquad \text{(v)}$$

$$\text{or} \qquad \Delta x = \frac{2\pi}{\Delta k}$$

$$\text{but} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{2\pi p}{h}$$

$$\Delta k = \frac{2\pi}{h}\Delta p$$

where Δp is the error (uncertainty) in the measurement of momentum p. Therefore, from Eq. (v)

$$\Delta x = \frac{2\pi h}{2\pi \Delta p} = \frac{h}{\Delta p}$$

$$\text{or} \qquad \Delta p \Delta x = h$$

However, more accurate measurements show that the product of uncertainties in momentum (Δp) and the position (Δx) cannot be less than $h/2\pi$. Therefore

$$\text{or} \qquad \Delta p \Delta x \geq \underset{2}{\hbar}$$

This is the Heisenberg's uncertainty principle.

15.2.2 Applications

Some important applications of uncertainty principle are discussed below.

15.2.2.1 Non-Existence of Electron in the Nucleus

The radius of the nucleus of an atom is of the order of 10^{-14} m. If an electron is confined within the nucleus, the uncertainty in its position must not be greater than 10^{-14} m. According to uncertainty principle for the lowest limit of accuracy

$$\Delta x \Delta p = \frac{h}{2\pi} \qquad \text{(i)}$$

where Δx is uncertainty in the position and Δp is the uncertainty in the momentum.

From Eq. (i),

$$\Delta p = \frac{h}{2\pi \Delta x} = \frac{6.625\times 10^{-34}}{2\times 3.14\times 2\times 10^{-14}} \qquad (\text{as } \Delta x = \text{diameter of nucleus})$$

$$\Delta p = 5.275\times 10^{-21}\,\text{kg m/sec}$$

This is the uncertainty in momentum of the electron. It means the momentum of the electron would not be less than Δp, rather it could be comparable to Δp. Thus

$$p = 5.275\times 10^{-21}\,\text{kg m/sec}$$

The kinetic energy of the electron can be obtained in terms of momentum as

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$$= \frac{(5.275\times10^{-21})^2}{2\times9.1\times10^{-31}} \text{ J}$$

$$= \frac{(5.275\times10^{-21})^2}{2\times9.1\times10^{-31}\times1.6\times10^{-19}} \text{ eV}$$

$$= 95.55\times10^{6}\text{eV}$$

$$\approx 96 \text{ MeV}$$

From the above result, it is clear that the electrons inside the nucleus may exist only when it possess the energy of the order of 96 MeV. However, the maximum possible kinetic energy of an electron emitted by radioactive nuclei is about 4 MeV. Hence, it is concluded that the electron cannot reside inside the nucleus.

15.2.2.2 *Radius of Bohr's First Orbit*

If Δx and Δp be the uncertainties in determining the position and momentum of the electron in the first orbit, then from the uncertainty principle

$$\Delta x \Delta p \approx \hbar$$

$$\text{or} \quad \Delta p \approx \frac{\hbar}{\Delta x} \qquad \text{(i)}$$

The uncertainty in kinetic energy (K.E.) of electron may be written as

$$\Delta T = \frac{(\Delta p)^2}{2m} \qquad \left[\text{K.E.} = T = \frac{p^2}{2m}\right] \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we have

$$\Delta T = \frac{1}{2m}\left[\frac{\hbar}{\Delta x}\right]^2$$

and the uncertainty in the potential energy of the same electron is given by

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(-e)}{\Delta x} \qquad \left[\because V = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(-e)}{x}\right]$$

The uncertainty in the total energy of electron together with Ze as the nucleus charge

$$\Delta E = \Delta T + \Delta V$$

$$= \frac{\hbar^2}{2m(\Delta x)^2} - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta x}$$

The condition for this uncertainty in the energy to be minimum is

$$\frac{d(\Delta E)}{d(\Delta x)} = 0$$

$$\text{or} \quad -\frac{\hbar^2}{m(\Delta x)^3} + \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{(\Delta x)^2} = 0$$

$$\Delta x = \frac{\hbar^2(4\pi\varepsilon_0)}{mZe^2}$$

Hence, the radius of first orbit

$$r = \Delta x = \frac{\hbar^2(4\pi\varepsilon_0)}{mZe^2}$$

or $$r = \frac{\varepsilon_0 h^2}{\pi m Z e^2}$$

This is the radius of first Bohr's orbit.

15.2.2.3 *Energy of a Particle in a Box or Infinite Potential Well*

Let us consider a particle having mass m in infinite potential well of width L. The maximum uncertainty in the position of the particle may be

$$(\Delta x)_{\max} = L$$

From the uncertainty principle

$$\Delta x \Delta p = \hbar$$

or $$\Delta p = \frac{\hbar}{\Delta x} = \frac{\hbar}{L}$$

Kinetic energy

$$T = \frac{p^2}{2m} = \frac{\hbar^2}{2mL^2}$$

$$T = \frac{\hbar^2}{2mL^2}$$

This is the minimum kinetic energy of the particle in an infinite potential well of width L.

15.2.2.4 *Ground State Energy of Linear Harmonic Oscillator*

The total energy E of a linear harmonic oscillator is the sum of its kinetic energy (K.E.) and potential energy (P.E.).

$$E = \text{K.E.} + \text{P.E.}$$

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad \text{(i)}$$

Let a particle of mass m executes a simple harmonic motion along X-axis. The maximum uncertainty in the determination of its position can be taken as Δx. From the uncertainty relation, the uncertainty in momentum is then given by

$$\Delta p = \frac{\hbar}{2\Delta x} \quad \left[\text{Taking } \Delta p \Delta x = \frac{\hbar}{2}\text{; more accuracy}\right] \quad \text{(ii)}$$

Then from Eqs. (i) and (ii), we have

$$E = \frac{(\Delta p)^2}{4m} + \frac{1}{4}k(\Delta x)^2 \qquad [\Delta p \approx p \quad \text{and} \quad \Delta x \approx x]$$

$$E = \frac{1}{4m}\left[\frac{\hbar}{\Delta x}\right]^2 + \frac{1}{4}k(\Delta x)^2$$

$$E = \frac{\hbar^2}{4m(\Delta x)^2} + \frac{1}{4}k(\Delta x)^2 \quad \text{(iii)}$$

For a minimum value of energy,

$$\frac{\partial E}{\partial(\Delta x)} = 0$$

then we get

$$-\frac{\hbar^2}{m(\Delta x)^3} + k(\Delta x) = 0$$

$$\text{or} \quad (\Delta x)^4 = \frac{\hbar^2}{mk}$$

$$\text{or} \quad (\Delta x) = \left(\frac{\hbar^2}{mk}\right)^{1/4} \tag{iv}$$

Substituting value of Δx in Eq. (iii) from Eq. (iv), we get

$$E_{min} = \frac{\hbar^2}{4m}\left(\frac{mk}{\hbar^2}\right)^{1/2} + \frac{1}{4}k\left(\frac{\hbar^2}{mk}\right)^{1/2}$$

$$E_{min} = \frac{\hbar}{2}\left(\frac{k}{m}\right)^{1/2}$$

But $\sqrt{\frac{k}{m}} = \omega =$ angular frequency. Therefore, the minimum energy of harmonic oscillator is expressed by the following relation

$$E_{min} = \frac{1}{2}\hbar\omega$$

15.3 WAVE FUNCTION AND ITS PHYSICAL SIGNIFICANCE

Waves in general are associated with quantities that vary periodically. In case of matter waves, the quantity that varies periodically is called *wave function*. The wave function, represented by ψ, associated with the matter waves has no direct physical significance. It is not an observable quantity. However, the value of the wave function is related to the probability of finding the particle at a given place at a given time. The square of the absolute magnitude of the wave function of a body evaluated at a particular time at a particular place is proportional to the probability of finding the particle at that place at that instant.

