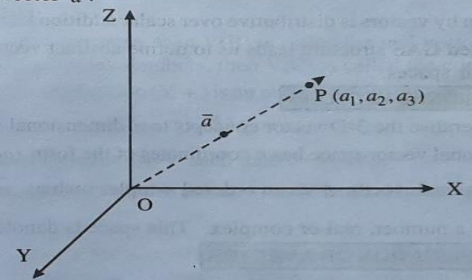
1

Vector Spaces

1.1. Let P be a point (a_1, a_2, a_3) with respect to the frame of reference OXYZ. Let $\overline{OP}$ be denoted by the vector $\bar{a}$. Now the vector $\bar{a}$ is associated with the point P given by the ordered trial (a_1, a_2, a_3). Conversely the ordered triad (a_1, a_2, a_3) defines a point P associated with the vector $\bar{a}$.



This shows that the set of all points in 3-D space has a one to one correspondence with the set of all vectors starting from the origin. Thus each vector is representable as an ordered triad of three real numbers. This enables us to write $\bar{a} = (a_1, a_2, a_3)$.

From this we can visualise the 3.dimensional space as an ordered set of triads (a_1, a_2, a_3) where a_1, a_2, a_3 are real numbers.

This space is denoted by R^3.

1.2. ALGEBRAIC STRUCTURE OF 3 - D VECTORS

Let V be the set of 3.D vectors and F be the field of scalars.

Now it is easy to verify the following algebraic structure with the vector addition (+) and scalar multiplication (•) of a vector.

G :(1) Closure : $\alpha+\beta \in V \; \forall \; \alpha, \beta \in V$

(2) Associativity : $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma \; \forall \; \alpha, \beta, \gamma \in V$

(3) Identity : $\alpha+\bar{O} = \bar{O}+\alpha = \alpha \; \forall \; \alpha \in V$ null vector $\bar{O}$ is the additive in V.

(4) Inverse : $\alpha+(-\alpha) = \bar{O} = (-\alpha)+\alpha$. Every vector in V has the additive inverse.

(5) Commutativity : $\alpha+\beta = \beta+\alpha \; \forall \; \alpha, \beta \in V$

$\therefore$ $(V, +)$ is an abelian group.

A : *Admission of scalar multiplication in V*

(1) $a\alpha \in V \ \forall \ a \in F$ and $\alpha \in V$

(2) $1.\alpha = \alpha \ \forall \ \alpha \in V$, where 1 is the unity of F.

S : *Scalar multiplication*

(1) $a\,(b\alpha) = (ab)\,\alpha \quad \forall \ a, b \in F$ and $\alpha \in V$

Scalar multiplication of vectors is associative

(2) $a\,(\alpha + \beta) = a\alpha + a\beta \quad \forall \ a \in F$ and $\alpha, \beta \in V$

Multiplication by scalars is distributive over vector addition.

(3) $(a + b)\,\alpha = a\alpha + b\alpha \quad \forall \ a, b \in F$ and $\alpha \in V$

Multiplication by vectors is distributive over scalar addition.

The above verified **GAS** structure leads us to define abstract vector spaces or simply vector spaces or linear spaces.

1.3. THE *n*-DIMENSIONAL VECTORS

Now we can generalise the 3-D vector concepts to n-dimensional vectors. Thereby, a point in the n-dimensional vector space has n coordinates of the form $(a_1, a_2, ..., a_n)$. Hence we define an n-dimensional vector $\bar{a}$ as an ordered n-tuple, such as $\bar{a} = (a_1, a_2, a_3, ..., a_n)$

where each a_i is a number, real or complex. This space is denoted by $\mathbf{R}^n$ or $\mathbf{C}^n$.

1.4. SCALAR MULTIPLICATION OF A VECTOR

Let $\alpha = (a_1, a_2, ..., a_n)$ be a vector in R^n and k be a scalar.

Then we define $k\alpha = (ka_1, ka_2, ..., ka_n)$.

1.5. ADDITION OF VECTORS

Let $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$ be two vectors of R^n.

Then we define $\alpha + \beta = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$

1.6. INTERNAL COMPOSITION

Let F be a set. If $aob \in F$ for all $a, b \in F$. Then 'o' is said to be an internal composition in the set F.

1.7. EXTERNAL COMPOSITION

Let V and F be two sets. If $a\,o\,\alpha \in V$ for all $a \in F$ and for all $\alpha \in V$, then 'o' is said the be an external composition in V over F. Here the resulting element $a\,o\,\alpha$ is an element of the set V.

1.8. VECTOR SPACES

Definition. *Let V be a non-empty set whose elements are called vectors. Let F be any set whose elements are called scalars where* $(F, +, \bullet)$ *is a field.*

The set V is said to be a vector space if

(1) *there is defined an internal composition in V called addition of vectors denoted by +, for which (V, +) is an abelian group.*

(2) *there is defined an external composition in V over F, called the scalar multiplication in which* $a\alpha \in V$ *for all* $a \in F$ *and* $\alpha \in V$.

(3) *the above two compositions satisfy the following postulates*

(*i*) $a\,(\alpha + \beta) = a\alpha + a\beta$ (*ii*) $(a + b)\alpha = a\alpha + b\alpha$ (*iii*) $(ab)\alpha = a\,(b\alpha)$ (*iv*) $1\alpha = \alpha$

$\forall \ a, b \in F$ and $\alpha, \beta \in V$ and 1 is the unity element of F. Instead of saying that V is a vector space over the field F, we simply say V(F) is a vector space. Sometimes if the field F is understood, then we simply say that V is a *'vector space'*.

1.9. VECTOR SPACE

If R is a field of real numbers, then V(R) is called the real vector space.

If C is the field of complex numbers, then V(C) is called the complex vector space.

Note.1. In the above definition (V, +) is an abelian group implies that for all $\alpha, \beta, \gamma \in V$

(*a*) $\alpha + \beta \in V$ (*b*) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

(*c*) there exists an element $\bar{O} \in V$ such that $\alpha + \bar{O} = \alpha$

(*d*) To every vector $\alpha \in V$ there exists $-\alpha \in V$ such that $\alpha + (-\alpha) = 0$

(*e*) $\alpha + \beta = \beta + \alpha$

2. The use of symbol + for two different compositions (for the addition in the field F and for the internal composition in V) should not cause any confusion. It should be understood according to the context in which it is used.

3. In vector space, two types of zero elements come into operation. One is the zero vector $\bar{O}$ of V and the other is the zero scalar 0 of the field F.

1.10. NULL SPACE OR ZERO VECTOR SPACE

The vector space having only one zero vector $\bar{O}$ is called the zero vector space or null space. It clearly satisfies all the postulates on any field of scalars.

Theorem : ***Let V(F) be a vector space and*** $O, \bar{O}$ ***be the zero scalar and zero vector respectively. Then***

(1) $a\bar{O} = \bar{O} \ \forall \ a \in F$ **(2)** $0\alpha = \bar{O} \ \forall \ \alpha \in V$

(3) $a(-\alpha) = -(a\alpha) \ \forall \ a \in F, \ \forall \alpha \in V$ **(4)** $(-a)\alpha = (-a\alpha) \ \forall \ a \in F, \ \forall \alpha \in V$

(5) $a\alpha = \bar{O} \Rightarrow a = 0$ or $\alpha = \bar{O}$ **(6)** $a\,(\alpha - \beta) = a\alpha - a\beta \ \forall \ a \in F, \ \forall \alpha, \beta \in V$

(7) $(-a)(-\alpha) = a\alpha \ \forall \ a \in F, \ \forall \alpha \in V$

Proof. (1) $a\bar{O} = a\,(\bar{O} + \bar{O}) = a\bar{O} + a\bar{O}$

$\therefore a\bar{O} + \bar{O} = a\bar{O} + a\bar{O}$ (Distributive Law)

$\bar{O} = a\bar{O}$ (Left Cancellation Law)

(2) $0\,\alpha=(0+0)\,\alpha=0\,\alpha+0\,\alpha$

$\therefore\ \bar{O}+0\,\alpha=0\,\alpha+0\,\alpha \Rightarrow \bar{O}=0\,\alpha$ (Cancellation Law)

(3) $a\,[\alpha+(-\alpha)]=a\bar{O} \Rightarrow a\,\alpha+a(-\alpha)=\bar{O}$

$\Rightarrow a(-\alpha)$ is the additive inverse of $a\,\alpha \Rightarrow a\,(-\alpha)=-(a\,\alpha)$

(4) $[a+(-a)]\,\alpha=0\,\alpha \Rightarrow a\,\alpha+(-a)\,\alpha=\bar{O}$

$\Rightarrow (-a)\,\alpha$ is the additive inverse of $a\,\alpha$

$\Rightarrow (-a)\,\alpha=-(a\,\alpha)$

(5) $a\alpha=\bar{O}$ and $a=0$. Then there is nothing to prove. $a\alpha=\bar{O}$ and $a\neq 0$

Now $a\neq 0, a\in F \Rightarrow$ there exists. $a^{-1}\in F$ such that $a\,a^{-1}=a^{-1}a=1$

Now $a\alpha=\bar{O} \Rightarrow a^{-1}(a\,\alpha)=a^{-1}\,\bar{O} \Rightarrow 1\alpha=\bar{O} \Rightarrow \alpha=\bar{O}$

$\therefore\ a\alpha=\bar{O} \Rightarrow a=0$ or $\alpha=\bar{O}$

(6) $a\,(\alpha-\beta)=a\,[\alpha+(-\beta)] \quad =a\,\alpha+a\,(-\beta)=a\,\alpha-a\,\beta$

(7) $(-a)\,(-\alpha)=-[a\,(-\alpha)] \quad =-[-(a\,\alpha)]=a\,\alpha$

1.11. Theorem. ***Let V(F) be a vector space***

(1) If $a,b\in F$ and $\alpha\in V$ where $\alpha\neq\bar{O}$ then $a\,\alpha=b\,\alpha \Rightarrow a=b$

(2) If $a\in F$ where $a\neq 0$ and $\alpha,\beta\in V$ then $a\,\alpha=a\,\beta \Rightarrow \alpha=\beta$.

Proof : (1) $a\,\alpha=b\,\alpha \Rightarrow a\,\alpha+(-b\,\alpha)=b\,\alpha+(-b\,\alpha) \Rightarrow (a-b)\,\alpha=\bar{O}$

$\Rightarrow a-b=0$ as $\alpha\neq\bar{O} \Rightarrow a=b$

(2) $a\,\alpha=a\,\beta \Rightarrow a\,\alpha-a\,\beta=\bar{O} \Rightarrow a\,(\alpha-\beta)=\bar{O} \Rightarrow \alpha-\beta=\bar{O}$ as $a\neq 0 \Rightarrow \alpha=\beta$.

SOLVED PROBLEMS

Ex. 1. *The set C_n of all n-tuples of complex numbers with addition as the external composition and scalar multiplication of complex numbers by complex numbers is a vector space over the field of complex numbers with the following definitions.*

(*i*) If $\alpha,\beta\in C_n$ and $\alpha=(a_1,a_2,\ldots,a_n)$ and $\beta=(b_1,b_2,\ldots,b_n)$ for all $a_k,b_k\in C$

$\alpha+\beta=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)\in C_n$

(*ii*) $x\,\alpha=(x\,a_1,x\,a_2,\ldots,x\,a_n)\ \forall\ x\in C$

Sol. (*i*) By definition $x+y$ is an n-tuple of complex numbers. Hence C_n is closed.

(*ii*) $(\alpha+\beta)+\gamma=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)+(c_1,\ldots,c_n)$

where $\gamma=(c_1,c_2,\ldots,c_n)$ with c_i's $\in C$

$=\{(a_1+b_1)+c_1,(a_2+b_2)+c_2,\ldots,(a_n+b_n)+c_n\}$

$=\{a_1+(b_1+c_1),a_2+(b_2+c_2),\ldots,a_n+(b_n+c_n)\}$ since + is associative in C

$=(a_1,a_2,\ldots,a_n)+(b_1+c_1,b_2+c_2,\ldots,b_n+c_n)=\alpha+(\beta+\gamma)$

$\therefore$ Vector addition is associative in C_n.

(*iii*) $\alpha+\bar{O}=(a_1,a_2,\ldots,a_n)+(0,0,\ldots,0)=(a_1+0,a_2+0,\ldots,a_n+0)=(a_1,a_2,\ldots,a_n)=\alpha$

Similarly $\bar{O}+\alpha=\alpha \Rightarrow \alpha+\bar{O}=\bar{O}+\alpha=\alpha$

$\therefore$ The n-tuple $\bar{O}=(0,0,\ldots,0)$ is the identity in C_n.

(*iv*) $\alpha+(-\alpha)=(a_1,a_2,\ldots,a_n)+(-a_1,-a_2,\ldots,-a_n)$

$=(a_1-a_1,a_2-a_2,\ldots,a_n-a_n)=(0,0,0,\ldots,0)=\bar{O}$. $\quad\therefore\ \alpha+(-\alpha)=\bar{O}$

$\Rightarrow (-\alpha)\in C$, is the additive inverse of α in C

(*v*) $\alpha+\beta=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)$

$=(b_1+a_1,a_2+b_2,\ldots,b_n+a_n)$ ($\because$ + is commutative in C)

$=\beta+\alpha$.

(*vi*) By definition $x\in C,\alpha\in C_n \Rightarrow x\,\alpha\in C_n$

(*vii*) Let $x,y\in C$. $\quad x\,(y\,\alpha)=x\,(ya_1,ya_2,\ldots,ya_n)=\{x\,(ya_1),x\,(ya_2),\ldots,x\,(ya_n)\}$

$=\{(xy)a_1,(xy)a_2,\ldots,(xy)a_n\}=xy(a_1,a_2,\ldots,a_n)=(xy)\,\alpha$

(*viii*) $x\in C$ and $\alpha,\beta\in C_n$

$\Rightarrow x\,(\alpha+\beta)=x\,(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)$

$=\{x(a_1+b_1),x(a_2+b_2),\ldots,x(a_n+b_n)\}=(xa_1+xb_1,xa_2+xb_2,\ldots,xa_n+xb_n)$

$=(xa_1,xa_2,\ldots,xa_n)+(xb_1,xb_2,\ldots,xb_n)=x\,\alpha+x\,\beta$

(*ix*) $x,y\in C$ and $\alpha\in C_n$

$\Rightarrow (x+y)\,\alpha=(x+y)\,(a_1,a_2,\ldots,a_n)=\{(x+y)a_1,(x+y)a_2,\ldots,(x+y)a_n\}$

$=\{(xa_1+ya_1),(xa_2+ya_2),\ldots,(xa_n+ya_n)\}$ (Since distributive law is true in C)

$=(xa_1,xa_2,xa_n)+(ya_1,ya_2,\ldots,ya_n)=x\,\alpha+y\,\alpha$

(*x*) $1.\alpha=1\,(a_1,a_2,\ldots,a_n)=(a_1,a_2,\ldots,a_n)=\alpha$

Since all the postulates are verified C_n (C) is a vector space.

Ex.2. *Prove that the set of all real valued continuous functions defined in the open interval (0,1) is a vector space over the field of real numbers, with respect to the operations of addition and scalar multiplication defined as*

$(f+g)\,(x)=f(x)+g(x)$ ... (1)

$(af)\,(x)=af(x).a$ is real ... (2) where $0<x<1$

Sol: Let V be the set of all such real values continuous functions and R be the field of real numbers.

(*i*) The sum of two continuous functions is again a continuous function

$(f+g)\in V\ \forall\ f,g\in V$.

(*ii*) $\{(f+g)+h\}\,(x)=(f+g)x+h(x)$ (by def. (1))

$= f(x)+g(x)+h(x) = f(x)+(g+h)(x) = \{f+(g+h)\}(x)$

$\Rightarrow (f+g)+h = f+(g+h)$. $\therefore$ V is associative

(*iii*) Let the function O be defined as $\bar{O}(x)=0$

$\therefore (0+b)(x)=\bar{O}(x)=f(x)$ (by def (1))

$=0+f(x)=f(x)$ as the real number 0 is the additive identity in R.

$\therefore \bar{O}+f=f,\ \forall f\in V$

(*iv*) $\{f+(-f)\}(x)=f(x)+(-f(x))\} = f(x)-f(x)=0=\bar{O}(x)$

The function $(-f)$ is the additive inverse of f

$\therefore f+(-f)=\bar{O}$ (the identity function)

(*v*) $(f+g)(x)=f(x)+g(x)$

$=g(x)+f(x)$ (real numbers are commutative under addition)

$=(g+f)(x)$ by definition (1)

Thus $f+g=g+f,\ \forall\ g,f\in V$. $\therefore$ (V, +) is an abelian group

(*vi*) For all $a\in R$ and $f,g\in V$, $a(f+g)=af+ag$

Now $[a(f+g)](x)=a[(f+g)(x)]=a[f(x)+g(x)] = af(x)+ag(x)$

$=(af)(x)+(ag)(x)$ $=(af+ag)(x)$ $\therefore a(f+g)=af+ag$

(*vii*) If $a,b\in R$ and $f\in V$ then, $\{(a+b)f\}(x)=(a+b)f(x)$ (by (2))

$=a f(x)+b f(x)$ (as $f(x)$ is real)

$=(a f)(x)+(b f)(x)=(af+bf)(x)$ (by (1))

$\Rightarrow (a+b) f = af+bf$

(*viii*) If $a,b\in R$ and $f\in V$ then, $\{a(bf)\}(x)$ (by (2))

$=a\{(bf)(x)\}=a\{bf(x)\}$

$=(a b) f(x)$ as $f(x)$ is real $=\{(a b) f\}(x) \Rightarrow a(bf)=(ab)f$

(*ix*) Since 1 is the identity of the field R and $f\in V$,

we have $(1 f)(x)=1 f(x)=f(x)$ $\therefore 1 f = f$

All the postulates of vector space are verified. Hence V (R) is a vector space.

Ex. 3. *V is the set of all $m\times n$ matrices with real entries and R is the field of real numbers. "Addition of matrices' is the internal composition and 'multiplication of a matrix by a real number' an external composition in V. Show that V (R) is a vector space.*

Sol: Let $\alpha,\beta,\gamma\in V$ and $x,y\in R$ where $\alpha=[a_{ij}]$, $\beta=[b_{ij}]$, $\gamma=[c_{ij}]$ for a's, b's, c's $\in R$

(*i*) Addition of two matrices is a matrix. $\therefore \alpha+\beta\in V\ \forall\ \alpha,\beta\in V$

(*ii*) $\alpha+(\beta+\gamma)=[a_{ij}]+[b_{ij}+c_{ij}]=[a_{ij}+(b_{ij}+c_{ij})] = [(a_{ij}+b_{ij})+c_{ij}]$

(Since real numbers are associative in R)

$=[a_{ij}+b_{ij}]+[c_{ij}] = (\alpha+\beta)+\gamma$ $\therefore$ V is associative.

(*iii*) If $O=[o_{ij}]$ is the null matrix then $\alpha+O=[a_{ij}]+[o_{ij}]=[a_{ij}+o_{ij}] = [a_{ij}]=\alpha$

Thus $\alpha+O=O+\alpha=\alpha$. $\Rightarrow$ The null matrix O is the additive identity in V.

(*iv*) $\alpha+(-\alpha)=[a_{ij}]+[-a_{ij}] = [a_{ij}-a_{ij}]=[-o_{ij}]=0$

$\therefore$ $(-\alpha)$ is the additive inverse of every α in V

(*v*) Clearly $\alpha+\beta=\beta+\alpha$ $\therefore$ (V, +) is an abelian group.

(*vi*) For $x\in R$ and $\alpha\in V$. $x\alpha=x[a_{ij}]=[xa_{ij}]\Rightarrow x\alpha\in V$

(*vii*) For $x,y\in R$ and $\alpha\in V$

$x(y\alpha)=x(y[a_{ij}])=x[ya_{ij}]=[xya_{ij}]=(xy)[a_{ij}]=(xy)\alpha$

(*viii*) $x(\alpha+\beta)=[a_{ij}+b_{ij}]=[x(a_{ij}+b_{ij})]=[xa_{ij}+xb_{ij}] = [xa_{ij}]+[xb_{ij}]$

$=x[a_{ij}]+x[b_{ij}]=x\alpha+x\beta$

(*ix*) $(x+y)\alpha=(x+y)[a_{ij}] = [(x+y)a_{ij}]=[x a_{ij}+y a_{ij}]$

$=[x a_{ij}]+[y a_{ij}] = x[a_{ij}]+y[a_{ij}]=x\alpha+y\alpha$

(*x*) $1.\alpha=1[a_{ij}]=[a_{ij}]=\alpha$ Hence V (R) is a vector space.

Ex.4. *Let V be the set of all pairs (a, b) of real numbers and R be the field of real numbers. Show that with the operation $(a_1,b_1)+(a_2,b_2)=(a_1+a_2,o)$, $c(a_1,b_1)=(ca_1,b_1)$ V (R) is not a vector space.*

Sol: If any one of the postulates of a vector space is not satisfied, then V (R) cannot be a vector space. Let (*x, y*) be the identity in V.

Now $(a,b)+(x,y)=(a,b)\ \forall\ a,b\in V \Rightarrow (a+x,0)=(a,b)$ ... (1)

If $b\neq 0$, then we cannot have the equality (1).

Thus there exists no element $(x,y)\in V$ such that $(a,b)+(x,y)=(a,b)\ \forall\ a,b\in V$

$\therefore$ Additive identity does not exist in V and hence V (R) is not a vector space.

EXERCISE 1 (a)

1. Show that the set of all triads (x_1,x_2,x_3) where x_1,x_2,x_3 are real numbers forms a vector space over the field of real numbers with respect to the operations of addition and scalar multiplication defined as

(*i*) $(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$ and

(*ii*) $c(x_1,x_2,x_3)=(c x_1,c x_2,c x_3)$ c is a real number.

2. Let P be the set of all polynomials in one indeterminate with real coefficients. Show that P (R) is vector space, if in P the addition of polynomials is taken as the internal composition and the multiplication of polynomial by a constant polynomial (*i.e.* by a element k) as scalar multiplication.

3. If F is a field, the F (F) is a vector space, if in F the addition of field F is taken as the internal composition and the multiplication of the field F is taken as the external composition in F over F.

4. Let R^n be the set of all ordered n-tuples of real numbers given by

$$R^n = [(x_1, x_2, ..., x_n) \mid x_i \in R]$$

The internal and external composition in R^n are defined by

(*i*) $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$

(*ii*) $a(x_1, x_2, ..., x_n) = (ax_1, ax_2, ..., ax_n)$, $a \in R$. Show that R^n (R) is a vector space.

5. Let F be any field and K any subfield of F. Then F (K) is a vector space, if the addition of the field F is taken as the internal composition in F, and the field multiplication is taken as the external composition in F over K.

6. Let V (R) be a vector space and $W = \{(x, y) \mid x, y \in V\}$. For (x_1, y_1) and (x_2, y_2) in W we have $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and if $\alpha = a_1 + ia_2 \in C$, we define

$a(x_1, y_1) = (a_1x_1 - a_2x_2, a_2x_1 + a_1x_2)$. Prove that W (C) is a vector space.

7. The set of all real valued differentiable or integrable functions defined in some interval [0,1] is a vector space.

8. Let V be the set of all pairs of real numbers and let F be the field of real numbers with the definition $(x_1, y_1) + (x_2, y_2) = (3y_1 + 3y_2, -x_1 - x_2)$, $c(x_1, y_1) = (3c\,y_1, -c\,x_1)$

Show that V (F) is not a field.

9. V is the set of all polynomials over real numbers of degree atmost one and F = R.

If $f(t) = a_0 + a_1 t$ and $g(t) = b_0 + b_1(t)$ in V ; define

$f(t) + g(t) = (a_0 + b_0) + (a_0 b_1 + a_1 b_0)\,t$ and $kf(t) = (ka_0) + (ka_1)\,t$, $k \in F$

Show that V (F) is not a vector space. **(Hint:** Additive identity does not exist)

10. C is the field of complex numbers and R is the field of real numbers. Show that C (R) is a vector space and R (C) is not a vector space.

11. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$, $c(x, y) = (cx, y)$

Show that V (F) is not a vector space.

1.12. VECTOR SUBSPACES

Definition. *Let V (F) be a vector space and* $W \subseteq V$. *Then W is said to be a subspace of V if W itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.*

Note.1. If W (F) is a subspace of V (F) then W is a sub-group of V.

Note.2. Let V (F) be a vector space. The zero vector space $\{\bar{O}\} \subseteq V$ and $V \subseteq V$.

$\therefore$ $\{\bar{O}\}$ and V are the trivial subspaces of V.

1.13. Theorem. ***Let V (F) be a vector space and let*** $W \subseteq V$. ***The necessary and sufficient conditions for W to be a subspace of V are***

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$ ***(ii)*** $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

Proof. *Conditions are necessary*

(*i*) W is a vector subspace of V

$\Rightarrow$ W is a subgroup of (V, +) $\Rightarrow$ (W, +) is a group

$\Rightarrow$ if $a, \beta \in W$ then $a - \beta \in W$.

(*ii*) W is a subspace of V

$\Rightarrow$ W is closed under scalar multiplication $\Rightarrow$ for $a \in F$, $\alpha \in W$; $a\alpha \in W$

Conditions are sufficient.

Let W be a nonempty subset of V satisfying the two given conditions

$\alpha \in W$, $\alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \bar{O} \in W$ (by (*i*))

$\therefore$ The zero vector of V is also the zero vector of W

$\bar{O} \in W$, $\alpha \in W \Rightarrow \bar{O} - \alpha \in W \Rightarrow (-\alpha) \in W$ (by (*i*))

$\Rightarrow$ additive inverse of each element of W is also in W

Again $\alpha \in W$, $\beta \in W \Rightarrow \alpha \in W$, $(-\beta) \in W \Rightarrow \alpha - (-\beta) \in W$ (by (*i*))

$\Rightarrow \alpha + \beta \in W$

i.e., W is closed under vector addition

As $W \subseteq V$, all the elements of W are also the elements of V. Thereby vector addition in W will be associative and commutative. This implies that (W, +) is an abelian group.

Further by (*ii*), W is closed under scalar multiplication and the other postulates of vector space hold in w as $W \subseteq V$.

$\therefore$ W itself is a vector space under the operations of V.

Hence W (F) is a vector subspace of V (F)

A more useful and abridged form of the above theorem is the following.

1.14. Theorem. ***Let V (F) be a vector space. A non-empty set*** $W \subseteq V$. ***The necessary and sufficient condition for W to be a subsapce of V is***

$a, b \in F$ ***and*** $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in W$ ... ***(I)*** ***(S.V. U. S97)***

Proof. *Condition is necessary.*

W (F) is a subspace of V (F) $\Rightarrow$ W (F) is a vector space

$\therefore$ $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$ and $b \in F$, $\beta \in W \Rightarrow b\beta \in W$

Now $a\alpha \in W$, $b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Condition is sufficient.

Let W be the non-empty subset of V satisfying the given condition

i.e., $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ ... (I)

Taking $a = 1, b = -1$ and $\alpha, \beta \in W \Rightarrow 1\alpha + (-1)\beta \in W$

$\Rightarrow \alpha - \beta \in W$ $\quad [\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

($H \subseteq G$ and $a, b \in H \Rightarrow a o b^{-1} \in H$ then (H, o) is subgroup of (G, 0)).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +).

$\Rightarrow$ (W, +) is an abelian group. Again taking $b = 0$

$a, 0 \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + 0\beta \in W \Rightarrow a\alpha \in W \Rightarrow a \in F$ and $\alpha \in W \Rightarrow a\alpha \in W$

$\therefore$ W is closed under scalar multiplication.

The remaining postulates of vector space hold in W as $W \subseteq V$.

$\therefore$ W (F) is a vector subspace of V (F).

1.15. Theorem. ***A non-empty set W is a subset of vector space V (F). W is a subspace of W if and only if $a \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + \beta \in W$.***

Proof. *Condition is necessary.* W is a subspace of V (F)

$\Rightarrow$ W (F) is a vector space. $\quad \therefore a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Further $a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$

Condition is sufficient. W is a non-empty subset of V satisfying the condition.

$a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

(*i*) Now taking $a = -1$, for $\alpha \in W$ we have $(-1)\alpha + \alpha \in W \Rightarrow \bar{O} \in W$

(*ii*) Again $a \in F, \alpha, \bar{O} \in W \Rightarrow a\alpha + \bar{O} \in W \Rightarrow a\alpha \in W$

$\therefore$ W is closed with respect to scalar multiplication.

(*iii*) $-1 \in F$ and $\alpha, \bar{O} \in W \Rightarrow (-1)\alpha + \bar{O} \in W \Rightarrow -\alpha \in W$. $\quad \therefore$ Inverse exists in W.

The remaining postulates of vector space hold good in W since they hold in V of which W is a subset.

Hence W is a subspace of V (F).

SOLVED PROBLEMS

Ex.1. *The set W of ordered triads (x, y, 0) where $x, y \in F$ is a subspace of $V_3(F)$.*

Sol: Let $\alpha, \beta \in W$ where $\alpha = (x_1, y_1, 0)$ and $\beta = (x_2, y_2, 0)$ for some $x_1, y_1, x_2, y_2 \in F$.

Let $a, b \in F$. $\quad \therefore a\alpha + b\beta = a(x_1, y_1, 0) + b(x_2, y_2, 0)$

$= (ax_1, ay_1, 0) + (bx_2, by_2, 0) = (ax_1 + bx_2, ay_1 + by_2, 0)$

Clearly $ax_1 + bx_2, ay_1 + by_2 \in F \Rightarrow a\alpha + b\beta \in W$ for all $a, b \in F$ and $\alpha, \beta \in W$

Hence W is subspace of V_3 (F).

Ex.2. *Let p, q, r be the fixed elements of a field F. Show that the set W of all triads (x, y, z) of elements of F, such that $px + qy + rz = 0$ is a vector subspace of $V_3(F)$.*

(S. V. U. S97)

Sol: By definition $W \neq \phi$. Let $\alpha, \beta \in W$ where $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in F$.

Then by definition $px_1 + qy_1 + rz_1 = 0$ ... (1) $px_2 + qy_2 + rz_2 = 0$ ... (2)

If $a, b \in F$, then we have $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $p(ax_1 + bx_2) + q(ay_1 + by_2) + r(az_1 + bz_2)$

$= a(px_1 + qy_1 + rz_1) + b(px_2 + qy_2 + rz_2) = a.0 + b.0 = 0$ (by (1) and (2))

$\therefore a\alpha + b\beta = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \in W$. Hence W is a subspace of V_3 (F).

Ex. 3. *Let R be the field of real numbers and W = { (x, y, z) / x, y, z are rational numbers }. Is W a subspace of V_3 (R).*

Sol: Let $\alpha = (2,3,4)$ be an element of W ; $a = \sqrt{7}$ is an element of R.

Now $a\alpha = \sqrt{7}(2,3,4) = (2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}) \notin W$.

($\therefore 2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}$ not rational numbers).

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of V_3 (R).

Ex. 4. *Let V be the vector space of all polynomials in an indeterminate x over the field F. Let W be a subset of V consisting of all polynomials of degree $\leq n$. Then W is a subspace of V.*

Sol: Let $\alpha, \beta \in W$. Then α, β are polynomials over F of degree $\leq n$.

If $a, b \in F$ then $a\alpha + b\beta$ will also be a polynomial of degree $\leq n$.

$\therefore a\alpha + b\beta \in W \Rightarrow$ W is a subspace of V.

Ex. 5. *Let V be the set of all $n \times n$ matrices and F be the field. If W is the subset of $n \times n$ symmetric matrices in V, show that W is a subspace of V (F).*

Sol: $O_{n\times n} = \begin{vmatrix} 0 & 0 & 0 & .... & 0 \\ 0 & 0 & .... & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & .... & .... & 0 \end{vmatrix}$ is a symmetric matrix

since $i - j$ th element of $O = 0 = j - i$ th element of 0

$\therefore W \neq \phi$. Let $a, b \in F$ and $P, Q \in W$ where $P = [p_{ij}], Q = [q_{ij}]$

$\because$ P, Q are symmetric, $p_{ij} = p_{ji}$ and $q_{ij} = q_{ji}$ ... (1)

$\therefore aP + bQ = a[p_{ij}] + b[q_{ij}] = [ap_{ij}] + [bq_{ij}] = [ap_{ij} + bq_{ij}]_{n\times n}$

The $i-j$ th element in $aP+bQ = ap_{ij}+bq_{ij} = ap_{ji}+bq_{ji}$ (by(1))

$= j-i$ th element of $aP+bQ$

$\Rightarrow aP+bQ$ is a symmetric $n \times n$ matrix $\Rightarrow aP+bQ \in W$

$\therefore$ W is a vector subspace of V (F).

Ex. 6. *Let F be a field and A be a $m \times n$ matrix over F.*

$F_{1\times m}$ is the set of all $1 \times m$ matrices defined over f forming the vector space $F_{1\times m}(F)$

Define $W = \{ X = [x_1, x_2, \ldots x_n] \in F_{1\times m} \; 1 \times A = O_{1\times n} \}$.

Prove that W is a subspace of $F_{1\times m}(F)$.

Sol: Let $X, Y \in W$ and $a, b \in F$. By def. of W, $XA = O_{1\times n}$ and $YA = O_{1\times n}$

Now $(aX+bY)A = (aX)A + (bY)A = a(XA) + b(YA) = a\,O_{1\times n} + b\,O_{1\times n} = O_{1\times n}$

$\therefore aX+bY \in W$ $\therefore$ W is a subspace of $F_{1\times m}(F)$.

Note. W is called a solution space of XA = O.

EXERCISE 1 (b)

1. Let R be the field of real numbers. Show the set of triads that
(1) $\{(x, 2y, 3z) / x, y, z \in R\}$ (2) $\{(x, x, x) / x \in R\}$ form the subspaces of $R^3(R)$.
2. Let $V = R^3 = \{ (x, y, z) / x, y, z \in R \}$ and W be the set of triads (x, y, z) such that $x-3y+4z=0$. Show that W is a subspace of V (R).
3. Show that the set W of the elements of the vector space $V_3(R)$ of the form $(x+2y, y, -x+3y)$ where $x, y \in R$ is a subspace of V_3 (R).
4. Prove that the set of solutions (x, y, z) of the equation $x+y+2z=0$ is a subspace of the space R^3 (R).
5. Show that the solutions of the differential equation $(D^2-5D+6)\,y=0$ is a subspace of the vector space of all real-valued continuous functions over R.
[**Hint.** Let $y=f(x)$ and $y=g(x)$ be two solutions.
$\therefore D^2f(x)-5Df(x)+6f(x)=0$ ... (1), $D^2g(x)-5Dg(x)+6g(x)=0$ ... (2)
(1) a + (2) b gives : $D^2[af(x)+bg(x)]-5D[af(x)+bg(x)]+6[af(x)+bg(x)]=0$
$\therefore af(x)+bg(x)$ is also a solution of D.E. etc.]
6. Show that the subset W defined below is not a subspace of R^3 (R).
(*i*) $W = \{ (a,b,c) \,|\, a,b,c$ are rationals $\}$ (*ii*) $W = \{ (a,b,c) \,|\, a^2+b^2+c^2 \le 1 \}$
7. Let V be the vector of $n \times n$ matrices over a field F. Show that W the set of matrices which commute with a given matrix T is a subspace of V.
$W = \{ A = [a_{ij}] \in V : AT = TA \}$

8. V is the vector space of 2×2 matrices over a field F. Show that W is not a subspace of V where (*i*) $W = \{ A \in V \,|\, \det. A = 0 \}$ (*ii*) $W = \{ A \in V \,|\, A^2 = A \}$
[**Hint.** (*i*) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in W.
$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with det. $(A+B) \neq 0$ etc.
[**Hint.** (*ii*) Clearly $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$. As $(2I)^2 \neq 2I$ scalar multiplication in W fails].
9. Let V be the set of all real valued functions defined on $[-1,1]$. Which of the following are subspaces of V (R).
(*i*) $W_1 = \{ f \in V \,|\, f(0)=0 \}$ (*ii*) $W_2 = \{ f \in V \,|\, f(1) = f(-1) \}$
(*iii*) $W_3 = \{ f \in V \,|\, f(x)=0$ if $x<0 \}$ (*iv*) $W_4 = \{ f \in V \,|\, f(x) = f(-x) \}$
10. W is the subset of C^3. Which of the following are the subsapces of C^3 (C).
(*i*) $W = \{ (\alpha_1, \alpha_2, \alpha_3) \in C^3 \,|\, \alpha_1$ is real $\}$
(*ii*) $W = \{ (\alpha_1, \alpha_2, \alpha_3) \in C^3 \,|$ either $\alpha_1 = 0$ or $\alpha_2 = 0 \}$
11. Let x be an indeterminate over the field F. Which of the following are subspaces of F [x] over F ?
(*i*) All monic polynomials of degree atmost 10.
(*ii*) All polynomials having α and β in F as roots. (*iii*) All polynomials divisible by x.
(*iv*) All polynomials $f(x)$ such that $2\,f(0) = f(1)$.
12. Prove that $W = \{ \lambda(1,1,1) \,|\, \lambda \in R \}$ is a subspace of R^3.

ALGEBRA OF SUBSPACES

1.16. Theorem. ***The intersection of any two subspaces W_1 and W_2 of vector space V (F) is also a subspace.***

Proof. W_1 and W_2 are subspaces V (F)

$\Rightarrow \bar{O} \in W_1$ and $\bar{O} \in W_2 \Rightarrow \bar{O} \in W_1 \cap W_2$ $\therefore W_1 \cap W_2 \neq \phi$

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$ $\therefore \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Now $a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

$\therefore a\alpha + b\beta \in W_1 \cap W_2$ $\therefore W_1 \cap W_2$ is a subspace of V (F)

1.17. Theorem. ***The intersection of any family of subspaces of a vector space is also a subspace.***

Proof. Let V (F) be a vector space.

Let $W_1, W_2, W_3 ....W_n$ be the n subspaces of V (F).

Let $W=\bigcap_{i=1}^{n} W_i = \{\alpha \in V : \alpha \in W_n \ \forall n\}$

$\bar{O}\in W_i$ for $i=1,2,....,n \Rightarrow \bar{O}\bigcap_{1}^{n} W_i$ and $\bigcap_{i=1}^{n} W_i \neq \phi$

Let $\alpha, \beta \in \bigcap_{i=1}^{n} W_i \Rightarrow \alpha, \beta \in W_n \ \forall n$

Since each W_n is a subspace we have $a, b \in F$ and $\alpha, \beta \in W_n \Rightarrow a\alpha + b\beta \in W_n \ \forall n$

$\Rightarrow a\alpha + b\beta \in \bigcap_{1}^{n} W_i$. Hence $\bigcap_{i=1}^{n} W_i$ is a subspace of V (F).

Note : The union of two subspaces of V (F) may not be a subspace of V (F).

e.g. Let W_1 and W_2 be two subspaces of V_3 (R) given by

$$W_1 = \{(0, y, 0) \mid y \in R\}, \quad W_2 = \{(0, 0, z) \mid z \in R\}$$

$\therefore \ W_1 \cup W_2 \ \{(0, y, 0) \cup (0, 0, z) \mid y, z \in R\}$

Now $(0, y, 0) + (0, 0, z) = (0, y, z) \notin W_1 \cup W_2$

Since neither $(0, y, z) \in W_1$ nor $(0, y, z) \in W_2$

Thus $W_1 \cup W_2$ is not closed under vector addition.

$\therefore \ W_1 \cup W_2$ is not a subspace of V (F).

1.18. Theorem. ***The union of two subspaces is a subspace if and only if one is contained in the other.***

Proof. Let W_1 and W_2 be two subspaces of V (F).

Condition is necessary. Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\therefore \ W_1 \cup W_2 = W_2$ or $W_1 \Rightarrow W_1 \cup W_2$ is a subspace of V (F)

Condition is sufficient. Let $W_1 \cup W_2$ be a subspace.

Let us suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

Now $W_1 \not\subset W_2 \Rightarrow$ there exists $x \in W_1$ and $x \notin | W_2$ ... (1)

$W_2 \not\subset W_1 \Rightarrow$ there exists $y \in W_2$ and $y \notin W_1$ ... (2)

$\therefore \ x \in W_1 \cup W_2$ and $y \in W_1 \cup W_2$

$\Rightarrow x + y \in W_1 \cup W_2$ $\quad$ ($\because$ $W_1 \cup W_2$ is a subspace)

$\Rightarrow x + y \in W_1$ or $x + y \in W_2$.

Now $x+y, x \in W_1$ (subspace) $\Rightarrow 1(x+y)+(-1)\ x \in W_1 \Rightarrow y \in W_1$ ... (3)

Similarly $x+y, y \in W_2$ (subspace) $\Rightarrow 1(x+y)+(-1)\ y \in W_2 \Rightarrow x \in W_2$ ... (4)

Thus (3) and (4) contradict (2) and (1). $\quad \therefore$ Either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

1.19. LINEAR SUM OF TWO SUBSPACES

Definition. *Let W_1 and W_2 be two subspaces of the vector space V (F). Then the linear sum of the subspaces W_1 and W_2, denoted by $W_1 + W_2$, is the set of all sums $\alpha_1 + \alpha_2$ such that $\alpha_1 \in W_1$, $\alpha_2 \in W_2$ i.e.,* $W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$

1.20. Theorem. ***If W_1 and W_2 are any two subspaces of a vector space (F) then***

(i) $W_1 + W_2$ is a subspace of V (F).

