Notes on Functional Analysis

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These notes are prepared from my reading of *Introductory Functional Analysis with Applications* by Erwin Kreyszig. They are intended to serve as an aid for my understanding the book content. I do not guarantee their accuracy, and they do not reflect the accuracy of the book.

Contents

1 Metric Spaces **3**

1.1 Metric Spaces: Definition and Examples 3

1.2 Set Theory, Continuity, and Separability 9

1.2.1 Set Theory 9

1.2.2 Continuity 11

1.2.3 Separability 12

1.3 Convergence, Cauchy Sequences, and Completeness 15

1.4 Examples of Complete Metric Spaces 21

1.5 Completing Metric Spaces 28

2 Normed Spaces, Banach Spaces **36**

2.1 A Review of Vector Spaces 36

1 Metric Spaces

1.1 Metric Spaces: Definition and Examples

Definition 1.1 (Metric Space): A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}_+$ is a **metric** on X. The metric d must satisfy the following for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

A metric can be thought of as a measure of distance between any two elements $x, y \in X$. To illustrate this point, we provide some examples of metric spaces:

Example 1.1 (The real line $\mathbb{R}$): Let $X = \mathbb{R}$ and $d(x, y) := |x - y|$. Then (X, d) is a metric space. This is our most classical notion of distance.

Example 1.2 (The complex plane $\mathbb{C}$): Similarly, we let $X = \mathbb{C}$ and $d(x, y) := |x - y|$ for $x, y \in \mathbb{C}$. Recall that for $x = a + bi$, the modulus of x is defined $|x| = \sqrt{a^2 + b^2}$. Then (X, d) is a metric space.

Example 1.3 (The Euclidean space $\mathbb{R}^n$, unitary space $\mathbb{C}^n$): For $X = \mathbb{R}^n$, we define $d(x, y) := \sqrt{(\xi_1 - \eta_1)^2 + \ldots + (\xi_n - \eta_n)^2}$, where $x = (\xi_1, \ldots, \xi_n), y = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$. (X, d) defines the Euclidean metric space.
Similarly, let $X = \mathbb{C}^n$ and define $d(x, y) := \sqrt{|\xi_1 - \eta_1|^2 + \ldots + |\xi_n - \eta_n|^2}$ for $x, y \in \mathbb{C}^n$. (X, d) defines the unitary metric space.

These are perhaps the most benign metric spaces that we can conjure from memory. Before presenting more exotic examples of metric spaces, we first discuss a few additional results:

Lemma 1.1 (Generalized Triangle Inequality): Let (X, d) be a metric space with $x_1, \ldots, x_n \in X$. Then it holds that

$$d(x_1, x_n) \leq d(x_1, x_2) + \ldots + d(x_{n-1}, x_n).$$

This follows from an elementary induction argument.

Definition 1.2 (Subspace): Let (X, d) be a metric space and $Y \subset X$ a subset of X. Then we can define a metric $\tilde{d}$ on Y according to $\tilde{d} = d|_{Y \times Y}$. This is called the metric induced on Y by d.

The following are important examples of metric spaces that will be relevant throughout the remainder of our studies:

Example 1.4 (The sequence space ℓ^p): For fixed $p \geq 1$, we define the space ℓ^p to consist of those complex sequences $x = (\xi_n)_{n=1}^{\infty} \subset \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |\xi_n|^p < \infty.$$

The metric defined on ℓ^p is

$$d(x, y) = \left(\sum_{n=1}^{\infty} |\xi_n - \eta_n|^p \right)^{1/p}, \qquad x = (\xi_n),\ y = (\eta_n) \in \ell^p.$$

One can verify that indeed $d(x, y) < \infty$ using the *Minkowski inequality*:

$$\left(\sum_{n=1}^{\infty} |\xi_n + \eta_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |\eta_n|^p \right)^{1/p}.$$

Then (ℓ^p, d) defines a metric space. In the special case of $p = 2$, this is a **Hilbert space** (see Section **??**).