The wave functions are usually complex. The probability in such a case is taken as $\psi^*\psi$, i.e. the product of the wave function with its complex conjugate, ψ^* being the complex conjugate. Since the probability of finding a particle somewhere is finite, we have the total probability over all space equal to unity. That is

$$\int_{-\infty}^{\infty} \psi^*\psi \, dV = 1 \tag{i}$$

where $dV = dxdydz$.

Eq. (i) is called the normalisation condition and a wave function that obeys this equation is said to be normalised. Further, ψ must be a single valued since the probability can have only one value at a particular place and time. Besides being normalisable, a further condition that ψ must obey is that it and its partial derivatives $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ and $\frac{\partial\psi}{\partial z}$ be continuous everywhere.

The important characteristics of the wave function are as follows.

(i) ψ must be finite, continuous and single valued everywhere.

(ii) $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ and $\frac{\partial\psi}{\partial z}$ must be finite, continuous and single valued.

(iii) ψ must be normalisable.

15.4 TIME INDEPENDENT SCHROEDINGER EQUATION

Consider a system of stationary waves associated with a moving particle. If the position coordinates of the particle are (x, y, z) and ψ be the periodic displacement for the matter waves at any instant of time t, then we can represent the motion of the wave by a differential equation as follows.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} \tag{i}$$

where u is the velocity of wave associated with the particle. The solution of Eq. (i) gives ψ as a periodic displacement in terms of time, i.e.

$$\psi(x,y,z,t)=\psi_0(x,y,z)e^{-i\omega t} \tag{ii}$$

where ψ_0 is the amplitude of the particle wave at the point (x, y, z) which is independent of time (t). It is a function of (x, y, z). i.e., the position r and not of time t, Here.

$$r = x\hat{i}+y\hat{j}+z\hat{k} \tag{iii}$$

Eq. (ii) may be expressed as

$$\psi(r,t)=\psi_0(r)e^{-i\omega t} \tag{iv}$$

Differentiating Eq. (iv) twice with respect to t, we get

$$\frac{\partial^2\psi}{\partial t^2}=-\omega^2\psi_0(r)e^{-i\omega t}$$

$$\text{or}\quad \frac{\partial^2\psi}{\partial t^2}=-\omega^2\psi \tag{v}$$

Substituting the value of $\frac{\partial^2\psi}{\partial t^2}$ from this equation in Eq. (i), we get

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{\omega^2}{u^2}\psi=0 \tag{vi}$$

where $\omega = 2\pi\nu = 2\pi(u/\lambda)$ [as $u = \lambda\nu$]

so that

$$\frac{\omega}{u} = \frac{2\pi}{\lambda} \tag{vii}$$

Also

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \nabla^2\psi \tag{viii}$$

where ∇^2 is known as Laplacian operator. Using Eqs. (vi), (vii) and (viii), we have

$$\nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0 \tag{ix}$$

Also from the deBroglie wave concept

$$\lambda = \frac{h}{mv}$$

Using this relation in Eq. (ix) gives

$$\nabla^2\psi + \frac{4\pi^2 m^2 v^2}{h^2}\psi = 0 \tag{x}$$

If E and V are respectively the total energy and potential energy of the particle then its kinetic energy is given by

$$\frac{1}{2}mv^2 = \mathrm{E} - \mathrm{V}$$

$$m^2v^2 = 2m(\mathrm{E} - \mathrm{V}) \tag{xi}$$

The use of Eq. (xi) in Eq. (x) gives

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(\mathrm{E} - \mathrm{V})\psi = 0$$

or $$\nabla^2\psi + \frac{2m}{\hbar^2}(\mathrm{E} - \mathrm{V})\psi = 0 \tag{xii}$$

This is the time independent Schroedinger equation, where the quantity ψ is known as *wave function.*

For a freely moving or free particle V = 0. Therefore, Eq. (xii) becomes

$$\nabla^2\psi + \frac{2m\mathrm{E}}{\hbar^2}\psi = 0 \tag{xiii}$$

This is called time independent Schroedinger equation for a free particle.

15.5 TIME DEPENDENT SCHROEDINGER EQUATION

In order to obtain a time dependent Schroedinger equation, we eliminate the total energy E from time independent Schroedinger equation. For this we differentiate Eq. (iv) w.r.t. t and obtain

$$\frac{\partial \psi}{\partial t} = -i\omega\psi_0(r)e^{-i\omega t}$$

$$= -i(2\pi\nu)\psi_0(r)e^{-i\omega t}$$

$$= -2\pi\nu i\psi = -2\pi i\frac{E}{h}\psi = -\frac{iE}{\hbar}\times\frac{i}{i}\psi \quad (2\pi)$$

h

$$\Rightarrow \frac{\partial \psi}{\partial t} = \frac{E\psi}{i\hbar}$$

$$\text{or} \quad E\psi = i\hbar\frac{\partial \psi}{\partial t} \tag{xiv}$$

Substituting the value of $E\psi$ from Eq. (xiv) in Eq. (xii), we have

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left[i\hbar\frac{\partial \psi}{\partial t} - V\psi\right] = 0$$

$$\text{or} \quad \nabla^2\psi = -\frac{2m}{\hbar^2}\left[i\hbar\frac{\partial \psi}{\partial t} - V\psi\right]$$

$$\text{or} \quad \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial \psi}{\partial t} \tag{xv}$$

This equation is known as *Schroedinger's time dependent wave equation*. The operator $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)$ is called *Hamilitonian operator* and is represented by H. If we see the RHS of Eq. (xv) and keep in mind Eq. (xiv), we notice that the operator $i\hbar\frac{\partial}{\partial t}$ operating on ψ gives E. Hence, Schroedinger equation can be written in operator form, as below

$$H\psi = E\psi$$

15.6 OPERATORS

In a physical system, there is a quantum mechanical operator that is associated with each measurable parameter. In quantum mechanics, we deal with waves (wave function) rather than discrete particles whose motion and dynamics can be described with the deterministic equations of Newtonian physics. Generally an operator is anything that is capable of doing something to a function. There is an operator corresponding to every observable quantity. However, the choice of operator is arbitrary in quantum mechanics. When an operator operates on a wave function it must give observable quantity times the wave function. It is a must condition for an operator.

