1 Introduction

The following basic 'zero model' serves as a proof-of-concept to demonstrate the basic idea and approach to the optimization of car fleet deployment within a specific feeding corridor zone A at the time of peak hour. For this purpose we can use data of O-D matrix, T_{AB} which gives number of trips between points A and B in one day. Based on those data, we can estimate $t_{AB}(t)$ as a number of traveling people per unit time from A to B at the time t so that $\int_{\text{day}} t_{AB}(t')dt' = T_{AB}$.

Further, in this analysis, we will consider the situation during a morning (or evening) peak hour, when majority of people commute to work from the residential zone A to a destination B. We assume that all of those taking the trip by mass public transport (MPT) make transfer to a backbone transport (e.g. light-rail or underground train) at the site E. We can describe the system as illustrated in Figure 1 and defined in Table 1.

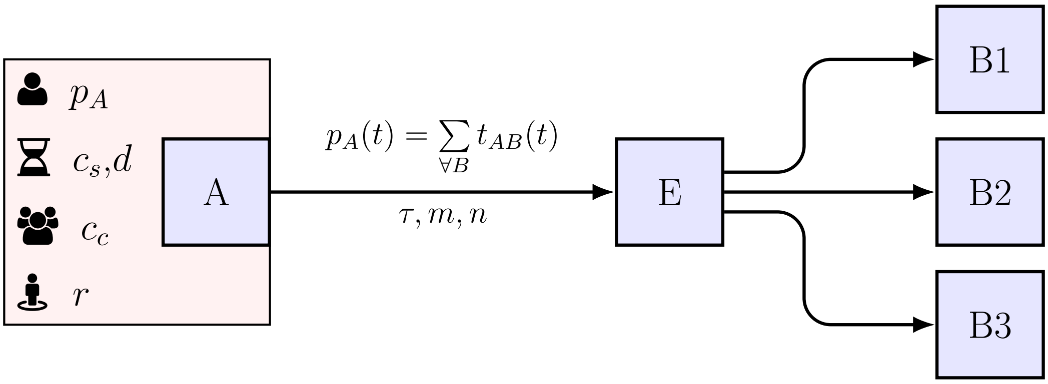

Figure 1: Scheme of transport along the $A \to E$ feeding corridor

We define the following symbols, in general, all of them would be time-dependent:

symbol	name	note
$t_{AB}(t)$	origin-destination (o-d) flow	Origin-destination flow of people per unit time from $A \to B$
$p_A(t)$	susceptible population flow	$p_A(t) = \sum_{\forall B} t_{AB}(t)$, prospect clients going $A \to E$ per unit time
$p_c(t)$	critical population flow	critical level, if $p_A > p_c$, people start to take taxi
$\Pi(t)$	taxi rate	probability of a single agent to take a taxi
$q(t)$	customer flow	number of customers transported per unit time
$c(t)$	personal cost	perceived monetary cost of taking MPT of an agent
$a(t)$	(deployed) car capacity	number of people possibly transported per unit time
$m(t)$	taxi price/fare	price of a taxi ride per person $A \to E$
$\tau(t)$	taxi roundtrip time	mean time for taxi to drive $A \to E \to A$
$s(t)$	driver salary	driver pay per unit time
$n(t)$	deployed cars	number of cars deployed for the trip $A \to E$
$d(t)$	waiting time	mean waiting time for MTP
r	walking time	mean walking time to reach MTP
c_s	time cost	mean monetary value of personal time ($\approx$ hourly wage)
c_c	crowding cost	mean monetary coefficient of crowd discomfort in MPT per person
c_M	car capacity	maximum number of passengers per taxi
$i(t)$	income/return rate	money earned (or lost) per unit time; net income
$i^+(t)$	collected fare/revenue rate	money collected per unit time; brute income
$i^-(t)$	operation costs	money spent on operations per unit time

Table 1: Definition of the descriptive parameters

Note, that most of the symbols are defined as rates, i.e. per unit time. Total quantity per interval T is then calculated by integrating over time, e.g.

$$Q = \int_T q(t')dt'$$

would be total number of clients Q served during the interval T. For time independent rates integration reduces to simple multiplication, e.g. $q(t) = q : Q = q \cdot T$.

2 Simplification

For sake of simplicity, we will treat only the morning peak time. We assume that during the morning peak period, the descriptors in Table 1 remain roughly constant (i.e. $p_A(t) = p_A, \Pi(t) = \Pi, s(t) = s$, etc.). We make the following simplifying assumptions:

1. we consider only two available modes of transport, MPT and taxi
2. people along the feeding corridor take transport from the given location A to the MPT exchange hub E, the population flow originating in A is p_A
3. the susceptible population flow p_A is much larger than the number of people in taxis or the taxi capacity ($p_A \gg a \geq q$) and so the amount of people taking taxi have no feedback on the system, or on the personal perceived cost of entering agents
4. return trips of taxi $E \to A$ are without clients, the travel time difference between a taxi and MPT is ignored
5. taxi drivers are paid flat rate per time unit s, and a taxi passenger pays a flat rate per seat m, independent of car occupancy
6. all taxi trips $A \to E$ are of equal length, time ($\frac{\tau}{2}$), and fare (m), cost of MPT is considered negligible
7. all n deployed taxi cars are effectively identical, with maximum capacity per car c_M
8. all traveling agents are identical (represented by constant mean values) and have perfect knowledge of the system (e.g. current descriptors), their MPT accessibility (d, r) is identical
9. agents are perfectly rational, they obey the cost function c.

