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INDETERMINATE FORMS

Art-1. Indeterminate forms

We have earlier pointed out in the beginning of this chapter that while evaluating limits, we may come across the situations $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 . \infty, 0^0, \infty^0, 1^\infty$, which are called indeterminate forms. For evaluating the form $\frac{0}{0}$, we use L' Hospital's Rule which is given below.

Art-2. L' Hospital's Rule

Statement. If f, g are two functions such that

(i) $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}}\, g(x) = 0$

(ii) $f'(x), g'(x)$ both exist and

$g'(x) \neq 0 \ \forall\ x \in (a-\delta, a+\delta),\ \delta > 0$

except possibly at $x = a$.

(iii) $\underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$ exists (finitely or infinitely),

then $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$.

Proof. Let us define two functions F and G such that

$$F(x) = \begin{cases} f(x), \ \forall\ x \in (a-\delta, a+\delta),\ x \neq a \\ 0\ , \quad x = a \end{cases}$$

$$G(x) = \begin{cases} g(x), \ \forall\ x \in (a-\delta, a+\delta),\ x \neq a \\ 0\ , \quad x = a \end{cases}$$

Let x be any real number such that $a < x < a + \delta$ then

1. F, G are both continuous in $[a, x]$

$$\left[\because \underset{x \to a}{\text{Lt}} F(x) = \underset{x \to a}{\text{Lt}} f(x) = 0 = F(a), \text{ etc.}\right]$$

2. F, G are both derivable in (a, x)
3. G' is not zero anywhere in (a, x).

$\therefore$ F and G satisfy all the conditions of Cauchy's Mean Value theorem

$\therefore$ there exists atleast one real number $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c)}{G'(c)} \quad \text{where } a < c < x$$

or $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \ldots(1)$$

Now when $x \to a+,\ c \to a+$

$\therefore$ from (1), we get,

$$\underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{c \to a+}{\text{Lt}} \frac{f'(c)}{g'(c)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)} \quad \ldots(2)$$

Similarly $$\underset{x \to a-}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a-}{\text{Lt}} \frac{f'(x)}{g'(x)} \quad \ldots(3)$$

From (2) and (3), we get,

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

Note 1. If $\underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$ again takes the form $\dfrac{0}{0}$, we repeat the process or use Taylor's Theorem.

Note 2. L' Hospital's Rule when $x \to \infty$

This rule holds even when $x \to \infty$. Its statement is :

If f, g are two functions such that

(i) $\underset{x \to \infty}{\text{Lt}} f(x) = \underset{x \to \infty}{\text{Lt}}\, g(x) = 0$

(ii) $f'(x), g'(x)$ both exist and $g'(x) \neq 0\ \forall\ x > 0$ except possibly at ∞

(iii) $\underset{x \to \infty}{\text{Lt}} \dfrac{f(x)}{g(x)}$ exists,

then $\underset{x \to \infty}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x \to \infty}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$

Same rule holds when $x \to -\infty$.

Note 3. Standard results

$$\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1,\quad \underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1,\quad \underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e \text{ etc.}$$

should be used before applying L' Hospital's Rule.

Note 4. Sometimes, we shall be using standard expansions in evaluating limits of the form $\dfrac{0}{0}$. The use of expansions reduces the labour of differentiating time and again. So readers are advised to remember the following expansions. We shall be using these in some of the illustrative examples.

Some Standard Expansions.

(i) $e^x = 1 + x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \ldots$

(ii) $e^{-x} = 1 - x + \dfrac{x^2}{\underline{|2}} - \dfrac{x^3}{\underline{|3}} + \ldots$

(iii) $\sin x = x - \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} - \ldots$

(iv) $\cos x = 1 - \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} - \ldots$

(v) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$

(vi) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{15} - \ldots$

(vii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

(viii) $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots$

(ix) $(1+x)^{\frac{1}{x}} = e - \dfrac{ex}{2} + \dfrac{11ex^2}{24} + \ldots$ (near $x = 0$)

Art-3. Indeterminate from $\dfrac{0}{0}$

We give some examples to explain the method.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos^2 x}{\sin x^2}$.