(ii) $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$. ***(N. U. 94 M)***

Proof. (*i*) Let $\alpha, \beta \in W_1 + W_2$. Then

$\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ where $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$.

If $a, b \in F$ then $a\alpha_1, b\beta_1 \in W_1$ $\quad$ ($\because$ W_1 subspace)

and $a\alpha_2, b\beta_2 \in W_2$ $\quad$ ($\because$ W_2 subspace)

Now $a\alpha + b\beta = a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) = (a\alpha_1, b\beta_1) + (a\alpha_2, b\beta_2) \in W_1 | W_2$

$\therefore \ a, b \in F$ and $\alpha, \beta \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V (F).

(*ii*) $\alpha_1 \in W_1$ and $\bar{O} \in W_2 \Rightarrow \alpha_1 + \bar{O} \in W_1 + W_2$. $\quad \therefore \ \alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$

$\Rightarrow W_1 \subseteq W_1 + W_2$. Similarly, $W_2 \subseteq W_1 + W_2$. $\quad$ Hence $W_1 \cup W_2 \subseteq W_1 + W_2$.

SOLVED PROBLEMS

Ex.1. *Let V be the vector space of all functions from R into R. Let S_e be the subset of even functions, $f(-x) = f(x)$ S_o be the subset of odd functions, $f(-x) = -f(x)$. Prove that* (1) *S_e and S_o are subspaces of V* (2) $S_e + S_o = V$ (3) $S_e \cap S_o = \{\bar{o}\}$

Sol: (1) Let $f_e, g_e \in S_e$ and $a, b \in R$

$\therefore \ (af_e + bg_e)(-x) = af_e(-x) + bg_e(-x) = af_e(x) + bg_e(x) = (af_e + bg_e)(x)$

$\Rightarrow af_e + bg_e$ is an even function.

$\therefore \ f_e, g_e \in S_e, a$ and $b \in R \Rightarrow af_e + bg_e \in S_e$

Hence S_e is a subspace of V. Similarly we can prove that S_o is a subspace of V.

(2) Since S_e and S_o are subspaces of V. $\quad S_e + S_o$ is also a subspace of V

$\therefore \ S_e + S_o \subseteq V$. $\quad$ Let $g_e(x) = \frac{1}{2}[f(x) + f(-x)]$, $\quad h_o(x) = \frac{1}{2}[f(x) - f(-x)]$

Clearly g_e is an even function and h_o is an odd function

Now $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = g_e(x) + h_o(x) = (g_e + h_o)(x)$

$\Rightarrow f = g_e + h_o$ where $g_e \in S_e, h_o \in S_o$

Thus $f \in V \Rightarrow f \in S_e + S_o$ $\quad \therefore \ V \subseteq S_e + S_o$. $\quad$ Hence $S_e + S_o = V$.

(3) Let $\bar{O}$ denote the zero function i.e., $\bar{O}(x)=0 \forall x \in R$

Let $f \in S_e \cap S_o$ then $f(-x)=f(x)$ and $f(-x)=-f(x)$

$\therefore f(x)=-f(x) \Rightarrow 2f(x)=0$

$\Rightarrow f(x)=0=\bar{O}(x) \Rightarrow f=\bar{O}$. Hence $S_e \cap S_o=\{\bar{O}\}$.

Ex.2. *Let* W_1 *and* W_2 *be subspaces of a vector space V such that* $W_1+W_2=V$ *and* $W_1 \cap W_2=\{\bar{O}\}$. *Prove that for each vector* α *in V there are unique vector* $\alpha_1 \in W_1, \alpha_2 \in W_2$ *such that* $\alpha=\alpha_1+\alpha_2$.

Sol: $\alpha \in V \Rightarrow \alpha \in W_1+W_2$ $(\because V=W_1+W_2)$

$\Rightarrow \alpha=\alpha_1+\alpha_2$ where $\alpha_1 \in W_1, \alpha_2 \in W_2$

If possible let $\alpha=\beta_1+\beta_2$ where $\beta_1 \in W_1, \beta_2 \in W_2$

$\therefore \alpha_1+\alpha_2=\beta_1+\beta_2$. $\alpha_1-\beta_1=\beta_2-\alpha_2$

Now $\alpha_1-\beta_1 \in W_1$ and $\beta_2-\alpha_2 \in W_2$

$\Rightarrow \alpha_1-\beta_1 \in W_1$ and $\alpha_1-\beta_1 \in W_2$

$\Rightarrow \alpha_1-\beta_1 \in W_1 \cap W_2 \Rightarrow \alpha_1-\beta_1=\bar{O}$ $(\because W_1 \cap W_2=\{\bar{O}\})$

$\Rightarrow \beta_2-\alpha_2=\bar{O}$

$\Rightarrow \alpha_1=\beta_1$ and $\alpha_2=\beta_2$. Hence $\alpha=\alpha_1+\alpha_2$ is unique.

1.21. LINEAR COMBINATION OF VECTORS

Definition. *Let V (F) be a vector space. If* $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ *then any vector* $\gamma=a_1\alpha_1+a_2\alpha_2+\ldots+a_n\alpha_n$ *where* $a_1, a_2, \ldots, a_n \in F$ *is called a linear combination of the vectors* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Note.1. γ is a vector belonging to V (F).

1.22. LINEAR SPAN OF A SET

Definition. *Let S be a non-empty subset of a vector space V (F). The linear span of S is the set of all possible linear combinations of all possible finite subsets of S.*

The linear span of S is denoted by L (S)

$\therefore L(S)=\{\gamma : \gamma=\sum a_i\alpha_i, a_i \in F, \alpha_i \in S\}$

Note.1. S may be a finite set but L (S) is infinite set.

L (S) is said to be generated or spanned by S

2. If S is an empty subset of V then we define L (S) = {0}

3. $S \subseteq L\{S\}$

1.23. Theorem. *The linear span L(S) of any subset S of a vector space V (F) is a subspace of v (F).* **(S. V. U. 94 S)**

Proof. Let $\alpha, \beta \in L(S)$ and $a, b \in F$

$\therefore \alpha=a_1\alpha_1+a_2\alpha_2+\ldots+a_m\alpha_m$, $\beta=b_1\beta_1+b_2\beta_2+\ldots+b_n\beta_n$

where $a'_i s, b'_i s \in F$ and $\alpha'_i s, \beta'_i s \in S$

$\therefore a\alpha+b\beta=a(a_1\alpha_1+a_2\alpha_2+\ldots+a_m\alpha_m)+b(b_1\beta_1+b_2\beta_2+\ldots+b_n\beta_n)$

$=(aa_1)\alpha_1+(aa_2)\alpha_2\ldots+(aa_m)\alpha_m+(bb_1)\beta_1+(bb_2)\beta_2+\ldots+(bb_n)\beta_n$

$\Rightarrow a\alpha+b\beta$ is a linear combination of finite set

$\alpha_1, \alpha_2 \ldots \alpha_m, \beta_1, \beta_2 \ldots \beta_n$ of the elements of S. $\Rightarrow (a\alpha+b\beta) \in L(S)$

Thus $a, b \in F$ and $\alpha, \beta \in L(S) \Rightarrow (a\alpha+b\beta) \in L(S)$ $\therefore$ L (S) is a subspace of V (F).

1.24. Theorem. *Let S be a non-empty subset of the vector space V (F). The linear span L (S) is the intersection of all subspaces of V which contain S.*

Proof. Let W be the subspace of V (F) containing S. Every linear combination of finite set of elements of S is an element of W as W is closed.

But the set of every linear combination of finite set of elements of S is L (S).

$\therefore L(S) \subseteq W$

$\therefore L(S) \subseteq$ Intersection of all subspaces of V containing S $(\because S \subseteq L(S))$

The intersection of all subspaces of V containing $S \subseteq L(S)$

Hence L (S) is the intersection of all subspaces of V containing S.

SOLVED PROBLEMS

Ex.1. *Express the vector* $\alpha=(1,-2,5)$ *as a linear combination of the vectors* $e_1=(1,1,1), e_2=(1,2,3)$ *and* $e_3=(2,-1,1)$

Sol: Let $\alpha=(1,-2,5)=x(1,1,1)+y(1,2,3)+z(2,-1,1)$

$=(x+y+2z, x+2y-z, x+3y+z)$

Hence $x+y+2z=1$, $x+2y-z=-2$, $x+3y+z=5$

Reducing to echelon form we get $\alpha+\beta+2\gamma=1$, $\beta-3\gamma=-3$, $5\gamma=10$

These equations are consistant and have a solution given by $\alpha=-6, \beta=3, \gamma=2$

Hence $\alpha=-6e_1+3e_2+2e_3$

Ex.2. *Show that the vector* $\alpha=(2,-5,3)$ *in* R^3 *cannot be expressed as a linear combination of the vectors* $e_1=(1,-3,2), e_2=(2,-4,-1)$ *and* $e_3=(1,-5,7)$

Sol: Let $\alpha=ae_1+be_2+ce_3$

i.e., $(2,-5,3)=a(1,-3,2)+b(2,-4,-1)+c(1,-5,7)$

$=(a+2b+c, -3a-4b-5c, 2a-b+7c)$

$\Rightarrow$ $a+2b+c=2$, $-3a-4b-5c=-5$, $2a-b+7c=3$

Now reducing to echelan form we get $a+2b+c=2$, $2b-2c=1$, $0=3$

As the system is inconsistant, the equations have no solution. Hence α cannot be expressed as a $l.c$ of e_1, e_2, e_3.

Ex.3. *The subset* $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ *of* $V_3(R)$ *generates the entire vector space* $V_3(R)$.

Sol: Let $(a, b, c) \in V$ then we can write $(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\Rightarrow (a, b, c) \in L(S) \Rightarrow V \subseteq L(S)$. But $L(S) \subseteq V$ Hence $L(S) = V$.

Ex.4. *In the vector space* $R^3(R)$. *Let* $\alpha = (1,2,1), \beta = (3,1,5), \gamma = (3,-4,5)$. *Show that subspace spanned by* $S = \{\alpha, \beta\}$ *and* $T = \{\alpha, \beta, \gamma\}$ *are the same.*

Sol: $L(T) = l.c.$ of the vectors of $T = \{a\alpha + b\beta + c\gamma \,/\, a, b, c \in R\}$

Let $\gamma = x\alpha + y\beta \Rightarrow (3,-4,5) = x(1,2,1) + y(3,1,5)$

$\therefore \; x + 3y = 3, \; 2x + y = -4, \; x + 5y = 5$

$x = -3, y = 2$ satisfy all the three equations. $\therefore (3,-4,5) = -3(1,2,1) + 2(3,1,5)$

$\therefore c(3,-4,5) = -3c(1,2,1) + 2c(3,1,5) \Rightarrow c\gamma = -3c\alpha + 2c\beta$

$\therefore a\alpha + b\beta + c\gamma = a\alpha + b\beta - 3c\alpha + 2c\beta$

$= (a-3c)\alpha + (b+2c)\beta = p\alpha + q\beta$ when $p, q \in R$

$\therefore$ But $p\alpha + q\beta \in L(S)$. $\therefore L(T) = L(S)$.

Ex.5. *In the vector space* $R^3(R)$ *let* $S = \{(1, 0, 0), (0, 1, 0)\}$. *Find* $L(S)$.

Sol: $L(S) = \{\alpha / \alpha = a(1,0,0) + b(0,1,0), a, b \in R\} = \{\alpha / \alpha = (a, b, 0) ; a, b \in R\}$

Geometrically the linear span is the plane $Z = 0$.

1.25. Theorem. ***If W_1 and W_2 are two subspaces of a vector space $V(F)$ then $L(W_1 \cup W_2) = W_1 + W_2$.***

Proof. Let $\alpha_1 \in W_1$ and $\bar{O} \in W_2$ $\therefore \alpha_1 \in W_1$ and $\bar{O} \in W_2 \Rightarrow \alpha_1 + \bar{O} \in W_1 + W_2$

$\therefore$ Every $\alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore W_1 \subseteq W_1 + W_2$. Similarly $W_2 \subseteq W_1 + W_2$. Hence $W_1 \cup W_2 \subseteq W_1 + W_2$

Also W_1 and W_2 subspaces of $V \Rightarrow W_1 \cup W_2$ subspace of $W_1 + W_2$.

$\therefore L(W_1 \cup W_2)$ is a subspace of $W_1 + W_2$. Again let $\alpha \in W_1 + W_2$

$\therefore$ By def. $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ $\therefore \alpha_1, \alpha_2 \in W_1 \cup W_2$

Now, $\alpha = \alpha_1 + \alpha_2 = 1.\alpha_1 + 1.\alpha_2 = l.c.$ of elements of $W_1 \cup W_2$

$\therefore \alpha \in L(W_1 \cup W_2)$ $\therefore W_1 + W_2 \subseteq L(W_1 \cup W_2)$ ... (*i*)

Also we know that $L(W_1 \cup W_2)$ is the smallest subsapce containing $W_1 \cup W_2$ and $W_1 \cup W_2 \subseteq W_1 + W_2$. $\therefore L(W_1 \cup W_2) \subseteq W_1 + W_2$ ... (*ii*)

From (*i*) and (*ii*) $L(W_1 \cup W_2) = W_1 + W_2$

1.26. Theorem. ***If S is a subset of a vector space V (F) then prove that (I) S is a subspace of V $\Leftrightarrow$ L (S) = S*** *(S. V. U. 2001/O)* ***(II) L (L (S)) = L (S).***

Proof. I. (*i*) Let S be a subspace of V,

Let $\alpha \in L(S)$. Then $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m$ where $a_1, a_2, \ldots a_m \in F$ and $\alpha_1, \alpha_2 \ldots \alpha_m \in S$

$\because$ S is a subspace of V, it is closed w.r. to scalar multiplication and vector addition.

$\therefore a_1\alpha_1 + a_2\alpha_2 \ldots + a_m\alpha_m = \alpha \in S$

Thus $\alpha \in L(S) \Rightarrow \alpha \in S$ $\therefore L(S) \subseteq S$

Let $\beta \in S$. Now $\beta = 1.\beta = l.c.$ of infinite elements of S

$\Rightarrow \beta \in L(S)$ $\therefore S \subseteq L(S)$. Hence $L(S) = S$.

(*ii*) Suppose $L(S) = S$.

We know that L (S) is a subspace of v. $\Rightarrow$ S is a subspace of V.

II. We know that L (S) is a subspace of V. By **I**, $L(L(S)) = L(S)$.

1.27. Theorem. ***If S, T are the subsets of a vector space V (F), then (I) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ (II) $L(S \cup T) = L(S) + L(T)$*** *(N. U. 95)*

Proof. (I) Let $\alpha \in L(S)$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where a_i's $\in F$ and α_i's $\in S$

$\because S \subseteq T$, then $\{\alpha_1, \alpha_2 \ldots \alpha_n\} \subseteq T$

$\therefore \alpha = l.c.$ of finite subset of $T \Rightarrow \alpha \in L(T)$

Hence $L(S) \subseteq L(T)$.

(II) Let $\alpha \in L(S \cup T)$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 \ldots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 \ldots + b_n\beta_n$ where $a_1, a_2 \ldots a_m, b_1, b_2 \ldots b_n \in F$

$\alpha_1, \alpha_2 \ldots \alpha_m \in S$ and $\beta_1, \beta_2 \ldots \beta_n \in T$.

(*i*) But $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \in L(S)$, $b_1\beta_1, b_2\beta_2 + \ldots + \beta_n \in L(T)$

$\therefore \alpha =$ an element of L (S) + an element of L (T)

$\therefore \alpha \in L(S) + L(T) \Rightarrow L(S \cup T) \subseteq L(S) + L(T)$ ... (1)

(*ii*) Let $\alpha \in L(S) + L(T)$

$\therefore \alpha = \gamma + \delta$ where $\gamma \in L(S)$, $\delta \in L(T)$ $\therefore \gamma = l.c.$ of a finite no. of elements of S

$\delta = l.c.$ of a finite no. of elements of T

$\therefore \alpha = \gamma + \delta = l.c.$ of a finite no. of elements of $S \cup T \Rightarrow \alpha = \gamma + \delta \in L(S \cup T)$

$\therefore L(S) + L(T) \subseteq L(S \cup T)$ ... (2)

Hence from (1) and (2) : $L(S \cup T) = L(S) + L(T)$.

EXERCISE 1 (c)

1. Show that each of the following set of vectors generates $R^3(R)$.

   (i) $\{(1,0,0), (1,1,0), (1,1,1)\}$ (ii) $\{(1,2,3), (0,1,2), (0,0,1)\}$ (iii) $\{(1,2,1), (2,1,0), (1,-1,2$

2. If $\alpha = (1,2,-1), \beta = (2,-3,2), \gamma = (4,1,3)$ and $\delta = (-3,1,2)$ are the vectors of $V_3(R)$ sho that $L(\{\alpha, \beta\}) \neq L(\{\gamma, \delta\})$.

3. Prove that the subspace spanned by a non-empty subset S of a vector space V is th set of all linear combinations of vectors in S. (N. U.)

1.28. LINEAR DEPENDENCE OF VECTORS

Definition. *Let V (F) be a vector space. A finite subset* $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *of vecto of V is said to be a linearly dependent (L.D.) set if there exist scalars* $\alpha_1, \alpha_2, ..., \alpha_n \in F$ *not all zero, such that* $a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O}$.

1.29. LINEAR INDEPENDENCE OF VECTORS

Definition. Let V (F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors o V is said to be linearly independent (L.I.) if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O}, a_i\text{'s} \in F$$

$$\Rightarrow a_1 = 0, a_2 = 0, ..., a_n = 0.$$

1.30. Theorem. ***Every superset of a linearly dependent (L.D) set of vectors i linearly dependent (L.D.).***

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a linearly dependent set of vectors.

$\therefore$ There exist scalars $a_1, a_2, ..., a_n \in F$, not all zero such that

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O} \quad ... (1)$$

Let $S' = \{\alpha_1, \alpha_2 ... \alpha_n, \beta_1, \beta_2 ... \beta_n\}$ be a super set of S. then (1) can be re-written as

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n + 0\beta_1, 0\beta_2 + ... + 0\beta_n = \bar{O} \quad ... (2)$$

In (2) all the scalars are not zero $\Rightarrow$ S' is linearly dependent (L.D.)

Hence any super set of an L.D. set is L.D.

1.31. Theorem. ***Every non-empty subset of a linearly independent (L.I.) set of vectors is linearly independent (L.I.)*** (N. U. 94 M)

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a L.I. set of vectors

Let us consider the subset $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ where $1 \le k \le m$.

Now $a_1\alpha_1 + a_2\alpha_2 + ... + a_k\alpha_k = \bar{O}$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... + a_k\alpha_k + 0\alpha_{k+1} + ... + 0\alpha_m = \bar{O}$$

$$\Rightarrow a_1 = 0, a_2 = 0, a_k = 0 \quad (\because \text{ S is L.I. set.})$$

Hence the subset $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ is L.I.

1.32. Theorem. ***A set of vectors which contains atleast one zero vector is linearly dependent.***

Proof. Let $\alpha_1 = \bar{O}$ and $\alpha_2, \alpha_3 ... \alpha_m \neq \bar{O}$

Then $1\alpha_1 + 0\alpha_2 + 0\alpha_3 ... + \bar{O}\alpha_m = \bar{O}$, $a_1 = 1 \in F$ with atleast one scalar $a_1 \neq 0$

$\Rightarrow \{\bar{O}, \alpha_2, ..., \alpha_m\}$ L.D. set.

Hence the set of vectors containing zero vector is linearly dependent (L.D.).

Note. A L.I. subset of a vector space V (F) does not contain zero vector.

1.33. Theorem. ***A single non - zero vector forms a linearly independent (L.I.) set.***

Proof. Let $L = \{\alpha\}$ be a subset of V (F) where $a \neq \bar{O}$

If $a \in F$ then $a\alpha = \bar{O} \Rightarrow a = 0$. $\therefore$ The set S is linearly independent.

Note. $\{\alpha\}$ is a L.D. set.

SOLVED PROBLEMS

Ex.1. *Show that the system of vectors* $(1,3,2), (1,-7,-8), (2,1,-1)$ *of* $V_3(R)$ *is linearly dependent.*

Sol. Let $a, b, c \in R$, then $a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = \bar{O}$

$\Rightarrow (a+b+2c, 3a-7b+c, 2a-8b-c) = (0,0,0)$

$\Rightarrow a+b+2c = 0, 3a-7b+c = 0, 2a-8b-c = 0$

$\Rightarrow a = 3, b = 1, c = -2$ $\therefore$ The given vectors are linearly dependent.

Ex.2. *Show that the system of vectors* $(1,2,0), (0,3,1), (-1,0,1)$ *of* $V_3(Q)$ *is L.I. where Q is the field of rational numbers.*

Sol. Let $x, y, z \in Q$ then $x(1,2,0) + y(0,3,1) + z(-1,0,1) = \bar{O}$

$\Rightarrow (x-z, 2x+3y, y+z) = (0,0,0)$

$\Rightarrow x-z = 0, 2x+3y = 0, y+z = 0 \Rightarrow x = 0, y = 0, z = 0$

Hence the system is L.I.

Ex.3. *If two vectors are linearly dependent, prove that one of them is a scalar multiple of the other.*

Sol: Let α, β be two L.D. vectors of V (F)

Then there exist $a, b \in F$, not both zero such that $a\alpha + b\beta = \bar{O}$

Let $a \neq 0$, then $\alpha = \left(-\frac{b}{a}\right)\beta$ $\Rightarrow \alpha$ is a scallar multiple of β

If $b \neq 0$, similarly we get $\beta = a$ scalar multiple of α.

Ex.4. *Prove that the vectors* (x_1, y_1) *and* (x_2, y_2) *of* $V_2(F)$ *are* L.D.

$x_1 y_2 - x_2 y_1 = 0$.

Sol. Let $a, b \in F$ then $a(x_1 y_1) + b(x_2 y_2) = (0,0)$

$\Rightarrow (a x_1 + b x_2, a y_1 + b y_2) = (0,0) \Rightarrow a x_1 + b x_2 = 0, a y_1 + b y_2 = 0$

These equations are consistant.

If det. $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = 0 \Rightarrow x_1 y_2 - x_2 y_1 = 0$

Ex.5. *In the vector space* $V_n(F)$, *the system of n vectors* $e_1 = (1,0,0 \ldots 0)$, $e_2 = (0,1,0 \ldots 0)$, ..., $e_n = (0,0,\ldots 1)$ *is linearly independent where 1 is unity of f.*

Sol. Let $a_1, a_2, \ldots, a_n \in F$, then $a_1 e_1 + a_2 e_2 + \ldots + a_n e_n = \bar{O}$

$\Rightarrow a_1(1,0,\ldots,0) + a_2(0,1,0\ldots)\ldots + a_n(0,0,\ldots,0,1) = \bar{O}$

$\Rightarrow (a_1, a_2, \ldots, a_n) = (0,0,\ldots,0) \Rightarrow a_1 = 0, a_2 = 0, \ldots a_n = 0$

$\Rightarrow$ The given set of vectors is linearly independent.

Ex.6. *Prove that the four vector* $\alpha = (1,0,0), \beta = (0,1,0), \gamma = (0,0,1), \delta = (1,1,1)$ *in* $V_3(C)$ *form L.D. set, but any three of them are L.I.*

Sol. Let $a, b, c, d \in C$ $\quad \therefore a\alpha + b\beta + c\gamma + d\delta = \bar{O}$

$\Rightarrow a(1,0,0) + b(0,1,0) + c(0,0,1) + d(1,1,1) = \bar{O}$

$\Rightarrow (a+d, b+d, c+d) = (0,0,0)$

$\Rightarrow a+d=0, b+d=0, c+d=0 \quad \therefore a=-d, b=-d, c=-d$

Thus if $d=-k$, then $a=k, b=k, c=k$, showing that $\alpha+\beta+\gamma-\delta=\bar{O}$

$\therefore$ The four vectors $\alpha, \beta, \gamma, \delta$ are L.D.

(*ii*) But $a\alpha + b\beta + c\delta = \bar{O} \Rightarrow a(1,0,0) + b(0,1,0) + c(1,1,1) = \bar{O}$

$\Rightarrow (a+c, b+c, c) = (0,0,0) \Rightarrow a+c=0, b+c=0, c=0$,

$\Rightarrow a=0, b=0, c=0 \Rightarrow$ the vectors α, β, δ are L.I.

Similarly we can show that any other three vectors are L.I.

Ex.7. *If* α, β, γ *are linearly independent vectors of* $V(R)$ *show that* $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ *are also L.I.* **(O. U. 2001/0)**

Sol. Let $a, b, c \in R$.

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = \bar{O} \Rightarrow (a+c)\alpha + (a+b)\beta + (b+c)\gamma = \bar{O}$

Given α, β, γ are linearly independent.

$\Rightarrow a + 0.b + c = 0, \quad a + b + 0.c = 0, \quad 0.a + b + c = 0$.

Now the coefficient matrix A of these equations is $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$,

where rank of A = 3 *i.e.*, equal to the no. of unknowns.

$\Rightarrow a=0, b=0, c=0$ is the only solution of the given equations.

$\therefore \alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also L.I.

Ex.8. *Let* $F(x)$ *be the vector space of all polynomials over the field F. Show that the infinite set* $S = \{1, x, x^2, \ldots\}$ *is linearly independent.*

Sol. Let $S' = \{x^{m_1}, x^{m_2}, \ldots, x^{m_n}\}$ be any finite subset of having a vectors, where $m_1, m_2, \ldots m_n$ are non-negative integers.

Let $a_1, a_2 \ldots a_n \in F$

$\therefore a_1 x^{m^1} + a_2 x^{m^2} + \ldots + a_n x^{m^n} = \bar{O}$ (*i.e.* zero polynomial)

Then by definition of equality of two vectors we have $a_1 = 0, a_2 = 0, \ldots\ldots, a_n = 0$

$\Rightarrow$ Every finite subset of S is L.I. and hence s is linearly independent.

Ex.9. *Let V be the vector space of* 2×3 *matrices over R. Show that the vectors A, B, C form L.I set where*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

Sol. Let $a, b, c \in R$ then for L.I.

$a\,A + b\,B = c\,C = 0 \quad \Rightarrow a=0, b=0, c=0$

Now $a\,A + b\,B = c\,C = 0$

$$\Rightarrow a\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c\begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a-2c & 4a+5b-3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad 2a+b+4c=0, \quad a+b-c=0, \quad -a-3b+2c=0$

$3a-2b+c=0, \quad -2a-2c=0, \quad 4a+5b-3c=0$

These equations have only one solution $a=0, b=0, c=0$.

Ex.10. *Prove that the set* $\{1, i\}$ *is L.D. in the vector space C (C) but is L.I. in the vector space C (R).*

Sol. Since $(-i)\,i = 1$ with $(-i) \in C$, (one vector is a multiple of other)

$\Rightarrow$ the set $\{1, i\}$ in C (C) is L.D.

There exists no real number a such that $a\, i = 1$.

Hence $\{1, i\}$ is not L.D and so L.I in C (R).

1.34. Theorem. *Let $V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of non-zero vectors of $V(F)$. Then S is linearly dependent if and only if some vector $\alpha_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceding vectors.* (N. U. 93S, O. U. 2001/O, S. V. U. 2001/O)

Proof. Given $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n\}$ is L.D. Then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + .. a_n\alpha_n = \bar{O}$

Let k be the greatest suffix of a for which $\alpha_k \neq 0$.

Then $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_n = \bar{O} \Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k = \bar{0}$

Now suppose $k = 1$, (1) $\Rightarrow a_1\alpha_1 = \bar{O}$

But $a_1 \neq 0$, so $\alpha_1 = 0$, which contradicts that each element of S is a non-zero vector.

Hence $k > 1$, *i.e.*, $2 \le k \le n$.

Again from (1) we have $a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 \ldots - a_{k-1}\alpha_{k-1}$

$\Rightarrow a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 \ldots - a_{k-1}\alpha_{k-1})$

$\Rightarrow \alpha_k = (-a_k^{-1}a_1)\,\alpha_1 + (-a_k^{-1}a_2)\,\alpha_2 + \ldots + (-a_k^{-1}\alpha_{k-1})\,\alpha_{k-1}$

= (L.C. of its preceding vectors).

Conversely. Let some $\alpha_p \in S$ be expressible as a linear combination of its preceding vectors *i.e.*, for $b_1, b_2, \ldots, b_{p-1} \in F$.

$\alpha_p = b_1\,\alpha_1 + b_2\,\alpha_2 + \ldots + b_{p-1}\alpha_{p-1} \Rightarrow b_1\,\alpha_1 + b_2\,\alpha_2 + \ldots + b_{p-1}\alpha_{p-1} + (-1)\,\alpha_p = \bar{O}$

$\Rightarrow$ the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is L.D. ... [(-1) is a non-zero coefft.]

Hence the superset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots, \alpha_n\}$ is L.D.

Note. If β is a linear combination of the set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the set of vectors $\{\beta, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is L.D.

1.35. Theorem. *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of the vector space $V(F)$. If $\alpha_i \in S$ is a linear combination of its preceding vectors then $L(S) = (S')$ where $S' = \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$*

Proof. Clearly $S' \subset S \Rightarrow L(S') \subset L(S)$. Let $\beta \in L(S)$ then for $a_1, a_2 \ldots a_n \in F$.

$\therefore\ \beta = a_1\,\alpha_1 + a_2\,\alpha_2 + \ldots + a_i\,\alpha_i + \ldots + a_n\,\alpha_n$

But given that $\alpha_i = b_1\,\alpha_1 + b_2\,\alpha_2 \ldots + b_{i-1}\,\alpha_{i-1}\ \ \forall\ b's \in F$

$\therefore\ \beta = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_i\,(b_1\alpha_1 + b_2\alpha_2 \ldots + b_{i-1}\,\alpha_{i-1}) + a_{i+1}\,\alpha_{i+1} + \ldots + a_n\,\alpha_n$

$= (a_1 + a_ib_1)\,\alpha_1 + (a_2 + a_ib_2)\,\alpha_2 + \ldots + (a_{i-1} + a_ib_{i-1})\,\alpha_{i-1} + a_{i+1}\alpha_{i+1} \ldots + a_n\alpha_n$

= L.C. of the elements of S'

$\therefore\ \beta \in L(S') \Rightarrow L(S) \subset L(S')$. Hence $L(S) = L(S')$.

EXERCISE 1 (*d*)

1. Determine whether the following sets of vectors are L.D. or L.I.

(*a*) $\{(2,-3), (6,-9)\}$ in $R^2(R)$ (*b*) $\{(4,3,-2), (2,-6,7)\}$ in $V_3(R)$.

(*c*) $\{A, B\}$ where $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 & 12 \\ 9 & 3 & -3 \end{bmatrix}$ in the vector space of 2×3 matrices over R.

(*d*) $\{p_1(x), p_2(x)\}$ where $p_1(x) = 1 - 2x + 3x^2$ and $p_2(x) = 2 - 3x$ in the vector space of all polynomials over R.

2. Determine whether the following sets of vectors in $R^3(R)$ are L.D or L.I.

(*a*) $\{(1,-2,1), (2,1,-1), (7,-4,1)\}$ (*b*) $\{(2,0,5), (3,-5,8), (4,-2,1), (0,0,1)\}$

(*c*) $\{(-1,2,1), (3,0,-1), (-5,4,3)\}$ **(*O. U., M 06*)**

3. Determine whether each of the following sets of vectors $V_4(Q)$ is or L.D. or L.I., Q is the field of rational numbers.

(*a*) $\{(2,1,1,1), (1,3,1,-2), (1,2,-1,3)\}$ (*b*) $\{(0,1,0,1), (1,2,3,-1), (1,0,1,0), (0,3,2,0)\}$

(*c*) $\{(1,2,-1,1), (0,1,-1,2), (2,1,0,3), (1,1,0,0)\}$ (*d*) $\{(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)\}$

4. Show that the set $\{1, x, x - x^2\}$ is L.I., set of vectors in F (x) over the field of real numbers.

5. If α, β, γ are the vectors of V (F), and $a, b \in F$ show that the set $\{\alpha, \beta, \gamma\}$ is L.D, if the set $\{\alpha + a\,\beta + b\,\gamma, \beta, \gamma\}$ is linearly dependent.

6. If $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is any linearly independent subset of a vector space V and $\beta \notin L(S)$, then prove that $\alpha_1, \alpha_2 \ldots \alpha_n, \beta\}$ is linearly independent.

7. Under what conditions on the scalar β is the subset $\{(\beta,1,0), (1,\beta,1), (0,1,\beta)\}$ of R^3, linearly dependent ?

8. If $f(x)$ is a polynomial of degree n with real coefficients prove that

$$\left\{ f(x), \frac{d}{dx} f(x), \frac{d^2}{dx^2} f(x), \ldots \frac{d^n}{dx^n} f(x) \right\}$$ is a linearly independent set.

9. If u_1, u_2, u_3 be the vectors of V (F) and $a, b \in F$, show that $\{u_1, u_2, u_3\}$ is L.D. if the set $\{u_1 + a\,u_2 + b\,u_3, u_2, u_3\}$ is L.D. set.

2

Basis and Dimension

2.1. BASIS OF VECTOR SPACE

Definition. *A subset S of a vector space V (F) is said to be the basis of V, if*

(*i*) *S is linearly independent.* (*ii*) *the linear span of* S is V *i.e.,* $L(S) = V$

Note: A vector space may have more than one basis.

2.2. FINITE DIMENSIONAL SPACE.

Definition. *A vector space V (F) is said to be finite dimensional if it has a finite basis.* (OR)

A vector space V (F) is said to be finite dimensional if there is a finite subset S in V such that L (S) = V.

Ex. 1. *Show that the set of n vectors* $e_1 = (1,0,\dots,0), e_2 = (0,1,0,\dots,0), e_n = (0,0,\dots,1)$ is a *basis of* $V_n(F)$.

Sol. Let $S = \{e_1, e_2, \dots, e_n\}$. (*i*) It can be easily verified that the set S is L.I.

(*ii*) Now any vector $\alpha = (a_1, a_2 \dots a_n) \in V_n(F)$ can be put in the form

$(a_1, a_2, \dots, a_n) = a_1(1,0,\dots,0) + a_2(0,1,\dots,0) \dots + a_n(0,\dots,1) = a_1e_1 + a_2e_2 + \dots + a_ne_n$

$=$ *l. c.* of elements of the set S $\Rightarrow \alpha \in L(S)$.

$\therefore \alpha \in V \Rightarrow \alpha \in L(S)$ $\quad \therefore V = L(S)$. Hence S is a basis of $V_n(F)$

Note 1. The set $S = \{e_1, e_2, \dots, e_n\}$ is called the standard basis of $V_n(F)$ or F^n

Note 2. The set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the standard basis of $V_3(R)$ or R^3.

Ex. 2. *Show that the infinite set* $S = \{1, x, x^2, \dots, x^n, \dots\}$ *is a basis of the vector space* $F[x]$ *over the field F.*

Sol. (*i*) The set S is clearly L. I. (*ii*) Let $f(x) = a_0 + a_1x + \dots + a_nx^n \in F[x]$

$\therefore$ where $a_0, a_1, \dots, a_n \in F$

$\therefore f(x) \in L(S) \Rightarrow F[x] = L(S)$ $\quad \therefore$ S is a basis of $F[x]$

Let $S' = \{1, x, x^2, \dots, x^m\}$ be a finite subset of S. If $g(x) = c_0 + c_1x + c_2x^2 \dots + c_nx^n$ and $n > m$ it is not possible to write $g(x)$ as a *l.c.* of the element of S'.

Hence the vector space $F[x]$ is not finite dimensional.

2.3 Theorem. *IF V(F) is a finite dimensional vector space, then there exists a basis set of V.* (*N. U. 95, S. V. U. S97, S. K. U. 2001/0*)

Proof. Since V (F) is finite dimensional, there exists a finite set S such that, L (S) = V

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

We may assume that S does not contain the $\bar{O}$ vector.

If S is L. I., then S is a basis set of V. If S is L. D. set, then there exists a vector $\alpha_i \in S$ which can be expressed as a linear combination of the preceding vectors,.

Omitting α_i from S, let $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n\} \Rightarrow S_1 \subset S$

By previous theorem $L(S_1) = L(S)$. Now $L(S) = V \Rightarrow L(S_1) = V$

If S_1 is L.I set then S_1 will be a basis of V.

If S_1 is linearly dependent, then proceeding as above for a finite number of steps, we will be left with a L. I. set S_k and $L(S_k) = V$. Hence S_k will be the basis of V (F)

Thus there exists a basis set for V (F)

2.4. BASIS EXTENSION

Theorem. *Let V (F) be a finite dimensional vector space and* $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ *a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.*

Proof. Given $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is L. I. subset of V.

Since V (F) is finite dimensional it has a finite basis B. Let $B = \{\beta_1, \beta_2 \dots \beta_n\}$

Now consider the set $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n\}$ written in this order. Clearly L (S) = V.

Each α can be expressed as a l.c. of β's as B is the basis of V $\Rightarrow S_1$ is L. D.

Hence some vector in S_1 can be expressed as a *l.c.* of its proceeding vectors. This vector cannot be any of α's, since S is L. I. So this vector must be some β_i.

Consider now the set.

$S_2 = \{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_{i-1}, \beta_{i+1}, \dots, \beta_n\} = S_1 - \{\beta_i\}$ obviously $L(S_n) = L(S_1) = V$.

If S_2 is L. I., then S_2 forms a basis of V and it is the extended set.

If S_2 is L. D. then continue this procedure till we get a set $S_k \subset S$ such that S_k is L. I.

$\therefore L(S_k) = L(S) = V$

S_k will be extended set of S forming a basis of V.

Note. 1. Every basis is a spanning set but every spanning set is not a basis.

2. If a basis of V(F) contains *n* elements, then $m \le n$.

2.5. Theorem. *Let* $S=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ *be a basis set of a finite dimensional vector space V (F). Then for every* $\alpha\in V$ *there exists a unique set of scalars* $a_1,a_2,\ldots,a_n\in F$ *such that* $\alpha=a_1\,\alpha_1+a_2\,\alpha_2+\ldots+a_n\,\alpha_n$

Proof. If possible let there exist another set of scalars $b_1,b_2\ldots b_n\in F$ such that

$\alpha=b_1\alpha_1+b_2\alpha_2+\ldots+b_n\alpha_n$

$\therefore\ a_1\,\alpha_1+a_2\,\alpha_2+\ldots+a_n\,\alpha_n=b_1\alpha_1+b_2\alpha_2+\ldots+b_n\alpha_n$

$\Rightarrow\ (a_1-b_1)\,\alpha_1+(a_2-b_2)\,\alpha_2\ldots+(a_n-b_n)\,\alpha_n=\bar{O}$

Since the set $S=\{\alpha_1,\alpha_2\ldots\alpha_n\}$ is L. I.

we have $a_1-b_1=0,\ a_2-b_2=0\ldots a_n-b_n=0\Rightarrow a_1=b_1,\ a_2=b_2,\ldots,a_n=b_n$

2.6. COORDINATES

Definition. *Let* $S=\{\alpha_1,\alpha_2\ldots\alpha_n\}$ *be the basis set of a finite dimensional vector space* $V(F)$. *Let* $\beta\in V$ *be given by* $\beta=a_1\alpha_1+a_2\alpha_2\ldots+a_n\alpha_n$ *for* $a_1,a_2,\ldots,a_n\in F$ *the the scalars* $(a_1,a_2,\ldots,a_n)$ *are called the coordinates.*

Note. Coordinates change with the change of basis.

SOLVED PROBLEMS

Ex. 1. *Show that the vector (1,1,2), (1,2,5), (5,3,4) of* $R^3(R)$ *do not form a basis set of* $R^3(R)$.

Sol. Writing these vectors as rows of a matrix we have $\begin{bmatrix}1&1&2\\1&2&5\\5&3&4\end{bmatrix}$

Reducing to echelon form, we get $\begin{bmatrix}1&1&2\\0&1&3\\0&-2&-6\end{bmatrix}\sim\begin{bmatrix}1&1&2\\0&1&3\\0&0&0\end{bmatrix}$

But in the last matrix there are only two non-zero rows, hence the given vectors are L.D. Therefore they ean not form a basis of $R^3(R)$.

Ex. 2. *Show that the set* $\{(1,0,0),(1,1,0),(1,1,1)\}$ *is a basis of* $C^3(C)$. *Hence find the coordinates of the vector* $(3+4i,6i,3+7i)$ *in* $C^3(C)$.

Sol. Let $S=\{(1,0,0),(1,1,0),(1,1,1)\}$

Keeping these vectors as rows of a matrix we get $A=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$

Reducing to echelen form, $A\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ *i.e.,* R_2-R_1 and R_3-R_2

$\therefore$ rank A = no. of unknowns. $\Rightarrow$ The given set is L. I.