3 Cost function

We define the function of perceived monetary costs of an agent stemming from the use of MPT.

$$c = \underbrace{c_s(d+r)}_{\text{MPT accessibility}} + \underbrace{c_c p_A d}_{\text{crowding}} \tag{1}$$

The first term represents the perceived cost in terms of time lost walking (r) and waiting (d), converted to monetary cost by the factor c_s, which can be estimated as an hourly wage of an agent. The second term refers to the crowding (and discomfort) effect in the MPT, $p_A\, d$ is a mean occupancy of a bus dispatch, weighted by a factor c_c, persons sensitivity to discomfort.

While many other factors play role, like tiredness, time of the day, weather, pedestrian infrastructure, MPT quality, reserved lanes for MPT, etc., we will neglect these in this analysis. Obviously, the variables that constitute the cost function would themselves be functions of time in a more general case. Note that the way the cost function is defined in Equation 1, it is constant for a given situation, and depends on the parameters of MPT (d, r) and demographics (p_A, c_c, c_s) of the zone A.

4 Personal choice and passenger flow

In the simplest choice model, there is a threshold effect for the agent to choose alternative transportation mode—taxi in our case. This occurs when the perceived cost of MPT is higher that the taxi cost, $c > m$. This implies existence of a critical population flow $p_c(m)$ as a function of fare m such, that the population in excess to this critical population flow $p_A > p_c$ opts for a taxi ride. That is, the probability of an entering agent to opt for a taxi is given as

$$\Pi = \begin{cases} 1 : c(p_A) > m \\ 0 : \text{otherwise} \end{cases} \tag{2}$$

The critical population flow is given as

$$\begin{aligned} c(p_A{=}p_c) &= m \\ m &= c_c p_c d + c_s(d+r) \\ p_c &= \frac{m - c_s(d+r)}{c_c d} \\ p_c(m) &= \underbrace{\frac{m}{c_c d}}_{\text{taxi cost}} - \underbrace{\frac{c_s(d+r)}{c_c d}}_{\text{MPT time cost}} \quad |p_c \in \mathbb{R}^+ \end{aligned} \tag{3}$$

Where the first term represents the demotivation effect of the taxi fare. The second term is the incentive of poor MPT accessibility—discomfort associated with reaching the MPT station. The critical flow p_c is modulated by the crowding effect; high sensitivity to crowds and infrequent MPT decrease the critical flow threshold and represent a discomfort associated with traveling by MPT.

The number of people taking a taxi per unit time, the passenger flow, is given as

$$q(m) = \Pi(m) \cdot (p_A - p_c(m)) \tag{4}$$
$$q(m) = \begin{cases} \min\left(-\frac{m}{c_c d} + p_A + \frac{c_s(d+r)}{c_c d}, p_A\right) & |\text{if } c > m \\ 0 \;\; |\text{otherwise} \end{cases}$$

We can see, that passenger flow $q(m)$ decreases with growing fare m up to a halt at $c < m$. The passenger flow in the taxis is limited by the taxi fleet capacity per unit time

$$a(n) = \frac{c_M\, n}{\tau}. \tag{5}$$

which depends only the number of cars n (other factors are predetermined).

5 Income analysis

The return rate is given as follows

$$i(m,n) = \underbrace{\min\left(q(m), a(n)\right) \cdot m}_{\text{collected fare};i^+} - \underbrace{s \cdot n}_{\text{operation cost};i^-} \tag{6}$$

under the predetermined zone parameters (e.g. p_A, d, r), operator can adjust two parameters: the fare m which determines the passenger flow $q(m)$, and the deployed capacity $a(n)$ by sending n cars. We denote the collected revenue per unit time i^+. The cost of operation per unit time, denoted i^-, is a sum of flat rates s of all the drivers. Total return rate is $i = i^+ - i^-$. This can be illustrated by the following (qualitative) Figure 2

