A Resource Bank for Writing Intensive Mathematics Courses

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Contents

Chapter 1

How do students learn to write proofs?

1.1 Introduction

At the University of Calgary, MATH 271 Discrete Mathematics, MATH 273 Numbers and Proofs, and MATH 311 Linear Methods II often serve as students' first introduction to writing mathematical proofs. The resources in this document are largely based on the topics covered in these classes. These three proof-based courses are important not only in terms of content but also in helping students develop their mathematical communication skills. A first course in writing precise mathematical proofs is a major milestone in students' mathematical careers and is often the time at which students have the opportunity to learn what *doing mathematics* is really like.

Learning to write rigorous, clear and concise logical arguments, in the midst of learning mathematical content, is challenging. In our experience teaching these courses, students often note that they "don't know where to start" writing a proof. Students also often make several typical mistakes of the novice proof writer (e.g., assuming that what you want to prove is true, improper use of the 'equals' symbol, confusion about how to prove that a statement is false, etc.) often despite explicit warnings about these mistakes.

Recent research has shown that an effective way for students to improve their mathematical writing is to read, analyze, and correct proofs that contain (serious) errors [1]. The approach to supporting students that we aim to facilitate with this resource is to provide students with an artificial sample proof that contains various mistakes and to have students identify the error and correct the proof. By asking students to grapple with proofs that they know are flawed, we intend to *show* our students what not to do (rather than tell them). This approach has the added benefit of encouraging critical, engaged reading of mathematical texts.

Combining error-correction activities, together with scaffolding, peer-assessment, and other techniques can provide a robust learning experience for students that goes beyond asking students to "prove it!" We hope that adding these resources, and proof-correction activities to your courses help your students too!

1.2 How to use this document

The resources in Part I of this document consist of

1. sample flawed proofs,
2. classifications and brief explanations of the major errors in each flawed proof, and
3. 'corrected'[1] versions of the proofs.

These resources are largely intended as prompts for discussion activities. We recommend that learners first engage with the flawed proof (with or without access to the corrected proof) and attempt to identify and correct the errors. Learners should then discuss their findings with peers, compare how they would correct the proof, and perhaps discuss how they would evaluate or assess the proof with a rubric. Of course, learners could also use this resource individually to practice reading, understanding, and correcting flawed mathematical writing.

In Part II of this document, we've included sample activities and assessments (Chapter 7) and sample rubrics (Chapter 8) for self- and peer-evaluation activities.

[1] Your version of a 'correct' proof may differ from ours, and this in part motivated our decision to create an open resource.

The error classifications (Section 1.3) classify the type and severity of the error, but do not include explanations of *why* the identified items are indeed errors. This detailed coding scheme is largely intended for use by instructors and teaching assistants. However, you may find it useful to introduce some of this language to your students to help them articulate the types of errors they are seeing (e.g. assertion, omission, false statement).

Of course, as this document is an open resource the error classifications and corrected proofs are available to students online. So, while the resources here will not be suitable for take-home or online assessments, we believe that the resources will be valuable in promoting student learning via independent study and discussion based activities.

1.3 Overview of error classification

For each of the Flawed Proofs in Part I, we include a classification of the errors contained therein using Strickland and Rand's Proof-Error Coding Scheme:

Figure 1.1: Strickland and Rand's Proof-Error Coding Scheme [2, Figure 1]

Error Type		Fundamental	Content	Rhetorical	Ambiguous
Basic Problems - Acategorical	Wrong Problem				
	Wrong Method				
	Scratch Work				
	Other				
Omissions	Error-caused Omission	An error caused large parts of the proof to be omitted. Must be used in conjunction with another code specifying the triggering error. See (1) below.			
	Assertions	Entire result is asserted	Local assertion. See (2) below.	Context requires more details. See (3) below.	
	Omitted Sections	Did not address one or more sections of the proof. (Trailed off halfway, skipped cases, etc.)			
	Local Omission	Omitted a single isolated step			
Importing known material	Misusing Theorem	Ex - applying the converse of a theorem	Misunderstanding known results or theorems. See (4) below.	Misphrasing a known theorem, while using it correctly	
	Vocabulary & Grammar	Rare or nonexistent. See (5) below.	Misuse of definition or vocabulary. See (6) below.	Prose is poorly written or missing, or student fails to follow mathematical conventions	
Flow of proof	Logical Order	Didn't proceed through the proof in a linear fashion, or ideas were not in logical order			
	Circular Argument	Uses conclusion in the course of the proof			
	False Implication	P does not imply Q. See (7) below.			
	Extraneous Detail	Attempted or proved irrelevant results. See (8) below.			
Miscellaneous	Locally Unintelligible	A single statement within the proof is completely unintelligible			
	Proof by Example	Usually this error is Fundamental.	Ex - Student confuses a universal proof for an existential proof.	Ex - Student fails to include "Without loss of generality..."	
	Notation	Notation is awkward/confusing/misleading. See (9) below.			
	Weakened Result	Proving a weaker result (eg - a result that only applies to a subset of the scope of original statement)			
	False Statement	Made a false statement or incorrect computation.		Apparent typo	

The rows of the coding scheme represent the type of error and the columns represent the severity of the error. Please refer to the Appendix of [2, Figure 1] for a more detailed description of these error classifications.

1.3.1 Abbreviated Error Codes

To streamline the use of the error classification we will use the following system for abbreviating the error classifications:

Figure 1.2: 'Severity of error' codes

F Fundamental

C Content

N Novice [2]

A Ambiguous

Figure 1.3: 'Type of error' codes

WP Wrong Problem

WM Wrong Method

SW Scratch Work

EO Error-caused Omission [3]

A Assertions

OS Omitted Sections

O Omission / Local Omission

MT Misusing Theorem

VG Vocabulary & Grammar

Log Logical Order

Cir Circular Argument

FI False Implication

ED Extraneous Detail

LU Locally Unintelligible

Eg Proof by Example

N Notation

WR Weakened Result

FS False Statement

[2] In this document, we use "Novice" in place of Strickland and Rand's descriptor "Rhetorical".

[3] Note that the Error-caused Omission code must always be used in conjunction with a second error type (and severity) code for the original error that caused the subsequent omission. We will use a hyphen/parentheses to indicate such an error. For example, if an omission occurred as a result of misunderstanding a known theorem. this would be an Error-caused Omission resulting from a Content Misusing Theorem error, and it would be abbreviated as EO-(C-MT).

For example, using these abbreviations, a Content Misusing Theorem error will be abbreviated by C-MT and a Fundamental Logical Order error will be abbreviated by F-Log. Using Strickland and Rand's scheme, the errors Wrong Problem, Wrong Method, and Scratch Work do not have severity ratings.

Part I

Flawed Proofs

Chapter 2

Elementary Topics

The topics covered in this chapter fit into the first halves of MATH 271 Discrete Mathematics and MATH 273 Numbers and Proofs at the University of Calgary, but are also common to many courses serving as an introduction to mathematical proof. Topics include

- divisibility, the gcd and modular arithmetic
- definition of even and odd integers, and
- basic properties of rational and real numbers.

In these exercises and proofs, learners are confronted with familiar notions from school mathematics in the unfamiliar setting of mathematical rigour.

Given its importance as proof technique, and its difficulty for the novice learner, Mathematical Induction, is covered in its own dedicated chapter (see Chapter 3).

2.1 Logic

Exercise 2.1.1: Converse of a Statement

Let n be an integer. Prove or disprove the converse and contrapositive of the following statement:

If n is a positive integer, then $n + 3$ is a positive integer.

Flawed Proof 2.1.1

The converse of the statement is "if $n+3$ is a positive integer, then n is a positive integer". This statement is true. If $n + 3$ is positive we can say $n + 3 > 0$, which means $n > 0$, so n is positive.

The contrapositive of the statement is "if $n + 3$ is not a positive integer, then n is a positive integer". This statement is false. We will provide a counterexample to show this. If $n = -4$, then $n + 3 = -1$ is not a positive integer, but $n = -4$ is also not a positive integer. $\square$

2.1.1 Error classification

There are several errors in the Flawed Proof 2.1.1.

F-FI The implication that $n + 3 > 0$ means $n > 0$ is false.

WP "If $n + 3$ is not a positive integer, then n is a positive integer" is not the contrapositive of the statement

Error codes

- Fundamental False Implication (F-FI)
- Wrong Problem (WP)

See Section 1.3 for more information about error classifications.

2.1.2 Corrected proof

The following is a corrected version of Flawed Proof 2.1.1.

Proof 2.1.1

Let $n \in \mathbb{Z}$.

The converse of the statement is "if $n + 3$ is a positive integer, then n is a positive integer". This statement is false. We will provide a counterexample. Let $n = -1$. We have that $n+3 = 2$ is positive, but $n = -1$ is negative.

The contrapositive of the statement is "if $n + 3$ is not a positive integer, then n is not a positive integer". This statement is true. Here is a proof:

Suppose that $n + 3$ is not a positive integer. Then $n + 3 \leq 0$. This means $n + 3 - 3 \leq 0 - 3$, so $n \leq -3 \leq 0$. Therefore n is not a positive integer. $\square$

2.2 Divisibility of a Product

Exercise 2.2.1: Divisibility of a Product

Let a, b and c be integers. Prove that if a divides b, then a divides bc.

Flawed Proof 2.2.1

Suppose a is even. Since a divides b this means b is even. Therefore bc is even, so that means a divides bc. $\square$

2.2.1 Error classification

There are multiple errors in the Flawed Proof 2.2.1.

WM There is no need to suppose the case that a is even.

EO-(WM) The flawed proof has not considered the case where a is odd.

N-A The assertion "since a divides b this means b is even" requires more details.

F-FI The implication "bc is even, so that means a divides bc" is false.

Error codes

- Wrong Method (WM)
- Error Caused Ommision Wrong Method (EO-(WM))
- Novice Assertion (N-A)
- Fundamental False Implication (F-FI).

See Section 1.3 for more information about error classifications.

2.2.2 Corrected proof

The following is a corrected version of Flawed Proof 2.2.1.

Proof 2.2.1

Suppose that $a, b \in \mathbb{Z}$ and suppose that a divides b. Since a divides b, there exists some integer k such that $ak = b$. Multiplying both sides by c gives us that $akc = bc$. We will let $kc = l$, and note that l is an integer. Thus $al = bc$, so by definition a divides bc.

$\square$

2.3 Contrapositive and Converse

Exercise 2.3.1

Let n be an integer. Write the contrapositive and the converse of the true statement

$$\text{“If } n > 2, \text{ then } n^2 > 4.\text{”}$$

Is the contrapositive true? Is the converse true? Explain why or why not.

Flawed Proof 2.3.1

1. Contrapositive: If $n^2 < 4$, then $n < 2$.
2. Converse: If $n^2 > 4$, then $n > 2$.

As the original statement is true, and the contrapositive is logically equivalent to the original statement, the contrapositive is true. Despite the fact that the converse is not logically equivalent to the original statement, in this case, the converse is true. Taking the square root of both sides of $n^2 > 4$

$$\sqrt{n^2} > \sqrt{4},$$

we see that this statement implies that $n > 2$. $\square$

2.3.1 Error classification

There are several errors in the Flawed Proof 2.3.1.

Error codes

C-O: Failed to recognize that 'not greater than' is equivalent to 'less than or equal to', which lead to the omission of the 'or equal to' from the statement of the contrapositive.

C-FS-EO: The statement that $\sqrt{n^2} > \sqrt{4}$ implies $n > 2$ is false. This error caused the omission of the case where n is negative, which is the key element of the proof.

Error codes

- Content Local Omission (C-O)
- Content False Statement Error Caused Omission (C-FS-EO)

See Section 1.3 for more information about error classifications.

2.3.2 Corrected proof

The following is a corrected version of Flawed Proof 2.3.1.

Proof 2.3.1

Let n be an integer.

1. Contrapositive: If $n^2 \leq 4$, then $n \leq 2$.

2. Converse: If $n^2 > 4$, then $n > 2$.

As the original statement is true, and the contrapositive is logically equivalent to the original statement, the contrapositive is true.

However, the converse is not logically equivalent to the original statement and in this case, the converse is false.

Indeed, the negation of the converse is: "$n^2 > 4$ but $n \leq 2$." Here is a proof: Consider $n = -5$. Then $n^2 = 25$, which is greater than 4, but -5 is not greater than 2. □

2.4 Properties of Integers

Exercise 2.4.1

Prove that the negative of any even integer is even.

Flawed Proof 2.4.1

Suppose that n is any even integer. By the definition of even, $n = 2k$ for some integer k. The negative of n is equal to $-n$. Thus, $-n = -2k$. By dividing both sides of the equation by -1, we get back the original equation of $n = 2k$. Therefore, the negative of any integer n is an even number. $\square$

2.4.1 Error classification

There are several errors in the Flawed Proof 2.4.1.

N-N: Awkward phrasing. Instead, write how the equation $-n = -2k$ was obtained.

C-FI: The equation $n = 2k$ does not inherently imply that $-n$ is even.

F-A: The entire result is asserted by stating that $n = 2k$ implies that $-n$ is even.

Error codes

- Novice Notation (N-N)
- Content False Implication (C-FI)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

2.4.2 Corrected proof

The following is a corrected version of Flawed Proof 2.4.1.

Proof 2.4.1

Suppose that n is an even integer. Then $n = 2k$ for some integer k. Multiplying both sides by $-1 \in \mathbb{Z}$, we obtain the equation $-n = -2k$. Moreover, we have

$$-n = -2k = 2(-k),$$

where $-k$ is an integer. Thus, $-n$ is also even. $\square$

2.5 The Rationals

Exercise 2.5.1

Prove or disprove the following statement. For all integers m and n, if $m, n > 1$, then $\frac{1}{m} + \frac{1}{n}$ is not an integer.

Flawed Proof 2.5.1

Suppose that m and n are integers and that $m, n > 1$. Since $\frac{1}{m}$ is the reciprocal of m and it is not an integer and $\frac{1}{n}$ is the reciprocal of n and it is not an integer, then there are no two distinct numbers such that $\frac{1}{m} + \frac{1}{n}$ can ever be an integer. $\square$

2.5.1 Error classification

There are several errors in the Flawed Proof 2.5.1.

Error codes

C-FI: The statement that $\frac{1}{m}$ and $\frac{1}{n}$ are not integers implies that $\frac{1}{m} + \frac{1}{n}$ is not an integer is false.

F-A: Asserted the entire result.

Error codes

- Content False Implication (C-FI)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

2.5.2 Corrected proof

The following is a corrected version of Flawed Proof 2.5.1.

Proof 2.5.1

This statement is false. Its negation is: there exist integers m and n so that $m > 1$, $n > 1$ and $\frac{1}{m} + \frac{1}{n}$ is an integer. Here's a proof:

Let $m, n = 2$. Then $m, n > 1$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{2} = 1 \in \mathbb{Z}$. $\square$

2.6 Divisibility

Exercise 2.6.1

Prove that for any integers a, b and c, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

Flawed Proof 2.6.1

We will consider two cases. First, let a be odd. Then we can randomly choose the values $a = 5$, $b = 30$ and $c = 10$ to show that $a \mid (b + c)$. We know that

$$\frac{30}{5} = 6 \quad \text{and} \quad \frac{10}{5} = 2.$$

Additionally, we have that $b + c = 30 + 10 = 40$. This implies

$$\frac{40}{5} = 8,$$

which means that $a \mid (b + c)$ when a is odd.

Now, let a be even. Then we can randomly choose the values $a = 2$, $b = 12$ and $c = 10$ to show that $a \mid (b + c)$. We know that

$$\frac{12}{2} = 6 \quad \text{and} \quad \frac{10}{2} = 5.$$

Additionally, we have that $b + c = 12 + 10 = 22$. This implies

$$\frac{22}{2} = 11,$$

which means that $a \mid (b + c)$ when a is even. □

2.6.1 Error classification

There are several errors in the Flawed Proof 2.6.1.

EO-F-Eg: Here, proof by example leads to the omission of nearly all cases, except the ones given in the example. Additionally, there is a fundamental error in the failure to recognize that 'randomly' choosing a specific example is not sufficient evidence to prove a statement with a universal quantifier.

F-WM: Dividing the proof into two cases is not the correct approach. Moreover, the definition of 'divides' is not explicitly used in the proof.

Error codes

- Fundamental Error-caused Omission – Fundamental – due to Proof by Example (EO-F-Eg)
- Fundamental Wrong Method (F-WM)

See Section 1.3 for more information about error classifications.

2.6.2 Corrected proof

The following is a corrected version of Flawed Proof 2.6.1.

Proof 2.6.1

Suppose that a, b and c are integers and that $a \mid b$ and $a \mid c$. By the definition of divisibility, we have that $b = ja$ and $c = ka$ for some integers j and k. Then

$$b + c = ja + ka = a(j + k),$$

where $j + k$ is also an integer, since it is the sum of two integers. Thus, by the definition of divisibility, it follows that $a \mid (b + c)$. $\square$

2.7 Divisibility and Modular Arithmetic

Exercise 2.7.1: Basic Divisibility Property

Let a, b be integers. Prove that $b \equiv 0 \mod a$ if and only if a divides b.

Flawed Proof 2.7.1

If $b \equiv 0 \mod a$, then 0 divides $b - a$. So there exists some integer n such that $b + 0 = na$. So a divides b. $\square$

2.7.1 Error classification

There are multiple errors in the Flawed Proof 2.7.1.

C-VG Incorrect understanding of what $b \equiv 0 \mod a$ means.

C-VG Incorrect understanding of divisibility.

F-OS Did not address one direction of the "if and only if" proof.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Fundamental Omitted Sections (F-OS)

See Section 1.3 for more information about error classifications.

2.7.2 Corrected proof

The following is a corrected version of Flawed Proof 2.7.1.

Proof 2.7.1

Let a, b be integers. Firstly suppose $b \equiv 0 \mod a$. By definition, this means a divides $b - 0$. But since $b - 0 = 0$, we conclude a divides b.

Conversely suppose a divides b. Then since $b = b - 0$, we can say a divides $b - 0$. Thus by definition, $b \equiv 0 \mod a$. $\square$

2.8 Proof by Contrapositive

Exercise 2.8.1

Let $x \in \mathbb{Z}$. Prove that if $x^2 - 6x + 5$ is even, then x is odd.

Flawed Proof 2.8.1

Suppose that $x^2 - 6x + 5$ is even. We want to prove that x is odd. By the definition of even, $x^2 - 6x + 5 = 2k$ for some integer k. So we have $x^2 - 6x + 5 = 2k$ and, rearranging, $x^2 - 6x + (5 - 2k) = 0$. Then, using the quadratic formula,

$$\begin{aligned} x &= \frac{6 \pm \sqrt{6^2 - 4(1)(5-2k)}}{2} \\ &= \frac{6 \pm \sqrt{36 - 20 + 8k}}{2} \\ &= \frac{6 \pm \sqrt{16 + 8k}}{2}. \end{aligned}$$

Now we must show that $x = \frac{6 \pm \sqrt{16+8k}}{2}$ is odd. That is, we must show that $\frac{6 \pm \sqrt{16+8k}}{2} = 2j + 1$ for some integer j. So

$$\begin{aligned} \frac{6 \pm \sqrt{16 + 8k}}{2} &= \frac{1}{2}\left(6 \pm \sqrt{16 + 8k}\right) \\ &= \frac{1}{2}\left(6 \pm \sqrt{16}\sqrt{8k}\right) \\ &= \frac{1}{2}\left(6 \pm 4\sqrt{8k}\right) \end{aligned}$$

... not enough time to finish. □

2.8.1 Error classification

There are several errors in the Flawed Proof 2.8.1.

F-WM: A direct proof may not be available with elementary methods. An indirect proof (by contradiction, or proving the contrapositive) will be more effective .

F-FS: $\frac{1}{2}\left(6 \pm \sqrt{16+8k}\right) \neq \frac{1}{2}\left(6 \pm \sqrt{16}\sqrt{8k}\right)$.

Error codes

- Fundamental Wrong Method (F-WM)
- Fundamental False Statement (F-FS)

See Section 1.3 for more information about error classifications.

2.8.2 Corrected proof

The following is a corrected version of Flawed Proof 2.8.1.

Proof 2.8.1

We will prove the contrapositive of this statement. That is, we prove that if x is even, then $x^2 - 6x + 5$ is odd. To do so, suppose that x is even and write $x = 2k$ for some integer k. Then

$$\begin{aligned} x^2 - 6x + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 2(2k^2 - 6k + 2) + 1. \end{aligned}$$

Now, since $2k^2-6k+2 \in \mathbb{Z}$, it follows that x^2-6x+5 is odd. Thus, since the contrapositive is true and it is logically equivalent to the original statement, we can conclude that the original statement is true. $\square$

2.9 Modular Arithmetic and Divisibility

Exercise 2.9.1

Let p be an integer and suppose that p is not divisible by 3. Prove that $p^2 \equiv 1 \pmod{3}$.

Flawed Proof 2.9.1

Since $3 \nmid p$, this implies that we have two cases. Namely, either $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

Case 1: $p \equiv 1 \pmod{3}$. By definition, $p = 3k + 1$ for some integer k. Then

$$\begin{aligned} p^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3\left(3k^2 + 2k + \frac{1}{3}\right) \end{aligned}$$

I'm not sure where to go from here...

Case 2: $p \equiv 2 \pmod{3}$. By definition, $p = 3k + 2$ for some integer k. Then

$$\begin{aligned} p^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3\left(3k^2 + 4k + \frac{4}{3}\right) \end{aligned}$$

I'm not sure where to go from here... $\square$

2.9.1 Error classification

There is only one error in the Flawed Proof 2.9.1.

C-OS: Failed to recognize that the first few terms in the expressions $9k^2 + 6k + 1$ and $9k^2 + 12k + 4$ can be factored independently of the last term, leading to the omission of the final steps in both cases.

Error codes

- Content Omitted Sections (C-OS)

See Section 1.3 for more information about error classifications.

2.9.2 Corrected proof

The following is a corrected version of Flawed Proof 2.9.1.

Proof 2.9.1

Since $3 \nmid p$, this implies that we have two cases. Namely, either $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$.

Case 1: $p \equiv 1 \pmod 3$. By definition, $p = 3k + 1$ for some integer k. Then

$$\begin{aligned} p^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \\ &= 3l + 1 \end{aligned}$$

where $l \coloneqq 3k^2 + 2k$ is an integer. Thus, $p^2 \equiv 1 \pmod 3$.

Case 2: $p \equiv 2 \pmod 3$. By definition, $p = 3k + 2$ for some integer k. Then

$$\begin{aligned} p^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &= 3l + 1 \end{aligned}$$

where $l \coloneqq 3k^2 + 4k + 1$ is an integer. Thus, $p^2 \equiv 1 \pmod 3$.

$\square$

2.10 GCD

Exercise 2.10.1: Multiple of gcd

Let a and b be positive integers. For any positive integer n prove $\gcd(na, nb) = n \gcd(a, b)$.

Flawed Proof 2.10.1

Let $\gcd(a, b) = d$. Then we can write $a = dk$ and $b = dl$ for some integers k, l. Let $\gcd(na, nb) = \gcd(ndk, ndl) = n \gcd(dk, dl) = n \gcd(a, b)$.

$\square$

2.10.1 Error classification

There are several errors in the Flawed Proof 2.10.1.

N-VG Incorrect use of the word "let" in the final statement. Something is being asserted, so "therefore" or "thus" should be used.

F-A The entire result is asserted in the final statement without proof.

Error codes

- Novice Vocabulary Grammar (N-VG)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

2.10.2 Corrected proof

The following is a corrected version of Flawed Proof 2.10.1.

Proof 2.10.1

Let $d = \gcd(a, b)$ and $c = \gcd(na, nb)$. We wish to show $nd = c$. By the definition of the greatest common divisor, we know $d \mid a$ and $d \mid b$. In particular, $a = dk$ and $b = d\ell$ for some (positive) integers $k, \ell \in \mathbb{Z}$. Moreover, $na = ndk$ and $nb = nd\ell$, so it follows that $nd \mid na$ and $nd \mid nb$. Since $c = \gcd(na, nb)$, there exist integers x and y so that $c = xna+ynb$. Substituting $na = ndk$ and $nb = nd\ell$ we obtain

$$c = xndk + ynd\ell = nd(xk + y\ell),$$

where $xk + y\ell \in \mathbb{Z}$, thus nd divides c. So we can write $c = ndt$ for some positive integer $t \in \mathbb{Z}$. Since $c = \gcd(na, nb)$ divides both na and nb, we can now say $ndt \mid na$ and $ndt \mid nb$. That is, there are integers p and q so that $na = ndtp$ and $nb = ndtq$. Dividing by n, we obtain $a = dtp$ and $b = dtq$, which implies $dt \mid a$ and $dt \mid b$. But by the definition of $\gcd(a, b)$ this means $dt \leq d$, which means $t = 1$ (because d and t are positive). Therefore $nd = c$. $\square$

2.11 The Parity Property

Exercise 2.11.1

Prove that any two consecutive integers have opposite parity.

Flawed Proof 2.11.1

Let $n \in \mathbb{Z}$ and suppose that n is even. By definition. $n = 2k$ and so $n + 1 = 2k + 1$, which is odd. Thus, any two consecutive integers have opposite parity. $\square$

2.11.1 Error classification

There are several errors in the Flawed Proof 2.11.1.

Error codes

C-OS: The case where n is odd was assumed to be true and therefore omitted from the proof.

C-N: Failed to define what the variable k is.

Error codes

- Content Omitted Sections (F-FS)
- Content Notation (C-N)

See Section 1.3 for more information about error classifications.

2.11.2 Corrected proof

The following is a corrected version of Flawed Proof 2.11.1.

Proof 2.11.1

Let $n \in \mathbb{Z}$. We have two cases:

1. n **is even:** By definition, $n = 2k$ for some integer k. Then $n+1 = 2k+1$, which is odd.
2. n **is odd:** By definition, $n = 2k+1$ for some integer k. Then

$$n + 1 = (2k + 1) + 1 = 2k + 2 = 2(k + 1),$$

which is even since it equals twice some integer $k + 1$.

Thus, any two consecutive integers have opposite parity. $\square$

2.12 Infinitude of Primes

Exercise 2.12.1: Infinitude of Primes

Prove there exist infinitely many prime numbers.

Flawed Proof 2.12.1

Let $p_1, ..., p_k$ be the set of all prime numbers, where k is some positive integer. Now let $N = p_1...p_k + 1$. Now we see that N is not divisible by any of $p_1, ..., p_k$. But we know all integers greater than 1 can be written as a product of primes. Therefore N is prime. This contradicts letting $p_1, ..., p_k$ being the set of all prime numbers. $\square$

2.12.1 Error classification

There are multiple errors in the Flawed Proof 2.12.1.

N-VG The flawed proof uses proof by contradiction. To set this up, the proof states "let $p_1, ..., p_k$ be the set of all prime numbers", intending to derive a contradiction. Mathematical convention is to use the word "assume", because we are assuming a false claim that will be contradicted.

C-FI The implication that "N is not divisible by any of $p_1, ..., p_k$ ... Therefore N is prime" is false.

Error codes

- Novice Vocabulary Grammar (N-VG)
- Content False Implication (C-FI).

See Section 1.3 for more information about error classifications.

2.12.2 Corrected proof

The following is a corrected version of Flawed Proof 2.12.1.

Proof 2.12.1

Assume there are finitely many primes, we will derive a contradiction. Let $p_1, ..., p_k$ be the set of all prime numbers, where k is some positive integer. Now let $N = p_1...p_k + 1$. Now we see that N is not divisible by any of $p_1, ..., p_k$. But we know all integers greater than 1 can be written as a product of primes. Therefore N has a prime factor that is not one of $p_1, ..., p_k$. So our list of all primes is not complete - a contradiction. Therefore there must be infinitely many primes. $\square$

Chapter 3

Mathematical Induction

Mathematical Induction is a fundamental tool in a mathematician's toolbox. At the University of Calgary, Mathematical Induction is first introduced to students in MATH 271 Discrete Mathematics and MATH 273 Numbers and Proof. The exercises here involve divisibility, summation formulas, inequalities, well-ordering, and recursively defined sequences.

3.1 Mathematical Induction with Divisibility

Exercise 3.1.1: Multiples of 8

Prove that $5^n - 4n - 1$ is divisible by 8 for all integers $n \geq 0$.

Flawed Proof 3.1.1

We argue by mathematical induction. Suppose that $n \in \mathbb{Z}$, $n \geq 0$.
Base case: Consider the case $n = 0$. We see that

$$5^n - 4n - 1 = 5^0 - 4(0) - 1 = 1 - 1 = 0 = 8 \cdot 0$$

is divisible by 8.
Induction Hypothesis: Suppose that 8 divides $5^k - 4k - 1$ for all integers k such that $k \geq 0$.

We must show that 8 divides $5^{k+1} - 4(k+1) - 1$. By the induction hypothesis, there exists an integer ℓ such that $5^k - 4k - 1 = 8\ell$. Now, we see that

$$\begin{aligned}
5^{k+1} - 4(k+1) - 1 &= 5^{k+1} - 4k - 4 - 1 \\
&= 5 \cdot 5^k - 4k - 5 \\
&= 5 \cdot 5^k - 4k - 16k + 16k - 5 && \text{(adding } 0 = -16k + 16k\text{)} \\
&= 5 \cdot 5^k - 20k - 5 + 16k \\
&= 5(5^k - 4k - 1) + 16k \\
&= 5(8\ell) + 16k && \text{(by the induction hypothesis)} \\
&= 8(5\ell + 2k),
\end{aligned}$$

and since $5\ell + 2k \in \mathbb{Z}$, we have that $5^{k+1} - 4(k+1) - 1$ is divisible by 8, as required.

We conclude, by the Principle of Mathematical Induction, that 8 divides $5^n - 4n - 1$, for all integers $n \geq 0$. $\square$

3.1.1 Error classification

There is only one error in the Flawed Proof 3.1.1.

F-A In the Induction Hypothesis, the entire result that we are attempting to prove is asserted.

Error codes

- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

3.1.2 Corrected proof

The following is a corrected version of Flawed Proof 3.1.1.