Let $z\in C^3$ be $z=(a,b,c)$ where $a,b,c\in C$

Now $(a,b,c)=p\,(1,0,0)+q\,(1,1,0)+r\,(1,1,1)$ for $p,q,r\in C$ $\quad=(p+q+r,\,q+r,\,r)$

$\Rightarrow\ a=p+q+r,\ \ b=q+r,\,c=r.\qquad\Rightarrow\ r=c,\,q=b-c,\,p=a-b$

$\therefore\ z=(a-b)\,(1,0,0)+(b-c)\,(1,1,0)+c\,(1,1,0)$

$=$ *l.c.* of the given vectors of S $\Rightarrow z\in L(S)$. $\qquad\therefore$ S is a basis of $C^3(C)$

Now if $(a,b,c)=(3+4i,6i,3+7i)$ then $p=3-2i,\,q=-3-i,\,r=3+7i$ which are the coordinates of the given vector.

Ex. 3. *The set* $S_4=\{\alpha,\beta,\gamma,\delta\}$ *where* $\alpha=(1,0,0),\beta=(1,1,0),\gamma=(1,1,1)$ *and* $\delta=(0,1,0)$ *is a spaning set of* $R^3(R)$ *but not a basis of set.*

Sol. Keeping the vectors in matrix form $\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\\0&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

δ - row reduced to zero row.

$\Rightarrow$ δ can be expressed as *l.c* of $\alpha,\beta,\gamma\Rightarrow$ S is L. D.

Consider $S_3=\{\alpha,\beta,\gamma\}\quad\therefore\quad\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Whenever $[\alpha\,\beta\,\gamma]$ can be reduced to the standard basis set, S_3 will be the basis of $V_3(F)$ *i.e.,* $L(S_3)=V$. As $S_3\subset S_4,\ L(S_3)=L(S_4)=V$

Thus S_4 is a spaning set of $V_3(F)$ but not a basis set.

Ex. 4. *If* $\alpha=(1,-1,0),\beta=(2,1,3)$ *find a basis for* R^3 *containing* α *and* β.

Sol. Let $e_1=(1,0,0),\,e_2=(0,1,0),\,e_3=(0,0,1)$ and $S=\{e_1,e_2,e_3\}$ we know that S is a standard basis for R^3

We have $\beta\in R^3$. Let $S_1=\{\beta,e_1,e_2,e_3\}$ then $L(S_1)=L(S)=R^3$

Now $\beta=2\,e_1+e_2+3\,e_3$

$\Rightarrow e_3 = \frac{1}{3}\beta - \frac{2}{3}e_1 - \frac{1}{3}e_2$ which shows that e_3 is a L. C. of β, e_1, e_2

Let $S_1 = \{\beta, e_1, e_2\}$ then we have $L(S_1) = L(S) = R^3$. Again $\alpha \in R^3$

Let $S_2 = \{\alpha, \beta, e_1, e_2\}$ then $L(S_2) = L(S_1) = R^3$

We have $\alpha = e_1 - e_2 \Rightarrow e_2 = -\alpha + 0\beta + e_1$ which show that e_2 is a L. C.of α, β, e_1

Remove e_2 from S_2 and let $S_3 = \{\alpha, \beta, e_1\}$ then $L(S_3) = L(S) = R^3$

Since the dimension of R^3 is 3. We have S_3 is a basis of R^3.

$\therefore \{(1,-1,0)\,(2,1,3)\,(1,0,0)\}$ is a basis of R^3.

Note: We can prove that $\{\alpha, \beta, e_2\}$ and $\{\alpha, \beta, e_3\}$ are also basis of R^3.

Ex. 5. *Find a basis for R^4 write $(3,2,5,4), (6,3,5,2)$ as numbers.*

Sol. Let $\alpha_1 = (3,2,5,4), \alpha_2 = (6,3,5,2)$

$e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)$

We know that $\{e_1, e_2, e_3, e_4\}$ is a basis of R^4

we have $\alpha_2 = 6e_1 + 3e_2 + 5e_3 + 2e_4$

$\therefore \alpha_2, e_1, e_2, e_3, e_4$ are L.D. *i.e.* e_4 is a L.C. of α_2, e_1, e_2, e_3

Suppose we remove e_4 from the set $\{\alpha_2, e_1, e_2, e_3, e_4\}$ then $\{\alpha_2, e_1, e_2, e_3\}$ is L.I. set

Consider $\{\alpha_1, \alpha_2, e_1, e_2, e_3\}$. Since the dimension of R^4 is 4 this must be a L.D. set

Consider $x_1\alpha_1 + x_2\alpha_2 + x_3e_1 + x_4e_2 + x_5e_3 = 0$

then $3x_1 + 6x_2 + x_3 = 0$ ... (1) $\quad 2x_1 + 3x_2 + x_4 = 0$ ... (2)

$5x_1 - 5x_2 + x_5 = 0$ ... (3) $\quad 4x_1 + 2x_2 = 0$ ... (4)

(4) $\Rightarrow x_2 = -2x_1$ substituting (I), (2), (3) we get

$$-9x_1 + x_3 = 0 \Rightarrow x_1 = \frac{1}{9}x_3;\quad -4x_1 + x_4 = 0 \Rightarrow x_1 = \frac{1}{4}x_4;\quad -5x_1 + x_5 = 0 \Rightarrow x_1 = \frac{1}{5}x_5$$

$\therefore x_1 = 1, x_2 = -2, x_3 = 9, x_4 = 4, x_5 = 5$

$\therefore \alpha_1 - 2\alpha_2 + 9e_1 + 4e_2 + 5e_3 = 0$ *i.e.,* $e_3 = \frac{-1}{5}\alpha_1 + \frac{2}{5}\alpha_2 - \frac{9}{5}e_1 - \frac{4}{5}e_2$

which proves that e_3 is a L. C. of $\alpha_1, \alpha_2, e_1, e_2$

We can prove that $\{\alpha_1, \alpha_2, e_1, e_2\}$ is L.I. set, which gives a basis for R^4.

Ex. 6. *If $\alpha_1 = (1,2,-1), \alpha_2 = (-3,-6,3), \alpha_3 = (2,1,3)$ and $\alpha_4 = (8,7,7)$ and if $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is such that $L(S) = W$, find a basis by reducing S.*

Sol. We observe $\alpha_2 = -3\alpha_1 = -3\alpha_1 + 0\alpha_3 + 0\alpha_4$

$\therefore \alpha_2$ is a L. C. of $\alpha_1, \alpha_3, \alpha_4$

Let $S_1 = \{\alpha_1, \alpha_3, \alpha_4\}$. We have $L(S_1) = L(S) = W$

Again $\alpha_4 = 2\alpha_1 + 3\alpha_3$. Thus α_4 is L. C. of α_1 and α_3

Let $S_2 = \{\alpha_1, \alpha_3\}$ then $L(S_2) = L(S_1) = W$

Now $a\alpha_1 + b\alpha_2 = 0 \Rightarrow a(1,2,-1) + b(2,1,3) = 0$

$\Rightarrow (a+2b, 2a+b, -a+3b) = (0,0,0) \Rightarrow a + 2b = 0$

$2a + b = 0, \quad -a + 3b = 0$

on solving : $a = 0 = b$ is the only solution which proves that S_2 is L.I. set

Thus S_2 is a basis for W

Note : In every step we can remove one vector which is a L. C. of other vectors. In the above problem any subset of S containing 2 vectors will become a basis of W.

EXERCISE 2 (*a*)

1. Show that the set $\{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $C^3(C)$ but not a basis of $C^3(R)$.
2. Find the coordinates of α with respect to the basis set $\{x, y, z\}$ where
   (*i*) $\alpha = (4,5,6), x = (1,1,1), y = (-1,1,1), z = (1,0,-1)$
   (*ii*) $\alpha = (1,0,-1), x = (0,1,-1), y = (1,1,0), z = (1,0,2)$
   (*iii*) $\alpha = (2,1,3), x = (1,1,1), y = (-1,1,0), z = (1,0,-1)$
3. Show that the four vectors $\{(2,1,0), (0,1,2), (-7,2,5), (8,0,0)\}$ is not a basis of $R^3(R)$
4. Show that if $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $R^3(R)$ then the set $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1\}$ is also a basis of $R^3(R)$
5. (*i*) Show that the set $\{(1,2,1), (2,1,0), (1,-1,2)\}$ forms a basis of $V_3(F)$ **(*S. V. U. S97*)**
   (*ii*) Show that the set of vectors $\{(2,1,4), (1,-1,2), (3,1,-2)\}$ form a basis for R^3. **(*S. V. U. S97*)**
6. Show that the set $\{(1,i,0), (2i,1,1), (0,1+i,1-i)\}$ is a basis of $V_3(C)$ and find the coordinates of the vectors $(1,0,0), (0,1,0), (0,0,1)$ with respect to this basis.
7. Show that the set $\{(3-i, 2+2i, 4), (2, 2+4i, 3), (1-i, -2i, 1)\}$ a basis of $V_3(C)$ and find the coordinates of the vectors (I, 0, 0) and (0, 0, I) with respect to this basis.

8. Show that the vectors (1,1,2), (1,2,5), (5,3,4) in $R^3(R)$ do not fonn a basis set of $R^3(R)$.

9. Find the coordinates of (*i*) $(2i, 3+4i, 5)$ (*ii*) $(6i, 7, 8i)$ (*iii*) $(3+4i, 6i, 3+7i)$ with respect to the basis set $\{(1,0,0), (1,1,0), (1,1,1)\}$ of $C^3(C)$.

10. Under what conditions on the scalar $a \in R$ is the set $\{(0,1,a), (a,0,1), (a,1,1+a)\}$ a basis of $R^3(R)$.

2.7 Theorem. ***Let V(F) be a finite dimensional vector space. Then any two bases of V have the same number of elements.*** **(S. V. U. S 97, S. V. U. O2001, O. U. M 06)**

Proof. Let S_m and S_n be the two bases of V(F) where $S_m = \{\alpha_1, \alpha_2, ..., \alpha_m\}$, $S_n = \{\beta_1, \beta_2, ..., \beta_n\}$

Obviously both S_m and S_n are L.I. subsets of V.

(*i*) Consider S_m as the basis of V and S_n and L. I. set.

$\Rightarrow$ $L(S_m) = V$ and $n(S_m) = m$

$\therefore$ S_n can be extended to be a basis of V $\Rightarrow n \le m$

(*ii*) Consider S_n as the basis of V and S_m as L.I. set $\Rightarrow$ $L(S_n) = V$ and $n(S_n) = n$

$\therefore$ S_m can be extended to form a basis of V $\Rightarrow m \le n$

But both S_m and S_n are bases of V. $\therefore$ $n \le m$ and $m \le n$ $\Rightarrow m = n$

Thus any two bases of V have the same number of elements.

2.8. DIMENSION OF A VECTOR SPACE

Definition. *Let V (F) be a finite dimensional vector space. The number of elements in any basis of V is called the dimension of V and is denoted by dim V.*

Note: The dimension of zero vector space $\{\bar{O}\}$ is said be zero.

e.g. The set S = {(1,0,0), (0,1,0), (0,0,1)} is a basis of $V_3(R)$

$\therefore$ dim V = no. of elements of S = 3.

2.9. Theorem. ***Every set of (n+1) or more vectors in an n dimensional vector space is linearly dependent.***

Proof. Let V (F) be the vector space with V = *n*

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_{n+1}\}$ be any subset consisting of $(n+1)$ vectors of V (F)

If S is L I. then S itself is the basis or can be extended to be a basis of V (F).

In any of these two cases S will have more than or equal to $(n+1)$ vectors. But every basis of V must contain exactly *n* vectors. Therefore S can not be L.I.

Hence S (*i.e.* every $n+1$ or more vectors in V) is L.D.

Note. The largest L.I. subset of a finite dimensional vector space of dimension *n* is a basis.

2. 10. Theorem. ***Let V (F) be a finite dimensional vector space of dimension n. Then any set of n linearly independent vectors in V forms a basis of V.***

Proof. Given dim V = *n*.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be L. I. set of *n* vectors in V.

If S is not a basis of V, then it can be extended to form a basis of V. In such a case the basis will contain more than *n* vectors.

But every basis of V must contain exactly *n* vectors. Therefore our presumption is wrong and S must be a basis of V.

2. 11. Theorem. ***Let V (S) be a finite dimensional vector space of dimension n. Let S be a set of n vectors of V such that L (S) = V. Then S is a basis of V (F).***

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the subset of vector space V (F).

If S is L.I. and because L (S) = V, the set S becomes the basis of V.

If S is L.D. then there exists a proper subset of S forming a basis of V. In such case we get a basis of V consisting less than a vectors.

As every basis of V must contain exactly n vectors, S cannot be L.D. Hence S is a basis of V.

Note. If V is a finite dimensional vector space of dimension *n*, then V cannot be generated by a set of vectors whose number of elements is less than *n*.

2.12.Theorem. ***V is a vector space which is spanned by a finite set of vectors*** $\beta_1, \beta_2, ..., \beta_n$ ***then any independent set of vectors in V is finite and contains no more than n elements.***

Proof. Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a linearly independent subset of vector space V (F)

Let $S = \{\beta_1, \beta_2, ..., \beta_n\}$ and L (S) = V.

Then any vector of V is *l.c.* of the elements of S.

Let $\alpha_m \in V$ be a linear combination of the elements of S.

$\Rightarrow$ $S' = \{\alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ is L. D.

$\therefore$ There exists a vector in S' which is a linear combination of its preceeding vectors. This vector must be one among β'*s*.

Let it be β_i. $\therefore$ β_i = *l.c.* of $\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}$

If $S_1 = \{\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$. Then $L(S_1) = L(S) = V$

Again the vector $\alpha_{m-1} \in V$ is a *l.c.*of the elements of S_1

$\therefore$ $\alpha_{m-1}, \alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n$ are L.D.

$\therefore$ There exists a vector in this set which is a *l.c.* of its predecessors.

Such vector is one of β's as α_{m-1}, α_m are L. I. Let it be β_k.

If $S_2 = \{\alpha_{m-1}, \alpha_m, \beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$ then

$\beta_k = l.c$ of the elements of S_3. $\therefore$ L (S) = L (S$_1$) = L (S) = V

Continuing this process further for $m-3$ times

We get $S_{m-1} = \{\alpha_2, \alpha_3, ..., \alpha_m, \beta_{i1}, \beta_{i2}, \beta_{in-m+1}\}$. So that $L(S_{m-1}) = V$.

This set consists of atleast one β_i. Otherwise $S_{m-1} = \{\alpha_2, \alpha_3 ... \alpha_m\}$

So that α_1 is a $l.c.$ of $\alpha_2, \alpha_3 ... \alpha_m$

This can not happen as the set $\alpha_2, \alpha_3 ..., \alpha_m$ is L. I.

Hence S_{m-1} consists of atleast one $\beta_i \Rightarrow n-m+1 \ge 1 \Rightarrow n \ge m$

$\therefore$ The no. of elements of the L. I. set in $V \le n$.

DIMENSION OF A SUBSPACE

2.13. Theorem. ***Let V(F) be a finite dimensional vector space of dimension n and W be the subspace of V. Then W is a finite dimensional vector space with dim W ≤ n.***

Proof. Dim V = $n \Rightarrow$ each $(n+1)$ or more vectors of V form an L.D. set.

Given W is a subspace of V(F)

$\Rightarrow$ each set of $(n+1)$ vectors in W is a subset of V and hence L.D.

Thus any LI. set of vectors in W can contain at the most n vectors.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the largest L.I. subset of W, where $m \le n$.

Now we shall prove that S is the basis of W.

For any $\beta \in W$ consider, $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta\}$

Since S is the largest set of L.I. vectors, S_1 is L.D.

$\therefore$ Therefore there exists $a_1, a_2, ..., a_m$, $b \in F$ not all zero such that

$a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b\beta = 0$

Let b = 0, then we have $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = 0$

$\Rightarrow a_1 = 0, a_2 = 0, ..., a_m = 0$ as S is L. I.

This proves that S_1 is L.I which is a contradiction.

$\therefore$ $b \ne 0$. Therefore there exists $b^{-1} \in F$ such that $bb^{-1} = 1$

$\therefore$ $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b\beta = 0 \Rightarrow b\beta = -a_1\alpha_1 - a_2\alpha_2 ... a_m\alpha_m$

$\Rightarrow \beta = (-b^{-1}a_1) + (-b^{-1}a_2)\alpha_2 + ... + (-b^{-1}a_m)\alpha_m$

$\Rightarrow \beta = a$ linear combination of elements of S. $\Rightarrow \beta \in L(S)$

Also S is L.I. Hence S is the basis of W.

$\therefore$ W is a finite dimensional vector space with dim $W \le n$.

2.14. Theorem. ***Let W_1 and W_2 be two subspaces of a finite dimensional vector space V (F). Then* dim (W_1 + W_2) = dim W_1 + dim W_2 − dim ($W_1 \cap W_2$)**

(O. U. M06, O. U. O01, S. V. U. O01, S. V. U. S97)

Proof. Since W_1 and W_2 are subspace of V, $W_1 + W_2$ and $W_1 \cap W_2$ are also subspace of V.

Let dim $(W_1 \cap W_2) = k$ and $S = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ be a basis of $W_1 \cap W_2$.

Clearly $S \subseteq W_1$ and $S \subseteq W_2$ and S is L.I.

Since S is L.I. and $S \subseteq W_1$, the set S can be extended to form a basis of W_1.

Let $B_1 = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1. $\therefore$ dim $W_1 = k + m$.

Again since S is L.I. and $S \subseteq W_2$, the set S can be extended to form a basis of W_2.

Let $B_2 = \{\gamma_1, \gamma_2, ..., \gamma_k, \beta_1, \beta_2, ..., \beta_t\}$ be a basis of W_2. $\therefore$ dim $W_2 = k + t$

$\therefore$ dim W_1 + dim W_2 − dim $(W_1 \cap W_2) = (k+m) + (k+t) - k = k + m + t$

Now we shall prove that the set

$S' = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_t\}$ is a basis of $W_1 + W_2$ and hence dim $(W_1 + W_2) = k + m + t$.

(i) To prove that S' is L.I.

Now $c_1\gamma_1 + c_2\gamma_2 + ... + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = 0$ ... (1)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = -c_1\gamma_1 - ... - c_k\gamma_k - a_1\alpha_1 - ... - a_m\alpha_m$

= $l.c.$ of elements of B_1 and $\therefore \in W_1$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1$

Again $0\gamma_1 + 0\gamma_2 + ... 0\gamma_k + b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t$

= $l.c.$ of elements of B_2 and $\therefore \in W_2$ $\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_2$

Therefore by (1) and (2) $\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1 \cap W_2$

Hence it can be expressed as a $l.c.$of the elements of the basis S of $W_1 \cap W_2$

Let $b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + ... + d_k\gamma_k$

$\therefore\ b_1\beta_1 + b_2\beta_2 \ldots + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 \ldots - d_k\gamma_k = \bar{O} \Rightarrow$ (*l.c.* of elements of basis B_2) $= \bar{O}$

$b_1 = 0, b_2 = 0 \ldots b_t = 0, d_1 = 0, \ldots k_k = 0$

Substituting in I we have $c_1\gamma_1 + c_2\gamma_2 + \ldots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 \ldots + a_m\alpha_m = \bar{O}$

$\Rightarrow$ (*l.c.* of elements of basis B_1) $= \bar{O} \Rightarrow c_1 = 0, c_2 = 0 \ldots c_k = 0, a_1 = 0, a_2 = 0 \ldots a_m = 0$

Thus the relation I implies that $c_1 = c_2 = \ldots = c_k = a_1 = a_2 = \ldots \alpha_m = o, b_1 = b_2 = \ldots = b_t = o$

$\therefore$ S' is L.I. set.

(*ii*) *To prove* $L(S') = W_1 + W_2$

Every vector of S' is a vector of $W_1 + W_2$. $\qquad \therefore\ L(S') \subseteq W_2 + W_2$

Let $\delta \in W_1 + W_2$. $\qquad \therefore\ \delta = \alpha + \beta$ where $\alpha \in W_1, \beta \in W_2$

$\delta =$ (*l.c.* of elements of B_1) + (*l.c.* of elements of B_2)

= (*l.c.* of γ's and α's) + (*l.c.* of γ's and β's) = *l.c.* of γ's, α's and β's.

= *l.c.* of elements of S'.

$\therefore\ \delta \in L(S') \quad \therefore\ W_1 + W_2 \subseteq L(S')$

$\therefore\ L(S') = W_1 + W_2$. $\qquad$ Hence S' is the basis of $W_1 + W_2$.

$\therefore\ \dim(W_1 + W_2) = k + m + t$. $\qquad$ Hence the theorem.

SOLVED PROBLEMS

Ex. 1. *Let* W_1 *and* W_2 *be two subspaces of* R^4 *given by* $W_1 = \{(a,b,c,d) : b - 2c + d = 0\}$, $W_2 = \{(a,b,c,d) : a = d, b = 2c\}$. *Find the basis and dimension of (i)* W_1 *(ii)* W_2 *(iii)* $W_1 \cap W_2$ *and hence find* $\dim(W_1 + W_2)$.

Sol. (i) Given $V = \{(a,b,c,d) : b - 2c - d\}$

Let $(a,b,c,d) \in V$ then $(a,b,c,d) = (a, 2c - d, c, d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$, $\qquad \therefore (a,b,c,d) =$ *l.c.* of L.I. set

$\{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$ which forms the basis of $W_1 \quad \therefore \dim W_1 = 3$.

(*ii*) Given $W_2 = \{(a,b,c,d) : a = d, b = 2c\}$.

Let $\alpha \in W_2 \Rightarrow \alpha = (a,b,c,d)$ where $a = d, b = 2c$

$\therefore\ \alpha = (d, 2c, c, d) = d(1,0,0,1), + c(0,2,1,0) \Rightarrow \alpha = l.c$ of L.I. set $\{(1,0,0,1), (0,2,1,0)\}$

$\therefore$ It forms a basis $\qquad \therefore \dim W_2 = 2$.

(*iii*) $W_1 \cap W_2 = \{(a,b,c,d) / b - 2c + d = 0, a = d, b = 2c\}$

Now $b - 2c + d = 0, a = d, b = 2c$ gives $b = 2c, a = 0, d = 0$

$\therefore\ (a,b,c,d) = (0,2c,c,0) = c(0,2,1,0) \quad \Rightarrow (a,b,c,d) =$ multiple of the vector $(0,2,1,0)$

$\therefore$ Basis of $W_1 \cap W_2 = (0,2,1,0) \quad \Rightarrow \dim W_1 \cap W_2 = 1$

(*iii*) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 = 3 + 2 - 1 = 4$

Ex. 2. *If W is the subspace of* $V_4(R)$ *generated by the vectors* $(1,-2,5,-3), (2,3,1,-4)$ *and* $(3,8,-3,-5)$ *find a basis of W and its dimension.*

Sol. Arranging the given vectors as rows of a matrix

Let $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$. Reducing to echelon form,

$R_2 - 2R_1, R_2 - 3R_1; A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$. Again by $R_3 - 2R_2$, $A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence the two non-zero rows *viz*; $(1,-2,5,-3), (0,7,-9,2)$ form the least L. I. set and hence a basis of W. $\quad \therefore \dim W = 2$

Ex. 3. *V is the space generated by the polynomials* $\alpha = x^3 + 2x^2 - 2x + 1$, $\beta = x^3 + 3x^2 - x + 4$, $\gamma = 2x^3 + x^2 - 7x - 7$. *Find a basis of V and its dimension.*

Sol. Now V is the polynomial space generated by $\{\alpha, \beta, \gamma\}$ given above

$\alpha = x^3 + 2x^2 - 2x + 1$

$\Rightarrow$ the coordinates of are $(1,2,-2,1)$ w. r. to the base $\{x^3, x^2, x, 1\}$

Similarly the coordinates of β and γ w. r. to the same base are $(1,3,-1,4)$ and $(2,1,-7,-7)$.

Forming the matrix A with these co-ordinate as rows

$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$ Reducing to echilon form,

by $R_2 - R_1, R_3 - 2R_1, \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$

again by $R_3 + 3R_1$, $A \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $W+\alpha \in L(S')$ $\quad\quad$ $\therefore$ $L(S') = V/W$

Therefore S' is the basis of V/W and hence $\dim V/W = l = \dim V - \dim W$.

Ex. *If W is a subspace of V (F) then for $a, b \in V$ show that*

(i) $\alpha \in (\beta + W)$ $\quad$ *(ii)* $\beta \in (\alpha + W)$ *are equivalent.*

Sol. $\alpha \in \beta + W \Rightarrow \alpha = \beta + \gamma$ for some $\gamma \in W$ $\quad \Rightarrow \alpha - \beta \in W$

For $(-1) \in F$, $(-1)(\alpha - \beta) \in W \Rightarrow \beta - \alpha \in W \Rightarrow (\beta - \alpha + \alpha) \in W + \alpha \Rightarrow \beta \in W + \alpha$.

3

Linear Transformations

3.1. VECTOR SPACE HOMOMORPHISM

Definition. *Let U and V be two vector spaces over the same field F. Thus the mapping $f: U \to V$ is called a homomorphism from U into V if*

(i) $f(\alpha+\beta) = f(\alpha) + f(\beta)\ \forall\ \alpha, \beta \in U$ $\quad\quad$ *(ii)* $f(a\alpha) = af(\alpha)\ \forall a \in F; \forall\ \alpha \in U$

Note. 1. If f is onto function then V is called the homomorphic image of f.

2. If f is one - one onto function then f is called an isomorphism. Thus it is said that U is isomorphic to V denoted by $U \cong V$.

3. The two conditions of homomorphism are combined into a single condition, called the linear property to define the linear transformation as below.

3.2. LINEAR TRANSFORMATION

Definition. *Let U(F) and V(F) be two vector spaces. Then the function. $T: U \to V$ is called a linear transformation of U into V if $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)\ \forall a, b \in F; \alpha, \beta \in U$.*

Clearly the vector space homomorphism is equivalent to linear transformation.

Linear Operator : Definition. *If $T: U \to U$ (i.e T transforms U into itself) then T is called a linear operator on U.*

Linear Functional : Definition. *If $T: U \to F$ (i.e. T transforms U into the field F) then T is called a linear functional on U.*

3.3. ZERO TRANSFORMATION

Theorem. ***Let U(F) and V(F) be two vector spaces. Let the mapping $T: U \to V$ be defined by $T(\alpha) = \hat{O}\ \forall\ \alpha \in U$ where $\hat{O}$ (zero crown) is the zero vector of V. Then T is a linear transformation.***

Proof. For $a, b \in F$ and $\alpha, \beta \in U \Rightarrow a\alpha + b\beta \in U$ $\quad$ ($\because$ U is V.S.)

By definition we have $T(a\alpha + b\beta) = \hat{O} = a\hat{O} + b\hat{O} = aT(\alpha) + bT(\beta)$

$\therefore$ By the definition of linearity T is a linear transformation.

Such a L.T. is called the zero transformation and is denoted by O.

3.4. IDENTITY OPERATOR

Theorem. ***Let V(F) be a vector spaee and the mapping $I: V \to V$ be defined by $I(\alpha) = \alpha\ \ \forall \alpha \in V$. Then, I is a linear operator from V into itself.***

Proof. $a, b \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in V$ $\quad$ ($\because$ V is L.S.)

By definition we have $\; I(a\alpha + b\beta) = a\alpha + b\beta = a\, I(\alpha) + b\, I(\beta)$ $\quad$ (by def.)

$\therefore$ I is a L. T from V into itself and I is called the **Identity Operator.**

3.5. NEGATIVE OF TRANSFORMATION

Theorem. *Let U (F) and V (F) be two vector spaces and* $T : U \to V$ *be a linear transformation. Then the mapping* $(-T)$ *defined by* $(-T)(\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ *is a linear transformation.*

Proof. $a, b \in F$ and $\alpha, \beta \in U \Rightarrow a\alpha + b\beta \in U$ $\quad$ ($\because$ U is V.S.)

Now by definition $(-T)(a\alpha + b\beta) = -[T(a\alpha + b\beta)]$

$= -[a\, T(\alpha) + b\, T(\beta)] = -a\, T(\alpha) - b\, T(\beta)$

$= a[-T(\alpha)] + b[-T(\beta)] = a[(-T)(\alpha)] + b[(-T)(\beta)]$

$\Rightarrow -T$ is a linear transformation.

PROPERTIES OF LINEAR TRANSFORMATIONS

3.6. Theorem. *Let* $T : U \to V$ *is a linear transformation from the vector space U(F) to the vector space V(F). Then (i)* $T(\bar{O}) = \hat{\bar{O}}$*, where* $\bar{O} \in U$ *and* $\hat{\bar{O}} \in V$

(ii) $T(-\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ $\qquad$ *(iii)* $T(\alpha - \beta) = T(\alpha) - T(\beta)\ \forall\ \alpha, \beta \in U$

(iv) $T(a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_n T(\alpha_n)\ \forall\ a_i \in F$ *and* α*'s* $\in U$.

Proof. (*i*) $\alpha, \bar{O} \in U \Rightarrow T(\alpha), T(\bar{O}) \in V$

Now $T(\alpha) + T(\bar{O}) = T(\alpha + \bar{O})$ $\quad$ (T is L.T.) $\quad = T(\alpha) = T(\alpha) + \hat{\bar{O}}$ $\quad$ ($\bar{O} \in V$)

By cancellation law $T(\bar{O}) = \hat{\bar{O}}$

(*ii*) $T(-\alpha) = T(-1 \,.\, \alpha) = (-1)\, T(\alpha) = -T(\alpha)$

(*iii*) $T(\alpha - \beta) = T[\alpha + (-1)\beta] = T(\alpha) + (-1) T(\beta) = T(\alpha) - T(\beta)$ $\quad$ ($\because$ T is L.T.)

(*iv*) For $n = 1$, $\; T(a_1\alpha_1) = a_1\, T(\alpha_1)$ $\quad$ ($\because$ T is L.T.)

$n = 2$, $\; T(a_1\alpha_1 + a_2\alpha_2) = a_1 T(\alpha_1) + a_2 T(\alpha_2)$

Let this be true for $n = m$

$\therefore T(a_1\alpha_1 + a_2\alpha_2 + a_m\alpha_m) = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_m T(\alpha_m)$ $\quad \ldots (1)$

Now $T[a_1\alpha_1 + \ldots + a_m\alpha_m) + a_{m+1}\alpha_{m+1}] = T(a_1\alpha_1 + \ldots + a_m\alpha_m) + T(a_{m+1}\alpha_{m+1})$

$= a\, T(\alpha_1) + \ldots + a_m T(\alpha_m) + a_{m+1} T(\alpha_{m+1})$ $\quad \therefore$ The relation is true for $n = m+1$

Hence it is true for all integral values of n.

DETERMINATION OF LINEAR TRANSFORMATION

3.7. Theorem. *Let U(F) and V(F) be two vector spaces and* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *be a basis of U. Let* $\{\delta_1, \delta_2, \ldots, \delta_n\}$ *be a set of n vectors in V. Then there exists a unique linear transformation* $T : U \to V$ *such that* $T(\alpha_i) = \delta_i$ *for i = 1, 2, ... n.*

Proof. Let $\alpha \in U$. Since $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a basis of U, there exist unique scalars $a_1, a_2 \ldots a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

(*i*) Existence of T, $\; \delta_1, \delta_2, \ldots, \delta_n \in V \Rightarrow (a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n) \in V$

We define $T : U \to V$, such that $T(\alpha) = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$

$\therefore$ T is a mapping from U into V.

Now $\alpha_i = 0 \,.\, \alpha_1 + 0 \,.\, \alpha_1 + 0 \,.\, \alpha_2 \ldots + 1 \,.\, \alpha_i + 0 \,.\, \alpha_{i+1} + \ldots + 0 \,.\, \alpha_n$

$\therefore$ by the definition of T mapping

$T(\alpha_i) = 0 \,.\, \delta_1 + 0 \,.\, \delta_1 + 0 \,.\, \delta_2 \ldots + \delta_i + 0 \,.\, \delta_{i+1} \ldots + 0 \,.\, \delta_n \Rightarrow T(\alpha_i) = \delta_i, \ \forall\ i = 1, 2, \ldots n$

(*ii*) To show that T is L.T:

Let $a, b \in F$ and $\alpha, \beta \in U$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$; $\qquad \beta = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

$\therefore T(\alpha) = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$; $\qquad T(\beta) = b_1\delta_1 + b_2\delta_2 + \ldots + b_n\delta_n$

$\therefore a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n)$ $\quad$ (by def.)

$= (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \ldots + (aa_n + bb_n)\alpha_n$

$\therefore T(a\alpha + b\beta) = T[(aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \ldots + (aa_n + bb_n)\alpha_n]$

$= (aa_1 + bb_1)\delta_1 + (aa_2 + bb_2)\delta_2 + \ldots + (aa_n + bb_n)\delta_n$

$= a(a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n) + b(b_1\delta_1 + \ldots + b_n\delta_n) = a\, T(\alpha) + b\, T(\beta)$

$\therefore$ T is a L.T.

(*iii*) To show that T is unique.

Let $T' : U \to V$ be another L.T. so that $T(\alpha_i) = \delta_i$ for $i = 1, 2, \ldots n$

If $\alpha = a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n$

then $T'(\alpha) = T'(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) = a_1 T'(\alpha_1) + a_2 T'(\alpha_2) + \ldots + a_n T'(\alpha_n)$

$= a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n = T(\alpha)$

$\therefore$ T' = T and hence T is unique.

Note. In determining the L.T. the assumption that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of U is essential.

3.8. Theorem. *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $S' = \{\beta_1, \beta_2, ..., \beta_n\}$ be two ordered bases of n, dimensional vector space V(F). Let $\{a_1, a_2, ..., a_n\}$ be an ordered set of n scalars such that $\alpha = a_1\alpha_1 + a_2\alpha_2, ..., a_n\alpha_n$ and $\beta = a_1\beta_1 + a_2\beta_2, ..., a_n\beta_n$. Show that $T(\alpha) = \beta$ where T is the linear operator on V defined by $T(\alpha_i) = \beta_i,\ i = 1, 2, ..., n$.*

Proof. Now $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$

$$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + ... + a_n T(\alpha_n) \qquad \text{(T is L.T.)}$$

$$= a_1\beta_1 + a_2\beta_2 + ... + a_n\beta_n = \beta$$

SOLVED PROBLEMS

Ex. 1. *The mapping $T : V_3(R) \to V_2(R)$ is defined by $T(x, y, z) = (x - y, x - z)$. Show that T is a linear transformation.*

Sol. Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be two vectors of $V_3(R)$

For $a, b \in R$

$T[a\alpha + b\beta] = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)] = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$$= ((ax_1 + bx_2) - (ay_1 + by_2),\ ax_1 + bx_2 - (az_1 + bz_2))$$

$$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$$

$$= (a(x_1 - y_1), a(x_1 - z_1) + (b(x_2 - y_2), b(x_2 - z_2))$$

$$= a(x_1 - y_1, x_1 - z_1) + b(x_2 - y_2, x_2 - z_2)$$

$$= a\,T(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2) = a\,T(\alpha) + b\,T(\beta)$$

$\Rightarrow$ T is a linear transformation from $V_3(R)$ to $V_2(R)$

Ex. 2. *The mapping $T : V_3(R) \to V_1(R)$ is defined by $T(a, b, c) = a^2 + b^2 + c^2$; Can T be a linear transformation ?* ***(O. U. 2001/O)***

Sol. Let $\alpha = (a, b, c)$ and $\beta = (x, y, z)$ be two vectors of $V_3(R)$.

For $p, q \in R$, $T(pa + qx, pb + qy, pc + qz) = (pa + qx)^2 + (pb + qy)^2 + (pc + qz)^2$

Now $= p\,T(\alpha) + q\,T(\beta) = p\,T(a, b, c) + q\,T(x, y, z)$

$$= p(a^2 + b^2 + c^2) + q(x^2 + y^2 + z^2) = T(p\alpha + q\beta) \neq p\,T(\alpha) + q\,T(\beta)$$

$\therefore$ T is not a L.T from $V_3(R)$ to $V_1(R)$.

Ex. 3. *Let V be the vector space of polynomials in the variable x over R. Let $f(x) \in V(R)$; show that*

(*i*) $D : V \to V$ *defined by* $Df(x) = \dfrac{df(x)}{dx}$

(*ii*) $I : V \to V$ *defined by* $If(x) = \int_0^x f(x)\,dx$ *are linear transformations.*

Sol. Let $f(x), g(x) \in V(R)$ and $a, b \in R$

(*i*) $D[af(x) + bg(x)] = \dfrac{d}{dx}[af(x) + bg(x)] = \dfrac{d}{dx}[af(x)] + \dfrac{d}{dx}[bg(x)]$

$$= a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)] = a\,D\,f(x) + b\,D\,g(x)$$

$\therefore$ D is a linear transformation and D is called a differential operator.

(ii) $I[af(x) + bg(x)] = \int_0^x [af(x) + bg(x)]\,dx = a\int_0^x f(x)\,dx + b\int_0^x g(x)\,dx = a\,I\,f(x) + b\,I\,g(x)$

$\therefore$ I is a linear transformation and is called Integral transformation.

Ex. 4. *Let $P_n(R)$ be the vector space of all polynomials of degree n over a field R. If a linear operator T on $P_n(R)$ is such that $Tf(x) = f(x+1)$, $f(x) \in P_n(R)$.*

Show that $T = 1 + \dfrac{D}{1!} + \dfrac{D^2}{2!} + \dfrac{D^3}{3!} + ... + \dfrac{D^n}{n!}$

Sol. Let $f(x) = a_0 + a_1x + a_2x^2 ... + a_nx^n \ \forall\ a_i \in R$

$$\left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + ... + \frac{D^n}{n!}\right] f(x) = a_0 + a_1x + a_2x^2 ... + a_nx^n$$

$$+ \frac{1}{1!}(0 + a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}) + \frac{1}{2!}(0 + 0 + 2a_2 + 6a_3 + ... + n(n-1)a_nx^{n-2})$$

$$+ ...... \ ...... \ ..... \ ..... + \frac{1}{n!}(0 + 0 + 0 ... + a_n\,n!)$$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + ... + a_n(x+1)^n$$

$$= f(x+1) = T\,f(x) \qquad \text{(by def.)}$$

$$\therefore\ T = \left(1 + \frac{D}{1!} + \frac{D^2}{2!} + ... + \frac{D^n}{n!}\right).$$

Thus T is a linear operator from $P_n(R)$ into $P_n(R)$.

Ex. 5. *Is the mapping $T : R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$ a linear transformation.*

Sol. We have $T:R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$

Let $\alpha, \beta \in R^3$ where $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

For $a, b \in R$, $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$\therefore T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = (|ax_1 + bx_2|, 0)$

And $aT(\alpha) + bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$

$= a(|x_1|, 0) + b(|x_2|, 0) = (a|x_1| + b|x_2|, 0)$

Clearly $T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$. Hence T is not a linear transformation.

Ex. 6. *Let T be a linear transformation on a vector space U into V. Prove that the vectors* $x_1, x_2, \dots x_n \in U$ *are linearly independent if* $T(x_1), T(x_2) \dots T(x_n)$ *are L.I.*

Sol. Given $T: U(F) \to V(F)$ is a L.T. and $x_1, x_2, \dots x_n \in U$.

Let there exist $a_1, a_2 \dots a_n \in F$ such that $a_1x_1 + a_2x_2 + \dots + a_nx_n = \bar{O}$ ... (1) $(\bar{O} \in U)$

$\therefore T(a_1x_1 + a_2x_2 + \dots + a_nx_n) = T(\bar{O}) \Rightarrow a_1 T(x_1) + a_2 T(x_2) \dots + a_n T(x_n) = \hat{\bar{O}}$ $(\bar{O} \in V)$

But $T(x_1), T(x_2), \dots, T(x_n)$ are L. I. $\therefore a_1 = a_2 = \dots = a_n = 0$.

$\therefore$ From (1) $x_1, x_2 \dots x_n$ are L. I.

Ex. 7. *Let V be a vector space of* $n \times n$ *matrices over the field F. M is a fixed matrix in V. The mapping* $T: V \to V$ *is defined by T (A) = AM + MA where* $A \in V$. *Show that T is linear.*

Sol. Let $a, b \in F$ and $A, B \in V$. Then T (A) = AM + MA and T (B) = BM + MB

$\therefore T(aA + bB) = (aA + bB)M + M(aA + bB) = a(AM + MA) + b(BM + MB)$

$= aT(A) + bT(B)$. T is a linear transformation.

Ex. 8. *Describe explicitly the linear transformation* $T: R^2 \to R$ *such that T (2,3) = (4,5) and T (1,0) = (0, 0).* **(S. V. U. S97)**

Sol. First of all we have to show that the vectors (2,3) and (1, 0) are L.I.

Let $= a(2,3) + b(1,0) = \bar{O}$

$\Rightarrow (2a + b, 3a + 0) = (0,0) \Rightarrow 2a + b = 0, 3a = 0 \Rightarrow 2a = 0, b = 0$

$\therefore S = \{(2,3), (1,0)\}$ is L.I. Let us prove that L (S) = R^2

Let $(x, y) \in R^2$ and $(x, y) = a(2,3) + b(1,0) = (2a + b, 3a)$

$\Rightarrow 2a + b = x,\ 3a = y \Rightarrow a = \frac{y}{3}; b = \frac{3x - 2y}{3}$. Hence S spans R^2

Now $T(x, y) = T\left[\frac{y}{3}(2,3) + \frac{3x - 2y}{3}(1,0)\right]$

$= \frac{y}{3}T(2,3) + \frac{3x - 2y}{3}T(1,0) = \frac{y}{3}(4,5) + \frac{3x - 2y}{3}(0,0) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$

$\therefore$ This is the required transformation.