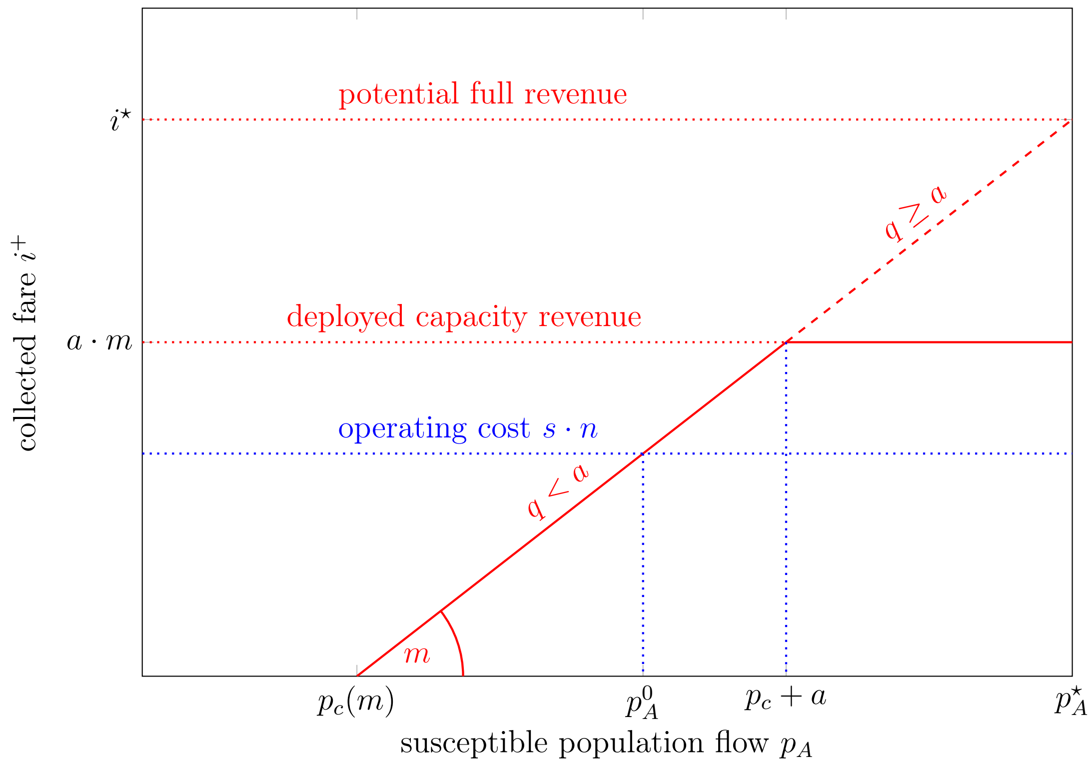

Figure 2: Illustration of the return rate i^+ dependence (red full line) on the susceptible population flow p_A. For optimally adjusted m and n, the full revenue rate $i^\star$ is reached, if all population flow $p_A^\star$ is served.

In the Figure 2, $a(n) \cdot m$ is the maximum revenue under the deployed fleet capacity, $i^\star$ is the maximal potential revenue and $p_A^\star$ is the corresponding susceptible population flow. p_A^0 is the value where the operation breaks even, i.e. $i(p_A^0) = 0$. The blue dotted line represent the operation cost of the deployed fleet.

Note that the collected fare i^+ (the full red line) depends on m. The fare m adjusts the participation threshold p_c (decreasing m motivates people to move into taxis and adjusts the position of p_c on the abscissa to the left) and the slope of the revenue line. p_c does not depend on n, but is determined by the fixed characteristics of the residential area (see Equation 3 for details).

As we can see in Figure 2, there are four distinct zones depending on the susceptible population flow p_A:

- zone I: $p_A \in (0, p_c)$, there is no traffic, people prefer to take MPT. $i = -s \cdot n < 0$
- zone II: $p_A \in (p_c, p_A^0)$, clients opt to take taxi, there are sufficient capacities to cater all the clients, operation is at loss. $i = q \cdot m - s \cdot n < 0$.
- zone III: $p_A \in (p_A^0, p_c + a)$, the same as II, but operation is at surplus. $i = q \cdot m - s \cdot n > 0$
- zone IV: $p \in (p_c + a, p_A^\star)$, transportation of clients is throttled by the insufficient deployed car capacity. $i = a \cdot m - s \cdot n > 0$.

Note that in the Figure 2, we assume the operating cost is below the collected revenue ($i^+ > i^-$) if the deployed fleet is fully occupied; in other words, adding another car, if it is full exploited, increases the return rate i; or, the cars are not subsidized. However, this does not need to be the case in general, and depends on the company adjustment of s and m. To avoid loss on fully exploited fleet (passenger flow exceeds the deployed capacity), the condition is

$$a \cdot m \geq s \cdot n$$
$$\frac{c_M\, n}{\tau} m \geq s \cdot n$$
$$\frac{c_M}{\tau} m \geq s$$

which means

$$m \geq \frac{s\tau}{c_M} \tag{7}$$

to assure feasibility of fully engaged cars.

We can now analyze the particular break points and optimal strategies of selection of parameters m and n.

5.1 Break even point

Assume we are running on a feasible choice of parameter m (Equation 7), and that the fleet capacity was not exceeded, $q \leq a$ (see Figure 3). In such case, the collected revenue is given as $i^+ = q \cdot m$. The costs associated are $i^- = s \cdot n$. If Equation 7 holds, then for fixed predetermined choice of m and n, there exist susceptible population flow p_A^0, and surpassing this flow yields positive revenue rate $i(p_A \geq p_A^0) \geq 0$, i.e. we break even, as noted in Figure 2. However, we can also invert the problem: that is, what parameter $m^\dagger$ do we need to set for a predetermined population flow $p_A^\dagger$[1], to break even.

$$\begin{aligned}
i^+ = q(p_A^\dagger, m^\dagger) \cdot m^\dagger &= s \cdot n \\
(p_A^\dagger - p_c) \cdot m^\dagger &= s \cdot n \quad |\text{using Equation 3} \\
\left[p_A^\dagger - \frac{m^\dagger}{c_c d} + \frac{c_s}{c_c d}(d + r)\right] m^\dagger &= s \cdot n \\
(m^\dagger)^2 - m^\dagger \underbrace{(p_A^\dagger c_c d + c_s(d + r))}_{\equiv c^\dagger} &= -snc_c d \\
(m^\dagger)^2 - m^\dagger c^\dagger + snc_c d &= 0 \\
m^\dagger_{<,>} &= \frac{1}{2}\left[c^\dagger \pm \sqrt{(c^\dagger)^2 - 4sc_c dn}\right]
\end{aligned}$$

This result tells us, that for a given n (costs depend on n) and a fixed $p_A^\dagger$, there are two pricing limits we can choose to break even. One such option involves fewer customers and higher price $m^\dagger_>$, the other would be pricing which attracts more customers at lower fare (which has a marketing benefit) at

$$m^\dagger_< = \frac{1}{2}\left[c^\dagger - \sqrt{(c^\dagger)^2 - 4sc_c dn}\right] \tag{8}$$

Actually, any price in the range $m^\dagger \in (m^\dagger_<, m^\dagger_>)$ yields feasible operation, which is visible, if we plot the graph of return rate as a function of $i(m)$, for $n : q(m^\dagger) \leq a(n)$, as shows Figure 3. From the graph, we can also read,

[1] We use dagger (†) to denote, that the variable is fixed to a particular value in this context. I.e. $p_A^\dagger$ is a predetermined population flow, and $m^\dagger$ is a fixed operations parameter to meet a specific goal.

that for non-throttled situation, the maximum return rate choice of ride price is half of the perceived MPT cost, $m = \frac{c^\dagger}{2}$.