If we consider an operator represented by A corresponding to the observable quantity a, then

$$A\psi = a\psi$$

Wave function that satisfies the above equation is called *eigen function* and corresponding observable quantity is called *eigen value* and the equation is called eigen value equation. Some of those operators are tabled below.

Classical Quantity	Quantum Mechanical Operator
Position $x, y, z,$	x, y, z
Momentum p	$-i\hbar\vec{\nabla}$
Momentum components p_x, p_y, p_z	$-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}$
Energy E	$i\hbar\frac{\partial}{\partial t}$
Hamiltonian (Time independent)	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(r)$
Kinetic energy	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}$

15.7 APPLICATIONS OF SCHROEDINGER EQUATION

Schroedinger's equation is extremely useful for investigating various quantum mechanical problems. With the help of this equation and boundary conditions, the expression for the wave function is obtained. Then the probability of finding the particle is calculated by using the wave function. In the following subsections, we discuss different quantum mechanical problems, viz. particle in a box, one-dimensional harmonic oscillator, step potential and step barrier.

15.7.1 Particle in a Box (Infinite Potential Well)

The simplest quantum mechanical problem is that of a particle trapped in a box with infinitely hard walls. Infinitely hard walls means the particle does not loose energy when it collides with such walls, i.e. its total energy remains constant. A physical example of this problem could be a molecule which is strictly confined to a box. Let us consider a particle restricted to move along the X-axis between $x = 0$ and $x = L$, by ideally reflecting, infinitely high walls of the infinite potential well, as shown in Fig. 15.2. Suppose that the potential energy V of the particle is zero inside the box, but rises to infinity outside, that is,

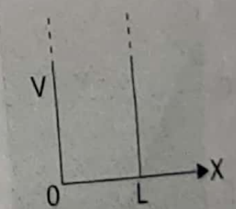

Fig. 15.2

$$V = 0 \quad \text{for} \quad 0 \le x \le L$$

$$V = \infty \quad \text{for} \quad x < 0 \quad \text{and} \quad x > L$$

In such a case, the particle is said to be moving in an infinitely deep potential well. In order to evaluate the wave function ψ in the potential well, Schoredinger equation for the particle within the well (V = 0) is written as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\psi = 0 \tag{i}$$

We put $\frac{8\pi^2 mE}{h^2} = k^2$ in the above equation for getting

$$\frac{\partial^2\psi}{\partial x^2} + k^2\psi = 0 \tag{ii}$$

The general solution of this differential equation is

$$\psi(x) = A \sin kx + B \cos kx \tag{iii}$$

where A and B are constants.

Applying the boundary condition $\psi(x) = 0$ at $x = 0$, which means the probability of finding particle at the wall $x = 0$ is zero, we obtain

$$A \sin (0) + B \cos (0) = 0 \quad \Rightarrow \quad B = 0$$

Again, we have $\psi(x) = 0$ at $x = L$, then

$$A \sin kL + B \cos kL = 0 \quad \Rightarrow \quad A \sin kL = 0$$

The above equation is satisfied when

$$kL = n\pi$$

$$\text{or} \quad \boxed{k = \frac{n\pi}{L}} \quad \text{where } n = 1, 2, 3, \ldots$$

$$\text{or} \quad k^2 = \frac{n^2\pi^2}{L^2} \tag{iv}$$

$$\text{or} \quad \frac{8\pi^2 mE}{h^2} = \frac{n^2\pi^2}{L^2} \tag{v}$$

or in general we can write Eq. (v) as

$$\boxed{E_n = \frac{n^2 h^2}{8mL^2}} \quad \text{where } n = 1, 2, 3, \ldots$$

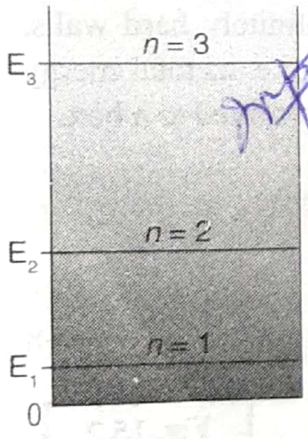

Fig. 15.3

Thus, it can be concluded that in an infinite potential well the particle cannot have an arbitrary energy, but can take only certain discrete energy values corresponding to $n = 1, 2, 3, \ldots$. Thses are called the *eigen values* of the particle in the well and constitutes the energy levels of the system. The integer n corresponding to the energy level E_n is called its *quantum number*, as shown in Fig. 15.3.

We can also calculate the momentum p of the particle or the eigen values of the momentum, as follows,

$$\text{Since } k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{p}{\hbar}$$

$$\boxed{p = \hbar k = \frac{n\pi\hbar}{L}}$$

The wave function (or eigen function) is given by Eq. (iii) along with the use of expression for k.

$$\psi_n(x) = A \sin \frac{n\pi x}{L}$$

To find the value of A, we use the normalisation condition.

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 \, dx = 1$$

As mentioned earlier, the above expression simply says that the probability of finding the particle is 1. In the present case, the particle is within the box i.e. between $0 < x < L$. So the normalisation condition becomes

$$A^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$$

$$A^2\left(\frac{L}{2}\right) = 1 \quad \text{or} \quad A = \sqrt{\frac{2}{L}}$$

The normalised eigen wave function of the particle is, therefore, given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

The first three eigen functions ψ_1, ψ_2, ψ_3 together with the probability densities $|\psi_1|^2, |\psi_2|^2, |\psi_3|^2$, are shown in Figs. 15.4 (a) and (b), respectively.

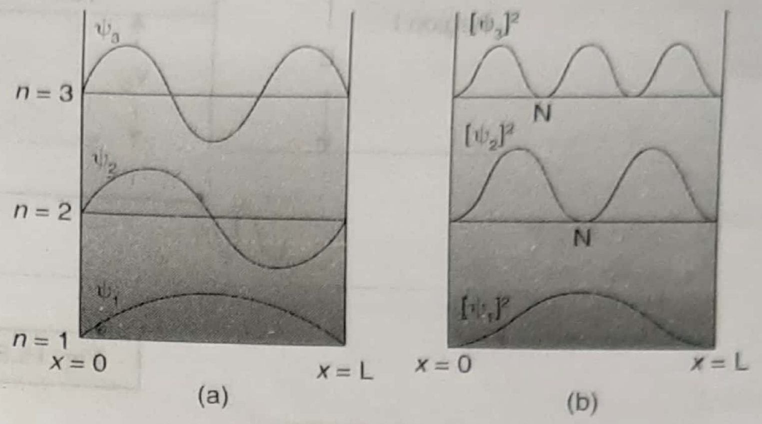

Fig. 15.4

Classical mechanics predicts the same probability for the particle being any where in the well. Wave mechanics, on the other hand, predicts that the probability is different at different points and there are points (nodes) where the particle is never found. Further, at a particular point, the probability of finding the particle is different for different energy states. For example, a particle in the lowest energy state ($n = 1$) is more likely to be in the middle of the box, while in the next energy state ($n = 2$) it is never there since $|\psi_2|^2$ is zero there. It is $|\psi_n|^2$ which provides the probability of finding the particle within the potential well.