Ex. 9. *Find* $T(x, y, z)$ *where* $T: R^3 \to R$ *is defined by T (1,1,1) = 3,* $T(0,1,-2) = 1$, $T(0,0,1) = -2$.

Sol. Let $S = \{(1,1,1), (0,1,-2), (0,0,1)\}$

(*i*) Let $a(1,1,1) + b(0,1,-2) + c(0,0,1) = \bar{O}$

$\Rightarrow (a, a + b, a - 2b + c) = (0,0,0)$ $(\because \bar{O} \in R^3)$

$\Rightarrow a = 0, a + b = 0, a - 2b + c = 0 \Rightarrow a = 0, b = 0, c = 0$ ($\therefore$ S is L.I. set)

(*ii*) Let $(x, y, z) \in R^3$

$(x, y, z) = a(1,1,1) + b(0,1,-2) + c(0,0,1) = (a, a + b, a - 2b + c)$

$\Rightarrow a = x, a + b = y, a - 2b + c = z \Rightarrow a = x, b = y - x, c = z + 2y - 3x$

$\therefore$ S spans R^3

Hence $T(x, y, z) = T[x(1,1,1) + (y - x)(0,1,-2) + (z + 2y - 3x)(0,0,1)]$

$= xT(1,1,1) + (y - x)T(0,1,-2) + (z + 2y - 3x)T(0,0,1)$

$= x(3) + (y - x)(1) + (z + 2y - 3x)(-2)$

$= 8x - 3y - 2z$ which is the required linear functional.

EXERCISE 3 (*a*)

1. Which of the following maps are linear transformations ,
   (*a*) $T: V_1 \to V_3$ defined by $T(x) = (x, 2x, 3x)$.
   (*b*) $T: R^2 \to R^2$ defined by $T(a,b) = (2a + 3b, 3a - 4b)$.
   (*c*) $T: R^3 \to R^2$ defined by $T(x, y, z) = (x - y, x - z)$.
   (*d*) $T: P \to P$ defined by $T(x) = x^2 + x$.
   (*e*) $T: P \to P$ defined by $T\,p(x) = p(0) + xp'(0) + \frac{x^2}{2!}p''(0)$.
   (*f*) $T: R^3 \to R^3$ defined by $T(x, y, z) = (x + 1, y, z)$.
   (*g*) $T: R^2 \to R^2$ defined by $T(x, y) = (x^3, y^3)$.

2. Let V be the space of $m \times n$ matrices over the field F. Let P be a fixed $m \times m$ matrix and Q is a fixed $n \times n$ matrix over F. $T: V \to V$ is defined by $T(A) = PAQ$. Then show that T is a linear transformation.

{ **Hint :** $T(aA + bB) = P(aA + bB)Q = (a\,PA + b\,PB)Q$

$= a\,PAQ + b\,PBQ = a\,T(A) + b\,T(B)$). Find a linear transformation.

3. $T: R^2 \to R^2$ such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$

4. $T: V_2 \to V_2$ such that $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$

5. $T: V_3 \to V_3$ such that $T(0,1,2) = (3,1,2)$ and $T(1,1,1) = (2,2,2)$.

6. $T: V_2(R) \to V_3(R)$ such that $T(1,2) = (3,-1,5)$ and $T(0,1) = (2,1,-1)$

7. $T: R^2 \to R^3$ such that $T(2,-5) = (-1,2,3)$ and $T(3,4) = (0,1,5)$

8. Find a linear transformation. $T: R^2 \to R^2$, such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$ Prove that T maps the square with vertices (0,0), (1,0), (1,1), (0,1) into a parallelogram. ***(S. V. U. 2001/0)***

SUM OF LINEAR TRANSFORMATONS

3.9. Definition. *Let* T_1 *and* T_2 *be two linear transformations from U (F) into V (F). Then their sum* $T_1 + T_2$ *is defined by* $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$

3.10. Theorem. ***Let U(F) and V(F) be two linear transformations. Let T_1 and T_2 be two linear transformations from U into V. Then the mapping $T_1 + T_2$ defined by $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$ is a linear tranformation.***

Proof. Given $T_1 : U \to V$ and $T_2 : U \to V$, $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$

$T_1(\alpha) \in V$ and $T_2(\alpha) \in V \Rightarrow T_1(\alpha) + T_2(\alpha) \in V$. Hence $(T_1 + T_2): U \to V$

Let $a, b \in F$ and $\alpha, \beta \in U$. Then $(T_1 + T_2)(a\alpha + b\beta) = T_1(a\alpha + b\beta) + T_2(a\alpha + b\beta)$ (by def.)

$= a\,T_1(\alpha) + b\,T_1(\beta) + a\,T_2(\alpha) + b\,T_2(\beta) = a\,[T_1(\alpha) + T_2(\alpha)] + b\,[T_1(\beta) + T_2(\beta)]$

$= a\,(T_1 + T_2)(\alpha) + b\,(T_1 + T_2)(\beta)$ $\quad\therefore$ $T_1 + T_2$ is a L. T. from U into V.

3.11. SCALAR MULTIPLICATION OF A L. T.

Theorem. ***Let $T: U(F) \to V(F)$ be a linear transformation and $a \in F$. Then the function aT defined by $(aT)(\alpha) = a\,T(\alpha)\ \forall\, \alpha \in U$ is a linear transformation.***

Proof. Given $T: U(F) \to V(F)$ and $(aT)(\alpha) = a\,T(\alpha)$, $a \in F$, $\alpha \in U$

Now $T(\alpha) \in V \Rightarrow a\,T(\alpha) \in V$. $\quad\therefore$ (aT) is a mapping from U into V

For $c, d \in F$ and $\alpha, \beta \in U$

$(aT)[c\alpha + d\beta] = a\,T(c\alpha + d\beta)$ (by def)

$= a\,[cT(\alpha) + d\,T(\beta)] = acT(\alpha) + ad\,T(\beta) = c\,(aT)(\alpha) + d\,(aT)(\beta)$

Hence aT is a L.T. from U into V.

SOLVED PROBLEMS

Ex. 1. *Let* $T: V_3(R) \to V_2(R)$ *and* $H: V_3(R) \to V_2(R)$ *be the two linear transformations defined by* $T(x,y,z) = (x-y,\ y+z)$ *and* $H(x,y,z) = (2x,\ y-z)$

Find (i) $H + T$ *(ii)* aH

Sol. *(i)* $(H+T)(x,y,z) = H(x,y,z) + T(x,y,z) = (2x,\ y-z) + (x-y,\ y+z) = (3x-y,\ 2y)$

(ii) $(aH)(x,y,z) = aH(x,y,z) = a(2x,\ y-z) = (2ax,\ ay-az)$

Ex. 2. *Let* $G: V_3 \to V_3$ *and* $H: V_3 \to V_3$ *be two linear operators defined by* $G(e_1) = e_1 + e_2$, $G(e_2) = e_3$, $G(e_3) = e_2 - e_3$ *and* $H(e_1) = e_3$, $H(e_2) = 2e_2 - e_3$, $H(e_3) = 0$ *where* $\{e_1, e_2, e_3\}$ *is the standard basis of* $V_3(R)$. *Find (i)* $G + H$ *(ii)* $2\,G$

Sol. Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $V_3(R)$ so that

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$G(e_1) = e_1 + e_2 \Rightarrow G(1,0,0) = (1,1,0)$, $G(e_1) = e_3 \Rightarrow G(0,1,0) = (0,0,1)$

$G(e_3) = e_2 - e_3 \Rightarrow G(0,0,0) = (0,1,-1)$

Again $H(e_1) = e_3 \Rightarrow H(1,0,0) = (0,0,1)$, $H(e_2) = 2e_2 - e_3 \Rightarrow H(0,1,0) = (0,2,-1)$

$H(e_3) = \bar{o} \Rightarrow H(0,0,1) = (0,0,0)$

(i) $(G+H)(e_1) = G(e_1) + H(e_1) = e_1 + e_2 + e_3 \Rightarrow (G+H)(1,0,0) = (1,1,1)$

$(G+H)(e_2) = G(e_2) + H(e_2) = 2e_2 \Rightarrow (G+H)(0,1,0) = (0,2,0)$

$(G+H)(e_3) = G(e_3) + H(e_3) = e_2 - e_3 \Rightarrow (G+H)(0,0,1) = (0,1,-1)$

(ii) $2G(e_1) = 2G(e_1) = 2e_1 + e_2$, $2G(e_2) = 2G(e_2) = 2e_3$, $2G(e_3) = 2G(e_3) = 2e_2 - 2e_3$ etc.

PRODUCT OF LINEAR TRANSFORMATIONS

3.12. Theorem. ***Let U(F), V (F) and W(F) are three vector spaces and $T: V \to W$ and $H: U \to V$ are two linear transformations. Then the composite function TH (called the product of linear transformations) defined by $(TH)(\alpha) = T[H(\alpha)]\ \forall\, \alpha \in U$ is a linear transformation from U into W.***

Proof. Given $H: U(F) \to V(F)$ and $T: V(F) \to W(F)$

For $a \in U \Rightarrow H(\alpha) \in V$

Again $H(\alpha) \in V \Rightarrow T[H(\alpha)] \in W \Rightarrow (TH)(\alpha) \in W$

$\therefore$ TH is a mapping from U into W

Now Let $a, b \in F$, $\alpha, \beta \in U$. Then $(TH)[a\alpha + b\beta] = T[H(a\alpha + b\beta)]$ (by def.)

(H is L.T.)

$= T[aH(\alpha) + bH(\beta)]$

$= a(TH)(\alpha) + b(TH)(\beta)$

$\therefore$ TH is a LT. from U to W.

Note. The range of H is the domain of T.

3.13. Theorem. *Let H, H' be two linear transformations from U(F) to V(F). Let T, T' be the linear transformations from V(F) to W(F) and $a \in F$.*

Then (*i*) $T(H+H') = TH + TH'$ (*ii*) $(T + T')H = TH + T'H$

(*iii*) $a(TH) = (aT)H = T(aH)$

Proof. (*i*) Let $\alpha \in U$. Then $T(H+H')(\alpha) = T[H(\alpha) + H'(\alpha)]$

$= TH(\alpha) + TH'(\alpha) = (TH + TH')(\alpha)$

(*ii*) Similar to (*i*)

(*iii*) Let at $\alpha \in U$. $a(TH)(\alpha) = aT[H(\alpha)] = [(aT)H](\alpha)$

Again $[T(aH)](\alpha) = T[aH(\alpha)] = [a(TH)](\alpha)$. $\therefore$ $a(TH)] = (aT)H = T(aH)$

3.14. ALGEBRA OF LINEAR OPERATORS

Theorem. *Let A, B, C be linear operators on a vector space V(F). Also let O be the zero operator and I the identity operator on V. Then*

(*i*) $AO = OA = O$ (*ii*) $AI = IA = A$

(*iii*) $A(B+C) = AB + AC$ (*iv*) $(A + B)C = AC + BC$

(*v*) $A(BC) = (AB)C$

Proof. Let $\alpha \in V$

(*i*) $AO(\alpha) = A[O(\alpha)] = A(\bar{O})$ (by def. of 0)

$= \bar{O}$ (A is L.T.) $= O(\alpha)$ $(\forall\ \alpha \in U)$

Similarly $OA(\alpha) = O[A(\alpha) = \bar{O} = O(\alpha) \Rightarrow OA = O$. Thus $AO = OA = O$.

(*ii*) Similar to (*i*)

(*iii*) $[A(B+C)](\alpha) = A[(B+C)(\alpha)] = A[B(\alpha) + C(\alpha)] = AB(\alpha) + AC(\alpha) = (AB + AC)(\alpha)$

$\therefore$ $A(B+C) = AB + AC$

(*iv*) Similar to (*iii*)

(*v*) $[A(BC)](\alpha) = A[(BC)(\alpha)] = A[B\{C(\alpha)\}] = (AB)[C(\alpha)] = [(AB)C](\alpha)$

$\therefore$ $A(BC) = (AB)C$

SOLVED PROBLEMS

Ex. 1. *Let* $T: R^3 \to R^2$ *and* $H: R^3 \to R^2$ *be defined by* $T(x, y, z) = (3x, y+z)$ *and* $H(x, y, z) = (2x - x, y)$. *Compute* (*i*) $T + H$ (*ii*) $4T - 5H$ (*iii*) TH (*iv*) HT

Sol. Since T and H map V, the linear transformations T + H and $4T - 5H$ are defined.

(*i*) $(T+H)(x, y, z) = T(x, y, z) + H(x, y, z) = (3x, y+z) + (2x - z, y) = (5x - z, 2y + z)$

(*ii*) $(4T - 5H)(x, y, z) = 4T(x, y, z) - 5H(x, y, z)$

$= 4(3x, y+z) - 5(2x - z, y) = (2x + 5z, -y + 4z)$

(*iii*) and (iv) both TH and HT are not defined because the range of T is not equal to the domain of H and vice versa.

Ex. 2. *Let* $T_1: R^3 \to R^2$ *and* $T_2: R^2 \to R^2$ *are two linear transformations defined by* $T_1(x, y, z) = (3x, 4y - z)$, $T_2(x, y) = (-x, y)$. *Compute* T_1T_2 *and* T_2T_1.

Sol. (*i*) Since the range of T_2 *i.e.* R^2 is not equal to the domain of T_1 *i.e.*, R^3, T_1T_2 is not defined.

(*ii*) But the range of T_1 *i.e.* R^3 is equal to the domain of T_2, T_2T_1 is defined.

$\therefore$ $(T_2T_1)(x, y, z) = T_2[T_1(x, y, z)] = T_2(3x, 4y - z) = (-3x, 4y - z)$

Ex. 3. *Let P(R) be the vector space of all polynomials in x and D, T be two linear operators on P defined by* $D[f(x)] = \frac{df}{dx}$ *and* $T[f(x)] = x\, f(x)\ \forall\ f(x) \in V$

Show (*i*) $TD \neq DT$ (*ii*) $(TD)^2 = T^2D^2 + TD$

Sol. (*i*) $(DT)\, f(x) = D[T\, f(x)] = D[x\, f(x)] = f(x) + x\, f'(x)$

$(TD)\, f(x) = T[D\, f(x)] = T\left[\frac{df}{dx}\right] = x\, f'(x)$

Clearly $DT \neq TD$. Also $(DT)\, f(x) - (TD)\, f(x) = f(x)$

$\Rightarrow (DT - TD)\, f(x) = I\, f(x)$ (I is identity) $\Rightarrow DT - TD = I$

(*ii*) $(TD)^2\, f(x) = (TD)\ [(TD)\, f(x)] = (TD)\left[x\frac{df}{dx}\right] = T\left[D\left(x\frac{df}{dx}\right)\right]$

$$= T\left[\frac{df}{dx} + x\frac{d^2f}{dx^2}\right] = x\frac{df}{dx} + x^2\frac{d^2f}{dx^2}$$

Now $(T^2D^2)\, f(x) = T^2D\,\{Df(x)\} = T^2\left[D\left(\frac{df}{dx}\right)\right] = T^2\left(\frac{d^2f}{dx^2}\right)$

$$= T\left[T\left(\frac{d^2f}{dx^2}\right)\right] = T\left[x\frac{d^2f}{dx^2}\right] = x^2\frac{d^2f}{dx^2}$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\, T_{ij}(\alpha_k)=\hat{O} \quad \Rightarrow \sum_{j=1}^{m} a_{kj}\, T_{kj}(\alpha_k)=\hat{O} \qquad (\hat{O}\in V)$$

$$\Rightarrow a_{k1}\,\alpha_1 + a_{k2}\,\alpha_2+\ldots+a_{km}\,\alpha_m=\hat{O} \Rightarrow a_{k1}=0, a_{k2}=0\ldots a_{km}=0 \qquad (\because B_1 \text{ is L.I.})$$

Hence $S=\{T_{ij}\}$ is an L.I. set.

(ii) *To show that* L (S) = L (U,V).

Let $T\in L(U,V)$. The vector $T(\alpha_1)\in V$ can be expressed as

$T(\alpha_1)=b_{11}\beta_1+b_{21}\beta_2+\ldots+b_{m1}\beta_m$.

In general for $i=1,2\ldots m$

$$T(\alpha_i)=b_{i1}\,\beta_1+b_{2i}\beta_2+\ldots+b_{mi}\,\beta_m \qquad \ldots (1)$$

Consider the linear transformation $H=\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}\, T_{ij}$

Clearly H is a linear combination of $S=\{T_{ij}\}$; therefore $H\in L(U,V)$.

Let $\alpha_k\in U$ for $k=1,2\ldots n$. Since $T_{ij}(\alpha_k)=\hat{O}$ for $k\neq i$, We have $T_{kj}(\alpha_k)=\beta_j$

Consider $H(\alpha_k)=\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}\, T_{ij}(\alpha_k)=\sum_{j=1}^{m} b_{kj}\, T_{kj}(\alpha_k) = \sum_{j=1}^{m} b_{kj}\,\beta_j$

i.e. $H(\alpha_k)=b_{k1}\,\beta_1+b_{k2}\,\beta_2+\ldots+b_{km}\,\beta_m=T(\alpha_k)$ [by (1)]

Hence $H(\alpha_k)=T(\alpha_k)$ for each *k*. $\therefore$ H = T

Thus T is a linear combination of elements of S. *i.e.,* L (S) = L (U, V).

$\therefore$ S is a basis set of L (U, V) $\therefore$ dim L(U, V) = *mn*.

EXERCISE 3 (*b*)

1. Let $T:R^3\to R^2$ and $H:R^2\to R^3$ be two linear transformation defined by $T(x,y,z)=(x-3y-2z,\ y-4z)$ and $H(x,y)=(2x, 4x-y, 2x+3y)$ Find HT and TH. Is product commutative ?
2. Define on R^2 linear operators H and T as follows $H(x,y)=(0,x)$ and $T(x,y)=(x,0)$ and show that $TH=0$, $HT\neq TH$ and $T^2=T$.
3. Give an example of a linear operator T on R^3 such that $T\neq O, T^2\neq O$ but $T^2=O$. **[Hint:** $T:R^3\to R^3$ where $T(x,y,z)=(0,x,y)$]

4. Let P be the polynomial space in one indeterminate x with real coefficients. Let $D:P\to P$ and $S:P\to P$ be two linear operators defined by .

   $D\,f(x)=\dfrac{df}{dx}$ and $S\,f(x)=\int_0^x f(x)\,dx \ \ \forall\, f(x)\in P$

   Show that DS = I and $SD\neq I$ where I is the identity transformation.
5. Let $T:V_3(R)\to V_3(R)$ be defined by $T(a,b,c)=(3a, a-b, 2a+b+c)$

   Prove that $(T^2-I)(T-3I)=O$.

3.17. RANGE AND NULL SPACE OF A LINEAR TRANSFORMATION

RANGE. Definition. *Let U (F) and V (F) be two vector spaces and let $T:U\to V$ be a linear transformation. The range of T is defined to be the set Range $(T)=R(T)=\{T(\alpha):\alpha\in U\}$.*

Obviously the range of T is a subset of V. *i.e.* $R(T)\subseteq V$.

3.18. Theorem. ***Let U(F) and V(F) be two vector spaces. Let $T:U(F)\to V(F)$ be a linear transformation. Then the range set R (T) is a subspace of V (F).***

(S. V. U. O 2000, S. K. U. S01, O. U. S01)

Proof. For $\bar{O}\in U \Rightarrow T(\bar{O})=O\in R(T)$ $\therefore$ R (T) is non-empty set and $R(T)\subseteq V$

Let $\alpha_1,\alpha_2\in U$ and $\beta_1,\beta_2\in R(T)$ be such that $T(\alpha_1)=\beta_1$ and $T(\alpha_2)=\beta_2$

For $a,b\in F$, $a\alpha_1+b\alpha_2\in U$ $(\because$ U is V.S.) $\Rightarrow T(a\alpha_1+b\alpha_2)\in R(T)$

But $T(a\alpha_1+b\alpha_2)=aT(\alpha_1)+bT(\alpha_2)=a\,\beta_1+b\,\beta_2\in R(T)$ $(\because$ T is L.T.)

Thus $a,b\in F$ and $\beta_1,\beta_2\in R(T) \Rightarrow a\,\beta_1+b\,\beta_2\in R(T)$

$\therefore$ R (T) is a subspace of V(F). R (T) is called the range space.

3.19. NULL SPACE OR KERNEL

Definition. *Let U (F) and V (F) be two vector spaces and $T:U\to V$ be a linear transformation. The null space denoted by N (T) is the set of all vectors $\alpha\in U$ such that $T(\alpha)=\bar{O}$ (zero vector of V).*

The null space of N (T) is also called the kernel of T *i.e.,* $N(T)=\{\alpha\in U:T(\alpha)=\bar{O}\in V\}$.

Obviously the null space $N(T)\subseteq U$.

3.20. Theorem. ***Let U (F) and V (F) be two vector spaces and $T:U\to V$ is a linear transformation. Then null space N (T) is a subspace of U (F).***

(S. V. U. O2000, S. K. U. 2001/O, O. U. 2001/O)

Proof. Let $N(T)=\{\alpha\in U:T(\alpha)=\bar{O}\in V\}$

$\because\ T(\bar{o}) = \hat{O} \Rightarrow \hat{O} \in N(T)$ $(\bar{O} \in U, \hat{O} \in V)$

$\therefore$ N(T) is a non-empty subset of U.

Now $\alpha, \beta \in N(T) \Rightarrow T(\alpha) = \hat{O}, T(\beta) = \hat{O}$

For $a, b \in F$. $T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta) = a.\hat{O} + b.\hat{O} = \hat{O}$ (T is L.T.)

$\therefore\ T(a\alpha + b\beta) = \hat{O}$

By definition $a\alpha + b\beta \in N(T)$

Thus $a, b \in F$ and $\alpha, \beta \in N(T) \Rightarrow a\alpha + b\beta \in N(T)$

$\therefore$ Null space N (T) is a subspace of U (F).

3.21. Theorem. ***Let $T: U(F) \to V(F)$ be a linear transformation. If U is finite dimensional then the range space (T) is a finite dimensional subspace of V (F).***

Proof. Given U is finite dimensional

$\therefore$ Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be the basis set of U (F).

Let $\beta \in R(T)$, the range space of T.

Then there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

$\therefore\ \alpha = a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n$ for $a's \in F$

$\Rightarrow T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n) \Rightarrow \beta = a_1T(\alpha_1) + a_2T(\alpha_2) \dots + a_nT(\alpha_n)$

But $S' = \{T(\alpha_1), T(\alpha_2) \dots T(\alpha_n)\} \in R(T)$

Now $\beta \in R(T)$ and $\beta = l.c$ of elements of $S' \Rightarrow \beta \in L(S')$.

Thus R(T) is spanned by a finite set S'.

$\therefore$ R(T) is finite dimensionalsubspace of V(F).

DIMENSION OF RANGE AND KERNEL

3.22. Definition. Let $T: U(F) \to V(F)$ be *a linear transformation where U is finite dimensional vector space.*

Rank : *Then the rank of T denoted by* $\rho(T)$ *is the dimension of range space R(T).*

$$\rho(T) = \dim R(T)$$

Nullity : *The nullity of T denoted by* $\nu(T)$ *is the dimension of null space N (T).*

$$\nu(T) = \dim N(T)$$

3.23. Theorem. ***Let U(F) and V(F) be two vector spaces and $T: U \to V$ be a linear transformation. Let U be finite dimensional then $\rho(T) + \nu(T) = \dim U$***

i.e. rank (T) + nullity (T) = dim U. (N. U. 94 M, S. V. U. A2000, S. K. U.2001/O)

Proof. The null space N (T) is a subspace of finite dimensional space U (F).

$\Rightarrow$ N (T) is finite dimensional

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_k\}$ be a basis set of N (T). $\therefore$ dim N(T) = ν (T) = k.

$\therefore\ T(\alpha_1) = \hat{O}, T(\alpha_2) = \hat{O} \dots T(\alpha_k) = \hat{O}$ ... (1) $(\hat{O} \in V)$

As S is L.I. it can be extended to form a basis of U (F)

Let $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_k, \theta_1, \theta_2 \dots \theta_m\}$ be the extended basis of U (F). $\therefore$ $\dim U = k + m$

Now we show that the set of images of additional vectors

$S_2 = \{T(\theta_1), T(\theta_2) \dots T(\theta_m)\}$ is a basis of R (T). Clearly $S_2 \subseteq R(T)$

(i) To prove S_2 is L. I.

Let $a_1, a_2 \dots a_m \in F$ be such that, $a_1T(\theta_1) + a_2T(\theta_2) + \dots + a_mT(\theta_m) = \hat{O}$

$\Rightarrow T[a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m] = \hat{O}$

$\Rightarrow a_1\theta_1 + a_2\theta_2 \dots + a_m\theta_m \in N(T)$, null space of T.

But each vector in N(T) is a *l.c.* of elements of basis S

$\therefore$ For some $b_1, b_2 \dots b_m \in F$, let $a_1\theta_1 + a_2\theta_2 \dots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \dots b_k\alpha_k$

$\Rightarrow a_1\theta_1 + \dots + a_m\theta_m - b_1\alpha_1 - \dots - b_k\alpha_k = \hat{O}$

$\Rightarrow a_1 = 0, a_2 = 0 \dots a_m = 0, b_1 = 0 \dots b_k = 0$ ($\because$ S_1 is L.I.) $\Rightarrow$ S_2 is L.I. set

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space R (T), then there exists $\alpha \in U$ such that $T(\alpha) = \beta$

Now $\alpha \in U \Rightarrow$ there exist $c's, d's \in F$ such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 \dots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 \dots + d_m\theta_m$$

$$\Rightarrow T(\alpha) = T[c_1\alpha_1 + \dots + c_k\alpha_k + d_1\theta_1 + \dots + d_m\theta_m]$$

$$= c_1T(\alpha_1) + \dots + c_kT(\alpha_k) + d_1T(\theta_1) + \dots + d_mT(\theta_m)$$

$\Rightarrow \beta = d_1T(\theta_1) + d_2T(\theta_2) \dots + d_mT(\theta_m)$ (by (1)) $\Rightarrow \beta \in L(S_2)$

$\therefore$ S_2 is a basis of R(T) and dim R(T) = m.

Thus dim R(T) + dim N(T) = $m + k$ = dim U *i.e.,* $\rho(T) + \nu(T) = \dim U$.

SOLVED PROBLEMS

Ex. 1. *If $T: V_4(R) \to V_3(R)$ is a linear transformation defined by $T(a,b,c,d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d)$ for $a, b, c, d \in R$, then verify $\rho(T) + \nu(T) = \dim V_4(R)$* **(O. U. M06)**

Sol. Let $S=\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ be the basis set of $V_4(R)$.

$\therefore$ The transformation T on B will be $T(1,0,0,0)=(1,1,1)$, $T(0,1,0,0)=(-1,0,1)$

$T(0,0,1,0)=(1,2,3)$, $T(0,0,0,1)=(1,-1,-3)$

Let $S_1=\{(1,1,1),(-1,0,1),(1,2,3),(1,-1,-3)\}$ $\therefore S_1\subseteq R(T)$

Now we verify whether S_1 is L.I. or not,. If not, we find the least L.I. set by forming

the matrix, $S_1=\begin{bmatrix}1&1&1\\-1&0&1\\1&2&3\\1&-1&-3\end{bmatrix}$ applying $R_2+R_1, R_3-R_1, R_4-R_1$

$S_1\sim\begin{bmatrix}1&1&1\\0&1&2\\0&1&2\\0&-2&-4\end{bmatrix}$ Again apply R_4+2R_3, R_3-R_2, $S_1\sim\begin{bmatrix}1&1&1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$

$\therefore$ The non - zero rows of vectors $\{(1,1,1),(0,1,2)\}$

Constitute the L.I. set forming the basis of R (T) $\Rightarrow$ $\dim R(T)=2$

Basis for null space of T. $\alpha\in N(T)\Rightarrow T(\alpha)=\hat{O}$

$\therefore T(a,b,c,d)=\hat{O}$ where $\hat{O}=(0,0,0)\in V_3(R)$

$\Rightarrow (a-b+c+d, a+2c-d, a+b+3c-3d)=(0,0,0)$

$\Rightarrow a-b+c+d=0$, $a+2c-d=0$

$a+b+3c-3d=0$, we have to solve these for a, b, c, d.

Coefficient matrix $=\begin{bmatrix}1&-1&1&1\\1&0&2&-1\\1&1&3&-3\end{bmatrix}$, by $R_2-R_1, R_3-R_1=\begin{bmatrix}1&-1&1&1\\0&1&1&-2\\0&2&2&-4\end{bmatrix}$

by R_3-2R_2, the echelon form is $\begin{bmatrix}1&-1&1&1\\0&1&1&-2\\0&0&0&0\end{bmatrix}$

$\therefore$ The equivalent system of equations are

$a-b+c+d=0, b+c-2d=0$ $\Rightarrow b=2d-c, a=d-2c$

The number of free variables is 2 namely c, d and the values of a, b depend on these and hence nullity (T) = dim N(T) = 2.

Choosing $c=1, d=0$, we get $a=-2, b=-1$, $\therefore (a,b,c,d)=(-2,-1,1,0)$

Choosing $c=0, d=1$, we get $a=1$, $\therefore (a,b,c,d)=(1,2,0,1)$

$\therefore \{(-2,-1,1,0),(1,2,0,1)\}$ constitute a basis of N(T)

$\therefore \dim R(T)+\dim N(T)=2+2=4=\dim V_4(R)$.

Ex. 2. *Find a linear transformation* $T:R^3\to R^3$ *whose range is spanned by* $(1,2,0,-4)$, $(2,0,-1,-3)$.

SoL Given R (T) is spanned by $\{(1,2,0,-4),(2,0,-1,-3)\}$

Let us include a vector (0, 0, 0) in this set which will not effect the spanning property so that

$S=\{(1,2,0,-4),(2,0,-1,-3),(0,0,0,0)\}$

Let $B=\{\alpha_1,\alpha_2,\alpha_3\}$ be the basis of R^3. We know there exists a transformation T such that

$T(\alpha_1)=(1,2,0,-4)$; $T(\alpha_2)=(2,0,-1,-3)$ and $T(\alpha_3)=(0,0,0,0)$

Now if $\alpha\in R^3\Rightarrow\alpha=(a,b,c)=a\,\alpha_1+b\,\alpha_2+c\,\alpha_3$

$\therefore T(a,b,c) = T(a\,\alpha_1+b\,\alpha_2+c\,\alpha_3)=a\,T(\alpha_1)+b\,T(\alpha_2)+c\,T(\alpha_3)$

$=a(1,2,0,-4)+b(2,0,-1,-3)+c(0,0,0,0)$

$\therefore T(a,b,c)=(a+2b, 2a, -b, -4a-3b)$ is the required transformation.

Ex. 3: *Find* $F:R^3\to R^4$ *is a linear transformation whose range is spanned by* $(1,-1,2,3)$ *and* $(2,3,-1,0)$.

Sol. Consider the standard basis for R^3 e_1,e_2,e_3 where $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$. Then $F(e_1)=(1,-1,2,3)$, $F(e_2)=(2,3,-1,0)$ and $F(e_3)=(0,0,0,0)$.

We know that, $(x,y,z)=xe_1+ye_2+ze_3$

$\therefore F(x,y,z)=F(xe_1+ye_2+ze_3)=xF(e_1)+yF(e_2)+zF(e_3)$ ($\because$ F is a linear transformation)

$=(x,-x,2x,3x)+(2y,3y,-y,0)+(0,0,0,0)$ $=(x+2y,-x+3y,2x-y,3x)$

Ex. 4. *Let V be a vector space of* 2×2 *matrices over reals. Let P be a fixed matrix of V ;* $P=\begin{bmatrix}1&-1\\-2&2\end{bmatrix}$ *and* $T:V\to V$ *be a linear operator defined by* $T(A)=PA$, $A\in V$. *Find the nullity T.*

Sol. Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in V$

The null space N (T) is the set of all 2×2 matrices whose T - image is $\hat{O}$

$\Rightarrow \; T(A) = PA = \hat{O} \;\Rightarrow\; T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a-c=0, b-d=0 \Rightarrow a=c\,, b=d\,,$

The free variables are c and d. Hence dim N(T) = 2.

Ex. 5. *Find the null space, range, rank and nullity of the transformation* $T : R^2 \to R^3$ *defined by* $T(x, y) = (x+y, x-y, y)$. **(S. V. U. S97, S. V. U. O2000)**

Sol. (*i*) Let $\alpha = (x, y) \in R^2$. Then $N(T) \Rightarrow T(a) = \hat{O}$ $\qquad (\hat{O} \in R^3)$

i.e., $T(x, y) = (0,0,0) \Rightarrow (x+y, x-y, y) = (0,0,0)$

$\Rightarrow x+y=0, x-y=0, y=0 \Rightarrow x=0, y=0 \qquad \therefore \alpha = (0,0) = (\hat{O} \in R^2)$

Thus the null space of T consists of only zero vector of R^2

$\therefore$ nullity T = dim N(T) = 0

(*ii*) Range space of $T = \{\beta \in R^2 : T(\alpha) = \beta$ for $\alpha \in R^2\}$

$\therefore$ The range space consists of all vectors of the type $(x+y, x-y, y)$ for all $(x, y) \in R^2$

(*iii*) dim R(T) + dim N(T) = dim R^2 $\Rightarrow$ dimR(T) + 0 = 2 $\Rightarrow$ rank of T = 2.

Ex. 6. *Let V (F) be a vector space and T be a linear operator on V. Prove that the following statements are true*

(*i*)*The intersection of the range of T and null space of T is the zero subspace of T.*

i.e., $R(T) \cap N(T) = \{\bar{O}\}$ $\qquad$ (*ii*) *If* $T[T(\alpha)] = \bar{O}$, *then* $T(\alpha) = \bar{O}$

Sol. (*i*) $\Rightarrow$ (*ii*)

Let $R(T) \cap N(T) = \{\bar{O}\}$. $\quad$ Let $T(\alpha) = \beta$ $\quad \therefore \beta \in R(T)$ $\qquad$ ... (1)

Now $T[T(\alpha)] = \bar{O} \Rightarrow T(\beta) = \bar{O} \Rightarrow \beta \in N(T)$ $\qquad$ ... (2)

From(1) and (2) $\beta \in R(T) \cap N(T)$. But $R(T) \cap N(T) = \{\bar{O}\} \Rightarrow \beta = \bar{O} \quad \Rightarrow T(\alpha) = \bar{O}$

Thus $T[T(\alpha)] = \bar{O} \Rightarrow T(\alpha) = \bar{O}$

(*ii*) $\Rightarrow$ (*i*) Given $T[T(\alpha)] = \bar{O} \Rightarrow T(\alpha) = \bar{O}$

Let $\beta \in R(T) \cap N(T)$. $\qquad \therefore \beta \in R(T)$ and $\beta \in N(T)$

Now $\beta \in R(T) \Rightarrow T(\alpha) = \beta$ for some $\alpha \in V$ and $\beta \in N(T) \Rightarrow T(\beta) = \bar{O} \Rightarrow T[T(\alpha)] = \bar{O}$

$\Rightarrow T(\alpha) = \bar{O} \Rightarrow \beta = \bar{O}$. $\qquad$ Thus $R(T) \cap N(T) = \bar{O}$

Ex. 7. *Verify the Rank-Nullity theorem for the linear map* $T : V_4 \to V_3$ *defined by* $T(e_1) = f_1 + f_2 + f_3$, $T(e_2) = f_1 - f_2 + f_3$, $T(e_3) = f_1$, $T(e_4) = f_1 + f_3$ *when* $\{e_1, e_2, e_3\}$ *and* $\{f_1, f_2, f_3\}$ *are standard basis* V_4 *and* V_3 *respectively* **(S. V. U. 2001/O)**

Sol.:Let $e_1 = (1,0,0,0), e_2 = (0,1,0,0)$ $e_3 = (0,0,1,0), e_4 = (0,0,0,1)$ and

$f_1 = (1,0,0),\ f_2 = (0,1,0),\ f_3 = (0,0,1)$

$\{e_1, e_2, e_3, e_4\}, \{f_1, f_2, f_3\}$ are the standard basis of V_4 and V_3 respectively.

we have $T(e_1) = f_1 + f_2 + f_3 \Rightarrow T(1,0,0,0) = (1,0,0) + (0,1,0) + (0,0,1) = (1,1,1)$

$T(e_2) = f_1 - f_2 + f_3 \Rightarrow T(0,1,0,0) = (1,0,0) - (0,1,0) + (0,0,1) = (1,-1,1)$

$T(e_3) = f_1 \Rightarrow T(0,0,1,0) = (1,0,0)$; $\quad T(e_4) = f_1 + f_3 \Rightarrow T(0,0,0,1) = (1,0,0) + (0,0,1)$

Let $\alpha \in V_4$. The α can be written as $\alpha = ae_1 + be_2 + ce_3 + de_4$

Then $T(\alpha) = a\,T(e_1) + b\,T(e_2) + c\,T(e_3) + d\,T(e_4)$

$= a(1,1,1) + b(1,-1,1) + c(1,0,0) + d(1,0,1) = (a+b+c+d, a-b, a+b+d)$

Consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_4 = R_4 - R_1 \\ R_3 = R_3 - R_1 \\ R_2 = R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$R_4 = R_4 + R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_1 + R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 - R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then $\{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ is a basis set for R(T).

Thus dim R(T) = 3. Suppose $T(\alpha) = \bar{0} \Rightarrow (a+b+c+d, a-b, a+b+d) = (0,0,0)$

Which gives $a+b+c+d=0, a-b=0, a+b+d=0$

From this we have $c=0, b=a$ and $d=-2a$

Thus $[a,b,c,d] = [1,1,0,-2]$

$\therefore$ Rank of the null space N(T) = 1.

$\therefore$ Rank of T = dim R(T) = 3

Nullity = dim N(T) = 1.

Dim $V_4 = 4$

Thus Rank + Nullity = Dimension, is verified

EXERCISE 3 (c)

1. Let $T: V_4 \to V_3$ be a linear transformation defined by $T(\alpha_1) = (1,1,1)$; $T(\alpha_2) = (1,-1,1)$; $T(\alpha_3) = (1,0,0)$, $T(\alpha_4) = (1,0,1)$. Then verify that $\rho(T) + \nu(T) = \dim V_4$

2. Describe explicitly the linear transformation $T: R^3 \to R^3$ whose range space is spanned by $\{(1,0,-1), (1,2,2)\}$.

3. Let $T: R^3 \to R^2$ be a linear transformation defined by $T(x, y, z) = (x+y, y+z)$
Find a basis, dimension of each of the range and null space of T.

4. Let $T: R^3 \to R^3$ be the linear mapping defined by
$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$
Find the rank, nullity and find a basis for each of the range and null space of T.

5. Let $T: R^3 \to R^3$ be defined by $T(x, y, z) = (x-y+2z, 2x+y-z, -x-2y)$
Find the null space of T.

6. $T: V_3(R) \to V_2(R)$ is defined by $T(a,b,c) = (a,b)\ \forall\ (a,b,c) \in R^3$
Prove that T is a linear transformation. Find the kernel of T.

7. Let V (F) be an n dimensional vector space and let T be a L.T. from V into V such that range and null space of T are identical. Prove that n is even.

4

Vector Space Isomorphism

4.1. Definition. *Let U(F) and V(F) be two vector spaces. The one-one onto transformation* $T: U \to V$ *is called the isomorphism and is denoted by* $U(F) \cong V(F)$.

Now we prove some more properties of vector space isomorphism.

4.2. Theorem. ***Two finite dimensional vector spaces U and V over the same field F are isomorphic if and only if they have the same dimension.*** **(S. K. U. M 07)**

i.e. $U(F) \cong V(F) \Leftrightarrow \dim U = \dim V$

Proof. Let U(F) and V(F) be finite dimensional and $U(F) \cong V(F)$. Then there exists an one-one onto transformation $T: U \to V$

To prove that $\dim U = \dim V$.

Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of U. $\quad \therefore \dim U = n$

Let $S' = \{T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)\}$ be the set of T-images of S $\Rightarrow S' \subseteq V$

(*i*) To show that S' is L.I.

Consider the equation $a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_n T(\alpha_n) = \hat{O}$ $\qquad a's \in F \quad \hat{O} \in V$

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n] = T(\bar{O})$ $\qquad$ (T is L.T. ; $\bar{O} \in U$)

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n = \bar{O}$ $\qquad$ ($\because$ T is one-one)

$\Rightarrow a_1 = 0, a_2 = 0, \ldots a_n = 0$ $\qquad$ ($\because$ S is L.I.)

$\Rightarrow$ S' is L.I. set.

(*ii*) To show that L (S') = V

Let $\beta \in V$. Since T is onto there exists; $\alpha \in U$ such that $T(\alpha) = \beta$

But $\alpha \in U \Rightarrow \alpha = l.c$ of elements of basis

$= b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n$ $\qquad$ ($b's \in F$)

$\Rightarrow T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n)$

$\Rightarrow \beta = b_1 T(\alpha_1) + b_2 T(\alpha_2) \ldots + b_n T(\alpha_n)$ $\qquad$ (T is L.T.)

