$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\, dy}\, dx}$$

Note 1. From cases (*i*) and (*ii*) above, we observe that *integration is to be performed w.r.t. that variable having variable limits first and then w.r.t. the variable with constant limits.*

Note 2. If $f(x, y)$ has discontinuities within or on the boundary of the region of integration, then the change of the order of integration does not result into the same integrals.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_3^4 \int_1^2 (xy + e^y)\, dx\, dy$.

Sol. $$\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_1^2 \left[\int_3^4 (xy + e^y)\, dy \right] dx$$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y \right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3 \right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3 \right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x \right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3$$

$$\int_3^4 \int_1^2 (xy + e^y)\, dx\, dy = \int_3^4 \left[\int_1^2 (xy + e^y)\, dx \right] dy = \int_3^4 \left[\frac{yx^2}{2} + xe^y \right]_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y \right) dy = \int_3^4 \left(\frac{3y}{2} + e^y \right) dy$$

$$= \left[\frac{3y^2}{4} + e^y \right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Hence the result.

Example 2. *Evaluate the following :*

(*i*) $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1 - x^2)(1 - y^2)}}$ (*ii*) $\int_0^1 \int_0^1 (x + 2)\, dy\, dx$ (P.T.U., Dec. 2005)

(*iii*) $\int_0^1 \int_0^3 (x + 5)\, dy\, dx$. (P.T.U. May. 2005)

Sol. (*i*) $$\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1 - x^2)(1 - y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - y^2)}} \right] dy = \int_0^1 \frac{1}{\sqrt{1 - y^2}} \left[\sin^{-1} x \right]_0^1 dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot \frac{\pi}{2}\, dy = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

(ii) $\int_0^1 \int_0^1 [(x+2)\, dy]\, dx = \int_0^1 (x+2)\, y \Big|_0^1 dx = \int_0^1 (x+2)\, dx$

$$= \frac{x^2}{2} + 2x \Big|_0^1 = \frac{1}{2} + 2 = \frac{5}{2}.$$

(iii) $\int_0^1 \int_0^3 (x+5)\, dy dx = \int_0^1 \left[\int_0^3 (x+5) dy\right] dx$

$$= \int_0^1 (x+5)\, y|_0^3\, dx = \int_0^1 3(x+5)\, dx$$

$$= 3\left(\frac{x^2}{2} + 5x\,|_0^1\right) = 3\left(\frac{1}{2}+5\right) = \frac{33}{2}.$$

Example 3. *Show that* $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx.$

Sol. **L.H.S.** $= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dy$

$$= \int_0^1 \left[2x \cdot \frac{(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1}\right]_0^1 dx = \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right]_0^1 dx$$

$$= \int_0^1 \left[\frac{-x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x}\right] dx = \int_0^1 \frac{-x+x+1}{(x+1)^2}\, dx = \int_0^1 \frac{1}{(x+1)^2}\, dx$$

$$= \left[-\frac{1}{x+1}\right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

R.H.S. $= \int_0^1 dy \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx = \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right] dx$

$$= \int_0^1 \left[\frac{(x+y)^{-1}}{-1} - 2y \cdot \frac{(x+y)^{-2}}{-2}\right]_0^1 dy = \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y}\right] dy = \int_0^1 \frac{-1-y+y}{(1+y)^2}\, dy = -\int_0^1 \frac{1}{(1+y)^2}\, dy$$

$$= \left[\frac{1}{1+y}\right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore$ The two integrals are not equal.

Example 4. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

(P.T.U., May 2006 ; Andhra, 1994 ; Kerala, 1990)

Sol.

$$I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\, dy \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0]\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log (x + \sqrt{1+x^2}) \right]_0^1 = \frac{\pi}{4} [\log (1+\sqrt{2}) - \log 1] = \frac{\pi}{4} \log (\sqrt{2}+1).$$

Example 5. *Evaluate the following :*

(i) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$ *(ii)* $\displaystyle\int_0^2 \int_0^{\sqrt{2x}} xy\, dy\, dx$ (P.T.U., Dec. 2003)

Sol. *(i)*

$$I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy.$$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1\, dy = \frac{\pi}{4} \int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right] = \frac{\pi a^3}{6}.$$

(ii)

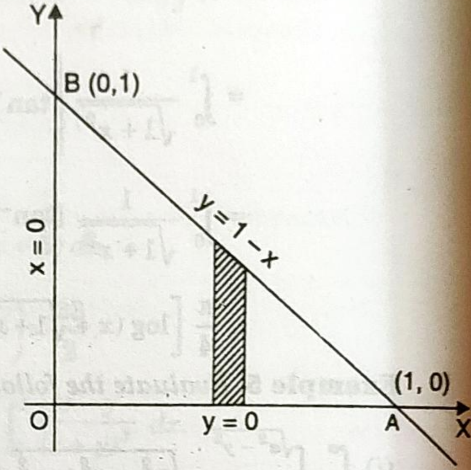
$$\int_0^2 \int_0^{\sqrt{2x}} xy\, dy\, dx = \int_0^2 \left[\int_0^{\sqrt{2x}} xy\, dy \right] dx = \int_0^2 x \frac{y^2}{2} \Bigg|_0^{\sqrt{2x}} dx$$

$$= \int_0^2 x \cdot \frac{2x}{2}\, dx = \int_0^2 x^2\, dx = \frac{x^3}{3} \Bigg|_0^2 = \frac{8}{3}.$$

Example 6. *(i) Evaluate* $\iint e^{2x+3y}\,dx\,dy$ *over the triangle bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$.

(ii) Evaluate $\int\int_S \sqrt{xy - y^2}\,dxdy$ *where S is the triangle with vertices (0, 0) (10, 1) and (1, 1).*

Sol. (*i*) The region R of integration is the triangle OAB. Here x varies from 0 to 1 and y varies from x-axis upto the line $x + y = 1$ *i.e.*, from 0 to $1 - x$.



$\therefore$ The region R can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1 - x$$

$$\therefore \quad \iint_R e^{2x+3y}\,dx\,dy = \int_0^1 \int_0^{1-x} e^{2x+3y}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}e^{2x+3y}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x})\,dx$$

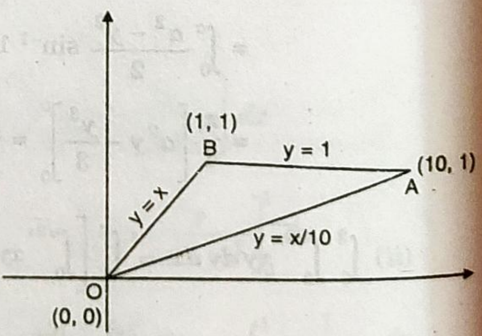
$$= \frac{1}{3}\left[-e^{3-x} - \frac{1}{2}e^{2x}\right]_0^1 = -\frac{1}{3}\left[\left(e^2 + \frac{1}{2}e^2\right) - \left(e^3 + \frac{1}{2}\right)\right]$$

$$= -\frac{1}{3}\left[-e^2(e-1) + \frac{1}{2}(e^2 - 1)\right]$$

$$= \frac{1}{6}(e-1)\,[2e^2 - (e+1)] = \frac{1}{6}(e-1)(2e^2 - e - 1)$$

$$= \frac{1}{6}(e-1)(e-1)(2e+1) = \frac{1}{6}(e-1)^2(2e+1).$$

(*ii*) Region R of integration is the region of the triangle OAB (shown in the figure). Here equation of the line OA is



$$y - 0 = \frac{1-0}{10-0}(x - 0)$$

or
$$y = \frac{1}{10}x$$

equation of OB is
$$y = x$$

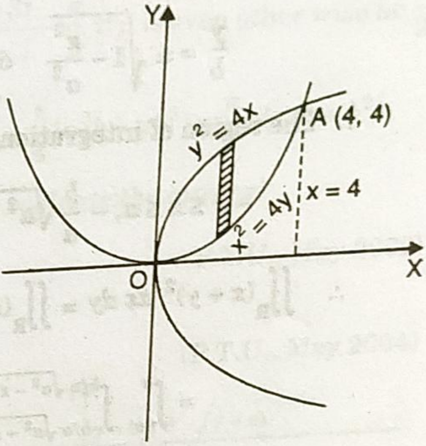
$$\therefore \quad R = \{(x, y)\ ;\ y \le x \le 10y\ ;\ 0 \le y \le 1\}$$

$$\therefore \quad \int\int_S \sqrt{xy - y^2}\,dxdy = \int_0^1\left[\int_y^{10y}\sqrt{xy - y^2}\,dx\right]dy$$

$$= \int_0^1 \left. \frac{(xy - y^2)^{3/2}}{\frac{3}{2}y} \right|_y^{10y} dy = \frac{2}{3}\int_0^1 \left[\frac{(9y^2)^{3/2}}{y} - 0\right] dy$$

$$= \frac{2}{3}\int_0^1 27y^2\, dy = \frac{2}{3}\left[27 \left.\frac{y^3}{3}\right|_0^1\right] = \frac{2}{3} . 9 = 6.$$

Example 7. *Evaluate* $\iint_R y\, dx\, dy$, *where R is the region bounded by the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.



Sol. Solving $y^2 = 4x$ and $x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \quad \text{or} \quad x(x^3 - 64) = 0$$

$\therefore$ $x = 0, 4$

When $x = 4,\ y = 4$

$\therefore$ Co-ordinates of A are (4, 4)

The region R can be expressed as

$$0 \le x \le 4,\ \frac{x^2}{4} \le y \le 2\sqrt{x}$$

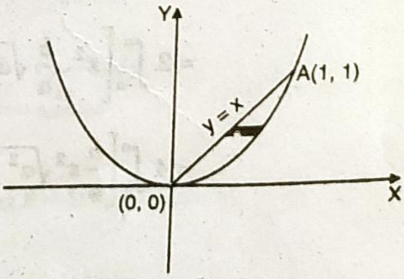
$$\therefore \quad \iint_R y\, dx\, dy = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx$$

$$= \int_0^4 \frac{1}{2}\Big[y^2\Big]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}.$$

Example 8. *Evaluate* $\iint xy(x + y)\, dxdy$ *over the area between* $y = x^2$ *and* $y = x$.

(Hamirpur 1994, Karnataka 1998)



Sol. Region of integration is

$$R = \{(x, y);\ y \le x \le \sqrt{y}\ ;\ 0 \le y \le 1\}$$

$$\therefore \quad \iint_R xy\,(x + y)\, dxdy$$

$$= \int_0^1 \left[\int_y^{\sqrt{y}} xy\,(x + y)\, dx\right] dy$$

$$= \int_0^1 \left. y\,\frac{x^3}{3} + y^2\,\frac{x^2}{2}\right|_y^{\sqrt{y}} dy = \int_0^1 \left(\frac{1}{3}y^{5/2} + \frac{1}{2}y^3 - \frac{y^4}{3} - \frac{y^4}{2}\right) dy$$

$$= \int_0^1 \left(\frac{1}{3}y^{5/2} + \frac{1}{2}y^3 - \frac{5}{6}y^4\right)dy = \frac{1}{3}\left(\frac{2}{7}y^{7/2}\right) + \frac{1}{2}\cdot\frac{y^4}{4} - \frac{5}{6}\frac{y^5}{5}\Bigg|_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{6} = \frac{16+21-28}{168} = \frac{9}{168} = \frac{3}{56}.$$

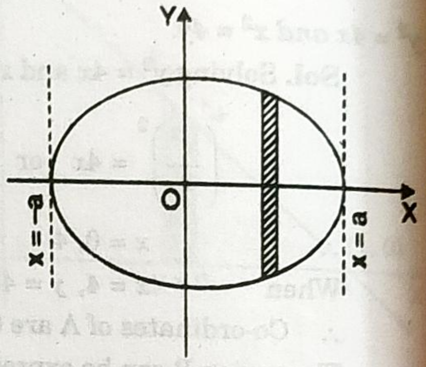
Example 9. *Evaluate* $\iint (x+y)^2\, dx\, dy$ *over the area bounded by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(Andhra, 1994 ; Madras, 1993)

Sol. For the ellipse

$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$



∴ The region of integration R can be expressed as

$$-a \le x \le a,\ -\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$$

$$\therefore \quad \iint_R (x+y)^2\, dx\, dy = \iint_R (x^2 + y^2 + 2xy)\, dx\, dy$$

$$= \int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy)\, dy\, dx$$

$$= \int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2)\, dy\, dx + \int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\, dy\, dx$$

$$= \int_{-a}^{a}\int_{0}^{b/a\sqrt{a^2-x^2}} 2(x^2 + y^2)\, dy\, dx + 0$$

[since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a}\left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2 \cdot \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\cdot\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right] dx$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right] dx$$

put $x = a\sin\theta$, $\quad\therefore\quad dx = a\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2}\left(\frac{b}{a}\cdot a^2\sin^2\theta\,.\,a\cos\theta + \frac{b^3}{3a^3}\cdot a^3\cos^3\theta\right) \times a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left(a^3b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right)d\theta$$

We know from integral calculus that

$$\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx = \frac{\{(p-1)(p-3)\ldots\}\{(q-1)(q-3)\ldots\}}{(p+q)(p+q-2)\ldots}\,\frac{\pi}{2}$$

if both p, q are even other wise no $\frac{\pi}{2}$

and $$\int_0^{\frac{\pi}{2}} \cos^p x\,dx = \int_0^{\frac{\pi}{2}} \sin^p x\,dx = \frac{(p-1)(p-3)\ldots}{p(p-2)-\ldots}\,\frac{\pi}{2}$$ if p is even other wise no $\frac{\pi}{2}$

$\therefore$ Given integral $= 4\left[a^3b\cdot\frac{1.1}{4.2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}\right] = \frac{\pi}{4}(a^3b + ab^3) = \frac{\pi}{4}ab(a^2+b^2).$

Example 10. (*i*) *Sketch the region of integration and evaluate the integral* (P.T.U., May 2002)

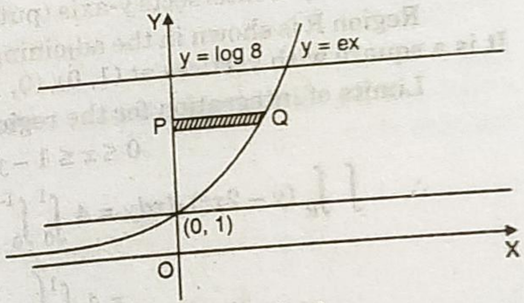
$$\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dxdy$$