A technical question remains: the determinant in the Equation 8 must remain positive. We can make order of magnitude realistic estimates of the parameters (with exception of crowd intolerance c_c, which is extremely subjective). For determinant to be positive, it is sufficient, if

$$\begin{aligned}
[p_A c_s(d+r)]c_c d &> 4snc_c d \\
p_A c_s(d+r) &> 4sn \;\; |s \sim c_s \\
p_A(d+r) &> 4n
\end{aligned}$$

And considering that the total number of people leaving by MTP each turn is larger than the number of deployed cars ($p_A \gg a$, see section 2), and the other terms of $(c^\dagger)^2$ expansion are always positive, we see that the determinant remains positive, and the solution with feasibility zone is guaranteed.

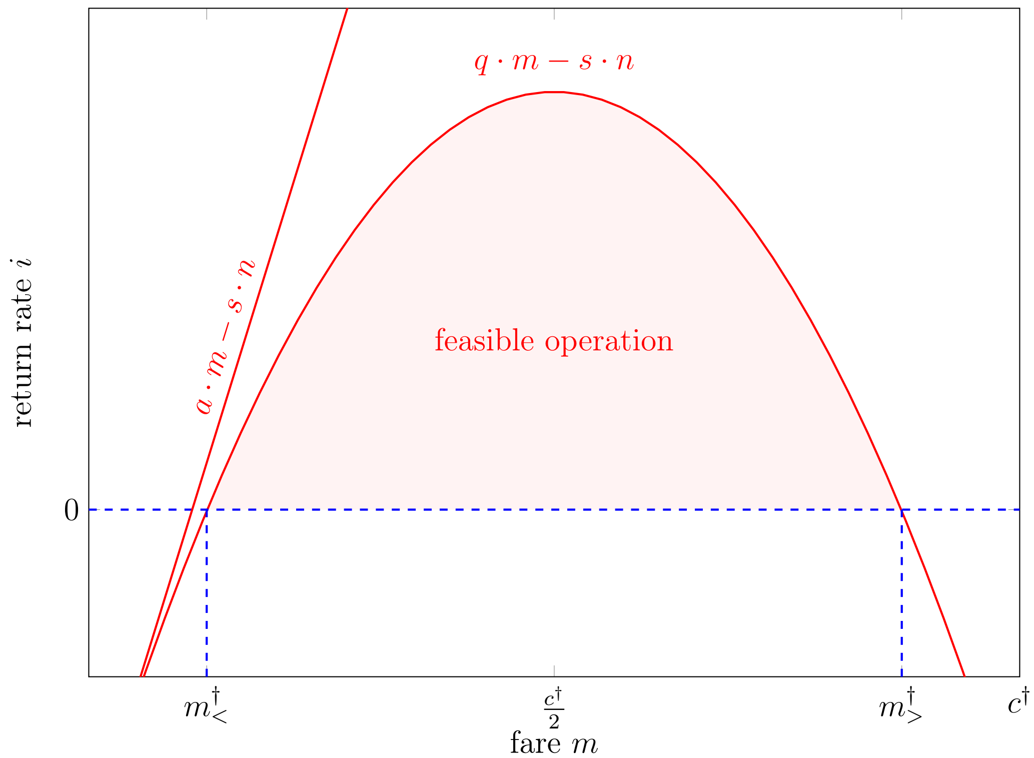

Figure 3: Illustration of the return rate dependence on the fare m for a passenger flow $q(m)$ (parabola) and potential revenue at fleet capacity (line). Note that the return rate (for a sufficient but fixed capacity n catering the passenger flow, $a(n) \geq q$) peaks at $m = \frac{c^\dagger}{2}$. It drops to the the full operation loss for free rides ($i(0) = -s \cdot n$) and prices above client cost function ($i(m > c^\dagger) = -s \cdot n$).

The analysis of the break-even points is meaningful for adaptive pricing. It shows the boundaries of feasible operations and as such gives predictable space for special fare discount or benefits for the drivers.

5.2 Maximal return rate

The last aspect of the model is maximizing the return rate $i(m,n)$. In the following sections, we assume a situation with no planned loss, as demanded by Equation 7, i.e. fully exploited car covers its operation costs. In the following, we will also assume a fixed susceptible population flow $p^\dagger_A$, which means a fixed cost function $c^\dagger$ (Equation 1).