$\Rightarrow$ = *l.c.* of elements of S' $\Rightarrow \beta \in L(S')$

$\therefore$ L(S') = V. As S' is also L.I. ; S' forms a basis of V. $\quad \therefore \dim V = n = \dim U$

Converse. Let dim U = dim V. To prove $U \cong V$.

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and $S' = \{\beta_1, \beta_2 \dots \beta_n\}$ be the bei of U and V respectively so that dim U = n = dim V.

$\therefore \alpha \in U \Rightarrow \alpha = c_1\alpha_1 + c_2\alpha_2 \dots + c_n\alpha_n$ for some $c's \in F$

Now define $T' : U \to V$ such that $T'(\alpha) = c_1\beta_1 + c_2\beta_2 \dots + c_n\beta_n = \sum c_i\beta_i$

(a) To show T' is one - one

Let $\theta \in U$ such that, $\theta = d_1\alpha_1 + \dots + d_n\alpha_n$ $\quad d's \in F$

$\Rightarrow T'(\theta) = d_1\beta_1 + \dots + d_n\beta_n$ (by def.)

Now $T'(\alpha) = T'(\theta) \Rightarrow c_1\beta_1 + c_2\beta_2 \dots + c_n\beta_n = d_1\beta_1 + d_2\beta_2 \dots + d_n\beta_n$

$\Rightarrow (c_1 - d_1)\beta_1 + \dots + (c_n - d_n)\beta_n = \hat{O}$ $\quad (\hat{O} \in V)$

$\Rightarrow c_1 - d_1 = 0, \dots c_n - d_n = 0$ (S' is L.I.)

$\Rightarrow c_1 = d_1, \dots c_n = d_n$

$\therefore c_1\alpha_1 + c_2\alpha_2 \dots + c_n\alpha_n = d_1\alpha_1 + d_2\alpha_2 \dots + d_n\alpha_n$

$\Rightarrow \alpha = \theta$ $\quad \therefore$ T' is one-one.

(b) To show T' is onto

For $\delta \in V$, we can express $\delta = c_1\beta_1 + c_2\beta_2 \dots + c_n\beta_n$ for $c's \in F$

If $\gamma = e_1\alpha_1 + e_2\alpha_2 \dots + e_n\alpha_n$ by definition $T'(\gamma) = \delta$

$\therefore$ For $\delta \in V$ there exists $\gamma \in U$ such that $T'(\gamma) = \delta$ $\quad \therefore$ T' is onto.

(c) To show that T' is L. T.

Let $a, b \in F$ and $\alpha, \theta \in U$, then $T'(a\alpha + b\theta) = T'[a \Sigma c_i\alpha_i + b \Sigma d_i\alpha_i]$

$= T'[\Sigma (a c_i + b \Sigma d_i)\alpha_i] = \Sigma (a c_i + b d_i)\beta_i$ (by def.)

$= a \Sigma c_i\beta_i + b \Sigma d_i\beta_i = a T'(\alpha) + b T'(\theta)$

$\therefore$ T' is a linear transformation.

Thus $T' : U \to V$ is one-one onto linear transformation.

Hence T' is an isomorphism $\quad \therefore U \cong V$

Corollary. ***The image set of a basis set under an isomorphism is a basis set.***

The proof of this statement is the first part of the above proof.

4.3. FUNDAMENTAL THEOREM OF HOMOMORPHISM

Theorem. ***Let U(F) and V(F) be two vector spaces and $T : U \to V$ is a onto linear transformation. N is the null space of T. Then $U/N \cong V$***

Proof. N is the null space of T $\Rightarrow N \subseteq U$

$\therefore$ The quotient space, $U/N = \{N + \alpha : \alpha \in U\}$ is the set of all cosets of N in U.

We know that $(N + \alpha_1) + (N + \alpha_2) = N + (\alpha_1 + \alpha_2)$ $\forall \alpha_i \in U$, $a(N + \alpha) = N + a\alpha$ $\forall a \in F$

Let a function $f : U/N \to V$ be defined such that $f(N + \alpha) = T(\alpha)$ $\forall \alpha \in U$

Now to prove f is an homomorphism.

Let $\alpha_1, \alpha_2 \in U$ and $a, b \in F$, then $f[a(N + \alpha_1) + b(N + \alpha_2)] = f[(N + a\alpha_1) + (N + b\alpha_2)]$

$= f[(N + (a\alpha_1 + b\alpha_2)] = T(a\alpha_1 + b\alpha_2)$

$= a\,T(\alpha_1) + b\,T(\alpha_2) = af(N + \alpha_1) + b(N + \alpha_2)$

$\therefore f$ is a homomorphism.

(i) To prove f is one-one

$f(N + \alpha_1) = f(N + \alpha_2) \Rightarrow T(\alpha_1) = T(\alpha_2) \Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{0}$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{0} \Rightarrow \alpha_1 - \alpha_2 \in N \Rightarrow N + \alpha_1 = N + \alpha_2$

(ii) To prove f is onto

Since $T : U \to V$ is onto for every $\beta \in V$, there exists some $\alpha \in U$ such that $T(\alpha) = \beta$

$\therefore$ For this $\alpha \in U$, $(N + \alpha) \in U/N$. Hence $f(N + \alpha) = T(\alpha) = \beta$.

$\therefore$ For all $\beta \in V$, there exists $N + \alpha \in U/N$, so that $f(N + \alpha) = \beta$

$\Rightarrow f$ is onto, thus f is one-one homomorphism

$\therefore f$ is an isomorphism from U/N to V *i.e.* $U/N \cong V$.

Note. In the above theorem, if T is not given as onto then the statement will be as :

Let U(F) and V(F) be two vector spaces and $T : U \to V$ be a linear transformation.

Then $\frac{U}{N} \cong T(U)$ where N is the null space of T.

4.4. Theorem. ***Every n-dimensional vector space V(F) is isomorphic to F^n.***

(S. K. U. M 07)

Proof. Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be the basis of the n-dimensional vector space V(F)

For $\alpha \in V$ there exist $a_1, a_2 \dots a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 \dots + a_n\alpha_n$

Let a mapping $T : U \to F^n$ be defined by $T(\alpha) = (\alpha_1, \alpha_2 \dots \alpha_n)$

i.e., then the T-image of α is the *n*-tuple of the coordinates of α

(i) To show that T is one-one

For $\alpha, \beta \in V$, $a's, b's \in F$

Let $\alpha = a_1\alpha_1 + a_2\alpha_2 \dots + a_n\alpha_n = \Sigma a_i\alpha_i$; $\quad \beta = b_1\alpha_1 + b_2\alpha_2 \dots + b_n\alpha_n = \Sigma b_i\alpha_i$

Now $T(\alpha) = T(\beta) \Rightarrow (a_1, a_2 \dots a_n) = (b_1, b_2 \dots b_n)$ $\therefore$ T is one-one

$\Rightarrow a_1 = b_1, a_2 = b_2, \dots a_n = b_n \Rightarrow \Sigma a_i\alpha_i = \Sigma b_i\alpha_i \Rightarrow \alpha = \beta$.

(ii) To show that T is onto

For given any $(c_1, c_2 \dots c_n) \in F^n$ there exists $\gamma \in V$ such that $\gamma = (c_1\alpha_1 + c_2\alpha_2 \dots + c_n\alpha_n)$

$\therefore T(\gamma) = (c_1, c_2 \dots c_n)$ (by def.) $\therefore$ T is onto

(iii) To show that T is linear

Let $a, b \in F$ and $\alpha, \beta \in V$

$T(a\alpha + b\beta) = [a\,\Sigma\, a_i\alpha_i + b\,\Sigma\, b_i\alpha_i] = T[\Sigma(aa_i + bb_i)\alpha_i]$

$= (aa_1 + bb_1, aa_2 + bb_2, \dots aa_n + bb_n)$

$= a(a_1, a_2 \dots a_n) + b(b_1, b_2 \dots b_n) = aT(\alpha) + bT(\beta)$

$\therefore$ T is a linear transformation.

Thus T is an isomorphism from U to F^n *i.e.*, $U \cong F^n$

SOLVED PROBLEMS

Ex. 1. *Let* $T_k : V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x, y, kz), k \neq 0, k \in R$ *show that* T_k *is an isomorphism. What about* T_o *if* $T_o : V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x, y, 0)$.

Sol. Let $\alpha, \beta \in V_3(R)$ where $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$\therefore T(\alpha) = (x_1, y_1, z_1), T(\beta) = (x_2, y_2, z_2)$

(i) For $a, b \in R$, $T_k(a\alpha + b\beta) = T_k[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$

$= T_k(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = (ax_1 + bx_2, ay_1 + by_2, k(az_1 + bz_2))$

$= (ax_1, ay_1, akz_1) + (bx_2, by_2, bkz_2) = a(x_1, y_1, kz_1) + b(x_2, y_2, kz_2)$

$= aT_k(\alpha) + bT_k(\beta)$. $\therefore$ T_k is a linear transformation

(ii) To prove T_k *is one one*

Now $T_k(x_1, y_1, z_1) = T_k(x_2, y_2, z_2) \Rightarrow (x_1, y_1, kz_1)(x_2, y_2, kz_2)$

$\Rightarrow kz_1 = kz_2 \Rightarrow z_1 = z_2$ $\because k \neq 0$

$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$ $\therefore$ T is one one.

(iii) To prove T is onto

Since $k \neq 0$, for every vector $(x_1, y_1, z_1) \in R^3$ there exists a vector $\left(x_1, y_1, \frac{z_1}{k}\right)$ in R^3 such that $T_k\left(x_1, y_1, \frac{z_1}{k}\right) = (x_1, y_1, z_1)$ $\therefore$ T is onto.

Thus T is one-one onto linear transformation from R^3 onto itself and hence t is an isomorphism.

Clearly T is L.T. $T_o(x_1, y_1, z_1) = T_o(x_2, y_2, z_2) \Rightarrow (x_1, y_1, 0) = (x_2, y_2, 0)$

does not $\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$ $\therefore$ T_o is not one one

Also T_o is not onto and hence T_o is not an isomorphism

Ex. 2. *If A and B are subspaces of a vector space V over a field F, then prove that* $\frac{A+B}{B} \cong \frac{A}{A \cap B}$.

Sol. We know that $A + B$ is a subspace of V containing B

$\therefore \frac{A+B}{B}$ is a vector space over F.

Also $A \cap B$ is a subspace of A. $\therefore \frac{A}{A \cap B}$ is a vector space over F.

An element of $\frac{A+B}{B}$ is of the form $\alpha + \beta + B$ where $\alpha \in A, \beta \in B$.

But $\beta + B = B$. $\therefore$ An element of $\frac{A+B}{B}$ is of the form $\alpha + \beta$

Define a map $T : A \to \frac{A+B}{B}$ by $T(\alpha) = \alpha + B \;\forall\; \alpha \in A$

Clearly T is well defined and onto.

Let $a, b \in F$ and $\alpha_1, \alpha_2 \in A$. $\therefore a\alpha_1 + b\alpha_2 \in A$

Now $T(a\alpha_1 + b\alpha_2) = (a\alpha_1 + b\alpha_2) + B = (a\alpha_1 + B) + (b\alpha_2 + B)$

$= a(\alpha_1 + B) + b(\alpha_2 + B) = aT(\alpha_1) + bT(\alpha_2)$

$\therefore$ T is a linear transformation.

Hence (By Theorem 4.3) $\frac{A}{\text{Ker } T} \cong \frac{A+B}{B}$

Now $\text{Ker } T = \left\{\alpha \in A / T(\alpha) = 0 \text{ of } \frac{A+B}{B}\right\} = \{\alpha \in A / \alpha + B = B\} = A \cap B$.

Thus $\frac{A}{A \cap B} \cong \frac{A+B}{B}$ *i.e.*, $\frac{A+B}{B} \cong \frac{A}{A \cap B}$

4.5. DIRECT SUMS

Definition. *Let* $U_1, \dots\dots, U_n$ *be subspaces of a vector space V (F). Then V is said to be the internal direct sum of* $U_1, \dots\dots, U_n$ *if every* $v \in V$ *can be written in one and only one way as* $v = u_1 + u_2 + \dots + u_n$ *where* $u_i \in U_i \;\forall\; i$.

Now we introduce another concept, known as the external direct sum as follows:

Let $V_1, ..., V_n$ *be any finite number of vector spaces over a field F. Let* $V = \{(v_1, ..., v_n) / v_i \in V_i\}$. *We take two elements* $(v_1, ..., v_n)$ *and* $(v'_1, ..., v'_n)$ *of V to be equal if and only if for each* $i, v_i = v'_i$. *We define addition on V as* $(v_1, ..., v_n) + (v'_1, ..., v'_n) = (v_1 + v'_1, ..., v_n + v'_n)$. *Finally we define scalar multiplication on V as* $a(v_1, ..., v_n) = (a\, v_1, ..., a\, v_n)$ *where* $a \in F$.

We can easily see that V is a vector space over F for the operations defined above (by checking the vector space axioms). V is called the external direct sum of $V_1, ..., V_n$ *is denoted by writing* $V = V \oplus ..... \oplus V_n$

Theorem. ***If V is the internal direct sum of*** $U_1, ..., U_n$, ***then*** $V \cong U_1 \oplus ..... \oplus U_n$.

Proof. Let $v \in V$. Since V is the internal direct sum of $U_1, ..., U_n$, v can be written in one and only one way as $v = u_1 + ... + u_n$ ... (1) where $u_i \in U_i \; \forall i$

Define $T : V \to U_1 \oplus ..... \oplus U_n$ by $T(v) = (u_1, ..., u_n)$

T is well defined since $v \in V$ has a unique representation of the form (1).

(*i*) If $(u'_1 + ... + u'_n)$ is any element of $U_1 + ... + U_n$ then

$v' = u'_1 + ... + u'_n \in V$ and is such that $T(v'_i) = (u'_1 \, ... \, u'_n)$ $\qquad \therefore$ T is onto.

(*ii*) Let $a, b \in F$ and $v, v' \in V$

Let $v = u'_1 + ... + u'_n$ and $v' = u'_1 + ... u'_n$ where $v'_i, u'_i \in U_i$.

Then $T(v) = T(v')$

$\Rightarrow (u_1, ..., u_n) = (u'_1, ..., u'_n) \Rightarrow u_1 = u'_1, ..., u_n = u'_n$

$\Rightarrow u_1 + ... + u_n = u'_1 + ... + u'_n \Rightarrow v = v'$ $\qquad \therefore$ T is 1–1.

(*iii*) Let $a, b \in F$ and $v, v' \in V$ $\qquad \therefore av + bv' \in V$

$\therefore T(av + bv') = T[a(u_1 + ... + u_n) + b(u'_1 + ... + u'_n)] \in V$

$= T[(au_1 + bu'_1) + ... + (au_n + bu'_n)] = (au_1 + bu'_1, ..., au_n + bu'_n)$

$= (au_1, ..., au_n) + (bu'_1, ..., bu'_n) = a(u_1, ..., u_n) + b(u'_1, ..., u'_n)$

$= aT(v) + bT(v')$. $\qquad \therefore$ T is a homomorphism.

Thus T is an isomorphism and hence $V \cong U_1 \oplus ..... \oplus U_n$.

Note. *Because of the isomorphism proved above, we shall henceforth merely refer to a direct sum, not qualifying that it be internal or external.*

4. 6. DIRECT SUM OF TWO SUBSPACES.

Definition. *Let* W_1 *and* W_2 *be two subspaces of the vector space* V(F). *Then V is said to be the direct sum of the subspaces* W_1 *and* W_2 *if every element* $\alpha \in V$ *can be uniquely written as* $\alpha = \alpha_1 + \alpha_2$ *where* $\alpha_1 \in W_1$ *and* $\alpha_2 \in W_2$.

Thus $V = W_1 \oplus W_2$ and every element of V can be uniquely written as sum of an element of W_1 and an element of W_2.

We denote "V is the direct sum of subspaces W_1, W_2" as $V = W_1 \oplus W_2$.

DISJOINT SUBSPACES :

Definition. *Two subspaces* W_1 *and* W_2 *of the vector space* V(F) *are said to be disjoint if their intersection is the zero subspace i.e. if* $W_1 \cap W_2 = \{\bar{O}\}$.

Theorem. ***The necessary and sufficient conditions for a vector space V(F) to be a direct sum of its subspaces*** W_1 ***and*** W_2 ***are that (i)*** $V = W_1 + W_2$ ***and (ii)*** $W_1 \cap W_2 = \{\bar{O}\}$ ***i.e.*** W_1 ***and*** W_2 ***are disjoint.***

Proof. The conditions are necessary.

V is the direct sum of its subspaces $\Rightarrow$ Each element of V can be uniquely written as sum of an element of W_1 and an element of $W_2 \Rightarrow V = W_1 + W_2$.

If possible, let $\bar{O} \neq \alpha \in W_1 \cap W_2$.

$\therefore \alpha \in W_1, \alpha \in W_2 \Rightarrow \alpha \in V$ and $\alpha = \bar{O} + \alpha$ where $\bar{O} \in W_1, \alpha \in W_2$

and $\alpha = \alpha + \bar{O}$ where $\alpha \in W_1, \bar{O} \in W_2$

Thus an element in V can be written in at least two different ways as sum of an element of W_1 and an element of W_2. This is a contradiction of the hypothesis. Hence $\bar{O}$ is the only element of V common to both W_1 and W_2 *i.e.* $W_1 \cap W_2 = \{\bar{O}\}$.

Thus the conditions (*i*) and (*ii*) are necessary.

The conditions are sufficient.

Let $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\bar{O}\}$

$V = W_1 + W_2 \Rightarrow$ Each element of *V* can be written as sum of an element of W_1 and an element of $W_2 \Rightarrow \alpha = \alpha_1 + \alpha_2$ where $\alpha \in V$ and $\alpha_1 \in W_1, \alpha_2 \in W_2$.

If posaible, let $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W_1, \beta_2 \in W_2$.

$\therefore \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \Rightarrow \alpha_1 - \beta_1 = \beta_2 - \alpha_2 \in W_1 \cap W_2$

Since $\alpha_1 \in W_1, \beta_1 \in W_1 \Rightarrow \alpha_1 - \beta_1 \in W_1$

$\alpha_2 \in W_2, \beta_2 \in W_2 \Rightarrow \alpha_2 - \beta_2 \in W_2$

Since $W_1 \cap W_2 = \{\bar{o}\}, \alpha_1 - \beta_1 = \bar{o} = \alpha_2 - \beta_2 \Rightarrow \alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$

$\Rightarrow \alpha \in V$ is uniquely written as an element of W_1 and an element of W_2

$\Rightarrow V = W_1 \oplus W_2$

Thus the conditions (*i*) and (*ii*) are sufficient.

SINGULAR AND NON-SINGULAR TRANSFORMATIONS

4. 7. SINGULAR TRANSFORMATION

Definition. *A linear transformation $T : U(F) \to V(F)$ is said to be singular if the null space of T consists of atleast one non-zero vector.*

i.e. If there exists a vector $\alpha \in U$ such that $T(\alpha) = \bar{O}$ for $\alpha \neq \bar{O}$ then T is singular.

4. 8. NON - SINGULAR TRANSFORMATION

Definition. *A linear transformation $T : U(F) \to V(F)$ is said to be non-singular if the null space consists of one zero vector alone.*

i.e., $\alpha \in U$ and $T(\alpha) = \hat{O} \Rightarrow \alpha = \hat{O} \Rightarrow N(T) = \{\bar{O}\}$

4. 9. Theorem. ***Let U(F) and V(F) be two vector spaces and $T : U \to V$ be a linear transformation. Then T is non-singular if f, the set of images of a linearly independent set, is linearly independent.***

Proof. (*i*) Let T be non-singular and let $S = \{\alpha_1, \alpha_2, ... \alpha_n\}$

be a L.I. subset of U. Then its T - images set be $S' = \{T(\alpha_1), T(\alpha_2), ... T(\alpha_n)\}$

Now to prove S' is L.I. For some $a_1, a_2 ... a_n \in F$

Let $a_1T(\alpha_1) + a_2T(\alpha_2) + ... + a_nT(\alpha_n) = \hat{O}$ $(\hat{O} \in V)$

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n] = \hat{O}$ ($\because$ T is L.T.)

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \hat{O}$ ($\because$ T is non- singular)

$\Rightarrow a_1 = 0, a_2 = 0, ..., a_n = 0$ ($\because$ S is L.I.)

Thus S' is linearly independent.

(*ii*) Let the T - image of any L.I. set be L.I., then to prove T is non-singular.

Let $\alpha \in U$ and $\alpha \neq \bar{O}$. Then the set $B = \{\alpha\}$ is L.I. set and its image set $B' = \{T(\alpha)\}$ given to be L.I.

$\Rightarrow T(\alpha) \neq \hat{O}$ ($\because \{\hat{O}\}$ vectors is L.D.)

Thus $\alpha \neq \bar{O} \Rightarrow T(\alpha) \neq \hat{O}$. $\therefore$ T is non-singular.

4.10. Theorem. ***Let U(F) and V(F) be two finite dimensional vector spaces. The linear transformation $T : U \to V$ is an isomorphism iff T is non singular.***

Proof. (*i*) Let $T : U(F) \to V(F)$ be an isomorphism so that T is one- one onto.

To prove T is non-singular

Let $\alpha \in U$, then $T(\alpha) = \hat{O} \Rightarrow T(\alpha) = T(\bar{O})$ ($\because T(\bar{O}) = \hat{O}$ in any L.T.)

$\Rightarrow \alpha = \bar{O}$ (T is one-one)

$\therefore$ T is non-singular

(*ii*) Let T be non-singular .

i.e. $\alpha \in U, T(\alpha) = \hat{O} \Rightarrow \alpha = \hat{O}$; $N(T) = \{\bar{O}\}$, $\therefore$ dim N(T) = 0

For $\alpha_1, \alpha_2 \in U, T(\alpha_1) = T(\alpha_2) \Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{O}$ $[\hat{O} \in V]$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{O}$ ($\because$ T is L.T.)

$\Rightarrow \alpha_1 - \alpha_2 = \bar{O} \Rightarrow \alpha_1 = \alpha_2$ ($\because$ T is non-singular)

$\therefore$ T is one-one

(*iii*) dim U = dim R(T) + dim N(T) = dim R(T) ($\because$ dim N(T) = 0)

Also $T : U \to V$ is one-one by (*ii*) $\Rightarrow V = R(T)$. $\Rightarrow$ T is an onto mapping

Again dim U = dim V. Hence T is an isomorphism.

SOLVED PROBLEMS

Ex. 1. *A linear mapping* $T : R^3 \to R^3$ *is defined by*

$T(x,y,z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z)$. *Show that T is non-singular.*

Sol. Let $T(x, y, z) = \hat{O}$

$\Rightarrow (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z) = (0,0,0) \Rightarrow x \cos\theta - y \sin\theta = 0$

$x \sin\theta + y \cos\theta = 0$, $z = 0$

Squaring and adding $x^2 + y^2 = 0 \Rightarrow x = 0, y = 0$

Thus $T(x, y, z) = \hat{O} \Rightarrow (x, y, z) = (0,0,0)$. $\therefore$ T is non singular.

Ex. 2. *Show that a linear transformation* $T : U \to V$ *over the field F is non-singular if and only if T is one-one.* **(S. K. U. M 07, S. U. M 07)**

Sol. (*i*) Let T be non-singular *i.e.*, $\alpha \in U, T(\alpha) = \bar{O} \Rightarrow \alpha = \bar{O}$

Now for $\alpha_1, \alpha_2 \in U, T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{O}$ $(\hat{O} \in V)$

$\Rightarrow T(\alpha_1-\alpha_2)=\hat{O}$ ($\because$ T is L.T.)

$\Rightarrow \alpha_1-\alpha_2=0$ ($\because$ T is non-singular)

$\Rightarrow \alpha_1=\alpha_2$

$\therefore$ T is one-one.

(*ii*) Let T be one-one

$\therefore$ zero element $\hat{O}$ of V is the T - image of only one element $\in$ U.

$\Rightarrow$ null space of U consists of only one element.

$\because$ null space $N(T)\subseteq U$, it must consist of $\bar{O}$.

$\Rightarrow$ null space N(T) consists of only one $\bar{O}$ element.

$\Rightarrow N(T)=\{\bar{O}\}$ $\Rightarrow$ T is non-singular.

Ex. 3. *Let $T:U\to V$ be a linear transformation of U (F) into V (F) where U (F) is finite dimensional. Prove that U and the range space of T have the same dimnesion iff T is non-singular.*

Sol. (*i*) Let dim U = dim (Range T) = dim R(T)

$\because$ dim U = dim R(T) + dim N(T) $\Rightarrow$ dim N(T) = 0

$\Rightarrow$ The null space of T is the zero space $\{\bar{O}\}$

$\Rightarrow$ Hence T is non-singular

(*ii*) Let T be non-singular. Then $N(T)=\{\bar{O}$ and nullity T = 0.

As dim U = dim R(T) + dim N(T) = dim R(T) + 0

$\Rightarrow$ dim U = dim R(T).

Ex. 4. *If U and V are finite dimensional vector spaces of the same dimension, then a linear mapping $T:U\to V$ is one-one iff it is onto.*

Sol. T is one-one $\Leftrightarrow N(T)=\{\bar{O}\}$ $\Leftrightarrow$ dim N(T) = 0

$\Leftrightarrow$ dim R(T) + dim N(T) = dim U = dim V

$\Leftrightarrow$ R (T) = V $\Leftrightarrow$ T is onto.

Note. In view of the above examples, we can state the following theorem.

Theorem. Let U and V are vector spaces of equal (finite) dimension, and let $T:U\to V$ is a linear transformation. Then the following are equivalent.

(*a*) T is one - to one (*b*) T is onto (*c*) rank (T) = dim (U)

We note that linearity of T is essential in the above theorems.

Ex. 5. Let $T:P_2(R)\to P_3(R)$ be the linear transformation

defined by $T\{f(x)\}=2f'(x)+\int_0^x 3f(t)\,dt$

Now $T(1)=0+\int_0^x 3\,dt=3x$, $T(x)=2+\int_0^x 3(t)\,dt=2+\frac{3}{2}x^2$, $T(x^2)=4x+\int_0^x 3t^2\,dt=4x+x^3$

$\therefore R(T)=\text{span}(\{T(1),T(x),T(x^2)\})=\text{span}\left(3x,2+\frac{3}{2}x^2,4x+x^3\right)$

Since $\left\{3x,2+\frac{3}{2}x^2,4x+x^3\right\}$ is linearly independent.

We have rank (T) = 3. Also $\dim P_3(R)=4$, T is not onto.

From Rank Nullity theorem, nullity (T) + 3 = 3. Thus nullity (T) = 0.

So N (T) = {0}. We have T is one to one.

Ex. 6. Let $T:F^2\to F^2$ be the linear transformation defined by $T(a_1,a_2)=(a_1+a_2,a_1)$.

We can easily see that $N(T)=\{0\}$. Then T is one - to - one. T must be onto.

4.11. INVERSE FUNCTION

Definition. *Let $T:U\to V$ be a one-one onto mapping. Then the mapping $T^{-1}:U\to V$ defined by $T^{-1}(\beta)=\alpha\Leftrightarrow T(\alpha)=\beta$, $\alpha\in U,\beta\in V$ is called the inverse mapping of T.*

Note: If $T:U\to V$ is one-one onto mapping then the mapping $T^{-1}:V\to U$ is also one-one onto.

4. 12. Theorem. *Let U (F) and V (F) be two vector spaces and $T:U\to V$ be a one-one onto linear transformation. Then T^{-1} is a linear transformation and that T is said to be invertihle.*

Proof. Let $\beta_1,\beta_2\in V$ and $a,b,\in F$

Since T is one-one onto function there exist unique vectors $\alpha_1,\alpha_2\in U$ such that

$T(\alpha_1)=\beta_1$ and $T(\alpha_2)=\beta_2$. Hence by the definition T^{-1},

we have $\alpha_1=T^{-1}(\beta_1)$ and $\alpha_2=T^{-1}(\beta_2)$. Also $\alpha_1,\alpha_2\in U$ and $a,b,\in F\Rightarrow a\,\alpha_1+b\,\alpha_2\in U$

$\therefore T(a\,\alpha_1+b\,\alpha_2)=a\,T(\alpha_1)+b\,T(\alpha_2)=a\,\beta_1+b\,\beta_2$ ($\because$ T is L.T.)

$\therefore$ By the definition of inverse $T^{-1}(a\,\beta_1+b\,\beta_2)=a\,\alpha_1+b\,\alpha_2=a\,T^{-1}(\beta_1)+b\,T^{-1}(\beta_2)$

$\therefore T^{-1}$ is a linear transformation from V into U.

4. 13. Theorem. *A linear transformation T on a finite dimensional vector space is invertible if and only if T is non-singular.*

Proof. Let U (F) and V (F) be two vector spaces.

Let $T:U\to V$ be a linear transformation.

(*i*) Let T be non-singular *i.e.* For $\alpha \in U, T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now to prove T is invertible, it is enough to show that T is one - one onto. For this refer proof of theorem 4.8.

(*ii*) Let T be invertible so that T is one - one onto. Now to prove T is non-singular.

For $\alpha \in U, T(\alpha) = \hat{0} = T(\bar{0})$ $\quad$ ($\because$ T is L. T.)

$\Rightarrow \alpha = \bar{0}$ $\quad$ T is one-one. $\quad \therefore$ T is non-singular.

4.14. Theorem. ***Let U(F) and V(F) be two finite dimensional vector spaces such that dim U = dim V. If $T : U \to V$ is a linear transformation then the following are equivalent : -***

(1) T is invertible $\quad$ (2) T is non-singular $\quad$ (3) The range of T is V

(4) If $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is any basis of U, then $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is a basis of V.

(5) There is some basis $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of U such that $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ basis of V.

Proof. Here we shall prove a series of implications

viz., (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1)

Now (1) $\Rightarrow$ (2)

If T is invertible then T is one-one and therefore $T(\alpha) = \hat{0} = T(\bar{O}) \Rightarrow \alpha = 0$.

Hence T is nonsingular.

(*ii*) $\Rightarrow$ (*iii*)

Let T be non -singular. $\quad$ Let $S = \{\alpha_1, \alpha_2 ... \alpha_n\}$ be a basis of U.

Then S is L.I. set. Since T is non-singular the set $S' = \{T(\alpha_1), T(\alpha_2) ... T(\alpha_n)\}$ is a linearly independent set in V. But dim V = *n*, hence the set S' is a basis of V. Then a vector $\beta \in V$ can be expressed as $\beta = a_1T(\alpha_1) + a_2T(\alpha_2) ... + a_nT(\alpha_n)$ for some a_i 's $\in F$

$= T[a_1\alpha_1 + a_2\alpha_2 ... + a_n\alpha_n] \;= T(\alpha),\ \alpha \in V$ $\quad$ ($\because$ T is L.T.)

$\Rightarrow \beta \in$ range of T.

Thus every vector in V is in the range of T. $\quad \therefore$ R (T) = V

(3) $\Rightarrow$ (4)

Let the range of T be V *i.e.* T is onto. If $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a basis of U, the T-images of these vectors $S' = \{T(\alpha_1), T(\alpha_2) ... T(\alpha_n)\}$ span the range of T *i.e.* V.

$\therefore$ L(S') = V.

Since S' is a L.I. set of *n* vectors and dim V = *n*, the set S' is a basis of V.

(4) $\Rightarrow$ (5). This is obvious in the above proof. $\quad$ (5) $\Rightarrow$ (1)

Let $S = \{\alpha_1, \alpha_2 ... \alpha_n\}$ be a basis of U such that $S' = \{T(\alpha_1), T(\alpha_2) ... T(\alpha_n)\}$ is a basis of V.

Since L (S') = R (T) it is clear that R (T) = V. *i.e.* T is onto.

Let $\alpha \in$ null space of T *i.e.* N(T) then $\alpha \in U$

$\therefore \alpha = b_1\alpha_1 + b_2\alpha_2 ... + b_n\alpha_n$ for some b_i 's $\in F$. $\quad$ Hence $T(\alpha) = \hat{\bar{O}}$.

$\Rightarrow T[b_1\alpha_1 + b_2\alpha_2 ... + b_n\alpha_n] = \hat{\bar{O}}$

$\Rightarrow b_1T(\alpha_1) + b_2T(\alpha_2) ... + b_nT(\alpha_n) = \hat{\bar{O}}$ $\quad$ ($\because$ T is L.T.)

$\Rightarrow b_1 = 0, b_2 = 0 ... b_n = 0 \Rightarrow \alpha = \bar{O}$ $\quad$ ($\because$ S' is L.I.)

$\therefore$ This shows that T is non-singular and one-one. Hence T is invertible.

SOLVED PROBLEMS

Ex. 1. ***If $T : R^3 \to R^3$ is invertible operator defined by***

$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. ***Find T^{-1}.*** $\quad$ ***(S. V. U. M 07)***

Sol. Since T is invertible. $\quad T(x, y, z) = (a, b, c) \Rightarrow T^{-1}(a, b, c) = (x, y, z)$

$\therefore (2x, 4x - y, 2x + 3y - z) = (a, b, c) \Rightarrow 2x = a, 4x - y = b, 2x + 3y - z = c$

Solving $x = \frac{a}{2}, y = 2a - b, z = 7a - 3b - c$. $\quad$ Hence $T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a - b, 7a - 3b - c\right)$

Ex. 2. ***The set $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(R)$. $T : V_3(R) \to V_3(R)$ is a linear operator defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. Show that T is non-singular and find its inverse.*** $\quad$ ***(S. K. U. S01)***

Sol. Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$

Now $T : R^3 \to R^3$ is defined by $\quad T(e_1) = e_1 + e_2 \Rightarrow T(1, 0, 0) = (1, 1, 0)$;

$T(e_2) = e_2 + e_3 \Rightarrow T(0, 1, 0) = (0, 1, 1)$; $\quad T(e_3) = e_1 + e_2 + e_3 \Rightarrow T(0, 0, 1) = (1, 1, 1)$

Let $\alpha = (x, y, z) \in V_3(R)$

$\therefore \alpha = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$\therefore T(\alpha) = x\,T(1, 0, 0) + y\,T(0, 1, 0) + z\,T(0, 0, 1) = x(1, 1, 0) + y(0, 1, 1) + z(1, 1, 1)$

$\therefore$ The transformation is given by $T(x, y, z) = (x + z, x + y + z, y + z)$

Now If $T(x, y, z) = \hat{\bar{O}}$, then $(x + z, x + y + z, y + z) = (0, 0, 0)$

$\Rightarrow x + z = 0, x + y + z = 0, y + z = 0 \Rightarrow x = y = z = 0$.

$\therefore T(\alpha) = \hat{\bar{O}} \Rightarrow \alpha = \bar{O}$. $\quad$ Hence T is non-singular and therfore T^{-1} exists.

Let $T(x, y, z) = (a, b, c)$ $\quad \Rightarrow (x + z, x + y + z, y + z) = (a, b, c)$

$\Rightarrow x+z=a$ Solving $x=b-c$

$x+y+z=b,\ y+z=c,\ y=b-a,\ z=a-b+c$

$\therefore\ T^{-1}(a,b,c)=(x,y,z)=(b-c,\ b-a,\ a-b+c)$

Ex. 3. *A linear transformation T is defined on* $V_3(C)$ *by* $T(a,b)=(a\alpha+b\beta,\ a\gamma+b\delta)$ *where* $\alpha,\beta,\gamma,\delta$ *are fixed elements of C. Prove that T is invertible if and only if* $\alpha\delta-\beta\gamma\neq 0$.

Sol. $T:V_2(C)\to V_2(C)$ is a L.T. and $\dim V_2(C)=2$

T is invertible if and only if T is one-one onto.

T is onto iff the range of T is the whole set V_2 *i.e.* $R(T)=V_2(C)$

Now $S=\{(1,0),(0,1)\}$ is a basis of $V_2\Rightarrow L(S)=V_2$.

$T(1,0)=(1.\alpha+0.\beta,\ 1.\gamma+0.\delta)=(\alpha,\gamma)$; $T(0,1)=(0.\alpha+1.\beta,\ 0.\gamma+1.\delta)=(\beta,\delta)$

$\therefore$ T is invertible iff $S'=\{(\alpha,\gamma),(\beta,\delta)\}$ span $V_2(C)$.

As $\dim V_2(C)=2$, the set S' containing two vectors will span V_2 if S' is L.I.

For $x,y\in C$, $x(\alpha,\gamma)+y(\beta,\delta)=(0,0)$

$$\Rightarrow (x\alpha+y\beta,\ x\gamma+y\delta)=(0,0)\Rightarrow x\alpha+y\beta=0,\ x\gamma+y\delta=0$$

These equations will have the only solution $x=0,\ y=0$ iff

$$\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0 \text{ i.e., } \alpha\delta-\beta\gamma\neq 0$$

$\therefore$ T is invertible $\Leftrightarrow \alpha\delta-\beta\gamma\neq 0$

EXERCISE 4

1. Show that each of the following linear operators T on R^3 is invertible and find T^{-1}.
   (a) $T(x,y,z)=(2x,\ 4x-y,\ 2x+3y-z)$ (b) $T(a,b,c)=(a-3b-2c,\ b-4c,\ c)$
   (c) $T(a,b,c)=(3a,\ a-b,\ 2a+b+c)$ **(S. V. U. S97)**
   (d) $T(x,y,z)=(x+y+z,\ y+z,\ z)$ (e) $T(a,b,c)=(a-2b-c,\ b-c,\ a)$
2. The set $\{e_1,e_2,e_3\}$ is the standard basis set of $V_3(R)$. The linear operator $T:R^3\to R^3$ is defined below. Show that T is invertible and find T^{-1}
   (i) $T(e_1)=e_1+e_2,\ T(e_2)=e_1-e_2+e_3,\ T(e_3)=3e_1+4e_3$
   (ii) $T(e_1)=e_1-e_2,\ T(e_2)=e_2,\ T(e_3)=e_1+e_2-7e_3$
   (iii) $T(e_1)=e_1-e_2+e_3,\ T(e_2)=3e_1-5e_3,\ T(e_3)=3e_1-2e_3$
3. Let $T:U\to V$ be a non. singular linear transformation then prove that $(T^{-1})^{-1}=T$.

4. 15. Theorem. ***The necessary and sufficient condition for a linear operator T on a vector space V (F) to be invertible is that there exists a linear transformation H on V such that TH = HT = I.***

Proof. Given $T:V(F)\to V(F)$ is a L.T.

(i) Let T be invertible. Hence T^{-1} exists and is one-one onto.

Let $\alpha\in V$ and $T(\alpha)=\beta$ so that $\alpha=T^{-1}(\beta)$

$$(TT^{-1})(\beta)=T[T^{-1}(\beta)]=T(\alpha)=\beta=I(\beta)\Rightarrow TT^{-1}=I$$

Again $(T^{-1}T)(\alpha)=T^{-1}[T(\alpha)]=T^{-1}(\beta)=\alpha=I(\alpha)\Rightarrow T^{-1}T=I$

$\therefore\ T^{-1}T=T^{-1}T=I$

Taking $H=T^{-1}$ we get $TH=HT=I$ such that $H:V\to V$ is one - one onto L.T.

(ii) Let there exist a linear operator $H:V\to V$ such that TH = HT = I.

To prove T is invertible.

For $\alpha_1,\alpha_2\in V$ $T(\alpha_1)=T(\alpha_2)$

$\Rightarrow H[T(\alpha_1)]=H[T(\alpha_2)]\Rightarrow (HT)(\alpha_1)=(HT)(\alpha_2)$

$\Rightarrow I(\alpha_1)=I(\alpha_2)\Rightarrow \alpha_1=\alpha_2$ ($\because$ HT = I = TH)

$\therefore$ T is one-one

For $\beta\in V$ there exists $\alpha\in V$ such that $H(\beta)=\alpha$ ($\because H:V\to V$)

$\Rightarrow TH(\beta)=T(\alpha)\Rightarrow I(\beta)=T(\alpha)\Rightarrow \beta=T(\alpha)$

$\therefore$ For any $\alpha\in V$ there exists $\beta\in V$ such that $T(\alpha)=\beta$, therefore T is onto.

Thus T is one-one onto and hence T is invertible.

4.16. UNIQUENESS OF INVERSE

Theorem. ***Let T be an invertible linear operator on a vector space V(F). Then T possesses unique inverse.***

Proof. Let Hand G be two inverses of T. Then

HT = TH = I and GT = TG = I

Now G (TH) = GI = G. $\therefore$ again G (TH) = (GT) H = IH = H

Since product of linear transformation,s is associative

$\therefore$ G = G (TH) = (GT) H = H $\Rightarrow$ G = H. Hence the inverse of T is unique.

4.17. Theorem. ***Let T be an invertible operator oa a vector space V (F). Then show that (i) a T is invertible linear operator. where*** $a\neq 0$ ***and*** $a\in F$.

(ii) $(aT)^{-1}=\frac{1}{a}T^{-1}$ ***(iii)*** T^{-1} ***is invertible and*** $(T^{-1})^{-1}=T$

Proof. T is an invertible operator T is one-one onto and $TT^{-1} = T^{-1}T = I$

For $a \neq 0$ and $a \in F \Rightarrow a^{-1} = \frac{1}{a} \in F$

T is a linear operator $\Rightarrow$ $(a\,T)$ is a linear operator.

