Introduction to Computer Vision

Background Review on

Linear Algebra

Some Definitions

An $m \times n$ (read "m by n") ***matrix***, denoted by **A**, is a rectangular array of entries or elements (numbers, or symbols representing numbers) enclosed typically by square brackets, where m is the number of rows and n the number of columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Definitions (Con't)

- **A** is ***square*** if $m = n$.
- **A** is ***diagonal*** if all off-diagonal elements are 0, and not all diagonal elements are 0.
- **A** is the ***identity matrix*** (**I**) if it is diagonal and all diagonal elements are 1.
- **A** is the ***zero*** or ***null matrix*** (**0**) if all its elements are 0.
- The ***trace*** of **A** equals the sum of the elements along its main diagonal.
- Two matrices **A** and **B** are ***equal*** iff they have the same number of rows and columns, and $a_{ij} = b_{ij}$.

Definitions (Con't)

- The ***transpose*** $\mathbf{A}^T$ of an $m \times n$ matrix **A** is an $n \times m$ matrix obtained by interchanging the rows and columns of **A**.
- A square matrix for which $\mathbf{A}^T$=**A** is said to be ***symmetric***.
- Any matrix X for which **XA**=**I** and **AX**=**I** is called the ***inverse*** of **A**.
- Let c be a real or complex number (called a ***scalar***). The ***scalar multiple*** of c and matrix **A**, denoted c**A**, is obtained by multiplying every elements of **A** by c. If $c = -1$, the scalar multiple is called the ***negative*** of **A**.

Definitions (Con't)

A ***column vector*** is an $m \times 1$ matrix:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

A ***row vector*** is a $1 \times n$ matrix:

$$\boxed{\mathbf{b} = [b_1, b_2, \cdots b_n]}$$

A column vector can be expressed as a row vector by using the transpose:

$$\boxed{\mathbf{a}^T = [a_1, a_2, \cdots, a_m]}$$

Some Basic Matrix Operations

- The ***sum*** of two matrices **A** and **B** (of equal dimension), denoted **A** + **B**, is the matrix with elements $a_{ij} + b_{ij}$.
- The ***difference*** of two matrices, **A**– **B**, has elements $a_{ij} - b_{ij}$.
- The ***product***, **AB**, of $m \times n$ matrix **A** and $p \times q$ matrix **B**, is an $m \times q$ matrix **C** whose (i,j)-th element is formed by multiplying the entries across the ith row of **A** times the entries down the jth column of **B**; that is,

$$\boxed{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{pj}}$$

Some Basic Matrix Operations (Con't)

The ***inner product*** (also called ***dot product***) of two vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is defined as

$$\boxed{\begin{aligned} \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} &= a_1b_1 + a_2b_2 + \cdots + a_mb_m \\ &= \sum_{i=1}^{m} a_ib_i. \end{aligned}}$$

Note that the inner product is a scalar.

Vectors and Vector Spaces

A ***vector space*** is defined as a nonempty set V of entities called *vectors* and associated scalars that satisfy the conditions outlined in A through C below. A vector space is *real* if the scalars are real numbers; it is *complex* if the scalars are complex numbers.

- Condition A: There is in V an operation called *vector addition*, denoted $\mathbf{x} + \mathbf{y}$, that satisfies:

  1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in the space.
  2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.
  3. There exists in V a unique vector, called the *zero vector*, and denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ and $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all vectors $\mathbf{x}$.
  4. For each vector $\mathbf{x}$ in V, there is a unique vector in V, called the *negation* of $\mathbf{x}$, and denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ and $(-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

Vectors and Vector Spaces (Con't)

- Condition B: There is in V an operation called *multiplication by a scalar* that associates with each scalar c and each vector $\mathbf{x}$ in V a unique vector called the *product* of c and $\mathbf{x}$, denoted by $c\mathbf{x}$ and $\mathbf{x}c$, and which satisfies:
    1. $c(d\mathbf{x}) = (cd)\mathbf{x}$ for all scalars c and d, and all vectors $\mathbf{x}$.
    2. $(c + d)\mathbf{x} = \mathrm{c}\mathbf{x} + \mathrm{d}\mathbf{x}$ for all scalars c and d, and all vectors $\mathbf{x}$.
    3. $\mathrm{c}(\mathbf{x} + \mathbf{y}) = \mathrm{c}\mathbf{x} + \mathrm{c}\mathbf{y}$ for all scalars c and all vectors $\mathbf{x}$ and $\mathbf{y}$.
- Condition C: $1\mathbf{x} = \mathbf{x}$ for all vectors $\mathbf{x}$.