15.7.2 Finite Potential Step

A physical example of this quantum mechanical problem can be thought as the neutron which is trying to escape nucleus. The potential function of a potential step may be represented as

$$V(x) = 0 \quad \text{for } x < 0 \text{ region I}$$

$$V(x) = V_0 \quad \text{for } x > 0 \text{ region II} \qquad \text{(i)}$$

We consider that a particle of energy E is incident from left on the potential step of height V_0 as shown in Fig. 15.5. Further, we assume that the energy of the incident particle is greater than the step barrier height i.e. $E > V_0$. Since $E > V_0$, according to classical theory there should be no reflection at the boundary of the step potential barrier. However, quantum mechanically this is not true. It means that there will be some reflection from the boundary of the potential step.

The wavelength of the particle suddenly changes from region I to region II and is given as follows

$$\lambda_1 = \frac{h}{p_1} = \frac{h}{\sqrt{2mE}} \qquad \text{(ii)}$$

and

$$\lambda_2 = \frac{h}{p_2} = \frac{h}{\sqrt{2m(E - V_0)}} \qquad \text{(iii)}$$

Hence, a small part of the wave associated with the particle is reflected due to this change in wavelength and the rest part is transmitted. This can be proved with the solution of Schroedinger wave equations for two regions. The Schroedinger equation for region I is written as

As mentioned earlier, the above expression simply says that the probability of finding the particle is 1. In the present case, the particle is within the box i.e. between $0 < x < L$. So the normalisation condition becomes

$$A^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$$

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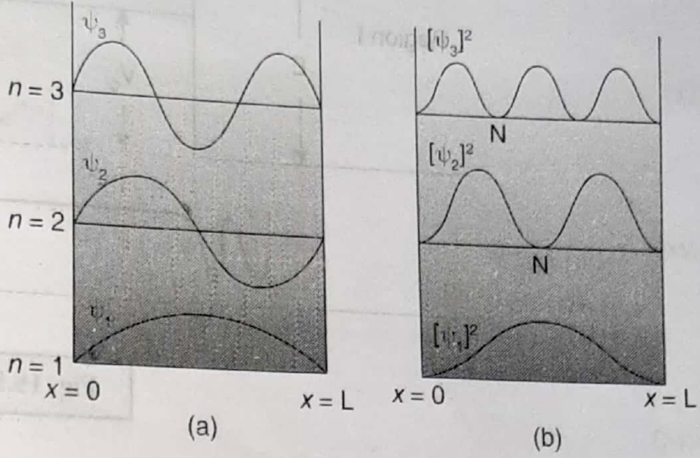

Fig. 15.4

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Hence, a small part of the wave associated with the particle is reflected due to this change in wavelength and the rest part is transmitted. This can be proved with the solution of Schroedinger wave equations for two regions. The Schroedinger equation for region I is written as

15.8 QUANTUM STATISTICS

Classical statistics, i.e. Maxwell-Boltzmann statistics, successfully explained the energy and velocity distribution of molecules of an ideal gas but it failed to explain the energy distribution of electrons in metals, for example, electron gas and the energy distribution of photons in a photon gas. These phenomena can be explained on the basis of quantum statistics, where the particles of the system are considered to be indistinguishable contrary to the consideration of particles as distinguishable in classical statistics. If n_i particles are distributed in the g_i cell of the i^{th} compartment in the phase space, then the number of particles per cell is defined as $n_i(E)/g_i(E)$. The factor $n_i(E)/g_i(E)$ is called *occupation index*. If $n_i(E)/g_i(E) \geq 0$ or 1 the particles are considered as indistinguishable which is the basic feature of the quantum statistics. If the indistinguishable particles have integral spin, we use the Bose-Einstein distribution function and if the particles have half-integral spins, then Fermi-Dirac distribution function is appropriate. The brief description of these statistics is given below.

15.8.1 Bose-Einstein Statistics

It is applicable to those systems which contain indentical, indistinguishable particles of zero or integral spins. Such particles are called *bosons*. Examples of bosons are photons, phonons etc. Pauli exclusion principle does not apply to the bosons. Bose-Einstein distribution law is given by

$$n_i(E) = \frac{g_i(E)}{e^{\alpha+\beta E} - 1} \qquad \text{(i)}$$

where $\alpha = e^{-E_F/kT}$ and $\beta = 1/kT$. This law is also applicable in the case of photon gas for which $\alpha = 0$ and $E = h\nu$. For the photon gas, then above equation reads

$$n_i(E) = \frac{g_i(E)}{e^{\beta E} - 1} \qquad \text{(ii)}$$

The plot of $n_i(E)/g_i(E)$ versus E is shown in Fig. 15.9 for two different temperatures with $T_2 > T_1$. If $E >> kT$, the exponential term in the above equation is very large and -1 may be dropped. It means

$$n_i(E) = g_i(E)e^{-E/kT}$$

The above relation represents Maxwell-Boltzmann statistics. So the Bose-Einstein statistics reduces to the Maxwell-Boltzmann statistics under the condition $E >> kT$.

At low energy, i.e. when $E << kT$, $e^{\beta E}$ can be neglected as -1 predominates. This makes $n_i(E)/g_i(E)$ much larger for Bose-Einstein statistics than for Maxwell-Boltzmann statistics at low energies.

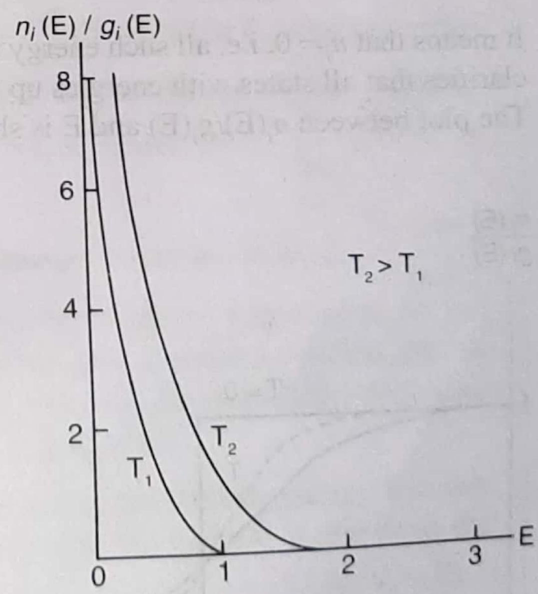

Fig. 15.9

15.8.2 Fermi-Dirac Statistics

This statistics is applicable to systems, which consist of identical, independent and indistinguishable particles of having half-integral spins. The particles, which obey Fermi-Dirac statistics, are called *fermions*. The examples of fermions are electrons, protons, neutrons, etc. The fermion must obey Pauli exclusion principle. In Fermi-Dirac statistics, interchange of two particles of the system leaves the resultant system in an antisymmetric state. That is, the wave function of the system gets changed only with minus sign. As it obeys the Pauli exclusion principle, in Fermi-Dirac statistics, there can be only one particle in each state. Hence, the total number of particles must be less than or equal to the total number of states available. Under these considerations, fermions lead to the following distribution law, named Fermi-Dirac distribution law, given by

$$n_i(E) = \frac{g_i(E)}{e^{\alpha+\beta E}+1} \tag{i}$$

where $\alpha = -E_F/kT$ and $\beta = 1/kT$.