(P.T.U., May 2004)

(*ii*) *Evaluate* $\int_0^2\int_0^{y^2} e^{x/y}\,dx\,dy$

Sol. (*i*) The given integral is

$$\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dxdy$$



Here x varies from 0 to $\log y$

i.e., $x = 0$ and $x = \log y$ or $y = e^x$

and y varies from 1 to log 8

i.e., $y = 1$ and $y = \log 8$

Region R is shown in the adjoining figure (shaded portion is the region)

$$\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy = \int_1^{\log 8}\int_0^{\log y} e^y\,.\,e^x\,dx\,dy$$

$$= \int_1^{\log 8}\left[\int_0^{\log y} e^y\,.\,e^x\,dx\right]dy = \int_1^{\log 8} e^y\,.\,e^x\Big|_0^{\log y}\,dy$$

$$= \int_1^{\log 8} e^y\,[e^{\log y} - e^0]\,dy = \int_1^{\log 8} \underset{\substack{\text{(2nd)}\\ \text{(function)}}}{e^y}\ \underset{\text{Ist function}}{(y-1)}\,dy$$

Integrate by parts

$$= (y-1)\,e^y\Big|_1^{\log 8} - \int_1^{\log 8} 1.e^y\,dy = (\log 8 - 1)\,e^{\log 8} - 0 - e^y\Big|_1^{\log 8}$$

$= (\log 8 - 1)\,.\,8 - [e^{\log 8} - e^1]$

$= 8\log 8 - 8 - 8 + e$

$= 8\log 8 - 16 + e = 8(\log 8 - 2) + e.$

(*ii*) $\int_0^2 \int_0^{y^2} e^{x/y}\, dx\, dy$

$$= \int_0^2 \left[\frac{e^{x/y}}{1/y} \Bigg|_0^{y^2} \right] dy = \int_0^2 (ye^y - y)\, dy = \int_0^2 ye^y - \frac{y^2}{2} \Bigg|_0^2$$

Integrating by parts,

$$= ye^y \Bigg|_0^2 - \int_0^2 1.e^y\, dy - 2 = 2e^2 - 0 - e^y \Bigg|_0^2 - 2$$

$$= 2e^2 - e^2 + 1 - 2 = e^2 - 1.$$

Example 11. (*i*) *Sketch the region of integration and evaluate* $\iint_R (y - 2x^2)\, dx\, dy$ *where R is the region inside the square* $|x| + |y| = 1$. (P.T.U., Dec. 2004)

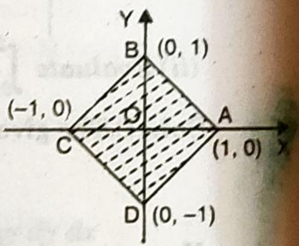
(*ii*) *Write the limits of integration in* $\iint_R xy\, dx\, dy$, *where R is the region inside the square* $|x| + |y| = 1$ *and hence evaluate.* (P.T.U., May 2003)

Sol. (*i*) The given region R is $|x| + |y| = 1$. It intersects x-axis (put $y = 0$) at $|x| = 1$ *i.e.*, $x = \pm 1$ *i.e.*, at $(-1, 0)$ and $(1, 0)$

Similarly it intersects y-axis (put $x = 0$) at $(0, -1)$ and $(0, 1)$

Region R is shown in the adjoining figure. (Shaded portion).

It is a square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$



Limits of integration for the region OAB are

$$0 \le x \le 1 - y \quad \text{and} \quad 0 \le y \le 1$$

$$\therefore \quad \iint_R (y - 2x^2)\, dxdy = 4 \int_0^1 \int_0^{1-y} (y - 2x^2)\, dxdy$$

$$= 4 \int_0^1 \left(yx - \frac{2x^3}{3} \right) \Bigg|_0^{1-y} dy = 4 \int_0^1 \left[y(1-y) - \frac{2}{3}(1-y)^3 \right] dy$$

$$= 4 \left\{ \frac{y^2}{2} - \frac{y^3}{3} - \frac{2}{3} \frac{(1-y)^4}{4(-1)} \Bigg|_0^1 \right\}$$

$$= 4 \left\{ \frac{1}{2} - \frac{1}{3} + \frac{1}{6}(-1) \right\} = 4\, \frac{3 - 2 - 1}{6} = 0$$

(*ii*) As proved in the 1st parts limits of integration of $\iint_R xy\, dxdy$ are

$$-1 \le x \le 1 \quad \text{and} \quad -1 \le y \le 1$$

$$\iint_R xy\, dxdy = 4 \int_0^1 \int_0^{1-y} xy\, dxdy$$

$$= 4 \int_0^1 \left[y \frac{x^2}{2} \Bigg|_0^{1-y} \right] dy = \frac{4}{2} \int_0^1 y(1-y)^2 dy$$

$$= 2 \int_0^1 y(1 - 2y + y^2)\, dy = 2 \int_0^1 (y - 2y^2 + y^3)\, dy$$

$$= 2\left\{\frac{y^2}{2} - 2\frac{y^3}{3} + \frac{y^4}{4}\Bigg|_0^1\right\} = 2\left(\frac{1}{2} - \frac{2}{3} + \frac{7}{4}\right)$$

$$= 2\,\frac{6-8+3}{12} = \frac{2}{12} = \frac{1}{6}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1—10) :

1. $\int_0^3\int_0^1 (x^2 + 3y^2)\, dy\, dx.$

2. $\int_0^3\int_1^2 xy\,(1 + x + y)\, dy\, dx.$

3. $\int_1^a\int_1^b \frac{dy\, dx}{xy}.$

4. $\int_1^2\int_0^x \frac{dy\, dx}{x^2 + y^2}.$

5. $\int_0^1 dx \int_0^x e^{y/x}\, dy.$

6. $\int_0^1\int_{x^2}^x (x^2 + 3y + 2)\, dy\, dx.$

7. (i) $\int_0^1\int_0^{x^2} e^{y/x}\, dy\, dx.$ (ii) $\int_0^2\int_0^{y^2} e^{x/y}\, dx\, dy$

8. $\int_0^1\int_y^{y^2+1} x^2 y\, dx\, dy.$

9. $\int_0^{2a}\int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dy\, dx.$

10. $\int_0^1\int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$

11. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the region in the positive quadrant for which $x + y \le 1.$
[**Hint.** $0 \le x \le 1 - y\,;\ 0 \le y \le 1$]

12. Evaluate $\iint x^2y^2\, dx\, dy$ over the circle $x^2 + y^2 = 1.$
[**Hint.** $\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}\,;\ -1 \le x \le 1$]

13. Evaluate $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2.$ *(Madras, 1996 ; A.M.I.E., 1997)*
[**Hint.** $0 \le x \le \sqrt{a^2 - y^2}\,;\ 0 \le y \le a$]

14. Compute the value of $\iint_R y\, dx\, dy$, where R is the region in the first quadrant bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

[**Hint.** $0 \le x \le a\sqrt{1 - \frac{y^2}{b^2}}\,;\ 0 \le y \le b$]

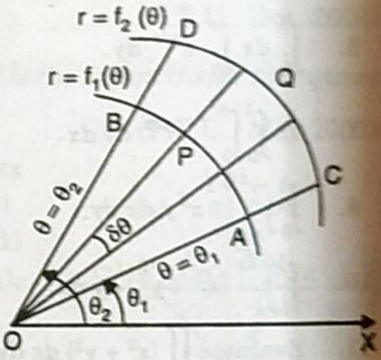
15. Evaluate $\iint_A xy\, dx\, dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay.$ *(Marathwada, 1993 ; A.M.I.E., 1990)*
[**Hint.** $0 \le y \le \frac{x^2}{4a}\,;\ 0 \le x \le 2a$]

Answers

1. 12
2. $30\frac{3}{4}$
3. $\log a \log b$
4. $\frac{\pi}{4}\log 2$
5. $\frac{1}{2}(e-1)$
6. $\frac{7}{12}$
7. (i) $\frac{1}{2}$ (ii) e^2-1
8. $\frac{67}{120}$
9. $\frac{3\pi a^4}{4}$
10. $\frac{3}{35}$
11. $\frac{1}{6}$
12. $\frac{\pi}{24}$
13. $\frac{a^4}{8}$
14. $\frac{ab^2}{3}$
15. $\frac{a^4}{3}$.

8.3. EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the straight lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$, we first integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ (treating θ as a constant). The resulting expression is then integrated w.r.t. θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$.



Geometrically, AB and CD are the curves $r=f_1(\theta)$ and $r=f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ so that ACDB is the region of integration. PQ is a wedge of angular thickness $\delta\theta$. Then $\int_{r=r_1}^{r=r_2} f(r,\theta)\,dr$ indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta.$

Sol.

$$I=\int_0^{\pi/2}\left[\int_0^{a\cos\theta}-\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta.$$

$$=\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta}d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta=-\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]=\frac{a^3}{18}(3\pi-4).$$

Example 2. *Evaluate* $\iint r^3\,dr\,d\theta$, *over the area bounded between the circles* $r=2\cos\theta$ *and* $r=4\cos\theta$.

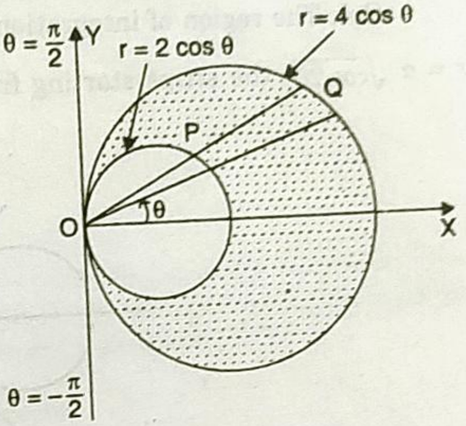
Sol. The region of integration R is shown shaded. Here r varies from $2\cos\theta$ to $4\cos\theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\therefore \iint_R r^3\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}(256\cos^4\theta - 16\cos^4\theta)\, d\theta$$

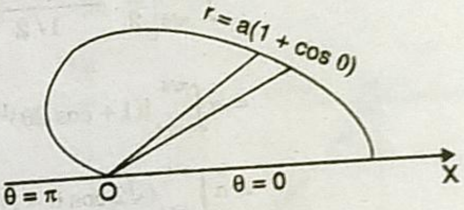
$$= 60 \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$



$$= 120 \int_0^{\pi/2} \cos^4\theta\, d\theta \qquad [\text{since } \cos^4\theta \text{ is an even function of } \theta]$$

$$= 120 \times \frac{3\times 1}{4\times 2}\cdot\frac{\pi}{2} = \frac{45}{2}\pi.$$

Example 3. *Evaluate* $\iint r \sin\theta\, dr\, d\theta$ *over the area of the cardioid* $r = a(1 + \cos\theta)$ *above the initial line.*



Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a(1 + \cos\theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.

$$\therefore \iint_R r\sin\theta\, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, dr\, d\theta.$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi} \sin\theta\,.\,a^2(1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta = 4a^2\int_0^{\pi}\sin\frac{\theta}{2}\cos^5\frac{\theta}{2}\, d\theta$$

$$\left[\text{Putting } \frac{\theta}{2} = \phi \text{ and } d\theta = 2d\phi\right]$$

$$= 4a^2\int_0^{\pi/2} 2\sin\phi\cos^5\phi\, d\phi$$

$$= -8a^2\int_0^{\pi/2}\cos^5\phi\,(-\sin\phi)\, d\phi = -8a^2\,.\left[\frac{\cos^6\phi}{6}\right]_0^{\pi/2}$$

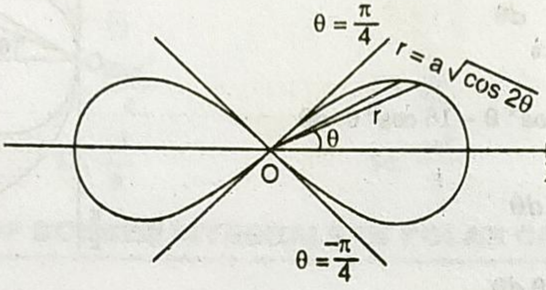
$$\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$$

$$= -\frac{4a^2}{3}(0-1) = \frac{4a^2}{3}.$$

Example 4. *Evaluate* $\iint \frac{r\, dr\, d\theta}{\sqrt{a^2+r^2}}$ *over one loop of the lemniscate* $r^2 = a^2\cos 2\theta$.

(Kuvempu, 1996 ; Madurai, M.E. 1990)

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\frac{\pi}{4}$ and ending at $\theta = \frac{\pi}{4}$.



$$\therefore \iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \tfrac{1}{2}(a^2+r^2)^{-1/2} . 2r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}\cdot\frac{(a^2+r^2)^{1/2}}{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4}[(a^2+a^2\cos 2\theta)^{1/2}-a]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}[(1+\cos 2\theta)^{1/2}-1]\,d\theta = a\int_{-\pi/4}^{\pi/4}[(2\cos^2\theta)^{1/2}-1]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}(\sqrt{2}\cos\theta-1)\,d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta-1)\,d\theta$$

∵ integrand is an even function

$$= 2a\left[\sqrt{2}\sin\theta-\theta\right]_0^{\pi/4} = 2a\left[\sqrt{2}\,.\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right] = 2a\left(1-\frac{\pi}{4}\right).$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1—4) :

1. $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$
2. $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$
3. $\int_0^{\pi/2}\int_{a(1+\cos\theta)}^{a} r\,dr\,d\theta$
4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$
5. Show that $\iint_R r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line. *(Mysore, 1987 S)*
6. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. *(Madras, 1996)*

Answers

1. $\frac{\pi a^2}{4}$
2. $\frac{a^2}{6}$
3. $-a^2\left(1+\frac{\pi}{8}\right)$
4. $\frac{5}{8}\pi a^3$
6. $\frac{45\pi}{2}$.