Proof 3.1.1

We argue by mathematical induction. Suppose that $n \in \mathbb{Z}$, $n \geq 0$.
Base case: Consider the case $n = 0$. We see that

$$5^n - 4n - 1 = 5^0 - 4(0) - 1 = 1 - 1 = 0 = 8 \cdot 0$$

is divisible by 8.
Induction Hypothesis: Suppose that there is an integer k, $k \geq 0$, such that 8 divides $5^k - 4k - 1$.

We must show that 8 divides $5^{k+1} - 4(k+1) - 1$. By the induction hypothesis, there exists an integer ℓ such that $5^k - 4k - 1 = 8\ell$. Now, we see that

$$\begin{aligned}
5^{k+1} - 4(k+1) - 1 &= 5^{k+1} - 4k - 4 - 1 \\
&= 5 \cdot 5^k - 4k - 5 \\
&= 5 \cdot 5^k - 4k - 16k + 16k - 5 && \text{(adding } 0 = -16k + 16k\text{)} \\
&= 5 \cdot 5^k - 20k - 5 + 16k \\
&= 5(5^k - 4k - 1) + 16k \\
&= 5(8\ell) + 16k && \text{(by the induction hypothesis)} \\
&= 8(5\ell + 2k),
\end{aligned}$$

and since $5\ell + 2k \in \mathbb{Z}$, we have that $5^{k+1} - 4(k+1) - 1$ is divisible by 8, as required.

We conclude, by the Principle of Mathematical Induction, that 8 divides $5^n - 4n - 1$, for all integers $n \geq 0$. $\square$

3.2 Summation Formula Induction

Exercise 3.2.1

Prove that
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\ ,$$
for all integers $n \geq 1$.

Flawed Proof 3.2.1

Base Case (n=1):

$$\begin{aligned}\sum_{i=1}^{1} i &= \frac{1+(1+1)}{2} \\ 1 &= \frac{2}{2} \\ 1 &= 1\end{aligned}$$

Induction Hypothesis: Suppose that $k \geq 1$ is an integer and that
$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\ .$$
Then
$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}\ .$$

So

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{(by the IH)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \, .
\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \, ,$$

for all integers $n \geq 1$. □

3.2.1 Error classification

There are several errors in the Flawed Proof 3.2.1.

C-FI/C-A: The base contains a false implication. Moreover, the base case begins with the assertion of what needs to be shown.

C-A: The induction hypothesis contains the assertion of the induction step.

Error codes

- Content False Implication (C-FI)
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

3.2.2 Corrected proof

The following is a corrected version of Flawed Proof 3.2.1.

Proof 3.2.1

Suppose that n is an integer and $n \geq 1$.
Base Case (n=1): Suppose $n = 1$. Then

$$\sum_{i=1}^{1} i = 1 = \frac{2}{2} = \frac{1(2)}{2} = \frac{1(1+1)}{2}.$$

Induction Hypothesis: Suppose that $k \geq 1$ is an integer and that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\ .$$

We want to prove that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}\ .$$

It follows that

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{(by the induction hypothesis)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}\ .
\end{aligned}$$

Thus, by induction,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\ ,$$

for all integers $n \geq 1$. $\square$

3.3 Binary Notation

Exercise 3.3.1

Prove that every positive integer can be written as a sum of distinct non-negative powers of 2.

Flawed Proof 3.3.1

We will proceed with strong induction.

Base Case If $n = 1$ then $1 = 2^0$, so the base case is satisfied.

Induction Hypothesis. Suppose n is an integer greater than or equal to 2 such that for all integers k with $1 \leq k \leq n$ we have that k can be written as distinct powers of 2.

Induction Step Firstly, if n is a power of 2 then of course n can be written as a sum of distinct powers of 2. So suppose n is not a power of 2. In particular this means $n > 1$. So let k be the largest positive integer such that $2^k < n$. Since n is not a power of 2 and k was the largest positive integer with this property, we can say $2^k < n < 2^{k+1}$. Thus

$$0 < n - 2^k < 2^{k+1} - 2^k = 2^k < n$$

So we can say $1 \leq n - 2^k \leq n$. So by the induction hypothesis, we can write $n - 2^k$ as distinct non-negative powers of 2. Since this value is less than 2^k, all these powers of 2 must be smaller than 2^k. So we can write, for some distinct non-negative integers $e_1, ..., e_t < k$:

$$n - 2^k = 2^{e_1} + ... + 2^{e_t}$$
$$\Downarrow$$
$$n = 2^{e_1} + ... + 2^{e_t} + 2^k$$

so we have written n as distinct non-negative powers of 2.

Therefore by the principal of strong induction, every positive integer can be written as a sum of distinct non-negative powers of 2. $\square$

3.3.1 Error classification

There are several errors in the Flawed Proof 3.3.1.

C-EO: In the induction, the case $n = 2$ was "missed". In the induction hypothesis where it states "for all integers k with $1 \leq k \leq n$" it should read "for all integers k with $1 \leq k < n$".

Error codes

- Content Error-caused Omission (C-EO)

See Section 1.3 for more information about error classifications.

3.3.2 Corrected proof

The following is a corrected version of Flawed Proof 3.3.1.

Proof 3.3.1

We will proceed with strong induction.

Base Case If $n = 1$ then $1 = 2^0$, so the base case is satisfied.

Induction Hypothesis. Suppose n is an integer greater than or equal to 2 such that for all integers k with $1 \leq k < n$ we have that k can be written as distinct powers of 2.

Induction Step Firstly, if n is a power of 2 then of course n can be written as a sum of distinct powers of 2. So suppose n is not a power of 2. In particular this means $n > 1$. So let k be the largest positive integer such that $2^k < n$. Since n is not a power of 2 and k was the largest positive integer with this property, we can say $2^k < n < 2^{k+1}$. Thus

$$0 < n - 2^k < 2^{k+1} - 2^k = 2^k < n$$

So we can say $1 \leq n - 2^k \leq n$. So by the induction hypothesis, we can write $n - 2^k$ as distinct non-negative powers of 2. Since this value is less than 2^k, all these powers of 2 must be smaller than 2^k. So we can write, for some distinct non-negative integers $e_1, ..., e_t < k$:

$$n - 2^k = 2^{e_1} + ... + 2^{e_t}$$
$$\Downarrow$$
$$n = 2^{e_1} + ... + 2^{e_t} + 2^k$$

so we have written n as distinct non-negative powers of 2.

Therefore by the principal of strong induction, every positive integer can be written as a sum of distinct non-negative powers of 2. $\square$

3.4 Inequality Induction

Exercise 3.4.1

Prove by induction that

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\ ,$$

for all integers $n \geq 2$.

Flawed Proof 3.4.1

Base Case (n=2):

$$\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}\ .$$

Induction Hypothesis: Suppose that $\displaystyle\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}\ .$

We want to prove that $\displaystyle\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}\ .$

From the inductive hypothesis,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} = \sqrt{k+1}\ .$$

Thus, for all integers $n \geq 2$,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\ .$$

$\square$

3.4.1 Error classification

There are several errors in the Flawed Proof 3.4.1.

N-N: The integer k is undefined and not quantified.

C-LO: Failed to state a range for k in the inductive step.

F-FI: The induction hypothesis was not used where it is claimed to be invoked.

FS: The claimed equality

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} = \sqrt{k+1}\,,$$

is not justified, and is in fact false.

Error codes

- Novice Notation (N-N)
- Content Local Omission (C-LO)
- Fundamental False Implication (F-FI)
- False Statement (FS)

See Section 1.3 for more information about error classifications.

3.4.2 Corrected proof

The following is a corrected version of Flawed Proof 3.4.1.

Proof 3.4.1

Suppose that $n \geq 2$ is an integer.
Base Case (n=2): Consider $n = 2$. We have

$$\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \, .$$

Induction Hypothesis: Suppose that $k \geq 2$ is an integer and suppose that $\displaystyle\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$.

We want to prove that $\displaystyle\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$.
From the induction hypothesis, it follows that

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\
&> \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&> \frac{k+1}{\sqrt{k+1}} \\
&= \sqrt{k+1} \, .
\end{aligned}$$

Thus, by mathematical induction, for all integers $n \geq 2$,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} \, .$$

□

3.5 Strong Induction

Exercise 3.5.1: Strong Induction with Multiple Base Cases

Let $n \geq 12$ be an integer. Prove there exists non-negative integers a and b such that $n = 4a + 5b$.

Flawed Proof 3.5.1

We will proceed with strong induction.
Base Case: If $n = 12$, then we can write $12 = 4(3) + 5(0)$. So our base case holds.
Induction Hypothesis: Suppose m is an integer greater than 12 such that for all integers k with $12 \leq k < m$ we have that k can be written as $k = 4a + 5b$ for non-negative integers a and b.
Induction Step: If we take m, note that $m - 4$ can be written as $m - 4 = 4a + 5b$ for some non-negative integers a and b. Rearranging we get $m = 4(a+1) + 5b$. Since a is non-negative, so is $a + 1$. Thus the statement holds for m.

Therefore by the principal of strong induction, every integer n greater than or equal to 12 can be written in the form $n = 4a + 5b$ for non-negative integers a and b $\square$

3.5.1 Error classification

There is one error in the Flawed Proof 3.5.1.

C - FI In the induction step, the induction hypothesis does not imply that $m - 4$ can be written as $m - 4 = 4a + 5b$ for non-negative integers a and b. (If $m = 15$, $m - 4 = 11$ cannot be written in this way).

Error codes

- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

3.5.2 Corrected proof

The following is a corrected version of Flawed Proof 3.5.1.

Proof 3.5.1

Let $n \in \mathbb{Z}$, $n \geq 12$. We will proceed with strong induction.
Base Case: If $n = 12$, then we can write $12 = 4(3) + 5(0)$. If $n = 13$, then we can write $13 = 4(2) + 5(1)$. If $n = 14$, then we can write $14 = 4(1) + 5(2)$. If $n = 15$, then we can write $15 = 4(0) + 5(3)$. So these four base cases hold.
Induction Hypothesis: Suppose m is an integer greater than 15 such that for all integers k with $12 \leq k < m$ we have that k can be written as $k = 4a + 5b$ for non-negative integers a and b.
Induction Step: We must prove that there exist non-negative integers a' and b' so that $m = 4a' + 5b'$. Since $m \geq 16$ and $m - 4 \geq 12$, the induction hypothesis implies that $m - 4$ can be written as $m - 4 = 4a + 5b$ for some non-negative integers a and b. Rearranging we get $m = 4(a+1) + 5b$. Since a is non-negative, so is $a + 1$. Thus the desired claim holds for m.

Therefore by the principal of strong induction, every integer n greater than or equal to 12 can be written in the form $n = 4a + 5b$ for some non-negative integers a and b $\square$

3.6 Induction Using Pascal's Identity

Exercise 3.6.1

Let $r \in \mathbb{Z}$, $r \geq 1$. Prove that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1},$$

for all integers $n \geq r$.

Flawed Proof 3.6.1

Base Case (n=r): $\displaystyle\sum_{k=r}^{r} \binom{k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1}$.

Inductive Step: Suppose that k is an integer and that

$$\sum_{k=r}^{k} \binom{k}{r} = \binom{k+1}{r+1}.$$

We want to prove that

$$\sum_{k=r}^{k+1} \binom{k}{r} = \binom{k+2}{r+1}.$$

Then

$$\begin{aligned}\sum_{k=r}^{k+1} \binom{k}{r} &= \sum_{k=r}^{k} \binom{k}{r} + \binom{k+1}{r} \\ &= \binom{k+2}{r+1}.\end{aligned}$$

Thus, $\displaystyle\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, for all integers $n \geq r$. $\square$

3.6.1 Error classification

There are several errors in the Flawed Proof 3.6.1.

N-N: The symbol k is used in two different ways.

C-O: Failed to specify that $k \geq r$ in the inductive step.

F-A: The statement

$$\sum_{k=r}^{k} \binom{k}{r} + \binom{k+1}{r} = \binom{k+2}{r+1}$$

is unjustified, and simply asserts the conclusion with no explanation.

Error codes

- Novice Notation (N-N)
- Content Local Omission (C-O)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

3.6.2 Corrected proof

The following is a corrected version of Flawed Proof 3.6.1.

Proof 3.6.1

Suppose that $r \in \mathbb{Z}$, $r \geq 1$. Suppose that $n \in \mathbb{Z}$.
Base Case (n=r):

$$\sum_{k=r}^{r} \binom{k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1}$$

Inductive Step: Suppose that $j \geq r$ is an integer and that

$$\sum_{k=r}^{j} \binom{k}{r} = \binom{j+1}{r+1}.$$

We want to prove that

$$\sum_{k=r}^{j+1} \binom{k}{r} = \binom{j+2}{r+1}.$$

Then

$$\begin{aligned}\sum_{k=r}^{j+1} \binom{k}{r} &= \sum_{k=r}^{j} \binom{k}{r} + \binom{j+1}{r} \\ &= \binom{j+1}{r+1} + \binom{j+1}{r} \\ &= \binom{j+2}{r+1}, \text{ by Pascal's Identity .}\end{aligned}$$

Thus,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1},$$

for all integers $n \geq r$. $\square$

3.7 Induction and Well-Ordering

Exercise 3.7.1: Induction Implies Well-Ordering

Prove that the principle of strong induction implies the Well-Ordering Principle.

Flawed Proof 3.7.1

We will use strong induction to prove the Well-Ordering Principle. Let $X \subseteq \mathbb{N}$ and define $P(n)$ as "if $n \in X$, then X has a least element". We will apply strong induction to $P(n)$ with $n_0 = 1$.

Base Case: Consider the case when $n = 1$. Since 1 is the smallest of all natural numbers, if $1 \in X$ then X has a least element, 1 itself. Thus $P(1)$ holds.
Induction Hypothesis: Suppose that $n > 1$ and it holds true for $n-1$.
Induction Step: Suppose $n \in X$. If X contains any element less than n, then by the induction hypothesis X contains a least element. On the other hand, if X does not contain any element less than n then n is its least element. Thus $P(n)$ holds.

Therefore by the principle of strong induction, $P(n)$ holds for all natural numbers n. This means that for any subset X of the natural numbers, if X contains some $n \in \mathbb{N}$ (that is, if X is non-empty) then X has a least element. $\square$

3.7.1 Error classification

There are several errors in the Flawed Proof 3.7.1.

WM The author is attempting to use induction, not strong induction.

C-LU The induction hypothesis is imprecise.

Error codes

- Wrong Method (WM)
- Content Locally Unintelligible (C-LU)

See Section 1.3 for more information about error classifications.

3.7.2 Corrected proof

The following is a corrected version of Flawed Proof 3.7.1.

Proof 3.7.1

We will use strong induction to prove the Well-Ordering Principle. Let $X \subseteq \mathbb{N}$ and define $P(n)$ to be the statement "if $n \in X$, then X has a least element". We will apply strong induction to $P(n)$ with $n_0 = 1$.

Base Case: Consider the case when $n = 1$. Suppose that $1 \in X$, since 1 is the smallest of all natural numbers, then X has a least element, 1 itself. Thus $P(1)$ holds.
Induction Hypothesis: Suppose that $n > 1$ and that for all $k \in \mathbb{N}$ with $1 \leq k < n$, if $k \in X$ then X has a least element. That is, suppose $P(k)$ holds for $1 \leq k < n$.
Induction Step: Suppose $n \in X$. If X contains any natural number less than n, then by the induction hypothesis X contains a least element. On the other hand, if X does not contain any element less than n then n is its least element. Thus $P(n)$ holds.

Therefore by the principle of strong induction, $P(n)$ holds for all natural numbers n. This means that for any subset X of the natural numbers, if X contains some $n \in \mathbb{N}$ (that is, if X is non-empty) then X has a least element.

□

3.8 A recursive sequence

Exercise 3.8.1

The sequence $\{b_n\}_{n=0}^{\infty} = \{b_0, b_1, b_2, \ldots\}$ is defined by: $b_0 = 1$, $b_1 = 8$, and for all integers $n \geq 2$,

$$b_n = b_{n-1} + 2b_{n-2}$$

Prove that $b_n = 3(2^n) - 2(-1)^n$, for all integers $n \geq 0$.

Flawed Proof 3.8.1

Suppose n is a non-negative integer.

Base case

If $n = 0$, then $b_0 = 3(2^0) - 2(-1)^0 = 3(1) - 2(1) = 3 - 2 = 1$.
If $n = 1$, then $b_1 = 3(2^1) - 2(-1)^2 = 3(2) - 2(-1) = 6 + 2 = 8$.
So the base cases hold.

Induction Hypothesis

Suppose that for all integers $k \geq 1$, we have $b_m = 3(2^m) - 2(-1)^m$, for all integers $0 \leq m \leq k$.

Inductive Step

We must show that $b_{k+1} = 3(2^{k+1}) - 2(-1)^{k+1}$.
Since $k + 1 \geq 2$, by definition of b_{k+1}, we see that

$$\begin{aligned} b_{k+1} &= b_k + 2b_{k-1} \\ &= 3(2^k) - 2(-1)^k + 2[3(2^{k-1}) - 2(-1)^{k-1}] \quad \text{(by the induction hypothesis)} \\ &= 3(2^{k+1}) - 2(-1)^{k+1}, \end{aligned}$$

as required.

We conclude, by the Principle of Mathematical Induction (Strong Form), that $b_n = 3(2^n) - 2(-1)^n$, for all integers $n \geq 0$.

□

3.8.1 Error classification

There are several errors in the Flawed Proof 3.8.1.

C-A Result is asserted in the base case.

F-A Entire result is asserted in the induction hypothesis.

N-A Calculation in inductions step could use more detail.

Error codes

- Content Assertion (C-A)
- Fundamental Assertion (F-A)
- Novice Assertion (N-A)

See Section 1.3 for more information about error classifications.

3.8.2 Corrected proof

The following is a corrected version of Flawed Proof 3.8.1.

Proof 3.8.1

Suppose that $n \in \mathbb{Z}$, $n \geq 0$.
Base case: Consider $n = 0$ and $n = 1$. If $n = 0$, then

$$3(2^n) - 2(-1)^n = 3(1) - 2(1) = 3 - 2 = 1 = b_0.$$

If $n = 1$, then

$$3(2^n) - 2(-1)^n = 3(2) - 2(-1) = 6 + 2 = 8 = b_1.$$

Induction Hypothesis: Suppose that there exists an integer $k \geq 1$ such that for all integers m if $0 \leq m \leq k$, then $b_m = 3(2^m) - 2(-1)^m$.
Inductive Step

We must show that $b_{k+1} = 3(2^{k+1}) - 2(-1)^{k+1}$.

Since $k + 1 \geq 2$, by definition of b_{k+1}, we see that

$$\begin{aligned}
b_{k+1} &= b_{k+1-1} + 2b_{k+1-2} \\
&= b_k + 2b_{k-1} \\
&= 3(2^k) - 2(-1)^k + 2[3(2^{k-1}) - 2(-1)^{k-1}] && \text{(by the induction hypothesis)} \\
&= 3(2^k) - 2(-1)^k + 3(2^k) - 4(-1)^{k-1} \\
&= 3(2^k) + 3(2^k) - 2(-1)^k - 4(-1)^{k-1} \\
&= 2 \cdot 3(2^k) - 2(-1)^{k-1}[(-1) + 2] \\
&= 3(2^{k+1}) - 2(-1)^{k-1}(1) \\
&= 3(2^{k+1}) - 2(-1)^{k-1}(-1)^2 && \text{(since } 1 = (-1)^2\text{)} \\
&= 3(2^{k+1}) - 2(-1)^{k+1},
\end{aligned}$$

as required.

We conclude, by the Principle of Mathematical Induction (Strong Form), that $b_n = 3(2^n) - 2(-1)^n$, for all integers $n \geq 0$.

$\square$

3.9 Strong Induction with Sequences

Exercise 3.9.1

Let a_0, a_1, a_2, $\cdots$ be the sequence defined by $a_0 = 12$, $a_1 = 29$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Prove that $a_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

Flawed Proof 3.9.1

Base Cases (n= 0,1):

$$a_0 = 12 = 5 + 7 = 5 \cdot 3^0 + 7 \cdot 2^0 \ .$$

$$a_1 = 29 = 15 + 14 = 5 \cdot 3^1 + 7 \cdot 2^1 \ .$$

Inductive Hypothesis: Let $k \geq 2$ be an integer and suppose that $a_m = 5 \cdot 3^m + 7 \cdot 2^m$ for all integers where $0 \leq m \leq k$. We want to prove that $a_k = 5 \cdot 3^k + 7 \cdot 2^k$.

Then

$$\begin{aligned}
a_k &= 5a_{k-1} - 6a_{k-2} \\
&= 5(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) - 6(5 \cdot 3^{k-2} + 7 \cdot 2^{k-2}) \\
&= 25 \cdot 3^{k-1} + 35 \cdot 2^{k-1} - 6 \cdot 5 \cdot 3^{k-2} - 6 \cdot 7 \cdot 2^{k-2}) \\
&= 25 \cdot 3^{k-1} + 35 \cdot 2^{k-1} - 10 \cdot 3^{k-1} - 21 \cdot 2^{k-1}) \\
&= (25 - 10)3^{k-1} + (35 - 21)2^{k-1} \\
&= (15)3^{k-1} + (14)2^{k-1} \\
&= 5 \cdot 3^k + 7 \cdot 2^k
\end{aligned}$$

Thus, $a_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$. $\square$

3.9.1 Error classification

There are several errors in the Flawed Proof 3.9.1.

C-OS: By letting $k \geq 2$ and then supposing that the inductive hypothesis holds for $0 \leq m \leq k$, the case where $n = 2$ is unverified. The correct statement is to suppose that the inductive hypothesis holds for $0 \leq m < k$ or $0 \leq m \leq k - 1$.

C-A: Asserting that the result is true for $n = 2$ without verification.

Error codes

- Content Omitted Section (C-OS)
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

3.9.2 Corrected proof

The following is a corrected version of Flawed Proof 3.9.1.

Proof 3.9.1

Base Cases (n= 0,1):

$$a_0 = 12 = 5 + 7 = 5 \cdot 3^0 + 7 \cdot 2^0 \ .$$

$$a_1 = 29 = 15 + 14 = 5 \cdot 3^1 + 7 \cdot 2^1 \ .$$

Inductive Hypothesis: Let $k \geq 2$ be an integer and suppose that $a_m = 5 \cdot 3^m + 7 \cdot 2^m$ for all integers where $0 \leq m < k$. We want to prove that $a_k = 5 \cdot 3^k + 7 \cdot 2^k$.

Then

$$\begin{aligned}
a_k &= 5a_{k-1} - 6a_{k-2} \\
&= 5(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) - 6(5 \cdot 3^{k-2} + 7 \cdot 2^{k-2}) \\
&= 25 \cdot 3^{k-1} + 35 \cdot 2^{k-1} - 6 \cdot 5 \cdot 3^{k-2} - 6 \cdot 7 \cdot 2^{k-2}) \\
&= 25 \cdot 3^{k-1} + 35 \cdot 2^{k-1} - 10 \cdot 3^{k-1} - 21 \cdot 2^{k-1}) \\
&= (25 - 10)3^{k-1} + (35 - 21)2^{k-1} \\
&= (15)3^{k-1} + (14)2^{k-1} \\
&= 5 \cdot 3^k + 7 \cdot 2^k
\end{aligned}$$

Thus, $a_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$. □

Chapter 4

Discrete Mathematics

The topics in this chapter are covered in MATH 271 Discrete Mathematics at the University of Calgary. Topics include

- elementary set theory
- basic properties of functions (one-to-one, onto, composition, etc.)
- elementary counting problems and probability, and
- equivalence relations and modular arithmetic.

4.1 Set Unions and Intersections

Exercise 4.1.1: De Morgan's Law

Let U be a set. Prove that for all subsets A and B of U,

$$(A \cup B)^c = A^c \cap B^c.$$

Here A^c denotes the complement of A in U, that is

$$A^c = U \setminus A = \{x \in U : x \notin A\}.$$

Flawed Proof 4.1.1

$x \in (A \cup B)^c$ means $x \notin A$ nor B means $x \notin A$ or $x \notin B$ means $x \in A^c$ or $x \in B^c$ means $x \in A^c \cap B^c$.

$x \in A^c \cap B^c$ means $x \in A^c$ or $x \in B^c$ means $x \notin A$ or $x \notin B$ means $x \notin A$ nor B means $x \in (A \cup B)^c$.

$\square$

4.1.1 Error classification

There are several errors in the Flawed Proof 4.1.1.

N-VG In general, the flawed proof is poorly written. The repetitive use of the word "means", and each paragraph being one run on sentence makes it difficult to read.

F-FA The assertion "$x \notin A$ nor B means $x \notin A$ or $x \notin B$" is incorrect.

F-FA The assertion "$x \in A^c$ or $x \in B^c$ means $x \in A^c \cap B^c$ is incorrect.

F-FA The assertion "$x \in A^c \cap B^c$ means $x \in A^c$ or $x \in B^c$" is incorrect.

F-FA The assertion "$x \notin A$ or $x \notin B$ means $x \notin A$ nor B" is incorrect.

Error codes

- Novice Vocabulary Grammar (N-VG)
- Fundamental False Assertion (F-FA)

See Section 1.3 for more information about error classifications.

4.1.2 Corrected proof

The following is a corrected version of Flawed Proof 4.1.1.

Proof 4.1.1

Suppose that A and B are subsets of U.

First, we will prove that $(A \cup B)^c$ is a subset of $A^c \cap B^c$. Suppose that $x \in (A \cup B)^c$. That is, $x \in U$ by $x \notin A \cup B$. Since $x \notin A \cup B$, it follows that $x \notin A$ and $x \notin B$. By definition of the complement, $x \in A^c$ and $x \in B^c$, and $x \in A^c \cap B^c$. Thus, $(A \cup B)^c$ is contained in $A^c \cap B^c$.

Now, we will prove that $A^c \cap B^c$ is a subset of $(A \cup B)^c$. Suppose that $y \in A^c \cap B^c$. Then $y \in A^c$ and $y \in B^c$. That is, $y \in U$ but $y \notin A$ and $y \notin B$. Moreover, $y \notin A \cup B$. Thus $y \in (A \cup B)^c$ and $A^c \cap B^c$ is contained in $(A \cup B)^c$.

Therefore, $(A \cup B)^c \subset (A^c \cap B^c)$ and $(A^c \cap B^c) \subset (A \cup B)^c$, which means $(A \cup B)^c = A^c \cap B^c$. $\square$

4.2 Equality of Sets

Exercise 4.2.1: Equality of Sets

Let $S = \{x \in \mathbb{R} \mid x^2 < x\}$ and let $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$.
Prove that $S = T$.

Flawed Proof 4.2.1

$0 < x < 1$ implies $0 < x^2 < x$, but squares are always positive so we just get that $x^2 < x$. $\square$

4.2.1 Error classification

There are several errors in the Flawed Proof 4.2.1.

N-A The claim "$0 < x < 1$ implies $0 < x^2 < x$" requires more justification.

F-OS The flawed proof only shows $T \subseteq S$, but does not show $S \subseteq T$.

N-VG In general, more detail is required and the proof does not follow good mathematical writing conventions; for instance x is undefined. It not clear what has been shown and what this has to do with the desired claim.

Error codes

- Novice Assertion (N-A)
- Fundamental Omitted Section (F-OS)
- Novice Vocabular & Grammar (N-VG)

See Section 1.3 for more information about error classifications.

4.2.2 Corrected proof

The following is a corrected version of Flawed Proof 4.2.1.

Proof 4.2.1

Firstly we will show $T \subseteq S$. Let $x \in T$. This means $0 < x < 1$. Since x is positive we can multiply the inequality by x, giving $0 < x^2 < x$. In particular, $x^2 < x$. So $x \in S$. Thus $T \subseteq S$.

Next we will show $S \subseteq T$. Let $x \in S$. This means $x^2 < x$. Rearranging and factoring we obtain

$$x(x-1) = x^2 - x < 0.$$

Since the product $x(x-1)$ is negative, one of x and $x-1$ is positive and the other is negative. If $x-1 > 0$, then $x > 1 > 0$. So, in order for x and $x-1$ to have opposite signs, we must have $x-1 < 0$ and $x > 0$, which implies $0 < x < 1$. So $x \in T$. Thus $S \subseteq T$.

Since $S \subseteq T$ and $T \subseteq S$, we can conclude that $S = T$. $\square$

4.3 Subsets and power sets

Exercise 4.3.1: Subsets and power sets

Prove that for all sets A and B, if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
Here, $\mathcal{P}(X)$ denotes the power set of a set X.

Flawed Proof 4.3.1

$A \in B \Rightarrow A \in \mathcal{P}(B) \Rightarrow \mathcal{P}(A) \in \mathcal{P}(B)$. $\square$

4.3.1 Error classification

There are multiple errors in the Flawed Proof 4.3.1.

C-N Misuse of $\in$ in place of $\subseteq$.

N-VG The intended logical progression (ignoring the notation error) is correct in spirit, but the proof is devoid of prose, detail and explanation.

Error codes

- Content Notation (C-N)
- Novice Vocabulary & Grammar (N-VG)

See Section 1.3 for more information about error classifications.

4.3.2 Corrected proof

The following is a corrected version of Flawed Proof 4.3.1.

Proof 4.3.1

Suppose that A and B are sets, and suppose that $A \subseteq B$. Let $S \in \mathcal{P}(A)$. We wish to show $S \in \mathcal{P}(B)$. Since $S \in \mathcal{P}(A)$ this means S is a subset of A. Since S is a subset of A, and A is a subset of B, it follows that $S \subseteq B$. This means $S \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. $\square$

4.4 Subsets and power sets again

Exercise 4.4.1: Subsets and power sets again

Let A and B be sets with $A \subseteq B$. Prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Here, $\mathcal{P}(X)$ deotes the power set of a set X.

Flawed Proof 4.4.1

Let A, B be sets with $A \subseteq B$. Let $S \subseteq \mathcal{P}(A)$. Then by definition $S \subseteq A \subseteq B$. Thus $S \subseteq \mathcal{P}(B)$. Thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. $\square$

4.4.1 Error classification

There are multiple errors in the Flawed Proof **??**.

C-VG Misunderstanding of the definition of $\mathcal{P}(X)$. It should read "Let $S \in \mathcal{P}(A)$" instead of "Let $S \subseteq \mathcal{P}(A)$", and similarly when arguing that $S \in \mathcal{P}(B)$.