So,

$$n_i(E) = \frac{g_i(E)}{e^{(E-E_F)/kT}+1} \tag{ii}$$

$$\frac{n_i(E)}{g_i(E)} = \frac{1}{e^{(E-E_F)/kT}+1} \tag{iii}$$

In the above equations, n_i is the number of particles in an energy state E, g_i is the statistical weight factor and E_F is the Fermi energy. Fermi energy is independent of temperature. The plots of $n_i(E)/g_i(E)$ versus E for different temperatures is shown in Fig. 15.10.

Now the different cases will be discussed.

(a) **At T = 0 K and when $E < E_F$, Eq. (iii)**

$$\frac{n_i(E)}{g_i(E)} = 1 \tag{iv}$$

It means that $n_i(E) = g_i(E)$, i.e. all the energy states will have one electron each.

(b) **At T = 0 K and $E > E_F$,**

$$\frac{n_i(E)}{g_i(E)} = 0 \tag{v}$$

It means that $n_i = 0$, i.e. all such energy states which have energies greater than Fermi energy are vacant. This clarifies that all states with energies up to E_F are filled while all states with energy greater than E_F are vacant. The plot between $n_i(E)/g_i(E)$ and E is shown in Fig. 15.11 for these conditions at T = 0 K.

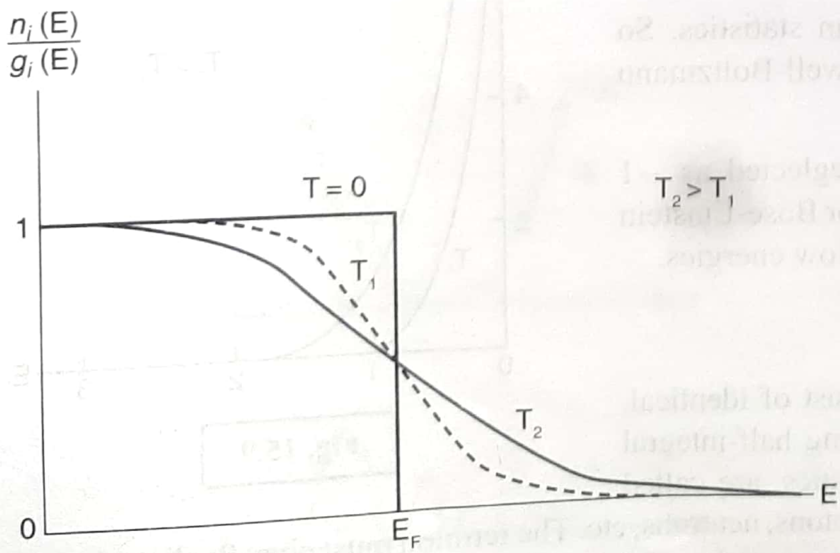

Fig. 15.10

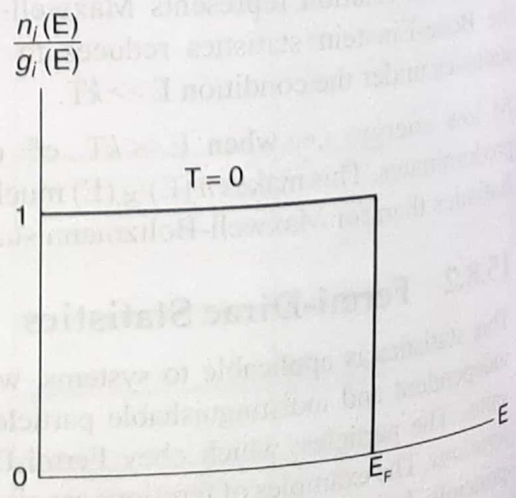

Fig. 15.11

Example 1 The position and momentum of a 1.0 keV electron are simultaneously measured. If the position is located within 1Å, what is the percentage of uncertainty in momentum?

Solution Given $\Delta x = 1.0 \times 10^{-10}$ m and $E = 1000 \times 1.6 \times 10^{-19} J = 1.6 \times 10^{-16}$ J.

Heisenberg's uncertainty principle says

$$\Delta x \Delta p = \frac{\hbar}{2} \quad \text{and} \quad p = \sqrt{2mE}$$

$$p = \sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-16}}$$

$$= 1.71 \times 10^{-23} \text{kg m/sec}$$

and

$$\Delta p = \frac{\hbar}{2\Delta x} = \frac{h}{2 \times 2\pi \times \Delta x} = \frac{6.62 \times 10^{-34}}{2 \times 2 \times 3.14 \times 1.0 \times 10^{-10}}$$

$$= 5.27 \times 10^{-25} \text{ kg m/sec}$$

Percentage of uncertainty in momentum

$$= \frac{\Delta p}{p} \times 100 = \frac{5.27 \times 10^{-25}}{1.71 \times 10^{-23}} \times 100$$

$$= \mathbf{3.1\%}$$

Example 2 The uncertainty in the location of a particle is equal to its deBroglie wavelength. Calculate the uncertainty in its velocity.

Solution Given $\Delta x = \frac{h}{p}$.

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$$\Delta x\,\Delta p = \frac{\hbar}{2} = \frac{h}{4\pi}$$

or $$\Delta p = \Delta(mv) = \frac{h}{4\pi}\frac{1}{\Delta x} = \frac{h}{4\pi}\frac{p}{h} = \frac{mv}{4\pi}$$

$$m\Delta v = \frac{mv}{4\pi}$$

or $$\Delta v = \frac{v}{4\pi}$$

Example 3 The position and momentum of 0.5 keV electron are simultaneously determined. If its position is located within 0.2 nm, what is the percentage uncertainty in its momentum?

Solution Given $E = 0.5 \times 10^3 \times 1.6 \times 10^{-19} = 0.8 \times 10^{-16}$ j and $\Delta x = 0.2 \times 10^{-9}$ m.