8.4. CHANGE OF ORDER OF INTEGRATION

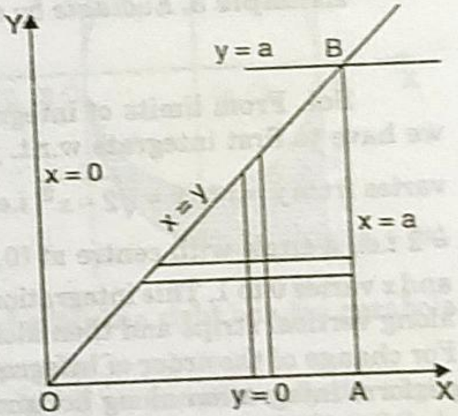
In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\int_c^d\int_a^b f(x, y)\,dx\,dy = \int_a^b\int_c^d f(x, y)\,dy\,dx.$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of integration helps in fixing the new limits of integration.

ILLUSTRATIVE EXAMPLES

Example 1. *Change the order of integration in* $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ *and hence evaluate the same.*



Sol. From the limits of integration, it is clear that the region of integration is bounded by $x = y, x = a, y = 0$ and $y = a$. Thus the region of integration is the ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become: y varies from 0 to x and x varies from 0 to a.

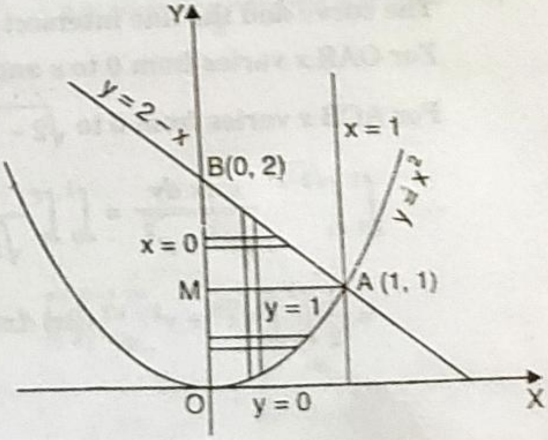
$$\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a\int_0^x \frac{x\,dy\,dx}{x^2+y^2}$$

$$= \int_0^a x\cdot\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}\cdot\left[x\right]_0^a = \frac{\pi a}{4}$$

Example 2. *Change the order of integration in* $I = \int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ *and hence evaluate the same.*

(P.T.U., Dec. 2002, May 2006 ; Madras, 1993 ; Andhra, 1991 ; Kerala, 1990 ; Ranchi, 1990 ; Mysore, 1994)



Sol. From the limits of integration, it is clear that we have to integrate first with respect to y which varies from $y = x^2$ to $y = 2 - x$ and then with respect to x which varies from $x = 0$ to $x = 1$. The region of integration (shown shaded) is divided into vertical strips. For changing the order of integration, we divide the region of integration into horizontal strips.

Solving $y = x^2$ and $y = 2 - x$, the co-ordinates of A are (1, 1). Draw AM ⊥ OY. The region of integration is divided into two parts, OAM and MAB.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $2 - y$ and y varies from 1 to 2.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

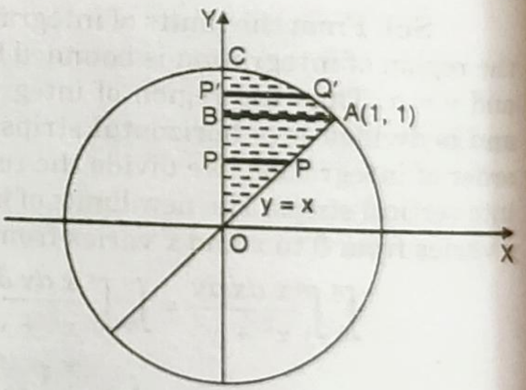
$$= \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \cdot \left[\frac{x^2}{2}\right]_0^{2-y} dy = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy$$

$$= \frac{1}{2} \cdot \left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy = \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right] = \frac{3}{8}.$$

Example 3. *Evaluate by changing the order of integration of* $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$.

(Rewa 1990)



Sol. From limits of integration it is clear that we have to first integrate w.r.t. y and then w.r.t. x y varies from $y = x$ to $y = \sqrt{2 - x^2}$ *i.e.*, $y^2 = 2 - x^2$ or $x^2 + y^2 = 2$ *i.e.*, a circle with centre at (0, 0) and radius = $\sqrt{2}$ and x varies 0 to 1. This integration is firstly performed along vertical strips and then along horizontal strips. For change of the order of integration we have to first perform integration along horizontal strips and then along vertical strips. The region of integration is shown shaded in the figure.

For horizontal strips the whole region is divided into two portion OAB and ACB. Let the horizontal strip in the portion OAB be PQ and that of in ACB be P′Q′.

The curve and the line intersect at A(1, 1).

For OAB x varies from 0 to x and y varies for 0 to 1.

For ACB x varies from 0 to $\sqrt{2 - y^2}$ and y varies for 1 to $\sqrt{2}$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}} = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2 + y^2}}\, dx\, dy + \int_0^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$$

$$= \frac{1}{2}\int_0^1 \int_0^y (x^2 + y^2)^{-1/2}\, (2x)\, dx\, dy + \frac{1}{2}\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} (2x)\, (x^2 + y^2)^{-1/2}\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \frac{(x^2+y^2)^{1/2}}{1/2}\Bigg|_0^y dy + \frac{1}{2}\int_1^{\sqrt{2}} \frac{(x^2+y^2)^{1/2}}{1/2}\cdot\Bigg|_0^{\sqrt{2-y^2}} dy$$

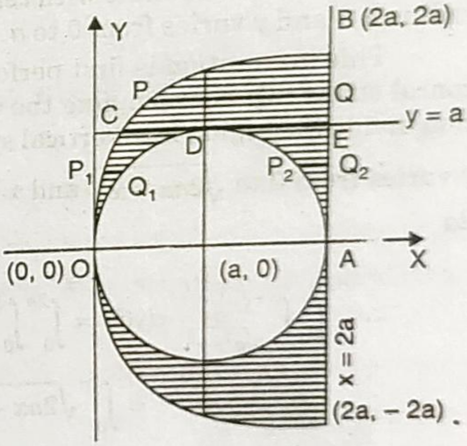
$$\Big| \text{ by using } \int [f(x)]\, f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$$

$$= \int_0^1 \left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right)dy$$

$$= \left(\frac{\sqrt{2}y^2}{2} - \frac{y^2}{2}\right)\Bigg|_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)\Bigg|_1^{\sqrt{2}} = \frac{\sqrt{2}-1}{2} + 2 - 1 - \sqrt{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left[\sqrt{2} - 1 + 4 - 2 - 2\sqrt{2} + 1\right] = \frac{1}{2}\left[2 - \sqrt{2}\right] = 1 - \frac{1}{\sqrt{2}}.$$

Example 4. *Change the order of integration of* $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dy\, dx$. (P.T.U., Dec. 2003)



Sol. From the limits of integration it is clear that we have to first integrate w.r.t. y which varies from $\sqrt{2ax-x^2}$ to $\sqrt{2ax}$ *i.e.*, $\sqrt{2ax-x^2} \le y \le \sqrt{2ax}$ *i.e.*, $2ax - x^2 \le y^2 \le 2ax$ and then w.r.t. x which varies from 0 to $2a$ *i.e.*, $0 \le x \le 2a$. This shows that integration is firstly performed along veritcal strips. For change of the order of integration we have to perform integration firstly along horizontal strips and then along vertical. We draw rough sketch of the region $y^2 = 2ax - x^2$ or $x^2 + y^2 - 2ax = 0$ which is a circle with centre at $(a, 0)$ and radius $= a$, $y = \sqrt{2ax}$ or $y^2 = 2ax$ is a right handed parabola. $x = 0$ represents y-axis and $x = 2a$ represents a line parallel to y-axis. The two curves Intersect at (0, 0). For horizontal strip PQ we see that only the region BCE is covered ∴ we divide the whole region into three portions namely : (*i*) BCE, ODC and AED

$$\because\ y^2 = 2ax - x^2 \text{ or } x^2 - 2ax + y^2 = 0$$

$$x = \frac{2a \pm \sqrt{4a^2 - 4y^2}}{2} = a \pm \sqrt{a^2 - y^2}$$

for BCE ;

x varies from $\frac{y^2}{2a}$ to $2a$

y varies from a to $2a$

for ODC ; x varies from $\frac{y^2}{2a}$ to $a + \sqrt{a^2 - y^2}$

y varies from 0 to a

for AED x varies from $a + \sqrt{a^2 - y^2}$ to $2a$

y varies from 0 to a

$$\therefore \int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dy\, dx = \int_a^{2a}\int_{y^2/2a}^{2a} V\, dx\, dy + \int_0^a \int_{y^2/2a}^{a+\sqrt{a^2-y^2}} V\, dx\, dy$$

$$+ \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} V\, dx\, dy$$

Example 5. *Change the order of integration in the following integral and evaluate :*

(*i*) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx$ (*ii*) $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$. (Mangalore, 1997)

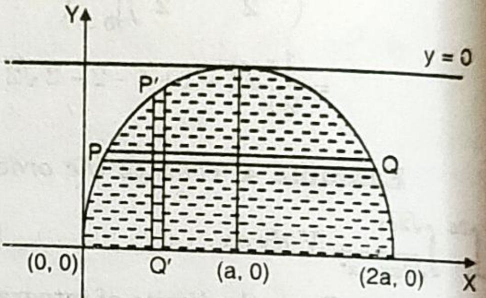
(P.T.U. May 2000)

Sol. (*i*) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx$

From the limits of integration it is clear that we have to first integrate w.r.t. x and then w.r.t. y

x varies from $a - \sqrt{a^2 - y^2}$ to $a + \sqrt{a^2 - y^2}$ *i.e.*, $x = a \pm \sqrt{a^2 - y^2}$ $(x - a)^2 = a^2 - y^2$ or $x^2 + y^2 - 2ax = 0$ *i.e.*, inside the circle with centre at $(a, 0)$ and radius $= a$ and y varies from 0 to a.

This integration is first perfomed along horizontal strips PQ. For changing the order of integration divide the region into vertical strips P′Q′ where y varies from 0 to $\sqrt{2ax - x^2}$ and x varies from 0 to $2a$

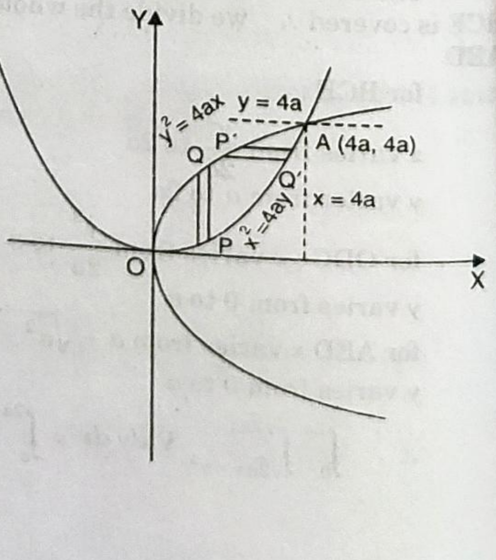


$$\therefore \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dydx = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\, dx = \int_0^{2a} y \Big|_0^{\sqrt{2ax-x^2}} dx$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\, dx = \int_0^{2a} \sqrt{a^2 - (x-a)^2}\, dx$$

$$= \frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{a} \Big|_0^{2a} = \frac{a^2}{2}\frac{\pi}{2} - \frac{a^2}{2}\left(-\frac{\pi}{2}\right) = \frac{a^2\pi}{2}.$$

(*ii*) From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and then w.r.t. x which varies from $x = 0$ to $x = 4a$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y = \frac{x^2}{4a}$ (*i.e.*, $x^2 = 4ay$) to the point Q on the parabola $y = 2\sqrt{ax}$ (*i.e.*, $y^2 = 4ax$). Then the strip slides from O to A $(4a, 4a)$, the point of intersection of the two parabolas.



For changing the order of integration, we divide the region of integration OPAQO into horizontal strips P′Q′ which extend from P′ on the parabola $y^2 = 4ax$ *i.e.*, $x = \dfrac{y^2}{4a}$ to Q′ on the parabola $x^2 = 4ay$ *i.e.*, $x = 2\sqrt{ay}$. Then this strip slides from 0 to A(4*a*, 4*a*), *i.e.*, varies from 0 to 4*a*.

$$\therefore \quad \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a}\Big[x\Big]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

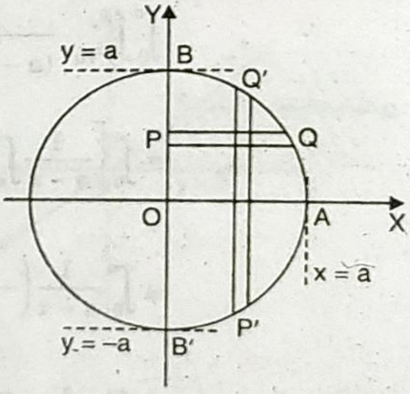
$$= \left[2\sqrt{a}\,.\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a}\,.(4a)^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}\,.8a^{3/2} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 6. *Change the order of integration in the integral* $\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy.$

(Andhra, 1990)

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. x which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and then w.r.t. y which varies from $y = -a$ to $y = a$. Thus integration is first performed along the horizontal strip PQ which extends from a point P on $x = 0$ (*i.e.*, *y*-axis) to the point Q on the circle $x = \sqrt{a^2 - y^2}$ (*i.e.*, $x^2 = a^2 - y^2$ or $x^2 + y^2 = a^2$). Then the strip slides from B′ to B.