Error codes

- Content Vocabulary and Grammar (C-VG)

See Section 1.3 for more information about error classifications.

4.4.2 Corrected proof

The following is a corrected version of Flawed Proof **??**.

Proof 4.4.1

Let $S \in P(A)$. We wish to show $S \in P(B)$. Since $S \in P(A)$ this means S is a subset of A. Since S is a subset of A and A is a subset of B, we have that $S \subseteq B$. This means $S \in P(B)$.

Therefore $P(A) \subseteq P(B)$. $\square$

4.5 Set Containment

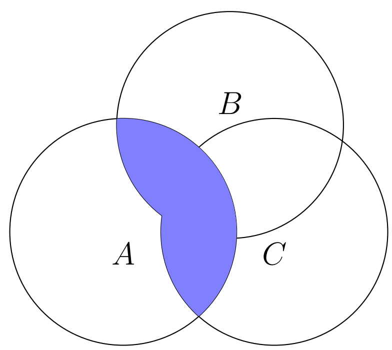
Exercise 4.5.1

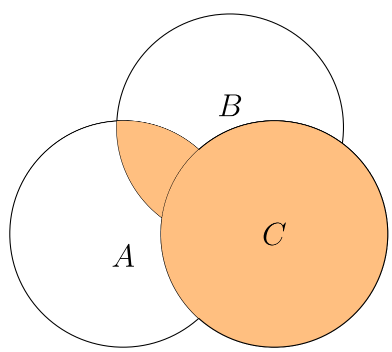
Prove or disprove the following statement. For all sets A, B and C,

$$A \cap (B \cup C) \subseteq (A \cap B) \cup C \; .$$

Flawed Proof 4.5.1

This statement is true. From the pictures below, we see that the shaded blue area (top diagram) covered by $A \cap (B \cup C)$ is clearly contained within the shaded orange area (bottom diagram) covered by $(A \cap B) \cup C$.





$\square$

4.5.1 Error classification

There are several errors in the Flawed Proof 4.5.1.

F-Eg: Although the logic expressed in the picture is correct, proof by picture is an invalid proof technique.

F-EO: The entirety of the proof is omitted due to the proof by example error.

Error codes

- Fundamental Proof by Example (F-Eg)
- Fundamental Error-Caused Omission (F-EO)

See Section 1.3 for more information about error classifications.

4.5.2 Corrected proof

The following is a corrected version of Flawed Proof 4.5.1.

Proof 4.5.1

Suppose A, B and C are sets. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we know that $x \in B$ or $x \in C$. Thus, we have two cases to consider:

Case 1: Suppose that $x \in B$. Then $x \in A$ and $x \in B$, so $x \in A \cap B$. Thus, $x \in (A \cap B) \cup C$.
Case 2: Suppose that $x \in C$. Since $x \in C$, then $x \in (A \cap B) \cup C$.

Thus, in both cases $x \in (A \cap B) \cup C$. Therefore, We can conclude that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup C\ .$$

$\square$

4.6 Set Containment Involving the Empty Set

Exercise 4.6.1

Recall that for any two sets X and Y, we define

$$X \setminus Y = \{x \in X : x \notin Y\}.$$

Prove or disprove the following statement.
For all sets A, B and C, if $A \setminus C = \varnothing$, then $A \setminus B \subseteq C$.

Flawed Proof 4.6.1

This statement is true. Suppose that $x \in A \setminus B$. We want to prove that $x \in C$. So since $x \in A \setminus B$, then $x \in A$ and $x \notin B$. Since $A \setminus C = \varnothing$, this means that $x \in A$ and $x \in C$ since if something is in A, then it cannot be in not C (because that is equal to the empty set). Thus, $x \in C$. $\square$

4.6.1 Error classification

There are several errors in the Flawed Proof 4.6.1.

N-VG: Failed to define A, B and C. Failed to assume that $A \setminus C = \varnothing$.

F-A: Entire result is asserted in the last sentence.

Error codes

- Novice Vocabulary and Grammar (N-VG)
- Fundamental Assertions (F-A)

See Section 1.3 for more information about error classifications.

4.6.2 Corrected proof

The following is a corrected version of Flawed Proof 4.6.1 that uses Proof by Contradiction.

Proof 4.6.1

Suppose that A, B and C are sets and that $A \setminus C = \varnothing$. Suppose that $x \in A \setminus B$. We want to prove that $x \in C$ and we will do this by contradiction. So suppose that $x \notin C$. Since $x \in A \setminus B$, we get that $x \in A$. Since $x \in A$ and $x \notin C$, we get $x \in A \setminus C$, but this contradicts the assumption that $A \setminus C = \varnothing$. Thus, $x \in C$ and we conclude that $A \setminus B \subset C$. □

The following is a corrected, direct version of Flawed Proof 4.6.1.

Proof 4.6.2

Suppose that A, B and C are sets and that $A \setminus C = \varnothing$. Suppose that $x \in A \setminus B$. In particular, $x \in A$. Since $x \in A$ and $A \setminus C = \varnothing$, it follows from the definition of the set-difference that $x \in C$. Thus, $A \setminus B \subset C$. □

4.7 Composition of Injective Functions

Exercise 4.7.1

Recall that "injective" means "one-to-one".
Prove or disprove the following statement. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f : A \to C$ is injective.

Flawed Proof 4.7.1

Suppose that $f : A \to B$ and $g : B \to C$ are injective functions. Let $a_1, a_2 \in A$. Then $f(a_1) = f(a_2)$ and so $g(f(a_1)) = g(f(a_2))$. Thus, $g \circ f$ is injective. $\square$

4.7.1 Error classification

There are several errors in the Flawed Proof 4.7.1.

C-VG: Misunderstanding the definition of what it means for the function $g \circ f : A \to C$ to be injective.

F-A: Asserted entire result by supposing that $a_1 = a_2$.

C-LO: Components of proof are not in logical order due to misuse of the C-VG error.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Fundamental Assertions (F-A)
- Content Logical Order (C-LO)

See Section 1.3 for more information about error classifications.

4.7.2 Corrected proof

The following is a corrected version of Flawed Proof 4.7.1.

Proof 4.7.1

Suppose $f : A \to B$ and $g : B \to C$ are injective functions. We want to prove that $g \circ f : A \to C$ is injective. So suppose that $a_1, a_2 \in A$ such that $(g \circ f)(a_1) = (g \circ f)(a_2)$. That is, $g(f(a_1)) = g(f(a_2))$. Since g is injective, it must be the case that $f(a_1) = f(a_2)$. Similarly, since f is injective, we have $a_1 = a_2$. Thus, $g \circ f$ is injective. $\square$

4.8 Surjections and Injections

Exercise 4.8.1: Divisors function

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers.
Define the function $\sigma : \mathbb{N} \to \mathbb{N}$ via

$$\sigma(n) = \sum_{d \in \mathbb{N} \text{ and } d|n} 1$$

Prove that σ is surjective but not injective.

Flawed Proof 4.8.1

The first thing to notice is that $\sigma(n)$ is simply counting the number of divisors of n. Let n be the product of the first m primes. Then $\sigma(n) = m$, so σ is surjective.

To show σ is not injective we need to find examples where $m = n$ but $\sigma(m) \neq \sigma(n)$. We can write $12 = 4 * 3$ which means $\sigma(12) = 2$ but also $12 = 2 * 2 * 3$ so $\sigma(12) = 3$. So σ is not injective. $\square$

4.8.1 Error classification

There are several errors in the Flawed Proof 4.8.1.

N-O m and n are not defined.

C-FS If n is the product of the first m primes, this does not mean $\sigma(n) = m$.

C-VG The statement "To show σ is not injective we need to find examples where $m = n$ but $\sigma(m) \neq \sigma(n)$" shows a fundamental misunderstanding of what it means for σ to be a function, as well as for a function to be injective.

C-FS Both computations of $\sigma(12)$ in the final paragraph are incorrect. The way in which the computations were justified shows a fundamental misunderstanding of the function σ.

Error codes

- Novice Omission (N-O)
- Content False Statement (C-FS)
- Content Vocabular and Grammar (C-VG)
- Content False Statement (C-FS)

See Section 1.3 for more information about error classifications.

4.8.2 Corrected proof

The following is a corrected version of Flawed Proof 4.8.1.

Proof 4.8.1

The first thing to notice is that if $k \in \mathbb{N}$, then $\sigma(k)$ is precisely the number of positive divisors of k.

To prove that σ is surjective, we have to show for each $m \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ such that $\sigma(n) = m$. Suppose that $m \in \mathbb{N}$. Let $n = 2^{m-1} \in \mathbb{N}$. By unique prime factorization, the only divisors of n are the powers of 2 less than or equal to n, namely $2^0, 2^1, ..., 2^{m-1}$. In particular, this means n has m divisors, so $\sigma(n) = m$. Thus $\sigma(n)$ is surjective.

To show σ is not injective we need to find two integers $m, n \in \mathbb{N}$ such that $\sigma(m) = \sigma(n)$ but $m \neq n$. Let $m = 3$ and $n = 2$. Both 2 and 3 are prime and thus have two positive divisors, so $\sigma(2) = 2 = \sigma(3)$, but of course $2 \neq 3$. $\square$

4.9 Injection and Surjections

Exercise 4.9.1: Bijection onto the Image 1

Recall that if $h : X \to Y$ is a function from a set X to a set Y, then the image of h is the set

$$\mathrm{Im}(h) = \{y \in Y : \exists\, x \in X \text{ so that } h(x) = y\}.$$

Let A and B be sets and let $f : A \to B$ be an injective function.
Define a function $g : A \to \mathrm{Im}(f)$ by $g(a) = f(a)$, for all $a \in A$.
Prove that g is a bijection.

Flawed Proof 4.9.1

Injective;

- $g = F$ and $F = injection$ so $G = injection$ so the next thing is
- surgect:
- The IM has all the targets the function hits so if the only target that G has is the targets that it hits then it hits all the targets it can so $G = SURJECTION$.

□

4.9.1 Error classification

There are several errors in the Flawed Proof 4.9.1.

N-N The capitalization of g to G and of f to F is confusing.

C-FS The statement "$g = F$" is false.

N-VG Using "$F = injection$", "$G = injection$" and later "$G = SURJECTION$" does not follow mathematical conventions.

N-VG In general, the proof is very hard to follow. It is not organized well, the use of bullet points is confusing and spelling/grammar is poor.

N-A Ultimately, the result is asserted without sufficient justification.

Error codes

- Novice Notation (N-N)
- Content False Statement (C-FS)
- Novice Vocabulary Grammar (N-VG)
- Novice Assertion (N-A)

See Section 1.3 for more information about error classifications.

Exercise 4.9.2: Bijection onto the Image 2

Recall that if $h : X \to Y$ is a function from a set X to a set Y, then the image of h is the set

$$\mathrm{Im}(h) = \{y \in Y : \exists\, x \in X \text{ so that } h(x) = y\}.$$

Let A and B be sets and let $f : A \to B$ be an injective function.
Define a function $g : A \to \mathrm{Im}(f)$ by $g(a) = f(a)$, for all $a \in A$.
Prove that g is a bijection.

Flawed Proof 4.9.2

Clearly $g(a) = f(a)$ for all elements a of the domain A, so $g = f$ and g is injective because f is injective. Similarly, $g = f$ but we shrunk the codomain to the image and g is obviously surjective. Thus, g is a bijection. $\square$

4.9.2 Error classification

There are several errors in the Flawed Proof 4.9.2.

C-FS The functions f and g are not equal, they have different codomains.

N-O/C-A The misunderstanding of what it means for two functions to be equal has led the author to claim that the desired statement is "obvious", omitting a detailed proof, and resulting in the assertion of the result.

Error codes

- Content False Statement (C-FS)
- Novice Omission (N-O) / Content Assertion (C-O)

See Section 1.3 for more information about error classifications.

4.9.3 Corrected proof

The following is a corrected version of Flawed Proofs 4.9.1 and 4.9.2.

Proof 4.9.1

We will show that g is both injective and surjective.

First, we prove that g is injective. Suppose that $a, a' \in A$ such that $g(a) = g(a')$. Then, by definition of the function g we have that

$$f(a) = g(a) = g(a') = f(a'),$$

and since f is injective we have that $a = a'$. So, g is injective.

Next, we prove that g is surjective. Suppose that $b \in \mathrm{Im}(f)$, that is, b is in the codomain of g. Since $b \in \mathrm{Im}(f)$, there exists some $a \in A$ such that $b = f(a)$. Then, by definition of g, $b = f(a) = g(a)$. Thus g is surjective.

We have shown that g is both injective and surjective; therefore, g is bijective. $\square$

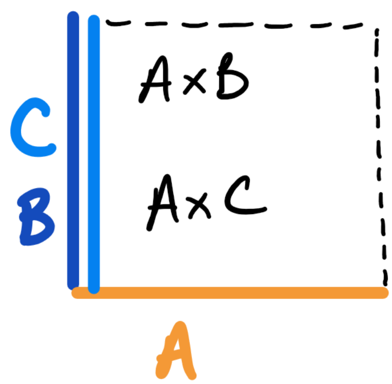
4.10 Cartesian Product Set Equality

Exercise 4.10.1

Prove or disprove the following statement. For all sets A, B and C, if $A \times B = A \times C$, then $B = C$.

Flawed Proof 4.10.1

Yes, this statement is true. By considering the picture below,



we see that if the area of $A \times B$ equals the area of $A \times C$, then it must be the case that $B = C$.

$\square$

4.10.1 Error classification

There are several errors in the Flawed Proof 4.10.1.

F-Eg: Proof by picture/example.

C-FS: The statement in question is false.

F-EO-FS: False statement caused omission of majority of the proof (i.e. a counterexample).

Error codes

- Fundamental Proof by Example (F-Eg)
- Content False Statement (C-FS)
- Fundamental Error-caused Omission resulting from False Statement (F-EO-FS)

See Section 1.3 for more information about error classifications.

4.10.2 Corrected proof

The following is a corrected version of Flawed Proof 4.10.1.

Proof 4.10.1

This statement is false. Consider the following counterexample. Let $A = \varnothing, B = \{1\}$ and $C = \{1, 2\}$. Then $A \times B = \varnothing$ and $A \times C = \varnothing$, but $B \neq C$ since $2 \in C$ and $2 \notin B$. $\square$

4.11 Cartesian Product Set Equality Again

Exercise 4.11.1

Prove or disprove the following statement. For all sets A, B and C, if $A \times B = A \times C$, then $B = C$.

Flawed Proof 4.11.1

The statement is false. We will consider some cases.

If A is empty, then $A \times B = \emptyset \times B = \emptyset$ and $A \times C = \emptyset \times C = \emptyset$ so $A \times B = A \times C$.

If A is non-empty, then let B, C both be empty. Then $A \times B = A \times \emptyset = \emptyset$ and $A \times C = A \times \emptyset = \emptyset$, so $A \times B = A \times C$, and of course $B = C$. $\square$

4.11.1 Error classification

There are several errors in the Flawed Proof 4.11.1.

F-WM: There is no need to consider cases nor deal with arbitrary sets; only a counter-example is needed.

Error codes

- Fundamental Wrong Method

See Section 1.3 for more information about error classifications.

4.11.2 Corrected proof

The following is a corrected version of Flawed Proof 4.11.1.

Proof 4.11.1

This statement is false. Consider the following counterexample. Let $A = \varnothing, B = \{1\}$ and $C = \{1, 2\}$. Then $A \times B = \varnothing$ and $A \times C = \varnothing$, but $B \neq C$ since $2 \in C$ and $2 \notin B$. $\square$

4.12 Relations

Exercise 4.12.1: Almost Rational

Let $\sim$ be a relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \sim (a', b')$ iff $a'b = ab'$. Prove $\sim$ is not an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

Flawed Proof 4.12.1

Need examples that contradict the definition of equivalence relation

Reflexive: ✓

Symmetric: ✓

Transitive: The error must be here. In class we did this but with $b \neq 0$ so it must have something to do with that.

$\square$

4.12.1 Error classification

There is one error in the Flawed Proof 4.12.1. which we have classified using the Coding Scheme Matrix in [2, pp. 919].

SW The flawed proof isnt a completed proof or even written as a proof - it is really just scratch work one might have been using in the process of discovering the proof.

Error codes

- Scratch Work (SW)

See Section 1.3 for more information about error classifications.

4.12.2 Corrected proof

The following is a corrected version of Flawed Proof 4.12.1.

Proof 4.12.1

We will disprove the statement by showing this relation is not transitive. Note that $(0,0)$, $(1,1)$, $(1,2) \in \mathbb{Z} \times \mathbb{Z}$. We can see that $(1,1) \sim (0,0)$ since $1 \cdot 0 = 0 \cdot 1$. Also we can see that $(0,0) \sim (1,2)$ since $0 \cdot 2 = 0 \cdot 1$. If this relation were transitive this would imply $(1,1) \sim (1,2)$ but $1 \cdot 2 \neq 1 \cdot 1$ so this is not true. Thus this relation is not transitive. $\square$

4.13 Constructing Rationals from Integers

Exercise 4.13.1

Let $\sim$ be the equivalence relation on $\mathbb{Z} \times \mathbb{Z}^{\times}$ given by,

$$(a, b) \sim (c, d) \text{ if } ad = bc \ .$$

Prove or disprove the statement,

$$(4, 6) \sim (48, 72) \ .$$

Flawed Proof 4.13.1

Yes, this statement is true since,

$$\frac{4}{6} = \frac{2}{3} = \frac{48}{72} \ .$$

$\square$

4.13.1 Error classification

There are several errors in the Flawed Proof 4.13.1.

F-A: The entire result is asserted by stating that

$$\frac{4}{6} = \frac{2}{3} = \frac{48}{72}\,.$$

That is, $ad = bc$ is asserted.

C-VG: Misuse/misunderstanding of the required equivalence relation.

Error codes

- Fundamental Assertion (F-A)
- Content Vocabulary and Grammar (C-VG)

See Section 1.3 for more information about error classifications.

4.13.2 Corrected proof

The following is a corrected version of Flawed Proof 4.13.1.

Proof 4.13.1

This statement is true. Recall that we can construct $\mathbb{Q}$ from $\mathbb{Z}$ using the equivalence relation on $\mathbb{Z} \times \mathbb{Z}^{\times}$ given by,

$$(a, b) \sim (c, d) \text{ if } ad = bc \ .$$

In our case, we have that $a = 4,\ b = 6,\ c = 48$ and $d = 72$. Then,

$$4(72) = 288 = 6(48) \ ,$$

which implies that $(4, 6) \sim (48, 72)$. $\square$

4.14 Counting Subsets

Exercise 4.14.1: Subsets containing even numbers

Let $n \geq 3$ be an integer, let $N = \{1, 2, ..., n\}$. Express the number of subsets of N containing at least one even number in terms of n.

Flawed Proof 4.14.1

If it has to contain an even number, then it is a subset of $\{2, 4, ..., n\}$ (the set of even numbers less than or equal to n). This set has $n/2$ elements. So the number of subsets of this set is $2^{n/2}$. $\square$

4.14.1 Error classification

There are multiple errors in the Flawed Proof 4.14.1

C-FI "it has to contain an even number" does not imply "it is a subset of $\{2, 4, ..., n\}$".

C-FS The set $\{2, 3, ..., n\}$ is not the set of even numbers less than or equal to n if n is odd.

C-FS The set $\{2, 4, ..., n\}$ does not have $n/2$ elements if n is odd.

Error codes

- Content False Implication (C-FI)
- Content False Statement (C-FS)

See Section 1.3 for more information about error classifications.

4.14.2 Corrected proof

The following is a corrected version of Flawed Proof 4.14.1.

Proof 4.14.1

Firstly, we know the total number of subsets of N is 2^n. A subset that contains no even numbers is a subset of $O = \{1, 3, ..., n\}$ if n is odd or $O = \{1, 3, ..., n-1\}$ if n is even. The size of O is $n/2$ if n is even, or $\lceil n/2 \rceil$ if n is odd. Either way, we can say the size of O is $\lceil n/2 \rceil$. Thus, the number of subsets of O is $2^{\lceil n/2 \rceil}$. Therefore, the number of subsets of N containing at least one even number would be $2^n - 2^{\lceil n/2 \rceil}$. $\square$

4.15 Card Combinations

Exercise 4.15.1

In a standard deck of 52 cards, how many different 5 card hands can be drawn?

Flawed Proof 4.15.1

$52 \times 51 \times 50 \times 49 \times 48$. □

4.15.1 Error classification

There are several errors in the Flawed Proof 4.15.1.

F-WM: Used incorrect probability method. More specifically, the answer to the exercise requires the concept of combinations.

N-VG: No explanation of chosen method provided.

Error codes

- Fundamental Wrong Method (F-WM)
- Novice Vocabulary and Grammar (N-VG)

See Section 1.3 for more information about error classifications.

4.15.2 Corrected proof

The following is a corrected version of Flawed Proof 4.15.1.

Proof 4.15.1

Since we need to choose 5 objects from a total number of 52, we obtain the following combination,

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} \ .$$

$\square$

4.16 Conditional Probability

Exercise 4.16.1

Suppose that two coins are tossed. What is the probability of observing exactly two heads given that at least one head is observed?

Flawed Proof 4.16.1

Let A denote the outcome of observing exactly two heads and B denote the outcome of observing at least one head. We want to find $P(A|B)$. Then,

$$\begin{aligned} P(A|B) &= \frac{P(A \cup B)}{P(B)} \\ &= \frac{n(A \cup B)}{n(B)} \\ &= \frac{3}{3} \\ &= 1 \; . \end{aligned}$$

□

4.16.1 Error classification

There is only one error in the Flawed Proof 4.16.1.

C-MT: The formula for conditional probability is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \ ,$$

not

$$P(A|B) = \frac{P(A \cup B)}{P(B)} \ .$$

Error codes

- Content Vocabulary and Grammar (C-VG)

See Section 1.3 for more information about error classifications.

4.16.2 Corrected proof

The following is a corrected version of Flawed Proof 4.16.1.

Proof 4.16.1

Let A denote the outcome of observing exactly two heads and B denote the outcome of observing at least one head. We want to find $P(A|B)$. Then,

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{n(A \cap B)}{n(B)} \\ &= \frac{1}{3}\,. \end{aligned}$$

$\square$

4.17 Properties of Functions: One-to-one and Onto

Exercise 4.17.1

Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(x) = 2x + 1 \ .$$

Is this function one-to-one? Is it onto? Prove or disprove.

Flawed Proof 4.17.1

This function is one-to-one. Let $a, b \in \mathbb{Z}$. Suppose that $f(a) = f(b)$. Then,

$$\begin{aligned} 2a + 1 &= 2b + 1 \\ 2a &= 2b \\ a &= b \ . \end{aligned}$$

Thus, f is one-to-one.

This function is onto. Let $y \in \mathbb{Z}$. We want to prove that $f(x) = 2x + 1 = y$. Isolating for x, we have that $x = \frac{y-1}{2}$. So choose $x = \frac{y-1}{2}$. Then $f(x)$ will always equal y. Thus, for all $x \in \mathbb{Z}$, $f(x) = y$ and so f is onto. $\square$

4.17.1 Error classification

There are several errors in the Flawed Proof 4.17.1.

F-WP: This proof would be correct if the function was defined on a different domain and codomain. In particular, the function f is both one-to-one and onto when $f : \mathbb{R} \to \mathbb{R}$.

F-FS: It is not true that $x = \frac{y-1}{2}$ will always be an integer.

Error codes

- Fundamental Wrong Problem (F-WP)
- Fundamental False Statement (F-FS)

See Section 1.3 for more information about error classifications.

4.17.2 Corrected proof

The following is a corrected version of Flawed Proof 4.17.1.

Proof 4.17.1

This function is one-to-one. Let $a, b \in \mathbb{Z}$. Suppose that $f(a) = f(b)$. Then,

$$\begin{aligned} 2a + 1 &= 2b + 1 \\ 2a &= 2b \\ a &= b\ . \end{aligned}$$

Thus, f is one-to-one.

This function is not onto. Consider the following counterexample. Choose $y = 0$, and assume that there exists $x \in \mathbb{Z}$ such that $f(x) = y$. Then $f(x) = 0 = 2x + 1$, which means that $x = \frac{-1}{2}$. But this is a contradiction since $x = -\frac{1}{2} \notin \mathbb{Z}$. Thus, for all $x \in \mathbb{Z}$, $f(x) \neq y$ and so f is not onto. □

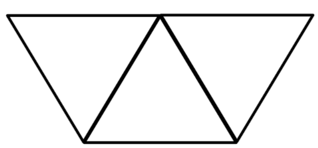
4.18 The Graph of a Matrix

Exercise 4.18.1

Draw a graph, G, associated to the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Flawed Proof 4.18.1



$\square$

4.18.1 Error classification

There are several errors in the Flawed Proof 4.18.1.

N-N: The vertices $v_1, ..., v_5$ are not labeled, which makes it difficult to interpret the graph.

N-VG: There are no explanations provided; more detail is required.

Error codes

- Novice Notation (N-N)
- Novice Vocabulary and Grammar (N-VG)

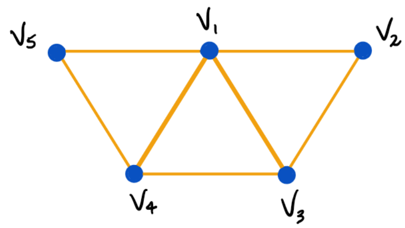
See Section 1.3 for more information about error classifications.

4.18.2 Corrected proof

The following is a corrected version of Flawed Proof 4.18.1.

Proof 4.18.1

Let • denote the vertices of the graph G.
Let — denote the edges of the graph G.
Then, from our matrix A, a possible graph, G, is given by



□

4.19 Addition and Multiplication of Congruence Relations

Exercise 4.19.1

Prove that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then

1. $(a + c) \equiv (b + d) \ (mod\ n)$,
2. $ac \equiv bd \ (mod\ n)$.

Flawed Proof 4.19.1

Suppose that $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$. By the definition of congruence, there exists integers m and k such that

$$a - b = mn \quad \text{and} \quad c - d = kn .$$

We have

$$\begin{aligned} (a + c) - (b + d) &= n(m + k) \\ a - b + c - d &= n(m + k) \end{aligned}$$

which implies that $(a + c) \equiv (b + d) \ mod\ (n)$.

We have

$$\begin{aligned} ac - bd &= n(cm + bk), \\ &= c(a - b) + b(c - d) \end{aligned}$$

which implies that $ac \equiv bd \ (mod\ n)$.

□

4.19.1 Error classification

There are several errors in the Flawed Proof 4.19.1.

C-Cir: Both portions of the proof use the conclusion in the course of the proof. In particular, the conclusions $(a+c)-(b+d)=n(m+k)$ and $ac-bd=n(cm+bk)$ are assumed at the beginning of the proof.

C-EO-Cir: Many of the important justification steps are omitted due to the arguments being circular.

F-A: The results are asserted as there are no justifications for the conclusions reached.

Error codes

- Content Circular Argument (C-Cir)
- Content Error-Caused Omission due to Circular Argument (C-EO-Cir)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

4.19.2 Corrected proof

The following is a corrected version of Flawed Proof 4.19.1.

Proof 4.19.1

Suppose that $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$. By the definition of congruence, there exists integers m and k such that

$$a - b = mn \quad \text{and} \quad c - d = kn\ .$$

First we prove that $(a + c) \equiv (b + d)\ (mod\ n)$.

We have

$$\begin{aligned} a - b + c - d &= n(m + k) \\ (a + c) - (b + d) &= n(m + k)\ , \end{aligned}$$

and so $n|(a + c) - (b + d)$ which implies that $(a + c) \equiv (b + d)\ mod\ (n)$.

Next, we prove that $ac \equiv bd\ (mod\ n)$. Here, we use that fact that $-bc + bc = 0$.

We have

$$\begin{aligned} ac - bd &= ac + 0 - bd \\ &= ac + (-bc + bc) - bd \\ &= c(a - b) + b(c - d) \\ &= c(mn) + b(kn) \\ &= n(cm + bk)\ , \end{aligned}$$

and so $n|(ac - bd)$ which implies that $ac \equiv bd\ (mod\ n)$.

□

4.20 Bayes Theorem

Exercise 4.20.1: Proof of Bayes Theorem

Let A and B be two events. Recall that $P(A \cap B) = P(A)P(B|A)$. Prove that

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)}$$

Flawed Proof 4.20.1

The information given that $P(A \cap B) = P(A)P(B|A)$ is obviously a red herring because the thing we are trying to show doesn't use intersections. It is obvious that $P(B)P(A|B) = P(A)P(B|A)$ because one side is just rearranging the other side. Now manipulating that equation means that obviously

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)}$$

like we wanted. $\square$

4.20.1 Error classification

There is only one error in the Flawed Proof 4.20.1.

F-A The assertion that $P(B)P(A|B) = P(A)P(B|A)$ is at the heart of the proof and needs an explanation.

Error codes

- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

4.20.2 Corrected proof

The following is a corrected version of Flawed Proof 4.20.1.

Proof 4.20.1

We were given that $P(A \cap B) = P(A)P(B|A)$. But note that in an intersection, A and B are interchangable. So we can also write $P(A \cap B) = P(B \cap A) = P(B)P(A|B)$. This means $P(B)P(A|B) = P(A)P(B|A)$. Rearranging this means

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)}$$

. $\square$

4.21 One-to-one Functions

Exercise 4.21.1: One-to-one from. $\mathbb{Z}$ to $\mathbb{N}$

Find a function $f : \mathbb{Z} \to \mathbb{N}$ that is one-to-one, and prove that it is one-to-one.

Flawed Proof 4.21.1

Let $f(n) = n$. Then this is one-to-one (because we did this in class) and it satisfies the things being asked. $\square$

4.21.1 Error classification

There is only one error in the Flawed Proof 4.21.1.

F-FS $f(n) = n$ does not work as this is not a well-defined function from $\mathbb{Z}$ to $\mathbb{N}$.

Error codes

- Fundamental False Statement (F-FS)

See Section 1.3 for more information about error classifications.

4.21.2 Corrected proof

The following is a corrected version of Flawed Proof 4.21.1.