We formally analyze the return rate maximum ($i^\star$) without capacity constrains, we consider parameter n chosen (and fixed) in such way, that $q(m) \leq a(n), \forall m$, that is, the capacity of the fleet of n cars can serve all the potential customers. This is represented in Figure 3 by the parabola being smaller than the line for all relevant m. Now, we want to maximize the collected fare i^+ (operation costs $i^-(n)$ are given and fixed by the choice of n) by selecting the maximizing $m^{\star 2}$. In absence of a capacity constraint, $m^\star$ maximizes the return rates for a given $p^\dagger_A$ and n.

$$
\begin{aligned}
m \overbrace{\left[p^\dagger_A - \frac{m}{c_c d} + \frac{c_s}{c_c d}(d+r)\right]}^{=q\leq a} &= i^+ \\
\frac{\partial i^+}{\partial m} = \left[p^\dagger_A + \frac{c_s}{c_c d}(d+r)\right] - 2\frac{m}{c_c d} &= 0 \\
m^\star = \frac{1}{2}\left[p^\dagger_A c_c d + c_s(d+r)\right] &= \frac{c^\dagger}{2} \quad |\text{as shown in Figure 3}
\end{aligned} \tag{9}
$$

In a case more general than presented in Figure 3, the total revenue rate corresponds to the transported passenger flow $q(m)$ under the fleet capacity constraint $a(n)$, and is given by $i^+ = \min(q(m), a(n)) \cdot m$ (see Equation 6 and illustrated in Figure 4). The operational cost $i^- = s \cdot n$ is constant for a given fleet size n_i, whether we fully use the capacity $a(n_i)$ or not. We maximize the revenue rate i, if we set the relationship between m and n to fully exploit the available capacity under the given fleet size n_i (illustrated in Figure 4), that is, under the constraint $a(n_i) = q(m^\star_i)$, the relationship yields

$$
\begin{aligned}
q(m^\star_i) &= a(n_i) \\
\underbrace{\left[p^\dagger_A - \frac{m^\star_i}{c_c d} + \frac{c_s}{c_c d}(d+r)\right]}_{\text{total transported flow}} &= \underbrace{\frac{c_M \cdot n_i}{\tau}}_{\text{fleet capacity}} \\
m^\star_i &= c^\dagger - \frac{c_M c_c d}{\tau} n_i \quad |\text{analogically to subsection 5.1}
\end{aligned} \tag{10}
$$

Note that Equation 10 is met only by a limited set of tuples $(n_i, m^\star_i)$, where $n_i \in \mathbb{N}$ (because we only have integer cars). In other words, for each fleet size n_i, there is exactly one pricing model $m^\star_i$ that yields the maximum return rate $i^\star_i$. Say we have 15 cars at hand, then there exist 15 distinct corresponding prices to optimize the revenue rate for each deployed fleet size (given by Equation 10).

Looking on the Equation 9 and Equation 10 and looking on the Figure 4, we note that the value of the optimal fare peaks at $m^\star = \frac{c^\dagger}{2}$, growing capacity beyond this point does not improve the revenue rate i^+ (but increases the costs i^-). We can write the optimal $m^\star$ in two equivalent forms.

$$
\begin{aligned}
m^\star_i &= \underset{m\in(\frac{c^\dagger}{2}, c^\dagger); n=n_i}{\arg\max} \{\min(a(n), q(m))\} \\
m^\star_i &= \max\left(\frac{c^\dagger}{2}, c^\dagger - \frac{c_M c_c d}{\tau} n_i\right)
\end{aligned} \tag{11}
$$

Under given circumstances of a market zone A (i.e. $p^\dagger_A, r, d, c_c$ etc.), we can select both $m^\star$ and $n^\star$ optimally, i.e. we can calculate the optimal fleet size $n^\star$ and the corresponding optimal pricing $m^\star$ for the given targeted zone A to guarantee maximal income rate i. We can take two approaches to solve this problem. First, we can make a real number estimate of the globally optimal tuple $(n^\star, m^\star) \in \mathbb{R}$ using optimization method with Lagrange multipliers and then find the nearest optimal integer solution $n^\star \in \mathbb{N}$. Second, we can look into the discrete differences of revenue rate as the fleet size is incremented. We will present both approaches in the following two sections.

[2] We use the star symbol $\star$ to denote the parameter value, which was chosen and fixed to optimize for maximum return rate.