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos^2 x}{\sin x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{2\sin x \cos x}{2x \cos x^2}$ [$\because$ of L' Hospital's Rule]

$= \underset{x \to 0}{\text{Lt}} \dfrac{\sin x \cos x}{x \cos x^2}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} \cdot \underset{x \to 0}{\text{Lt}} \dfrac{\cos x}{\cos x^2} = (1) \cdot \dfrac{1}{1}$

$\left[\because \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1, \ \underset{x \to 0}{\text{Lt}} \cos x = 1\right]$

$= 1.$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

(Pbi. U. 1998)

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \dfrac{x}{\sin x}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1$

$\left(\because \underset{x \to 0}{\text{Lt}} \dfrac{x}{\sin x} = 1\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x + e^{-x} - \dfrac{2}{1+x}}{2x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} + \dfrac{2}{(1+x)^2}}{2} = \dfrac{1-1+\dfrac{2}{1}}{2}$

$= \dfrac{2}{2} = 1.$

Example 3. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x \tan^2 x}$.

(P.U. 1989)

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x \tan^2 x}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3} \left(\dfrac{x}{\tan x}\right)^2$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3} \times (1)^2$

$\left[\because \underset{x \to 0}{\text{Lt}} \dfrac{x}{\tan x} = 1\right]$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\cos x + \sin x - \dfrac{1}{1-x}}{3x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-\sin x + \cos x - \dfrac{1}{(1-x)^2}}{6x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-\cos x - \sin x - \dfrac{2}{(1-x)^3}}{6}$

$= \dfrac{-1-0-2}{6} = -\dfrac{3}{6} = -\dfrac{1}{2}.$

Example 4. Prove that $\underset{x\to 0}{\text{Lt}} \dfrac{xe^x - \log(1+x)}{x^2} = \dfrac{3}{2}$

Sol. $\underset{x\to 0}{\text{Lt}} \dfrac{xe^x - \log(1+x)}{x^2}$

$\left[\text{It is of the form } \dfrac{0}{0} \text{ but let us use expansion}\right]$

$= \underset{x\to 0}{\text{Lt}} \dfrac{x\left(1+x+\dfrac{x^2}{\lfloor 2}+\ldots\right)-\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots\right)}{x^2}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{(x-x)+\left(x^2+\dfrac{x^2}{2}\right)+\left(\dfrac{x^3}{2}-\dfrac{x^3}{3}\right)+\ldots}{x^2}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\dfrac{3x^2}{2}+\dfrac{x^3}{6}+\ldots}{x^2} = \underset{x\to 0}{\text{Lt}} \dfrac{x^2\left(\dfrac{3}{2}+\dfrac{x}{6}+\ldots\right)}{x^2}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{x^2\left(\dfrac{3}{2}+\text{terms containing } x \text{ and its higher powers}\right)}{x^2}$

$= \underset{x\to 0}{\text{Lt}} \left(\dfrac{3}{2}+\text{terms containing } x \text{ and its higher powers}\right)$

$= \dfrac{3}{2}+0 = \dfrac{3}{2}$.

Example 5. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{(1+\sin x)^{\frac{1}{3}} - (1-\sin x)^{\frac{1}{3}}}{x}$

(Pbi.U. 1995 ; G.N.D.U. 1998)

Sol. $\underset{x\to 0}{\text{Lt}} \dfrac{(1+\sin x)^{\frac{1}{3}} - (1-\sin x)^{\frac{1}{3}}}{x}$

$(1+x)^n = 1+nx+\dfrac{n(n-1)}{\lfloor 2}x^2+\ldots$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\left[1+\dfrac{1}{3}\sin x+\dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]-\left[1-\dfrac{1}{3}\sin x+\dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]}{x}$

$\dfrac{1}{3}\left(\dfrac{1-3}{3}\right) = -\dfrac{2}{9}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\dfrac{2}{3}\sin x-\dfrac{1}{9}\sin^2 x+\ldots}{x}$