Proof 4.21.1

Define $f : \mathbb{Z} \to \mathbb{N}$ as follows:

$$f(n) = \begin{cases} 2|n| + 1 & x < 0 \\ 2n & x \geq 0 \end{cases}$$

Now to show that this is one-to-one, let $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$. Now firstly note that under f, all negative integers get mapped to odd integers and all non-negative integers get mapped to even integers. So if $f(a) = f(b)$ then a, b are either both negative or both non-negative. We will consider each case

- If $a, b < 0$ then $f(a) = f(b)$ means $2|a|+1 = 2|b|+1$, which means $|a| = |b|$. Since both are negative, this means $a = b$.
- If $a, b > 0$ then $f(a) = f(b)$ means $2a = 2b$, which means $a = b$.

Therefore $f(a) = f(b)$ implies $a = b$, so f is one-to-one. $\square$

4.22 Relations

Exercise 4.22.1: Transitive Relation

For this question, $0 \notin \mathbb{N}$. Let $<$ be the relation on $\mathbb{N}$ (the natural numbers) defined by

for all $m, n \in \mathbb{N}$, $n < m$ iff there exists some $p \in \mathbb{N}$ such that $n + p = m$

Prove that this is a transitive relation, but not a symmetric relation.

Flawed Proof 4.22.1

Let $a < b$ and $b < c$. Then of course $a < c$ so this is transitive.

I can show this isnt symmetric because its not possible for $a < b$ and $b < a$. $\square$

4.22.1 Error classification

There is only one error in the Flawed Proof 4.22.1.

F-A Entire result was asserted.

N-VG Use of the first person pronoun "I" is typically bad form in a mathematical proof.

C-N a, b and c are not defined.

Error codes

- Fundamental Assertion (F-A)
- Novice Vocabulary and Grammar (N-VG)
- Content Notation (C-N)

See Section 1.3 for more information about error classifications.

4.22.2 Corrected proof

The following is a corrected version of Flawed Proof 4.22.1.

Proof 4.22.1

Let $a, b, c \in \mathbb{N}$ such that $a < b$ and $b < c$. We need to show this implies $a < c$. Since $a < b$, there exists some $p \in \mathbb{N}$ such that $a + p = b$. Also since $b < c$ there exists some $q \in \mathbb{N}$ such that $b+q = c$. Now substituting gives $a + p + q = c$. Now since $p, q \in \mathbb{N}$, $p + q \in \mathbb{N}$. So let $r = p + q$. Then $a + r = c$, so $a < c$.

To show this is not a symmetric relation, suppose there exists $a, b \in \mathbb{N}$ such that $a < b$ and $b < a$. Then there exists natural numbers p, q such that $a + q = b$ and $b + p = a$. Substituting gives $b + p + q = b$, or that $p + q = 0$. This is not possible for natural numbers p, q. Therefore, this relation is not symmetric. $\square$

4.23 Equivalence Relations

Exercise 4.23.1: Modular Congruence is Transitive

Prove that modular congruence is transitive; that is, for integers a, b, c, n, prove that if $a \equiv b \mod n$ and $b \equiv c \mod n$, then $a \equiv c \mod n$.

Flawed Proof 4.23.1

Since $a \equiv b \mod n$, then $a - b = nk$. Now if we let $n = c - b$ and $k = 1$, then $a - b = c - b = nk$, or $a - c = nk$. So $a \equiv c \mod n$. □

4.23.1 Error classification

There are several errors in the Flawed Proof 4.23.1.

N-N k is not defined.

C-Eg The statement needs to be proven for all n, but the proof attempts to prove the statement for a specific n

F-FI That $a - b = c - b = nk$ implies $a - c = nk$ is false.

Error codes

- Novice Notation (N-N)
- Content Proof by Example (C-Eg)
- Fundamental False Implication (F-FI)

See Section 1.3 for more information about error classifications.

4.23.2 Corrected proof

The following is a corrected version of Flawed Proof 4.23.1.

Proof 4.23.1

For integers a, b, c, n suppose that $a \equiv b \mod n$ and $b \equiv c \mod n$. This means there exists integer k, l such that $a - b = nk$ and $b - c = nl$. Solving for b in the second equation gives $b = nl + c$. Substituting into the second equation means $a - nl - c = nk$, so $a - c = nl + nk = n(l + k)$. Thus n divides $a - c$, so by definition $a \equiv c \mod n$. $\square$

4.24 Set Containment Version Two

Exercise 4.24.1

Prove or disprove the following statement. For all sets A, B and C,

$$A \cap (B \cup C) \subseteq (A \cap B) \cup C\ .$$

Flawed Proof 4.24.1

Suppose A, B and C are sets. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we can arbitrarily choose to consider $x \in B$. Suppose that $x \in B$. Then $x \in A$ and $x \in B$, so $x \in A \cap B$. Thus, $x \in (A \cap B) \cup C$. Therefore, We can conclude that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup C\ .$$

□

4.24.1 Error classification

There are several errors in the Flawed Proof 4.24.1.

C-VG: Misunderstanding of the definition of the union operation, evidenced by the statement: 'Since $x \in B \cup C$, we can arbitrarily choose to consider $x \in B$.' Both cases must be considered.

C-OS: The case where $x \in C$ is omitted.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Content Omitted Section (C-OS)

See Section 1.3 for more information about error classifications.

4.24.2 Corrected proof

The following is a corrected version of Flawed Proof 4.24.1.

Proof 4.24.1

Suppose A, B and C are sets. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we know that $x \in B$ or $x \in C$. Thus, we have two cases to consider:

Case 1: Suppose that $x \in B$. Then $x \in A$ and $x \in B$, so $x \in A \cap B$. Thus, $x \in (A \cap B) \cup C$.
Case 2: Suppose that $x \in C$. Since $x \in C$, then $x \in (A \cap B) \cup C$.

Thus, in both cases $x \in (A \cap B) \cup C$. Therefore, We can conclude that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup C\ .$$

□

4.25 Surjection

Exercise 4.25.1

Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{1\}$ defined by

$$f(x) = \frac{x + 15}{x} .$$

Is f surjective? Prove or disprove.

Flawed Proof 4.25.1

Yes, f is surjective since for all $y \in \mathbb{R} - \{1\}$, there exists an $x \in \mathbb{R} - \{0\}$ such that $y = f(x)$. Namely,

$$\begin{aligned} y =& f(x) \\ y =& \frac{x + 15}{x} \\ yx =& x + 15 \\ yx - x =& 15 \\ x(y - 1) =& 15 \\ x =& \frac{15}{y - 1} . \end{aligned}$$

Thus, we can see that f is surjective. $\square$

4.25.1 Error classification

There are several errors in the Flawed Proof 4.25.1.

EO-F-A: Although the method of finding x is correct, the entirety of the proof is omitted (that is, verifying that $f(x) = y$ with the particular x that was calculated) due to the assumption that $y = f(x)$.

C-VG: Misunderstanding the definition of surjective.

Error codes

- Error-Caused Omission Fundamental Assertion (EO-F-A)
- Content Vocabulary and Grammar

See Section 1.3 for more information about error classifications.

4.25.2 Corrected proof

The following is a corrected version of Flawed Proof 4.25.1.

Proof 4.25.1

Yes, f is surjective. To prove this, we begin by finding a suitable $x \in \mathbb{R} - \{0\}$:

$$\begin{aligned} y =& f(x) \\ y =& \frac{x+15}{x} \\ yx =& x+15 \\ yx - x =& 15 \\ x(y-1) =& 15 \\ x =& \frac{15}{y-1} \ . \end{aligned}$$

So choose $x = \frac{15}{y-1}$ with $x \in \mathbb{R} - \{0\}$. Then,

$$f\left(\frac{15}{y-1}\right) = \frac{\frac{15}{y-1} + 15}{\frac{15}{y-1}} = \frac{15 + 15(y-1)}{15} = \frac{15 + 15y - 15}{15} = y \ .$$

Thus, we can see that f is surjective since for all $y \in \mathbb{R} - \{1\}$, there exists an $x \in \mathbb{R} - \{0\}$ such that $f(x) = y$. $\square$

4.26 Injection

Exercise 4.26.1

Consider $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 3x - 4$ and $g(n) = n^4$ for all $n \in \mathbb{Z}$. Is f injective? Is g injective? Prove or disprove.

Flawed Proof 4.26.1

Yes, f is injective. Suppose that $a, b \in \mathbb{R}$ and that $f(a) = f(b)$. Then:

$$\begin{aligned} f(a) &= f(b) \\ 3a - 4 &= 3b - 4 \\ 3a &= 3b \\ a &= b\,. \end{aligned}$$

Thus, f is injective.

Yes, g is injective. Suppose that $a, b \in \mathbb{Z}$ and that $g(a) = g(b)$. Then:

$$\begin{aligned} g(a) &= g(b) \\ \sqrt[4]{a^4} &= \sqrt[4]{b^4} \\ a &= b\,. \end{aligned}$$

Thus, g is injective. $\square$

4.26.1 Error classification

There is only one error in the Flawed Proof 4.26.1.

EO-C-FS The entirely of the proof was omitted due to the false statement that $\sqrt[4]{a^4} = \sqrt[4]{b^4} \implies a = b$. Namely, it was assumed that a and b are positive.

Error codes

- Error-cased Omission Content False Statement

See Section 1.3 for more information about error classifications.

4.26.2 Corrected proof

The following is a corrected version of Flawed Proof 4.26.1.

Proof 4.26.1

Yes, f is injective. Suppose that $a, b \in \mathbb{R}$ and that $f(a) = f(b)$. Then:

$$\begin{aligned} f(a) &= f(b) \\ 3a - 4 &= 3b - 4 \\ 3a &= 3b \\ a &= b\ . \end{aligned}$$

Thus, f is injective.

No, g is not injective. To prove this, we will consider the following counterexample. Let $a = 2$ and $b = -2$. Then:

$$\begin{aligned} g(a) &= g(b) \\ 16 &= 16 \\ 2^4 &= (-2)^4\ , \end{aligned}$$

and so $g(a) = g(b) = 16$, but $a = 2 \neq -2 = b$. Thus, g is not injective.

$\square$

Chapter 5

Linear Algebra

The material in this chapter is designed to fit within a typical second course in abstract linear algebra and is based on the material covered in MATH 311: Linear Methods II at the University of Calgary.

Topics include: subspaces, span, linear independence, bases and dimension in finite dimensional real vector spaces; eigenvalues, eigenvectors and orthogonality in $\mathbb{R}^n$, row and column spaces, the Rank-Nullity theorem, the Gram-Schmidt Algorithm, similar matrices, orthogonal diagonalization; linear transformations, kernel, image, injections and surjections, isomorphisms, inverses, the matrix of a linear transformation and change of basis.

The following topics, which may appear in your course, are omitted: complex vector spaces, inner products, positive definite matrices, canonical matrix forms (i.e., Jordan Canonical Form, etc.), infinite dimensional vector spaces, and many others.

5.1 Skew-Symmetric Matrices

Exercise 5.1.1

Decide whether the following statement is TRUE or FALSE. If it's true, then prove it. If it's false, then find an explicit counterexample.

Every 2×2 skew-symmetric matrix has a determinant of zero.

Flawed Proof 5.1.1

The statement is false. To see this, suppose we had a 2×2 skew-symmetric matrix A. Then we can write $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ for some $a \in \mathbb{R}$. A simple computation gives $\det(A) = a^2$, and squares are always positive. $\square$

5.1.1 Error classification

There are several errors in the Flawed Proof 5.1.1.

WM The student is trying to prove that the statement is always false, but the prompt says to find an explicit counterexample if the statement is false.

EO-(F-FS): An explicit counterexample is omitted due to the false statement that a^2 is always positive.

C-A: The claim that every 2×2 skew-symmetric matrix A can be written in that specific form requires more justification.

Error codes

- Wrong Method (WM)
- Error-caused Omission due to Fundamental False Statement (EO-(F-FS))
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

5.1.2 Corrected proof

The following is a corrected version of Flawed Proof 5.1.1.

Proof 5.1.1

This statement is false. Indeed, consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This matrix A is skew-symmetric since $A = -A^T$, but $\det(A) = 1 \neq 0$. $\square$

5.2 Intersection of Subspaces

Exercise 5.2.1: Intersection of Subspaces

Let V and W be subspaces of $\mathbb{R}^n$. Prove that $V \cap W$ is a subspace of $\mathbb{R}^n$.

Flawed Proof 5.2.1

To show that $V \cap W$ is a subspace, we will show that it contains the zero vector, is closed under addition and is closed under scalar multiplication.

Firstly, since V and W are subspaces of $\mathbb{R}^n$, we know that $\mathbf{0} \in V$ and $\mathbf{0} \in W$. This means $\mathbf{0} \in V \cap W$.

Secondly, let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V \cap W$. Since V and W are subspaces this means $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in V \cap W$.

Thirdly, let $\boldsymbol{v} \in V \cap W$ and $k \in \mathbb{R}$. Since V and W are subspaces this means $k\boldsymbol{v} \in V \cap W$.

$\square$

5.2.1 Error classification

There are several errors in the Flawed Proof 5.2.1.

C-A The assertion "let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V \cap W$. Since V and W are subspaces this means $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in V \cap W$" is true, but has not been sufficiently proven.

C-A The assertion "let $\boldsymbol{v} \in V \cap W$ and $k \in \mathbb{R}$. Since V and W are subspaces this means $k\boldsymbol{v} \in V \cap W$" is true, but has not been sufficiently proven.

Error codes

- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

5.2.2 Corrected proof

The following is a corrected version of Flawed Proof 5.2.1.

Proof 5.2.1

To show that $V \cap W$ is a subspace, we will show that it contains the zero vector, is closed under addition and is closed under scalar multiplication.

Firstly, since V and W are subspaces of $\mathbb{R}^n$, we know that $\mathbf{0} \in V$ and $\mathbf{0} \in W$. This means $\mathbf{0} \in V \cap W$.

Secondly, let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V \cap W$. This means $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in W$. Since V and W are subspaces they are both closed under addition, so $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in V$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in W$. This means $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in V \cap W$.

Thirdly, suppose $\boldsymbol{v} \in V \cap W$. So $\boldsymbol{v} \in V$ and $\boldsymbol{v} \in W$. Let $k \in \mathbb{R}$. Since V and W are subspaces they are both closed under scalar multiplication, so $k\boldsymbol{v} \in V$ and $k\boldsymbol{v} \in W$. This means $k\boldsymbol{v} \in V \cap W$. $\square$

5.3 Sum of Subspaces

Exercise 5.3.1

Prove or disprove the following statement. If V and W are both subspaces of $\mathbb{R}^n$, then $V + W$ is also a subspace of $\mathbb{R}^n$.

Flawed Proof 5.3.1

This statement is true. We will use the definition of a subspace to prove that $V + W$ is a subspace.

1. Contains $\vec{0}$: Take $\vec{0} \in V$ and $\vec{0} \in W$. Then $\vec{0} + \vec{0} = \vec{0} \in V + W$.

2. Closed Under Addition: Let $\vec{v} \in V$ and $\vec{w} \in W$. Then $\vec{v} + \vec{w} \in V + W$, which means that $V + W \in \mathbb{R}^n$.

3. Closed Under Scalar Multiplication: Let $\vec{v} \in V + W$ and $w \in \mathbb{R}$. Then $\vec{v}(w) = \vec{v}w \in V + W$.

$\square$

5.3.1 Error classification

There are several errors in the Flawed Proof 5.3.1.

EO-(C-VG): Misunderstanding what it means to be 'closed' under addition and scalar multiplication leading to omission of major parts of the proof. Start with elements that are in $V + W$ and show that they are still in $V + W$ after these operations are performed.

N-N: To avoid confusing notation, choose a different letter for the scalar.

Error codes

- Error-Caused Omission due to Content Vocabulary & Grammar (EO-(C-VG))
- Novice Notation (N-N)

See Section 1.3 for more information about error classifications.

5.3.2 Corrected proof

The following is a corrected version of Flawed Proof 5.3.1.

Proof 5.3.1

This statement is true. We will use the definition of a subspace to prove that $V + W$ is a subspace.

1. Contains $\vec{0}$: Since both V and W are subspaces, take $\vec{0} \in V$ and $\vec{0} \in W$. Then $\vec{0} + \vec{0} = \vec{0} \in V + W$.

2. Closed Under Addition: Let $\vec{x}, \vec{y} \in V + W$. By definition, there exists $\vec{v_1}, \vec{v_2} \in V$ and $\vec{w_1}, \vec{w_2} \in W$ such that $\vec{x} = \vec{v_1} + \vec{w_1}$ and $\vec{y} = \vec{v_2} + \vec{w_2}$. Then

$$\vec{x} + \vec{y} = (\vec{v_1} + \vec{w_1}) + (\vec{v_2} + \vec{w_2}) = (\vec{v_1} + \vec{v_2}) + (\vec{w_1} + \vec{w_2}).$$

   Since V is a subspace, it is closed under addition and so $(\vec{v_1} + \vec{v_2}) \in V$. Similarly, $(\vec{w_1} + \vec{w_2}) \in W$. This means that $(\vec{v_1} + \vec{v_2}) + (\vec{w_1} + \vec{w_2}) \in V + W$. Thus, $\vec{x} + \vec{y} \in V + W$ and so $V + W$ is closed under addition.

3. Closed Under Scalar Multiplication: Let $\vec{x} \in V + W$ and $k \in \mathbb{R}$. By definition, there exists $\vec{v} \in V$ and $\vec{w} \in W$ such that $\vec{x} = \vec{v} + \vec{w}$. Then

$$k\vec{x} = k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}.$$

   Since V is a subspace, it is closed under scalar multiplication and so $k\vec{v} \in V$. Similarly, $k\vec{w} \in W$. This means that $k\vec{v} + k\vec{w} \in V + W$. Thus, $k\vec{x} \in V + W$ and so $V + W$ is closed under scalar multiplication.

$\square$

5.4 Subspaces and Nullspace

Subspace Test: A set $U \subseteq \mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if it satisfies the following:

- Zero Vector: $\vec{0} \in U$.
- Closed Under Addition: $\vec{x}, \vec{y} \in U \Rightarrow \vec{x} + \vec{y} \in U$.
- Closed Under Scalar Multiplication: $\vec{x} \in U \Rightarrow k\vec{x} \in U \, \forall \, k \in \mathbb{R}$.

Exercise 5.4.1

Let A be an $m \times n$ matrix. Prove that null(A) is a subspace of $\mathbb{R}^n$ by using the definition of a subspace (i.e. the Subspace Test).

Flawed Proof 5.4.1

$$\text{null}(A) = \{\vec{x} \,|\, A\vec{x} = \vec{0}\}.$$

For a set to be a subspace 3 conditions must hold:

1. closed under addition
2. closed under multiplication
3. must contain $\vec{0}$

Members of nul(A) will have $\dim(\text{null}(A)) = n$ since A is an $m \times n$ matrix.

Thus $(\vec{x} + \vec{y}) \in \mathbb{R}^n$ for any arbitrary $\vec{y} \in \mathbb{R}^n$ holds.

And $(k\vec{x}) \in \mathbb{R}$ for any arbitrary scalar $\in \mathbb{R}$ holds.

And $\vec{x} = \vec{0}$ is implied by the line above where $k = 0$.

$\square$

5.4.1 Error classification

There are several errors in the Flawed Proof 5.4.1.

C-FI: The matrix A having size $m \times n$ does not imply $\dim(\text{null}(A)) = n$.

EO-(C-VG): Misunderstanding of what it means for a set to be closed under addition and closed under scalar multiplication leads to omission of major parts of the proof.

C-A Claiming closed under scalar multiplication implies contains the zero vector, but hasn't shown null(A) is nonempty.

N-O The variable $\vec{x}$ is not defined.

N-N Using $\in$ in the middle of a sentence: "for any arbitrary scalar $\in \mathbb{R}$".

N-FI The claim $(k\vec{x}) \in \mathbb{R}$ is false.

WM Not showing that the zero vector is contained in null(A) using the definition of a subspace as the prompt requires.

Error codes

- Content False Implication (C-FI)
- Error-Caused Omission due to Content Vocabulary & Grammar (EO-(C-VG)) (EO-(C-VG))
- Content Assertion (C-A)
- Novice Local Omission (N-O)
- Novice Notation (N-N)
- Novice False Implication
- Wrong Method (WM)

See Section 1.3 for more information about error classifications.

5.4.2 Corrected proof

The following is a corrected version of Flawed Proof 5.4.1.

Proof 5.4.1

By definition,

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

The zero vector $\vec{0} \in \text{null}(A)$, since $A\vec{0} = \vec{0}$.

Suppose $\vec{x}, \vec{y} \in \text{null}(A)$. By definition, this means $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$. Hence,

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A\vec{x} + A\vec{y} \\ &= \vec{0} + \vec{0} \\ &= \vec{0}, \end{aligned}$$

and hence $\vec{x} + \vec{y} \in \text{null}(A)$.

Let $k \in \mathbb{R}$ and $\vec{x} \in \text{null}(A)$. Then

$$\begin{aligned} A(k\vec{x}) &= k(A\vec{x}) \\ &= k\vec{0} \\ &= \vec{0}, \end{aligned}$$

and hence $k\vec{x} \in \text{null}(A)$.

Therefore $\text{null}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

5.5 Linear Independence

Let $\underline{\mathbf{0}}$ denote the $m \times n$ zero matrix.

Exercise 5.5.1

Let C be a nonzero $m \times n$ matrix and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell$ be nonzero vectors in $\mathbb{R}^n$. Prove that if $\{C\boldsymbol{v}_1, C\boldsymbol{v}_2, \ldots, C\boldsymbol{v}_\ell\}$ is linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent.

Flawed Proof 5.5.1

Suppose $\{C\boldsymbol{v}_1, C\boldsymbol{v}_2, \ldots, C\boldsymbol{v}_\ell\}$ is linearly independent. Then

$$a_1(C\boldsymbol{v}_1) + a_2(C\boldsymbol{v}_2) + \ldots + a_\ell(C\boldsymbol{v}_\ell) = \vec{0}_m$$

implies that $a_1 = a_2 = \ldots = a_\ell = 0$. Now, we have that

$$\vec{0}_m = a_1(C\boldsymbol{v}_1) + a_2(C\boldsymbol{v}_2) + \ldots + a_\ell(C\boldsymbol{v}_\ell) = C(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell)$$

and, since $C \neq \underline{\mathbf{0}}$, it follows that $\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$. We know that $a_1 = a_2 = \ldots = a_\ell = 0$. This implies that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent. $\square$

5.5.1 Error classification

There are several errors in the Flawed Proof 5.5.1.

F-Log: The Flawed Proof 5.5.1 incorrectly begins by considering a linear combination of the vectors $C\boldsymbol{v}_1, C\boldsymbol{v}_2, \ldots, C\boldsymbol{v}_\ell$. In order to prove that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent, we must prove that: if $\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$ for some scalars $a_i \in \mathbb{R}$, $1 \le i \le \ell$, then $a_i = 0$ for all $1 \le i \le \ell$. In order to prove this statement, we must begin with the assumption that "$\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$ for some scalars $a_i \in \mathbb{R}$, $1 \le i \le \ell$".

N-O: The coefficients $a_1, a_2, \ldots, a_\ell$ are undefined.

C-FI: The implication "since $C \neq \underline{\mathbf{0}}$, it follows that $\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$" is false. In this setting, the claim that $\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$ is equivalent to stating that $\text{null}(C) = \{\vec{0}_n\}$. But the fact that C is a nonzero matrix does not imply that the nullspace of C is equal to $\{\vec{0}_n\}$. For example, the nonzero matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has nullspace $\text{null}(C) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

C-FI: The implication "We know that $a_1 = a_2 = \ldots = a_\ell = 0$. This implies that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent." is false. The fact that $\vec{0}_n = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \ldots + 0\boldsymbol{v}_\ell$ does not imply that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent. To prove that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent, one must show that $a_1 = a_2 = \ldots = a_\ell = 0$ is the **only solution** to the equation $\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$, where $a_i \in \mathbb{R}$, $1 \le i \le \ell$.

Error codes

- Fundamental Logical Order (F-Log)
- Novice Local Omission (N-O)
- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

5.5.2 Corrected proof

The following is a corrected version of Flawed Proof 5.5.1.

Proof 5.5.1

Let C be a nonzero $m \times n$ matrix and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell$ be nonzero vectors in $\mathbb{R}^n$. Suppose that $\{C\boldsymbol{v}_1, C\boldsymbol{v}_2, \ldots, C\boldsymbol{v}_\ell\}$ is linearly independent. Suppose that

$$\vec{0}_n = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell$$

for some scalars $a_i \in \mathbb{R}$, $1 \leq i \leq \ell$. Multiplying this equation by the $m \times n$ matrix C we obtain

$$\begin{aligned}
\vec{0}_m &= C\,\vec{0}_n \\
&= C(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_\ell\boldsymbol{v}_\ell) \\
&= a_1(C\boldsymbol{v}_1) + a_2(C\boldsymbol{v}_2) + \ldots + a_\ell(C\boldsymbol{v}_\ell).
\end{aligned}$$

Since $\{C\boldsymbol{v}_1, C\boldsymbol{v}_2, \ldots, C\boldsymbol{v}_\ell\}$ is linearly independent, it must be the case that $a_i = 0$ for all $1 \leq i \leq \ell$. Thus, the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_\ell\}$ is linearly independent. $\square$

5.6 Linear Independence of Subsets

Exercise 5.6.1

Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a subset of $\mathbb{R}^m$. Let $T = \{\vec{v}_1, ..., \vec{v}_k\}$ where $k < n$. Prove that if S is linearly independent, then T is linearly independent.

Flawed Proof 5.6.1

Suppose that $\{v_1, v_2, ..., v_n\}$ is a linearly independent set and that $\{v_1, ..., v_k\}$ is a subset where $k < n$. Suppose

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0 \ ,$$

for scalars $a_1, ..., a_n$. Since $\{v_1, ..., v_k\}$ is linearly independent, this implies that $a_1 = \cdots = a_n = 0$. Consider

$$a_1 v_1 + a_2 v_2 + \cdots a_k v_k = 0.$$

Since $a_1 = \cdots = a_n = 0$, we know that $a_1 = \cdots = a_k = 0$. Hence $\{v_1, ..., v_k\}$ is linearly independent. $\square$

5.6.1 Error classification

There are several errors in the Flawed Proof 5.6.1.

EO-(F-Log): No progress is made towards the proof due to a logical error: the proof begins incorrectly by supposing that $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$.

N-N: The zero vector in the equation $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$ should have a vector hat to distinguish it from the scalar 0. It would also be good to give the v_i's vector hats.

Error codes

- Error-Caused Omission due to Fundamental Logical Order (EO-(F-Log))
- Novice Notation (N-N)

See Section 1.3 for more information about error classifications.

5.6.2 Corrected proof

The following is a corrected version of Flawed Proof 5.6.1.

Proof 5.6.1

Suppose that $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is a linearly independent subset of $\mathbb{R}^m$. Let $T = \{\vec{v}_1, ..., \vec{v}_k\}$, where $k < n$. If

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_k\vec{v}_k = \vec{0} \ ,$$

for scalars $a_1, ..., a_k \in \mathbb{R}$, then we have

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_k\vec{v}_k + (0\vec{v}_{k+1} + 0\vec{v}_{k+2} + ... + 0\vec{v}_n) = \vec{0} \ .$$

Since $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is linearly independent, we must have that $a_1 = \cdots = a_k = 0$. Thus, $T = \{\vec{v}_1, ..., \vec{v}_k\}$ is linearly independent. $\square$

5.7 Uniqueness of Basis Representation

Exercise 5.7.1: Uniqueness of Basis Representation

Let $S = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ be a basis of $\mathbb{R}^n$. Prove that every vector $\boldsymbol{x} \in \mathbb{R}^n$ can be expressed in the form $\boldsymbol{x} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n$ in exactly one way, where $c_1, ..., c_n \in \mathbb{R}$.

Flawed Proof 5.7.1

By definition of a basis, we can write every vector $\boldsymbol{x} \in \mathbb{R}^n$ in the form $\boldsymbol{x} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n$ for $c_1, ..., c_n \in \mathbb{R}^n$. Suppose we had another basis $T = \{\boldsymbol{w}_1, ..., \boldsymbol{w}_n\}$ such that $\boldsymbol{x} = c_1\boldsymbol{w}_1 + ... + c_n\boldsymbol{w}_n$. This means that $c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n = c_1\boldsymbol{w}_1 + ... + c_n\boldsymbol{w}_n$. Rearranging means

$$c_1(\boldsymbol{v}_1 - \boldsymbol{w}_1) + ... + c_n(\boldsymbol{v}_n - \boldsymbol{w}_n) = \vec{0}.$$

Because these are bases they are linearly independent, so $c_1 = ... = c_n = 0$. This means that $\boldsymbol{x} = 0$. So the only vector that can be written in multiple ways is the zero vector. $\square$

5.7.1 Error classification

There are several errors in the Flawed Proof 5.7.1.

WM Introducing a second basis T is irrelevant to the problem.

F-N The double use of $c_1, ..., c_n$ as coefficients, which should be in $\mathbb{R}$ and not $\mathbb{R}^n$, in linear combinations is incorrect; different coefficients should be used for each basis.

C-FI The assertion that $c_1(\boldsymbol{v}_1 - \boldsymbol{w}_1) + ... + c_n(\boldsymbol{v}_n - \boldsymbol{w}_n) = \vec{0}$ implies $c_1 = ... = c_n = 0$ is incorrect. Although S and T are bases, we cannot say anything about $\boldsymbol{v}_1 - \boldsymbol{w}_1, ..., \boldsymbol{v}_n - \boldsymbol{w}_n$.

Error codes

- Wrong Method (WM)
- Fundamental Notation (F-N)
- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

5.7.2 Corrected proof

The following is a corrected version of Flawed Proof 5.7.1.