Example 1.5 (The sequence space ℓ^∞): Similar to the space considered in Example 1.4, we let ℓ^∞ be the set of all bounded sequences in $\mathbb{C}$. That is,

$$\ell^\infty = \{x = (\xi_n) \subset \mathbb{C} \mid |\xi_n| \ \leq c_x \text{ for some } c_x \in \mathbb{R}_+\}.$$

The metric we define on ℓ^∞ is $d(x, y) = \sup_{n \in \mathbb{N}} |\xi_n - \eta_n|$. (ℓ^∞, d) defines a metric space.

Example 1.6 (The function space $C[a, b]$): Let X be the set of all real-valued continuous functions defined on the interval $[a, b]$. Then for each $x(t), y(t) \in X$, we define the metric $d(x, y) := \max_{t \in [a,b]} |x(t) - y(t)|$.
One should notice that we use "max" rather than "sup" in our definition of the metric d. This is because $f(t) = |x(t) - y(t)|$ is continuous on the closed interval $[a, b]$ and thus attains its maximum on this interval (the Weierstrass Extreme Value Theorem).
This defines the metric space $C[a, b]$.

Example 1.7 (The discrete metric space): For any set X, define the metric d such that

$$d(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

Then (X,d) is called the **discrete metric space**.

Exercise 1.1: If A is the subspace of ℓ^∞ consisting of all zeros and ones, what is the induced metric on X?
The induced metric $\tilde{d}$ on A is the discrete metric (see Example 1.7).

Proof. Let $x = (\xi_n), y = (\eta_n) \in A$. Then if $x \neq y$, it must be the case that $|\xi_j - \eta_j| = 1$ for some $j \in \mathbb{N}$. By the definition of (ℓ^∞, d), this implies $\tilde{d}(x,y) = 1$. Also, by the properties of a metric space, we know $\tilde{d}(x,x) = 0$. We deduce that $\tilde{d}$ is the discrete metric. □

Exercise 1.2: Show that a metric on the space X from Example 1.6 is

$$\tilde{d}(x,y) = \int_a^b |x(t) - y(t)| dt.$$

Proof. First, notice that since $x, y \in X$, then $|x - y| \in X$. That is, $|x(t) - y(t)|$ is continuous over $[a,b]$, and is thus Riemann integrable. This means that $\tilde{d}$ is well-defined. Similarly, we have

1. $|x(t) - y(t)| \geq 0,\ t \in [a,b] \implies \tilde{d}(x,y) = \int_a^b |x(t) - y(t)| dt \geq 0$
2. $|x(t) - y(t)| \leq c,\ \forall t \in [a,b]$ for some $c \in \mathbb{R}_+ \implies \tilde{d}(x,y) \leq c \cdot (b-a) \in \mathbb{R}_+$

Therefore, it is indeed the case that $\tilde{d} : X \times X \to \mathbb{R}_+$.
To verify axiom (1), we clearly see that $x = y$ implies $\tilde{d}(x,y) = 0$. For the opposite implication, suppose $x(t_0) \neq y(t_0)$ for some $t_0 \in [a,b]$. Without loss of generality, suppose $t_0 \notin \{a,b\}$. Then by the continuity of x and y, this implies that $|x(t) - y(t)| > 0$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \subseteq [a,b]$, where $\varepsilon > 0$ is chosen to be sufficiently small. Accordingly, we have

$$\begin{aligned} \tilde{d}(x,y) &= \underbrace{\int_a^{t_0-\varepsilon} |x(t) - y(t)| dt}_{\geq 0} + \underbrace{\int_{t_0-\varepsilon}^{t_0+\varepsilon} |x(t) - y(t)| dt}_{>0} + \underbrace{\int_{t_0-\varepsilon}^{b} |x(t) - y(t)| dt}_{\geq 0} \\ \implies \tilde{d}(x,y) &> 0. \end{aligned}$$

Symmetry (2) of the metric $\tilde{d}$ follows from the properties of the absolute value.

Finally, to prove the triangle inequality (3), we have that for x, y, z continuous functions on $[a, b]$

$$
\begin{aligned}
\tilde{d}(x,z) &= \int_a^b |x(t) - z(t)| dt \leq \int_a^b \Big(|x(t) - y(t)| + |y(t) - z(t)| \Big) dt \\
&= \int_a^b |x(t) - y(t)| dt + \int_a^b |y(t) - z(t)| dt \\
&= \tilde{d}(x,y) + \tilde{d}(y,z).
\end{aligned}
$$

□

Exercise 1.3: Show that nonnegativity of a metric follows from axioms (1) - (3).