$= \underset{x\to 0}{\text{Lt}} \left[\dfrac{2}{3}\dfrac{\sin x}{x}-\dfrac{1}{9}\dfrac{\sin x}{x}\sin x+\ldots\right]$

$= \dfrac{2}{3}.1-\dfrac{1}{9}.1.0+0 = \dfrac{2}{3}-0 = \dfrac{2}{3}$

Example 6. Evaluate the following limits :

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin ax}{\sin bx}$ (ii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-1}{x}$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{a^x-1}{b^x-1}$ (iv) $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^n-1}{x}$

(v) $\underset{x\to 1}{\text{Lt}} \dfrac{\log x}{x-1}$ (vi) $\underset{x\to 0}{\text{Lt}} \dfrac{x-\tan x}{x-\sin x}$

(vii) $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2}$ (viii) $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{x^3}$

(ix) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-e^{-x}-2x}{x-\sin x}$

(x) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x)-x}{1-\cos x}$

(xi) $\underset{x\to 0}{\text{Lt}} \log(1-x)\cot\dfrac{\pi x}{2}$

(xii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$

(xiii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x\sin x-x-x^2}{x^3}$ (xiv) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-e^{\sin x}}{x-\sin x}$

(xv) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log\cos x}$ (xvi) $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{(x\sin x)^{\frac{3}{2}}}$

(xvii) $\underset{x\to 0}{\text{Lt}} \dfrac{x-\tan^{-1}x}{x-\sin x}$

(xviii) $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x-\log(1+x)}{x^2}$

(xix) $\underset{x\to a}{\text{Lt}} \dfrac{a^x-x^a}{a^a-x^x}$ (H.P.U. 1996)

(xx) $\underset{x\to 1}{\text{Lt}} \dfrac{x^x-x}{1-x+\log x}$.

Sol. (i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin ax}{\sin bx}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{a\cos ax}{b\cos bx}$ [$\because$ of L' Hospital's Rule]

$= \dfrac{a\cos 0}{b\cos 0} = \dfrac{a\times 1}{b\times 1} = \dfrac{a}{b}$

(ii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-1}{x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - 0}{1} \quad [\because \text{ of L'Hospital's Rule}]$$

$$= \underset{x \to 0}{\text{Lt}}\ e^x = e^0 = 1.$$

(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{b^x - 1}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{a^x \log a}{b^x \log b} \quad [\because \text{ of L' Hospital's Rule}]$$

$$= \frac{a^0 \log a}{b^0 \log b} = \frac{\log a}{\log b} = \log_b a.$$

(iv) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{n(1+x)^{n-1}}{1} \quad [\because \text{ of L' Hospital's Rule}]$$

$$= n(1+0)^{n-1} = n.$$

(v) $\underset{x \to 1}{\text{Lt}} \dfrac{\log x}{x - 1}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 1}{\text{Lt}} \frac{\frac{1}{x}}{1} \quad [\because \text{ of L' Hospital's Rule}]$$

$$= \underset{x \to 1}{\text{Lt}} \frac{1}{x} = \frac{1}{1} = 1.$$

(vi) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \tan x}{x - \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \sec^2 x}{1 - \cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2\sec x . \sec x \tan x}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\frac{2}{\cos x} . \frac{1}{\cos x} . \frac{\sin x}{\cos x}}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2}{\cos^3 x} = \frac{-2}{\cos^3 0} = -\frac{2}{1} = -2.$$

(vii) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{2x} = \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = \frac{1}{2} \times 1 = \frac{1}{2}.$$

(viii) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{3x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{6x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cos x}{6} = \frac{1}{6}.$$

(ix) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + e^{-x} - 2}{1 - \cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x}}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2.$$

(x) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x) - x}{1 - \cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{1+x} - 1}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\frac{1}{(1+x)^2}}{\cos x} = \frac{-\frac{1}{(1+0)^2}}{1} = -1.$$