Also $(aT)(a^{-1}T^{-1}) = a[T(a^{-1}T^{-1})] = a[a^{-1}(TT^{-1})] = (aa^{-1})(TT^{-1}) = 1.I = I$

For $\alpha_1, \alpha_2 \in F$ $(a\,T)(\alpha_1) = (a\,T)(\alpha_2)$

$\Rightarrow a\,T(\alpha_1) = a\,T(\alpha_2) \Rightarrow T(a\alpha_1) = T(a\alpha_2)$

$\Rightarrow a\,\alpha_1 = a\,\alpha_2 \Rightarrow \alpha_1 = \alpha_2$ $(\because$ T is is one-one)

$\therefore$ a T is one-one. Also T is onto $\Rightarrow$ a T is onto.

Hence $(a\,T)$ is invertible operator. For $\alpha, \beta \in V$ and $c, d \in F$

$(aT)(c\,\alpha + d\,\beta) = aT(c\,\alpha + d\,\beta) = T(ac\,\alpha + ad\,\beta)$

$= acT(\alpha) + adT(\beta) = c\,(aT)(\alpha) + d\,(aT)(\beta)$

Hence $(a\,T)$ is a linear transformation.

Thus $(a\,T)$ is a one-one onto linear operator on the vector space V (F).

Hence $(a\,T)$ is invertible.

(ii) Since $TT^{-1} = T^{-1}T = I$

we have $(aa^{-1})(TT^{-1}) = (a^{-1}a)(T^{-1}T) = I$ $(\because aa^{-1} = a^{-1}a = 1)$

$\Rightarrow (aT)(a^{-1}T^{-1}) = (a^{-1}T^{-1})(aT) = I$ $\Rightarrow (aT)^{-1} = a^{-1}T^{-1} = \frac{1}{a}T^{-1}$

(iii) Again $TT^{-1} = T^{-1}T = I$

$\Rightarrow$ that the inverse of T^{-1} is T. *i.e.* $(T^{-1})^{-1} = T$

4.18. Theorem. ***If T and U are invertible linear operators on a vector space V(F) then show that TH is invertible and*** $(TH)^{-1} = H^{-1}T^{-1}$

Proof. Given T and H are invertible $\Rightarrow$ T^{-1}, H^{-1} exist and $TT^{-1} = T^{-1}T = I$

$$HH^{-1} = H^{-1}H = I$$

Now $(H^{-1}T^{-1})(TH) = H^{-1}(T^{-1}T)\,H = H^{-1}I\,H = H^{-1}H = I$

Again $(TH)(H^{-1}T^{-1}) = T\,(H\,H^{-1})T^{-1} = T\,I\,T^{-1} = TT^{-1} = I$

Thus $(TH)(H^{-1}T^{-1}) = (H^{-1}T^{-1})(TH) = I \Rightarrow (TH)^{-1} = H^{-1}T^{-1}$

$\therefore$ T H is invertible and $(TH)^{-1} = H^{-1}T^{-1}$

SOLVED PROBLEMS

Ex. 1. *If A, B, C are linear transformations on a vector space V (F) such that* $AB = CA = I$ *then show that A is invertible and* $A^{-1} = B = C$

Sol. *(i) To prove* A^{-1} *exists*

For $\alpha_1, \alpha_2 \in V$ $A(\alpha_1) = A(\alpha_2) \Rightarrow$ $C\,A(\alpha_1) = C\,A(\alpha_2)$

$\Rightarrow I(\alpha_1) = I(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$ $\therefore$ A is one one.

Let $\beta \in V$. Since $B : V \to V$, then $B(\beta) \in V$

For some $\alpha \in V$ let $B(\beta) = \alpha$ $\Rightarrow A[B(\beta)] = A(\alpha) \Rightarrow (AB)(\beta) = A(\alpha)$

$\Rightarrow$ $I(\beta) = A(\alpha)$ $\Rightarrow \beta = A(\alpha)$

Thus for some $\beta \in V$ there exists $\alpha \in V$ such that $A(\alpha) = \beta$. $\therefore$ A is onto

Thus A is one one onto $\Rightarrow A^{-1}$ exists.

(ii) To prove $A^{-1} = B = C$

Now $AB = I$. $\Rightarrow A^{-1}(AB) = A^{-1}I \Rightarrow (A^{-1}A)\,B = A^{-1}$ $\Rightarrow IB = A^{-1} \Rightarrow B = A^{-1}$

Again $C\,A = I \Rightarrow (CA)\,A^{-1} = I\,A^{-1} \Rightarrow C\,(AA^{-1}) = A^{-1} \Rightarrow C\,I = A^{-1} \Rightarrow C = A^{-1}$

Hence $A^{-1} = B = C$.

Ex. 2. *If T is a linear operator on a vector space V (F) such that* $T^2 - T + I = 0$, *then show that T is invertible.*

Sol. If $T^2 - T + I = 0$, then $T - T^2 = I$

(i) To prove T is one one

Let $\alpha_1, \alpha_2 \in V$, then $T(\alpha_1) = T(\alpha_2)$ .... (1)

$\Rightarrow T[T(\alpha_1)] = T[T(\alpha_2)] \Rightarrow T^2(\alpha_1) = T^2(\alpha_2)$ .... (2)

Subtracting (2) from (1) $\Rightarrow$ $T(\alpha_1) - T^2(\alpha_1) = T(\alpha_2) - T^2(\alpha_2)$

$\Rightarrow (T - T^2)(\alpha_1) = (T - T^2)(\alpha_2)$ $\Rightarrow I(\alpha_1) = I(\alpha_2)$ $\Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one one

(ii) To prove T is onto

For $\alpha \in V$, $T(\alpha) \in V$ and $\alpha - T(\alpha) \in V$

Now $T\,[\alpha - T(\alpha)] = T(\alpha) - T^2(\alpha) = (T - T^2)(\alpha) = I(\alpha) = \alpha$

Thus for $\alpha \in V$ there $\alpha - T(\alpha) \in V$ such that $T\,[\alpha - T(\alpha)] = \alpha$

$\therefore$ T is onto. Thus T is one one onto and hence invertible.

Ex. 3. *If A and B are linear transformations on a finite dimensional vector space V (F) and if A B = I , then show that A and B are invertible.*

Sol. *(i) To prove B is invertible*

Let $\alpha_1, \alpha_2 \in V(F)$, then $B(\alpha_1) = B(\alpha_2)$

$\Rightarrow A[B(\alpha_1)] = A[B(\alpha_2)] \Rightarrow (AB)(\alpha_1) = (AB)(\alpha_2)$

$\Rightarrow I(\alpha_1) = I(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$ $\quad\therefore$ B is one one.

Since B is a. L.T on a finite dimensional veotor space such that B is one-one, B will be onto.

Now B is one-one onto $\Rightarrow$ B is invertible $\Rightarrow B^{-1}$ exists

(ii) To prove A is invertible

Now $AB = I \Rightarrow (AB)B^{-1} = IB^{-1} \Rightarrow A(BB^{-1}) = B^{-1}$

$\Rightarrow AI = B^{-1} \Rightarrow A = B^{-1}$

Hence $B^{-1}B = BB^{-1} = I \Rightarrow AB = BA = I$

$\Rightarrow$ A is invertible and $A^{-1} = B$.

Ex. 4. *Let $A = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ and $B = \{\beta_1, \beta_2 \ldots \beta_n\}$ be two ordered bases of a finite dimensional vector space V (F). Prove that there exists an invertible linear operator T on V such that $T(\alpha_i) = \beta_i$.*

Sol. Already we have proved in a previous theorem that there exists a linear transformation T on V such that $T(\alpha_i) = \beta_i$ for $i = 1, 2, \ldots n$.

Now we prove that T is invertible. This is equivalent to proving T is non-singular.

For $\alpha \in V \Rightarrow \alpha = l.c$ of elements of A.

$= a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n$ for $a's \in F$ $\qquad \therefore T(\alpha) = \bar{0}$

$\Rightarrow T(a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n) = \bar{0} \quad \Rightarrow a_1T(\alpha_1) + a_2T(\alpha_2) \ldots + a_nT(\alpha_n) = \bar{0}$

$\Rightarrow a_1\beta_1 + a_2\beta_2 \ldots + a_n\beta_n = \bar{0} \qquad (\because T(\alpha_i) = \beta_i)$

$\Rightarrow a_1 = 0, a_2 = 0 \ldots a_n = \bar{0} \Rightarrow \alpha = \bar{0} \qquad (\because$ B is L.I.)

Thus $T(\alpha) = \bar{0} \Rightarrow \alpha = \bar{0}$. $\quad\therefore$ T is non-singular and hence invertible.

5

Matrix of Linear Transformation

5.1. Let U(F) and V(F) be two vector spaces so that dim U = n and dim V = m.Let $T: U \to V$ be a linear transformation.

Let $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the ordered basis of U. and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be the ordered basis of V.

For every $\alpha \in U \Rightarrow T(\alpha) \in V$ and $T(\alpha)$ can be expressed as a $lc.$ of elements of the basis B_2. Let there exist $a's \in F$ such that

$$\left.\begin{array}{ll} T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{m1}\beta_m, & T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \ldots + a_{m2}\beta_m \\ \ldots \quad \ldots \quad \ldots, & T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \ldots + a_{mj}\beta_m \\ \ldots \quad \ldots \quad \ldots, & T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \ldots + a_{mn}\beta_m \end{array}\right\} \quad \ldots (A)$$

Writing the coordinates of $T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)$ successively as columns of a matrix

we get $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$

This matrix represented as $[a_{ij}]_{m\times n}$ is called the matrix of the linear transformation T *w.r.* to the bases B_1 and B_2. Symbolically $[T : B_1, B_2]$ or $[T] = [a_{ij}]_{m\times n}$

Thus the matrix $[a_{ij}]_{m\times n}$ completely determines the linear transformation through the relations given in (A). Hence the matrix $[a_{ij}]_{m\times n}$ represents the transformation T.

Note. Let $T : V \to V$ be the linear operator so that dim V = n.

If $B_1 = B_2 = B$ (say) then the above said matrix is called the matrix of T relative to the ordered basis B. It is denoted as $[T ; B] = [T]_B = [a_{ij}]_{n\times n}$.

5.2. Corollary. ***Let V (F) be an n dimensional vector space for which B is a basis. Show that***

(i) $[I:B]=I_{n\times n}$ (ii) $[O:B]=O_{n\times n}$ *where I and 0 are the identity and zero transformations on V, respectively.*

Proof. Let $B=\{\alpha_1, \alpha_2 ... \alpha_n\}$

(i) $I(\alpha_1)=1.\alpha_1+0.\alpha_2 ...+0.\alpha_n$, $I(\alpha_2)=0.\alpha_1+1.\alpha_2 ...+0.\alpha_n$

... ... , $I(\alpha_n)=0.\alpha_1+0.\alpha_2 ...+1.\alpha_n$

$\therefore$ The matrix is the unit matrix

$$I_{n\times n}=\begin{bmatrix}1 & 0 & 0 & ... & 0\\ 0 & 1 & 0 & ... & 0\\ ... & ... & ... & ... & ...\\ 0 & 0 & 0 & ... & 1\end{bmatrix}_{n\times n}=[\delta_{ij}]_{n\times n}\text{; Kronecker delta }\begin{bmatrix}\delta_{ij}=1, i=j\\ =0, i\neq j\end{bmatrix}$$

(ii) Now $O(\alpha_1)=\bar{O}=0.\alpha_1+0.\alpha_2 ...+0.\alpha_n$, $O(\alpha_2)=\bar{O}=0.\alpha_1+0.\alpha_2 ...+0.\alpha_n$

... ... , $O(\alpha_n)=\bar{O}=0.\alpha_1+0.\alpha_2 ...+0.\alpha_n$

$$\therefore \text{ The matrix of zero transformation } [O;B]=\begin{bmatrix}0 & 0 & ... & 0\\ 0 & 0 & ... & 0\\ ... & & ... & \\ 0 & 0 & .. & 0\end{bmatrix}=O_{n\times n}$$

SOLVED PROBLEMS

Ex. 1. *Let* $T:V_2\to V_3$ *be defined by* $T(x,y)=(x+y, 2x-y, 7y)$ *Find* $[T:B_1, B_2]$ *where* B_1 *and* B_2 *are the standard bases of* V_2 *and* V_3. **(S. K. U. M 07)**

Sol. B_1 is the standard basis of V_2 and B_2 that of V_3

$\therefore$ $B_1=\{(1,0),(0,1)\}$ $B_2=\{(1,0,0),(0,1,0),(0,0,1)\}$

$\therefore$ $T(1,0)=(1,2,0) =1(1,0,0)+2(0,1,0)+0(0,0,1)$

$T(0,1)=(1,-1,7) =1(1,0,0)-1(0,1,0)+7(0,0,1)$

$$\therefore \text{ The matrix of T relative to } B_1 \text{ and } B_2 \text{ is } [T:B_1,B_2]=\begin{bmatrix}1 & 1\\ 2 & -1\\ 0 & 7\end{bmatrix}$$

Ex. 2. *Let* $T:R^3\to R^2$ *be the linear transformations defined by* $T(x,y,z)=(3x+2y-4z, x-5y+3z)$. *Find the matrix of T relative to the bases* $B_1=\{(1,1,1),(1,1,0),(1,0,0)\}$, $B_2=\{(1,3),(2,5)\}$

Sol. Let $(a,b)\in R^2$ and let $(a,b)=p(1,3)+q(2,5)=(p+2q, 3p+5q)$

$\Rightarrow p+2q=a$, $3p+5q=b$. Solving $p=-5a+2b$, $q=3a-b$

$\therefore$ $(a,b)=(-5a+2b)(1,3)+(3a-b)(2,5)$

Now $T(1,1,1)=(1,-1)=-7(1,3)+4(2,5)$, $T(1,1,0)=(5,-4)=-33(1,3)+19(2,5)$,

$T(1,0,0)=(3,1)=-13(1,3)+8(2,5)$

$$\therefore \text{ The matrix of L. T. relative to } B_1 \text{ and } B_2. \quad [T:B_1,B_2]=\begin{bmatrix}-7 & -33 & -13\\ 4 & 19 & 8\end{bmatrix}$$

Ex. 3. *If the matrix of a linear transformation T on* $V_3(R)$ *w.r. to the standard basis is* $\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{bmatrix}$ *what is the matrix of T w.r.to the basis.* $\{(0,1,-1),(1,-1,1),(-1,1,0)\}$

Sol. (i) Let the standard basis of $V_3(R)$ be $B=\{(1,0,0),(0,1,0),(0,0,1)\}$

$$\text{Let } \alpha_1=(1,0,0), \alpha_2=(0,1,0), \alpha_3=(0,0,1). \quad \therefore \text{ Given } [T]_B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{bmatrix}$$

$\therefore$ $T(\alpha_1)=0\,\alpha_1+1.\alpha_2+(-1)\alpha_3=(0,1,-1)$

$T(\alpha_2)=1.\alpha_1+0.\alpha_2-1.\alpha_3=(1,0,-1)$, $T(\alpha_3)=1.\alpha_1-1.\alpha_2+0.\alpha_3=(1,-1,0)$

Let $(a,b,c)\in V_3(R)$ then $(a,b,c)=a(1,0,0)+b(0,1,0)+c(0,0,1)=a\,\alpha_1+b\,\alpha_2+c\,\alpha_3$

$\therefore$ $T(a,b,c)=a\,T(\alpha_1)+b\,T(\alpha_2)+c\,T(\alpha_3)=a(0,1,-1)+b(1,0,-1)+c(1,-1,0)$

$=(b+c), a-c, -a-b)$ which is the required transformation.

(ii) Let $B_2=\{\beta_1,\beta_2,\beta_3\}$ where $\beta_1=(0,1,-1), \beta_2=(1,-1,1), \beta_3=(-1,1,0)$

Using the transformation $T(a,b,c)=(b+c, a-c, -a-b)$ we have

$T(\beta_1)=T(0,1,-1)=(0,1,-1)$; $T(\beta_2)=T(1,-1,1)=(0,0,0)$, $T(\beta_3)=T(-1,1,0)=(1,-1,0)$

Now Let $(a,b,c)=x\beta_1+y\beta_2+z\beta_3=x(0,1,-1)+y(1,-1,1)+z(-1,1,0)$

$=(y-z, x-y+z, -x+y)$

$y-z=a, x-y+z=b, -x+y=c$ $\Rightarrow x=a+b, y=a+b+c, z=b+c$

$\therefore$ $(a,b,c)=(a+b)\beta_1+(a+b+c)\beta_2+(b+c)\beta_3$

$\therefore$ $T(\beta_1)=(0,1,-1)=1.\beta_1+0.\beta_2+0.\beta_3$

$\therefore\ T(\beta_2) = (0,0,0) = 0.\beta_1 + 0.\beta_2 + 0.\beta_3$, $\qquad \therefore\ T(\beta_3) = (1,-1,0) = 0.\beta_1 + 0.\beta_2 - 1.\beta_3$

$$\therefore\ [T;B_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Ex. 4. *Let* $D: P_3 \to P_2$ *be the polynomial differential transformation* $D(p) = \frac{dp}{dx}$.

Find the matrix of D relative to the standard bases. $B_1 = \{1, x, x^2, x^3\}$ *and* $B_2 = \{1, x, x^2\}$

Sol. $D(1) = 0 = 0.1 + 0.x + 0.x^2$, $\quad D(x) = 1 = 1.1 + 0.x + 0.x^2$

$D(x^2) = 2x = 0.1 + 2.x + 0.x^2$, $\quad D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2$

$\therefore$ The matrix of D relative to B_1 and B_2 is $[T:B_1;B_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

5.3. Theorem. *Let U(F) and V(F) be two vector spaces such that dim U = n and dim V = m. Then corresponding to every matrix* $[a_{ij}]_{m\times n}$ *of mn scalars belonging to F there corresponds a unique linear transformation* $T:U \to V$ *such that* $[T:B;B'] = [a_{ij}]_{m\times n}$ *where B and B' are the ordered bases of U and V respectively.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

$$\text{Given } [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Now $\alpha_j \in B \Rightarrow T(\alpha_j) \in V$. $\qquad$ Let $T: U \to V$ be defined by

$$T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \ldots + a_{ij}\beta_i + \ldots + a_{mj}\beta_m = \sum_{i=1}^{m} a_{ij}\beta_i$$

Let $\alpha \in U$ be any vector then $\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

$\therefore\ T(\alpha) = b_1T(\alpha_1) + b_2T(\alpha_2) + \ldots + b_nT(\alpha_n)$

$$= b_1\sum_{i=1}^{m} a_{i1}\beta_i + b_2\sum_{i=1}^{m} a_{i2}\beta_i \ldots + b_n\sum_{i=1}^{m} a_{in}\beta_i = \sum_{j=1}^{n} b_j\left(\sum_{i=1}^{m} a_{ij}\beta_i\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}b_j\right)\beta_i$$

$$= \sum_{i=1}^{m} c_i\beta_i \text{ where } c_i = \sum_{i=1}^{n} a_{ij}b_j = c_1\beta_1 + c_2\beta_2 + \ldots + c_m\beta_m$$

$\Rightarrow T(\alpha)$ is expressible uniquely as a *l.c.* of the elements of B' $\quad (\because \beta$'*s* are unique)

$\Rightarrow$ T is unique.

Thus for all $\sum a_{ij}\beta_i \in V$ there is a L.T. from U to V such that

$$T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\beta_i \qquad (j = 1, 2 \ldots n)$$

Thus corresponding to the matrix $[a_{ij}]_{m\times n}$ there corresponds a transformation $T: U \to V$.

5.4. Theorem. 1. *Let U(F) and V(F) be two linear transformations so that* dim U = *n and* dim V = *m. Let B and B' be the ordered bases for U and V respectively. Let* $T:U \to V$ *then for all* $\alpha \in U$, $[T:B;B'][\alpha]_B = [T(\alpha)]_{B'}$ $\qquad$ *(S. V. U. O,01)* *where* $[\alpha]_B$ *is the coordinate matrix of* α *with respect to the basis B and* $[T(\alpha)]_{B'}$ *is the coordinate matrix of* $T(\alpha) \in V$ *with respect to B'.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

Let $A = [a_{ij}]$ be the matrix of T relative to B and B'. Then $[T:B,B'], A = [a_{ij}]_{m\times n}$

$$\Rightarrow \text{For } \alpha_j \in B,\ T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 \ldots + a_{ij}\beta_i + \ldots + a_{mj}\beta_m = \sum_{i=1}^{m} a_{1j}\beta_i$$

For any $\alpha \in U$, $\alpha = b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n \quad \forall\ b's \in F$

$$\therefore \text{ The coordinate matrix of } \alpha = [\alpha]_B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\therefore\ T(\alpha) = b_1T(\alpha_1) + b_2T(\alpha_2) \ldots + b_nT(\alpha_n) = \sum_{j=1}^{m} b_jT(\alpha_j)$$

$$= \sum_{j=1}^{n} b_j\left(\sum_{j=1}^{m} a_{ij}\beta_i\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} b_ja_{ij}\right)\beta_i$$

$$= \sum_{i=1}^{m} c_i\beta_i \quad \text{where } c_i = \sum_{j=1}^{n} b_j a_{ij} \quad \Rightarrow \quad T(\alpha) = c_1\beta_1 + c_2\beta_2 + \dots + c_m\beta_m$$

$\therefore$ The coordinate matrix of $T(\alpha)$ *w.r.* to the basis of B' is

$$[T(\alpha)]B' = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \dots \\ c_m \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \dots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \dots + a_{2n}b_n \\ \dots \quad \dots \quad \dots \\ a_{m1}b_1 + a_{m2}b_2 + \dots + a_{mn}b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} = A\,[\alpha]_B$$

Thus $[T(\alpha)]_{B'} = [T : B, B'][T(\alpha)]_B$

Theorem. 2. ***If V is an n dimensional vector space over F and B is an ordered basis of V, then prove that for any linear operator T on V and $\alpha \in V$, $[T(\alpha)]_B = [T]_B\,[\alpha]_{B'}$.***

Proof. Let $T : V \to V$ is a linear transformation and B is an ordered basis of V.

Let $B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$.

Let $A = [a_{ij}]_{n \times n}$ be the matrix of T relative to B.

Then $[T]_B = A = [a_{ij}]_{n \times n}$

$$\therefore \ \forall\ \alpha_j \in B, \ T(\alpha_j) = a_{1j}\alpha_1 + a_{2j}\alpha_2 + \dots + a_{ij}\alpha_i + \dots + a_{nj}\alpha_n = \sum_{i=1}^{n} a_{ij}\,\alpha_i$$

For any $\alpha \in V$, $\alpha = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$ for every $b \in F$.

$$\therefore \text{ The coordinate matrix of } \alpha = [\alpha]_B = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix}$$

$$\therefore \ T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n) = b_1T(\alpha_1) + b_2T(\alpha_2) + \dots + b_nT(\alpha_n)$$

$$= \sum_{j=1}^{n} b_jT(\alpha_j) = \sum_{j=1}^{n} b_j\left(\sum_{i=1}^{n} a_{ij}\,\alpha_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_ja_{ij}\right)\alpha_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\,b_j\right)\alpha_1$$

$$= \sum_{i=1}^{n} c_i\alpha_i \quad \text{where } c_i = \sum_{j=1}^{n} a_{ij}\,b_j$$

$\therefore$ The coordinate matrix of $T(\alpha)$ *w.r.t.* basis B is

$$[T(\alpha)]_B = \begin{bmatrix} c_1 \\ c_2 \\ : \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\,b_j \\ \sum_{j=1}^{n} a_{2j}\,b_j \\ \dots\dots \\ \dots\dots \\ \sum_{j=1}^{n} a_{nj}\,b_j \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \dots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \dots + a_{2n}b_n \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{n1}b_1 + a_{n2}b_2 + \dots + a_{nn}b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} = A[(\alpha)]_B = [T]_B\,[\alpha]_B$$

5.5. Theorem. ***Let U(F) and V(F) be two vector spaces so that dim U = n and dim V = m. Let T and H be the linear transformations from U to V. If B and B' are the ordered bases of U and V respectively then***

(i) [T + H : B : B′] = [T : B ; B′] + [H : B ; B′ ;]

(ii) [c T : B ; B′] = c [T : B ; B′] where $c \in F$.

Proof. Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and $B' = \{\beta_1, \beta_2 \dots \beta_m\}$

Let $[a_{ij}]_{m \times n}$ be the matrix [T : B ; B']. $\quad \therefore \ T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\beta_i, \ j = 1, 2 \dots n$

Let $[b_{ij}]_{m \times n}$ be the matrix [H : B ; B']. $\quad \therefore \ T(\alpha_j) = \sum_{i=1}^{m} b_{ij}\beta_i, \ j = 1, 2 \dots n.$

(*i*) Now $(T + H)(\alpha_j) = T(\alpha_j) + H(\alpha_j) \quad j = 1, 2 \dots n$

$$= \sum_{i=1}^{n} a_{ij}\beta_i + \sum_{i=1}^{m} b_{ij}\beta_i = \sum_{i=1}^{n} (a_{ij} + b_{ij})\,\beta_i$$

$\therefore$ Matrix of [T + H : B ; B'] = [T : B, B'] + [H : B ; B']

(*ii*) $(cT)(\alpha_j) = cT(\alpha_j) \quad (j = 1, 2 \dots n) \qquad = c \sum_{i=1}^{m} a_{ij}\beta_i = \sum_{i=1}^{m} (c\,a_{ij})\,\beta_i$

$\therefore$ The matrix of cT relative to B and B' is

$$[cT:B;B'] = [c\,a_{ij}]_{m\times n} = c\,[a_{ij}]_{m\times n} = c\,[T:B;B']$$

Note. If U = V and the bases are such that B = B'

Then (*i*) $[T+H] = [T]+[H]$ (*ii*) $[c\,T] = c\,[T]$

5.6. Theorem. ***Let T and H be linear operators on an n-dimensional vector space V(F). If B is the basis of V then [TH] = [T] [H]***

Proof. Let $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$. Let $[a_{ik}]_{n\times n}$ be the matrix [T]

$$T(\alpha_k) = \sum_{i=1}^{n} a_{ik}\,\alpha_i \quad k = 1, 2 \ldots n.$$

Let $[b_{kj}]_{n\times n}$ be the matrix [H]. Then $H(\alpha_j) = \sum_{k=1}^{n} b_{kj}\,\alpha_k \quad j = 1, 2 \ldots n.$

$\therefore$ Now $(T\,H)(\alpha_j) = T\,[H(\alpha_j)]$

$$= T\left(\sum_{k=1}^{n} b_{kj}\,\alpha_k\right) \qquad = \sum_{k=1}^{n} b_{kj}\,T(\alpha_k) \qquad [\text{T is L.T.}$$

$$= \sum_{k=1}^{n} b_{kj}\left(\sum_{i=1}^{n} a_{ik}\alpha_i\right) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)\alpha_i = \sum_{i=1}^{n} c_{ij}\,\alpha_i \quad \text{where } c_i = \sum_{k=1}^{n} a_{ik}\,b_{kj}$$

$\therefore$ The matrix $[TH] = [c_{ij}]_{n\times n} = [\Sigma\, a_{ik}\, b_{kj}]_{n\times n} = [a_{ik}]\;[b_{kj}] = [T]\;[H].$

5.7. Theorem. ***Let T be a linear operator on an n-dimensional vector space V and let B be an ordered basis for V. Then T is invertible iff*** $[T]_B$ ***is an invertible matrix and*** $[T^{-1}]_B = [[T]_B]^{-1}$

Proof. (*i*) Let T be invertible. $\therefore$ T^{-1} exists and $T^{-1}T = I = TT^{-1}$

$\Rightarrow$ $[T^{-1}T]_B = [I]_B = [TT^{-1}]_B$ $\Rightarrow$ $[T^{-1}]_B\,[T]_B = I = [T]_B\,[T^{-1}]_B$

$\Rightarrow$ $[T]_B$ is invertible and $[[T]_B]^{-1} = [T^{-1}]_B$.

(*ii*) Let $[T]_B$ be an invertible matrix. $[T]_{B^{-1}}$ exists.

$\therefore$ There exists a linear operator H on V such that $[T]^{-1} = [H]$

$\Rightarrow$ $[H][T] = [I] = [T][H]$ $\Rightarrow$ $[HT] = [I] = [TH]$

$\Rightarrow$ HT = I = TH $\Rightarrow$ T is invertible.

CHANGE OF BASIS

5.8. Transition Matrix. Let V(F) be an n-dimensional vector space and $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_n\}$ be two arbitrary bases of V.

Let us suppose

$\beta_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots a_{n1}\alpha_n$; $\quad \beta_2 = a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots a_{n2}\alpha_n$

....; $\quad \beta_n = a_{1n}\alpha_1 + a_{2n}\alpha_2 + \ldots a_{nn}\alpha_n$

The matrix of transformation from B to B' is $P = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n}$

$\therefore$ The matrix P is called the transition matrix from B to B'.

Note. Since $B' = \{\beta_1, \beta_2, \ldots \beta_n\}$ is an L.I. set, the matrix P is invertible and P^{-1} is a transition matrix from B' to B.

5.9. Theorem. ***Let P be the transition matrix from a basis B to a basis B' in an n-dimensional vector space V(F). Then for*** $\alpha \in V$ (*i*) $P[\alpha]_{B'} = [\alpha]_B$ (*ii*) $[\alpha]_{B'} = P^{-1}[\alpha]_B$

Proof. Let $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_n\}$ be two ordered bases of $V_n(F)$

Let the transition matrix $P = [a_{ii}]_{n\times n}$. Then there exists a unique linear transformation T in V such that $T(\alpha_j) = \beta_j \quad j = 1, 2 \ldots n.$

Since T is an onto function, then T is invertible.

$\Rightarrow$ $[T]_B = P$ is a unque matrix and hence invertible.

(*i*) Now $T(\alpha_j) = \beta_j = a_{1j}\alpha_1 + a_{2j}\alpha_2 + \ldots a_{nj}\alpha_n = \sum_{i=1}^{n} a_{ij}\alpha_i \qquad (j = 1, 2 \ldots n)$

For $\alpha \in V$, $\alpha = l.c.$ of elements $B' = b_1\beta_1 + b_2\beta_2 \ldots + b_n\beta_n = \sum_{j=1}^{n} b_j\,\beta_j \qquad (b's \in F)$

$\Rightarrow$ The coordinate matrix of $[\alpha]_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Also $\alpha = \sum_{j=1}^{n} b_j\,\beta_j = \sum_{j=1}^{n} b_j\left(\sum_{i=1}^{n} a_{ij}\alpha_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_j\right)\alpha_i = \sum_{i=1}^{n}(a_{i1}b_1 + a_{i2}b_2 \ldots + a_{in}b_n)\,\alpha_i$

$\therefore$ The coordinate matrix of α relative to B

$$[\alpha]_B = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \dots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \dots + a_{2n}b_n \\ \dots \quad \dots \quad \dots \\ a_{n1}b_1 + a_{n2}b_2 + \dots + a_{nn}b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \qquad \therefore [\alpha]_B = P[\alpha]_{B'}$$

(*ii*) Pre multiplying by P^{-1}. $\qquad P^{-1}[\alpha]_B = P^{-1}P[\alpha]_{B'}$

$\Rightarrow P^{-1}[\alpha]_B = I[\alpha]_{B'} \qquad \Rightarrow P^{-1}[\alpha]_B = [\alpha]_{B'}$.

5.10. Theorem. *Let P be the transition matrix from a basis B to a basis B' in a vector space $V_n(F)$. Then for any linear operator T on V, $[T ; B]P = P[T ; B]$*

Proof. Let $\alpha \in V$ be arbitary vector. Then by the previous theorems we have

$P[\alpha]_{B'} = [\alpha]_B$ ... (1) and $[T]_B[\alpha]_B = [T(\alpha)]_B$ ... (2)

Now pre-multiplying (1) by $P^{-1}[T]_B$ we get $P^{-1}[T]_B P[\alpha]_{B'} = P^{-1}[T]_B[\alpha]_B$

$= P^{-1}[T(\alpha)]_B$ (by (2)) $\qquad = [T(\alpha)]_{B'}$ (by (1))

$\therefore P^{-1}[T]_B P[\alpha]_{B'} = [T]_{B'}[\alpha]_{B'} \Rightarrow P^{-1}[T]_B P = [T]_{B'}$

Premultiplying with P $[T]_B P = P[T]_{B'}$ or $[T : B]P = P[T : B']$

SOLVED PROBLEMS

Ex. 1. *Let T be the linear operator on* R^2 *defined by* $T(x, y) = (4x - 2y, 2x + y)$

Find the matrix of T w.r. to the basis $T\{(1,1), (-1,0)\}$.

Also verify $[T]_B[\alpha]_B = [T](\alpha)_B \quad \forall\ \alpha \in R^2$

Sol. Let $(a, b) \in R^2$. Then $(a, b) = p(1,1) + q(-1,0) \quad = (p - q, p)$

$\Rightarrow a = p - q$ and $b = p \qquad \Rightarrow p = b$ and $q = b - a$

$\therefore (a, b) = b(1,1) + (b - a)(-1,0)$ ... (1)

(*i*) Given transformation is $T(x, y) = (4x - 2y, 2x + y)$

$\therefore T(1,1) = (2,3) \quad = 3(1,1) + 1(-1,0)$ .... (by (1))

$T(-1,0) = (-4,-2) \quad = -2(1,1) + 2(-1,0) \qquad \therefore [T : B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

(*ii*) Let $\alpha \in R^2$ where $\alpha = (a, b)$

$\therefore \alpha = (a, b) = b(1,1) + (b - a)(-1,0) \qquad \therefore [\alpha]_B = \begin{bmatrix} b \\ b - a \end{bmatrix}$

$\therefore [T]_B[\alpha]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ b - a \end{bmatrix} = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix}$ ... (1)

Again $T(\alpha) = T(a, b) = (4a - 2b, 2a + b)$

Let $(4a - 2b, 2a + b) = x(1,1) + y(-1,0) \quad = (x - y, x)$

$\Rightarrow 4a - 2b = x - y$ and $2a + b = x \qquad \Rightarrow x = 2a + b$ and $y = -2a + 3b$

Hence $T(\alpha) = T(a, b) \quad = (2a + b)(1,1) + (-2a + 3b)(-1,0)$

$\therefore$ The matrix of $T(\alpha)$ *w.r.* to the base B is $[T(\alpha)]_B = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix}$ ... (2)

$\therefore$ From (1) and (2) $[T]_B[\alpha]_B = [T(\alpha)]_B$.

Ex. 2. *Let T be a linear operator on* $V_3(R)$ *defined by*

$T(x, y, z) = (3x + z, -2x + y, -x + 2y + z)$. *Prove that T is invertible and find* T^{-1}.

Sol. Consider the standard basis $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

Given transformation is $T(x, y, z) = (3x + z, -2x + y, -x + 2y + z)$

$\therefore T(1,0,0) = (3,-2,-1) = 3(1,0,0) - 2(0,1,0) - 1(0,0,1)$

$\therefore T(0,1,0) = (0,1,2) = 0(1,0,0) + 1(0,1,0) + 2(0,0,1)$

$\therefore T(0,0,1) = (1,0,4) = 1(1,0,0) + 0(0,1,0) + 4(0,0,1)$

$\therefore [T]_B = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} = P$ (say) $\qquad$ T is invertible if $[T]_B$ is invertible.

$\det P = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{vmatrix} = 9 \qquad \because \det P \neq 0$, the matrix P is inversible

Calculating $P^{-1} = \dfrac{adj\ P}{\det P}$ we get $P^{-1} = \dfrac{1}{9} \begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & -2 \\ -3 & -6 & 3 \end{bmatrix}$. $\qquad \therefore P^{-1} = [T]_B^{-1} = [T^{-1}]_B$

To find the transformation T^{-1} take $\alpha \in V$ where $\alpha = (a, b, c)$

$\Rightarrow \alpha = a\,(1,0,0) + b\,(0,1,0) + c\,(0,0,1)$ $\Rightarrow (\alpha)_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

We know that $[T^{-1}(\alpha)]_B = [T^{-1}]_B\,[\alpha]_B = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & -2 \\ -3 & -6 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\therefore\ [T^{-1}(\alpha)]_B = \begin{bmatrix} 4a+2b-c \\ 8a+13b-2c \\ -3a-6b+3c \end{bmatrix}$

$\Rightarrow T^{-1}(\alpha) = T^{-1}(a,b,c) = (4a+2b-c,\ 8a+13b-2c,\ -3a-6b+3c)$

Ex. 3. *Let* $B = \{(1,0),(0,1)\}$ *and* $B' = \{(1,3),(2,5)\}$ *be the bases of* R^2. *Find the transition matrices from B to* B' *and* B' *to B.*

Sol. (*i*) Given $B = \{(1,0),(0,1)\}$ and $B' = \{(1,3),(2,5)\}$

Now $(1,3) = 1\,(1,0) + 3\,(0,1);\ (2,5) = 2\,(1,0) + 5\,(0,1)$

$\therefore$ The transition matrix from B to $B' = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

(*ii*) Again $(1,0) = a\,(1,3) + b\,(2,5) = (a+2b, 3a+5b)$

$\therefore\ 1 = a+2b$ and $0 = 3a+5b$ $\Rightarrow a = -5$ and $b = 3$

Similarly $(0,1) = x\,(1,3) + y\,(2,5) = (x+2y, 3x+5y) \Rightarrow x+2y = 0$ and $3x+5y = 1$

$\Rightarrow x = 2$ and $y = -1$. Hence $(1,0) = -5\,(1,3) + 3\,(2,5);\ (0,1) = 2\,(1,3) - 1\,(2,5)$.

Ex. 4. *Let T be the linear operator on* $V_3(R)$ *defined by* $T(x,y,z) = (2y+z, x-4y, 3x)$

(*i*) *Find the matrix of T relative to the basis* $B = \{(1,1,1),(1,1,0),(1,0,0)\}$

(*ii*) *Verify* $[T(\alpha)]_B = [T]_B\,[\alpha]_B$

Sol. Let $\alpha = (a,b,c) \in V_3(R)$

$\therefore\ \alpha = (a,b,c) = p\,(1,1,1) + q\,(1,1,0) + r\,(1,0,0) = (p+q+r, p+q, p)$

$\therefore\ a = p+q+r, b = p+q, c = p$ $\Rightarrow p = c, q = b-c, r = a-b$

$\therefore \alpha\,(a,b,c) = c\,(1,1,1) + (b-c)\,(1,1,0) + (a-b)\,(1,0,0)$

$\Rightarrow [\alpha]_B = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$. Given $T(x,y,z) = (2y+z, x-4y, 3x)$

$\therefore\ T(1,1,1) = (3,-3,3) = l\,(1,1,1) + m\,(1,1,0) + n\,(1,0,0) = (l+m+n, l+m, l)$

$\Rightarrow l+m+n = 3, l+m = -3$ and $l = 3$ $\Rightarrow l = 3, m = -6, n = 6$

$\therefore\ T(1,1,1) = 3\,(1,1,1) - 6\,(1,1,0) + 6\,(1,0,0)$

Similarly $T(1,1,0) = (2,-3,3) = 3\,(1,1,1) - 6\,(1,1,0) + 5\,(1,0,0)$

$T(1,0,0) = (0,1,3) = 3\,(1,1,1) - 2\,(1,1,0) - 1\,(1,0,0)$ $\therefore\ [T]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & 1 \end{bmatrix}$

Now $(T)_B(\alpha)_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & 1 \end{bmatrix}\begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix} = \begin{bmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{bmatrix}$ .... (1)

Again $T(\alpha) = T(a,b,c) = (2b+c, a-4b, 3a)$

$= 3a\,(1,1,1) + (a-4b-3a)\,(1,1,0) + (2b+c-a+4b)\,(1,0,0)$

$= 3a\,(1,1,1) - (2a+4b)\,(1,1,0) + (-a+6b+c)\,(1,0,0)$

$\therefore\ [T(\alpha)]_B = \begin{bmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{bmatrix}$. Hence $[T(\alpha)]_B = [T]_B\,[\alpha]_B$.