For changing the order of integration, we divide the region of integration B′AQBPB′ into vertical strips P′Q′ which extend from P′ on the circle $y = -\sqrt{a^2 - x^2}$ to Q′ on the circle $y = \sqrt{a^2 - x^2}$. $[x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$; for points in the 4th quadrant, $y = -\sqrt{a^2 - x^2}$ and for points in the first quadrant, $y = \sqrt{a^2 - x^2}]$. Then this strip slides from *y*-axis ($x = 0$) to A, where $x = a$.

$$\therefore \quad \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

$$= \int_0^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx\,.$$

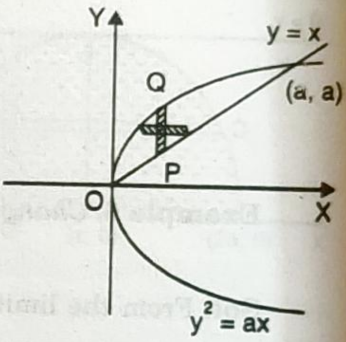
Example 7. *Evaluate the following*

(*i*) $\displaystyle\int_0^{a}\int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$ (P.T.U., May 2002)

(*ii*) $\displaystyle\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx\,.$ (P.T.U., May 2004 ; Nagpur 1997 ; Mysore 1997)

Sol. (*i*) $\displaystyle\int_0^a\int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$

Here limits of integration are given by $0 \le y \le a$ and $\dfrac{y^2}{a} \le x \le y$. The limits of x are depending on y where as that of y are constant. ∴ We have to integrate first w.r.t. x and then w.r.t. y but we cannot easily integrate in this order, integral of $\dfrac{y}{(a-x)\sqrt{ax-y^2}}$ w.r.t. y is simpler. ∴ We change the order of integration for this we first sketch the region given by $x = \dfrac{y^2}{a}$ or $y^2 = ax$ (a right handed parabola) and $x = y$ a straight line and these two intersect at points given by $x^2 = ax$ or $x = 0, x = a$ *i.e.*, at (0, 0) and (a, a).



To change the order of integration we need limits of y as variable and that of x as constant.

∴ We divide the region into vertical strips (PQ)

∴ y varies from x to $\sqrt{ax}$

and x varies from 0 to a

$$\therefore \quad \int_0^a\int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}} = \int_0^a\left[\int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dy\right]dx$$

$$= \int_0^a\left[\frac{1}{a-x}\int_x^{\sqrt{ax}}\left(-\frac{1}{2}\right)(-2y)(ax-y^2)^{-1/2}\,dy\right]dx$$

$$= \int_0^a \frac{1}{a-x}\left(-\frac{1}{2}\right)\left[\frac{(ax-y^2)^{1/2}}{1/2}\bigg|_x^{\sqrt{ax}}\right]dx = \int_0^a \frac{-1}{2(a-x)}[(0-2(ax-x^2)^{1/2}]\,dx$$

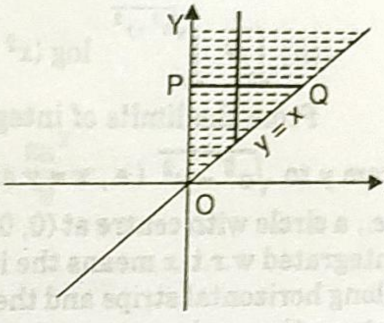
$$= \int_0^a \frac{1}{a-x}\sqrt{x}\,\sqrt{a-x}\,dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\,dx$$

Put $x = a\sin^2\theta$; $dx = 2a\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{\sqrt{a}\sin\theta}{\sqrt{a}\cos\theta}\,2a\sin\theta\cos\theta\,d\theta = 2a\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 2a\,\frac{1}{2}\,\frac{\pi}{2} = \frac{a\pi}{2}.$$

(*ii*) $\displaystyle\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$



Here the limits of integrations of y are from x to ∞ and the limits of integrations of x are from 0 to ∞. As $\frac{e^{-y}}{y}$ cannot be integrated w.r.t. y, $\therefore$ we change the order of integration. So we first sketch the region and then divide it into horizontal strips (Region is the shaded portion).

For horizontal strip x varies from 0 to y and y varies from 0 to ∞

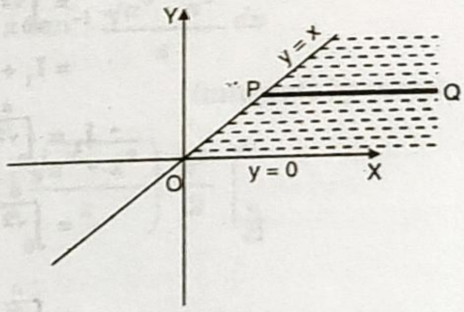
$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dydx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y}\, dx\right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, x\Big|_0^y\, dy = \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy = \int_0^\infty e^{-y}\, dy = -e^{-y}\Big|_0^\infty = 0 + 1 = 1.$$

Example 8. *Evaluate the following integrals by changing the order of integration*

(i) $\int_0^\infty \int_0^x xe^{-x^2/y}\, dydx$ (Bhopal 1999, Gorakhpur 1991)

(ii) $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\, dy\ ;\ a > 0.$ (Bhopal 1998)



Sol. (i) $\int_0^\infty \int_0^x xe^{-x^2/y}\, dy\, dx$

limits of integration show that the region is bounded by $y = 0, y = x$ and $x = 0, x = \infty$. Region is firstly integrated w.r.t. y and then w.r.t. x. So the region is divided into vertical strips. For change of integration divide the region into horizontal strips as shown in the figure.

For horizontal strip x varies from y to ∞ and y varies from 0 to ∞

$$\therefore \quad \int_0^\infty \int_0^x xe^{-x^2/y}\, dy\, dx = \int_0^\infty \int_y^\infty xe^{-x^2/y}\, dx\, dy$$

To integrate $xe^{-x^2/y}$ w.r.t. x

Put $\frac{x^2}{y} = t \quad \therefore \quad 2x\, dx = -y\, dt$ and t varies from y to ∞

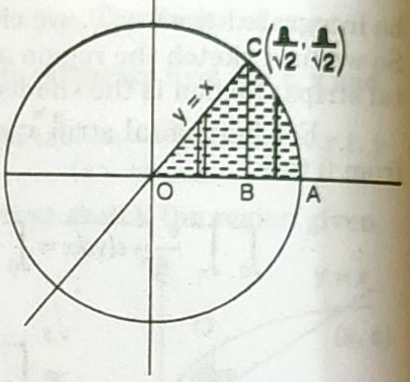
$$= \int_0^\infty \left[\int_y^\infty e^{-t}\, y\, \frac{dt}{2}\right] dy = \int_0^\infty \frac{y}{2} \cdot \frac{e^{-t}}{-1}\Big|_y^\infty dy = \int_0^\infty -\frac{y}{2}\,(0 - e^{-y})\, dy$$

$$= \frac{1}{2}\int_0^\infty ye^{-y}\, dy = \frac{1}{2}\left\{y\,\frac{e^{-y}}{-1} - \int_0^\infty 1 \cdot \frac{e^{-y}}{-1}\, dy\right\} = \frac{1}{2}\left\{-ye^{-y} + \frac{e^{-y}}{-1}\Big|_0^\infty\right\}$$

$$= \frac{1}{2}\{-0 + 0 + 0 + 1\} = \frac{1}{2}.$$

(*ii*) $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$

From the limits of integration it is clear that x varies from y to $\sqrt{a^2-y^2}$ *i.e.*, $x = y$ and $x = \sqrt{a^2-y^2}$ or $x^2+y^2 = a^2$ *i.e.*, a circle with centre at (0, 0) and radius a. Region is firstly integrated w.r.t. x means the integration is firstly performed along horizontal strips and then along vertical. For change of integration we have to perform integration w.r.t. y *i.e.*, vertical strips first and then horizontal strips. Region is shown in the figure.



The whole region of integration is divided into two parts OBC and ACB. For OBC ; y varies from 0 to x and x varies from 0 to $\frac{a}{\sqrt{2}}$.

For ACB ; y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from $\frac{a}{\sqrt{2}}$ to a.

$$\therefore \int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \int_0^{x} \log(x^2+y^2)\,dy\,dx + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} \log(x^2+y^2)\,dy\,dx$$

$$= I_1 + I_2 \qquad \text{...(1)}$$

$$I_1 = \int_0^{\frac{a}{\sqrt{2}}} \int_0^{x} 1\,.\log(x^2+y^2)\,dy\,dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \left\{ \log(x^2+y^2)\,.\,y \Big|_0^x - \int_0^x \frac{2y}{x^2+y^2}\,.\,y\,dy \right\} dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} x \log 2x^2 - 2\int_0^x \frac{y^2+x^2-x^2}{y^2+x^2}\,dy \Bigg]\,dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \left\{ x\log 2x^2 - 2\left(y - x^2\,\frac{1}{x}\tan^{-1}\frac{y}{x} \right)\Bigg|_0^x \right\} dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} (x\log 2x^2 - 2x + 2x\tan^{-1} 1)\,dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \left\{ x\log 2x^2 - 2x + 2x\,\frac{\pi}{4} \right\} dx \text{ integrate by parts}$$

$$= (\log 2x^2)\,\frac{x^2}{2}\Bigg|_0^{\frac{a}{\sqrt{2}}} - \int_0^{\frac{a}{\sqrt{2}}} \frac{1}{2x^2}\,.\,4x\,.\,\frac{x^2}{2}\,dx - x^2 + \frac{\pi x^2}{4}\Bigg|_0^{\frac{a}{\sqrt{2}}}$$

$$= \frac{a^2}{4}\log a^2 - \int_0^{\frac{a}{\sqrt{2}}} x\,dx - \frac{a^2}{2} + \frac{\pi}{4}\,.\,\frac{a^2}{2}$$

$$= \frac{a^2}{4}\log a^2 - \frac{x^2}{2}\Bigg|_0^{\frac{a}{\sqrt{2}}} - \frac{a^2}{2} + \frac{\pi a^2}{8}$$

$$= \frac{a^2}{2}\log a - \frac{a^2}{4} - \frac{a^2}{2} + \frac{\pi a^2}{8} = \frac{a^2\log a}{2} - \frac{3a^2}{4} + \frac{\pi a^2}{8}$$

$$I_2 = \int_{\frac{a}{\sqrt{2}}}^{a}\left\{\int_0^{\sqrt{a^2-x^2}} 1.\log(x^2+y^2)\,dy\right\}dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a}\left[\log(x^2+y^2).y\,\Big|_0^{\sqrt{a^2-x^2}} - \int_0^{\sqrt{a^2-x^2}}\frac{1}{x^2+y^2}.2y.y\,dy\right]dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a}\left[\sqrt{a^2-x^2}\,\log a^2 - 2\int_0^{\sqrt{a^2-x^2}}\frac{y^2+x^2-x^2}{y^2+x^2}\,dy\right]dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a}\left[(\log a^2)\sqrt{a^2-x^2} - 2\left(y - \frac{x^2}{x}\tan^{-1}\frac{y}{x}\Bigg|_0^{\sqrt{a^2-x^2}}\right)\right]dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a}\left[(\log a^2)\sqrt{a^2-x^2} - 2\sqrt{a^2-x^2} + 2x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\right]dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a}(2\log a - 2)\sqrt{a^2-x^2}\,dx + 2\int_{\frac{a}{\sqrt{2}}}^{a} x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\,dx$$

(integrate by parts)

$$= (2\log a - 2)\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx + 2\left\{\left(\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\right).\frac{x^2}{2}\Bigg|_{\frac{a}{\sqrt{2}}}^{a}\right.$$

$$\left. - \int_{\frac{a}{\sqrt{2}}}^{a}\frac{1}{1+\frac{a^2-x^2}{x^2}}\,\frac{x\cdot\frac{-2x}{2\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}}{x^2}\times\frac{x^2}{2}\,dx\right.$$

$$= (2\log a - 2)\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx + 2\left(-\frac{a^2}{4}.\frac{\pi}{4}\right) - 2\int_{\frac{a}{\sqrt{2}}}^{a}\frac{x^2}{a^2} - \frac{(x^2+a^2-x^2)}{x^2\sqrt{a^2-x^2}}.\frac{x^2}{2}\,dx$$

$$= (2\log a - 2)\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx - \frac{\pi a^2}{8} + \int_{\frac{a}{\sqrt{2}}}^{a}\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= (2\log a - 2)\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx - \frac{\pi a^2}{8} - \int_{\frac{a}{\sqrt{2}}}^{a}\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= (2\log a - 2)\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx - \frac{\pi a^2}{8} - \int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx + a^2\int_{\frac{a}{\sqrt{2}}}^{a}\frac{dx}{\sqrt{a^2-x^2}}$$

$$= (2\log a - 3)\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx - \frac{\pi a^2}{8} + a^2 \int_{\frac{a}{\sqrt{2}}}^{a} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= (2\log a - 3)\left\{\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\} + a^2 \sin^{-1}\frac{x}{a}\Bigg|_{\frac{a}{\sqrt{2}}}^{a} - \frac{\pi a^2}{8}$$

$$= (2\log a - 3)\left(\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a}{2\sqrt{2}}\,\frac{a}{\sqrt{2}} - \frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right) + a^2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{\pi a^2}{8}$$

$$= (2\log a - 3)\,a^2\left(\frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8}\right) + a^2\,\frac{\pi}{4} - \frac{\pi a^2}{8}$$

$$= a^2(2\log a - 3)\left(\frac{\pi}{8} - \frac{1}{4}\right) + \frac{\pi a^2}{8}$$

Substituting the values of I_1 and I_2 in (1)

$$\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} \log(x^2 + y^2)\,dx\,dy = \frac{a^2\log a}{2} - \frac{3a^2}{4} + \frac{\pi a^2}{8} + 2a^2\log a\left(\frac{\pi}{8} - \frac{1}{4}\right)$$

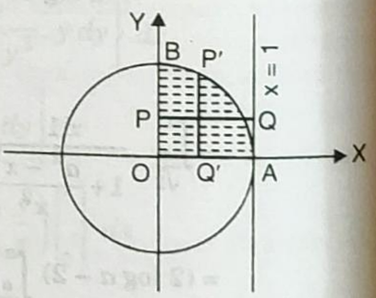
$$- \frac{3a^2\pi}{8} + \frac{3}{4}a^2 + \frac{\pi a^2}{8}$$

$$= a^2\log a\left[\frac{1}{2} + \frac{\pi}{4} - \frac{1}{2}\right] - \frac{\pi a^2}{8} = \frac{\pi a^2}{4}\left[\log a - \frac{1}{2}\right]$$

Example 9. *Evaluate the following integrals by changing the order of integration*

(*i*) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$ (P.T.U., Dec. 2000, 2004)

(*ii*) $\int_0^1 \int_{4y}^{4} e^{x^2}\,dx\,dy$ (P.T.U., Dec. 2005)



Sol. (*i*) From the limits of integration, it is clear that of region of integration is bounded by the lines $x = 0, x = 1, y = 0$ and the circle $x^2 + y^2 = 1$ ($\because$ $y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$). Thus the region of integration is OAB. As in the given integral we have to first integrate w.r.t. y and then w.r.t. x. So the region is divided into vertical strips. To change the order of integration we have to divide the region into horizontal strips.