(xi) $\underset{x \to 0}{\text{Lt}} \log(1-x) \cot \dfrac{\pi x}{2}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log(1-x)}{\tan \frac{\pi x}{2}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{-1}{1-x}}{\frac{\pi}{2}\sec^2 \frac{\pi x}{2}} = \frac{\frac{-1}{1-0}}{\frac{\pi}{2}\sec^2 0} = \frac{-1}{\frac{\pi}{2}} = -\frac{2}{\pi}.$$

(xii) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{2x - \frac{x}{1-x} + \log(1-x)} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x - 2}{2 - \frac{(1-x).1 - x(-1)}{(1-x)^2} - \frac{1}{1-x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{2 - \frac{1}{(1-x)^2} - \frac{1}{1-x}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2(e^x \cos x - e^x \sin x)}{-\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}} = \frac{2(1-0)}{-2-1} = -\frac{2}{3}.$$

(xiii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{3x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x - 2}{6x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{6x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2(e^x \cos x - e^x \sin x)}{6} = \frac{2(1-0)}{6} = \frac{1}{3}.$$

(xiv) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{e^x - e^{\sin x}.\cos x}{1 - \cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{e^x - e^{\sin x}.\cos^2 x + e^{\sin x} \sin x}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{e^x - e^{\sin x}.\cos^3 x + e^{\sin x}.2\cos x \sin x + e^{\sin x} \sin x \cos x + e^{\sin x} \cos x}{\cos x}$$

$$= \frac{1-1+0+0+1}{1} = 1.$$

(xv) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log \cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{-2x}{1-x^2}}{-\frac{\sin x}{\cos x}} = \underset{x\to 0}{\text{Lt}} \frac{\frac{-2x}{1-x^2}}{-\tan x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{(1-x^2)(-2) - 2x(-2x)}{(1-x^2)^2}}{-\sec^2 x} = \frac{\frac{-2-0}{1}}{-1} = 2.$$

(xvi) $\underset{x\to 0}{\text{Lt}} \dfrac{x - \sin x}{(x\sin x)^{\frac{3}{2}}} = \underset{x\to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3.\left(\frac{\sin x}{x}\right)^{\frac{3}{2}}}$

$$= \underset{x\to 0}{\text{Lt}} \frac{x - \sin x}{x^3} \quad \left[\because \underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - \cos x}{3x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\sin x}{6x} = \underset{x\to 0}{\text{Lt}} \frac{\cos x}{6} = \frac{1}{6}.$$

(xvii) $\underset{x\to 0}{\text{Lt}} \dfrac{x - \tan^{-1} x}{x - \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - \frac{1}{1+x^2}}{1 - \cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{2x}{(1+x^2)^2}}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{(1+x^2)^2.2 - 2x.2(1+x^2).2x}{(1+x^2)^4}}{\cos x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{2(1+x^2) - 8x^2}{(1+x^2)^3}}{\cos x} = \frac{\frac{2-0}{1}}{1} = 2.$$

(xviii) $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-x\cos x - \sin x - \sin x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{-0-0-0+1}{2} = \frac{1}{2}.$$

(xix) $\underset{x\to a}{\text{Lt}} \dfrac{a^x - x^a}{a^a - x^x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to a}{\text{Lt}} \frac{a^x \log a - a x^{a-1}}{0 - x^x(1+\log x)}$$

$\left[\text{Put } y = x^x, \ \therefore \log y = x\log x \Rightarrow \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x.1 \quad \therefore \frac{dy}{dx} = x^x(1+\log x)\right]$

$$= \frac{a^a \log a - a.a^{a-1}}{-a^a(1+\log a)} = \frac{a^a \log a - a^a}{-a^a(1+\log a)}$$

$$= \frac{\log a - 1}{-(1+\log a)} = -\frac{\log a - 1}{1+\log a}.$$

...LUSTRATIVE EXAMPLES

...le 1. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\log \sin x}{\cot x}$.