Ex. 5. *Let* $T: R^3 \to R^2$ *be a linear transformation defined by*

(*i*) *Obtain the matrix of T relative to the bases*

$B_1 = \{(1,1,1),(1,1,0),(1,0,0)\}$; $B_2 = \{(1,3),(1,4)\}$

(*ii*) *Verify for any vector* $\alpha \in R^3$ $[T:B_1,B_2]\,[\alpha:B_1] = [T(\alpha):B_2]$

Sol. Let $(a,b) \in R^2$ and let $(a,b) = p\,(1,3) + q\,(1,4) \Rightarrow (p+q, 3p+4q)$

$\Rightarrow a = p+q$ and $b = 3p+4q$ $\Rightarrow p = 4a-b$ and $q = b-3a$

$\therefore\ (a,b) = (4a-b)\,(1,3) + (b-3a)\,(1,4)$ .... (1)

Given $T(x,y,z) = (2x+y-z, 3x-2y+4z)$

$\therefore\ T(1,1,1) = (2,5) = (4.2-5)\,(1,3) + (5-3.2)\,(1,4) = 3\,(1,3) + (-1)\,(1,4)$

$T(1,1,0) = (3,1) = 11\,(1,3) - 8\,(1,4)$

(*ii*) Let $\alpha = (x,y,z) \in R^3$

$\therefore\ (x,y,z) = l\,(1,1,1) + m\,(1,1,0) + n\,(1,0,0) = (l+m+n, l+m, l)$

$\Rightarrow x = l+m+n, y = l+m, z = l$ $\Rightarrow l = z, m = y-z, n = x-y$

$\therefore\ \alpha = (x,y,z) = z\,(1,1,1) + (y-z)\,(1,1,0) + (x-y)\,(1,0,0)$

$$\therefore [\alpha]_{B_1} = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} \qquad \therefore [T:B_1;B_2](\alpha_{B_1}) = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} = \begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$$

Also $T(\alpha) = T(x, y, z) = (2x+y-z, 3x-2y+4z)$

$= \{4(2x+y-z)-(3x-2y+4z)\,(1,3)\} + \{(3x-2y+4z)-3(2x+y-z)\}\,(1,4)$

$= (5x+6y-8z)\,(1,3) + (-3x-5y+7z)\,(1,4)$

$$\therefore [T(\alpha)]_{B_2} = \begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}. \qquad \text{From I and II : } [T:B_1,B_2]\,[\alpha:B_1] = [T(\alpha):B_2]$$

Ex. 6. *Let V(F) be a vector space of polynomials in x of degree atmost 3 and D be the differential operator on V.*

If the basis for V(F) is $B = \{1, x, x^2, x^3\}$ verify $[D:B]\,[\alpha:B] = [D(\alpha):B]$.

Sol. The basis $B = \{1, x, x^2, x^3\}$. $\therefore D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$

$D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$; $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$

$D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$

$$\therefore [D:B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \text{Let } \alpha = f(x) = a + bx + cx^2 + dx^3 \in V$$

$$\therefore [\alpha:B] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \therefore [D:B]\,[\alpha:B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}$$

Again $D(\alpha) = D\{f(x)\} = b + 2cx + 3dx^2$

$$[D(\alpha):B] = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}. \qquad \text{Hence } [D:B]\,[\alpha:\beta] = [D(\alpha):B]$$

Ex. 7. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Let T be a linear operator on* R^2 *defined by* $T(\alpha) = A\alpha$, *where α is written as a column vector. Find the matrix of T relative to the basis* $\{(1,0), (0,1)\}$

Sol. Let $T(1,0) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\,(1,0) + 3\,(0,1)$

$T(0,1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\,(1,0) + 4\,(0,1)$ $\qquad\therefore$ The matrix of $T = [T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Ex. 8. *If C(R) is a vector space having the bases* $B_1 = \{1, i\}$ *and* $B_2 = \{1+i, 1+2i\}$, *find the transition matrix of T from* B_1 *to* B_2

Sol. Now $(1+i) = 1\,(1) + 1\,(i)$; $(1+2i) = 1\,(1) + 2\,(i)$

$\therefore$ The transition matrix from B_1 to B_2 is $[T:B_1;B_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

5.11. We complete this section with the introduction of the "Left-Multiplication Transformation" L_A, where A is $m \times n$ matrix. This transformation is most useful in transfering properties about transformations to analogous properties about matrices and vice versa.

5.12. Def : *Let A be an* $m \times n$ *matrix with elements from a field F. We denote by* L_A, *the mapping* $L_A : F^n \to F^m$ *defined by* $L_A(x) = Ax$ *(the matrix product of A and x) for each column vector* $x \in F^n$. *We call* L_A *as "Left - Multiplication transformation".*

Ex. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Then $A \in M_{2\times 3}(R)$ and $L_A : R^3 \to R^2$

Take $x = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ then $L_A(x) = Ax = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

5.13. Theorem. ***Let A be an*** $m \times n$ ***matrix with elements from F. Then the left multiplication transformation*** $L_A : F^n \to F^m$ ***is linear. Further, if B is any other*** $m \times n$ ***matrix (with elements from F) and*** β, γ ***are the standard ordered bases for*** F^n ***and*** F^m, ***respectively then we have***

(i) $[L_A]_\beta^\gamma = A$ **(ii)** $L_A = L_B$ **iff** $A = B$ **(iii)** $L_{A+B} = L_A + L_B$, $L_{aA} = aL_A \forall a \in F$

(iv) If $T : F^n \to F^m$ **is linear, then there exists a unique** $m \times n$ **matrix** c **such that** $T = L_C$ **in fact** $C = [T]_\beta^\gamma$

(v) If E is an $n \times p$ **matrix then** $L_{AE} = L_A L_E$ **(vi) If** $m = n$, **then** $LI_n = I_{F^n}$.

Proof. Let $a, b \in F$ and x, y are two vectors then we have $L_A(ax+by) = A(ax+by)$

$= A(ax) + A(by) = a(Ax) + b(Ay)$

$= a L_A(x) + b L_A(y)$

Thus L_A is a linear transformation.

(*i*) The j^{th} column of $[L_A]_\beta^\gamma = L_A(e_j)$.

However, $L_A(e_j) = A(e_j)$. This is the j^{th} column of A.

Thus $[L_A]_\beta^\gamma = A$

(*ii*) If $L_A = L_B$, we have by (*i*) $[L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$ converse is trivial.

(*iii*) $L_{A+B}(x) = (A+B)(x) = Ax + Bx = L_A(x) + L_B(x)$

$\therefore L_{A+B} = L_A + L_B$

$L_{aA}(x) = (aA)\, x = a\,(A\, x) = a\, L_A(x)$

$\therefore L_{aA} = a\, L_A$

(*iv*) Let $C = [T]_\beta^\gamma$. We know that $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$ or $T(x) = C_x = L_C(x)\ \forall\, x \in F^n$

Thus $T = L_C$.

Uniqueness of *c* follows from (*iii*), (*v*), (*vi*) are simple to follow.

As remarked earlier we use the definition of L_A, to prove the properties of matrices. Here we prove the associativity of matrix multiplication.

Theorem. Let A, B, C be the matrices such that A (BC) is defined. Thus (AB) C is also defined and A(BC) = (AB) C. *i.e.* matrix multiplication is associative.

Proof. We can use properties of matrices to show that (AB) C is defined.

Now $L_{A(BC)} = L_A(L_{BC}) = L_A(L_B L_C)$

$= (L_A L_B) L_C = (L_{AB}) L_C = L_{(AB)C}$

EXERSICE 5

1. Find the matrix of linear transformation T on R^3 defined by $T(x, y, z) = (2y+z, x-4y, 3x)$ with respect to the ordered basis $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

2. If the matrix of T on R^2 relative to the standard basis is $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$ find the matrix of T relative to the basis $\{(1,1), (1,-1)\}$.

3. If the matrix of T on R^3 relative to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ find the matrix relative to the basis $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$

4. If the matrix of transformation T on $V_3(R)$ relative to the standard basis is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ find the matrix of T relative to the basis $S = \{(1,2,2), (1,1,2), (1,2,1)\}$

5. Let T be a linear operator on R^3 defined by $T(x, y) = (2y, 3x-y)$ Find the matrix of T relative to the basis $\{(1,3), (2,5)\}$

6. Let T be a linear transformation from R^3 to R^2 defined by $T(x, y, z) = (x+y, 2z-x)$ Find the matrix of T from the basis $\{(1,0,-1), (1,1,1), (1,0,0)\}$ to the base $\{(0,1), (1,0)\}$

7. If T be linear operation on R^3 defined by $T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 + x_2, -x_1, 2x_2 + 4x_3)$ determine the matrix of T relative to the standard basis of R^3

8. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and T be the linear operator on R^2 defined by $T(\alpha) = A\,\alpha$, where α is written as a column vector. Find the matrix of T relative to the basis $\{(1,3), (2,5)\}$

9. If C(R) is a vector space having the basis as $B_1 = \{1, i\}$ and $B_2 = \{1+i, 1+2i\}$ Find the transition matrix from B_2 to B_1

10. Let the matrix of T relative to the bases B_1 and B_2 be Find the linear transformation relative to the bases B_1 and B_2 where
    (*i*) B_1 and B_2 are standard bases of V_2 and V_3 respectively.
    (*ii*) $B_1 = \{(1,1), (-1,1)\}; B_2 = \{(1,1,1), (1,-1,1), (0,0,1)\}$
    (*iii*) $B_1 = \{(1,2), (-2,1)\}; B_2 = \{(1,-1,-1), (1,2,3), (-1,0,2)\}$

11. Given $[T : B_1; B_2] = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ find the linear transformation when
    (*i*) B_1 and B_2 are standard bases of V_3 and V_2 respectively.
    (*ii*) $B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$; $B_2 = \{(1,1), (1,-1)\}$
    (*iii*) $B_1 = \{(1,-1,1), (1,2,0), (0,-1,0)\}$; $B_2 = \{(1,0), (2,-1)\}$

12. The matrix of linear transformation $T : V_4 \to V_3$

8.3. INNER PRODUCT SPACE

Definition. 1. *Let $V(F)$ be a vector space where F is the field of real numbers or the field of complex numbers. The vector space $V(F)$ is said to be an inner product space if there is defined for any two vectors $\alpha, \beta \in V$ an element $\langle \alpha, \beta \rangle \in F$ such that*

(1) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ (conjugate symmetry)

(2) $\langle \alpha, \alpha \rangle > 0$ (zero element in F) for $\alpha \neq \bar{0}$, the zero element in V (Positivity) and

(3) $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$ (Distributivity) for any $\alpha, \beta, \gamma \in V$ and $a, b \in F$.

(*K.U. 08O. U. 04, 05, N. U. 03, 04, 08*)

A function $f: V \times V \to F$ satisfying the above properties is called an **inner product.** If f is the **inner product** function then $f\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle$ or (α, β) $\forall$ $\alpha, \beta \in V$.

Note. 1. The vectors $\alpha, \beta \in V$ while the inner product is a scalar $\langle \alpha, \beta \rangle \in F$.

2. In the property (3) : the '+' sign' in $a\alpha + b\beta$ is addition in V while '+' sign' in $a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$ is addition in F.

3. If F is the field of complex numbers $\overline{\langle \beta, \alpha \rangle}$ denotes the complex conjugate of $\langle \beta, \alpha \rangle \in F$.

4. If F is the field of real numbers, then $\langle \beta, \alpha \rangle$ is a real number and $\overline{\langle \beta, \alpha \rangle} = \langle \beta, \alpha \rangle$.

5. By conjugate symmetry property; $\langle \alpha, \alpha \rangle = \overline{\langle \alpha, \alpha \rangle}$ for every $\alpha \in V$ $(Z = \overline{Z})$ and hence $\langle \alpha, \alpha \rangle$ **is always real.** Thus the order property (2) makes sense.

Definition. 2. *Let $V(F)$ be a vector space where F is the field of real numbers or the field of complex numbers. The vector space $V(F)$ is said to be inner product space if there is defined for any two vectors $\alpha, \beta \in V$ an element $\langle \alpha, \beta \rangle \in F$ such that*

(1) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ (conjugate symmetry)

(2) $\langle \alpha, \alpha \rangle > 0$ (zero element in F) for $\alpha \neq \bar{0}$ the zero element in V (Positivity)

(3) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$ (Distributivity) and

(4) $\langle a\alpha, \gamma \rangle = a\langle \alpha, \gamma \rangle$ for any $\alpha, \beta, \gamma \in V$ and $a \in F$.

Note. Definition (1) and Definition (2) are equivalent.

From the definition it is clear that a vector space V over F endowed with a specific inner product is an inner product space.

If $F = R$ the field of real numbers then $V(F)$ is called **Euclidean space or Real inner product space.**

If $F = C$, the field of complex numbers then $V(F)$ is called **Unitary space or Complex inner product space.**

An inner product space having only zero vector is called zero space or null space.

If $V(F)$ is an inner product space then $V(F)$ is a vector space. A sub space $W(F)$ of the vector space $V(F)$ is also inner product space with the same inner product as in $V(F)$.

ILLUSTRATIONS.

1. If $\alpha = (a_1, a_2, a_3), \beta = (b_1, b_2, b_3)$ are the elements of a vector space R^3, then $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + a_3b_3$ defines an inner product on R^3.

Let $\alpha = (a_1, a_2, a_3), \beta = (b_1, b_2, b_3)$ and $\gamma = (c_1, c_2, c_3) \in R^3$.

Then $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in R$.

(1) $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$ ($\because$ a's and b's are real numbers)

$$= \langle \beta, \alpha \rangle = \overline{\langle \beta, \alpha \rangle}.$$

(2) $\langle \alpha, \alpha \rangle = a_1a_1 + a_2a_2 + a_3a_3 = a_1^2 + a_2^2 + a_3^2$

If $\alpha = (a_1, a_2, a_3) \neq (0,0,0)$ then at least one of a_1, a_2, a_3 is not zero.

So, $\langle \alpha, \alpha \rangle = a_1^2 + a_2^2 + a_3^2 > 0$.

(3) For $a, b \in R$ and $\alpha, \beta, \gamma \in R^3$ we have

$$a\alpha + b\beta = a\,(a_1, a_2, a_3) + b\,(b_1, b_2, b_3) = (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$$

$$\langle a\alpha + b\beta, \gamma \rangle = (aa_1 + bb_1)\, c_1 + (aa_2 + bb_2)\, c_2 + (aa_3 + bb_3)\, c_3$$

$$= (aa_1c_1 + aa_2c_2 + aa_3c_3) + (bb_1c_1 + bb_2c_2 + bb_3c_3)$$

$$= a\,(a_1c_1 + a_2c_2 + a_3c_3) + b\,(b_1c_1 + b_2c_2 + b_3c_3) = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$$

$\therefore$ The product $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + a_3b_3$ is an inner product on the vector space R^3.

Hence R^3 is an inner product space with the above inner product and $R^3(R)$ is real inner product space.

Note. 1. The inner product of α and β namely, $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + a_3b_3$ is called the dot product of α and β and is denoted by $\alpha \,.\, \beta$. This is called the **standard inner product** in R^3.

Note. 2. If $\alpha = (a_1, a_2, ..., a_n), \beta = (b_1, b_2, ..., b_n)$ are any two vectors of the vector space $V_n(R)$ or R^n then $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + ... + a_nb_n$ is called the standard inner product.

2. If $\alpha = (a_1, a_2, ..., a_n)$ and $\beta = (b_1, b_2, ..., b_n)$ are the elements of the vector space $V_n(C)$ where C is the field of complex numbers, then

$\langle \alpha, \beta \rangle = a_1\overline{b_1} + a_2\overline{b_2} + ... + a_n\overline{b_n} = \sum_{i=1}^{n} a_i\overline{b_i}$ defines an inner product on $V_n(C)$.

Let $a, b \in C$ and $\alpha, \beta, \gamma \in V_n$ so that $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$, $\gamma = (c_1, c_2, ..., c_n)$ where a's, b's and c's are complex numbers.

(If Z is a complex number we know that $\overline{\overline{Z}} = Z$ and $Z\overline{Z} = |Z|^2$, $\overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2$)

(1) $\overline{\langle \beta, \alpha \rangle} = \overline{b_1\overline{a_1} + b_2\overline{a_2} + ... + b_n\overline{a_n}} = \overline{b_1\overline{a_1}} + \overline{b_2\overline{a_2}} + ... + \overline{b_n\overline{a_n}}$

$= \overline{b_1}\overline{\overline{a_1}} + \overline{b_2}\overline{\overline{a_2}} + ... + \overline{b_n}\overline{\overline{a_n}} = \overline{b_1}a_1 + \overline{b_2}a_2 + ... + \overline{b_n}a_n = a_1\overline{b_1} + a_2\overline{b_2} + ... + a_n\overline{b_n} = \langle \alpha, \beta \rangle$.

(2) $\langle \alpha, \alpha \rangle = a_1\overline{a_1} + a_2\overline{a_2} + ... + a_n\overline{a_n} = |a_1|^2 + |a_2|^2 + ... + |a_n|^2$

If $\alpha \neq 0$ then at least one of $a_1, a_2, ..., a_n$ is non-zero complex number.

So $\langle \alpha, \alpha \rangle = |a_1|^2 + |a_2|^2 + ... + |a_n|^2 > 0$.

(3) $a\alpha + b\beta = a\,(a_1, a_2, ..., a_n) + b\,(b_1, b_2, ..., b_n) = (aa_1 + bb_1, aa_2 + bb_2, ..., aa_n + bb_n)$

$\langle a\alpha + b\beta, \gamma \rangle = (aa_1 + bb_1)\,\overline{c_1} + (aa_2 + bb_2)\,\overline{c_2} + ... + (aa_n + bb_n)\,\overline{c_n}$

$= (aa_1\overline{c_1} + aa_2\overline{c_2} + ... + aa_n\overline{c_n}) + (bb_1\overline{c_1} + bb_2\overline{c_2} + ... + bb_n\overline{c_n})$

$= a\,(a_1\overline{c_1} + a_2\overline{c_2} + ... + a_n\overline{c_n}) + b\,(b_1\overline{c_1} + b_2\overline{c_2} + ... + b_n\overline{c_n}) = a\langle \alpha, \gamma \rangle + b\,\langle \beta, \gamma \rangle$.

$\therefore$ the product $\langle \alpha, \beta \rangle = a_1\overline{b_1} + a_2\overline{b_2} + ... + a_n\overline{b_n}$ is an inner product on $V_n(C)$.

$\therefore$ $V_n(C)$ or $C^n(C)$ is the unitary space.

Note. 1. The product above defined is called **standard inner product** on $V_n(C)$.

2. We can always define an inner product on a finite dimensional vector space of real or complex numbers as follows :

Let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of $V_n(F)$.

$\alpha, \beta \in V_n \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ and $\beta = b_1\alpha_1 + b_2\alpha_2 + ... + b_n\alpha_n$ where $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in F$

Then $<\alpha, \beta> = a_1\overline{b_1} + a_2\overline{b_2} + ... + a_n\overline{b_n} = \sum_{i=1}^{n} a_i\overline{b_i}$ defines inner product on V_n.

3. If $\alpha = (1+i, 4)$ and $\beta = (2-3i, 4+5i)$ in $V_2(C)$ or C^2 then

$\langle \alpha, \beta \rangle = (1+i)\,(\overline{2-3i}) + 4\,(\overline{4+5i}) = (1+i)\,(2+3i) + 4\,(4-5i) = 15-15i$

$\langle \alpha, \alpha \rangle = (1+i)\,(\overline{1+i}) + 4\,(\overline{4}) = (1+i)\,(1-i) + 4(4) = 18$

$\overline{\langle \beta, \alpha \rangle} = \{\overline{(2-3i)\,(1-i) + (4+5i)\,4}\} = \{\overline{(-1-5i) + (16+20i)}\} = \overline{15+15i} = 15-15i = \langle \alpha, \beta \rangle$.

3. Let $V(C)$ be the vector space of all continuous complex valued functions on the closed interval $[0,1]$. For $f, g \in V$ if $\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}$, then V is an inner product space.

(O. U. 04, 05)

f is a continuous complex - valued function on $[0,1]$ means that $f(t)$ is a complex number for $t \in [0,1]$.

Let $a, b \in C$ and $f, g, h \in V$ where $t \in [0,1]$.

(1) $\overline{\langle g, f \rangle} = \overline{\left\{ \int_0^1 g(t)\,\overline{f(t)}\,dt \right\}} = \int_0^1 \overline{g(t)\overline{f(t)}}\,dt$. $\qquad \left(\because \overline{\int_0^1 f(t)\,dt} = \int_0^1 \overline{f(t)}\,dt \right)$

$= \int_0^1 \overline{g(t)}\,\overline{\overline{f(t)}}\,dt = \int_0^1 \overline{g(t)} f(t)\,dt = \int_0^1 f(t)\overline{g(t)}\,dt = \langle f, g \rangle$. $\qquad (\because \overline{\overline{Z}} = Z)$

(2) $\langle f, f \rangle = \int_0^1 f(t)\overline{f(t)}\,dt = \int_0^1 |f(t)|^2\,dt$ $\qquad (\because Z\overline{Z} = |Z|^2)$

If $f(t) \neq 0$ then $|f(t)|^2 > 0$ *i.e.,* $\langle f, f \rangle > 0$.

(3) $\langle af + bg, h \rangle = \int_0^1 \{af(t) + bg(t)\}\,\overline{h(t)}\,dt = \int_0^1 \{af(t)\,\overline{h(t)} + bg(t)\,\overline{h(t)}\}\,dt$

$= \int_0^1 af(t)\,\overline{h(t)}\,dt + \int_0^1 bg(t)\,\overline{h(t)}\}\,dt = a\int_0^1 f(t)\,\overline{h(t)}\,dt + b\int_0^1 g(t)\,\overline{h(t)}\}\,dt$

$= a\langle f, h \rangle + b\langle g, h \rangle$ $\qquad \therefore$ $V(C)$ is an inner product space.

Note. We can define more than one inner product on a given vector space. Let $V(R)$ be the vector space of continuous real valued functions. We can define two inner products

(*i*) $\langle f, g \rangle = \int_0^1 f(t)\,g(t)\,dt$ and (*ii*) $\langle f, g \rangle = \int_{-1}^1 f(t)\,g(t)\,dt$ which result in two distinct inner product spaces.

Theorem. 1. *Definition (1) and Definition (2) are equivalent.*

Proof. The properties of conjugate symmetry and positivity are same in both the definitions.

(*i*) Definition (1) $\Rightarrow$ Definition (2)

For any $\alpha, \beta, \gamma \in V$ and $a, b \in F$ we have $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$.

If we take $a = 1, b = 1 \in F$ then

$\langle 1\,\alpha + 1\,\beta, \gamma \rangle = 1\langle \alpha, \gamma \rangle + 1\langle \beta, \gamma \rangle \Rightarrow \langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

If we take $b = 0 \in F$ the zero element, then $\langle a\alpha + 0\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + 0\,\langle \beta, \gamma \rangle$

$\Rightarrow \langle a\alpha + \overline{0}, \gamma \rangle = a\langle \alpha, \gamma \rangle + 0 \Rightarrow \langle a\alpha, \gamma \rangle = a\langle \alpha, \gamma \rangle$.

$(\because 0 \in F, \beta \in V \Rightarrow 0\beta = \overline{0} \in V$ and $a\alpha + \overline{0} = a\alpha \in V)$

Then $\alpha = \begin{bmatrix} i \\ 2 \end{bmatrix}, \beta = \begin{bmatrix} 1+2i \\ 3+4i \end{bmatrix}$ and $\beta^* = [1-2i \quad 3-4i]$

$\langle \alpha, \beta \rangle = \beta^* \alpha = [1-2i \quad 3-4i] \begin{bmatrix} i \\ 2 \end{bmatrix} = (1-2i)i + (3-4i)2 = 8-7i \in C$.

SOLVED PROBLEMS

Ex. 1. *If α, β are vectors in a complex inner product space prove that $\langle \alpha, \beta \rangle = \text{Re}\langle \alpha, \beta \rangle + i\,\text{Re}\langle \alpha, i\beta \rangle$.*

Sol. For $\alpha, \beta \in V$, $\langle \alpha, \beta \rangle \in C$, the field of complex numbers.

Let $\langle \alpha, \beta \rangle = x + iy$ so that $x = \text{Re}\langle \alpha, \beta \rangle$ and $y = \text{Im}\langle \alpha, \beta \rangle$.

$-i\langle \alpha, \beta \rangle = -i(x+iy) = y - ix$ so that $y = \text{Re}\{-i\langle \alpha, \beta \rangle\} = \text{Re}\{\bar{i}\langle \alpha, \beta \rangle\} = \text{Re}\langle \alpha, i\beta \rangle$

$\therefore \langle \alpha, \beta \rangle = \text{Re}\langle \alpha, \beta \rangle + i\,\text{Re}\langle \alpha, i\beta \rangle$.

Ex. 2. *In the innerproduct space $M_2(C)$, the vector space of 2×2 matrices over the complex field if $\langle A, B \rangle = tr(B^* A)\ \forall\, A, B \in M_2$ find $\langle A, B \rangle$ given $A = \begin{bmatrix} 1 & 2+i \\ 3 & i \end{bmatrix}$ and $B = \begin{bmatrix} 1+i & 0 \\ i & -i \end{bmatrix}$.*

Sol. $B^* = \begin{bmatrix} 1-i & -i \\ 0 & i \end{bmatrix}$ = the conjugate transpose of B.

$$B^* A = \begin{bmatrix} 1-i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 2+i \\ 3 & i \end{bmatrix} = \begin{bmatrix} 1-i-3i & 2+i-2i+1+1 \\ 0+3i & 0-1 \end{bmatrix} = \begin{bmatrix} 1-4i & 4-i \\ 3i & -1 \end{bmatrix}$$

so that $tr(B^* A) = 1 - 4i - 1 = -4i$. $\quad \therefore \langle A, B \rangle = -4i \in C$.

Ex. 3. *In the inner product space C^2 for $x, y \in C^2$ and $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ if the inner product $\langle x, y \rangle = x A y^*$ compute $\langle x, y \rangle$ for $x = [1-i, 2+3i]$ and $y = [2+i, 3-2i]$.*

Sol. $y = [2+i, 3-2i] \Rightarrow y^* = \begin{bmatrix} 2-i \\ 3+2i \end{bmatrix}$. $\quad x A = [1-i, 2+3i] \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

$= [1-i-2i+3, i+1+4+6i] = [4-3i, 5+7i]$

$\therefore \langle x, y \rangle = x A y^* = [4-3i, 5+7i] \begin{bmatrix} 2-i \\ 3+2i \end{bmatrix} = (4-3i)(2-i) + (5+7i)(3+2i)$

$= (5-10i) + (1+31i) = 6+2i \in C$, the field.

Ex. 4. *For $\alpha = (a,b), \beta = (c,d) \in R^2$, the vector space, if $\langle \alpha, \beta \rangle = ac - bd$ then prove that it is not an inner product.*

Sol. Consider $\alpha = (3,4) \in R^2$ so that $\alpha \neq (0,0)$.

$\langle \alpha, \alpha \rangle = (3)(3) - (4)(4) = 9 - 16 = -7 \not> 0$. $\therefore$ the property of positivity is not true.

Ex. 5. *For $f(x), g(x) \in P(R)$, the vector space of polynomials over the field R defined on $[0,1]$ if $\langle f(x), g(x) \rangle = \int_0^1 f'(t)\, g(t)\, dt$ then prove that it is not an inner product.*

Sol. Take $f(x) = x$, $g(x) = x^2$ on $[0,1]$. $\langle f, g \rangle = \int_0^1 1.t^2\, dt = \frac{1}{3}$ and $\langle g, f \rangle = \int_0^1 2t\,.\,t\, dt = \frac{2}{3}$

$\therefore \langle f, g \rangle \neq \langle g, f \rangle$ and hence conjugate symmetry is not true.

Ex. 6. *In the vector space $V_2(C)$, the set of all ordered pairs over the field of complex numbers, $\langle \alpha, \beta \rangle = 2x_1\bar{y}_1 + x_1\bar{y}_2 + x_2\bar{y}_1 + x_2\bar{y}_2\ \forall\, \alpha = (x_1, x_2), \beta = (y_1, y_2) \in V_2$. Prove that $V_2(C)$ is inner product space.*

Sol. Conjugate symmetry.

$\overline{\langle \beta, \alpha \rangle} = \overline{2y_1\bar{x}_1 + y_1\bar{x}_2 + y_2\bar{x}_1 + y_2\bar{x}_2} = 2\bar{y}_1\bar{\bar{x}}_1 + \bar{y}_1\bar{\bar{x}}_2 + \bar{y}_2\bar{\bar{x}}_1 + \bar{y}_2\bar{\bar{x}}_2$

$= 2\bar{y}_1 x_1 + \bar{y}_1 x_2 + \bar{y}_2 x_1 + \bar{y}_2 x_2 = 2x_1\bar{y}_1 + x_1\bar{y}_2 + x_2\bar{y}_1 + x_2\bar{y}_2 = \langle \alpha, \beta \rangle$.

Positivity . $\langle \alpha, \alpha \rangle = 2x_1\bar{x}_1 + x_1\bar{x}_2 + x_2\bar{x}_1 + x_2\bar{x}_2 = x_1\bar{x}_1 + (x_1 + x_2)(\bar{x}_1 + \bar{x}_2)$

$= x_1\bar{x}_1 + (x_1 + x_2)\overline{(x_1 + x_2)} = |x_1|^2 + |x_1 + x_2|^2$

For $\alpha = (x_1, x_2) \neq (0,0)$ either $x_1 \neq 0$ or $x_2 \neq 0$. $\quad \therefore \langle \alpha, \alpha \rangle = |x_1|^2 + |x_1 + x_2|^2 > 0$.

Distributivity. Let $a, b \in C$ and $\gamma = (z_1, z_2) \in V_2$

$a\alpha + b\beta = a(x_1, x_2) + b(y_1, y_2) = (ax_1 + by_1, ax_2 + by_2)$

$\langle a\alpha + b\beta, \gamma \rangle = 2(ax_1 + by_1)\bar{z}_1 + (ax_1 + by_1)\bar{z}_2 + (ax_2 + by_2)\bar{z}_1 + (ax_2 + by_2)\bar{z}_2$

$= a(2x_1\bar{z}_1 + x_1\bar{z}_2 + x_2\bar{z}_1 + x_2\bar{z}_2) + b(2y_1\bar{z}_1 + y_1\bar{z}_2 + y_2\bar{z}_1 + y_2\bar{z}_2)$

$= a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$. $\quad \therefore V_2(C)$ is an inner product space.

Ex. 7. *Let $V(R)$ be the vector space of real valued continuous functions on $[0,1]$. For $f, g \in V$ if we define $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$ then prove that V is an inner product space.*

Sol. $f \in V$ is a continuous real valued function on $[0,1]$

$\Rightarrow f(t)$ is a real number $\forall\ t \in [0,1] \Rightarrow \overline{f(t)} = f(t)$.

Conjugate symmetry.

$$\overline{\langle g, f\rangle} = \overline{\int_0^1 g(t)f(t)\,dt} = \int_0^1 \overline{g(t)f(t)}\,dt = \int_0^1 g(t)f(t)\,dt = \int_0^1 f(t)g(t)\,dt = \langle f, g\rangle$$

Positivity. Let $f \neq O$, the zero function defined by $O(t) = 0 \in R$ for $t \in [0,1]$.

$$\langle f, f\rangle = \int_0^1 f(t)f(t)\,dt = \int_0^1 f(t)^2\,dt > 0 \quad (\because f \text{ is continuous on } [0,1])$$

Distributivity. Let $a, b \in R$ and $h \in V$. $\langle af + bg, h\rangle = \int_0^1 \{af(t) + bg(t)\}h(t)\,dt$

$$= \int_0^1 a\,f(t)h(t)\,dt + \int_0^1 b\,g(t)h(t)\,dt = a\int_0^1 f(t)h(t)\,dt + b\int_0^1 g(t)h(t)\,dt = a\langle f, h\rangle + b\langle g, h\rangle$$

$\therefore V(R)$ is an inner product space.

Ex. 8. *Let $V = M_2(R)$ be the vector space of 2×2 matrices over the field of real numbers. For $A, B \in V$ we define $\langle A, B\rangle = tr\,(B^T A)$. Prove that V is an inner product space.*

Sol. Let $A, B, C \in V$. Then A, B, C are 2×2 matrices with real numbers as elements.

Therefore for any matrix A, $\overline{A} = A$, $\overline{A^T} = A^T$.

Conjugate Symmetry. $\overline{\langle B, A\rangle} = \overline{tr\,(A^T B)} = tr\,(A^T B) = tr\,(BA^T)$

$$= tr\,(BA^T)^T = tr\,(B^T (A^T)^T) = tr\,(B^T A) = \langle A, B\rangle$$

Positivity. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $a_{11}, a_{12}, a_{21}, a_{22} \in R$ and $A \neq O$.

Then atleast one of $a_{11}, a_{12}, a_{21}, a_{22}$ is not zero.

$\langle A, A\rangle = tr\,(A^T A) = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 > 0$. $\therefore$ Positivity is true.

Distributivity. For $a, b \in R$; $\langle aA + bB, C\rangle = tr\,(C^T\{aA + bB\})$

$= tr\,(C^T(aA) + C^T(bB))$ (Distributity in matrices)

$= tr\,(a\,(C^T A) + b\,(C^T B)) = tr\,(a\,(C^T A) + tr(b\,(C^T B))$

$= a\,tr\,(C^T A) + b\,tr\,(C^T B) = a\langle A, C\rangle + b\langle B, C\rangle$

Hence $V = M_2(R)$ is an inner product space.

Ex. 9. *Let $P_n(x)$ be the vector space of all real valued polynomials of degree atmost n. Prove that $P_n(x)$ is an inner product space for the product $\langle f, g\rangle = a_0b_0 + a_1b_1 + \ldots + a_nb_n$ for $f(x) = a_0 + a_1x + \ldots + a_nx^n$, $g(x) = b_0 + b_1x + \ldots + b_mx^m \in P_n(x)$.*

Sol. Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$, $g(x) = b_0 + b_1x + \ldots + b_mx^m$

and $h(x) = c_0 + c_1x + \ldots + c_px^p \in P_n(x)$ so that $m \le n$, $p \le n$.

Conjugate Symmetry. $\langle f, g\rangle = a_0b_0 + a_1b_1 + \ldots + a_nb_n = b_0a_0 + b_1a_1 + \ldots + b_na_n$

$= \langle g, f\rangle$ ($\because a_i, b_i$'s are real numbers) $\Rightarrow \overline{\langle g, f\rangle} = \langle f, g\rangle$

Positivity. $\langle f, f\rangle = a_0a_0 + a_1a_1 + \ldots + a_na_n = a_0^2 + a_1^2 + \ldots + a_n^2 > 0$ for $f \neq O$.

Distributivity. For $\alpha, \beta \in R$;

$\langle \alpha f + \beta g, h\rangle = (\alpha a_0 + \beta b_0)\,c_0 + (\alpha a_1 + \beta b_1)\,c_1 + \ldots + (\alpha a_n + \beta b_n)\,c_n$

$= \alpha\,(a_0c_0 + a_1c_1 + \ldots + a_nc_n) + \beta\,(b_0c_0 + b_1c_1 + \ldots + b_nc_n)$

$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ ($\because$ real numbers are commutative, associative and distributive)

$\therefore P_n(x)$ with respect to the product is an inner product space.

EXERCISE 8 (a)

1. If α, β, γ are vectors in the inner product space $V(F)$ prove that

   (i) $\langle \alpha, \beta \pm \gamma\rangle = \langle \alpha, \beta\rangle \pm \langle \alpha, \gamma\rangle$ (ii) $\langle \alpha - \beta, \alpha - \beta\rangle = \langle \alpha, \alpha\rangle - \langle \alpha, \beta\rangle - \langle \beta, \alpha\rangle + \langle \beta, \beta\rangle$

   (iii) $\langle a\alpha - b\,\beta, a\alpha - b\beta\rangle = a\bar{a}\langle \alpha, \alpha\rangle - a\bar{b}\langle \alpha, \beta\rangle - b\bar{a}\langle \beta, \alpha\rangle + b\bar{b}\langle \beta, \beta\rangle$.

2. $\alpha = (1,2), \beta = (-1,1)$ are two vectors in the vector space R^2 with standard inner product. If γ is a vector such that $\langle \alpha, \gamma\rangle = -1$ and $\langle \beta, \gamma\rangle = 3$ find γ.

3. If $\alpha = (2, 1+i, i), \beta = (2-i, 2, 1+2i)$ are two vectors in C^3 with standard inner product find $\langle \alpha, \beta\rangle$.

4. If $M_2(R)$ is the vector space of 2×2 matrices over real field and $A, B \in M_2$ prove that $\langle A, B\rangle = tr\,(A + B)$ is not an inner product.

5. If $\alpha = (x_1, x_2, \ldots, x_n)$ and $\beta = (y_1, y_2, \ldots, y_n)$ are vectors in R^n vector space prove that $\langle \alpha, \beta\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$ defines inner product in R^n.

6. In R^2 show that $\langle \alpha, \beta\rangle = x_1 + x_2 + y_1 + y_2$ where $\alpha = (x_1, x_2), \beta = (y_1, y_2)$ does not define inner product.

7. In R^2 if we define $\langle \alpha, \beta \rangle = x_1 y_1 - x_2 y_1 + 4 x_2 y_2$ for $\alpha = (x_1, x_2), \beta = (y_1, y_2)$ prove that R^2 is an inner product space.

8. In the vector space $R^3(R)$ if $\bar{x} = (x_1, x_2, x_3), \bar{y} = (y_1, y_2, y_3)$ are two vectors such that $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ then prove that it is not an inner product.

9. If f, g are two inner products defined on a vector space V(F) then show that (*i*) $(f+g)$ is also an inner product on V(F) (*ii*) (nf) where $n \in$ N is also an inner product and (*iii*) $(f-g)$ need not be an inner product on V(F).

ANSWERS

2. $\left(-\frac{7}{3}, \frac{2}{3}\right)$ 3. $8+5i$

8.4. NORM OR LENGTH OF A VECTOR

In what follows we generalise the concept of length in R^2 and R^3 to inner product space.

Definition. *Let V be an inner product space over the field F. The norm (length) of $\alpha \in V$ denoted by $\|\alpha\|$ is defined as the positive square root of $\langle \alpha, \alpha \rangle$.*

Norm or length of $\alpha \in V = \|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} \Rightarrow \|\alpha\|^2 = \langle \alpha, \alpha \rangle$.

Note. 1. For $\alpha \in V, \langle \alpha, \alpha \rangle$ is a non- negative real number and hence the norm of α is always a non-negative real number.

2. $\alpha = \bar{0} \Leftrightarrow \|\alpha\| = 0$

e.g.1. In the inner product space $V_2(R) = R^2(R)$; if $\alpha = (a, b) \in V_2$

then $\|\alpha\| = \|(a,b)\| = \sqrt{a^2 + b^2} = \sqrt{\langle \alpha, \alpha \rangle}$.

2. In the inner product space $V_3(R) = R^3(R)$; if $\alpha = (a, b, c)$

then $\|\alpha\| = \|(a,b,c)\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{\langle \alpha, \alpha \rangle}$.

3. In the inner product space $V_n(C) = C^n$ if $\alpha = (a_1, a_2, \ldots, a_n)$

then $\|\alpha\| = \|(a_1, a_2, \ldots, a_n)\| = \sqrt{|a_1|^2 + |a_2|^2 + |a_n|^2} = \sqrt{\sum_{i=1}^{n} |a_i|^2} = \sqrt{\langle \alpha, \alpha \rangle}$.

Theorem. 1. ***In an inner product space V(F) (1) $\|\alpha\| > 0$ if $\alpha \neq 0$ and (2) $\|a\alpha\| = |a| \, \|\alpha\|$ where $0, a \in F$ and $\bar{0}, \alpha \in V$.*** **(S. K. D. 08, O.U. 03)**

Proof. (1) If $\alpha \neq \bar{0}$ then $\langle \alpha, \alpha \rangle > 0$.

$\therefore \|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} > 0$. For any $\alpha \in V$, $\|\alpha\| \geq 0$.

(2) By the definition of norm, $\|a\alpha\|^2 = \langle a\alpha, a\alpha \rangle$

$= a \langle \alpha, a\alpha \rangle = a\bar{a} \langle \alpha, \alpha \rangle$ (By (3) of the theorem)

$= |a|^2 \|\alpha\|^2 = (|a| \|\alpha\|)^2$ $(\because Z\bar{Z} = |Z|^2)$ $\therefore \|a\alpha\| = |a| \, \|\alpha\|$.

Note. If $\alpha \in V$ and $\alpha \neq \bar{0}$ by the above theorem $\|\alpha\| > 0$. Since $\|\alpha\| (>0) \in F$ and F is a field, there exists $\frac{1}{\|\alpha\|} \in F$ such that $\|\alpha\| \frac{1}{\|\alpha\|} = 1$. Now, for $\frac{1}{\|\alpha\|} \in F$ and $\alpha \in V$ we have $\frac{1}{\|\alpha\|} \alpha \in V$, such that $\left\langle \frac{1}{\|\alpha\|} \alpha, \frac{1}{\|\alpha\|} \alpha \right\rangle = \frac{1}{\|\alpha\|} \left(\overline{\frac{1}{\|\alpha\|}} \right) \langle \alpha, \alpha \rangle = \left(\frac{1}{\|\alpha\|} \right) \left(\frac{1}{\|\alpha\|} \right) \|\alpha\|^2 = 1$.

Hence, for $\alpha \in V$ and $\alpha \neq \bar{0}$, $\frac{1}{\|\alpha\|} \alpha \in V$ is a vector of length 1.

Definition. *Let $V(F)$ be an inner product space. $\alpha \in V$ is called a unit vector if $\|\alpha\| = 1$. If $\alpha \in V$ then $\frac{1}{\|\alpha\|} \alpha \in V$ is unit vector.*

e.g.1. In the inner product space $R^2, i = (1,0), j = (0,1)$ are unit vectors.

e.g.2. In the inner product space R^3 with standard inner product $i = (1,0,0), j = (0,1,0)$ and $k = (0,0,1)$ are vectors of each length 1.

3. In the inner product space R^3, if $\alpha = (a_1, a_2, a_3)$ then $\|\alpha\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

and $\frac{1}{\|\alpha\|} \alpha = \left(\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \right)$

Theorem. 2. ***In an inner product space V(F), $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$ for all $\alpha, \beta \in V$.***

Cauchy - Schwarz's inequality.