Proof 5.7.1

By definition of a basis, we can write every vector $\boldsymbol{x} \in \mathbb{R}^n$ in the form $\boldsymbol{x} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n$ for $c_1, ..., c_n \in \mathbb{R}$. Suppose we could also write it as $\boldsymbol{x} = d_1\boldsymbol{v}_1 + ... + d_n\boldsymbol{v}_n$ for $d_1, ..., d_n \in \mathbb{R}$. Equating the two equations gives

$$c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n = d_1\boldsymbol{v}_1 + ... + d_n\boldsymbol{v}_n$$

and rearranging gives

$$(c_1 - d_1)\boldsymbol{v}_1 + ... + (c_n - d_n)\boldsymbol{v}_n = \vec{0}.$$

Because S is linearly independent, this implies $c_1 - d_1 = 0, ..., c_n - d_n = 0$, which further implies $c_1 = d_1, ..., c_n = d_n$. So the two ways of writing $\boldsymbol{x}$ are the same.

$\square$

5.8 Orthogonality and Linear Independence

Exercise 5.8.1

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$. Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Flawed Proof 5.8.1

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Since this set is orthogonal, then for any vectors in the set, their dot product is equal to zero. Now suppose that $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$. We want to prove that $t_1 = t_2 = t_3 = 0$. We will do this by taking dot products. So we have

$$\begin{aligned}
(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_1 = t_1\vec{v}_1 \cdot \vec{v}_1 \implies t_1 = 0\ , \\
(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_2 = t_2\vec{v}_2 \cdot \vec{v}_2 \implies t_2 = 0\ , \\
(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_3 = t_3\vec{v}_3 \cdot \vec{v}_3 \implies t_3 = 0\ .
\end{aligned}$$

Thus, it must be the case that $t_1 = t_2 = t_3 = 0$ and so we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

□

5.8.1 Error classification

There are several errors in the Flawed Proof 5.8.1.

C-VG: The definition of orthogonal set is not correctly stated. The flawed proof states "for any vectors in the set, their dot product is equal to zero", but they should have stated "$\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$".

N-O: Did not define t_1, t_2, t_3. Moreover, both sides of the equation $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$ should have been dotted with $\vec{v}_i$ for $1 \leq i \leq 3$, however only the right-hand side was written.

C-A: The assertion $t_i\vec{v}_i \cdot \vec{v}_i = 0$ implies $t_i = 0$ requires justification. This holds because we know that the vectors in our set are nonzero, by definition of an orthogonal set.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Novice Local Omission (N-O)
- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

5.8.2 Corrected proof

The following is a corrected version of Flawed Proof 5.8.1.

Proof 5.8.1

Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Suppose that $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$ for some $t_1, t_2, t_3 \in \mathbb{R}$. By the definition of orthogonality, we know

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = \vec{0}$$

and

$$\vec{v}_1, \vec{v}_2, \vec{v}_3 \neq \vec{0}.$$

Since these vectors are nonzero, we know $\|\vec{v}_i\|^2 \neq 0$ for all $1 \leq i \leq 3$. Hence, it follows that

$$\begin{aligned}
0 = \vec{0} \cdot \vec{v}_1 &= (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_1 = t_1\vec{v}_1 \cdot \vec{v}_1 = t_1\|\vec{v}_1\|^2 \implies t_1 = 0\ , \\
0 = \vec{0} \cdot \vec{v}_2 &= (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_2 = t_2\vec{v}_2 \cdot \vec{v}_2 = t_2\|\vec{v}_2\|^2 \implies t_2 = 0\ , \\
0 = \vec{0} \cdot \vec{v}_3 &= (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_3 = t_3\vec{v}_3 \cdot \vec{v}_3 = t_3\|\vec{v}_3\|^2 \implies t_3 = 0\ .
\end{aligned}$$

Therefore, we have $t_1 = t_2 = t_3 = 0$, and so we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. $\square$

5.9 Similar Matrices

Exercise 5.9.1: Similar Matrices and Eigenvalues

Prove similar matrices have the same set of eigenvalues.

Flawed Proof 5.9.1

Since they are similar this means there exists some invertible matrix P such that $A = P^{-1}BP$. The characteristic polynomial for B is given by $\det(B - I\lambda)$. Now making substitutions we can get

$$\begin{aligned}
\det(B - I\lambda) &= \det(P^{-1}AP - II\lambda) \\
&= \det(P^{-1}AP) - \det(P^{-1}PI\lambda) \\
&= \det(P^{-1})\det(A)\det(P) - \det(P^{-1})\det(P)\det(I\lambda) \\
&= \det(P^{-1})\big(\det(A) - \det(I\lambda)\big)\det(P) \\
&= \det(P^{-1})\det(A - I\lambda)\det(P) \\
&= \det(P)^{-1}\det(P)\det(A - I\lambda) \\
&= \det(A - I\lambda).
\end{aligned}$$

Thus $\det(A-I\lambda) = \det(B-I\lambda)$, so A and B have the same characteristic polynomial and hence the same eigenvalues. $\square$

5.9.1 Error classification

There are several errors in the Flawed Proof 5.9.1.

N-O A and B are not defined.

N-N Initially the proof writes $A = P^{-1}BP$, but later the proof makes the substitution $B = P^{-1}AP$.

C - MT The flawed proof assumes that determinants distribute over addition/subtraction, which is not the case.

Error codes

- Novice Local Omission (N-O)
- Novice Notation (N-N)
- Content Misusing Theorem (C-MT)

See Section 1.3 for more information about error classifications.

5.9.2 Corrected proof

The following is a corrected version of Flawed Proof 5.9.1.

Proof 5.9.1

Let A and B be similar matrices. We will show that they have the same characteristic polynomial, and hence the same eigenvalues. Since A and B are similar this means there exists some invertible matrix P such that $A = P^{-1}BP$. The characteristic polynomial for A is given by $\det(A - I\lambda)$. Now making substitutions we have

$$\begin{aligned}
\det(A - I\lambda) &= \det(P^{-1}BP - II\lambda) \\
&= \det(P^{-1}BP - P^{-1}PI\lambda) \\
&= \det(P^{-1}BP - P^{-1}I\lambda P) \\
&= \det(P^{-1}(B - I\lambda)P) \\
&= \det(P^{-1})\det(B - I\lambda)\det(P) \\
&= \frac{1}{\det(P)}\det(B - I\lambda)\det(P) \\
&= \det(B - I\lambda).
\end{aligned}$$

Thus $\det(A-I\lambda) = \det(B-I\lambda)$, so A and B have the same characteristic polynomial and hence the same eigenvalues. $\square$

5.10 Column Space

Exercise 5.10.1: Column Space Containment

Let A and B be $n \times n$ matrices. Prove that $Col(AB) \subseteq Col(A)$.

Flawed Proof 5.10.1

The column space of AB is the set of linear combinations of the columns of AB. The column space of A is the set of linear combinations of A. One will notice that the columns of AB are linear combinations of the columns of A. So linear combinations of the columns of AB are really linear combinations of columns of A. This means that $Col(AB) \subseteq Col(A)$. $\square$

5.10.1 Error classification

There are several errors in the Flawed Proof 5.10.1.

N-A Although the proof has the right idea, more explicit details are needed. It is just a "sketch" of a proof.

N-FS The statement "The column space of A is the set of linear combinations of A" is an apparent typo. It should be the set of linear combinations of the **columns** of A.

Error codes

- Novice Assertion (N-A)
- Novice False Statement (N-FS)

See Section 1.3 for more information about error classifications.

5.10.2 Corrected proof

The following is a corrected version of Flawed Proof 5.10.1.

Proof 5.10.1

Let $\boldsymbol{v} \in Col(AB)$. Recall that this means there exists some vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $AB\boldsymbol{x} = \boldsymbol{v}$. We know $\boldsymbol{v}$ would be in the column space of A if there exists some $\boldsymbol{y}$ such that $A\boldsymbol{y} = \boldsymbol{v}$. Take $\boldsymbol{y} = B\boldsymbol{x} \in \mathbb{R}^n$. We get $\boldsymbol{v} = AB\boldsymbol{x} = A\boldsymbol{y}$, so this means $\boldsymbol{v} \in Col(A)$.

This means any element of $Col(AB)$ is in $Col(A)$, so $Col(AB) \subseteq Col(A)$.

$\square$

5.11 Properties of Column Space

Exercise 5.11.1

Let $Col(A)$ denote the column space of an $m \times n$ matrix A. Let U be an $n \times n$ invertible matrix. Prove that $Col(AU) = Col(A)$.

Flawed Proof 5.11.1

$$\begin{aligned} b &= AUx \\ &= A(Ux) \end{aligned}$$

So $b \in Col(A)$ and $b \in Col(AU)$. $\square$

5.11.1 Error classification

There are several errors in the Flawed Proof 5.11.1.

C-WR: Proved a weaker result. Namely, proved that $Col(AU) \subseteq Col(A)$ instead of proving that $Col(AU) = Col(A)$.

C-OS: Omitted the section $Col(A) \subseteq Col(AU)$.

N-O: Did not define a single element of the proof (e.g. what are x and b?).

N-VG: Full sentences should be used to help guide the reader through the proof. (e.g. At the beginning it would be helpful to say, "First we will show that $Col(AU) \subseteq Col(A)$.")

Error codes

- Content Weakened Result (C-WR)
- Content Omitted Sections (C-OS)
- Novice Local Omission (N-O)
- Novice Vocabulary and Grammar (N-VG)

See Section 1.3 for more information about error classifications.

5.11.2 Corrected proof

The following is a corrected version of Flawed Proof 5.11.1.

Proof 5.11.1

First we'll show that $Col(AU) \subseteq Col(A)$. Suppose that $\vec{b} \in Col(AU)$. Then, there exists an $\vec{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} \vec{b} &= (AU)\vec{x} \\ &= A(U\vec{x}) \ , \end{aligned}$$

and so $\vec{b} \in Col(A)$. Thus, $Col(AU) \subseteq Col(A)$.

Next we show that $Col(A) \subseteq Col(AU)$. Suppose that $\vec{b} \in Col(A)$. Then, there exists an $\vec{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} \vec{b} &= A\vec{x} \\ &= AUU^{-1}\vec{x} \\ &= (AU)(U^{-1}\vec{x}) \ , \end{aligned}$$

and so $\vec{b} \in Col(AU)$. Thus, $Col(A) \subseteq Col(AU)$.

Since $Col(AU) \subseteq Col(A)$ and $Col(A) \subseteq Col(AU)$, we can conclude that $Col(AU) = Col(A)$. $\square$

5.12 Definition of a Vector space

Exercise 5.12.1

Let $V = \mathbb{R}^2$ equipped with the usual scalar multiplication of $\mathbb{R}^2$ and addition defined as:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} .$$

Is V is a vectorspace? If so, prove it. If not, specify which axioms hold and which fail.

Flawed Proof 5.12.1

V is a vectorspace.

1. Closure Under Addition: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} ,$$

for some $a, b, c, d \in \mathbb{R}$. Then:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} \in V.$$

2. Commutative Under Addition: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} ,$$

for some $a, b, c, d \in \mathbb{R}$. Then:

$$\vec{u} + \vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} = \begin{pmatrix} ca \\ db \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \vec{v} + \vec{u} .$$

3. Associative Under Addition: Let $\vec{u}, \vec{v}, \vec{w} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \ , \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} e \\ f \end{pmatrix} \ ,$$

   for some $a, b, c, d, e, f \in \mathbb{R}$. Then:

$$\vec{u} + (\vec{v} + \vec{w}) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} ce \\ df \end{pmatrix} = \begin{pmatrix} ace \\ bdf \end{pmatrix} \ ,$$

$$(\vec{u} + \vec{v}) + \vec{w} = \begin{pmatrix} ac \\ bd \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ace \\ bdf \end{pmatrix} \ .$$

4. Contains Additive Identity:
   Suppose $\vec{0} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then for $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, we need:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \ .$$

   In particular, this implies

$$\begin{pmatrix} w_1 a \\ w_2 b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \ .$$

   If $a \neq 0$, then $w_1 = \frac{a}{a} = 1$.
   If $a = 0$, then $w_1 a = a$ holds for any w_1.
   Similarly, $w_2 b = b \implies w_2 = 1$.
   Thus,

$$\vec{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ .$$

5. Additive Inverses: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} \ ,$$

for some $a, b, c, d \in \mathbb{R}$. Set $c = \frac{1}{a}$ and $d = \frac{1}{b}$. Then:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix} = \begin{pmatrix} a\left(\frac{1}{a}\right) \\ b\left(\frac{1}{b}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0}\,.$$

6. Closure Under Scalar Multiplication: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}\ ,$$

for some $a, b \in \mathbb{R}$. Then for $k \in \mathbb{R}$:

$$k\vec{u} = k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} \in V.$$

7. Distributive Under Multiplication: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}\ ,$$

for some $a, b, c, d \in \mathbb{R}$. Then for $k \in \mathbb{R}$:

$$k(\vec{u} + \vec{v}) = k\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} kac \\ kbd \end{pmatrix} \in V.$$

8. Vector Distributivity: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}\ ,$$

for some $a, b \in \mathbb{R}$. Then for $k, m \in \mathbb{R}$:

$$(k+m)\,\vec{u} = (k+m)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} + \begin{pmatrix} ma \\ mb \end{pmatrix} = k\vec{u} + m\vec{u}\,.$$

9. Associative Under Multiplication:: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}\ ,$$

for some $a, b \in \mathbb{R}$. Then for $k, m \in \mathbb{R}$:

$$k\,(m\vec{u}) = k\begin{pmatrix} ma \\ mb \end{pmatrix} = \begin{pmatrix} kma \\ kmb \end{pmatrix} \;,$$

$$(km)\,\vec{u} = km\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} kma \\ kmb \end{pmatrix} \;.$$

10. Multiplicative Identity: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \;,$$

for some $a, b \in \mathbb{R}$. Then

$$1 * \vec{u} = 1 * \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} * 1 = \vec{u} * 1 \;.$$

□

5.12.1 Error classification

There are several errors in the Flawed Proof 5.12.1.

F-O: In axiom 5 forgot to consider the case when a and b are zero, which led to an erroneous conclusion.

C-OS: In axiom 7, omitted the section $k\vec{u} + k\vec{v}$, which led to an erroneous conclusion.

C-FS: In axiom 8, incorrectly concluded that

$$(k+m)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} + \begin{pmatrix} ma \\ mb \end{pmatrix}.$$

Error codes

- Fundamental Local Omission (F-O)
- Content Omitted Section (C-OS)
- Content False Statement (C-FS)

See Section 1.3 for more information about error classifications.

5.12.2 Corrected proof

The following is a corrected version of Flawed Proof 5.12.1.

Proof 5.12.1

V is not a vectorspace because it violates three of the vectorspace axioms. Namely, it violates axioms $5, 7$ and 8.

1. Closure Under Addition: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} ,$$

   for some $a, b, c, d \in \mathbb{R}$. Then:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} \in V.$$

2. Commutative Under Addition: Let $\vec{u}, \vec{v} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} ,$$

   for some $a, b, c, d \in \mathbb{R}$. Then:

$$\vec{u} + \vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix} = \begin{pmatrix} ca \\ db \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \vec{v} + \vec{u} \,.$$

3. Associative Under Addition: Let $\vec{u}, \vec{v}, \vec{w} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} , \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} e \\ f \end{pmatrix} ,$$

   for some $a, b, c, d, e, f \in \mathbb{R}$. Then:

$$\vec{u} + (\vec{v} + \vec{w}) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} ce \\ df \end{pmatrix} = \begin{pmatrix} ace \\ bdf \end{pmatrix} ,$$

$$(\vec{u}+\vec{v})+\vec{w} = \begin{pmatrix} ac \\ bd \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ace \\ bdf \end{pmatrix} .$$

4. Contains Additive Identity:
   Suppose $\vec{0} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then for $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, we need:

   $$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} .$$

   In particular, this implies

   $$\begin{pmatrix} w_1 a \\ w_2 b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} .$$

   If $a \neq 0$, then $w_1 = \frac{a}{a} = 1$.
   If $a = 0$, then $w_1 a = a$ holds for any w_1.
   Similarly, $w_2 b = b \implies w_2 = 1$.
   Thus,

   $$\vec{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

5. Additive Inverses: Consider $\vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We will show that $\vec{u}$ does not have an additive inverse. Suppose $(-\vec{u}) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then

   $$\vec{u} + (-\vec{u}) = \vec{0} \iff \begin{pmatrix} 0 \cdot v_1 \\ 0 \cdot v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

   which is impossible. Thus, this axiom fails.

6. Closure Under Scalar Multiplication: Let $\vec{u} \in V$ where

   $$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} ,$$

for some $a, b \in \mathbb{R}$. Then for $k \in \mathbb{R}$:

$$k\vec{u} = k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix} \in V.$$

7. Distributive Under Multiplication: Consider

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Then for $k = 2$,

$$k(\vec{u} + \vec{v}) = 2\begin{pmatrix} 1 \cdot 1 \\ 0 \cdot 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} .$$

On the other hand,

$$k\vec{u} + k\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 \\ 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} .$$

Thus, this axiom fails.

8. Vector Distributivity: Consider

$$\vec{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} .$$

Then for $k = m = 1$,

$$(k + m)\vec{u} = (1 + 1)\begin{pmatrix} 3 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} .$$

On the other hand,

$$k\vec{u} + k\vec{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 3 \\ 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} .$$

Thus, this axiom fails, and so V is not a vectorspace.

9. Associative Under Multiplication:: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} ,$$

for some $a, b \in \mathbb{R}$. Then for $k, m \in \mathbb{R}$:

$$k\left(m\vec{u}\right) = k\begin{pmatrix} ma \\ mb \end{pmatrix} = \begin{pmatrix} kma \\ kmb \end{pmatrix} ,$$

$$\left(km\right)\vec{u} = km\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} kma \\ kmb \end{pmatrix} .$$

10. Multiplicative Identity: Let $\vec{u} \in V$ where

$$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix} ,$$

for some $a, b \in \mathbb{R}$. Then

$$1 * \vec{u} = 1 * \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} * 1 = \vec{u} * 1 .$$

$\square$

5.13 Definition of a Subspace

Exercise 5.13.1

Consider the vectorspace P_3, where P_3 is the space of polynomials with degree less than or equal to 3. Let

$$S_1 = \{p(x) : p(1) = 0\} \subseteq P_3 \ .$$

Is S_1 a subspace of P_3?

Flawed Proof 5.13.1

Yes, S_1 is a subspace of P_3. We will prove this using the Subspace Test.

1. $\vec{0} \in P_3$ **is in** S_1 : In P_3, $\vec{0}$ is given by the constant polynomial $\vec{0} = q(x) = 0$. By definition of S_1, we have $q(1) = 0$, which implies that $\vec{0} \in S_1$.

2. S_1 **is closed under addition:** Let $p(x), q(x) \in S_1$. Then, by definition of $S_1, p(1) = q(1) = 0$. Thus,

   $$(p + q)(1) = p(1) + q(1) = 0 + 0 = 0 \ ,$$

   which implies $(p + q)(x) \in S_1$ and so S_1 is closed under addition.

3. S_1 **is closed under multiplication:** Let $p(x), q(x) \in S_1$. Then, by definition of $S_1, p(1) = q(1) = 0$. Thus,

   $$p(1)q(1) = 0 \times 0 = 0 \ ,$$

   which implies $(pq)(x) \in S_1$ and so S_1 is closed under multiplication.

Since $S_1 \subseteq P_3$ and satisfies the conditions of the Subspace Test, we can conclude that it is a subspace. $\square$

5.13.1 Error classification

There is one error in the Flawed Proof 5.13.1.

EO-(F-MT): The third condition of the Subspace Test is closure under scalar multiplication (not closure under multiplication). This led to the verification of closure under scalar multiplication to be omitted from the proof.

Error codes

- Error-caused Omission due to Fundamental Misusing Theorem (EO-(F-MT))

See Section 1.3 for more information about error classifications.

5.13.2 Corrected proof

The following is a corrected version of Flawed Proof 5.13.1.

Proof 5.13.1

Yes, S_1 is a subspace of P_3. We will prove this using the Subspace Test.

1. $\vec{0} \in P_3$ **is in** S_1 : In P_3, $\vec{0}$ is given by the constant polynomial $\vec{0} = q(x) = 0$. Since $\vec{0} = q(x) = 0$ sends everything to 0, we have $q(1) = 0$, which implies that $\vec{0} \in S_1$.

2. S_1 **is closed under addition:** Let $p(x), q(x) \in S_1$. Then, we know $p(1) = q(1) = 0$. Thus,

$$(p + q)(1) = p(1) + q(1) = 0 + 0 = 0 \ ,$$

   which implies $(p + q)(x) \in S_1$ and so S_1 is closed under addition.

3. S_1 **is closed under scalar multiplication:** Let $p(x) \in S_1$. Then, we know $p(1) = 0$. Further, consider $k \in \mathbb{R}$. Then,

$$(kp)(1) = kp(1) = k(0) = 0 \ ,$$

   which implies $(kp)(x) \in S_1$ and so S_1 is closed under scalar multiplication.

Since $S_1 \subseteq P_3$ and satisfies the conditions of the Subspace Test, we can conclude that S_1 is a subspace of P_3. $\square$

5.14 Span

Exercise 5.14.1: Span is Smallest Subspace

Let V be a real vector space and let S be a subset of V. Prove that if W is a subspace of V with $S \subseteq W$, then $\operatorname{span}(S) \subseteq W$.

Flawed Proof 5.14.1

Let $\boldsymbol{v} \in \operatorname{span}(S)$. We can write $\boldsymbol{v}$ as a linear combination. So we will write $\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$. Since W is a subspace it contains linear combinations, so $\boldsymbol{v} \in W$.

$\square$

5.14.1 Error classification

There are several errors in the Flawed Proof 5.14.1.

C-N It is incorrect to assume we can write "$\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$". In general we should write a general linear combination with a variable number of vectors, not just 2.

N-O $a_1, a_2, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are undefined.

C-A The assertion "since W is a subspace is contains linear combinations" is incorrect. A subspace does not contain ALL linear combinations within a larger vector space.

Error codes

- Content Notation (C-N)
- Novice Local Omission (N-O)
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

5.14.2 Corrected proof

The following is a corrected version of Flawed Proof 5.14.1.

Proof 5.14.1

Let $\boldsymbol{v} \in \text{span}(S)$. We need to show $\boldsymbol{v} \in W$. Since $\boldsymbol{v} \in \text{span}(S)$ we can write $\boldsymbol{v}$ as a linear combination of elements in S. So we will write $\boldsymbol{v} = a_1\boldsymbol{v}_1 + ... + a_n\boldsymbol{v}_n$ for $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in S$ and $a_1, ..., a_n \in \mathbb{R}$. Now since $S \subseteq W$, we have $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in W$. Since W is a subspace, W is closed under addition and scalar multiplication. Hence $\boldsymbol{v} = a_1\boldsymbol{v}_1 + ... + a_n\boldsymbol{v}_n \in W$.

Therefore $\text{span}(S) \subseteq W$. $\square$

5.15 Zero is in the Span

Exercise 5.15.1: Zero is in the Span

Decide whether the following statement is TRUE or FALSE. If it's true, then prove it. If it's false, then find an explicit counterexample.

Let V be a vector space. If $S \subseteq V$ is non-empty, then $\mathbf{0} \in \operatorname{span}(S)$.

Flawed Proof 5.15.1

The statement is false and we will give a counterexample. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Then clearly $\mathbf{0} \notin \operatorname{span}(S)$ since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\square$

5.15.1 Error classification

There is one error in the Flawed Proof 5.15.1.

EO-(C-VG) No progress is made towards the proof, due to misunderstanding what the span of a set is.

Error codes

- Error-Caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))

See Section 1.3 for more information about error classifications.

5.15.2 Corrected proof

The following is a corrected version of Flawed Proof 5.15.1.

Proof 5.15.1

The statement is true. Indeed, let V be a vector space and S a non-empty subset of V. Since S is non-empty, there exists a $v \in S$. Hence, $\mathbf{0} = 0v \in \operatorname{span}(S)$, since $\operatorname{span}(S)$ is the set of all linear combinations of elements in S. $\square$

5.16 Basis of a Polynomial Space

Exercise 5.16.1

Is $S = \{x^3 + 2,\ x^2 - 3x + 1,\ 5\}$ a basis of $U = \text{span}(S)$?

Flawed Proof 5.16.1

Since U is the span of S, then by definition of a basis, S is trivially a basis of U. $\square$

5.16.1 Error classification

There is one error in the Flawed Proof 5.16.1.

EO-(C-VG): Misunderstanding the definition of basis caused omission of the linear independence section of the proof. A basis S of a vector space U must span U and be linearly independent.

Error codes

- Error-caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))

See Section 1.3 for more information about error classifications.

5.16.2 Corrected proof

The following is a corrected version of Flawed Proof 5.16.1.

Proof 5.16.1

In order for S to be a basis of U, it must satisfy two conditions:

First, S must span U. In this case, since we have that $U = \text{span}(S)$, we know that S spans U.

Secondly, the set S must be linearly independent. Since the three polynomials in S have different degrees, this implies that they are linearly independent.

Thus, since S spans U and is linearly independent, we can conclude that S is a basis of U. $\square$

5.17 Basis

Exercise 5.17.1: Basis of Polynomial Subspace

Let P_n denote the vector space of polynomials of degree at most n with real coefficients, for some integer $n \geq 1$.

Suppose S is a subset of P_n with $n+1$ distinct polynomials. Suppose $p(0) = 0$ for all $p(x) \in S$. Is it possible for S to be a basis for P_n? Explain.

Flawed Proof 5.17.1

From class I know a basis has two things. It has the same number of vectors as the dimension and it is linearly independent. The dimension of P_n is n, but S has $n + 1$ vectors. So it can't be a basis. $\square$

5.17.1 Error classification

There are multiple errors in the Flawed Proof 5.17.1

N-VG Writing in the first person (the use of the word "I") in a proof is typically not conventional in mathematics.

EO-(F-FS) The vector space P_n has dimension $n+1$ (not n). This error led to no substantial progress being made towards the proof.

Error codes

- Novice Vocabulary and Grammar (N-VG)
- Error-caused Omission due to Fundamental False Statement (EO-(F-FS))

See Section 1.3 for more information about error classifications.

5.17.2 Corrected proof

The following is a corrected version of Flawed Proof 5.17.1.

Proof 5.17.1

It is not possible for S to be a basis for P_n.

Suppose $S = \{p_0(x), ..., p_n(x)\}$ for $p_0(x), ..., p_n(x) \in P_n$. Since $p_i(x) \in S$, we know that $p_i(0) = 0$ for all $1 \leq i \leq n$.

If S were a basis for P_n, then S spans P_n. Since $x - 1 \in P_n$, this would mean that we could write $x-1$ as a linear combination of elements in S. Namely we could write

$$x - 1 = a_0 p_0(x) + ... + a_n p_n(x)$$

for some $a_0, ..., a_n \in \mathbb{R}$. Evaluating both sides of this equation at $x = 0$ we have:

$$0 - 1 = a_0(0) + ... + a_n(0) = 0,$$

which implies $-1 = 0$. This is a contradiction.

Therefore, S cannot be a basis for P_n. $\square$

5.18 Linear Transformations and Matrices

Exercise 5.18.1: Matrices are Linear Transformations

Let A be an $m \times n$ matrix. Prove the map

$$\begin{aligned} T &: \mathbb{R}^n \to \mathbb{R}^m \\ T(\vec{v}) &= A\vec{v}, \text{ for all } \vec{v} \in V \end{aligned}$$

is a linear transformation.

Flawed Proof 5.18.1

We have to show T is a linear transformation
$T(x+y) = T(x) + T(y)$
$T(rx) = rT(x)$ $\square$

5.18.1 Error classification

There are multiple errors in the Flawed Proof 5.25.1

F-A Definition of a linear transformation was recalled, but the fact that T satisfies this was asserted without proof.

N-O x, y and r were not defined.

Error codes

- Fundamental Assertion (F-A)
- Novice Omission (N-O)

See Section 1.3 for more information about error classifications.

5.18.2 Corrected proof

The following is a corrected version of Flawed Proof 5.18.1.

Proof 5.18.1

To show T is linear we must show that T preserves addition and scalar multiplication. Let $\vec{v}_1, \vec{v}_2 \in V$ and let $k \in \mathbb{R}$. Using the rules of matrix multiplication we have:

$$T(\vec{v}_1 + \vec{v}_2) = A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = T(\vec{v}_1) + T(\vec{v}_2),$$

and

$$T(k\vec{v}_1) = A(k\vec{v}_1) = kA\vec{v}_1 = kT(\vec{v}_1).$$

Therefore, T is a linear transformation. $\square$

5.19 Linear Transformations

Exercise 5.19.1

Let A be an $n \times n$ matrix. Consider $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $T(A) = \text{Tr}(A)$. Is T a linear transformation? Explain.

Flawed Proof 5.19.1

Yes, T is a linear transformation. Consider the following arbitrary $n \times n$ matrices A and B defined as,

$$A = \begin{bmatrix} a & & \\ & \ddots & \\ & & y \end{bmatrix} \quad , \quad B = \begin{bmatrix} b & & \\ & \ddots & \\ & & z \end{bmatrix}$$

First, we show that $\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$.

$$\begin{aligned} \text{Tr}(A+B) &= \text{Tr}\left(\begin{bmatrix} a & & \\ & \ddots & \\ & & y \end{bmatrix} + \begin{bmatrix} b & & \\ & \ddots & \\ & & z \end{bmatrix}\right) \\ &= \text{Tr}\begin{bmatrix} a+b & & \\ & \ddots & \\ & & y+z \end{bmatrix} \\ &= \text{Tr}(A) + \text{Tr}(B) \ . \end{aligned}$$

Next, we show that $\text{Tr}(kA) = \ k\text{Tr}(A)$.

$$\begin{aligned} \text{Tr}(kA) &= k\text{Tr}\begin{bmatrix} a & & \\ & \ddots & \\ & & y \end{bmatrix} \\ &= k\text{Tr}(A) \ . \end{aligned}$$

□

5.19.1 Error classification

There are several errors in the Flawed Proof 5.19.1.

N-N: Denoting the matrix diagonal entries by $a, \ldots, y$ and $b, \ldots, z$ is imprecise. It would be better to denote them by $a_{11}, \ldots, a_{nn}$ and $b_{11}, \ldots, b_{nn}$ since these are $n \times n$ matrices.