Proof. Let (X, d) be a metric space and $x, y \in X$ arbitrary points. Then by (3) we have

$$d(x,x) \leq d(x,y) + d(y,x).$$

Property (1) implies

$$0 \leq d(x,y) + d(y,x).$$

And property (2) implies

$$0 \leq 2d(x,y) \iff 0 \leq d(x,y).$$

□

Exercise 1.4: Define s to be the space of all (bounded or unbounded) sequences of complex numbers and the metric d defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}, \qquad x = (\xi_n),\ y = (\eta_n) \in s.$$

Show that we can obtain another metric by replacing $(1/2^n)$ with (μ_n), $\mu_n > 0$ such that $\sum \mu_n$ converges.

Proof. First, we point out that $\mu_n \cdot \frac{|\xi_n - \eta_n|}{1+|\xi_n - \eta_n|} \geq 0$ for each $n \in \mathbb{N}$, and so $\tilde{d}(x,y) = \sum_{n=1}^{\infty} \mu_n \cdot \frac{|\xi_n - \eta_n|}{1+|\xi_n - \eta_n|} \geq 0$. Also we notice that $\mu_n \cdot \underbrace{\frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}}_{<1} < \mu_n$ for each $n \in \mathbb{N}$, and so $\tilde{d}(x,y) = \sum_{n=1}^{\infty} \mu_n \cdot \frac{|\xi_n - \eta_n|}{1+|\xi_n - \eta_n|} < \sum_{n=1}^{\infty} \mu_n < \infty$.

Now, for (1) we have

$$\tilde{d}(x,y)=0 \iff \sum_{n=1}^{\infty}\underbrace{\mu_n\cdot\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}}_{\geq 0}=0 \iff \mu_n\cdot\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}=0,\ n\in\mathbb{N}$$

$$\iff \frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}=0,\ n\in\mathbb{N} \iff x=y.$$

(2) follows simply from the properties of the modulus function.
Finally, for the triangle inequality (3), we let $z=(\gamma_n)\in s$ and use the result from pp. 10 - 11:

$$\begin{aligned}
&\frac{|\xi_n-\gamma_n|}{1+|\xi_n-\gamma_n|}\leq\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}+\frac{|\eta_n-\gamma_n|}{1+|\eta_n-\gamma_n|},\quad n\in\mathbb{N}\\
\iff&\mu_n\cdot\frac{|\xi_n-\gamma_n|}{1+|\xi_n-\gamma_n|}\leq\mu_n\cdot\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}+\mu_n\cdot\frac{|\eta_n-\gamma_n|}{1+|\eta_n-\gamma_n|},\quad n\in\mathbb{N}\\
\implies&\sum_{n=1}^{\infty}\mu_n\cdot\frac{|\xi_n-\gamma_n|}{1+|\xi_n-\gamma_n|}\leq\sum_{n=1}^{\infty}\mu_n\cdot\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}+\sum_{n=1}^{\infty}\mu_n\cdot\frac{|\eta_n-\gamma_n|}{1+|\eta_n-\gamma_n|}\\
\iff&\tilde{d}(x,z)\leq\tilde{d}(x,y)+\tilde{d}(y,z).
\end{aligned}$$

We deduce $(s,\tilde{d})$ is a metric space. □

Exercise 1.5: The **diameter** $\delta(A)$ of a nonempty set A in a metric space (X,d) is defined to be

$$\delta(A)=\sup_{x,y\in A}d(x,y).$$

A is said to be **bounded** if $\delta(A)<\infty$. Prove each of the following:

1. $A\subset B$ implies $\delta(A)\leq\delta(B)$.
2. $\delta(A)=0\iff A$ consists of a single point.