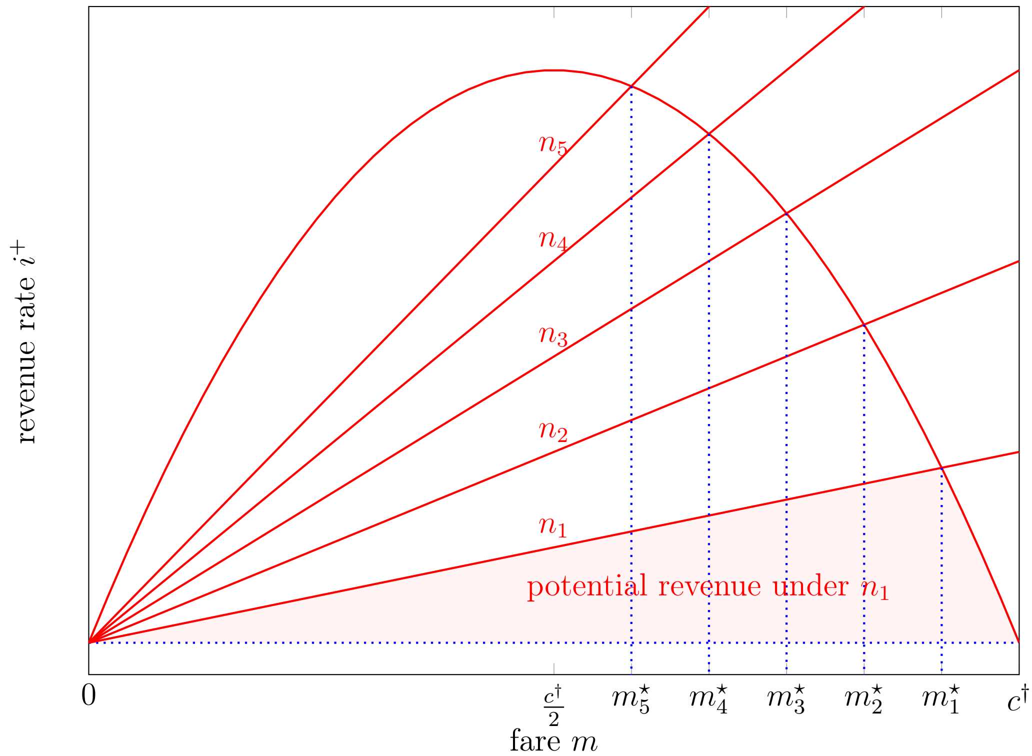

Figure 4: The parabola represents the collected revenue rate curve for a given fare without any constraint of deployed fleet capacity. The lines labeled n_i represent the revenue rates upper limits given by the fleet capacity $a(n_i)$. Points labeled $m_i^\star$ label the optimal fare for the given fleet size n_i, i.e. $m_i^\star = \underset{\substack{m \in (\frac{c^\dagger}{2}, c^\dagger) \\ n = n_i}}{\arg\max} \{\min(a(n_i), q(m))\}$. Note that the $m_i^\star$ are equidistant.

Lagrange multiplier optimization

We will find the optimal choice of pricing $m^\star$ and fleet size $n^\star$ for the revenue rate function (see Equation 6) in the following non-throttled form (represented by the mesh in Figure 5)

$$i(m,n) = -\frac{m^2}{c_c d} + \frac{c^\dagger}{c_c d} m - s \cdot n \tag{12}$$

under the the fleet capacity constraint with optimal pricing (see $m - n$ relation in Equation 11, represented by the red constraining solid line in Figure 5)

$$g(m,n) = m - \max\left(\frac{c^\dagger}{2}, c^\dagger - \frac{c_M c_c d}{\tau} n\right) = 0 \tag{13}$$

For a given susceptible population flow $p_A^\dagger$, there is a (real approximation of the) threshold fleet size $n_T \in \mathbb{R}$ (see Figure 5). After such threshold, adding more cars $n_k > n_T$ does not increase the maximal return rate, $\max(i(n_k)) < \max(i(n_T))$, and the pricing $m^\star = \frac{c^\dagger}{2}$ maximizes the revenue rate, $\underset{m}{\arg\max}\, i(m, n_k) = \frac{c^\dagger}{2}$. The threshold is given as

$$\begin{aligned} \frac{c^\dagger}{2} &= c^\dagger - \frac{c_M c_c d}{\tau} n_T \\ n_T &= \frac{c^\dagger}{2} \frac{\tau}{c_M c_c d} \end{aligned} \tag{14}$$

$n < n_T$	$n \geq n_T$
$m = c^\dagger - \frac{c_M c_c d}{\tau} n$	$m = \frac{c^\dagger}{2}$
$g(m,n) = m - c^\dagger + \frac{c_M c_c d}{\tau} n = 0$	$g(m,n) = m - \frac{c^\dagger}{2} = 0$

Table 2: Two domains for optimization, depending on fleet size threshold n_T.

We split our calculation into two domains, for $n < n_T$ and $n \geq n_T$ as sketched in Table 2.

For each of the two domains in Table 2, we solve the system

$$\begin{aligned} \nabla_{m,n}\left[i(m,n) - \lambda g(m,n)\right] &= 0 \\ g(m,n) &= 0 \end{aligned}$$

applying the technique of Lagrange multipliers, where λ is a Lagrange multiplier.