Solving $y = \sqrt{1-x^2}$ gives $x = \sqrt{1-y^2}$

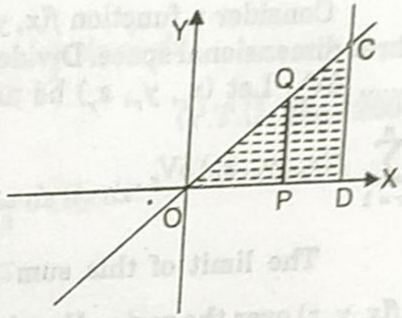
$\therefore$ for horizontal strip x varies from 0 to $\sqrt{1-y^2}$ and y varies from 0 to 1.

$$\therefore \int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_0^1\int_0^{\sqrt{1-y^2}} y^2\,dx\,dy = \int_0^1 y^2 x\Big|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2\sqrt{1-y^2}\,dx \quad \text{Put } y = \sin\theta \quad dy = \cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta = \frac{1.1}{4.2}\,\frac{\pi}{2} = \frac{\pi}{16}.$$

(ii) $\int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy$



From the limits of integration, it is clear that the region of integration is $x = 4y$, $x = 4$, $y = 0$, $y = 1$. As in the given integral we have to first integrate w.r.t. x and then w.r.t. y, so region is divided into horizontal strips. To change order of integration we have to divide the region ODC (shown in figure) into vertical strips.

For the region ODC y varies from 0 to $\frac{x}{4}$ and x varies for 0 to 4.

$$\therefore \quad \int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx = \int_0^4 e^{x^2}\, y \Big|_0^{x/4} dx = \int_0^4 \frac{x}{4} e^{x^2}\, dx$$

$$= \frac{1}{4} \int_0^4 x e^{x^2}\, dx \qquad \text{Put } x^2 = t\ ;\ 2\,x dx = dt$$

$$= \frac{1}{4} \int_0^{16} e^t \frac{dt}{2} = \frac{1}{8} e^t \Big|_0^{16} = \frac{e^{16} - 1}{8}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals by changing the order of integration :

1. $\int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2}$

2. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x\, dx\, dy$

3. $\int_0^a \int_{x^2/a}^{2a - x} xy\, dy\, dx$

4. $\int_0^b \int_0^{a/b\sqrt{b^2 - y^2}} xy\, dx\, dy$ *(Delhi, 1997)*

5. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2 x^2}}$ *(Kerala, 1995 ; Mysore, 1994 S ; Andhra, 1992)*

6. $\int_0^3 \int_1^{\sqrt{4 - y}} (x + y)\, dx\, dy$

7. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$.

Answers

1. π
2. $\frac{\sqrt{2}a^3}{6}$
3. $\frac{3a^4}{8}$
4. $\frac{1}{8}a^2 b^2$
5. $\frac{\pi a^2}{6}$
6. $\frac{241}{60}$
7. $\frac{a^3}{28} + \frac{a}{20}$

8.5. TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ which is continuous at every point of a finite region V of three dimensional space. Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \ldots\ldots, \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r .$$

The limit of this sum as $n \to \infty$ and $\delta V_r \to 0$, if it exists, is called the *triple integral* of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\, dV$.

For purposes of evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz , \qquad \ldots(1)$$

the order of integration depending upon the limits.

Let x_1, x_2 be function of y, z ; y_1, y_2 be function of z and z_1, z_2 be constants, *i.e.*,

Let $x_1 = f_1(y, z), x_2 = f_2(y, z), y_1 = \phi_1(z), y_2 = \phi_2(z)$ and $z_1 = a, z_2 = b$.

Then the integral (1) is evaluated as follows :

$$\boxed{\int_{z_1=a}^{z_2=b} \boxed{\int_{y_1=\phi_1(z)}^{y_2=\phi_2(z)} \boxed{\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\, dx}\; dy}\; dz}$$

First $f(x, y, z)$ is integrated w.r.t. x (keeping y and z constant) between the limits x_1 and x_2. The resulting expression, which is a function of y and z is then integrated w.r.t. y (keeping z constant) between the limits y_1 and y_2. The resulting expression, which is a function of z only is then integrated w.r.t. z between the limits z_1 and z_2. The order of integration is from the innermost rectangle to the outermost rectangle.

Limits involving two variables are kept innermost, then the limits involving one variable and finally the constant limits.

If $x_1, x_2 ; y_1, y_2$ and z_1, z_2 are all constants, then the order of integration is immaterial, provided the limits are changed accordingly. Thus

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x,y,z)\, dy\, dz\, dx .$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dz\, dy\, dx , \text{ etc.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$

(P.T.U., May 2004)

Sol.
$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz\, dy\, dx$$
$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$
$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \int_0^1 \frac{\pi}{2}\Big[y\Big]_0^{\sqrt{1-x^2}} dx$$
$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \frac{\pi}{4}[\sin^{-1} 1] = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

(P.T.U., May 2006)

Example 2. *Evaluate* $\int_0^2\int_1^2\int_0^{yz} xyz\, dx\, dy\, dz.$

Sol. $\int_0^2\int_1^2\left[\int_0^{yz} xyz\, dx\right] dy\, dz$
$$= \int_0^2\int_1^2 yz\,\frac{x^2}{2}\Bigg|_0^{yz} dy\, dz = \int_0^2\left[\int_1^2 \frac{y^3z^3}{2}\, dy\right] dz$$
$$= \int_0^2 \frac{z^3}{2}\cdot\frac{y^4}{4}\Bigg|_1^2 dz = \int_0^2 \frac{z^3}{8}(16-1)\, dz = \frac{15}{8}\cdot\frac{z^4}{4}\Bigg|_0^2$$
$$= \frac{15}{8}\cdot\frac{16}{4} = \frac{15}{2}.$$

Example 3. *Evaluate* $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\, dz\, dx\, dy.$

Sol.
$$I = \int_1^e\int_1^{\log y}\left[\int_1^{e^x} \log z\, dz\right] dx\, dy$$

Since $\int_1^{e^x} \log z\, dz = \int_1^{e^x} \log z\, .\, 1\, dz$

Integrating by parts $= \Big[\log z\, .\, z\Big]_1^{e^x} - \int_1^{e^x} \frac{1}{z}\, .\, z\, dz$
$$= e^x \log e^x - 0 - \Big[z\Big]_1^{e^x} = xe^x - e^x + 1 = (x-1)\, e^x + 1$$

$$\therefore \qquad I = \int_1^e \int_1^{\log y} [(x-1)\,e^x + 1]\, dx\, dy$$

Now $\displaystyle\int_1^{\log y} [(x-1)\,e^x + 1]\, dx = \int_1^{\log y} (x-1)\,e^x\, dx + \Big[\, x \,\Big]_1^{\log y}$

$$= \Big[(x-1)\,e^x\Big]_1^{\log y} - \int_1^{\log y} 1\,.\,e^x\, dx + \log y - 1 \qquad \text{(Integration by parts)}$$

$$= (\log y - 1)\,e^{\log y} - \Big[\,e^x\,\Big]_1^{\log y} + \log y - 1$$

$$= y(\log y - 1) - (e^{\log y} - e) + \log y - 1 \qquad [\because \quad e^{\log y} = y]$$

$$= y(\log y - 1) - y + e + \log y - 1 = (y+1)\log y - 2y + e - 1$$

$$\therefore \qquad I = \int_1^e [\log y\,.\,(y+1) - 2y + e - 1]\, dy$$

$$= \left[\log y\,.\left(\frac{y^2}{2} + y\right)\right]_1^e - \int_1^e \frac{1}{y}\left(\frac{y^2}{2} + y\right) dy - \Big[\,y^2\,\Big]_1^e + (e-1)\Big[\,y\,\Big]_1^e$$

$$= \frac{e^2}{2} + e - \int_1^e \left(\frac{y}{2} + 1\right) dy - (e^2 - 1) + (e-1)^2$$

$$= \frac{e^2}{2} + e - \left[\frac{y^2}{4} + y\right]_1^e - 2e + 2 = \frac{e^2}{2} + e - \left[\left(\frac{e^2}{4} + e\right) - \left(\frac{1}{4} + 1\right)\right] - 2e + 2$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4}(e^2 - 8e + 13).$$

Example 4. *Evaluate* $\displaystyle\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

Sol. $\displaystyle\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y}\,.\,e^z\, dz\, dy\, dx = \int_0^{\log 2}\int_0^x e^{x+y}\,.\,e^z \Big|_0^{x+\log y} dy\, dx$

$$= \int_0^{\log 2}\int_0^x e^{x+y}\,(e^{x+\log y} - 1)\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^x e^{x+y}\,(e^x\,.\,e^{\log y} - 1) dy\, dx = \int_0^{\log 2}\int_0^x e^{x+y}\,(ye^x - 1)\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^x [e^{2x}(ye^y) - e^x\,.\,e^y]\, dy\, dx$$

$$= \int_0^{\log 2} \left\{e^{2x}(y-1)\,e^y - e^x\, e^y \Big|_0^x\right\} dx$$

$$\Big| \because \quad \int y\, e^y\, dy = y\, e^{2y} - \int 1\,.\,e^y\, dy = (y-1)\,e^y$$

$$= \int_0^{\log 2} \{e^{2x}\,(x-1)\,e^x - e^{2x} + e^{2x} + e^x\}\, dx$$

$$= \int_0^{\log 2} \{(x-1)\,e^{3x} + e^x\}\, dx$$

$$= (x-1)\,\frac{e^{3x}}{3}\Bigg|_0^{\log 2} - \int_0^{\log 2} 1.\,\frac{e^{3x}}{3}\,dx + e^x\Big|_0^{\log 2}$$

$$= (\log 2 - 1)\,.\,\frac{1}{3}\,e^{3\log 2} + \frac{1}{3} - \frac{e^{3x}}{9}\Bigg|_0^{\log 2} + (e^{\log 2} - 1)$$

$$= \frac{8}{3}\,(\log 2 - 1) + \frac{1}{3} - \frac{e^{3\log 2}}{9} + \frac{1}{9} + 2 - 1 \qquad \Big|\ \because\ e^{3\log 2} = e^{\log 2^3} = 8$$

$$= \frac{8}{3}\log 2 - \frac{8}{3} + \frac{1}{3} - \frac{8}{9} + \frac{1}{9} + 1 = \frac{8}{3}\log 2 - \frac{19}{9}.$$

Example 5. *Evaluate* $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta.$

Sol. $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} rz\,\Big|_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{(a^2-r^2)^{3/2}}{3/2}\Bigg|_0^{a\cos\theta} d\theta \quad \text{using} \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} (a^3\sin^3\theta - a^3)\,d\theta$$

$$= -\frac{a^3}{3}\left\{\int_0^{\frac{\pi}{2}} \sin^3\theta\,d\theta - \int_0^{\frac{\pi}{2}} 1.\,d\theta\right\}$$

$$= -\frac{a^3}{3}\left\{\frac{2}{3.1} - \frac{\pi}{2}\right\} \quad \text{using} \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta = \frac{(n-1)(n-3)\ldots}{n(n-1)\ldots} \text{ when } n \text{ is odd}$$

$$= \frac{a^3}{3}\left\{\frac{\pi}{2} - \frac{2}{3}\right\}.$$

Example 6. *Evaluate* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *over the tetrahedron bounded by the coordinate planes and the plane* $x + y + z = 1.$ (P.T.U. Dec. 2003)

Sol. $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx = \int_0^1\int_0^{1-x} \frac{(x+y+z+1)^{-2}}{-2}\Bigg|_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} (x+y+1+1-x-y)^{-2} - (x+y+1)^{-2}]\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} \frac{1}{2^2} - (x+y+1)^{-2}]\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\Bigg|_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + 2^{-1} - (x+1)^{-1}\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\Bigg|_0^1\right]$$

$$= -\frac{1}{2}\cdot\left(\frac{3}{4} - \frac{1}{8} - \log 2\right) = -\frac{1}{2}\left(\frac{5}{8} - \log 2\right)$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1—12) :

1. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz.$

2. $\int_0^a\int_0^a\int_0^a (yz + zx + xy)\,dx\,dy\,dz.$

3. $\int_{-1}^1\int_0^z\int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz.$

4. $\int_1^3\int_{1/x}^1\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx.$

5. $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx.$

6. $\int_0^1\int_{y^2}^{1-x}\int_0^{1-x} x\,dz\,dx\,dy.$ (*Karnataka, 1988*)

7. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$

8. $\int_0^{\pi/2} d\theta\int_0^{a\sin\theta} dr\int_0^{(a^2-r^2)} r\,dz.$ (*Marathwada, 1993*)

9. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx.$ (*Andhra, 1994 ; Ranchi, 1990*)

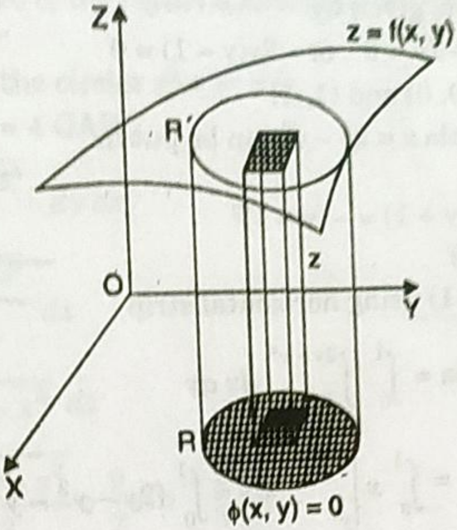
10. $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta.$ (*Mysore, 1997*)

11. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$

[Hint See S.E. 6]

Answers

1. $(e-1)^3$	2. $\frac{3}{4}a^5$	3. 0	4. $\frac{13}{9} - \frac{1}{6}\log 3$
5. $\frac{1}{720}$	6. $\frac{4}{35}$	7. $\frac{1}{48}$	8. $\frac{5\pi a^3}{64}$



(*b*) *Cylindrical Co-ordinates.* Let the equation of the surface be $z = f(r, \phi)$. Replacing $dx\,dy$ by $r\,dr\,d\phi$, we get $V = \iint_R zr\,dr\,d\phi$.