Proof. (1) This follows from the properties of the the supremum. Namely,

$$\begin{aligned}
&C_1=\{d(x,y)\mid x,y\in A\}\subset C_2=\{d(x,y)\mid x,y\in B\}\\
\implies&\sup C_1\leq\sup C_2\\
\iff&\delta(A)\leq\delta(B).
\end{aligned}$$

(2)

$$\delta(A)=0\iff\sup_{x,y\in A}d(x,y)=0\iff d(x,y)=0,\ \forall x,y\in A$$

$$\iff \text{A consists of a single point}$$

□

Exercise 1.6: The **distance** $D(A,B)$ between two nonempty subsets A and B of a metric space (X,d) is defined to be

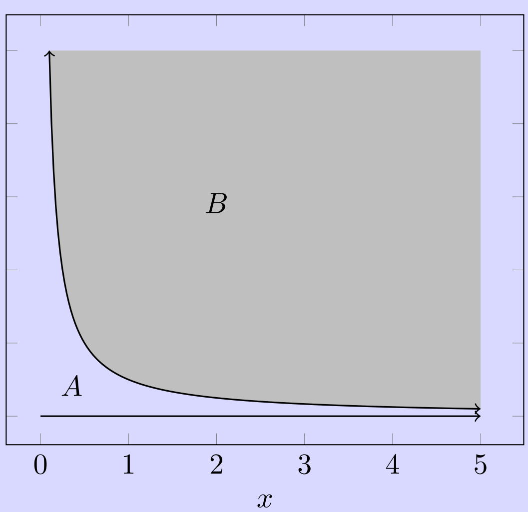
$$D(A,B) = \inf_{\substack{a\in A\\ b\in B}} d(a,b).$$

Show each of the following:

1. D does *not* define a metric on the power set of X.

2. If $A \cap B \neq \varnothing$, then $D(A,B) = 0$. What about the converse?

Proof. (1) Let $P, Q \in 2^X$ such that $P \neq Q$ and $P \cap Q \neq \varnothing$. Notice that d is a metric, and so $d(p,q) \geq 0$, $\forall p \in P$, $\forall q \in Q$. This implies $D(P,Q) \geq 0$. Moreover, $P \cap Q \neq \varnothing$ implies $D(P,Q) \leq 0$. Altogether, we have shown $D(P,Q) = 0$ but $P \neq Q$. Therefore, we conclude D does not define a metric.
(2) Since $A \cap B \neq \varnothing$, then $0 \in \{d(a,b) \mid a \in A,\ b \in B\}$. This tells us that $D(A,B) \geq 0$. And by our argument from (1), $d(a,b) \geq 0$, $\forall a \in A$, $\forall b \in B$, which implies $D(A,B) = \inf_{\substack{a\in A\\ b\in B}} d(a,b) \geq 0$. We conclude $D(A,B) = 0$.
The converse is not true. Consider the Euclidean metric space $(\mathbb{R}^2, d)$. Let $A = \{(x,0) \mid x > 0\}$ be the positive x-axis and $B = \{(x,y) \mid x > 0,\ y \geq x^{-1}\}$ the region bounded by the curve $y = x^{-1}$. Since $x^{-1} > 0$ for every $x > 0$, then $A \cap B = \varnothing$. Consider the sequence $(a_n) \subset A$ defined $a_n = (n,0)$ as well as the sequence $(b_n) \subset B$ defined $b_n = (n, n^{-1})$. Then $d(a_n, b_n) = n^{-1}$ for all $n \in \mathbb{N}$, which proves that $D(A,B) = \inf_{\substack{a\in A\\ b\in B}} d(a,b) \leq 0$. By nonnegativity of the metric d, we conclude $D(A,B) = 0$ but $A \cap B = \varnothing$. $\square$



Exercise 1.7: The **distance** $D(x, B)$ from a point x to a nonempty subset B of a metric space (X, d) is defined to be

$$D(x, B) = \inf_{b \in B} d(x, b).$$

Show that for any $x, y \in X$, $|D(x, B) - D(y, B)| \leq d(x, y)$.

Proof. For each element $b \in B$, it holds that $d(x, b) \leq d(x, y) + d(y, b)$ by the triangle inequality. Consequently, we get the inequality

$$\inf_{b \in B} d(x, b) \leq d(x, y) + \inf_{b \in B} d(y, b) \iff D(x, B) - D(y, B) \leq d(x, y).$$