F-A: The closed under addition and closed under scalar multiplication portions of the proof are asserted. Indeed, no explanation is given to justify why

$$\operatorname{Tr}\begin{bmatrix} a+b & & \\ & \ddots & \\ & & y+z \end{bmatrix} = \operatorname{Tr}(A) + \operatorname{Tr}(B).$$

And similarly, the scalar k is pulled outside of $\operatorname{Tr}(A)$ without justification.

Error codes

- Novice Notation (N-N)
- Fundamental Assertions (F-A)

See Section 1.3 for more information about error classifications.

5.19.2 Corrected proof

The following is a corrected version of Flawed Proof 5.19.1.

Proof 5.19.1

Yes, T is a linear transformation. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices and $k \in \mathbb{R}$.

First, we will show that T is closed under addition. We have

$$\mathrm{Tr}(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(B) \; .$$

Next, we will show that T is closed under scalar multiplication. We have

$$\mathrm{Tr}(kA) = \sum_{i=1}^{n}(ka_{ii}) = k\sum_{i=1}^{n} a_{ii} = k\mathrm{Tr}(A) \; .$$

Thus, T is a linear transformation.

$\square$

5.20 Linear Maps on Complex Vector Spaces

Exercise 5.20.1: Operator on $\mathbb{C}$ is scalar

Consider $\mathbb{C}$ as a vector space over itself. Let $T : \mathbb{C} \to \mathbb{C}$ be a linear transformation. Prove that there exists some $c \in \mathbb{C}$ such that $T(x) = cx$ for all $x \in \mathbb{C}$.

Flawed Proof 5.20.1

If $c \in \mathbb{C}$, then for $x \in \mathbb{C}$

$$T(x) = \frac{cT}{c}x$$

so let $c = \frac{cT}{c}$, so

$$T(x) = cx$$

$\square$

5.20.1 Error classification

There are multiple errors in the Flawed Proof 5.20.1

F-FS Writing $T(x) = \frac{cT}{c}x$ and treating this as a constant demonstrates a misunderstanding of what T is.

C-N Reusing the variable c as $c = \frac{cT}{c}$ is confusing.

Error codes

- Fundamental False Statement (F-FS)
- Content Notation (C-N)

See Section 1.3 for more information about error classifications.

5.20.2 Corrected proof

The following is a corrected version of Flawed Proof 5.20.1.

Proof 5.20.1

Let $T : \mathbb{C} \to \mathbb{C}$ be a linear operator over $\mathbb{C}$ as a vector space over itself. Suppose $T(1) = c$. Then for any $x \in \mathbb{C}$

$$T(x) = T(1x) = xT(1) = xc = cx.$$

$\square$

5.21 One-to-One and Onto

Exercise 5.21.1: Differentiation Map

Let P be the infinite vector space consisting of all polynomials with real coefficients. Let $T : P \to P$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 ... + na_nx^{n-1}$$

for $a_0, ..., a_n \in \mathbb{R}$. Is T is one-to-one? Is T is onto? Explain.

Flawed Proof 5.21.1

$T(a) = T(b) = a = b = T(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = T(b_0 + a_1x + b_2x^2 + ... + b_nx^n) = a_0 + a_1x + a_2x^2 + ... + a_nx^n = b_0 + b_1x + b_2x^2 + ... + b_nx^n$.

$b = T(a) = b_0 + b_1x + b_2x^2 + ... + b_nx^n = T(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$. □

5.21.1 Error classification

There is one error in the Flawed Proof 5.21.1.

F-LU The proof in general isn't readable. Nothing is defined and no explanations are given.

Error codes

- Fundamental Locally Unintelligible (F-LU)

See Section 1.3 for more information about error classifications.

5.21.2 Corrected proof

The following is a corrected version of Flawed Proof 5.21.1.

Proof 5.21.1

The linear transformation T is not one-to-one since $T(1) = 0$ and $T(0) = 0$, but $1 \neq 0$.

To show T is onto, we will show $P \subseteq \text{im}(T)$. Let $p(x) = a_0 + a_1x + ... + a_nx^n \in P$. Now consider $q(x) = a_0x + \frac{a_1}{2}x^2 + ... + \frac{a_n}{n+1}x^{n+1} \in P$. We have

$$T(q(x)) = T(a_0x + \frac{a_1}{2}x^2 + ... + \frac{a_n}{n+1}x^{n+1}) = a_0 + a_1x + ... + a_nx^n = p(x).$$

Hence, $p(x) \in \text{im}(T)$. Therefore, T is onto. $\square$

5.22 Composition of Linear Transformations

Exercise 5.22.1

Let V, W and U be vector spaces. Prove that the composition of two linear transformations $T : V \to W$ and $S : W \to U$ is a linear transformation.

Flawed Proof 5.22.1

Since

$$\begin{aligned} S \circ T(\vec{u} + \vec{v}) &= S(T(\vec{u} + \vec{v})) \\ &= S(T(\vec{u}) + T(\vec{v})) \\ &= S(T(\vec{u})) + S(T(\vec{v})) \\ &= S \circ T(\vec{u}) + S \circ T(\vec{v}) \end{aligned}$$

and

$$\begin{aligned} S \circ T(k\vec{u}) &= S(T(k\vec{u})) \\ &= S(kT(\vec{u})) \\ &= kS(T(\vec{u})) \\ &= kS \circ T(\vec{u}) \end{aligned}$$

then $S \circ T$ is linear. □

5.22.1 Error classification

There are several errors in the Flawed Proof 5.22.1.

N-O: Did not define $\vec{u}$, $\vec{v}$, and k.

C-A: Results should be justified by explaining that they follow because T and S are linear.

Error codes

- Novice Local Omission (N-O)
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

5.22.2 Corrected proof

The following is a corrected version of Flawed Proof 5.22.1.

Proof 5.22.1

We want to prove that the composition $S \circ T : V \to U$ is linear. Since T is linear, we know that

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \ , \ \text{and}$$

$$T(k\vec{u}) = kT(\vec{u}) \ ,$$

for all $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$. Similarly, since S is linear, we know that

$$S(\vec{u} + \vec{v}) = S(\vec{u}) + S(\vec{v}) \ , \ \text{and}$$

$$S(k\vec{u}) = kS(\vec{u}) \ ,$$

for all $\vec{u}, \vec{v} \in W$ and $k \in \mathbb{R}$. Let $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$. Then

$$\begin{aligned} S \circ T(\vec{u} + \vec{v}) &= S(T(\vec{u} + \vec{v})) \\ &= S(T(\vec{u}) + T(\vec{v})) \\ &= S(T(\vec{u})) + S(T(\vec{v})) \\ &= S \circ T(\vec{u}) + S \circ T(\vec{v}), \end{aligned}$$

and

$$\begin{aligned} S \circ T(k\vec{u}) &= S(T(k\vec{u})) \\ &= S(kT(\vec{u})) \\ &= kS(T(\vec{u})) \\ &= kS \circ T(\vec{u}). \end{aligned}$$

Thus, by the definition of linearity, $S \circ T : V \to U$ is linear. □

5.23 Isomorphism of a Linear Transformation

Exercise 5.23.1

Prove that the linear map $T : P_1 \to \mathbb{R}^2$ defined by

$$T(a+bx) = \begin{pmatrix} a \\ a+b \end{pmatrix} ,$$

is an isomorphism and find T^{-1}. (You do not need to confirm that T is linear.)

Flawed Proof 5.23.1

We prove that $\ker(T) = \{\vec{0}\}$. Suppose that $a + bx \in \ker(T)$. Then

$$\begin{pmatrix} a \\ a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

which implies that $a = b = 0$, and so $\ker(T) = \{\vec{0}\}$. This means that T is injective. Thus, T is an isomorphism.

Finally, we must find T^{-1}. We need $T(T^{-1}) = I$. We have

$$T(a+bx) = \begin{pmatrix} a \\ a+b \end{pmatrix} .$$

So then

$$T(T^{-1}) = \begin{pmatrix} a \\ a+b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which means that $a = 1$ and $a+b = 1$. In particular, $a = 1$ and $b = 1-a$. Thus,

$$T^{-1} = \begin{pmatrix} 1 \\ 1-a \end{pmatrix} .$$

$\square$

5.23.1 Error classification

There are several errors in the Flawed Proof 5.23.1.

C-OS: Omitted surjectivity in proving that T is an isomorphism.

EO-(C-VG): No progress is made towards finding the inverse due to misunderstanding the definition of an inverse.

C-FS: The claim that

$$T(T^{-1}) = \begin{pmatrix} a \\ a+b \end{pmatrix} ,$$

is incorrect.

Error codes

- Content Omitted Sections (C-OS)
- Error-caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))
- Content False Statement (C-FS)

See Section 1.3 for more information about error classifications.

5.23.2 Corrected proof

The following is a corrected version of Flawed Proof 5.23.1.

Proof 5.23.1

Since $\dim(P_1) = \dim(\mathbb{R}^2) = 2$, to show T is an isomorphism it suffices to show that T is injective.

Since T is linear, we know that T is injective exactly when $\ker(T) = \{\vec{0}\}$. Suppose that $a + bx \in \ker(T)$. Then

$$T(a+bx) = \begin{pmatrix} a \\ a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

which implies that $a = b = 0$, and so $\ker(T) = \{\vec{0}\}$. This means that T is injective. Therefore, T is an isomorphism.

Now, we must find T^{-1}. Suppose that

$$T(a+bx) = \begin{pmatrix} c \\ d \end{pmatrix} .$$

This implies

$$\begin{pmatrix} a \\ a+b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} ,$$

and so $a = c$ and $a + b = d$. In particular, $a = c$ and $b = d - c$. Then

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = T^{-1}(T(a+bx)) = a + bx = c + (d-c)x .$$

Thus,

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = c + (d-c)x .$$

□

5.24 Inverse Linear Transformation

Exercise 5.24.1: Inverse of Composition

Let V be a vector space, and let $T : V \to V$ and $S : V \to V$ be isomorphisms. Prove $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$.

Flawed Proof 5.24.1

This is simple algebra.

$$(T \circ S)^{-1} = \frac{1}{T \circ S} = \frac{1}{S} \circ \frac{1}{T} = S^{-1} \circ T^{-1}$$

$\square$

5.24.1 Error classification

There is one error in the Flawed Proof 5.24.1.

EO-(C-VG): No progress is made towards the proof due to misunderstanding the definitions of inverse and composition.

Error codes

- Error-caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))

See Section 1.3 for more information about error classifications.

5.24.2 Corrected proof

The following is a corrected version of Flawed Proof 5.24.1.

Proof 5.24.1

Let $\vec{v} \in V$. Then we have

$$\begin{aligned}(T \circ S) \circ (S^{-1} \circ T^{-1})(\vec{v}) &= T \circ S(S^{-1}(T^{-1}(\vec{v}))) \\ &= T(T^{-1}(\vec{v})) \\ &= \vec{v},\end{aligned}$$

and

$$\begin{aligned}(S^{-1} \circ T^{-1}) \circ (T \circ S)(\vec{v}) &= S^{-1} \circ T^{-1}(T(S(\vec{v}))) \\ &= S^{-1}(S(\vec{v})) \\ &= \vec{v}.\end{aligned}$$

Therefore, $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$. $\square$

5.25 Orthogonal Sets

Exercise 5.25.1: Intersection of Orthogonal Sets

Let S and T be subsets of $\mathbb{R}^n$. Suppose S and T are orthogonal; that is, suppose $\boldsymbol{s} \cdot \boldsymbol{t} = 0$ for all $\boldsymbol{s} \in S$ and all $\boldsymbol{t} \in T$. Prove that either $S \cap T = \{\mathbf{0}\}$ or $S \cap T = \emptyset$

Flawed Proof 5.25.1

If $S \cap T = \{\mathbf{0}\}$, then $\boldsymbol{s} \cdot \boldsymbol{t} = 0$ since $\mathbf{0} \cdot \mathbf{0} = 0$.

If $S \cap T = \emptyset$, then there is nothing to take the dot product of so it is vacuously true. $\square$

5.25.1 Error classification

There are multiple errors in the Flawed Proof 5.25.1

WP: Misunderstanding of what needed to be proven.

NO: Variables $\boldsymbol{s}$ and $\boldsymbol{t}$ are not defined.

Error codes

- Wrong Problem (WP)
- Novice Omission (NO)

See Section 1.3 for more information about error classifications.

5.25.2 Corrected proof

The following is a corrected version of Flawed Proof 5.25.1.

Proof 5.25.1

We will show that if there exists some $\boldsymbol{v} \in S \cap T$, then $\boldsymbol{v} = \mathbf{0}$. This will show that either the intersection is empty, or it only contains the zero vector.

Let $\boldsymbol{v} \in S \cap T$. Then $\boldsymbol{v} \in S$ and $\boldsymbol{v} \in T$. Since all elements of S are orthogonal to all elements of T, this means that $\boldsymbol{v} \cdot \boldsymbol{v} = 0$, which implies that $||\boldsymbol{v}|| = 0$. This is only possible if $\boldsymbol{v} = \mathbf{0}$. Therefore, $S \cap T = \{\mathbf{0}\}$ or $S \cap T = \emptyset$. $\square$

5.26 Gram-Schmidt Procedure in a Polynomial Space

Exercise 5.26.1

A basis for P_2 is given by $\{f, g, h\}$ where $f(x) = 1$, $g(x) = x$ and $h(x) = x^2$, and the inner product is defined as

$$\langle f, g \rangle = f(0)g(0) + f(1)g(1) + f(2)g(2) \; .$$

Use the Gram-Schmidt Orthogonalization Algorithm to find an orthonormal basis of P_2.

Flawed Proof 5.26.1

We start by finding v_1, v_2 and v_3 using the Gram-Schmidt process.

$$\begin{aligned}
v_1 &= f = 1 \\
v_2 &= g - \frac{\langle g, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\
&= x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \\
&= x - \frac{0(1) + 1(1) + 2(1)}{1 + 1 + 1} \\
&= x - \frac{3}{3} \\
&= x - 1 \\
v_3 &= h - \frac{\langle h, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle h, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\
&= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, x-1 \rangle}{\langle x-1, x-1 \rangle}(x-1) \\
&= x^2 - \frac{0(1) + 1(1) + 4(1)}{3} - \frac{0(-1) + 1(0) + 4(1)}{2}(x-1) \\
&= x^2 - \frac{5}{3} - \frac{4}{2}(x-1) \\
&= x^2 - 2x + \frac{1}{3}
\end{aligned}$$

Next, we normalize $v_1,\ v_2$ and v_3.

$$\begin{aligned}
\|v_1\| &= \sqrt{0 + 0 + 1^2} = 1 \\
\|v_2\| &= \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2} \\
\|v_3\| &= \sqrt{\langle v_3, v_3 \rangle} = \sqrt{1^2 + (-2)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{46}{9}}\ .
\end{aligned}$$

Then our orthonormal basis vectors are,

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= \frac{1}{\sqrt{2}} v_2 \\ u_3 &= \sqrt{\frac{9}{46}} v_3 \ . \end{aligned}$$

Thus, $\{u_1, u_2, u_3\}$ is an orthonormal basis of P_2. $\square$

5.26.1 Error classification

There are several errors in the Flawed Proof 5.26.1.

C-VG: Used incorrect inner product during the normalization calculations (i.e. did not use the correct inner product definition given in the question)

C-FS: Incorrect computations in the normalization procedure.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Content False Statements (C-FS)

See Section 1.3 for more information about error classifications.

5.26.2 Corrected proof

The following is a corrected version of Flawed Proof 5.26.1.

Proof 5.26.1

We start by finding v_1, v_2 and v_3 using the Gram-Schmidt process.

$$\begin{aligned}
v_1 &= f = 1 \\
v_2 &= g - \frac{\langle g, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\
&= x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \\
&= x - \frac{0(1) + 1(1) + 2(1)}{1 + 1 + 1} \\
&= x - \frac{3}{3} \\
&= x - 1 \\
v_3 &= h - \frac{\langle h, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle h, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\
&= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, x - 1 \rangle}{\langle x - 1, x - 1 \rangle}(x - 1) \\
&= x^2 - \frac{0(1) + 1(1) + 4(1)}{3} - \frac{0(-1) + 1(0) + 4(1)}{2}(x - 1) \\
&= x^2 - \frac{5}{3} - \frac{4}{2}(x - 1) \\
&= x^2 - 2x + \frac{1}{3}
\end{aligned}$$

Next, we normalize $v_1,\ v_2$ and v_3.

$$\begin{aligned}
\|v_1\| &= \sqrt{\langle v_1, v_1\rangle} = \sqrt{1^2+1^2+1^2} = \sqrt{3} \\
\|v_2\| &= \sqrt{\langle v_2, v_2\rangle} = \sqrt{-1^2+0^2+1^2} = \sqrt{2} \\
\|v_3\| &= \sqrt{\langle v_3, v_3\rangle} = \sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{-2}{3}\right)^2+\left(\frac{1}{3}\right)^2} = \sqrt{\frac{2}{3}}\ .
\end{aligned}$$

Then our orthonormal basis vectors are,

$$\begin{aligned}
u_1 &= \frac{1}{\sqrt{3}}v_1 \\
u_2 &= \frac{1}{\sqrt{2}}v_2 \\
u_3 &= \sqrt{\frac{3}{2}}v_3\ .
\end{aligned}$$

Thus, $\{u_1, u_2, u_3\}$ is an orthonormal basis of P_2.

$\square$

5.27 Orthogonal Matrices

Exercise 5.27.1: Orthogonal Matrix Inverses

Let M be an $n \times n$ matrix that is orthogonal (i.e. $MM^T = M^T M = I$). Prove M is invertible and M^{-1} is orthogonal.

Flawed Proof 5.27.1

Transposes work across inverses so

$$M{-}1(M{-}1)^T = I$$

and

$$(M{-}1)^T M{-}1 = I$$

which shows it satisfies the definition of being orthogonal. □

5.27.1 Error classification

There are several errors in the Flawed Proof 5.27.1.

F-A The entire result is asserted without proof.

C-N The "-1" should be superscripted when indicating an inverse.

Error codes

- Fundamental Assertion (F-A)
- Content Notation (C-N)

See Section 1.3 for more information about error classifications.

5.27.2 Corrected proof

The following is a corrected version of Flawed Proof 5.27.1.

Proof 5.27.1

Since M is orthogonal, we know $MM^T = I$ and $M^TM = I$. This implies that $M^{-1} = M^T$ since M^T satisfies the definition of the inverse of M.

Moreover, M^{-1} is orthogonal, since

$$M^{-1}(M^{-1})^T = M^T(M^T)^T = M^TM = I$$

and

$$(M^{-1})^TM^{-1} = (M^T)^TM^T = MM^T = I.$$

$\square$

5.28 Orthogonal Diagonalization

Exercise 5.28.1

Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix} ,$$

and indicate what the corresponding diagonal matrix is.

Flawed Proof 5.28.1

First, we find the eigenvalues of the matrix A:

$$C_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & 2 \\ 2 & \lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) ,$$

and so the eigenvalues of A are given by $\lambda_1 = 4$ and $\lambda_2 = -1$.

Next, we find the eigenvectors corresponding to λ_1 and λ_2.

To find the eigenvector associated with $\lambda_1 = 4$, we solve the equation $A - 4I = 0$.

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} .$$

To find the eigenvector associated with $\lambda_2 = -1$, we solve the equation $A - (-1)I = 0$.

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

Finally, the matrix P which diagonalizes A is given by

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

where P diagonalizes A to

$$P^{-1}AP = D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} .$$

$\square$

5.28.1 Error classification

There are several errors in the Flawed Proof 5.28.1.

N-FS: Apparent typo when solving for the eigenvalues: the first entry should be $\lambda - 3$ (not $\lambda - 1$).

EO-(C-VG): An orthogonal matrix which diagonalizes A is not found due to misunderstanding the definition of orthogonal diagonalization (i.e. the solution diagonalizes A, but does not orthogonally diagonalize A).

N-N: Minor notational mistakes. In particular, to say that we are solving the equations $A - 4I = 0$ and $A - (-1)I = 0$ is imprecise. We are solving the equations $(A - 4I)\vec{x} = \vec{0}$ and $(A - (-1)I)\vec{x} = \vec{0}$. Moreover, it is customary to give vectors, v_1 and v_2, vector hats.

Error codes

- Novice False Statement (N-FS)
- Error-caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))
- Novice Notation (N-N)

See Section 1.3 for more information about error classifications.

5.28.2 Corrected proof

The following is a corrected version of Flawed Proof 5.28.1.

Proof 5.28.1

First, we find the eigenvalues of the matrix A:

$$C_A(\lambda) = \det \begin{pmatrix} \lambda - 3 & 2 \\ 2 & \lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) \ ,$$

and so the eigenvalues of A are given by $\lambda_1 = 4$ and $\lambda_2 = -1$.

Next, we find the eigenvectors corresponding to λ_1 and λ_2.

To find the eigenvector associated with $\lambda_1 = 4$, we solve the homogeneous equation $(A - 4I)\vec{x} = \vec{0}$:

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \in E_4(A).$$

To find the eigenvector associated with $\lambda_2 = -1$, we solve the homogeneous equation $(A - (-1)I)\vec{x} = \vec{0}$:

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in E_{-1}(A).$$

We know that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal since

$$\vec{v}_1 \cdot \vec{v}_2 = -2(1) + 1(2) = 0 \ .$$

Next, we normalize $\vec{v}_1$ and $\vec{v}_2$:

$$\|\vec{v}_1\| = \sqrt{5} = \|\vec{v}_2\| \ .$$

Hence, the orthogonal matrix P which diagonalizes A is given by

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \ ,$$

and

$$P^{-1}AP = D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} .$$

$\square$

5.29 Change of Basis Computation

Exercise 5.29.1

Consider the linear map $T : P_2 \to \mathbb{R}^2$ defined by

$$T(a + bx + cx^2) = \begin{pmatrix} a+b \\ c \end{pmatrix} .$$

Let $\alpha = \{1, x, x^2\}$ be a basis for P_2 and $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find $M_{\beta\alpha}(T)$.

Flawed Proof 5.29.1

First, we find $C_\alpha(T(1))$. We have

$$T(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\alpha(T(1)) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Next, we find $C_\alpha(T(x))$. We have

$$T(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\alpha(T(x)) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Finally, we find $C_\alpha(T(x^2))$. We have

$$T(x^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\alpha\left(T(x^2)\right) = \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Thus, $M_{\beta\alpha}(T) = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\1 & 1 & 1\end{pmatrix}$. $\square$

5.29.1 Error classification

There is only one error in the Flawed Proof 5.29.1.

C-VG: All steps of the solution are correct, but the coordinate vector C_α should be written as C_β.

Error codes

- Content Vocabulary and Grammar (C-VG)

See Section 1.3 for more information about error classifications.

5.29.2 Corrected proof

The following is a corrected version of Flawed Proof 5.29.1.

Proof 5.29.1

First, we find $C_\beta\left(T(1)\right)$. We have

$$T(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\beta\left(T(1)\right) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Next, we find $C_\beta\left(T(x)\right)$. We have

$$T(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\beta\left(T(x)\right) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Finally, we find $C_\beta\left(T(x^2)\right)$. We have

$$T(x^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

which implies that

$$C_\beta\left(T(x^2)\right) = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} .$$

Thus, $M_{\beta\alpha}(T) = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

$\square$

Chapter 6

Advanced Topics

This chapter contains more advanced topics, some of which are covered in MATH 273 Numbers and Proofs at the University of Calgary, and many topics that might be covered in a first course in real analysis. Topics include the Well-Ordering Principle, the definition of convergence for sequences, the notion of a Cauchy sequence, elementary topology of $\mathbb{R}$, basic properties of complex numbers and polynomials.

6.1 Well-Ordering Principle

Exercise 6.1.1: Product of Primes

Without the use of induction, prove every integer greater than 1 can be factored as a product of primes.

Flawed Proof 6.1.1

We will use the well-ordering principle. Let M be the set of all integers greater than one that cannot be factored as a product of primes. Assume that M is not empty. Then by the well-ordering principle, there exists some least element $n \in M$. We know that n is not prime, since if it was n would be a product of one prime and hence would not be in M. So n is composite.

This means we can write $n = xy$ for some positive integers x and y with $1 < x, y < n$. But, since $x < n$ this means x is minimal, not n. We assumed that n was the least element, so this is a contradiction.

Thus M does not have a least element, so it must be empty. Therefore all integers can be factored as a product of primes.

□

6.1.1 Error classification

There is only one error in the Flawed Proof 6.1.1.

F-FI The implication "since $x < n$ this means x is minimal, not n" is false. Minimality in this case refers to the set M, but $x \notin M$.

Error codes

- Fundamental False Implication (F-FI).

See Section 1.3 for more information about error classifications.

6.1.2 Corrected proof

The following is a corrected version of Flawed Proof 6.1.1.

Proof 6.1.1

We will use the well-ordering principle. Let M be the set of all integers greater than one that cannot be factored as a product of primes. In order to derive a contradiction, assume that M is not empty. Then by the well-ordering principle, there exists some least element $n \in M$. We know that n is not prime, since if it was n would be a product of one prime and hence would not be in M. So n is composite.

This means we can write $n = xy$ for some positive integers x and y with $1 < x, y < n$. Since n is minimal in M, this means $x, y \notin M$. So we can write x and y as a product of primes, namely we can say $x = p_1 ... p_k$ and $y = q_1 ... q_l$ for primes $p_1, ..., p_k, q_1, ..., q_l$, where $k, l \in \mathbb{N}$. But now this means $n = xy = p_1 ... p_k q_1 ... q_l$ so n can be written as a product of primes. Thus n cannot be in M, which is a contradiction.

Thus M does not have a least element, so it must be empty. Therefore all integers can be factored as a product of primes.

□

6.2 Conjugation Identities

Exercise 6.2.1

Let z and w be complex numbers. Prove the following identities:

1. $\overline{z} + \overline{w} = \overline{z+w}$
2. $\overline{z}\,\overline{w} = \overline{zw}$

Flawed Proof 6.2.1

Let $z = a + bi$ and $w = c + di$.

First we verify $\overline{z} + \overline{w} = \overline{z+w}$:

$$\begin{aligned}\overline{z} + \overline{w} &= \overline{(a+bi)} + \overline{(c+di)}\\ &= (a-bi) + (c-di)\\ &= \overline{z+w}\ .\end{aligned}$$

Next, we verify $\overline{z}\,\overline{w} = \overline{zw}$:

$$\begin{aligned}\overline{z}\,\overline{w} &= \overline{(a+bi)}\,\overline{(c+di)}\\ &= (a-bi)(c-di)\\ &= ac - adi - bci - bd\\ &= \overline{ac - adi - bci - bd}\\ &= \overline{(a+bi)(c+di)}\\ &= \overline{zw}\ .\end{aligned}$$

□

6.2.1 Error classification

There are several errors in the Flawed Proof 6.2.1.

N-A: The statement $(a - bi) + (c - di) = \overline{z + w}$ requires additional justification.

N-FS: There is an apparent typo in the statement

$$ac - adi - bci - bd = \overline{ac - adi - bci - bd} = \overline{(a + bi)(c + di)}\ .$$

Instead, it should read

$$ac - adi - bci - bd = \overline{ac + adi + bci - bd} = \overline{(a + bi)(c + di)}\ .$$

Error codes

- Novice Assertion (N-A)
- Novice False Statement (N-FS)

See Section 1.3 for more information about error classifications.

6.2.2 Corrected proof

The following is a corrected version of Flawed Proof 6.2.1.

Proof 6.2.1

Let $z = a + bi$ and $w = c + di$ be complex numbers.

First we verify $\overline{z} + \overline{w} = \overline{z + w}$:

$$\begin{aligned}\overline{z} + \overline{w} &= \overline{(a + bi)} + \overline{(c + di)} \\ &= (a - bi) + (c - di) \\ &= (a + c) - i(b + d) \\ &= \overline{z + w}\ .\end{aligned}$$

Next, we verify $\overline{z}\,\overline{w} = \overline{zw}$:

$$\begin{aligned}\overline{z}\,\overline{w} &= \overline{(a + bi)}\,\overline{(c + di)} \\ &= (a - bi)(c - di) \\ &= ac - adi - bci - bd \\ &= \overline{ac + adi + bci - bd} \\ &= \overline{(a + bi)(c + di)} \\ &= \overline{zw}\ .\end{aligned}$$

$\square$

6.3 Conjugation of a Complex Polynomial

Exercise 6.3.1

Let $p(z)$ be a complex polynomial with real coefficients. Prove that

$$\overline{p(z)} = p(\overline{z}) \ .$$

Flawed Proof 6.3.1

$$\begin{aligned} \overline{p(z)} &= \overline{az^n + bz^{n-1} + ... + ez + f} \\ &= \overline{az^n} + \overline{bz^{n-1}} + ... + \overline{ez} + \overline{f} \\ &= p(\overline{z}) \ . \end{aligned}$$

□

6.3.1 Error classification

There are several errors in the Flawed Proof 6.3.1.

C-A: The last equality in the proof requires further justification.

N-O The coefficients a, b, e, f and variables z^n, z^{n-1}, z are not defined.

N-N: The notation used for the coefficients in the polynomial $p(z) = az^n + bz^{n-1} + ... + ez + f$ does not follow mathematical conventions.

N-VG There are no sentences in this proof.

Error codes

- Content Assertion (C-A)
- Novice Omission (N-O)
- Novice Notation (N-N)
- Novice Vocabulary and Grammar (N-VG)

See Section 1.3 for more information about error classifications.

6.3.2 Corrected proof

The following is a corrected version of Flawed Proof 6.3.1.

Proof 6.3.1

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 + a_0$ where $a_0, ..., a_n \in \mathbb{R}$ and $z \in \mathbb{C}$. Then,

$$\begin{aligned}\overline{p(z)} &= \overline{a_n z^n + a_{n-1} z^{n-1} + ... + a_1 + a_0} \\ &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + ... + \overline{a_1 z} + \overline{a_0} \\ &= \overline{a_n}\ \overline{z^n} + \overline{a_{n-1}}\ \overline{z^{n-1}} + ... + \overline{a_1}\ \overline{z} + \overline{a_0} \\ &= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + ... + a_1 \overline{z} + a_0 \\ &= p(\overline{z})\ .\end{aligned}$$

Thus, $\overline{p(z)} = p(\overline{z})$. □

6.4 Factoring Complex Polynomials

Exercise 6.4.1

For each of the following polynomials $p(z)$. Prove that there exists a polynomial $h(z)$ such that

$$p(z) = (z - z_0)h(z) + p(z_0).$$

- $p(z) = z^2$ and $z_0 = i$,
- $p(z) = z^3 + z^2 + z$ and $z_0 = -1$.