ILLUSTRATIVE EXAMPLES

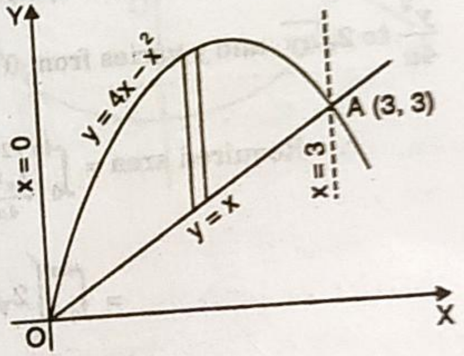
Example 1. (*a*) *Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.* (P.T.U. May 2003 ; Hamirpur, 1995)

(*b*) *Find by double integration the area bounded by $x = 2y - y^2$ and $x = y^2$.* (P.T.U., May 2002)

(*c*) *Find the area between the parabolas $y^2 = 4ax$, $x^2 = 4ay$.* (P.T.U., Dec. 2003)

Sol. (*a*) The two curves intersect at points whose abscissae are given by $4x - x^2 = x$

or $x^2 - 3x = 0$ *i.e.*, $x = 0, 3$.



Using vertical strips, the required area lies between $x = 0, x = 3$ and $y = x, y = 4x - x^2$.

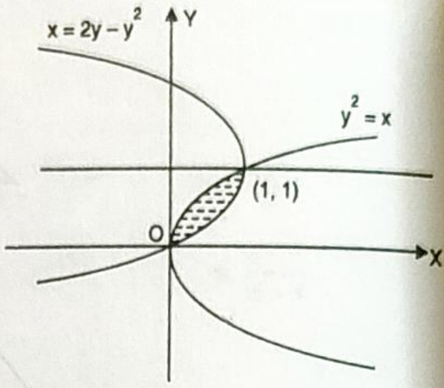
$$\therefore \text{ Required area} = \int_0^3 \int_x^{4x - x^2} dy\,dx$$

$$= \int_0^3 \Big[y \Big]_x^{4x - x^2} dx$$

$$= \int_0^3 (4x - x^2 - x)\,dx = \int_0^3 (3x - x^2)\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = 4.5.$$

(*b*) Equations of the curves are $x = 2y - y^2$ and $x = y^2$. Both are parabolas.



They intersect at points given by
$y^2 = 2y - y^2$ or $2y^2 - 2y = 0$ or $2y(y-1) = 0$
or $y = 0, y = 1$ or at (0, 0) and (1, 1)

Equation of the parabola $x = 2y - y^2$ can be put in the form $x = -(y^2 - 2y)$

$$x - 1 = -(y^2 - 2y + 1) = -(y-1)^2$$

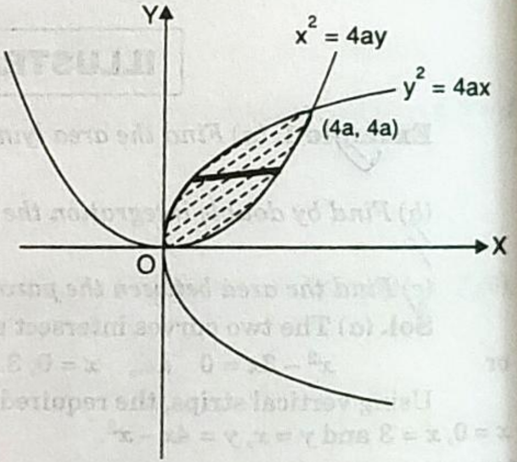
or $$(y-1)^2 = -(x-1)$$

parabola having vertex at (1, 1) using horizontal strip

$$\text{Required area} = \int_0^1 \int_{y^2}^{2y-y^2} dx\, dy$$

$$= \int_0^1 x \Big|_{y^2}^{2y-y^2} dy = \int_0^1 (2y - y^2 - y^2)\, dy = \int_0^1 (2y - 2y^2)\, dy$$

$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1\right] = 2 . \left[\frac{1}{2} - \frac{1}{3}\right] = 2 . \frac{1}{6} = \frac{1}{3}.$$



(c) The two curves $y^2 = 4ax$

and $x^2 = 4ay$

intersect at $\frac{x^4}{16a^2} = 4ax$

or $x^4 - 64a^3x = 0$

or $x(x^3 - 64a^3) = 0$

i.e., at $x = 0$ and $x = 4a$

i.e., at $(0, 0)$ and $(4a, 4a)$

Using horizontal strips ; x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$ and y varies from 0 to $4a$.

$$\therefore \quad \text{Required area} = \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy = \int_0^{4a} x \Big|_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

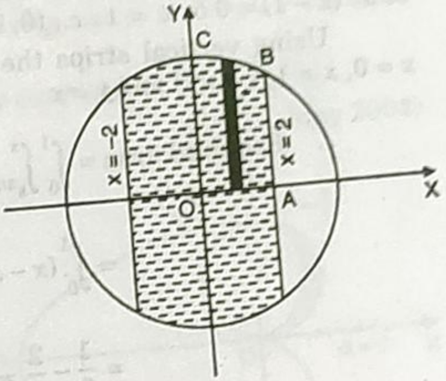
$$= 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\Big|_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{4}{3}\sqrt{a} \,.\, 8a^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

Example 2. *Find the area of the region bounded by the lines $x = -2$, $x = 2$ and the circle $x^2 + y^2 = 9$.* (P.T.U. May 2007)

Sol. The equation on of the circles $x^2 + y^2 = 9$



Required area = 4 OABC

$$= 4 \int_0^2 \int_0^{\sqrt{9-x^2}} dy\, dx$$

$$= 4 \int_0^2 \left| y \right|_0^{\sqrt{9-x^2}} dx$$

$$= 4 \int_0^2 \sqrt{9-x^2}\, dx$$

$$= 4 \cdot \left\{ \frac{x\sqrt{9-x^2}}{2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \right|_0^2 \Bigg\}$$

$$= 4 \left\{ \frac{2\sqrt{5}}{2} + \frac{9}{2} \sin^{-1} \frac{2}{3} \right\}$$

$$= 4\sqrt{5} + 18 \sin^{-1} 2/3.$$

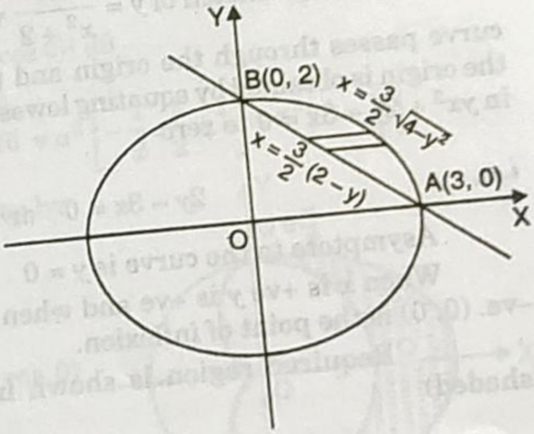
Example 3. *Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.*

Sol. The equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad \text{...(1)}$$

and the line is

$$\frac{x}{3} + \frac{y}{2} = 1 \qquad \text{...(2)}$$



Both meet x-axis at A(3, 0) and y-axis at B(0, 2).

Using horizontal strips, the required area lies between

$$x = \frac{3}{2}(2-y),\ x = \frac{3}{2}\sqrt{4-y^2} \text{ and } y = 0, y = 2.$$

$$\therefore \quad \text{Required area} = \int_0^2 \int_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dx\, dy$$

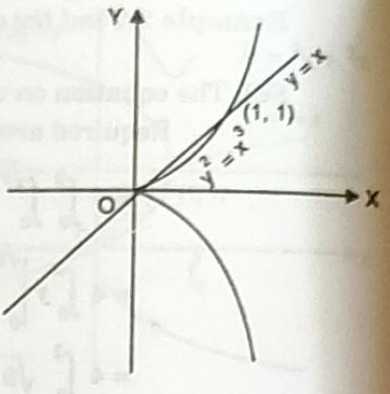
$$= \int_0^2 \Big[x \Big]_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dy$$

$$= \int_0^2 \frac{3}{2} [\sqrt{4-y^2} - (2-y)]\, dy$$

$$= \frac{3}{2} \left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \frac{y}{2} - 2y + \frac{y^2}{2} \right]_0^2$$

$$= \frac{3}{2} [2 \sin^{-1} 1 - 4 + 2] = \frac{3}{2}\left(2 \cdot \frac{\pi}{2} - 2\right) = \frac{3}{2}(\pi - 2).$$

Example 4. *Using double integration, Find the area enclosed by the curves* $y^2 = x^3$ *and* $y = x$. (P.T.U., Dec. 2005)



Sol. The two curves intersect at points given by $x^2 = x^3$ or $x^2(x-1) = 0$ or $x = 1$ *i.e.,* (0, 0) and (1, 1)

Using vertical strips the required area lies between $x = 0, x = 1, y = x^{3/2}$ and $y = x$

$$\therefore \quad \text{Required area} = \int_0^1 \int_{x^{3/2}}^{x} dy\,dx = \int_0^1 y\Big|_{x^{3/2}}^{x} dx$$

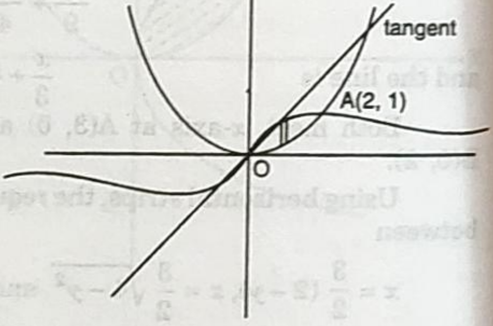
$$= \int_0^1 (x - x^{3/2})dx = \frac{x^2}{2} - \frac{x^{5/2}}{5/2}\Bigg|_0^1$$

$$= \frac{1}{2} - \frac{2}{5} = \frac{1}{10}.$$

Example 5. *Find the area enclosed by the curves* $y = \dfrac{3x}{x^2+2}$ *and* $4y = x^2$.

Sol. Equations of the curves are $y = \dfrac{3x}{x^2+2}$ ...(1)

and $4y = x^2$ ...(2)



For rough sketch of $y = \dfrac{3x}{x^2+2}$ we see that the curve passes through the origin and the tangent at the origin is obtained by equating lowest degree terms in $yx^2 + 2y - 3x = 0$ to zero

i.e., $2y - 3x = 0$ or $y = \dfrac{3}{2}x$

Asymptote to the curve is $y = 0$

When x is +ve y is +ve and when x is –ve, y is –ve. (0, 0) is the point of inflexion.

$\therefore$ Required region is shown in the figure (shaded)

[(1) and (2) intersect at $\dfrac{x^2}{4} = \dfrac{3x}{x^2+2}$ *i.e.,* $x = 0$ or $x(x^2+2) = 12$

or $x^3 + 2x - 12 = 0$ or $(x-2)(x^2+2x+6) = 0$

i.e., at (0, 0) and (2, 1)]

$$\text{Area OAO} = \int_{x=0}^{2} \int_{x^4/4}^{3x/x^2+2} dy\,dx \qquad \text{(vertical strip is taken)}$$

$$= \int_0^2 y\Bigg|_{\frac{x^4}{4}}^{\frac{3x}{x^2+4}} dx = \int_0^2 \left(\frac{3x}{x^2+4} - \frac{x^2}{4}\right) dx$$

$$= \frac{3}{2}\log(x^2+2) - \frac{x^3}{12}\Big|_0^2 = \frac{3}{2}\log 6 - \frac{3}{2}\log 2 - \frac{8}{12} = \frac{3}{2}\log 3 - \frac{2}{3}.$$

Example 6. (*i*) *Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.*

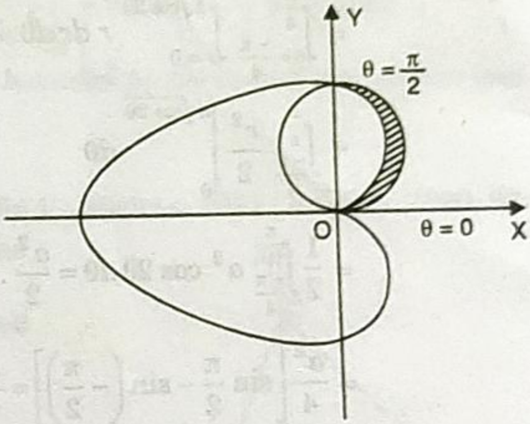
(*ii*) *Find the area outside the circle $r = a$ and inside the cardioid $r = a(1+\cos\theta)$.*

(P.T.U., May 2004)

Sol. (*i*) Eliminating r between the equations of two curves, $\sin\theta = 1 - \cos\theta$ or $\sin\theta + \cos\theta = 1$

Squaring $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$

$\therefore \quad 2\theta = 0$ or π *i.e.*, $\theta = 0$ or $\frac{\pi}{2}$



For the required area, r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required area

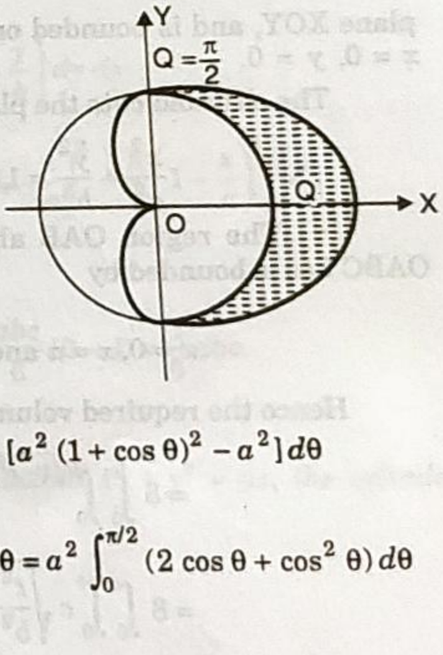
$$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} a^2\,[\sin^2\theta - (1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(-2\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left[-\frac{1}{2}.\frac{\pi}{2}+1\right] = a^2\left(1-\frac{\pi}{4}\right).$$

(*ii*) Intersection of $r = a$ and $r = a(1+\cos\theta)$ is given by $a = a(1+\cos\theta)$ *i.e.*, $1 + \cos\theta = 1$ *i.e.*, $\cos\theta = 0$.