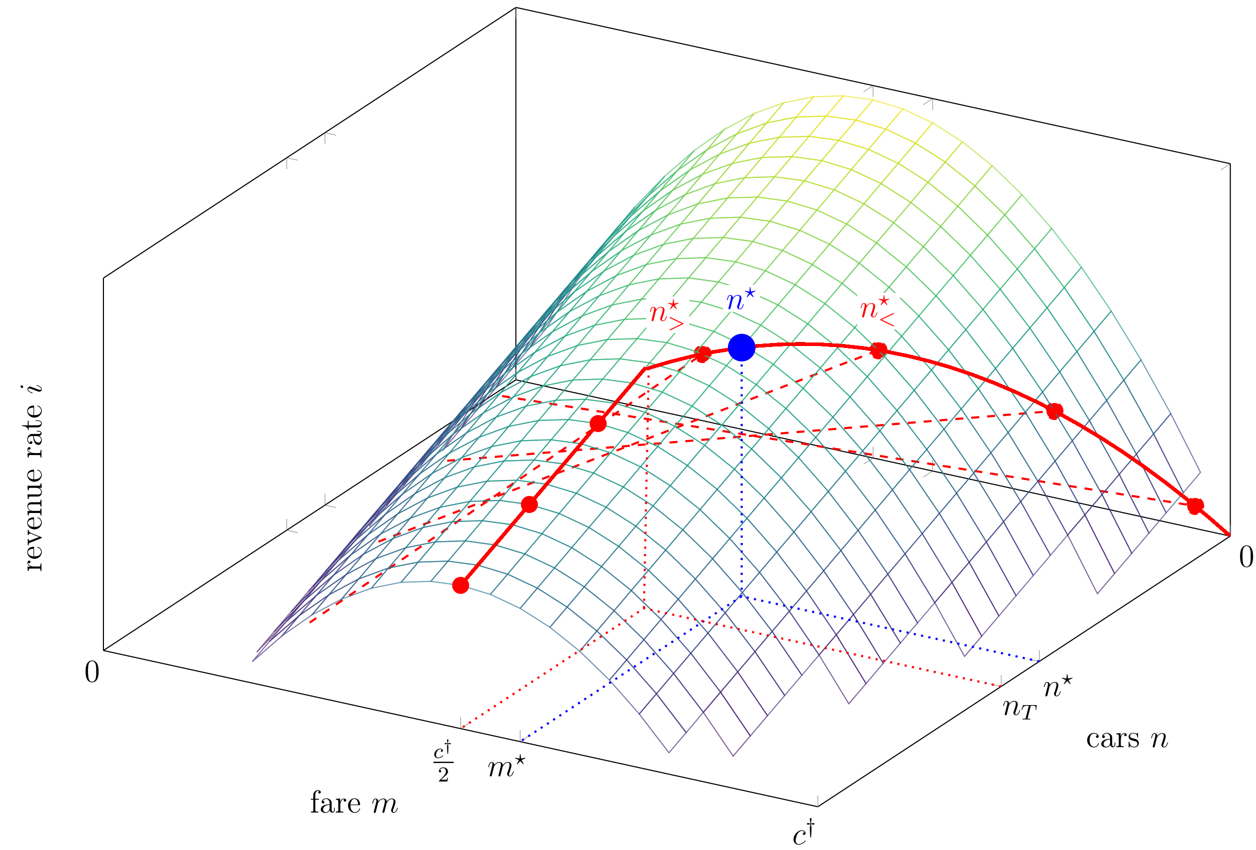

Figure 5: Illustration of the Lagrange multiplier solution. The mesh illustrates the revenue rate i for the given car pool n and pricing m. The red thick curve is the constraint of the optimal pricing given car fleet size—note that after the point n_T, there is no reason to increase the fleet size. The highest revenue rate $i^\star$ is marked by the blue dot. The nearest integer values are annotated as $n^\star_<$ and $n^\star_>$. The straight red dashed lines represent the revenue rate for a fully exploited fleet of a given size, analogical to Figure 4.

We have the following solution

$n < n_T$	$n \geq n_T$
$\lambda = -s\frac{\tau}{c_M c_c d}$	λ is undefined
$m^\star = \frac{c^\dagger}{2} + \frac{s\tau}{2c_M}$	$m^\star = \frac{c^\dagger}{2}$
$n^\star = \left[\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right]\frac{\tau}{c_M c_c d}$	$n^\star = n_T = \frac{c^\dagger}{2}\frac{\tau}{c_M c_c d}$
$i^\star\vert_{n<n_T} = \frac{1}{c_c d}\left[\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right]^2$	$i^\star\vert_{n>n_T} = \frac{1}{c_c d}\left[\frac{c^\dagger}{2}\right]\left[\frac{c^\dagger}{2} - \frac{s\tau}{c_M}\right]$

Table 3: The optimization shows that there is a local maximum for i if $n < n_T$, and that a local extrema does not exist for $n \geq n_T$, and so it is maximal at the point $\arg\max_{(m,n)} i(m,n) = (\frac{c^\dagger}{2}, n_T)$. If we compare the return rates, we see that $i^\star|_{n<n_T} > i^\star|_{n>n_T}$.

where we note, that $n^\star \in \mathbb{R}$. And that $i^\star|_{n<n_T} > i^\star|_{n>n_T}$, specifically

$$i^\star|_{n<n_T} = i^\star|_{n>n_T} + \left[\frac{s\tau}{2c_M}\right]^2.$$

We will therefore consider

$$n^\star = \left[\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right]\frac{\tau}{c_M c_c d} < n_T$$

to be the correct global real estimate of the optimal integer size of the fleet $n^\star_i$.

To correctly account for the discrete feature of $n^\star$, we define two integers

$$n^\star_{<,>} = \left\{ \left\lfloor \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} \right\rfloor, \left\lceil \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} \right\rceil \right\} \tag{15}$$

where we used *floor* $\lfloor . \rfloor$ and *ceil* $\lceil . \rceil$ functions respectively. To each of those correspond $m^\star_<$ and $m^\star_>$ according to Equation 11. The optimal of the two is given by

$$(m^\star, n^\star) = \underset{\substack{m\in\{m^\star_<, m^\star_>\} \\ n\in\{n^\star_<, n^\star_>\}}}{\arg\max} \{i(m,n)\} \tag{16}$$

Finite differences

Less formal and more direct way to determine the optimal tuple $(m^\star, n^\star)$ is looking into the changing discrete values of revenue rate $i_k(m^\star_k, n_k)$. In this approach, we assume that there is one local maximum which is also the global maximum $i^\star$, and that the series of i_k monotonically increases with k (and so n_k) towards the maximum and then decreases. In such situation, we can iterate over the tuples $(m^\star_k, n_k)$, increasing $n_{k+1} = n_k + 1$, until the point where the revenue keeps increasing, $\Delta i_k = i_k - i_{k-1} > 0$. The last (highest) index n_k labels the $n^\star = n_k$. The difference form of the Equation 6 and Equation 11 is

$$m^\star_k = \max\left(\frac{c^\dagger}{2}, c^\dagger - \frac{c_M c_c d}{\tau} n_k\right) \tag{17}$$

$$i_k = -\frac{m_k^2}{c_c d} + \frac{c^\dagger}{c_c d} m_k - s n_k \tag{18}$$

with the threshold as in Equation 14

$$n_k < \frac{c^\dagger}{2} \frac{\tau}{c_M c_c d}$$

Then, we combine the equations and look for the optimal $n^\star$. The difference in revenue from n_{k-1} to n_k is given as

$$\begin{aligned}
\Delta i_k &= -\frac{1}{c_c d}\left[m_k^2 - m_{k-1}^2 - c^\dagger m_k + c^\dagger m_{k-1}\right] - s \\
&= -\frac{1}{c_c d}(m_k^2 - m_{k-1}^2) - \frac{c^\dagger c_M}{\tau} - s \\
&= -c_c d \left(\frac{c_M}{\tau}\right)^2 (2n_k - 1) + \frac{c^\dagger c_M}{\tau} - s
\end{aligned}$$