$$\underset{x\to 0}{\text{Lt}} \frac{\log \sin x}{\cot x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \sin x \cos x$$

$$= -\underset{x\to 0}{\text{Lt}} \sin x \,.\, \underset{x\to 0}{\text{Lt}} \cos x$$

$$= -0 \times 1 = 0.$$

...ple 2. Evaluate $\underset{x\to 0+}{\text{Lt}} \log_{\sin 2x} \sin x$.

$$\underset{x\to 0+}{\text{Lt}} \log_{\sin 2x} \sin x$$

Sin2x

$$= \underset{x\to 0+}{\text{Lt}} \frac{\log \sin x}{\log \sin 2x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\frac{1}{\sin x}.\cos x}{\frac{1}{\sin 2x}.2\cos 2x}$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{\sin 2x}{2\cos 2x}$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{2\sin x \cos x}{2\cos 2x}$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\cos^2 x}{\cos 2x} = \frac{1}{1} = 1.$$

...ample 3. Evaluate $\underset{x\to \infty}{\text{Lt}} \dfrac{x^n}{e^x}$, $n \in \mathbf{N}$.

$$\underset{x\to \infty}{\text{Lt}} \frac{x^n}{e^x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{n x^{n-1}}{e^x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{n(n-1)x^{n-2}}{e^x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{n(n-1)(n-2)\ldots 2.1}{e^x}$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{\lfloor n}{e^x} = \lfloor n \,.\, \underset{x\to \infty}{\text{Lt}} \frac{1}{e^x} = 0.$$

$$\left[\because \underset{t\to \infty}{\text{Lt}} \frac{1}{e^t} = \underset{t\to \infty}{\text{Lt}} e^{-t} = 0\right]$$

Example 4. Evaluate the following limits

(*i*) $\underset{x\to \frac{\pi}{2}+}{\text{Lt}} \dfrac{\tan x}{\tan 3x}$ (*ii*) $\underset{x\to 1}{\text{Lt}} \dfrac{\log(1-x^2)}{\cot \pi x}$

(*iii*) $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\log x^2}{\cot^2 x}\right)$ (*iv*) $\underset{x\to 0}{\text{Lt}} \log_{\tan^2 x} \tan^2 2x$.

Sol. (*i*) $$\underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{\tan x}{\tan 3x} = \underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{\sec^2 x}{\sec^2 3x}$$

$$= \underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 3x}} = \frac{1}{3} \underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{\cos^2 3x}{\cos^2 x}$$

$$= \frac{1}{3} \underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{(4\cos^3 x - 3\cos x)^2}{\cos^2 x}$$

$$= \frac{1}{3} \underset{x\to \frac{\pi}{2}+}{\text{Lt}} (4\cos^2 x - 3)^2 = \frac{1}{3}(4 \times 0 - 3)^2 = 3.$$

(*ii*) $$\underset{x\to 1}{\text{Lt}} \frac{\log(1-x^2)}{\cot \pi x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to 1}{\text{Lt}} \frac{\frac{-2x}{1-x^2}}{-\pi \operatorname{cosec}^2 \pi x} = \frac{2}{\pi} \underset{x\to 1}{\text{Lt}}\, x.\frac{\sin^2 \pi x}{1-x^2}$$

$$= \frac{2}{\pi} \underset{x\to 1}{\text{Lt}}\, x \times \underset{x\to 1}{\text{Lt}} \frac{\sin^2 \pi x}{1-x^2}$$

$$= \frac{2}{\pi} \times 1 \times \underset{x\to 1}{\text{Lt}} \frac{2\pi \sin \pi x \cos \pi x}{-2x} = \frac{2}{\pi} \times 1 \times 0 = 0.$$

(*iii*) $$\underset{x\to 0}{\text{Lt}} \frac{\log x^2}{\cot^2 x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{2}{x}}{-2\cot x \operatorname{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \frac{\sin^3 x}{x \cos x}$$

$$= -\underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^3 \times \frac{x^2}{\cos x} = -\underset{x\to 0}{\text{Lt}} \frac{x^2}{\cos x}$$

$$= -\frac{0}{1} = 0.$$