Flawed Proof 6.4.1

First, we find $h(z)$ when $p(z) = z^2$ and $z_0 = i$:

$$\begin{aligned}
p(z) &= (z - z_0)h(z) + p(z_0) \\
&= (z + i)h(z) + p(i) \\
&= (z + i)h(z) - 1 \\
&= (z + i)(z - i) - 1 \\
&= z^2 - zi + zi - i^2 - 1 \\
&= z^2 + 1 - 1 \\
&= z^2 ,
\end{aligned}$$

and so $h(z) = z - i$.
Next, we find $h(z)$ when $p(z) = z^3 + z^2 + z$ and $z_0 = -1$:

$$\begin{aligned}
p(z) &= (z - z_0)h(z) + p(z_0) \\
&= (z + 1)h(z) + p(-1) \\
&= (z + 1)(z^2 + 1) - 1 \\
&= z^3 + z^2 + z + 1 - 1 \\
&= z^3 + z^2 + z ,
\end{aligned}$$

and so $h(z) = z^2 + 1$. $\square$

6.4.1 Error classification

There are several errors in the Flawed Proof 6.4.1.

N-FS: There is an apparent typo in the first calculation: the polynomial $p(z)$ should be written as $p(z) = (z - i)h(z) + p(i)$ as opposed to $p(z) = (z + i)h(z) + p(i)$.

N-Log: In the first calculation, they use $h(z) = z - i$ and then claim that their calculation implies that $h(z) = z - i$. A similar mistake is made in the second calculation.

Error codes

- Novice False Statement (N-FS)
- Novice Logical Order (N-Log)

See Section 1.3 for more information about error classifications.

6.4.2 Corrected proof

The following is a corrected version of Flawed Proof 6.4.1.

Proof 6.4.1

First consider $p(z) = z^2$ and $z_0 = i$. Choose $h(z) = z + i$. Indeed, if $h(z) = z + i$, then we have

$$\begin{aligned} p(z) &= (z - z_0)h(z) + p(z_0) \\ &= (z - i)(z + i) + p(i) \\ &= (z - i)(z + i) - 1 \\ &= z^2 - zi + zi - i^2 - 1 \\ &= z^2 + 1 - 1 \\ &= z^2 \,. \end{aligned}$$

Now, consider $p(z) = z^3 + z^2 + z$ and $z_0 = -1$. Choose $h(z) = z^2 + 1$. Indeed, if $h(z) = z^2 + 1$ then we have:

$$\begin{aligned} p(z) &= (z - z_0)h(z) + p(z_0) \\ &= (z + 1)(z^2 + 1) + p(-1) \\ &= (z + 1)(z^2 + 1) - 1 \\ &= z^3 + z^2 + z + 1 - 1 \\ &= z^3 + z^2 + z \,. \end{aligned}$$

□

6.5 Complex Trigonometric Functions

Exercise 6.5.1

Prove the following identity for all $z \in \mathbb{C}$:

$$\sin(z) = -i \sinh(iz) \ .$$

Flawed Proof 6.5.1

$$\begin{aligned} -i \sinh(z) &= -i \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \\ &= \frac{e^{iz} - e^{-iz}}{2} \\ &= \sin(z) \ . \end{aligned}$$

□

6.5.1 Error classification

There are several errors in the Flawed Proof 6.5.1.

C-VG: Misunderstanding the definitions of $\sin(z)$ and $\sinh(iz)$.

N-FS: The multiplication

$$-i\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{e^{iz}-e^{-iz}}{2}$$

is incorrect.

N-VG There are no sentences in this proof.

Error codes

- Content Vocabulary and Grammar (C-VG)
- Novice False Statement (N-FS)
- Novice Vocabulary and Grammar (N-VG)

See Section 1.3 for more information about error classifications.

6.5.2 Corrected proof

The following is a corrected version of Flawed Proof 6.5.1.

Proof 6.5.1

Suppose that $z \in \mathbb{C}$. By definition of $\sin(z)$ and $\sinh(z)$ we have:

$$\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
&= \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{2}\right) \\
&= \frac{1}{i}\left(\sinh(iz)\right) \\
&= \frac{i}{i}\frac{1}{i}\left(\sinh(iz)\right) \\
&= -i\sinh(iz)\ .
\end{aligned}$$

Thus, $\sin(z) = -i\sinh(iz)$. $\square$

6.6 Definition of an Infinite Limit

Exercise 6.6.1

Let $n \in \mathbb{Z}$. Prove that

$$\lim_{n\to\infty} \sqrt{4n^2-1} - 2n = 0.$$

Flawed Proof 6.6.1

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{4n^2-1} - 2n &= \lim_{n\to\infty} \sqrt{4n^2} - 2n \\
&= \lim_{n\to\infty} 2n - 2n \\
&= 0
\end{aligned}$$

$\square$

6.6.1 Error classification

There are several errors in the Flawed Proof 6.6.1.

EO-(WM): Omitted crucial steps of the proof due to a misunderstanding that to "prove" in this context means to use the formal ε definition of an infinite limit.

C-A: The equality $\lim_{n\to\infty} \sqrt{4n^2-1} - 2n = \lim_{n\to\infty} \sqrt{4n^2} - 2n$ requires justification.

Error codes

- Error-Caused Omission due Wrong Method (EO-(WM))
- Content Assertion (C-A)

See Section 1.3 for more information about error classifications.

6.6.2 Corrected proof

The following is a corrected version of Flawed Proof 6.6.1.

Proof 6.6.1

Suppose $n \in \mathbb{Z}$. Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{2\varepsilon}$. Then for $n \geq N$, we have:

$$\begin{aligned}
\left|\sqrt{4n^2-1}-2n-0\right| &= \left|\left(\sqrt{4n^2-1}-2n\right)\frac{\sqrt{4n^2-1}+2n}{\sqrt{4n^2-1}+2n}\right| \\
&= \left|\frac{(4n^2-1)-4n^2}{\sqrt{4n^2-1}+2n}\right| \\
&= \left|\frac{-1}{\sqrt{4n^2-1}+2n}\right| \\
&= \frac{1}{\sqrt{4n^2-1}+2n} \\
&\leq \frac{1}{2n} \, ,
\end{aligned}$$

since $\sqrt{4n^2-1} \geq 0 \Rightarrow \sqrt{4n^2-1}+2n \geq 2n \Rightarrow \dfrac{1}{2n} \geq \dfrac{1}{\sqrt{4n^2-1}+2n}$.

Hence, we have

$$\left|\sqrt{4n^2-1}-2n-0\right| \leq \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon \, .$$

Note that the second inequality follows from the fact that $n \geq N$ implies that $\frac{1}{2N} \geq \frac{1}{2n}$. The third inequality follows from the fact that $N > \frac{1}{2\varepsilon}$ implies that $\frac{1}{2N} < \varepsilon$.

Thus, $\lim\limits_{n\to\infty} \sqrt{4n^2-1}-2n = 0$ by the definition of convergence. $\square$

6.7 Convergent Sequences: Unique Limit

Exercise 6.7.1: Unique Limits

Let $\{a_n\}$ be a sequence of real numbers. Let L and L' be limits of this sequence, that is, suppose that $\{a_n\}$ converges to L and L'. Prove $L = L'$. That is, if a sequence converges, then the limit is unique.

Flawed Proof 6.7.1

By the triangle inequality, we have that

$$\begin{aligned} |L - L'| &= |L - a_n + a_n - L'| \\ &\leq |L - a_n| + |L' - a_n| \\ &< \epsilon. \end{aligned}$$

So $|L - L'| < \epsilon$. But we can make ϵ as small as we want, so the difference between L and L' is infinitely small, and thus they are equal. $\square$

Flawed Proof 6.7.2

Suppose a_n converges to both L and L'. Then,

$$\begin{aligned} |L - L'| &= \lim a_n - \lim a_n \\ &= \lim(a_n - a_n) \\ &= 0\ . \end{aligned}$$

Thus, the limit is unique. $\square$

6.7.1 Error classification

There are multiple errors in the Flawed Proof 6.7.1.

C-O The variable n needs to be defined in relation to L, L' and ϵ.

C-A The final statement has the right idea, but should be made more rigorous.

There is one error in the Flawed Proof 6.7.2.

EO-(F-MT): No progress is made towards this proof due to misusing a theorem: the properties of convergent sequences cannot be used without first establishing the uniqueness of a limit.

Error codes

- Content Omission (C-O)
- Content Assertion (C-A)
- Error-Caused Omission due to Fundamental Misusing Theorem (EO-(F-MT))

See Section 1.3 for more information about error classifications.

6.7.2 Corrected proof

The following is a corrected version of Flawed Proof 6.7.1.

Proof 6.7.1

Suppose $\{a_n\} \subseteq \mathbb{R}$ is a convergent sequence. Suppose that L and L' are limits of the sequence $\{a_n\}$. In order to derive a contradiction, suppose $L \neq L'$. Let $\epsilon = \frac{1}{2}|L - L'|$. Note that this implies that $\epsilon > 0$ and $2\epsilon = |L - L'|$.

Since $\{a_n\}$ converges to L, we know that there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - L| < \epsilon$. Similarly, since $\{a_n\}$ converges to L' there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - L'| < \epsilon$. Let $N = \max\{N_1, N_2\}$. By the triangle inequality we can see that for all $n \geq N$ we have

$$\begin{aligned}
|L - L'| &= |L - a_n + a_n - L'| \\
&\leq |L - a_n| + |L' - a_n| \\
&< \epsilon + \epsilon \\
&= |L - L'|.
\end{aligned}$$

So $|L - L'| < |L - L'|$, which is a contradiction. Thus we must have that $L = L'$. $\square$

6.8 Convergent Sequences: Bounded

Exercise 6.8.1: Convergent Sequences are Bounded

Let (a_n) be a convergent sequence of real numbers. Prove that the sequence is bounded; that is, prove there exists some positive real number M such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Flawed Proof 6.8.1

We know $|a_n - L| < \epsilon$. So by the triangle inequality this means $|a_n| < |L| + \epsilon$. So it is bounded.

□

6.8.1 Error classification

There are several errors in the Flawed Proof 6.8.1.

N-O The variables n, L, and ϵ are not defined.

EO-(F-MT) No progress is made in the proof due to using the triangle inequality incorrectly.

Error codes

- Novice Omission (N-O)
- Error-Caused Omission due to Fundamental Misusing Theorem (EO-(F-MT))

See Section 1.3 for more information about error classifications.

6.8.2 Corrected proof

The following is a corrected version of Flawed Proof 6.8.1.

Proof 6.8.1

Suppose $(a_n) \subseteq \mathbb{R}$ is a convergent sequence. Since (a_n) converges to $L \in \mathbb{R}$. This means that there exists a natural number N such that for all $n \geq N$ we have $|a_n - L| < 1$. This means that for all $n \geq N$ we have

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{n-1}|, |L| + 1\} + 1$. Then for all n, $|a_n| < M$.

$\square$

6.9 Equivalence Classes of Real Numbers

Exercise 6.9.1

Consider the Cauchy sequences $a_n = \frac{1}{3}$ and $b_n = 0.\underbrace{333\cdots 3}_{n-\text{many}}$ for all n. Show that the sequences a_n and b_n are in the same equivalence class.

Flawed Proof 6.9.1

Notice that $b_n \to \frac{1}{3}$ and so $b_n = a_n$. Then,

$$|a_n - b_n| = 0.$$

□

6.9.1 Error classification

There is one error in the Flawed Proof 6.9.1.

C-FI: The fact that $b_n \to \frac{1}{3}$ does not imply that $b_n = a_n$ (i.e. if two sequences have the same limit, this does not mean that the sequences themselves are equal).

Error codes

- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

6.9.2 Corrected proof

The following is a corrected version of Flawed Proof 6.9.1.

Proof 6.9.1

Consider the sequences (a_n) and (b_n) defined above. We have

$$\begin{aligned}|a_n - b_n| &= 0.\underbrace{000\cdots0}_{n-\text{many}}333\ldots \\ &< 0.\underbrace{000\cdots0}_{(n-1)-\text{many}}3 \\ &= \frac{3}{10^n}.\end{aligned}$$

We have $\lim_{n\to\infty} \frac{3}{10^n} = 0$. Therefore, the sequence $a_n - b_n$ converges to 0.

Thus a_n and b_n are in the same equivalence class. $\square$

6.10 Convergent Sequences are Cauchy

Exercise 6.10.1

Prove that if a sequence a_n converges, then a_n is Cauchy.

Flawed Proof 6.10.1

Suppose that $a_n \to L$. Then

$$\begin{aligned}\lim_{n,m\to\infty}(a_n - a_m) &= L - L \\ &= 0 \\ &< \varepsilon\ ,\end{aligned}$$

and so a_n is Cauchy. □

6.10.1 Error classification

There is one error in the Flawed Proof 6.10.1.

EO-(C-VG): No progress is made in this proof due to misunderstanding the definitions of Cauchy, convergence and limits.

Error codes

- Error-Caused Omission due to Content Vocabulary and Grammar (EO-(C-VG))

See Section 1.3 for more information about error classifications.

6.10.2 Corrected proof

The following is a corrected version of Flawed Proof 6.10.1.

Proof 6.10.1

Suppose that $a_n \to L$. Fix $\varepsilon > 0$. Since $a_n \to L$ we know that there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies that $|a_n - L| < \frac{\varepsilon}{2}$.

If $m, n \geq N$ then by the triangle inequality,

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\ . \end{aligned}$$

Therefore, a_n is Cauchy. $\square$

6.11 Cauchy Sequences

Exercise 6.11.1: Almost Real

Let C be the set of all Cauchy sequences of rational numbers. Let $\sim$ be a relation on C given by $(a_n) \sim (b_n)$ if and only if $|a_n^2 - b_n^2| \to 0$. Prove $\sim$ is an equivalence relation.

Flawed Proof 6.11.1

Suppose $(a_n) \sim (b_n)$, which means $|a_n^2 - b_n^2| \to 0$. Well factoring the left means $|a_n - b_n||a_n + b_n| \to 0$, which means $|a_n - b_n| \to 0$. In class we showed that this relation is an equivalence relation, so this one is too.

$\square$

6.11.1 Error classification

There are multiple errors in the Flawed Proof 6.11.1.

C-FI "$|a_n - b_n||a_n + b_n| \to 0$" does not imply "$|a_n - b_n| \to 0$".

N-VG The relation they are referring to from class should be explicitly stated.

F-A Referring to some other relation and saying this implies the result side-steps the entire problem itself.

Error codes

- Content False Implication (C-FI)
- Novice Vocabulary Grammar (N-VG)
- Fundamental Assertion (F-A)

See Section 1.3 for more information about error classifications.

6.11.2 Corrected proof

The following is a corrected version of Flawed Proof 6.11.1.

Proof 6.11.1

Let $(a_n), (b_n), (c_n)$ be Cauchy sequences of rational numbers. We will show $\sim$ satifies the definition of an equivalence relation.

- Reflexive:
  Note that $|a_n^2 - a_n^2| = |0|$, and this constant sequence converges to 0. So $(a_n) \sim (a_n)$.

- Symmetric:
  Suppose $(a_n) \sim (b_n)$. This means $|a_n^2 - b_n^2| \to 0$. But we can interchange the terms under absolute value sign, so this means that $|b_n^2 - a_n^2| \to 0$. So $(b_n) \sim (a_n)$.

- Transitive
  Suppose $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Let $\epsilon > 0$. Since $(a_n) \sim (b_n)$ we know that there exists some N_1 such that for all $n \geq N_1$, $|a_n^2 - b_n^2| < \epsilon/2$. Similarly, since $(b_n) \sim (c_n)$ we know that there exists some N_2 such that for all $n \geq N_2$, $|b_n^2 - c_n^2| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Let $n \geq N$. We have

  $$\begin{aligned} |a_n^2 - c_n^2| &= |a_n^2 - b_n^2 + b_n^2 - c_n^2| \\ &\leq |a_n^2 - b_n^2| + |b_n^2 - c_n^2| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

  So by the definition of a convergent sequence, this means that $|a_n^2 - c_n^2| \to 0$, and hence $(a_n) \sim (c_n)$.

Therefore $\sim$ is an equivalence relation. $\square$

6.12 Interior and Boundary of a Subset of $\mathbb{R}$

Exercise 6.12.1

Suppose that $S \subseteq \mathbb{R}$. Prove that $S \subseteq \operatorname{int}(S) \cup \operatorname{bd}(S)$.

Flawed Proof 6.12.1

Suppose that $x \in S$. If $x \in \operatorname{int}(S)$, then we are done. If $x \notin \operatorname{int}(S)$, then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \nsubseteq S$. Since $x \in S$, then $x \in S \cap N_\varepsilon(x)$, so $S \cap N_\varepsilon(x) \neq \varnothing$. Since $N_\varepsilon(x) \nsubseteq S$, there must also be some $y \in N_\varepsilon(x)$ not in S. This implies that $N_\varepsilon(x) \cap (\mathbb{R} \setminus S) \neq \varnothing$. Thus, $x \in \operatorname{bd}(S)$. $\square$

6.12.1 Error classification

There is only one error in the Flawed Proof 6.12.1.

C-VG: Existential qualifier was used instead of universal qualifier in the definition of $\text{int}(S)$.

Error codes

- Content Vocabulary and Grammar (C-VG)

See Section 1.3 for more information about error classifications.

6.12.2 Corrected proof

The following is a corrected version of Flawed Proof 6.12.1.

Proof 6.12.1

Suppose that S is a subset of $\mathbb{R}$. Suppose that $x \in S$. If $x \in \text{int}(S)$, then we are done. If $x \notin \text{int}(S)$, then for all $\varepsilon > 0$, $N_\varepsilon(x) \nsubseteq S$. Since $x \in S$, then $x \in S \cap N_\varepsilon(x)$, so $S \cap N_\varepsilon(x) \neq \varnothing$. Since $N_\varepsilon(x) \nsubseteq S$, there must exists a $y \in N_\varepsilon(x)$ such that $y \notin S$. This implies that $N_\varepsilon(x) \cap (\mathbb{R} \setminus S) \neq \varnothing$. Thus, $x \in \text{bd}(S)$. $\square$

6.13 Compactness

Exercise 6.13.1

Show that a non-empty closed subset S of a compact metric space X is compact.

Flawed Proof 6.13.1

Suppose that X is compact and that $S \subseteq X$ is closed. We want to show that S is compact. So consider $\{x_n\} \subseteq S$. By Bolzano Weierstrass, $\{x_n\}$ converges and so it has a converging subsequence. Since S is closed, then it contains all of its accumulation points and so S is sequentially compact. $\square$

6.13.1 Error classification

There are several errors in the Flawed Proof 6.13.1.

C-MT: Misused Bolzano-Weierstrass Theorem.

C-FI: Bolzano-Weierstrass does not give convergence of $\{x_n\}$.

Error codes

- Content Misusing Theorem (C-MT)
- Content False Implication (C-FI)

See Section 1.3 for more information about error classifications.

6.13.2 Corrected proof

The following is a corrected version of Flawed Proof 6.13.1.

Proof 6.13.1

Let S be a non-empty closed subset of a compact metric space X. To show S is compact we will show that each sequence in S has a convergent subsequence whose limit is in S.

Let $\{x_n\}$ denote a sequence in S. Since X is compact and $\{x_n\} \subseteq X$, we know that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to $x \in X$. Since S is closed and $\{x_{n_k}\}$ is a convergent sequence in S, we must have that $x \in S$. Therefore, S is compact. $\square$

6.14 Least Upper Bounds

Exercise 6.14.1: $\mathbb{Q}$ is not complete

Let $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$. Prove A does not have a least upper bound in $\mathbb{Q}$.

Flawed Proof 6.14.1

Let $\alpha = LUB(A)$.
$\alpha < \sqrt{2} \;\Rightarrow\; \exists \beta \in \mathbb{Q}$ s.t. $\alpha < \beta < \sqrt{2} \;\Rightarrow\; \alpha \neq LUB(A)$
$\alpha > \sqrt{2} \;\Rightarrow\; \exists \beta \in \mathbb{Q}$ s.t. $\sqrt{2} < \beta < \alpha \;\Rightarrow\; \alpha \neq LUB(A)$
$\Rightarrow \alpha = \sqrt{2} \Rightarrow \alpha \notin \mathbb{Q}$

$\square$

6.14.1 Error classification

There are multiple errors in the Flawed Proof 6.14.1.

N-N $LUB(A)$ to indicate the least upper bound of a set A is not standard notation. If non-standard notation is used, it should be defined.

N-VG More explanation for the method being used (contradiction) and explanation for the steps being taken should be given.

N-A Justifying the existence of β in the proof either requires explicit construction, or density of $\mathbb{Q}$ in $\mathbb{R}$ which should be cited.

Error codes

- Novice Notation (N-N)
- Novice Vocabulary and Grammar (N-VG)
- Novice Assertion (N-A)

See Section 1.3 for more information about error classifications.

6.14.2 Corrected proof

The following is a corrected version of Flawed Proof 6.14.1.

Proof 6.14.1

Let $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$. Suppose A does have a least upper bound $\alpha \in \mathbb{Q}$. We will derive a contradiction by showing that α must equal the irrational number $\sqrt{2}$. To do this, we will show that if $\alpha < \sqrt{2}$ or $\alpha > \sqrt{2}$, then α is not a least upper bound.

Firstly, suppose $\alpha < \sqrt{2}$. We know $1 \in A$ since $1^2 < 2$. So $1 < \alpha < \sqrt{2}$. This means $\alpha^2 < 2$, so $\alpha \in A$. In order to show that α is not a least upper bound, we wish to show that there exists some other $\beta \in A$ with $\alpha < \beta$. To construct β, we wish to find some positive $\epsilon \in \mathbb{Q}$ such that $\beta = \alpha + \epsilon$ and $\beta^2 = (\alpha + \epsilon)^2 < 2$. Since $\alpha < 2$, if we choose $\epsilon < 1$, we have $\beta^2 = (\alpha + \epsilon)^2 = \alpha^2 + 2\alpha\epsilon + \epsilon^2 < \alpha^2 + 5\epsilon$. Hence $\beta^2 < 2$ exactly when $\epsilon < \frac{2-\alpha^2}{5}$. So if $\epsilon = \frac{2-\alpha^2}{6}$, then $\beta > \alpha$ and $\beta \in A$. Hence, if $\alpha < \sqrt{2}$, then α is not a least upper bound for A.

Now suppose $\alpha > \sqrt{2}$. We know that $\alpha < 2$, since 1.5 is an upper bound for A. So $\sqrt{2} < \alpha < 2$. In order to show that α is not a least upper bound, we wish to show there exists some positive $\beta' \in \mathbb{Q}$ such that $\alpha > \beta'$ but $(\beta')^2 > 2$. This would imply that β' is an upper bound, since if $r \in A$ and $r \geq \beta' > 0$, then this would imply that $r^2 \geq (\beta')^2 > 2$, which cannot occur. To construct β' we wish to find some positive $\epsilon \in \mathbb{Q}$ such that $\beta' = \alpha - \epsilon$ and $(\beta')^2 = (\alpha - \epsilon)^2 > 2$. Using the fact that $\alpha < 2$ we have $(\beta')^2 = \alpha^2 - 2\alpha\epsilon + \epsilon^2 > \alpha^2 - 4\epsilon + \epsilon^2 > \alpha^2 - 4\epsilon$. We have $(\beta')^2 > \alpha^2 - 4\epsilon > 2$ if and only if $\epsilon < \frac{\alpha^2-2}{4}$. So choosing $\epsilon = \frac{\alpha^2-2}{6}$ gives $\beta' < \alpha$ and $(\beta'^2) > 2$. Moreover, we have $\beta' > 0$, since $\alpha < 2$ implies $\epsilon = \frac{\alpha^2-2}{6} < \frac{1}{3}$, and so $\alpha > \sqrt{2}$ implies that $\beta' = \alpha - \epsilon > 0$. Hence, β' is a smaller upper bound than α. Therefore, if $\alpha > \sqrt{2}$, then α is not a least upper bound.

We have shown that if $\alpha > \sqrt{2}$ or if $\alpha < \sqrt{2}$, then α cannot be a least upper bound. So the only possible option is $\alpha = \sqrt{2}$. But this number is not rational, so A cannot have a least upper bound in $\mathbb{Q}$. $\square$

6.15 Gaussian Integers

Exercise 6.15.1: Invertible Gaussian Integers

Let $G = \{a + bi \mid a, b \in \mathbb{Z}\}$. Let

$$U = \{\alpha \in G \mid \text{there exists some } \beta \in G \text{ such that } \alpha\beta = 1\}.$$

Prove that $U = \{1, -1, i, -i\}$.

Hint: use the modulus of a complex number.

Flawed Proof 6.15.1

Let $\alpha \in G$. Now let $\beta = \frac{1}{\alpha}$. Then $\alpha\beta = \alpha\frac{1}{\alpha} = 1$, so $\alpha \in U$.

$\square$

6.15.1 Error classification

There are several errors in the Flawed Proof 6.15.1.

WP It looks like the flawed proof is attempting to show $G = U$.

F-FI The fact that $\alpha\frac{1}{\alpha} = 1$ does not show $\alpha \in U$, since for this to be true we need $\frac{1}{\alpha} \in G$ and determining when this happens is the essence of the problem.

Error codes

- Wrong Problem (WP)
- Fundamental False Implication (F-FI)

See Section 1.3 for more information about error classifications.

6.15.2 Corrected proof

The following is a corrected version of Flawed Proof 6.15.1.

Proof 6.15.1

Let G and U be defined as above. First we will show that $\{1, -1, i, -i\} \subseteq U$. We have $1 \cdot 1 = 1$, $-1 \cdot -1 = 1$, $i \cdot -i = 1$ and $-i \cdot i = 1$. Therefore, each of these four elements are in U, and so $\{1, -1, i, -i\} \subseteq U$.

Next we will show $U \subseteq \{1, -1, i, -i\}$. Suppose $\alpha \in U$. Then we can write $\alpha = a + bi$ for some $a, b \in \mathbb{Z}$. Moreover, since $\alpha \in U$ we know there exists a $\beta \in G$ such that $\alpha\beta = 1$. Taking the modulus of each side of $\alpha\beta = 1$ and squaring we have

$$|\alpha|^2 |\beta|^2 = 1.$$

Note that $|\alpha|^2 = a^2 + b^2 \in \mathbb{Z}$, and similarly, $|\beta|^2 \in \mathbb{Z}$. So, both $|\alpha|^2$ and $|\beta|^2$ are integers dividing 1. Since the modulus of a nonzero complex number is positive, we must have $|\alpha|^2 = |\beta|^2 = 1$. In particular, $|\alpha|^2 = a^2 + b^2 = 1$. Since a, b are integers, this implies that either $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$. Hence, $a + bi \in \{1, -1, i, -i\}$, and so $U \subseteq \{1, -1, i, -i\}$.

Therefore $U = \{1, -1, i, -i\}$.

$\square$

6.16 Primitive Roots of Unity

Exercise 6.16.1: Primitive Roots of Unity

Let $n \in \mathbb{N}$. Let

$$P = \{\beta \in \mathbb{C} \mid \beta^n = 1 \text{ but } \beta^k \neq 1 \text{ for every positive integer } k < n\}.$$

Let

$$Q = \{e^{2i\pi k/n} \mid k \in \{1, ..., n\} \text{ and } \gcd(k, n) = 1\}.$$

Show $P = Q$.

Flawed Proof 6.16.1

Let $n \in \mathbb{N}$. In polar form, we can write $1 = e^{0\pi i}$. So, its n-th roots, or all the complex numbers whose n-th powers are equal to 1, are of the form $e^{2ki\pi/n}$, for positive integers $k < n$. So elements in P must be of this form. We just need to show that they all satisfy $\gcd(k, n) = 1$.

Let $\beta \in Q$. We can write $\beta = e^{2ki\pi/n}$ for some positive integer $k < n$ with $\gcd(n, k) = 1$. Then $\beta^n = e^{2ki\pi} = 1$. So $\beta \in P$.

Suppose $\beta \in P$. If it were in Q then we could write it as $\beta = e^{2ki\pi/n}$ for some positive integer $k < n$ with $\gcd(n, k) = 1$, which means $\beta^n = 1$, so $\beta \in P$ as we assumed. So there is no contradiction. So $\beta \in Q$.

Thus $P = Q$. $\square$

6.16.1 Error classification

There are several errors in the Flawed Proof 6.16.1.

EO-(C-FI) In the second paragraph, $\beta^n = 1$ does not imply $\beta \in P$. This error causes the omission of part of the proof.

F-FI The third paragraph uses faulty logic, and again $\beta^n = 1$ does not imply $\beta \in P$.

Error codes

- Error-Caused Omission due to Content False Implication (EO-(C-FI))
- Fundamental False Implication (F-FI)

See Section 1.3 for more information about error classifications.

6.16.2 Corrected proof

The following is a corrected version of Flawed Proof 6.16.1.

Proof 6.16.1

Let $n \in \mathbb{N}$.

We will first show that $Q \subseteq P$. Let $\beta \in Q$. We know that $\beta = e^{2ki\pi/n}$ for a positive integer $k \leq n$ with $\gcd(n,k) = 1$. We have $\beta^n = e^{2ki\pi} = 1$. So, it suffices to show that if $\beta^r = 1$ for a positive integer r, then $r \geq n$. Suppose $\beta^r = 1$ for a positive integer r. Then $e^{2kri\pi/n} = 1$, which means that kr/n is an integer. Hence, $n|kr$. Since $\gcd(n,k) = 1$ this means that $n|r$. Therefore $r \geq n$ since r is a positive integer. Hence, $\beta \in P$, and therefore $Q \subseteq P$.

We will now show that $P \subseteq Q$. Suppose $\beta \in P$. Since $\beta^n = 1$, we know that $\beta = e^{2ki\pi/n}$ for some positive integer $k \leq n$. In order to derive a contradiction, suppose $\beta \notin Q$. This would imply that $\gcd(k,n) = d > 1$. Since $d|k$, since implies that $\beta^{n/d} = e^{2ki\pi/d} = 1$. However, $0 < n/d < n$, which implies that $\beta \notin P$, which is a contradiction. Hence, $\beta \in Q$, and so $P \subseteq Q$.