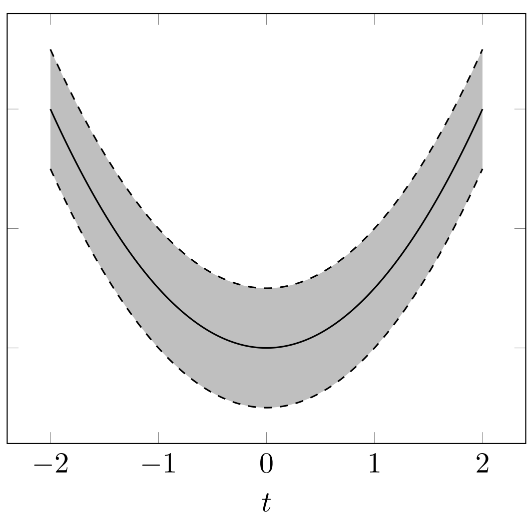
By starting with $d(y, b) \leq d(y, x) + d(x, b)$, we similarly have $D(y, b) - D(x, b) \leq d(x, y)$. And so we conclude $|D(x, B) - D(y, B)| \leq d(x, y)$, as desired. $\square$

1.2 Set Theory, Continuity, and Separability

1.2.1 Set Theory

Definition 1.3 (Ball and sphere): Let (X, d) be a metric space. Given a point $x_0 \in X$ and a real number $r > 0$, we define three sets:

1. **Open ball**: $B_r(x_0) = \{x \in X | d(x, x_0) < r\}$
2. **Closed ball**: $\tilde{B}_r(x_0) = \{x \in X | d(x, x_0) \leq r\}$
3. **Sphere**: $S_r(x_0) = \{x \in X | d(x, x_0) = r\}$



The open unit ball $B_{r=1}(x)$ for $x(t) = t^2$ in the function space $C[-2, 2]$

Definition 1.4 (Open set, closed set): A subset M of a metric space X is said to be **open** if it contains a (open) ball about each of its points. A subset K is said to be **closed** if its complement $X \setminus K$ is open.

Example 1.8: In an arbitrary metric space X, the subsets $\varnothing$ and X are both open and closed.

Example 1.9: In a discrete metric space X, every subset is both open and closed.

Proof. It suffices to show that for each $M \in 2^X$, then M is open. One will recall that $\varnothing$ is open from Exercise 1.8, so we suppose $|M| > 0$. Let $m \in M$ be an arbitrary point in M. Then for each $r \in (0, 1]$, the open ball $B_r(x_0) = \{m\}$ is contained in the set M. This proves that M is open, and we conclude that every subset of a discrete metric space is both open and closed. $\square$

Definition 1.5: Let X be a metric space and M a (nonempty) subset of X. Then $x_0 \in M$ is an **interior point** of the set M if there exists some $r > 0$ such that $B_r(x_0) \subset M$. The **interior** of M, denoted $\text{Int}(M)$ is the set of all interior points of M.

Immediately, we see that every point of an open set M is an interior point, per our definition of an open set, and so $\text{Int}(M) = M$. Moreover, we have the following result which explains why we care about set interiors:

Theorem 1.1: Let X be a metric space and M a nonempty subset of X. Then $\text{Int}(M)$ is the largest open set contained in M.

Proof. First we prove that $\text{Int}(M)$ is a subset of M and is open. By our definition, it is clear that $\text{Int}(M) \subseteq M$. Toward contradiction, suppose that $\text{Int}(M)$ is not open, meaning there exists some $x_0 \in \text{Int}(M)$ such that $B_r(x_0) \not\subset \text{Int}(M)$ for each $r > 0$. But this mean that $B_r(x_0) \not\subset M$ for each $r > 0$, which contradicts our assumption that $x_0 \in \text{Int}(M)$.
We proceed to show that $\text{Int}(M)$ is the largest open set contained in M. Let U be an arbitrary open set contained in M. Then for each $u \in U$ we have $u \in B_r(u) \subset U \subseteq M$, which implies $u \in \text{Int}(M)$. And so we conclude $U \subseteq \text{Int}(M)$. $\square$

Having formally defined an open set for a metric space (X, d), we give a more general notion of a space that satisfies the properties of open sets:

Definition 1.6: A **topological space** $(X, \mathscr{F})$ is a set X and a collection $\mathscr{F}$ of subsets of X such that $\mathscr{F}$ satifies the following axioms:

1. $\varnothing, X \in \mathscr{F}$
2. $\{F_n\}_{n \in \mathcal{I}} \subseteq \mathscr{F} \implies \bigcup_{n \in \mathcal{I}} F_n \in \mathscr{F}$ ($\mathscr{F}$ is closed under arbitrary unions)
3. $\{F_n\}_{n=1}^N \subseteq \mathscr{F} \implies \bigcap_{n=1}^N F_n \in \mathscr{F}$ ($\mathscr{F}$ is closed under finite intersections)

It is evident that each metric space X is a topological space with $\mathscr{F}$ the collection of all open subsets of X.