And now, we search for the k, where $i^\star(n_k)$ reaches maximum and begins to decline, i.e. the last increase, the highest n_k with $\Delta i_k > 0$.

$$\begin{aligned}
0 &< -c_c d \left(\frac{c_M}{\tau}\right)^2 (2n_k - 1) + \frac{c^\dagger c_M}{\tau} - s \\
2n_k - 1 &< \left[\frac{c^\dagger c_M}{\tau} - s\right] \left[\frac{\tau}{c_M}\right]^2 \frac{1}{c_c d} \\
n_k &< \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} + \frac{1}{2}
\end{aligned} \tag{19}$$

The threshold inequality Equation 14 is automatically satisfied for if

$$\begin{aligned}
n_k &< \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} + \frac{1}{2} \leq \frac{c^\dagger}{2} \frac{\tau}{c_M c_c d} \\
\frac{c_M c_c d}{\tau} &\leq \frac{s\tau}{c_M}
\end{aligned} \tag{20}$$

The condition Equation 20 (which is a stronger condition than Equation 14) is independent on n_k. So if the condition Equation 20 holds, it is a sufficient condition which guarantees, that the largest integer n_k meeting the requirement Equation 5.2 is really the global maximum $n^\star$, i.e.

$$\begin{aligned}
n^\star &= \left\lfloor \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} + \frac{1}{2} \right\rfloor \\
n^\star &= \left\lfloor \left(\frac{c^\dagger}{2} - \frac{s\tau}{2c_M}\right) \frac{\tau}{c_M c_c d} \right\rceil
\end{aligned} \tag{21}$$

which is basically equivalent to the Equation 15. The notation $\lfloor . \rceil$ of Equation 21 means selecting the nearest integer. $m^\star$ is easily computed from $n^\star$ using Equation 17. We will denote such coordinates of maximum return rate $i^\star(m^\star, n^\star)$, meeting the condition of Equation 20, as a tuple $(m^\star_{\text{loc}}, n^\star_{\text{loc}})$.

If Equation 20 is not met, there is a possibility, under a certain configuration of parameters, that $n^\star \geq n_T$. Then, the maximum $i^\star$ would be realized for the first integer higher than n_T (see Figure 5),

$$n^\star = \lceil n_T \rceil = \left\lceil \frac{c^\dagger}{2} \frac{\tau}{c_M c_c d} \right\rceil. \tag{22}$$

We will denote such maximum coordinates $n^\star_T$ and $m^\star_T$, related by Equation 17.

In general, if Equation 20 is not met, the optimal car fleet is given as

$$(m^\star, n^\star) = \underset{\substack{n \in \{n^\star_T, n^\star_{\text{loc}}\} \\ m \in \{m^\star_T, m^\star_{\text{loc}}\}}}{\arg\max} \{i(m, n)\} \tag{23}$$

similarly to the Equation 16. So, Equation 20 gives us a sufficient condition for $(m^\star, n^\star)$ to be given by Equation 21 and Equation 17. If this condition doesn't hold, we must verify, using the Equation 23.

6 Conclusion

Only the very basic model was presented, which describes one particular feeding corridor during peak time window, when most of the descriptors can be approximated as time-independent. The peak time is also the time of most intensive movement, and so the most interesting part of the day. The model is deterministic and simplistic. Simple models give us initial analytical insight into more complex versions of those problems.

We have shown that the transport system described by a short list of quasi-static parameters (see Table 1) and their simple inter-relation can serve as a basic model of a feeding corridor, considering its demographic factors (both intensive preferences of the agents like c_s, c_c, and extensive intrinsic area values like p_A, d), geometric factors (τ, r), business parameters (n, s, m, c_M). These results allow to compare the performance over various locations in a pre-election process. Most of the parameters from Table 1 can be realistically estimated using freely available data. The intensive parameters, particularly c_c are more difficult to grasp.

Analysis of the model allowed us to estimate the optimal values and ranges for business parameters, assuming some of the parameters are fixed, i.e. $p^\dagger_A$, $c^\dagger$. Thus getting estimates for $m^\dagger_{<,>}$ and optimal fare $m^\star$, and the necessary corresponding size of the fleet $n^\star$. While at the same time, it gave us more general feasibility limits on fares, driver pay, fleet size, etc.

As we can see from the subsection 5.2, the parameters of the optimal return rate $i^\star$, $(m^\star, n^\star)$, depend on an interplay of many factors. Our model predicts a single global optimal tuple for a given set of system parameters. Such optimum can be easily located using explicit equations from Table 3 or using nearly identical Equation 21. However, in some situations of particular system parameters (as mandated by Equation 20) or accounting for the discrete character of n (as treated in Equation 15), we may need to double check the optimum comparing returns at integer values, or at values close to the fleet size threshold point n_T (given by Equation 14), using Equation 16 and Equation 23.