(S.K. D. 08, A. U. 08, S. V. U. 07, N. U. 07, A. U. 08, O. U. 03,05, K. U. 04)

Proof. Case (1). Let $\alpha = \bar{0}$. Then $\langle \alpha, \beta \rangle = \langle \bar{0}, \beta \rangle = 0$ and $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = \langle \bar{0}, \bar{0} \rangle = 0$.

$\therefore |\langle \alpha, \beta \rangle| = 0$ and $\|\alpha\| \, \|\beta\| = 0 \; (\|\beta\|) = 0$. $\therefore |\langle \alpha, \beta \rangle| = \|\alpha\| \, \|\beta\|$.

Case (2). Let $\alpha \neq \bar{0}$. Then $\|\alpha\| > 0$ so that $\frac{1}{\|\alpha\|} > 0$.

EXERCISE 8 (b)

1. If (i) $\alpha=(1,-2,5)\in R^3(R)$ (ii) $\alpha=(3+i,4i,-4)\in C^3(C)$ find $\|\alpha\|$ and unit vector of α. **(S. V. U. 08)**
2. If $\alpha=(1-i,2+3i),\beta=(2-5i,3-i)$ are two vectors in $V(C)=C^2(C)$ with standard inner product find $\langle\alpha,\beta\rangle$ and $\|\alpha\|,\|\beta\|$.
3. If α,β are two vectors in Euclidean space $V(R)$ such that $\|\alpha\|=\|\beta\|$ prove that $\langle\alpha+\beta,\alpha-\beta\rangle=0$. **(S. V. U. 07)**
4. If α,β are two vectors in a unitary space find $\|\alpha+i\beta\|$ and $\|\alpha-i\beta\|$.
5. If α,β are two vectors in a unitary space prove that $\|\alpha+\beta\|^2-\|\alpha-\beta\|^2=4\,\mathrm{Re}\,\langle\alpha,\beta\rangle$.
6. If $M_2(R)$ is the inner product space and for $A=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}, B=\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}\in M_2$ the inner product is defined as $\langle A,B\rangle=a_1b_1+a_2b_2+a_3b_3+a_4b_4$ find the angle between $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$.
7. If $\alpha=(4,3,1,-2),\beta=(-2,1,2,3)$ be two vectors in the vector space $V_4(R)$ with standard inner product then find the angle between α and β.
8. In a real inner product space if α,β are two vectors such that $\|\alpha+\beta\|=\|\alpha\|+\|\beta\|$ then prove that α,β are linearly dependent.

ANSWERS

1. $\sqrt{30},\sqrt{42},\frac{1}{\sqrt{30}}(1,-2,5),\frac{1}{\sqrt{42}}(3+i,4i,-4)$ 2. $10+14i,\sqrt{27},\sqrt{39}$

4. $\sqrt{\|\alpha\|^2-i<\alpha,\beta>+i<\beta,\alpha>+\|\beta\|^2}$ $\sqrt{\|\alpha\|^2+i<\alpha,\beta>-i<\beta,\alpha>+\|\beta\|^2}$

6. $\pi/2$ 7. $\theta=\mathrm{Cos}^{-1}\sqrt{(3/20)}$

9

Orthogonality

9.1. We now introduce the concept of orthogonality in an inner product space which is analogous to perpendicularity in Geometry. In 3-D geometry we know that the three unit vectors $\bar{i},\bar{j}$ and $\bar{k}$ along the three mutually perpendicular coordinate axes form a base for any vector. Just as bases are building blocks of vector spaces, the basis vectors forming an orthonormal set are the building blocks of inner product spaces. By Gram - Schmidt orthogonalisation process we now establish that every finite dimensional inner product space has an orthonormal basis.

9.2. ORTHOGONAL AND ORTHONORMAL VECTORS

Definition. *Let $V(F)$ be an inner product space and $\alpha,\beta\in V$. α is said to be orthogonal to β if $\langle\alpha,\beta\rangle=0$.* **(O. U. 00)**

α is orthogonal to $\beta\Leftrightarrow\langle\alpha,\beta\rangle=0\Leftrightarrow\langle\beta,\alpha\rangle=\overline{\langle\alpha,\beta\rangle}=0$ ($\because$ the conjugate of 0 is 0)

$\Leftrightarrow\beta$ is orthogonal to α.

So, we say that $\alpha,\beta\in V$ are orthogonal, if and only if, $\langle\alpha,\beta\rangle=0$.

Note. 1. If $\bar{0}\in V$ is the zero vector then $(\bar{0},\alpha)=0$ for all $\alpha\in V$.

Therefore the zero vector is orthogonal to every vector in V and $\bar{0}$ is the only vector in V with this property.

2. Since $(\bar{0},\bar{0})=0$ for $\bar{0}\in V$, the zero vector in V is orthogonal to itself.

e.g. 1. Let $\alpha=(0,-2,3),\beta=(7/2,3,2)$ be two vectors of the inner product space R^3 with standard inner product.

Then $\langle\alpha,\beta\rangle=(0)(7/2)+(-2)(3)+(3)(2)=0$. $\therefore\ \alpha,\beta$ are orthogonal vectors.

2. Consider the inner product space $V(C)$ of continuous complex valued functions on [0,1] with the inner product defined as : $\langle f,g\rangle=\int_0^1 f(t)\overline{g(t)}\,dt$ for $f(t),g(t)\in V$.

For $f(t)=2\sin\pi t,\ g(t)=\cos\pi t$ we have

$$\langle f,g\rangle=\int_0^1 2\sin\pi t\cos\pi t\,dt=\int_0^1\sin 2\pi t\,dt=-\frac{1}{2\pi}[\cos 2\pi t]_0^1=0.$$

$\therefore\ 2\sin\pi t,\cos\pi t$ are orthogonal vectors.

9.3. ORTHOGONAL AND ORTHONORMAL SETS OF INNER PRODUCT SPACE

Definition. *Let S be a non-empty subset of an inner product space $V(F)$.*

The set S is said to be an orthogonal set if every pair of distinct vectors are orthogonal.

Or, $S \subset V$ is an orthogonal set if $\langle \alpha_i, \alpha_j \rangle = 0 \ \forall \ \alpha_i, \alpha_j \in S$ and $i \neq j$. **(S.K.D. 05)**

Definition. *Let S be a non-empty subset of an inner product space $V(F)$.*

The set S is said to be an orthonormal set if (i) $\| \alpha_i \| = 1$ for each $\alpha_i \in S$ and (ii) $(\alpha_i, \alpha_j) = 0$ for $\alpha_i, \alpha_j \in S, i \neq j$.

$S \subset V$ is an orthonormal set if $(\alpha_i, \alpha_j) = 1$ for $i = j$ and $(\alpha_i, \alpha_j) = 0$ for $i \neq j$ where $\alpha_i, \alpha_j \in S$. That is $\langle \alpha_i, \alpha_j \rangle = \delta_{ij}$ (Kronecker delta). **(S. K. D. 05, N. U. 97)**

Note. 1. $S \subset V$ is an orthonormal set $\Leftrightarrow S$ contains mutually orthogonal unit vectors.

2. An orthonormal set is an orthogonal set with the property that each vector is of length 1.

3. An orthonormal set does not conatin zero vector.

e.g.1. If $\alpha \in V$ is a nonzero vector then $\frac{1}{\| \alpha \|} \alpha$ is unit vector. Therefore $\left\{ \frac{1}{\| \alpha \|} \alpha \right\}$ is an orthonormal set in V. Thus, every inner product space which is not a null space has an orthonormal subset.

e.g. 2. The standard basis of the inner product space R^3 or $V_3(R)$ is the set $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$. **(S. K. D. 08)**

Then $\| e_1 \| = 1, \| e_2 \| = 1, \| e_3 \| = 1$ and $\langle e_1, e_2 \rangle = 0, \langle e_2, e_3 \rangle = 0, \langle e_1, e_3 \rangle = 0$.

Thus the standard basis form an orthonormal set.

Theorem 1. ***In an inner product space, any orthogonal set of non-zero vectors is linearly independent.*** **(S. V. U. 07)**

Proof. Let S be a finite or infinite orthogonal set of non-zero vectors in an inner product spare $V(F)$.

Then $\alpha_i, \alpha_j \in S \Rightarrow \alpha_i \neq \bar{0}, \alpha_j \neq \bar{0}$ and $(\alpha_i, \alpha_j) = 0$ for $i \neq j$.

Let $T = \{ \alpha_1, \alpha_2, ..., \alpha_m \}$ be a finite subset of S having m distinct vectors.

Let there exist $a_1, a_2, ..., a_m \in F$ such that $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = \bar{0}$.

For any $i \in \{1, 2, ..., m\}$, $(a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m, \alpha_i)$

$= a_1 (\alpha_1, \alpha_i) + a_2 (\alpha_2, \alpha_i) + ... + a_i (\alpha_i, \alpha_i) + ... + a_m (\alpha_m, \alpha_i)$

$= a_i (\alpha_i, \alpha_i)$ since $(\alpha_j, \alpha_i) = 0$ for $j \neq i$

$\therefore a_1, a_2, ..., a_m \in F$ and $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = \bar{0}$

$\Rightarrow \langle \bar{0}, \alpha_i \rangle = a_i \langle \alpha_i, \alpha_i \rangle$ for each $i \in \{1, 2, ..., m\}$

$\Rightarrow 0 = a_i \langle \alpha_i, \alpha_i \rangle$ for each $i \in \{1, 2, ..., m\}$

$\Rightarrow a_i = 0$ for each $i \in \{1, 2, ..., m\}$ $\quad (\because \alpha_i \neq \bar{0}, \langle \alpha_i, \alpha_i \rangle \neq 0)$

Hence the set T is linearly independent.

Thus every non-zero finite subset T of S is linearly independent.

$\therefore$ The set S is linearly independent.

Note. If $\beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m$ then for each $i \in \{1, 2, ..., m\}$ we have $\langle \beta, \alpha_i \rangle = a_i (\alpha_i, \alpha_i) = \alpha_i \| \alpha_i \|^2$ so that $a_i = \frac{\langle \beta, \alpha_i \rangle}{\| \alpha_i \|^2}$.

Therefore $\beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = \sum_{i=1}^{m} a_i\alpha_i = \sum_{i=1}^{m} \frac{\langle \beta, \alpha_i \rangle}{\| \alpha_i \|^2} \alpha_i$.

Corollary. ***In an inner product space any orthonormal set of vectors is linearly independent.*** **(S. V. U. M 07, S. K. U. M 07, O. U. 99)**

Proof. Let S be a finite or infinite orthonormal set of vectors in an inner product space $V(F)$.

Then $\alpha_i, \alpha_j \in S \Rightarrow (\alpha_i, \alpha_j) = 1$ for $i = j$ and $(\alpha_i, \alpha_j) = 0$ for $i \neq j$.

Let $T = \{ \alpha_1, \alpha_2, ..., \alpha_m \}$ be a finite subset of S having m distinct vectors.

$\therefore a_1, a_2, ..., a_m \in F$ and $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = \bar{0}$

$\Rightarrow (\bar{0}, \alpha_i) = a_i$ for each $i \in \{1, 2, ..., m\}$

$\Rightarrow a_i = 0 \ \forall \ i \in \{1, 2, ..., m\} \Rightarrow T$ is linearly independent

$\therefore S$ is linearly independent.

Corollary. 2. ***If S is an orthonormal set of vectors of an inner product space $V(F)$, then $\beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n \Rightarrow a_i = \langle \beta, \alpha_i \rangle$ for $i = 1, 2, ..., n$.*** **(N. U. 97)**

Proof. S is an orthonormal set $\Rightarrow$ for $\alpha_i, \alpha_j \in S$

$\| \alpha_i \| = 1, \| \alpha_j \| = 1$ and $(\alpha_i, \alpha_j) = 0$ when $i \neq j$.

Then, for each $i = 1, 2, ..., n$; $(\beta, \alpha_i) = (a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n, \alpha_i) = a_i(\alpha_i, \alpha_i) = a_i$

$\therefore \beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \sum_{i=1}^{n} a_i \alpha_i = \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle \alpha_i$.

Theorem. 2. ***Let $V(F)$ be an inner product space and $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be an orthogonal subset of V consisting non - zero vectors. If $\beta \in$ span $S = L(S)$ then $\beta = \sum_{i=1}^{m} \frac{\langle \beta, \alpha_i \rangle}{\| \alpha_i \|^2} \alpha_i$.*** **(K. U. 07, 08)**

9. 4. GRAM - SCHMIDT ORTHOGONALISATION PROCESS.

For a vector space we have seen earlier the special role of basis. We have learnt that, if S is a basis of vector space $V(F)$ then S is linearly independent and $L(S) = V$.

Definition. *A basis of an inner product space $V(F)$ which is also orthonormal is called "orthonormal basis" of the inner product space.*

e. g. 1. The basis $S = \{(1,0), (0,1)\}$ of the inner product space $R^2(R)$ is also orthonormal. So, S is an orthonormal basis of $R^2(R)$.

e. g. 2. The basis $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of the inner product space $R^3(R)$ is also orthonormal. So, S is an orthonormal basis of $R^3(R)$

e.g. 3. For the inner product space $R^2(R), S = \left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)\right\}$ is also an orthonormal basis.

Definition. *A finite dimensional vector space, in which an inner product is defined, is called a finite dimensional inner product space.*

We now establish that every finite dimensional inner product space possesses an orthonormal basis. If S is a basis of the finite dimensional inner product space $V(F)$, we construct an orthonormal set S' from S such that $L(S) = L(S') = V$.

e.g. Let $V_2(F)$ be a 2 - dimensional inner product space and $S = \{\beta_1, \beta_2\}$ be a basis. Then S is linearly independent and $L(S) = V$.

Consider $S' = \{\alpha_1, \alpha_2\}$ where $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2 - a\beta_1$ where $a \in F$ chosen such that α_2 is orthogonal to β_1 .

Then $0 = \langle \alpha_2, \beta_1 \rangle = \langle \beta_2 - a\beta_1, \beta_1 \rangle = \langle \beta_2, \beta_1 \rangle - a\langle \beta_1, \beta_1 \rangle \Rightarrow a = \dfrac{\langle \beta_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \dfrac{\langle \beta_2, \beta_1 \rangle}{\|\beta_1\|^2}$.

Thus $S' = \left\{\beta_1, \beta_2 - \dfrac{\langle \beta_2, \beta_1 \rangle}{\|\beta_1\|^2}\beta_1\right\}$ is an orthogonal set of non-zero vectors such that $L(S) = L(S') = V$.

Observe that each vector in $S' = \{\alpha_1, \alpha_2\}$ is a linear combination of β_1, β_2 of S.

Theorem. 1. ***Every finite dimensional inner product space has an orthonormal basis.*** **(N. U. 07,08, K. U. 05, 03, A. U. 04, 03, O. U. 04, 03)**

Proof. Let $V(F)$ be an inner product space of dimension n and $B = \{\beta_1, \beta_2, ..., \beta_n\}$ be a basis for V.

Now we shall obtain an orthonormal set $B' = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ in $V(F)$ such that $L(B') = L(B) = V$ with the help of the vectors in B by means of a construction known as Gram - Schmidt orthogonalization process.

The main idea behind this construction is that each α_i is a linear combination of $\beta_1, \beta_2, ..., \beta_n$.

We now prove this construction by using induction on the dimension n of V.

Since B is a basis of V, B is linearly independent $\Rightarrow \beta_i \neq \bar{0}$ for $i = 1, 2, ..., n$.

Let $\alpha_1 = \dfrac{\beta_1}{\|\beta_1\|}$.

Then we have $\langle \alpha_1, \alpha_1 \rangle = 1$ and α_1 is a linear combination of β_1

Thus $\{\alpha_1\}$ is an ortho-normal set which is a basis of V such that $L(\{\alpha_1\}) = L(\{\beta_1\}) = V$ when dim $V = n = 1$.

Suppose that $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ where $1 \le m < n$ has been constructed as an ortho-normal basis of V when dim $V = m$.

Then $L(\{\alpha_1, \alpha_2, ..., \alpha_m\}) = L(\{\beta_1, \beta_2, ..., \beta_m\}) = V$ and each α_i $(i = 1, 2, ..., m)$ is a linear combination of $\beta_1, \beta_2, ..., \beta_m$.

Consider the vector

$$\gamma_{m+1} = \beta_{m+1} - \langle \beta_{m+1}, \alpha_1 \rangle \alpha_1 - \langle \beta_{m+1}, \alpha_2 \rangle \alpha_2 - ... - \langle \beta_{m+1}, \alpha_m \rangle \alpha_m \quad ... (1)$$

By theorem (3) of Art. 9.3; γ_{m+1} is orthogonal to each of $\alpha_1, \alpha_2, ..., \alpha_m$.

Suppose that $\gamma_{m+1} = \bar{0}$. Then $\beta_{m+1} = \langle \beta_{m+1}, \alpha_1 \rangle \alpha_1 + \langle \beta_{m+1}, \alpha_2 \rangle \alpha_2 + ... + \langle \beta_{m+1}, \alpha_m \rangle \alpha_m$ so that β_{m+1} is a linear combination of $\alpha_1, \alpha_2, ..., \alpha_m$.

By our induction hypothesis, each α_i $(i = 1, 2, ..., m)$ is a linear combination of $\beta_1, \beta_2, ..., \beta_m$.

Therefore, β_{m+1} is a linear combination of $\beta_1, \beta_2, ..., \beta_m$.

This is a contradiction, as $\{\beta_1, \beta_2, ..., \beta_m, \beta_{m+1}\}$ is linearly independent.

$\therefore \gamma_{m+1} \neq \bar{0}$, Take $\alpha_{m+1} = \dfrac{\gamma_{m+1}}{\|\gamma_{m+1}\|}$.

Then we have $\|\alpha_{m+1}\| = 1$ and α_{m+1} is orthogonal to each of $\alpha_1, \alpha_2, ..., \alpha_m$.

Also, $\alpha_{m+1} \neq \alpha_i$ for $i = 1, 2, ..., m$.

Otherwise, from (1) we see that β_{m+1} will become a linear combination of $\beta_1, \beta_2, ..., \beta_m$.

Further from (1) we see that α_{m+1} is a linear combination of $\beta_1, \beta_2, ..., \beta_m, \beta_{m+1}$.

Thus $\{\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}\}$ is an orthonormal basis of V such that $L(\{\alpha_1, \alpha_2, ..., \alpha_{m+1}\}) = L(\{\beta_1, \beta_2, ..., \beta_{m+1}\}) = V$ when $\dim V = m+1$.

Therefore, by induction, $B' = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is an orthonormal basis of V such that $L(B) = L(B') = V$ when $\dim V = n$.

The above theorem can also be stated as follows :

"Let $V(F)$ be an inner product space of dimension n and $B = \{\beta_1, \beta_2, ..., \beta_n\}$ be a linearly independent subset of V. If $B' = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 = \beta_1$ and

$$\alpha_k = \beta_k - \sum_{i=1}^{k-1} \frac{\langle \beta_k, \alpha_i \rangle}{\| \alpha_i \|^2} \alpha_i \text{ for } 2 \le k \le n$$ then B' is an orthogonal set of non-zero vectors such that $L(B') = L(B) = V$."

WORKING METHOD FOR FINDING ORTHOGONAL BASIS

Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be the given L.I. basis of $V(F)$.

The orthogonal basis of $V(F)$, namely $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ is given by the following relations:

$$\gamma_1 = \beta_1,\ \gamma_2 = \beta_2 - \frac{\langle \beta_2, \gamma_1 \rangle}{\| \gamma_1 \|^2} \gamma_1,\ \gamma_3 = \beta_3 - \frac{\langle \beta_3, \gamma_1 \rangle}{\| \gamma_1 \|^2} \gamma_1 - \frac{\langle \beta_3, \gamma_2 \rangle}{\| \gamma_2 \|^2} \gamma_2 .....$$

$$.... \gamma_n = \beta_n - \frac{\langle \beta_n, \gamma_1 \rangle}{\| \gamma_1 \|^2} \gamma_1 - .... - \frac{\langle \beta_n, \gamma_{n-1} \rangle}{\| \gamma_{n-1} \|^2} \gamma_{n-1}.$$

WORKING METHOD FOR FINDING ORTHONORMAL BASIS

Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be a given basis of a finite dimensional inner product space $V(F)$.

The vectors $\alpha_1, \alpha_2, ..., \alpha_n$ of the orthonormal basis of $V(F)$ are given by

$$\alpha_1 = \frac{\beta_1}{\| \beta_1 \|},\ \alpha_2 = \frac{\gamma_2}{\| \gamma_2 \|} \text{ where } \gamma_2 = \beta_2 - \langle \beta_2, \alpha_1 \rangle \alpha_1,\ \alpha_3 = \frac{\gamma_3}{\| \gamma_3 \|}$$

where $\gamma_3 = \beta_3 - \langle \beta_3, \alpha_1 \rangle \alpha_1 - \langle \beta_3, \alpha_2 \rangle \alpha_2, ..., \alpha_n = \dfrac{\gamma_n}{\| \gamma_n \|}$

where $\beta_n = \beta_n - \langle \beta_n, \alpha_1 \rangle \alpha_1 - \langle \beta_n, \alpha_2 \rangle \alpha_2 - ... - \langle \beta_n, \alpha_{n-1} \rangle \alpha_{n-1}$.

SOLVED PROBLEMS

Ex. 1. *Given* $\{(2,1,3), (1,2,3), (1,1,1)\}$ *is a basis of* R^3; *construct an orthonormal basis.* ***(N. U. 91, S. K. U. 00)***

Sol. Let $\beta_1 = (2,1,3), \beta_2 = (1,2,3), \beta_3 = (1,1,1)$.

By Gram - Schmidt orthogonalization process an ortho-normal basis $= \{\alpha_1, \alpha_2, \alpha_3\}$ is given by $\alpha_1 = \dfrac{\beta_1}{\| \beta_1 \|}, \alpha_2 = \dfrac{\gamma_2}{\| \gamma_2 \|}$ where $\gamma_2 = \beta_2 - \langle \beta_2, \alpha_1 \rangle \alpha_1$ and $\alpha_3 = \dfrac{\gamma_3}{\| \gamma_3 \|}$.

where $\gamma_3 = \beta_3 - \langle \beta_2, \alpha_1 \rangle \alpha_1 - \langle \beta_3, \alpha_2 \rangle \alpha_2$.

$\| \beta_1 \|^2 = \langle \beta_1, \beta_1 \rangle = 2^2 + 1^2 + 3^2 = 14$. $\quad \therefore \alpha_1 = \dfrac{1}{\| \beta_1 \|} \beta_1 = \dfrac{1}{\sqrt{14}}(2,1,3) = \left(\dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right)$

$$\langle \beta_2, \alpha_1 \rangle = 1 . \frac{2}{\sqrt{14}} + 2 . \frac{1}{\sqrt{14}} + 3 . \frac{3}{\sqrt{14}} = \frac{13}{\sqrt{14}}$$

$$\gamma_2 = (1,2,3) - \frac{13}{\sqrt{14}} \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) = \left(\frac{-12}{14}, \frac{15}{14}, \frac{3}{14} \right)$$

$$\| \gamma_2 \|^2 = \langle \gamma_2, \gamma_2 \rangle = \left(\frac{-12}{14} \right)^2 + \left(\frac{15}{14} \right)^2 + \left(\frac{3}{14} \right)^2 = \frac{378}{196}.$$

$$\therefore \alpha_2 = \frac{1}{\| \gamma_2 \|} \gamma_2 = \frac{14}{\sqrt{378}} \left(\frac{-12}{14}, \frac{15}{14}, \frac{3}{14} \right) = \left(\frac{-12}{\sqrt{378}}, \frac{15}{\sqrt{378}}, \frac{3}{\sqrt{378}} \right).$$

$$\langle \beta_3, \alpha_1 \rangle = 1 . \frac{2}{\sqrt{14}} + 1 . \frac{1}{\sqrt{14}} + 1 . \frac{3}{\sqrt{14}} = \frac{6}{\sqrt{14}},\ \langle \beta_3, \alpha_2 \rangle = 1 . \frac{-12}{\sqrt{378}} + 1 . \frac{15}{\sqrt{378}} + 1 . \frac{3}{\sqrt{378}} = \frac{6}{\sqrt{378}}.$$

$$\gamma_3 = (1,1,1) - \frac{6}{\sqrt{14}} \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) - \frac{6}{\sqrt{378}} \left(\frac{-12}{\sqrt{378}}, \frac{15}{\sqrt{378}}, \frac{3}{\sqrt{378}} \right)$$

$$= (1,1,1) - \left(\frac{12}{14}, \frac{6}{14}, \frac{18}{14} \right) - \left(\frac{-72}{378}, \frac{90}{378}, \frac{18}{378} \right) = \left(\frac{126}{378}, \frac{126}{378}, \frac{-126}{378} \right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3} \right)$$

$$\| \gamma_3 \|^2 = \langle \gamma_3, \gamma_3 \rangle = \left(\frac{1}{3} \right)^2 + \left(\frac{1}{3} \right)^2 + \left(\frac{-1}{3} \right)^2 = \frac{1}{3}.$$

$$\therefore \alpha_3 = \frac{1}{\| \gamma_3 \|} \gamma_3 = \sqrt{3} \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3} \right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)$$

Hence orthonormal basis $\left\{ \left(\dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right), \left(\dfrac{-12}{\sqrt{378}}, \dfrac{15}{\sqrt{378}}, \dfrac{3}{\sqrt{378}} \right), \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}} \right) \right\}$.

Ex. 2. *Find an orthonormal basis of a vector space* $V(R)$ *of all real polynomials of degree not greater than 2, in which the inner product is defined as* $\langle f(x), g(x) \rangle = \int_0^1 f(x)\, g(x)\, dx \ \forall\ f(x), g(x) \in V$ *and* $x \in [0,1]$, *from a given basis* $\{1, x, x^2\}$.

Sol. Let $\beta_1 = 1, \beta_2 = x, \beta_3 = x^2$.

$$\| \beta_1 \|^2 = \langle \beta_1, \beta_1 \rangle = \int_0^1 (1)\,(1)\, dx = 1. \qquad \therefore \alpha_1 = \frac{1}{\| \beta_1 \|} \beta_1 = 1.$$

$$\langle \beta_2, \alpha_1 \rangle = \int_0^1 (x)\,(1)\, dx = \int_0^1 x . dx = \frac{1}{2}. \qquad \gamma_2 = \beta_2 - \langle \beta_2, \alpha_1 \rangle \alpha_1 = x - \frac{1}{2}(1) = x - \frac{1}{2}.$$

$$\|\gamma_2\|^2=\langle\gamma_2,\gamma_2\rangle=\int_0^1\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)dx=\left[\frac{\left(x-\frac{1}{2}\right)^3}{3}\right]_0^1=\frac{1}{12}\quad\therefore\ \alpha_2=\frac{1}{\|\gamma_2\|}\gamma_2=\sqrt{12}\left(x-\frac{1}{2}\right)$$

$$\langle\beta_3,\alpha_1\rangle=\int_0^1 x^2\,.\,1\,dx=\frac{1}{3}\text{ and }\langle\beta_3,\alpha_2\rangle=\int_0^1 x^2\,.\,\sqrt{12}\left(x-\frac{1}{2}\right)dx=\sqrt{12}\left[\frac{x^4}{4}-\frac{1}{2}\frac{x^3}{3}\right]_0^1=\frac{1}{\sqrt{12}}$$

$$\gamma_3=\beta_3-(\beta_3,\alpha_1)\,\alpha_1-(\beta_3,\alpha_2)\alpha_2=x^2-\frac{1}{3}.1-\frac{1}{\sqrt{12}}\sqrt{12}\left(x-\frac{1}{2}\right)=x^2-x+\frac{1}{6}$$

$$\|\gamma_3\|^2=\langle\gamma,\gamma_3\rangle=\int_0^1\left(x^2-x+\frac{1}{6}\right)\left(x^2-x+\frac{1}{6}\right)dx$$

$$=\int_0^1\left(x^4-2x^3+\frac{4x^2}{3}-\frac{x}{3}+\frac{1}{36}\right)dx=\left[\frac{x^5}{5}-\frac{x^4}{2}+\frac{4x^3}{9}-\frac{x^2}{9}+\frac{x}{36}\right]_0^1=\frac{1}{180}$$

$$\therefore\ \alpha_3=\frac{1}{\|\gamma_3\|}\gamma_3=\sqrt{180}\left(x^2-x+\frac{1}{6}\right).$$

Hence orthonormal basis $\{\alpha_1,\alpha_2,\alpha_3\}=\left\{1,\sqrt{12}\left(x-\frac{1}{2}\right),3\sqrt{20}\left(x^2-x+\frac{1}{6}\right)\right\}$

Theorem. 2. ***Let*** $B=\{\alpha_1,\alpha_2,\ldots\ldots,\alpha_k\}$ ***be an orthonormal set in n - dimensional inner product space V(F). Then B can be extended to an orthonormal basis*** $B'=\{\alpha_1,\alpha_2,\ldots\ldots,\alpha_k,\alpha_{k+1},\ldots\ldots,\alpha_n\}$ ***for V.***

Proof. V(F) is an inner product space $\Rightarrow$ V(F) is a vector space.

By basis extension theorem of vector space, B can be extended to the set $B_1=\{\alpha_1,\alpha_2,\ldots\ldots,\alpha_k,\beta_{k+1},\ldots\ldots,\beta_n\}$. So as to form the basis of V(F).

By applying Gram - Schmidt orthogonalisation process to B_1 we can obtain orthonormal basis $B'=\{\alpha_1,\alpha_2,\ldots\ldots,\alpha_k,\alpha_{k+1},\ldots\ldots,\alpha_n\}$ so that $L(B')=V$ of the inner product space.

Theorem. 3. ***If V(F) is a non- zero finite dimensional inner product space of dimension n then*** $\gamma=\sum_{i=1}^{n}\langle\gamma,\alpha_i\rangle\alpha_i$ ***where*** $\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma\in V$.

Proof. Let $B=\{\beta_1,\beta_2,\ldots,\beta_n\}$ be a linearly independent subset of V such that $L(B)=V$.

By Gram - Schmidt orthogonalisation process, we can obtain

$B'=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, an orthonormal basis of V such that $L(B')=L(B)=V$

$$\gamma\in V\Rightarrow\gamma\in L(B')\Rightarrow\gamma=\sum_{i=1}^{n}a_i\,\alpha_i\ \text{ for }a_i\in F$$

For $1\le j\le n\,;\,\langle\gamma,\alpha_j\rangle=\left\langle\sum_{i=1}^{n}a_i\,\alpha_i,\alpha_j\right\rangle=\sum_{i=1}^{n}a_i\langle\alpha_i,\alpha_j\rangle$

$=a_j\langle\alpha_j,\alpha_j\rangle=a_j\|\alpha_j\|^2\quad(\because\ \langle\alpha_i,\alpha_j\rangle=0\text{ for }i\ne j)$

$\therefore\ a_j=\dfrac{\langle\gamma,\alpha_j\rangle}{\|\alpha_j\|^2}=\langle\gamma,\alpha_j\rangle\qquad(\because\ \|\alpha_j\|=1).$ Hence $\gamma=\sum_{i=1}^{n}\langle\gamma,\alpha_i\rangle\alpha_i$.

Corollary. ***Let V(F) be a finite dimensional vector space with an orthonormal basis*** $B=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. ***Let T be a linear operator on V and*** $A=[T]_B$. ***Then for any i and j,*** $a_{ij}=\langle T(\alpha_j),\alpha_i\rangle$.

Proof. From the above theorem,

$$T(\alpha_j)=\sum_{i=1}^{n}\langle T(\alpha_j),\alpha_i\rangle\alpha_i\ \text{ and hence }a_{ij}=\langle T(\alpha_j),\alpha_i\rangle.$$

Definition. (Fourier Coefficients) *If B is an orthonormal basis of a finite dimensional inner product space V(F) and $\beta\in V$ then the Fourier coefficients of β relative to basis B are the scalars $\langle\beta,\alpha\rangle$ where $\alpha\in B$.*

e.g. Let $B=\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)\right\}$ be an orthonormal basis of $R^2(R)$.

Then the Fourier coefficients of $\beta=(3,4)\in R^2$ relative to B are as below :

Let $\alpha_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\alpha_2=\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$

The Fourier coefficients of β with respect to B are $\langle\beta,\alpha_1\rangle$ and $\langle\beta,\alpha_2\rangle$.

$$\langle\beta,\alpha_1\rangle=3\left(\frac{1}{\sqrt{2}}\right)+4\left(\frac{1}{\sqrt{2}}\right)=\frac{7}{\sqrt{2}}\,;\quad\langle\beta,\alpha_2\rangle=3\left(\frac{1}{\sqrt{2}}\right)+4\left(\frac{-1}{\sqrt{2}}\right)=\frac{-1}{\sqrt{2}}.$$

9.5. BESSEL'S INEQUALITY AND PARSEVAL'S IDENTITY

Theorem. 1. (Bessel's inequality) ***If*** $S=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ ***is any orthonormal subset in an inner product space V(F) and*** β ***is any vector in V, then*** $\sum_{i=1}^{n}|\langle\beta,\alpha_i\rangle|^2\le\|\beta\|^2$.

Further equality holds if and only if β ***is in the subspace generated by*** $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ ***or if and only if*** $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ ***is a basis for V.***

(K. U. 05, 07, N. U. 00, 08, S. K. D. 01)

Proof. $S=\{\alpha_1,\alpha_2,\ldots\ldots\ldots,\alpha_n\}$ is an orthonormal set

From the definition, $\langle \alpha, \gamma \rangle = 0, \langle \beta, \gamma \rangle = 0 \ \forall \ \gamma \in W$.

For each $\gamma \in W; \langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle = a0 + b0 = 0 \Rightarrow a\alpha + b\beta \in W^{\perp}$

$\therefore$ $W^{\perp}$ is a subspace of V.

Note. The non-empty subset W need not be a subspace of V.

Theorem. 2. ***If W is a subspace of the inner product space $V(F)$ then $W^{\perp}$ is also subspace of V and $W \cap W^{\perp} = \{\bar{0}\}$.***

Proof. W is a subspace of $V \Rightarrow$ W is a non - empty subset of V

$\Rightarrow W^{\perp}$ is subspace of V (By theorem(1)). Let $\bar{0} \in V$ be the zero element.

Then $\bar{0} \in W$ and $\bar{0} \in W^{\perp}$ as $W, W^{\perp}$ are both subspaces of V. $\therefore$ $\bar{0} \in W \cap W^{\perp}$.

Let $\alpha \in W \cap W^{\perp}$. $\alpha \in W^{\perp} \Rightarrow \langle \alpha, \beta \rangle = 0 \ \forall \ \beta \in W$

As $\alpha \in W$ we have $\langle \alpha, \alpha \rangle = 0 \Rightarrow \alpha = \bar{0}$

$\therefore$ Every $\alpha \in W \cap W^{\perp}$ is such that $\alpha = \bar{0}$. Hence $W \cap W^{\perp} = \{\bar{0}\}$.

SOLVED PROBLEMS

Ex. 5. *Let $V = P_3(R)$ be the inner product space of atmost 3rd degree polynomials continuous on $[-1,1]$ let the innerproduct be defined as $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$ where $f, g \in V$. If $W = P_2(R)$ is the subspace of V with standard basis $B = \{1, x, x^2\}$, by Gram - Schmidt process obtain its orthonormal basis.*

Sol. Let $B = \{\beta_1, \beta_2, \beta_3\}$ so that $\beta_1 = 1, \beta_2 = x, \beta_3 = x^2$

Let $B' = \{\alpha_1, \alpha_2, \alpha_3\}$ be the orthonormal basis.

$$\|\beta_1\|^2 = \langle \beta_1, \beta_1 \rangle = \int_{-1}^{1} (1)\,(1)\, dt = t]_{-1}^{1} = 2 \Rightarrow \alpha_1 = \frac{1}{\|\beta_1\|}\beta_1 = \frac{1}{\sqrt{2}}$$

$$\langle \beta_2, \alpha_1 \rangle = \int_{-1}^{1} t.\frac{1}{\sqrt{2}}\, dt = 0 . \qquad \gamma_2 = \beta_2 - \langle \beta_2, \alpha_1 \rangle \alpha_1 = x - 0\alpha_1 = x$$

$$\|\gamma_2\|^2 = \langle \gamma_2, \gamma_2 \rangle = \int_{-1}^{1} t\,.\,t\, dt = 2\int_{0}^{1} t^2 = \frac{2}{3} \Rightarrow \alpha_2 = \frac{1}{\|\gamma_2\|}\gamma_2 = \sqrt{\frac{3}{2}}\,.\,x$$

$$\langle \beta_3, \alpha_1 \rangle = \int_{-1}^{1} t^2.\frac{1}{\sqrt{2}}\, dt = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\right) = \frac{\sqrt{2}}{3}; \ \langle \beta_3, \alpha_2 \rangle = \int_{-1}^{1} t^2.\sqrt{\frac{3}{2}}\, dt = \sqrt{\frac{3}{2}}\int_{-1}^{1} t^3 dt = 0$$

$$\gamma_3 = \beta_3 - \langle \beta_3, \alpha_1 \rangle \alpha_1 - \langle \beta_3, \alpha_2 \rangle \alpha_2 = x^2 - \frac{\sqrt{2}}{3}.\frac{1}{\sqrt{2}} - 0.\sqrt{\frac{3}{2}}x = x^2 - \frac{1}{3}$$

$$\|\gamma_3\|^2 = \int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt = \int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right)dt = 2\int_{0}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right)dt = 2\left(\frac{1}{5} - \frac{2}{3}.\frac{1}{3} + \frac{1}{9}\right) = \frac{8}{45}$$

$$\therefore \ \alpha_3 = \frac{1}{\|\gamma_3\|}\gamma_3 = \left(x^2 - \frac{1}{3}\right)\sqrt{\frac{45}{8}} = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$$

Hence orthonormal basis of the subspace $W = \{\alpha_1, \alpha_2, \alpha_3\} = \left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\right\}$

Ex. 6. *Represent the polynomial $f(x) = 1 + 2x + 3x^2 \in P_2(R)$ as linear combination of the vectors in the orthonormal basis $\{\alpha_1, \alpha_2, \alpha_3\}$ in Ex. 10.*

Sol. The linear combination of $f(x) = \sum_{i=1}^{3} \langle f, \alpha_i \rangle \alpha_i$

$$\langle f, \alpha_1 \rangle = \int_{-1}^{1} (1 + 2t + 3t^2)\,\frac{1}{\sqrt{2}}\, dt = \frac{2}{\sqrt{2}}\left\{\int_{0}^{1} 1\, dt + 3\int_{0}^{1} t^2 dt\right\} = \sqrt{2}\{1 + 1\} = 2\sqrt{2}$$

$$\langle f, \alpha_2 \rangle = \int_{-1}^{1} (1 + 2t + 3t^2)\,\sqrt{3/2}\; t\, dt = 2\,(\sqrt{3/2})\left\{\int_{0}^{1} 2t^2\, dt\right\} = 2\sqrt{\frac{3}{2}}.\frac{2}{3} = \frac{2\sqrt{6}}{3}$$

$$\langle f, \alpha_3 \rangle = \int_{-1}^{1} (1 + 2t + 3t^2)\,\sqrt{5/8}\,(3t^2 - 1)\, dt = 2\,(\sqrt{5/8})\left\{\int_{0}^{1} (-1 + 9t^4)\, dt\right\} = \frac{2\sqrt{10}}{5}.$$

$$\therefore \ f(x) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{2\sqrt{6}}{3}\left(\sqrt{\frac{3}{2}}x\right) + \frac{2\sqrt{10}}{5}\left(\sqrt{\frac{5}{8}}(3x^2 - 1)\right)$$

EXERCISE 9 (*b*)

1. Normalize the vectors (*a*) $\{(1,1,0), (1,-1,1), (-1,1,2)\}$ (*b*) $\{(1,0,1), (1,0,-1), (0,3,0)\}$ in $R^3(R)$.
2. Find an orthonormal basis of $R^2(R)$ from the basis (*a*) $\{(1,-3), (2,2)\}$ (*b*) $\{(1,0), (3,-5)\}$
3. Applying Gram - Schmidt process obtain an orthonormal basis of $R^3(R)$ from the basis
   (*a*) $\{(1,0,1), (1,0,-1), (0,3,4)\}$ **(*N.U.* 01, *S. K. D.* 08)**
   (*b*) $\{(2,0,1), (3,-1,5), (0,4,2)\}$ **(*N.U.* 03, 08)**
   (*c*) $\{(1,1,0), (-1,1,0), (1,2,1)\}$ **(*K.U.* 04)**
4. Applying Gram - Schmidt process compute the orthogonal set and then normalize to obtain an orthonormal set of $\{(1,0,1,0), (1,1,1,1), (0,1,2,1)\}$ in $R^4(R)$.
5. Find an orthonormal basis of the subspace $W = L(\{(1,0,i), (2,1,1+i)\})$ of $C^3(C)$.
6. If $\langle f, g \rangle = \int_{0}^{1} f(t)\, g(t)\, dt$ is the inner product in $P_2(R)$ with a basis $S = \{1, x, x^2\}$ find the Fourier coefficients of $h(x) = 1 + x$ relative to an orthonormal basis.