$\therefore \quad \theta = \frac{\pi}{2}, -\frac{\pi}{2}$

For the required area r varies from a to $a(1+\cos\theta)$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$\therefore$ Required area

$$= \int_{-\frac{\pi}{2}}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2}\frac{r^2}{2}\Bigg|_a^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2}[a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= \frac{1}{2}.2a^2\int_0^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta = a^2\int_0^{\pi/2}(2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 2a^2\Big[\sin\theta\Big]_0^{\pi/2} + a^2\,\frac{1}{2}\,\frac{\pi}{2}$$

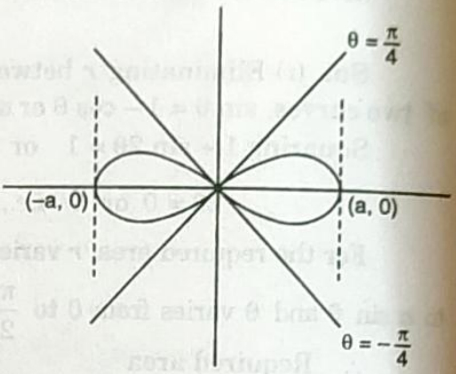
$$= 2a^2 \,.\, 1 + \frac{a^2\pi}{4} = \frac{a^2}{4}\,(8 + \pi).$$

Example 7. *Find by double integration, the area of one loop of the laminscate* $r^2 = a^2 \cos 2\theta$.

(Hamirpur 1995)

Sol. For details of figure see S.E.6 art. 4.16

Area of one loop



$$= \int_{\theta=\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left.\frac{r^2}{2}\right|_0^{a\sqrt{\cos 2\theta}} d\theta$$

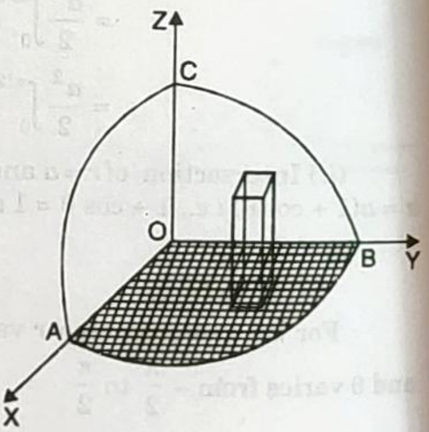
$$= \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta = \frac{a^2}{2} . \left.\frac{\sin 2\theta}{2}\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{a^2}{4} . 2 = \frac{a^2}{2}.$$

Example 8. *Find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(P.T.U. May 2003, Dec. 2000 ; Kerala, 1990 S ; Andhra, 1990 ; Madras, 1994 S)

Sol. On account of symmetry, the required volume is 8 times the volume of the ellipsoid in the positive octant. The volume OABC in the positive octant lies between the ellipsoid $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ and the plane XOY, and is bounded on the sides by the planes $x = 0$, $y = 0$.



The ellipsoid cuts the plane XOY in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$

∴ The region OAB above which the volume OABC lies is bounded by

$$x = 0, x = a \text{ and } y = 0, y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Hence the required volume of the ellipsoid

$$= 8\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} z\,dy\,dx = 8\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a \int_0^t c\sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}}\,dy\,dx \quad \text{where } \sqrt{1 - \frac{x^2}{a^2}} = \frac{t}{b}.$$

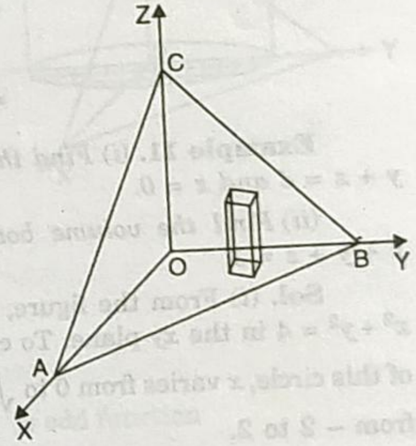
$$= 8\int_0^a \int_0^t \frac{c}{b}\sqrt{t^2 - y^2}\, dy\, dx = 8\int_0^a \frac{c}{b}\left[\frac{y\sqrt{t^2 - y^2}}{2} + \frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t dx$$

$$= \frac{4c}{b}\int_0^a t^2 \sin^{-1} 1\, dx = \frac{2\pi c}{b}\int_0^a t^2\, dx$$

$$= \frac{2\pi c}{b}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a = 2\pi bc\left(a - \frac{a}{3}\right) = \frac{4}{3}\pi abc.$$

Example 9. *Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(P.T.U., Dec. 2003 ; Mysore, 1994 S ; Madras, 1992 ; Andhra, 1990, 92)



Sol. The required volume OABC lies between the plane $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ and the plane XOY, and is bounded on the sides by the planes $x = 0, y = 0$.

The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the plane XOY in the line $\frac{x}{a} + \frac{y}{b} = 1, z = 0$

∴ The region OAB above which the volume OABC lies is bounded by $x = 0, x = a$ and $y = 0$,

$$y = b\left(1 - \frac{x}{a}\right)$$

Hence the required volume of the tetrahedron

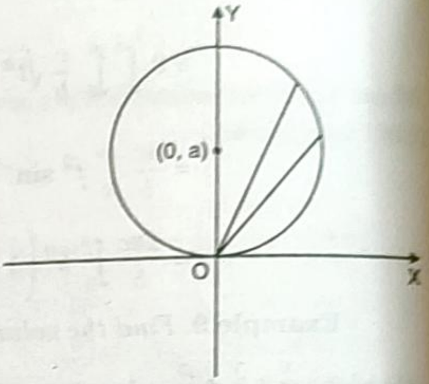
$$= \int_0^a \int_0^{b(1-x/a)} z\, dy\, dx = \int_0^a \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= \int_0^a c\left[\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx = c\int_0^a \left[b\left(1 - \frac{x}{a}\right)^2 - \frac{b}{2}\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1 - \frac{x}{a}\right)^2 dx = \frac{bc}{2}\cdot\left[\frac{\left(1 - \frac{x}{a}\right)^3}{-\frac{3}{a}}\right]_0^a = -\frac{abc}{6}(0 - 1) = \frac{1}{6}abc.$$

Example 10. *Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.*

Sol. Required volume $= \iint z\, dx\, dy$



$$= 2\iint \frac{x^2+y^2}{a}\,dx\,dy$$

(twice ∵ volume in both parts of the circle is change it to polar coordinates same)

$$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr d\theta$$

When $x^2 + y^2 = 2ay$, we have in polar coordinates

$$r^2 = 2ar\sin\theta \quad \therefore \quad r = \theta \quad \text{or} \quad r = 2a\sin\theta$$

is r varies from 0 to $2a \sin\theta$ and θ veries from 0 to $\frac{\pi}{2}$

$$\therefore \quad \text{Reqduired volume} = 2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2a\sin\theta} \frac{r^2}{a}\,r\,dr\,d\theta$$

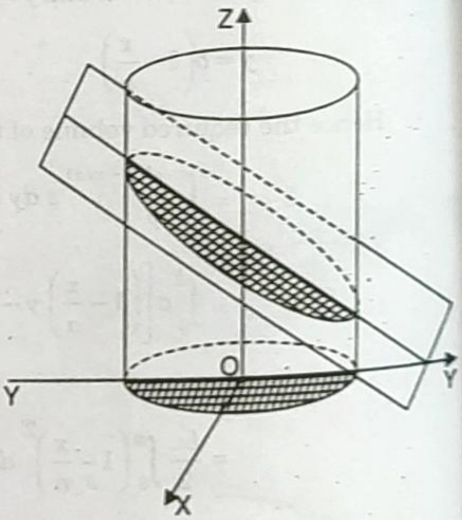
$$= \frac{2}{a}\int_0^{\frac{\pi}{2}} \left.\frac{r^4}{4}\right|_0^{2a\sin\theta} d\theta = \frac{1}{2a}\int_0^{\frac{\pi}{2}} 16a^4 \sin^4\theta\,d\theta$$

$$= 8a^3\,\frac{3.1}{4.2}\,\frac{\pi}{2} = \frac{3a^3\pi}{2}.$$

Example 11. *(i) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*

(ii) Find the volume bounded by the xy plane the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. (P.T.U., Dec. 2005)

Sol. (*i*) From the figure, it is clear that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{4-y^2}$ while y varies from -2 to 2.



∴ Required volume

$$= 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} (4-y)\Big[x\Big]_0^{\sqrt{4-y^2}} dy$$

$$= 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

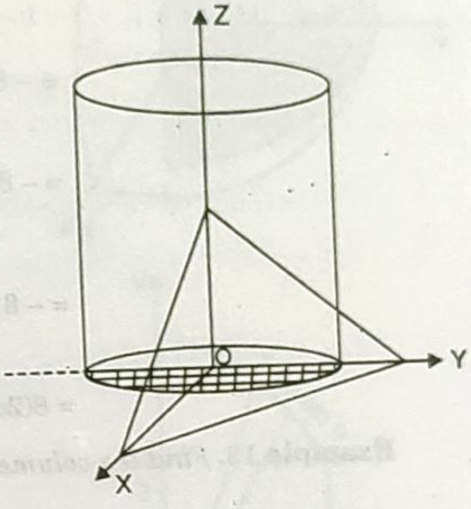
$$= 8\int_{-2}^{2} \sqrt{4-y^2}\,dy$$

$\left[\begin{array}{l}\text{the second integral is zero since}\\ y\sqrt{4-y^2} \text{ is an odd function of } y\end{array}\right.$

$$= 16 \int_0^2 \sqrt{4-y^2}\, dy \qquad [\because \sqrt{4-y^2} \text{ is an even function of } y]$$

$$= 16\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2 = 16\,[2\sin^{-1} 1] = 32 \times \frac{\pi}{2} = 16\pi.$$

(ii) From the figure it is clear that $z = 3 - x - y$ is to be integrated over the circle $x^2 + y^2 = 1$ over xy plane for which x varies from 0 to $\sqrt{1-y^2}$ and y varies from -1 to 1.



∴ Required volume

$$= 2\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} (3 - x - y)\, dx\, dy$$

$$= 2\int_{-1}^{1}\left[3x - \frac{x^2}{2} - yx\right]_0^{\sqrt{1-y^2}} dy$$

$$= 2\int_{-1}^{1}\left[3\sqrt{1-y^2} - \frac{1-y^2}{2} - y\sqrt{1-y^2}\right] dy$$

$$= 6\int_{-1}^{1}\sqrt{1-y^2}\, dy - \frac{2}{2}\left(y - \frac{y^3}{3}\Big|_{-1}^{1}\right) - 2\int_{-1}^{1} y\sqrt{1-y^2}\, dy$$

$$= 6\left\{\frac{y\sqrt{1-y^2}}{2} + \frac{1}{2}\sin^{-1} y\right\}_{-1}^{1} - \left(\frac{4}{3}\right) - 0 \quad \because y\sqrt{1-y^2} \text{ is odd function}$$

$$= 6\left\{\frac{1}{2}\, 2\, \frac{\pi}{2}\right\} - \frac{4}{3} = 2\left(\frac{3\pi}{2} - \frac{2}{3}\right) = \left(3\pi - \frac{4}{3}\right).$$

Example 12. *Prove that the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{128\, a^3}{15}$.*

Sol. The section of the cylinder is a circle $x^2 + y^2 - 2ax = 0$ with centre $(a, 0)$ in the xy plane is $z = \sqrt{2ax}$ is to be integrated over the circle $x^2 + y^2 - 2ax = 0$.

∴ In 1st quadrant y varies from 0 to $\sqrt{2ax - x^2}$ and x varies from 0 to $2a$.

∴ Required volume $= 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} z\, dydx$

$$= 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \sqrt{2ax}\, dydx$$

$$= 4\int_0^{2a} \sqrt{2ax}\, y\Big|_0^{\sqrt{2ax-x^2}} dx = 4\int_0^{2a} \sqrt{2ax}\,\sqrt{2ax - x^2}\, dx$$

$$= 4\sqrt{2a}\int_0^{2a} x\sqrt{2a - x}\, dx$$

Put $2a - x = t^2$ $\therefore$ $dx = -\,2t\, dt$

$$= 4\sqrt{2a}\int_{\sqrt{2a}}^{0} (2a - t^2)\,.\,t\,(-\,2t)\, dt \qquad \left|\begin{array}{l} \text{when } x = 0,\ t^2 = 2a \\ t = \sqrt{2a}\text{, when } x = 2a,\ t = 0 \end{array}\right.$$

$$= -\,8\sqrt{2a}\int_{\sqrt{2a}}^{0} (2at^2 - t^4)\, dt$$

$$= -\,8\sqrt{2a}\left[\frac{2at^3}{3} - \frac{t^5}{5}\Big|_{\sqrt{2a}}^{0}\right]$$

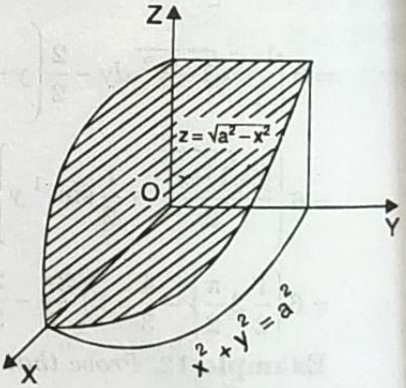
$$= -\,8\sqrt{2a}\left[-\frac{2a}{3}(2a)^{3/2} + \frac{(2a)^{5/2}}{5}\right] = -\,8\sqrt{2a}\left(\frac{1}{3} - \frac{1}{5}\right)(2a)^{5/2}$$

$$= 8(2a)^3\,\frac{2}{15} = \frac{16}{15}\;8a^3 = \frac{128}{15}a^3.$$

Example 13. *Find the volume common to the cylinders* $x^2 + y^2 = a^2$ *and* $x^2 + z^2 = a^2$.

(P.T.U., May 2004 ; Andhra, 1990 ; Kerala, 1990)

Sol. The section of the cylinder $x^2 + y^2 = a^2$ is the circle $x^2 + y^2 = a^2$ in the xy-plane. In the figure, only one-eighth (in positive octant) of the required volume is shown.



From the figure, it is evident that $z = \sqrt{a^2 - x^2}$ is to be evaluated over the quadrant of the circle $x^2 + y^2 = a^2$ in the first quadrant for which x varies from 0 to a and y varies from 0 to $\sqrt{a^2 - x^2}$.

$$\therefore \quad \text{Required volume} = 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} z\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2}\, dy\, dx = 8\int_0^a \sqrt{a^2 - x^2}\,\Big[\,y\,\Big]_0^{\sqrt{a^2 - x^2}} dx$$

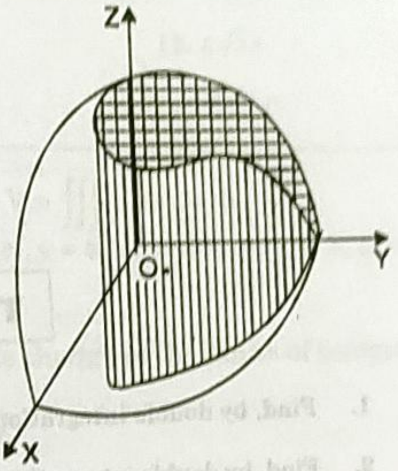
$$= 8\int_0^a (a^2 - x^2)\, dx = 8\left[a^2x - \frac{x^3}{3}\right]_0^a = 8\left(a^3 - \frac{a^3}{3}\right) = \frac{16a^3}{3}.$$

Example 14. *Find the volume common to the sphere* $x^2 + y^2 + z^2 = a^2$ *and the cylinder* $x^2 + y^2 = ay$.

(Kerala, 1990)

Sol. The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder. On account of the symmetry of the sphere, half of it lies above the plane XOY and half below it.

∴ Required volume $= 2\iint z\, dy\, dx$



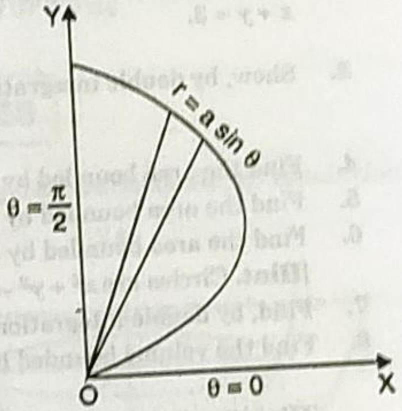
where $z = \sqrt{a^2 - x^2 - y^2}$, and the region of integration is the area inside the circle $x^2 + y^2 = ay$ in the xy-plane. ...(1)

On account of symmetry, the volumes above the two parts of circle (1) in the first and the second quadrants are equal. (The figure shows only the part in the first quadrant).

∴ Required volume $= 2 \times 2 \iint_R \sqrt{a^2 - x^2 - y^2}\, dy\, dx$

where R is the half of the circle (1) lying in the first quadrant.

Changing to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$



so that $x^2 + y^2 = r^2$, equation (1) becomes

$$r^2 = ar\sin\theta \text{ or } r = a\sin\theta.$$

The region of integration is bounded by

$$r = 0,\ r = a\sin\theta \text{ and } \theta = 0,\ \theta = \frac{\pi}{2}.$$

∴ Required volume

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta} \sqrt{a^2 - r^2}\,.\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r)\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} -\frac{1}{2}\cdot\left[\frac{(a^2 - r^2)^{3/2}}{3/2}\right]_0^{a\sin\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}(a^3\cos^3\theta - a^3) = -\frac{4a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right] = \frac{2}{9}a^3(3\pi - 4).$$

Example 15. *A triangular prism is formed by the planes whose equations are $ay = bx$, $y = 0$ and $x = a$, prove that the volume of this prism between the plane $z = 0$ and the surface $z = c + xy$ is $\frac{ab}{8}(4c + ab)$.*

Sol. Prism is bounded between $z = 0$ and $z = c + xy$ and in $z = 0$ the planes and $ay = bx$, $y = 0$ and $x = a$.

∴ Required volume $= \iint z\, dydx$ where $z = c + xy$ and y varies from 0 to $\frac{bx}{a}$ and x varies from 0 to a.

∴

$$\text{Volume} = \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}}(c + xy)\, dydx$$

$$= \int_0^a cy + \frac{xy^2}{2}\bigg|_0^{\frac{bx}{a}} dx = \int_0^a\left(\frac{bc}{a}x + \frac{x}{2}\cdot\frac{b^2x^2}{a^2}\right)dx$$

$$= \int_0^a \left(\frac{bc}{a} x + \frac{b^2}{2a^2} x^3 \right) dx = \frac{bc}{a} \cdot \frac{x^2}{2} + \frac{b^2}{2a^2} \cdot \frac{x^4}{4} \Bigg|_0^a$$

$$= \frac{bc}{2a} \cdot a^2 + \frac{b^2}{8a^2} \cdot a^4 = \frac{abc}{2} + \frac{a^2 b^2}{8} = \frac{ab}{8}(4c + ab).$$

TEST YOUR KNOWLEDGE

1. Find, by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. Find, by double integration, the smaller of the areas bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.
3. Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3} a^2$. *(Mysore, 1994 ; Karnataka, 1993)*
4. Find the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.
5. Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.
6. Find the area bounded by the circles $r = 2 \sin \theta$ and $r = 4 \sin \theta$.
   [**Hint.** Circles are $x^2 + y^2 - 2y = 0$ and $x^2 + y^2 - 4y = 0$]
7. Find, by double integration, the volume of the sphere $x^2 + y^2 + z^2 = 9$.
8. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. *(Hamirpur, 1995)*
   [**Hint.** See S.E. 10]
9. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
10. Find the volume bounded by the plane $z = 0$, surface $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.
    [**Hint.** Req. volume $= 4 \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} (x^2 + y^2 + 2)\, dy\, dx$]
11. Find the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.
    [**Hint.** Req. volume $= \int_{x=0}^{3} \int_{y=0}^{\frac{2}{3}\sqrt{9-x^2}} (x + y)\, dy\, dx$]
12. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
    [**Hint.** See S.E. 10]
13. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Answers

1. πab **2.** $\frac{9}{4}(\pi - 2)$ **4.** $10\frac{2}{3}$ **5.** 8

6. 3π **7.** 36π **8.** 4π **9.** $\frac{8a^4}{3}$

10. 16π 11. 10 12. $\frac{3}{4}\pi a^3$ 13. $4\sqrt{3}\pi$.

9. VOLUME AS A TRIPLE INTEGRAL

The volume V of a three dimensional region is given by $V = \iiint_V dx\,dy\,dz$.

If the region is bounded by $x = f_1(y, z), x = f_2(y, z)$; $y = \phi_1(z), y = \phi_2(z)$ and $z = a, z = b$, then

$$V = \int_a^b \int_{\phi_1(z)}^{\phi_2(z)} \int_{f_1(y,z)}^{f_2(y,z)} dx\,dy\,dz$$

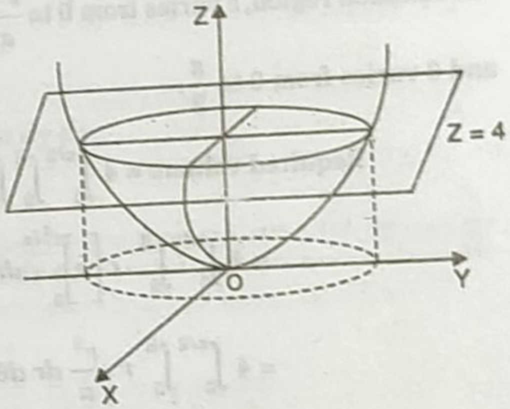
The order of integration may be changed with a suitable change in the limits of integration.

In cylindrical co-ordinates, we have $V = \iiint_V r\,dr\,d\phi\,dz$

In spherical polar co-ordinates, we have $V = \iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find by triple integration, the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$.*



Sol. By symmetry, the required volume is 4 times the volume in the positive octant. The volume in the positive octant is bounded on the sides by the zx and yz-planes ; from above by the plane $z = 4$ and below by the curved surface $x^2 + y^2 = 4z$.

The section of the paraboloid by the plane $z = 4$ is the circle $x^2 + y^2 = 16$, $z = 4$ and its projection on the xy-plane is the circle $x^2 + y^2 = 16$, $z = 0$.

The volume in the positive octant is bounded by $z = \frac{x^2 + y^2}{4}, z = 4, y = 0, y = \sqrt{16 - x^2}$ and $x = 0, x = 4$.

∴ Required volume

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \int_{(x^2+y^2)/4}^4 dz\,dy\,dx = 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \Big[z \Big]_{\frac{x^2+y^2}{4}}^4 dy\,dx$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \left(4 - \frac{x^2 + y^2}{4}\right) dy\,dx = 4 \int_0^4 \left[\left(4 - \frac{x^2}{4}\right) y - \frac{1}{4} \cdot \frac{y^3}{3}\right]_0^{\sqrt{16-x^2}} dx$$

$$= 4 \int_0^4 \left[\left(4 - \frac{x^2}{4}\right)\sqrt{16 - x^2} - \frac{1}{12}(16 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^4 \left[\frac{1}{4}(16-x^2)\sqrt{16-x^2} - \frac{1}{12}(16-x^2)^{3/2}\right] dx$$

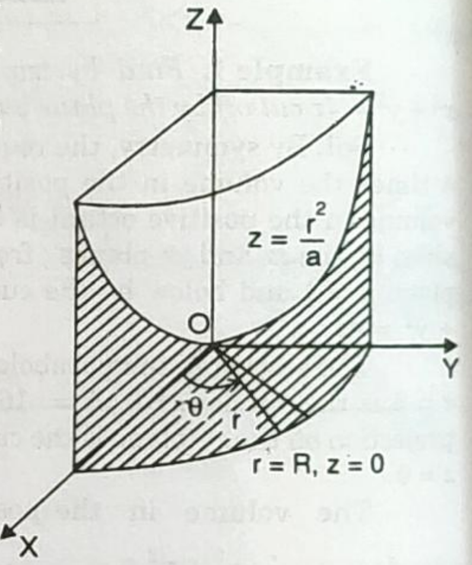
$$= 4\int_0^4 \frac{1}{6}(16-x^2)^{3/2}\, dx = \frac{2}{3}\int_0^4 (16-x^2)^{3/2}\, dx$$

$$= \frac{2}{3}\int_0^{\pi/2} (16)^{3/2} . \cos^3\theta . 4\cos\theta\, d\theta, \text{ where } x = 4\sin\theta$$

$$= \frac{512}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{512}{3} . \frac{3.1}{4.2} . \frac{\pi}{2} = 32\pi.$$

Example 2. *Find, by triple integration, the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = R^2$.*

Sol. Changing to cylindrical co-ordinates, by putting $x = r\cos\theta$, $y = r\sin\theta$ the equation of the paraboloid becomes $az = r^2$ or $z = \frac{r^2}{a}$ and the equation of the cylinder becomes $r^2 = R^2$ or $r = R$. On account of symmetry, the required volume is four times the volume in the positive octant. Thus, in the common region, z varies from 0 to $\frac{r^2}{a}$, r varies from 0 to R and θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \quad \text{Required volume} = 4\int_0^{\pi/2}\int_0^R\int_0^{r^2/a} r\, dz\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^R r\Big[z\Big]_0^{r^2/a} dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^R r . \frac{r^2}{a}\, dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{r^4}{4a}\right]_0^R d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} R^4\, d\theta = \frac{R^4}{a} . \frac{\pi}{2} = \frac{\pi R^4}{2a}.$$

Example 3. *Find, by triple integration, the volume of a sphere of radius a.*

(P.T.U. May 2003)

Sol. Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

The equation of a sphere of radius a in cartesian co-ordinates is $x^2 + y^2 + z^2 = a^2$

The same equation in spherical polar co-ordinates is $r^2 = a^2$ or $r = a$.

On account of symmetry, the required volume is 8 times the volume of the sphere in the positive octant for which r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \text{Required volume} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta . \left[\frac{r^3}{3}\right]_0^a d\theta\, d\phi = 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3} \sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2} -\frac{a^3}{3}\Big[\cos\theta\Big]_0^{\pi/2} d\phi = \frac{8}{3}a^3\int_0^{\pi/2} d\phi = \frac{8}{3}a^3 . \frac{\pi}{2} = \frac{4}{3}\pi a^3.$$

Example 4. *Find the volume of the region enclosed by the surfaces* $z = x^2 + 3y^2$, $z = 8 - x^2 - y^2$. (P.T.U., May 2003)

Sol. Equations of the surfaces are

$$z = x^2 + 3y^2,\ z = 8 - x^2 - y^2$$

Their intersections are given by

$$x^2 + 3y^2 = 8 - x^2 - y^2 \quad \text{or} \quad 2x^2 + 4y^2 = 8 \quad \text{or} \quad x^2 + 2y^2 = 4$$

$\therefore$ z varies from $x^2 + 3y^2$ to $8 - x^2 - y^2$

y varies from $-\sqrt{\frac{4-x^2}{2}}$ to $\sqrt{\frac{4-x^2}{2}}$

and x varies from -2 to 2

$$\text{Volume of the Region} = \int_{-2}^{2}\int_{-\sqrt{\frac{4+x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - x^2 - y^2 - x^2 - 3y^2)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} 8y - 2x^2y - \frac{4y^3}{3}\Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx = \int_{-2}^{2} (8 - 2x^2)\, 2\sqrt{\frac{4-x^2}{2}} - \frac{4}{3} . 2\left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \int_{-2}^{2} \frac{8}{3\sqrt{2}}(4 - x^2)^{3/2}\, dx \quad \text{Put } x = 2\sin\theta\,;\ dx = 2\cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3\sqrt{2}}(4 - 4\sin^2\theta)^{3/2} . 2\cos\theta\, d\theta$$

$$= \frac{128}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{128}{3\sqrt{2}} 2\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{256}{3\sqrt{2}}\, \frac{3.1}{4.2}\, \frac{\pi}{2} = \frac{16}{\sqrt{2}}\pi = 8\sqrt{2}\,\pi.$$