Now

$$\Delta x \Delta p = \frac{\hbar}{2} \text{ and momentum } p = \sqrt{2mE}$$

so $$p = \sqrt{2 \times 9.1 \times 10^{-31} \times 0.8 \times 10^{-16}} = 12.06 \times 10^{-24}$$

or $$p = 1.21 \times 10^{-23} \text{ kg m/sec}$$

or $$\Delta p = \frac{\hbar}{2}\frac{1}{\Delta x} = \frac{h}{4\pi}\frac{1}{0.2 \times 10^{-9}}$$

$$\Delta p = \frac{6.62 \times 10^{-34}}{4 \times 3.14 \times 0.2 \times 10^{-9}} = 2.635 \times 10^{-25} \text{kg m/sec}$$

∴ Percentage uncertainty in momentum

$$\frac{\Delta p}{p} \times 100 = \frac{2.635 \times 10^{-25}}{1.21 \times 10^{-23}} \times 100$$

$$= \frac{2.635 \times 10^{-23}}{1.21 \times 10^{-23}} = \mathbf{2.18\%}$$

Example 4 Wavelengths can be determined with accuracies of one part in 10^6. What is the uncertainty in the position of a 1Å X-ray photon when its wavelength is simultaneously measured?

Solution Given $\lambda = 10^{-10}$ m.

By uncertainty principle,

$$\Delta x \Delta p = \frac{\hbar}{2} = \frac{h}{4\pi} \tag{i}$$

and $\lambda = \dfrac{h}{p}$ or $p\lambda = h$

By differentiating

$$p\Delta\lambda + \lambda\Delta p = 0$$

$$\text{or} \quad \Delta p = -\frac{p\Delta\lambda}{\lambda} = -\frac{h\Delta\lambda}{\lambda^2} \qquad \left[\because p = \frac{h}{\lambda}\right] \qquad \text{(ii)}$$

By using Eqs. (i) and (ii), we get

$$\Delta x \frac{h\Delta\lambda}{\lambda^2} = \frac{h}{4\pi}$$

$$\text{or} \quad \Delta x \Delta\lambda = \frac{\lambda^2}{4\pi} \qquad \text{(iii)}$$

Wavelength can be measured with accuracy of one part in 10^6, it means the uncertainty in wavelength is

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{10^6} = 10^{-6} \qquad \text{(iv)}$$

By putting this value in Eq. (iii), then

$$\Delta x \frac{\Delta\lambda}{\lambda} = \frac{\lambda}{4\pi} \quad \text{or} \quad \Delta x \times 10^{-6} = \frac{\lambda}{4\pi}$$

$$\text{or} \quad \Delta x = \frac{10^6 \times \lambda}{4\pi} = \frac{10^6 \times 10^{-10}}{4 \times 3.14} = \mathbf{7.96\ \mu m}$$

Example 5 Calculate the uncertainty in measurement of momentum of an electron if the uncertainty in locating it is 1Å.

Solution Given $\Delta x = 1.0 \times 10^{-10}$ m.

Formula used is

$$\Delta x \Delta p = \frac{\hbar}{2} = \frac{h}{4\pi}$$

$$\Delta p = \frac{h}{4\pi}\frac{1}{\Delta x} = \frac{6.62 \times 10^{-34}}{4 \times 3.14} \times \frac{1}{10^{-10}}$$

$$\Delta p = \mathbf{5.27 \times 10^{-25}\ kg\ m/sec}$$

Example 6 An electron has a momentum 5.4×10^{-26} kg m/sec with an accuracy of 0.05%. Find the minimum uncertainty in the location of the electron.

Solution Given $p = 5.4 \times 10^{-26}$ kg m/sec.

The uncertainty in the measurement of momentum

$$\Delta p = \frac{5.4 \times 10^{-26} \times 0.05}{100}$$

$$= 2.7 \times 10^{-29}\ \text{kg m/sec}$$

$$\Delta x \Delta p = \frac{\hbar}{2} = \frac{h}{4\pi}$$

$$\therefore \ \Delta x = \frac{h}{4\pi}\frac{1}{\Delta p} = \frac{6.62\times10^{-34}}{4\times3.14}\times\frac{1}{2.7\times10^{-29}}$$
$$= 1.952\times10^{-6}\ \text{m}$$
$$= \mathbf{1.952\ \mu m}$$

Example 7 A hydrogen atom is 0.53 Å in radius. Use uncertainty principle to estimate the minimum energy an electron can have in this atom.

Solution Given $\Delta x_{max} = 0.53$ Å.

Heisenberg's uncertainty principle

$$\Delta x \Delta p = \frac{\hbar}{2} = \frac{h}{4\pi}$$

$$(\Delta x)_{max}(\Delta p)_{min} = \frac{h}{4\pi}$$

and $$(\text{K.E.})_{min} = \frac{p^2_{min}}{2m} = \frac{(\Delta p)^2_{min}}{2m} \qquad [\because p_{min} = \Delta p_{min}] \qquad \text{(i)}$$

$$(\Delta p)_{min} = \frac{h}{4\pi}\frac{1}{\Delta x} = \frac{6.62\times10^{-34}}{4\times3.14}\frac{1}{0.53\times10^{-10}}$$
$$= 0.9945\times10^{-24}$$
$$= 9.945\times10^{-25}\ \text{kg m/sec}$$

and $$(\text{K.E.})_{min} = \frac{(\Delta p)^2_{min}}{2m} = \frac{(9.945\times10^{-24})^2}{2\times9.1\times10^{-31}}$$
$$= \mathbf{5.434\times10^{-17}\ J}$$

Example 8 The speed of an electron is measured to be 5.0×10^3 m/sec to an accuracy of 0.003%. Find the uncertainty in determining the position of this electron.

Solution Given $v = 5.0\times10^3$ m/sec.

Formula used is

$$\Delta x \Delta p = \frac{\hbar}{2} = \frac{h}{4\pi}$$

$$\Delta v = v\times\frac{0.003}{100} = 5.0\times10^3\times\frac{0.003}{100} = 0.15\ \text{m/sec}$$

and $$\Delta p = m\Delta v = 9.1\times10^{-31}\times0.15 = 1.365\times10^{-31}\ \text{kg m/sec}$$

$$\Delta x = \frac{6.62\times10^{-34}}{4\times3.14}\frac{1}{\Delta p} = \frac{6.62\times10^{-34}}{4\times3.14}\frac{1}{1.365\times10^{-31}}$$
$$= \mathbf{3.861\times10^{-4}\ m}$$

Example 9 An electron has speed of 6.6×10^4 m/sec with an accuracy of 0.01%. Calculate the uncertainty in position of an electron. Given mass of an electron as 9.1×10^{-31} kg and Planck's constant h as 6.6×10^{-34} J sec.

Solution Given $v = 6.6\times10^4$ m/sec and $\Delta v = 6.6\times10^4\times\frac{0.01}{100}$ m/sec

$$= 6.6\ \text{m/sec}.$$

Formula used is

$$\Delta x\,\Delta p=\frac{\hbar}{2}=\frac{h}{4\pi} \quad \text{or} \quad \Delta x=\frac{h}{4\pi}\frac{1}{\Delta p}$$

$$\Delta p=m\Delta v=9.1\times10^{-31}\times6.6$$

$$\text{or} \quad \Delta x=\frac{h}{4\pi}\frac{1}{\Delta p}=\frac{6.6\times10^{-34}}{4\times3.14\times9.1\times10^{-31}\times6.6}$$

$$\Delta x=\mathbf{8.75\times10^{-6}\,m}$$

Example 10 Calculate the smallest possible uncertainty in the position of an electron moving with a velocity 3×10^{7} m/sec.

Solution Given $v=3\times10^{7}$ m/sec.

Formula used is

$$\Delta x\Delta p=\frac{\hbar}{2}=\frac{h}{4\pi}$$

$$\Delta p_{\min}\approx p=mv=\frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

$$\therefore \quad \Delta x=\frac{h}{4\pi}\frac{1}{\Delta p}=\frac{h}{4\pi}\left[\frac{\sqrt{1-v^2/c^2}}{m_0 v}\right]$$

$$=\frac{6.62\times10^{-34}}{4\times3.14}\left[\frac{\sqrt{1-\left(\frac{3\times10^{7}}{3\times10^{8}}\right)^2}}{9.1\times10^{-31}\times3\times10^{7}}\right]$$

$$=\mathbf{1.92\times10^{-12}\,m}$$

Example 11 If an excited state of hydrogen atom has a life-time of 2.5×10^{-14} sec, what is the minimum error with which the energy of this state can be measured? Given $h=6.62\times10^{-34}$ J sec.

Solution Given $\Delta t=2.5\times10^{-14}$ sec.

Formula used is

$$\Delta E\Delta t=\frac{\hbar}{2}=\frac{h}{4\pi}$$

$$\Delta E=\frac{h}{4\pi}\frac{1}{\Delta t}=\frac{6.62\times10^{-34}}{4\times3.14}\times\frac{1}{2.5\times10^{-14}}=0.211\times10^{-20}\,\text{J}$$

$$\Delta E=\mathbf{2.11\times10^{-21}\,J}$$

Example 12 An excited atom has an average life-time of 10^{-2} sec. During this time period it emits a photon and returns to the ground state. What is the minimum uncertainty in the frequency of this photon?

Solution Given $\Delta t=10^{-8}$ sec.

Formula used is

$$\Delta E \Delta t = \frac{\hbar}{2} = \frac{h}{4\pi}$$

As $E = h\upsilon$ or $\Delta E = \Delta(h\upsilon) = h\Delta\upsilon$

so $h\Delta\upsilon\Delta t = \frac{h}{4\pi}$ or $\Delta\upsilon\Delta t = \frac{1}{4\pi}$

or $\Delta\upsilon = \frac{1}{4\pi}\frac{1}{\Delta t} = \frac{1}{4\times 3.14}\times\frac{1}{10^{-8}}$

$$\Delta\upsilon = \mathbf{7.96\times 10^{6}\ sec}$$

Example 13 Compare the uncertainties in velocity of a proton and an electron contained in a 20 Å box.

Solution Given $\Delta x = 2.0\times 10^{-9}$ m.

Formula used is

$$\Delta p\Delta x = \frac{\hbar}{2} = \frac{h}{4\pi} \quad \text{or} \quad \Delta p = \frac{h}{4\pi\Delta x}$$

10^{-7}

10^{3}

As uncertainty in momentum for electron and proton does not depend upon mass, we have

0. 015 77X

$$\Delta p = \Delta(mv) = m\Delta v \quad \text{or} \quad \Delta v = \frac{\Delta p}{m}$$

As $\Delta p_p = \Delta p_e$

$$\Delta v_p = \frac{\Delta p_p}{m_p} \quad \text{and}$$

$$\Delta v_e = \frac{\Delta p_e}{m_e}$$

$$\frac{\Delta v_p}{\Delta v_e} = \frac{\Delta p_p}{\Delta p_e}\frac{m_e}{m_p} = \frac{m_e}{m_p} \quad [\because \Delta p_p = \Delta p_e]$$

$$= \frac{9.1\times 10^{-31}}{1.67\times 10^{-27}} = \mathbf{5.45\times 10^{-4}}$$

Example 14 Find the energy of an electron moving in one dimension in an infinitely high potential box of width 1.0 Å. Given $m = 9.1\times 10^{-31}$ kg and $h = 6.62\times 10^{-34}$ J sec.

Solution Given $l = 1.0\times 10^{-10}$ m, $m = 9.1\times 10^{-31}$ kg and $h = 6.62\times 10^{-34}$ J sec.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2}$$

$$= \frac{n^2(6.62\times 10^{-34})^2}{8\times 9.1\times 10^{-31}\times(1.0\times 10^{-10})^2}$$

$$= 0.602\times 10^{-17}\,n^2\ \text{J}$$

for $n = 1$,

$$E_1 = 6.02\times 10^{-18}\ \text{J}$$

and for $n=2$,

$$E_2 = 6.02\times10^{-18}\times 4\text{ J}$$
$$= 2.408\times10^{-17}\text{ J}$$
$$= \mathbf{2.41\times10^{-17}\ J}$$

Example 15 Calculate the energy difference between the ground state and the first excited state for an electron in a box of length 1.0 Å.

Solution Given $L=1.0\times10^{-10}$ m.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2}$$

Put $n=1$ for ground state and $n=2$ for first excited state

$$E_2 - E_1 = \frac{h^2}{8mL^2}[2^2-1^2] = \frac{(6.62\times10^{-34})^2\times3}{8\times9.1\times10^{-31}\times(1.0\times10^{-10})^2}$$
$$= \mathbf{1.81\times10^{-17}\ J}$$

Example 16 Compute the energy of the lowest three levels for an electron in a square well of width 3Å.

Solution Given $L=3\times10^{-10}$ m.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2}$$

Put $n=1, 2, 3$ for first three levels, then

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.62\times10^{-34})^2}{8\times9.1\times10^{-31}\times(3\times10^{-10})^2}$$
$$= 6.688\times10^{-19}\text{ J}$$
$$= \mathbf{6.7\times10^{-19}\ J}$$
$$E_2 = 4E_1 = \mathbf{2.68\times10^{-18}\ J} \quad \text{and}$$
$$E_3 = 9E_1 = \mathbf{6.03\times10^{-18}\ J}$$

Example 17 An electron is bound in one dimensional potential box which has a width 2.5×10^{-10} m. Assuming the height of the box to be infinite, calculate the lowest two permitted energy values of the electron.

Solution Given $L=2.5\times10^{-10}$ m.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2}$$

For lowest two permitted energy values of electrons, put $n = 1$ and 2. Then

for $n = 1$,

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.62\times10^{-34})^2}{8\times9.1\times10^{-31}\times(2.5\times10^{-10})^2}$$
$$= \mathbf{9.63\times10^{-19}\ J}$$

for $n = 2$,

and $$E_2 = \frac{(2)^2\times(6.62\times10^{-34})^2}{8\times9.1\times10^{-31}\times(2.5\times10^{-10})^2}$$
$$= \mathbf{3.853\times10^{-18}\ J}$$

Example 18 Compute the lowest energy of a neutron confined to the nucleus which is considered as a box with a size of 10^{-14} m.

Solution Given $L = 10^{-4}$ m, $h = 6.62\times10^{-34}$ J sec and $m = 1.67\times10^{-27}$ kg.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2}$$

For lowest energy $n = 1$

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.62\times10^{-34})^2}{8\times1.67\times10^{-27}\times(10^{-14})^2}$$
$$E_1 = \mathbf{3.28\times10^{-13}\ J}$$

Example 19 State the values of momentum and energy of a particle in one dimensional box with impenetr able walls. Find their values for an electron in a box of length 1.0 Å for $n = 1$ and $n = 2$ energy states. Given $m = 9.1\times10^{-31}$ kg and $h = 6.63\times10^{-34}$ J sec.

Solution Given $L = 1.0\times10^{-10}$ m, $n = 1$ and 2, $m = 9.1\times10^{-31}$ kg and $h = 6.63\times10^{-34}$ J sec.

The Formulae used are

$$p_n = \frac{nh}{2L} \quad \text{(i)}$$

$$E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{8mL^2} \quad \text{(ii)}$$

Momentum for $n = 1$ and 2 are

$$p_1 = \frac{1\times6.63\times10^{-34}}{2\times10^{-10}} = 3.315\times10^{-24}\ \text{kg m/sec}$$

and $$p_2 = \frac{2\times6.63\times10^{-34}}{2\times10^{-10}} = 6.63\times10^{-24}\ \text{kg m/sec}$$

Energy for $n = 1$ and 2 are

$$E_1 = \frac{n^2h^2}{8mL^2} = \frac{(1)^2(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(10^{-10})^2}$$
$$= \mathbf{6.04\times10^{-18}\,J}$$

For $n=2$,

$$E_2 = \frac{n^2h^2}{8mL^2} = (2)^2E_1 = 2.416\times10^{-17}$$
$$= \mathbf{2.42\times10^{-17}\,J}$$

Example 20 An electron is constrained to move in a one dimensional box of length 0.1 nm. Find the first three energy eigen values and the corresponding deBroglie wave lengths. Given $h = 6.63\times10^{-34}$ J sec.

Solution Given $L = 1.0\times10^{-10}$ m.

Formulae used are

$$E_n = \frac{n^2h^2}{8mL^2} \quad \text{and} \quad p_n = \frac{nh}{2L}$$

for $n=1$

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times(9.1\times10^{-31})\times(10^{-10})^2}$$
$$= 6.04\times10^{-18}\,J$$

Similarly, for $n=2$, is equal to

$$E_2 = (2)^2E_1 = 24.16\times10^{-18}\,J$$

and for $n=3$,

$$E_3 = (3)^2E_1 = 54.36\times10^{-18}\,J$$

As we know

$$\lambda_n = \frac{h}{p_n} \quad \text{and} \quad p_n = \frac{nh}{2L}$$

$$\text{or} \quad \lambda_n = \frac{2L}{n}$$

For $n=1$

$$\lambda_1 = 2L = 2.0\times10^{-10}\,m = 2\,Å$$

For $n=2$,

$$\lambda_2 = \frac{2L}{2} = l = 1.0\times10^{-10}\,m = 1.0\,Å$$

For $n=3$,

$$\lambda_3 = \frac{2L}{3} = 0.667\times10^{-10}\,m = \mathbf{0.667\,Å}$$

Example 21 The minimum energy possible for a particle entrapped in a one dimensional box is 3.2×10^{-18} J. What are the next three energies in eV the particle can have?

Solution Given $E_1 = 3.2 \times 10^{-18}$ J.

Formula used is

$$E_n = \frac{n^2h^2}{8mL^2} \quad \text{or} \quad E_n \propto n^2$$

Now energy in eV $= \dfrac{32.0 \times 10^{-19}}{1.6 \times 10^{-19}}$

$$E_1 = 20 \text{ eV}$$

Next three values of energy can be obtained by putting $n = 2$, 3 and 4.

$$E_2 = n^2E_1 = (2)^2E_1 = 4 \times 20 \text{ eV} = \mathbf{80\ eV}$$

$$E_3 = (3)^2E_1 = 9 \times E_1 = 9 \times 20 \text{ eV} = \mathbf{180\ eV}$$

and $E_4 = (4)^2E_1 = 16 \times 20 \text{ eV} = \mathbf{320\ eV}$

Example 22 The energy of an electron constrained to move in a one dimensional box of length 4.0 Å is 9.664×10^{-17} J. Find out the order of excited state and the momentum of the electron in that state. Given $h = 6.63 \times 10^{-34}$ J sec.

Solution Given $E_n = 9.664 \times 10^{-17}$ J and $L = 4 \times 10^{-10}$ *m*.

Formulae used are

$$E_n = \frac{n^2h^2}{8mL^2} \quad \text{and} \quad p_n = \frac{nh}{2L}$$

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (4 \times 10^{-10})^2}$$

$$= 3.774 \times 10^{-19} \text{ J}$$

$$E_n = \frac{n^2h^2}{8mL^2} = n^2E_1$$

or $n^2 = \dfrac{E_n}{E_1} = \dfrac{966.4 \times 10^{-19} \text{ J}}{3.774 \times 10^{-19} \text{ J}}$

or $n = 16$ (order of excited state)

Momentum of electron for $n = 16$

$$p_n = \frac{nh}{2L} = \frac{16 \times 6.63 \times 10^{-34}}{2 \times 4 \times 10^{-10}}$$

$$= \mathbf{13.26 \times 10^{-24}\ kg\ m/sec}$$

Example 23 Evaluate the first three energy levels of an electron enclosed in a box of width 10Å. Compare it with those of glass marble of mass 1.0 gm, contained in a box of width 20 cm. Can these levels of the marble be measured experi mentally?

Solution Given for an electron $n = 1$ and $L = 1.0 \times 10^{-9}$ m and for glass marble $n = 1$, $L = 0.2$ m and $m = 1.0 \times 10^{-3}$ kg.

Formula used is $E_n = \dfrac{n^2h^2}{8mL^2}$

For electron

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.62\times10^{-34})^2}{8\times9.1\times10^{-31}\times(1.0\times10^{-9})^2}$$
$$= \mathbf{6.02\times10^{-20}\,J}$$

Similarly

$$E_2 = (2)^2E_1 = 4\cdot E_1 = \mathbf{24.08\times10^{-20}\,J}$$

and $E_3 = (3)^2E_1 = 9\times6.02\times10^{-20}\,J$

$$= \mathbf{54.18\times10^{-20}\,J}$$

For glass marble

$$E_1 = \frac{(6.62\times10^{-34})^2}{8\times10^{-3}\times(0.2)^2}$$
$$= 1.3695\times10^{-63}\,J$$
$$= \mathbf{1.37\times10^{-63}\,J}$$

Similarly,

$$E_2 = (2)^2E_1 = \mathbf{5.48\times10^{-63}\,J}$$

and $E_3 = (3)^2E_1 = 9E_1 = \mathbf{12.33\times10^{-63}\,J}$

It is clear that the levels in case of marble are very small and are nearly zero. So it is not possible to measure them experimentally.