Thus $P = Q$. $\square$

6.17 Fundamental Theorem of Algebra

Exercise 6.17.1: Fundamental Theorem of Algebra

The fundamental theorem of algebra states that every polynomial with complex coefficients has at least one complex root. Show that this implies that every complex polynomial of the form $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ for n a positive integer and $a_{n-1}, ..., a_0 \in \mathbb{C}$ can be factored as $p(z) = (z - c_1)(z - c_2)...(z - c_n)$ for some $c_1, ..., c_n \in \mathbb{C}$.

Flawed Proof 6.17.1

$p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$. By the fundamental theorem of algebra, $p(z)$ has a complex root so $p(z) = (z^{n-1} + a_{n-1}z^{n-2} + ... + a_1)(z - a_0)$. Repeating this process n times means $p(z) = (z - a_0)(z - a_1)...(z - a_{n-1})$. $\square$

6.17.1 Error classification

There are several errors in the Flawed Proof 6.17.1.

N-O The coefficients $a_0, ..., a_{n-1}$ were not defined.

F-FS Stating that $p(z) = (z^{n-1} + a_{n-1}z^{n-2} + ... + a_1)(z - a_0)$ is false.

Error codes

- Novice Omission (N-O)
- Fundamental False Statement (F-FS)

See Section 1.3 for more information about error classifications.

6.17.2 Corrected proof

The following is a corrected version of Flawed Proof 6.17.1.

Proof 6.17.1

We will proceed by induction on n.

Base Case: If $n = 1$, then $p(z) = z + a_0$ for $a_0 \in \mathbb{C}$. Hence, $p(z) = z - (-a_0)$, as desired.

Induction Hypothesis: Suppose that for some positive integer n, every complex polynomial of the form $p(z) = z^{n-1} + a_{n-2}z^{n-2} + ... + a_0$ with $a_{n-2}, ..., a_0 \in \mathbb{C}$ can be factored as $p(z) = (z - c_1)(z - c_2)...(z - c_{n-1})$ for some $c_1, ..., c_{n-1} \in \mathbb{C}$.

Induction Step: Consider a complex polynomial of the form $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ with $a_{n-1}, ..., a_0 \in \mathbb{C}$. By the fundamental theorem of algebra $p(z)$ has a complex root, c_n. By the quotient-remainder theorem we can write

$$p(z) = (z - c_n)q(z) + r$$

for some $r \in \mathbb{C}$ and some $(n-1)$-th degree polynomial $q(z)$. Since $p(c_n) = 0$, we have $(c_n - c_n)q(c_n) + r = 0$, which implies that $r = 0$. Thus $p(z) = (z - c_n)q(z)$. Note that since the leading coefficient of $p(z)$ is equal to 1, the coefficient in front of z^{n-1} in $q(z)$ must also be 1. Therefore, by the induction hypothesis, we can write $q(z)$ in the form $(z - c_1)(z - c_2)...(z - c_{n-1})$ for some $c_1, ..., c_{n-1} \in \mathbb{C}$. Hence $p(z) = (z - c_1)(z - c_2)...(z - c_{n-1})(z - c_n)$.

Therefore, by induction, every complex polynomial of the form $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ with $a_{n-1}, ..., a_0 \in \mathbb{C}$ and n a positive integer can be factored as $p(z) = (z - c_1)(z - c_2)...(z - c_n)$ for some $c_1, ..., c_n \in \mathbb{C}$.

$\square$

6.18 Ring Theory

Exercise 6.18.1: Zero Divisor or Unit

Let R be a finite ring with unity. Prove that every element of R is either a unit or a zero divisor.

Flawed Proof 6.18.1

Let $r \in R$. Thus by Lagrange's theorem $r^{|R|} = 1$. Thus $r(r^{|R|-1}) = 1$, thus r is a unit.

$\square$

6.18.1 Error classification

There is one error in the Flawed Proof 6.18.1.

C-MT Misuse of Lagrange's theorem.

Error codes

- Content Misusing Theorem (C-MT)

See Section 1.3 for more information about error classifications.

6.18.2 Corrected proof

The following is a corrected version of Flawed Proof 6.18.1.

Proof 6.18.1

Let R be a finite ring with unity. Let 1 denote the multiplicative identity of R. Fix $r \in R$. Consider the map

$$\begin{aligned} m_r : R &\to R \\ x &\mapsto rx. \end{aligned}$$

We have two possibilities: either m_r is injective or it is not injective. If m_r is injective, then it is also surjective because R is finite. Hence, there exists some $x \in R$ such that $1 = rx$, and thus r is a unit. On the other hand, if m_r is not injective, then there exists $x, x' \in R$ with $x \neq x'$ such that $rx = rx' \implies r(x - x') = 0$, and so r is a zero divisor.

□

Part II

Additional Resources

Chapter 7

Activities and Assessments

In this chapter, we include some sample activities that can be adapted to use the resources from the earlier chapters.

The simplest in-person activity, suitable for all of the above resources, would be a Think-Pair-Share or (small) Group-Discussion of the flawed proof. Such an activity could then be debriefed by an instructor or teaching assistant, or a variety of follow-up activities (like a quiz where students are asked to write a correct proof of a similar statement).

7.1 Online Discussion Board Activity

Learning Outcomes

1. Create proofs of mathematical statements; establish clear and consistent notation; write a clear and organized logical argument.

2. Work with precise definitions and reason in an abstract setting.

3. Verify that an abstract mathematical object satisfies a given definition, or is a counterexample.

4. Read and critique a mathematical proof, and use a rubric to provide constructive feedback to a peer.

Instructions

1. Consider the **Flawed Proof** the (true) statement below. *The proof contains both mathematical and writing errors.*

2. Use the Peer Evaluation Rubric (§8.1) for Mathematical Writing to score the proof in four categories: Notation, Language & Clarity, Logic, and Completeness.

3. Briefly (300-400 words) describe the mathematical and writing errors that you find.

4. Read the evaluations given by your group members. Did you give the proof the same scores? Did you find the all the same errors? o What (if any) are the differences between your evaluations? Post replies (50-100 words) to each of your group members discussing your observations.

*** **Remember to assume positive intent; keep your comments constructive and professional.**

Statement: *Statement text.*

Flawed Proof. *Proof text.* □

7.2 Peer Evaluation Activity

Learning Outcomes

1. Create proofs of mathematical statements; establish clear and consistent notation; write a clear and organized logical argument.

2. Work with precise definitions and reason in an abstract setting.

3. Verify that an abstract mathematical object satisfies a given definition, or is a counterexample.

4. Read and critique a mathematical proof, and use a rubric to provide constructive feedback to a peer.

*** **To achieve these outcomes, and to receive valuable feedback on your writing, is important for you to write your proof on your own, without aids, and without discussing the proof with your peers in advance.**

Instructions

1. Formulate a precise statement of, and write a complete and detailed proof of, the following statement (below).

2. Post your proof to the Writing Assignment # N discussion board; you can add your proof as a PDF attachment or type it directly using the LaTeX environment.

3. Read the proofs written by your group members and use the **Peer Evaluation Rubric for Mathematical Writing** (§8.1) to score your peers proofs as “Beginning, Developing” or “Accomplished” in four categories: Notation, Language & Clarity, Logic, and Completeness.

4. You will also provide brief written comments to explain your scoring.

*** **Remember to assume positive intent; keep your comments constructive and professional.**

Statement

If the composition of two functions is surjective and the second function is injective, then the first function must be surjective.
Instructor note: try to keep the statement general so that students have to formulate it precisely and introduce their own notation.

7.3 Proof-Writing Reflection

In the following activity, students attempt to prove a statement, post their proof attempt on the Discussion Board, then provide positive and constructive feedback to their peers using a simplified version of Strickland and Rand's error-coding scheme (Section 1.3). The simplified error list used for this activity can be found in Section 8.2. At the end of the semester, students submit a final reflection where they reflect on the experience and summarize how their proof-writing has evolved throughout the semester. In preparation for this activity, students practice identifying errors in class using Section 5's flawed proofs.

In order create a low-stakes safe environment for practicing proof writing, students are not graded based on the mathematical correctness of their proofs or feedback. In practice, we've found that many students are very good at pointing out communication errors, but often struggle to identify logical errors if they have not yet mastered the material. As such, we would recommend that an instructor or teaching assistant also provide feedback on the Discussion Board if possible.

Overview:

This *Proof-Writing Assignment* accounts for 15% of your final grade. It is designed to help you develop your proof-writing skills by individually attempting proofs, providing and receiving formative feedback from your peers, and reflecting on the experience in a *Final Reflection*.

To complete this task you must:

1. Complete 5 proof activities throughout the semester. To complete the proof activity corresponding to Topic x you must:
    - Submit your proof attempt to the Topic x Discussion Board by 11:59PM MST on Wednesday of Week x.
    - Provide constructive feedback to *two* other Topic x proof attempts by 11:59PM MST on Wednesday of Week $(x + 1)$. Please see *Constructive Feedback Guidelines* below.
2. Submit your *Final Reflection* to the D2L Dropbox by 11:59PM MST on Wed. Mar. 31st.

Note: There are 10 proof prompts, but you only need to complete 5. If you find these activities helpful and would like to complete more than 5 you're welcome to do so, but please only include 5 in your *Final Reflection*.

Final Reflection Instructions:

Your *Final Reflection* (due Wed. Mar. 31st) should have the following format:

- At the beginning of your document, include a screenshot of your five proofs and your 10 replies to other people's proofs. Please organize these by topic and include the date in the screenshot.

- Write a ∼750-word reflection about what you learned during this proof-writing activity. You may also reflect more generally about how your proof-writing skills developed throughout the course.

  You may put the entire reflection at the end of your screenshots, or you may organize it by writing a ∼150-word reflection after each set of screenshots (or a combination of the two).

  Here are a few prompts to help guide your reflection. Please don't feel like you need to respond to each of these; they are provided to help spark ideas to get you started.

  - What did you learn from the feedback you received from your peers? What did you do in light of this feedback?
  - What did you learn from reading your peer's proofs and providing constructive feedback? How did this impact your proof-writing?
  - How did this experience make you feel (positively and/or negatively)?
  - Did you have any "aha!" moments during these proof-writing activities?

Grading Rubric:

Only the *Final Reflection* document will be graded. It will account for 15% of your course grade and will be graded out of 100pts:

- **30 points:** Posting your five proofs by the deadlines in the appropriate forums.
- **20 points:** Providing constructive feedback to 10 students (i.e. 2 students per topic) by the deadlines in the appropriate forums.
- **10 points:** The feedback given followed the *Constructive Feedback Guidelines*.
- **5 points:** The *Final Reflection* document was organized and followed the format guidelines described about (word count, organized by topic).
- **35 points:** The reflection was thoughtful and robust. (*In past semesters, I have only rarely removed marks for a reflection piece.*)

Note that you will **not** be graded based on the quality of your proofs nor based on whether or not the feedback you provided was mathematically correct.

Constructive Feedback Guidelines:

- When providing feedback, prioritize posts which do not have any replies yet. (...this won't always be possible, but if there exist posts with no replies, please provide feedback to these posts first. We want to ensure that everyone receives feedback.)
- Provide at least one piece of positive feedback praising the student for what they have done right.
- Provide at least one piece of feedback intended to help the student improve. This constructive feedback does not need to give the solution (a key will be provided). The feedback should help empower the student to understand how to improve their work.
- Describe errors using our *error-coding scheme* (e.g. assertion, false implication, misusing theorem, etc.).
- Be respectful.

Tip: You may want to make personal notes as you go about what you learned when providing and receiving feedback. These notes will make the process of writing your final reflection easier.

Chapter 8

Rubrics

In this final chapter, we provide sample self- and peer-evaluation rubrics to be used along with the activities and resources outlined in earlier chapters. We also provide a sample summative rubric that uses language inspired by Strickland and Rand's error-coding scheme (Section 1.3).

8.1 Peer-Evaluation Rubrics

Here we provide two peer-evaluation rubrics. The first is tailored to a discrete mathematics or "intro-to-proof" course, and the second is tailored to a course in linear algebra. Both of the rubrics refer to a "proof writing handbook" which we include here to provide additional context.

8.1.1 The Proof Writing Handbook

Let P and Q be statement variables. When needed, suppose that $P = P(x)$ depends on a variable x. The symbol "$\forall$" means "for all" or "for any". The symbol "$\exists$" means "there exists".

Type of statement	**What we must do to prove that it is true**
(1) If P, then Q (2) $\forall\, P$, Q	Suppose that P is true. Prove that Q is true.
(3) $\exists\ x\ P(x)$ such that Q	Choose** x so that $P(x)$ is true. Prove that Q is true.

You **do not need to explain how you find x, nor do you need to construct **all possible** x.

Rule #1: To prove that a statement is **false**, you must **write out the negation of the statement and prove that.**

Five Common Mistakes to Avoid

- When proving any of the types of statements (1), (2), or (3):

  1. **You cannot** suppose that Q is true.
  2. **You should not** overuse symbols nor violate the rules of grammar.†

  † You **must** write in full sentences and use symbols correctly.

- When proving a statement of the form (2) "$\forall\, P$, Q":

  3. **You cannot** "choose" or exhibit an example in place of a proof.

- When proving a statement of the form (3) "$\exists\ x\ P(x)$ such that Q":

  4. **You should not** attempt to construct all possible x so that $P(x)$ and Q are true.

- When proving a statement **by contradiction**:

  5. **You cannot** claim a contradiction has been reached without explanation.††

  †† You **must** clearly identify the contradiction being made by making a statement of the form "P and NOT P, which is a contradiction".

PEER-EVALUATION OF MATHEMATICAL WRITING

Why are we doing this?

Communicating mathematical ideas clearly in writing is a skill that takes practice. It is challenging to learn how to write mathematics well while you are immersed in learning the concepts themselves. Evaluating the clarity of your own writing is also difficult because *you know what you mean to say*.

Our goals in evaluating the mathematical writing of ours peers are to

(1) provide useful formative feedback to our peers so that they may improve their writing,
(2) discover that we face challenges that our peers are facing as they learn to write mathematics,
(3) improve our own mathematical writing by analyzing and constructively critiquing the writing of others.[1]

The Rubric

Rating	**Beginning (1)**	**Developing (2)**	**Accomplished (3)**
Notation	Some variables and/or new/non-standard are not defined; <u>consistent</u> mistakes are made with standard notation, e.g., Z instead of $\mathbb{Z}$; $=$ used incorrectly	Variables are all defined; <u>few mistakes</u> are made with standard notation; $=$ is used correctly; for example, may make mistakes with $\in$ and $\subset$	Variables are all defined; all standard notation is used correctly; new notation is efficient and helpful
Language & Clarity	Arguments are unclear or confusing; contains many spelling and grammar mistakes; overuse of symbols and equations instead of sentences	Sometimes challening to read or understand arguments, contains <u>few</u> spelling and grammar mistakes; full sentences used to clarify equations	Easy to read and understand arguments; contains <u>very few</u> spelling and grammar mistakes; use of full sentences balanced well with equations and symbols adding to the clarity;
Logic	Contains *false* statements; <u>begins with</u> or <u>assumes</u> the conclusion; quantifiers "for all", "there exists" used incorrectly; hard to understand logical connections	Follows the proof handbook; <u>most</u> logical connections clear; <u>few</u> missing/implied quantifiers; all assumptions and conclusion present	Follows proof handbook; connections between statements are clear; quantifiers "for all", "there exists" are present and used correctly; easy to follow argument
Completeness	Many details are missing; Theorems are used without checking assumptions; uses words like "obvious" to cover up missing details	Few missing details; most assumptions of Theorems used are checked; things called "obvious" really are true and easy to check	All necessary details are present; the assumptions of Theorems used are checked; nothing "obvious" is left out

Date: May 6, 2021.

[1] Recent research has shown that the best way to improve your mathematical writing is to read, analyze, and correct proofs that contain (serious) errors. See, for instance: A. Selden and J. Selden, *Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem?*. Journal for Research in Mathematics Education, Vol. 34, No. 1 (2003), pp. 4–36. (stable link).

Some Common Errors

Notation.

- Declaring a variable, but forgetting to say what it is, i.e., working with $n = 2k$ but forgetting to say that n and k are integers.
- Reusing one variable for two different purposes
- Interchanging $\in$ and $\subset$
- Misuse of the number zero 0 and the empty set $\emptyset$
- Misuse of the notation $a \mid b$ for "a divides b", i.e., do **not** write " $2 \mid 6 = 3$". Instead write: "2 divides 6", "6 divided by 2 is 3", or "$\frac{6}{2} = 3$."

Language & Clarity.

- Overuse of symbols. Often "$\Rightarrow$" is overused and used incorrectly.
- Please, please, please **don't** use $\therefore$ and $\because$ See, it looks terrible!
- Using "Obvious", "Clearly", etc. to cover up missing details.

Logic & Mathematical Errors.

- Including false implications in a proof. For instance, writing: "Since $x \in \mathbb{R}$ is nonzero, x must be positive", instead of the correct statement "Since $x \in \mathbb{R}$ is nonzero, we know that either x is positive or x is negative."
- To prove "if P then Q", you cannot assume that Q is true – you must suppose that P is true and argue that Q is true.
- The negation of $P \Rightarrow Q$ is **not** "NOT $P \Rightarrow$ NOT Q". The negation of $P \Rightarrow Q$ is in fact the statement "P and NOT Q".
- Interchanging "there exists $= \exists$" and "for all $= \forall$"
- Writing "for all $k \in \mathbb{Z}$" as part of an induction hypothesis.
- Writing "Suppose" when you mean "Choose" (when proving an existence statement)
- Writing "Choose" when you should "Suppose" (when proving a "for all" statement)
- The number 0 is even (don't make the mistake of saying that "0 is neither even nor odd").
- The number 1 is not prime (by definition primes are greater than 1).

Completeness.

- Forgetting to write out the negation when you are proving that a statement is false.
- Forgetting to indicate that your proof is an argument by contradiction.
- Forgetting to indicate that your proof is an argument by mathematical induction.
- Assuming that a proof must be written at a level appropriate for the instructor, and thus omitting important details. Think of proof writing as an opportunity to demonstrate your clear understanding of the topic.
- My advice: write your proof for your "future-self", i.e., write down all the details that you might forget later, or write your proof for an audience of your peers.
- More advice: if you are using a definition that we gave in class, make sure that you carefully prove that an object satisfies the definition, i.e., to show that 6 is even write "$6 = 2 \times 3$ and since $3 \in \mathbb{Z}$ the number 6 is even."

PEER-EVALUATION OF MATHEMATICAL WRITING FOR LINEAR ALGEBRA

Why are we doing this?

Communicating mathematical ideas clearly in writing is a skill that takes practice. It is challenging to learn how to write mathematics well while you are immersed in learning the concepts themselves. Evaluating the clarity of your own writing is also difficult because *you know what you mean to say*. Our goals in evaluating the mathematical writing of ours peers are to

(1) provide useful formative feedback to our peers so that they may improve their writing,
(2) discover that we face challenges that our peers are facing as they learn to write mathematics,
(3) improve our own mathematical writing by analyzing and constructively critiquing the writing of others.[1]

The Rubric

Rating	**Beginning (1)**	**Developing (2)**	**Accomplished (3)**
Notation	Some variables and/or new/non-standard are not defined; <u>consistent</u> mistakes are made with standard notation, e.g., Rn instead of $\mathbb{R}^n$; $=$ used incorrectly	Variables are all defined; <u>few mistakes</u> are made with standard notation; $=$ is used correctly; for example, may make mistakes with $\in$ and $\subset$	Variables are all defined; all standard notation is used correctly; new notation is efficient and helpful
Language & Clarity	Arguments are unclear or confusing; contains many spelling and grammar mistakes; overuse of symbols and equations instead of sentences	Sometimes challening to read or understand arguments, contains <u>few</u> spelling and grammar mistakes; full sentences used to clarify equations	Easy to read and understand arguments; contains <u>very few</u> spelling and grammar mistakes; use of full sentences balanced well with equations and symbols adding to the clarity;
Logic	Contains *false* statements; <u>begins with</u> or <u>assumes</u> the conclusion; quantifiers "for all", "there exists" used incorrectly; hard to understand logical connections	Follows the proof handbook; <u>most</u> logical connections clear; <u>few</u> missing/implied quantifiers; all assumptions and conclusion present	Follows proof handbook; connections between statements are clear; quantifiers "for all", "there exists" are present and used correctly; easy to follow argument
Completeness	Many details are missing; Theorems are used without checking assumptions; uses words like "obvious" to cover up missing details	Few missing details; most assumptions of Theorems used are checked; things called "obvious" really are true and easy to check	All necessary details are present; the assumptions of Theorems used are checked; nothing "obvious" is left out

Date: May 6, 2021.

[1]Recent research has shown that the best way to improve your mathematical writing is to read, analyze, and correct proofs that contain (serious) errors. See, for instance: A. Selden and J. Selden, *Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem?*. Journal for Research in Mathematics Education, Vol. 34, No. 1 (2003), pp. 4–36. (stable link).

Some Common Errors

Notation.

- Declaring a variable, but incorrectly stating where it lives, e.g., $\boldsymbol{x} \in \mathbb{R}$ when you should say that $\boldsymbol{x} \in U$, where U is a subset of $\mathbb{R}$.
- Reusing one variable for two different purposes
- Interchanging $\in$ and $\subset$
- Misuse of the zero vector $\vec{0}$ and the zero subspace $\{\vec{0}\}$
- Misuse of the number zero 0 and the empty set $\emptyset$

Language & Clarity.

- Overuse of symbols. For instance, $\therefore$ and $\because$ start to look a lot alike but have different meaning.[2]

Logic & Mathematical Errors.

Logic.

- Including false implications in a proof. For instance, writing: "Since $\{\boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent, for $a = 0, b = 0$ we have $a\boldsymbol{v} + b\boldsymbol{w} = \vec{0}$" is <u>incorrect</u>. We could correct this to say: "Since $\{\boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent, if $a\boldsymbol{v} + b\boldsymbol{w} = \vec{0}$ then $a = 0$ and $b = 0$."
- To prove "if P then Q", you cannot assume that Q is true – you must suppose that P is true and argue that Q is true.
- The negation of $P \Rightarrow Q$ is **not** "NOT $P \Rightarrow$ NOT Q". The negation of $P \Rightarrow Q$ is in fact the statement "P and NOT Q".
- Interchanging "there exsits = $\exists$" and "for all = $\forall$"
- Writing "Suppose" when you mean "Choose" (when proving an existence statement)
- Writing "Choose" when you should "Suppose" (when proving a "for all" statement)

Linear algebra.

- Misusing the scalar 0 and the vector $\vec{0}$ or $\mathbf{0}$.
- Using "span" incorrectly. For instance, $\{v_1, \ldots, v_k\} \neq \operatorname{span}\{v_1, \ldots, v_k\}$, and $\operatorname{span}\{e_1, \ldots, e_n\}$ is **not** a basis of $\mathbb{R}^n$ – the finite set $\{e_1, \ldots, e_n\}$ is a basis.

Calculus.

- Dropping "lim" from the calculation of a limit.
- Dropping "$\sum$" from manipulations involving series.
- Confusing a series $\sum_{n=1}^{\infty} a_n$ with the sequence of its terms $\{a_n\}_{n=1}^{\infty}$.

Completeness.

- Assuming that a proof must be written at a level appropriate for the instructor, and thus omitting important details. Think of proof writing as an opportunity to demonstrate your clear understanding of the topic.
- My advice: write your proof for your "future-self", i.e., write down all the details that you might forget later, or write your proof for an audience of your peers.

[2]Please, please, please **don't** use $\therefore$ and $\because$ See, it looks terrible!

8.2 Summative Rubrics

Here is a sample summative rubric for a linear algebra course. Breaking the rubric into a *Mathematics Score* and a *Communication Score* helps to emphasize that mathematical communication is very important and will contribute to their grade.

In practice, I usually provide this sample rubric to my students as a guide, but then fine-tune it for each individual question. In particular, for each question I break down the *Mathematics Score* into concrete pieces (e.g. 1pt for showing the set is linearly independent, etc.). I grade the *Communication Score* holistically. The clause that the *Communication Score* may not exceed the *Mathematics Score* is included so that students who trivialize the problem (e.g., submits a single sentence that contains no communication errors) do not receive a higher score than students who made several communication errors in the process of making good progress towards the problem. In practice, this clause is meant to be taken as a guideline and is not applied strictly.

MATH 311

Proof-Writing Rubric

Communicating mathematical ideas clearly in writing is a skill that takes practice. Writing a complete solution also challenges you to place the problem in context and develop a deep understanding.

Our goals in evaluating your written solutions are to

1. provide useful feedback so that you may improve your mathematical writing skills, and
2. provide opportunities to engage with challenging problems and practice problem-solving.

Communication Score: 3pts

Attribute	Criteria	Excellent (3pts)	Very Good (2pts)	Good (1pt)	Needs Improvement (0pts)
Communication Score	Submission is well articulated using proper **notation** (e.g. all variables are defined) **prose** are used to help the reader navigate (e.g. solution is organized, uses full sentences, computations are introduced, results are clearly articulated), and there is **no extraneous detail** (e.g. no irrelevant results or computations). The Communication Score may not exceed the Mathematics Score.	The submission is **clearly** communicated and follows **all** of the communication criteria (notation, prose, extraneous detail).	The submission is **clearly** communicated and follows **most** of the communication criteria (notation, prose, extraneous detail).	The submission is **unclearly** communicated and follows **some** of the communication criteria (notation, prose, extraneous detail).	The submission is **unclearly** communicated and follows **few** of the communication criteria (notation, prose, extraneous detail).

Mathematics Score: 4 pts

Attribute	Criteria	Excellent (4pts)	Very Good (3pts)	Good (2pt)	Needs Improvement (0-1pts)
Mathematics Score	Submission is mathematically **correct** (e.g. no false statements, calculations are accurate, understanding of theory is evident), contains an **appropriate amount of detail** (e.g. assumptions of theorems are checked, claims are justified), and **solves the problem**.	Submission follows **all** of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows **most** of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows **some** of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows **few** of the mathematics criteria (correct, appropriate amount of detail, solves the problem).

Proof-Writing Checklist

To make sure your solution is complete, it may help to ask yourself the following questions:

Communication Score

Notation

- Are all variables defined? (e.g., did you say what **x** represents?)
- Am I using the proper notation? (e.g., a finite set versus a span.)

Prose

- Is my solution written in full sentences and organized in paragraphs (or does it look like a string of implications and computations)?
- Are my computations introduced so that the reader knows what I am about to do? (e.g., before jumping into a matrix algebra computation, we need to tell the reader how this algebra ties back to the problem at hand.)
- Do I have an introductory and concluding sentence so that it's clear what I am about to do and what my result is?

Extraneous Detail

- Have I included details that are irrelevant and do not contribute to my result? (e.g. scrap work that isn't necessary to the problem)
- Is my explanation concise, or can I communicate this is an easier way?

Mathematics Score

Correctness

- Is the terminology I am using accurate? (content vocabulary & grammar)
- If I used a theorem, am I using it correctly? (not misusing a theorem)
- Are all of my statements logically correct? (no false statements.)

Appropriate Amount of Detail (no assertions)

- If a fellow classmate who knows a little less than me was reading my solution, would they understand all steps, or should I provide more details?
- If I used a theorem, am I explaining why all assumptions are satisfied?

Solves the Problem

- Is my proof missing any parts? (no omitted sections.)
- Have I reread the prompt to make sure that I read it correctly and didn't miss any parts if it is a multipart question?
- Have I doubled-checked to make sure that my single PDF contains all of my work, and I haven't accidentally missed a page?

Proof-Writing Tips:

I strongly recommend first working out your thoughts on scrap paper, then writing a good copy (the way you initially solve a problem usually isn't the most optimal way to explain it to others).

Your "audience" should be another student in the class who knows a little less than you. They should be able to understand what you wrote during the first read through; if they need to read it several times then it is not communicated clearly.

- For example, suppose you claim that a statement is false and provide an explicit counterexample. You must clearly explain why the given counterexample satisfies the hypotheses of the statement, and why it does not satisfy the conclusion. (Even if the counterexample works, if you have not provided this justification, you will not receive full points.)
- Suppose a prompt asks you, "Is P true? Explain." If you just answer "Yes" or "No" without proper explanation, you will not receive any points.
- Be careful about notation. For example, {<1,0,0>,<0,1,0>} is a finite set with two vectors. On the other hand, span{<1,0,0>,<0,1,0>} is an infinite set. Do no write a finite set when you mean to write a span, and vice versa.
- If you are doing a matrix algebra computation, be sure to first clearly explain to the reader what you are doing and why you are doing it. There are often multiple ways to solve a problem, so it's important that you let you reader know what you're about to do. (e.g. "To show this set spans U I need to show that this system is consistent because… . To show this system is consistent, I need to ensure the rank equals…. *Now it's fine to do your matrix algebra because we know what your goal is and why.*) If you just show a bunch of matrix algebra with little or no explanation, you will not receive many points (because it can be terribly confusing to read and understand!).

Error Types:

Here are the errors we've seen so far in class. Keep the three "Writing Clarity" error types in mind when you proof-read your work on tests.

Mathematical Errors:

- False Implication
- False Statement (a statement that's false, but it's not an implication)
- Content Vocabulary & Grammar (misuse of definition or vocabulary)
- Assertion (result stated without proper justification)
- Omitted Section (e.g., a case is missing)
- Misusing a theorem (e.g., applying the converse of a theorem)
- Wrong Method (e.g., if the prompt says to use a certain method and they don't)
- Wrong Problem (e.g., told to prove the converse, but prove something else)

Writing Clarity Errors:

- Notation
- Rhetorical Vocabulary & Grammar (prose is poorly written)
- Extraneous Detail (attempted or proved irrelevant results)

Bibliography

[1] Annie Selden and John Selden, *Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem?*, Journal for Research in Mathematics Education **34** (2003), no. 1, 4–36.

[2] S. Strickland and B. Rand, *A framework for identifying and classifying undergraduate student proof errors*, PRIMUS **26** (2016), no. 10, 905–921.