Note 1.1: One should be careful in recognizing that open sets, and thus topological spaces, *are not closed under countable intersections.* For a straightforward counterexample, let $(\mathbb{R}, d)$ be the Euclidean metric space and take $\{F_n\}_{n\in\mathbb{N}}$ where each $F_n := (-n^{-1}, n^{-1})$ is an open set. Then $\cap_{n\in\mathbb{N}} F_n = \{0\}$ is not open.

Definition 1.7: Let X be a metric space and M a (nonempty) subset of X. Then a point x of X, which may or may not be a point of M, is called an **accumulation point** or **limit point** of M if for every $\varepsilon > 0$, $B_\varepsilon(x)$ contains a point $m_\varepsilon \in M$ distinct from x. The set consisting of the points of M and the limit points of M is called the **closure** of M and is denoted by $\bar{M}$.

Theorem 1.2: Let X be a metric space and M a nonempty subset of X. Then $\bar{M}$ is the smallest closed set containing M.

Proof. Similar to the structure of our proof of Theorem 1.1, we first show that $\bar{M}$ contains M and is closed. The first statement is a consequence of the definition of the closure. As for the latter, let $x_0 \in X \setminus \bar{M}$ be an arbitrary point. Suppose towards contradiction that there exists no $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset X \setminus \bar{M}$. Then for every $\varepsilon > 0$, there exists some $m_\varepsilon \in M$ such that $m_\varepsilon \in B_\varepsilon(x_0)$. However, this means that x_0 is a limit point of the set M, and so $x_0 \in \bar{M}$, a contradiction.
Now we show that $\bar{M}$ is the smallest closed set containing M. In particular, let us consider an arbitrary closed set U containing M. And choose an arbitrary point $m_0 \in \bar{M}$. Suppose towards contradiction that $m_0 \notin U \implies m_0 \in X \setminus U$. Also notice that since $M \subseteq U$, then the point m_0 cannot be contained in M. Thus, m_0 must be a limit point of M. This, however, is a contradiction, since it implies that $B_\varepsilon(m_0)$ contains a point $m_\varepsilon \in M \subseteq U$ for each $\varepsilon > 0$, and so $X \setminus U$ cannot be open. Thus, it must be the case that $\bar{M} \subseteq U$ for each closed set U containing M. ☐

1.2.2 Continuity

Definition 1.8 (Continuous mapping): Let (X, d) and $(Y, \tilde{d})$ be metric spaces. A mapping $T : X \longrightarrow Y$ is said to be **continuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \implies \quad \tilde{d}(Tx, Tx_0) < \varepsilon.$$

T is said to be **continuous** if it is continuous at all points $x \in X$.

Continuous mappings have an important connection with open sets as summarized by the following theorem:

Theorem 1.3 (Continuous Mapping): A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

Proof. $\implies$ Suppose T is continuous, and let A be any open subset of Y. Further, let $B = T^{-1}A$ be the inverse image of A. Consider an arbitrary $b_0 \in B$. Then $Tb_0 \in A$. By assumption that A is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(Tb_0) \subset A$. And since T is continuous, there exists some $\delta > 0$ such that $d(b, b_0) < \delta \implies \tilde{d}(Tb, Tb_0) < \varepsilon$. But because $B_\varepsilon(Tb_0) \subset A$, this implies that the ball $B_\delta(b_0)$ is contained in B.
$\impliedby$ Conversely, suppose the inverse image of any open subset of Y is an open subset of X. Let us take $x_0 \in X$ to be arbitrary. Then for each $\varepsilon > 0$, $B_\varepsilon(Tx_0)$ is an open set. By our assumption, this means that the inverse image of $B_\varepsilon(Tx_0)$, denoted A, is open. Consequently, by the definition of A an open set, there exists some $\delta > 0$ such that $B_\delta(x_0) \subset A$. