Solution to Problem H.4

$y = X\beta + \epsilon$, X is $n \times (p+1)$ matrix.
Rank of $X = p+1$

$\hat{\mu} = X\hat{\beta}$, $\hat{\mu}$ is fitted values, $\hat{\beta}$ is vector of least square estimates

$H = X(X'X)^{-1}X'$ → hat matrix.

a) Mean vector and covariance matrix of $\hat{\mu}$

i) Mean vector of $\hat{\mu}$

$$E(\hat{\mu}) = E(X\hat{\beta})$$
$$= X\,E(\hat{\beta})$$

$$E(\hat{\beta}) = E\left[(X'X)^{-1}X'Y\right]$$
$$= (X'X)^{-1}X'\,E[Y]$$
$$= (X'X)^{-1}X'\,E[X\beta + \epsilon]$$
$$= (X'X)^{-1}X'\left[X\,E[\beta] + E(\epsilon)\right]$$
$$= (X'X)^{-1}X'X\beta \qquad (\text{As } E(\epsilon) = 0)$$
$$= \beta \qquad \left[(X'X)^{-1}X'X = I\right]$$

$$\therefore\ E(\hat{\mu}) = X\beta$$

ii) Covariance matrix of $\hat{\mu}$

$$\mathrm{cov}(\hat{\mu}) = \mathrm{cov}(X\hat{\beta})$$
$$= X\,\mathrm{cov}(\hat{\beta})\,X'$$
$$= X\left[(X'X)^{-1}X'\,\mathrm{Var}(y)\,X(X'X)^{-1}\right]X' \qquad \left[\hat{\beta} = (X'X)^{-1}X'y\right]$$
$$= X\left[(X'X)^{-1}X'\,\mathrm{Var}(\epsilon)\,X(X'X)^{-1}\right]X'$$
$$= X\left[(X'X)^{-1}X'\,\sigma^2 I\,X(X'X)^{-1}\right]X'$$

$$= X\left[\sigma^2 I (X'X)^{-1}\right]X' \qquad \left[(X'X)^{-1}X'X = I\right]$$

$$= \sigma^2 I X (X'X)^{-1} X'$$

$$= \sigma^2 I H \qquad \left[\text{As } H = X(X'X)^{-1}X'\right]$$

$$= \sigma^2 H$$

$$\therefore \operatorname{cov}(\hat{u}) = \underline{\underline{\sigma^2 H}}$$

b) Show that

$$\frac{1}{n}\sum_{i=1}^{n} \operatorname{Var}(\hat{u}_i) = \frac{\sigma^2(p+1)}{n}.$$

~~can~~ Now, $\operatorname{cov}(\hat{u})$ can be written as $\operatorname{cov}(u_i, u_j)$ where $i, j \in 1, 2, \dots n$

$$\operatorname{cov}(\hat{u}) = \begin{bmatrix} \operatorname{var}(\hat{u}_1) & \operatorname{cov}(\hat{u}_1, \hat{u}_2) & \cdots & \operatorname{cov}(\hat{u}_1, \hat{u}_n) \\ \operatorname{cov}(\hat{u}_2, \hat{u}_1) & \operatorname{var}(\hat{u}_2) & \cdots & \operatorname{cov}(\hat{u}_2, \hat{u}_n) \\ \vdots & & & \\ \operatorname{cov}(\hat{u}_n, \hat{u}_1) & \operatorname{cov}(\hat{u}_n, \hat{u}_2) & \cdots & \operatorname{var}(\hat{u}_n) \end{bmatrix}$$

$$\therefore \sum_{i=1}^{n} \operatorname{var}(\hat{u}_i) = \operatorname{tr}\left[\operatorname{cov}(\hat{u})\right]$$

$$= \operatorname{tr}(\sigma^2 H) \qquad \left(\text{From previous part } \operatorname{cov}(\hat{u}) = \sigma^2 H\right)$$

$$= \operatorname{tr}(\sigma^2 X (X'X)^{-1} X')$$

$$= \sigma^2 \operatorname{tr}(X'X(X'X)^{-1}) \qquad \left[\text{As } \operatorname{tr}(AB) = \operatorname{tr}(BA)\right]$$

$$= \sigma^2 \, tr\left[I_{(p+1)\times(p+1)} \right]$$

$$= \sigma^2 (p+1)$$

$$\therefore \frac{1}{n} \sum_{i=1}^{n} Var(\hat{u}_i) = \frac{1}{n} \times \sigma^2 (p+1)$$

$$= \frac{\sigma^2 (p+1)}{n}$$

c) To show H is symmetric and idempotent.

i) Symmetric property

$A' = A \rightarrow$ symmetric property.

$\therefore$ To show $H' = H$.

$$H = X(X'X)^{-1}X'$$

$$H' = \left(X(X'X)^{-1}X' \right)'$$

$$= X\left[(X'X)^{-1} \right]' X' \quad \text{using } (AB)' = B'A'$$

$$= X\left[(X'X)' \right]^{-1} X' \quad \text{using } (AB^{-1})' = ((AB)')^{-1}$$

$$= X(X'X)^{-1}X' \quad \text{using } (AB)' = B'A'$$

$$= H$$

$\therefore H' = H$ H is symmetric.

ii) Idempotent property $AA = A$

To show H is idempotent

$\therefore HH = H$.

$$\cancel{HH}\ HH = X(X'X)^{-1}X'X(X'X)^{-1}X'$$

$$= X\left[(X'X)^{-1}X'X\right](X'X)^{-1}X'$$

$$= X\, I\, (X'X)^{-1}X'$$

$$= X(X'X)^{-1}X'$$

$$= H$$

$\therefore$ $HH = H$ $\quad$ H is idempotent

iii) To show diagonal elements h_{ii} of H is between 0 and 1.

Now

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ \vdots & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & \cdots & \cdots & h_{nn} \end{bmatrix}$$

As H is idempotent $\Rightarrow$ $HH = H$.

$\therefore$

$$h_{11} = h_{11}^2 + h_{12}^2 \cdots h_{1n}^2$$ [As H is symmetric $\therefore h_{12} = h_{21}$]

$\therefore$ As h_{11} is a sum of squared entities

So minimum value of $h_{11} = 0$

As $h_{11} = h_{11}^2 + \cdots$

$\therefore$ maximum value of h_{11} should be equal to 1.

So $0 \leq h_{11} \leq 1$

Same with $h_{22}, h_{33} \cdots h_{nn}$

$\therefore$ we can say diagonal elements h_{ii} of H is between 0 and 1

d) We parametrise the model by centring the predictor variables, i.e $x_{ij} - \bar{x}_j$, $j = 1 \dots p$

$\therefore H_c = X(X_c' X_c)^{-1} X_c'$

$$H_c = \begin{bmatrix} x_{11} - \bar{x}_1 & \cdots & x_{1n} - \bar{x}_1 \\ \vdots & & \\ x_{n1} - \bar{x}_n & \cdots & x_{nn} - \bar{x}_n \end{bmatrix}$$

$y = \alpha 1 + X_c' \beta + \epsilon$

$\therefore \hat{y} = \hat{\alpha} 1 + X_c' \hat{\beta}$

$= \bar{y} + X_c' \hat{\beta}$

$\hat{y} = \bar{y} + X_c'(X_c' X_c)^{-1} X_c' y$

$= \left[\frac{1}{n} I' y\right] + H_c y$

$= \left[\frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + H_c \right] y$

$= H y$

$\therefore H = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + H_c$

$\therefore h_{ii} > \frac{1}{n}$ as H_c is a positive definite matrix