Vectors and Vector Spaces (Con't)

We are interested particularly in real vector spaces of real $m \times 1$ column matrices. We denote such spaces by $\Re^m$, with vector addition and multiplication by scalars being as defined earlier for matrices. Vectors (column matrices) in $\Re^m$ are written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Vectors and Vector Spaces (Con't)

Example

The vector space with which we are most familiar is the two-dimensional real vector space $\Re^2$, in which we make frequent use of graphical representations for operations such as vector addition, subtraction, and multiplication by a scalar. For instance, consider the two vectors

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

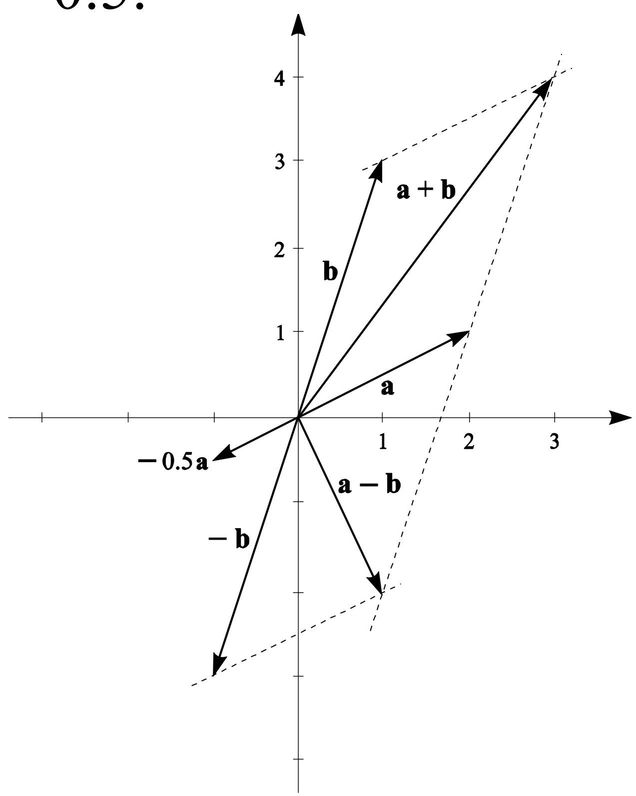
Using the rules of matrix addition and subtraction we have

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \mathbf{a} - \mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Vectors and Vector Spaces (Con't)

Example (Con't)

The following figure shows the familiar graphical representation of the preceding vector operations, as well as multiplication of vector **a** by scalar $c = -0.5$.



Vectors and Vector Spaces (Con't)

Consider two real vector spaces V_0 and V such that:

- Each element of V_0 is also an element of V (i.e., V_0 is a *subset* of V).
- Operations on elements of V_0 are the same as on elements of V. Under these conditions, V_0 is said to be a ***subspace*** of V.

A ***linear combination*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an expression of the form

$$\boxed{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n}$$

where the α's are scalars.

Vectors and Vector Spaces (Con't)

A vector $\mathbf{v}$ is said to be ***linearly dependent*** on a set, S, of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if and only if $\mathbf{v}$ can be written as a linear combination of these vectors. Otherwise, $\mathbf{v}$ is ***linearly independent*** of the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Vectors and Vector Spaces (Con't)

A set S of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in V is said to ***span*** some subspace V_0 of V if and only if S is a subset of V_0 and every vector $\mathbf{v}_0$ in V_0 is linearly dependent on the vectors in S. The set S is said to be a ***spanning set*** for V_0. A ***basis*** for a vector space V is a linearly independent spanning set for V. The number of vectors in the basis for a vector space is called the ***dimension*** of the vector space. If, for example, the number of vectors in the basis is n, we say that the vector space is n-dimensional.

Vectors and Vector Spaces (Con't)

An important aspect of the concepts just discussed lies in the representation of any vector in $\Re^m$ as a ***linear combination*** of the basis vectors. For example, any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in $\Re^3$ can be represented as a linear combination of the basis vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vector Norms

A ***vector norm*** on a vector space V is a function that assigns to each vector **v** in V a nonnegative real number, called the ***norm*** of **v**, denoted by $\|\mathbf{v}\|$. By definition, the norm satisfies the following conditions:

(1) $\|\mathbf{v}\| > 0$ for $\mathbf{v} \neq \mathbf{0}$; $\|\mathbf{0}\| = 0$,

(2) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ for all scalars c and vectors $\mathbf{v}$, and

(3) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Vector Norms (Con't)

There are numerous norms that are used in practice. In our work, the norm most often used is the so-called ***2-norm***, which, for a vector **x** in real $\Re^m$, space is defined as

$$\boxed{\|\mathbf{x}\| = [x_1^2 + x_2^2 + \cdots + x_m^2]^{1/2}}$$

which is recognized as the *Euclidean distance* from the origin to point **x**; this gives the expression the familiar name Euclidean norm. The expression also is recognized as the length of a vector **x**, with origin at point **0**. From earlier discussions, the norm also can be written as

$$\boxed{\|\mathbf{x}\| = [\mathbf{x}^T\mathbf{x}]^{1/2}}$$

Vector Norms (Con't)

The ***Cauchy-Schwartz*** inequality states that

$$\boxed{|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|}$$

Another well-known result used in the book is the expression

$$\boxed{\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}}$$

where θ is the angle between vectors **x** and **y**. From these expressions it follows that the inner product of two vectors can be written as

$$\boxed{\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta}$$

Thus, the inner product can be expressed as a function of the norms of the vectors and the angle between the vectors.

Vector Norms (Con't)

From the preceding results, two vectors in $\Re^m$ are ***orthogonal*** if and only if their inner product is zero. Two vectors are ***orthonormal*** if, in addition to being orthogonal, the length of each vector is 1.

From the concepts just discussed, we see that an arbitrary vector $\mathbf{a}$ is turned into a vector $\mathbf{a}_n$ of unit length by performing the operation $\mathbf{a}_n = \mathbf{a}/\|\mathbf{a}\|$. Clearly, then, $\|\mathbf{a}_n\| = 1$.

A ***set of vectors*** is said to be an ***orthogonal*** set if every two vectors in the set are orthogonal. A ***set of vectors*** is ***orthonormal*** if every two vectors in the set are orthonormal.

Some Important Aspects of Orthogonality

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal or orthonormal basis in the sense defined in the previous section. Then, an important result in vector analysis is that any vector v can be represented with respect to the orthogonal basis B as

$$\boxed{\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n}$$

where the coefficients are given by

$$\boxed{\begin{aligned} \alpha_i &= \frac{\mathbf{v}^T \mathbf{v}_i}{\mathbf{v}_i^T \mathbf{v}_i} \\ &= \frac{\mathbf{v}^T \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \end{aligned}}$$

Orthogonality (Con't)

The key importance of this result is that, if we represent a vector as a linear combination of orthogonal or orthonormal basis vectors, we can determine the coefficients directly from simple inner product computations. It is possible to convert a linearly independent spanning set of vectors into an orthogonal spanning set by using the well-known ***Gram-Schmidt*** process. There are numerous programs available that implement the Gram-Schmidt and similar processes, so we will not dwell on the details here.

Eigenvalues & Eigenvectors

Definition: The ***eigenvalues*** of a real matrix $\mathbf{M}$ are the real numbers λ for which there is a nonzero vector $\mathbf{e}$ such that

$$\boxed{\mathbf{M}\mathbf{e} = \lambda\, \mathbf{e}.}$$

The ***eigenvectors*** of $\mathbf{M}$ are the nonzero vectors $\mathbf{e}$ for which there is a real number λ such that $\mathbf{M}\mathrm{e} = \lambda\, \mathbf{e}$.

If $\mathbf{M}\mathrm{e} = \lambda\, \mathbf{e}$ for $\mathbf{e} \neq 0$, then $\mathbf{e}$ is an ***eigenvector*** of $\mathbf{M}$ associated with ***eigenvalue*** λ, and vice versa. The eigenvectors and corresponding eigenvalues of $\mathbf{M}$ constitute the ***eigensystem*** of $\mathbf{M}$.

Numerous theoretical and truly practical results in the application of matrices and vectors stem from this beautifully simple definition.

Eigenvalues & Eigenvectors (Con't)

Example: Consider the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

It is easy to verify that $\mathbf{M}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1$ and $\mathbf{M}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2$ for $\lambda_1 = 1$, $\lambda_2 = 2$ and

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In other words, $\mathbf{e}_1$ is an eigenvector of **M** with associated eigenvalue λ_1, and similarly for $\mathbf{e}_2$ and λ_2.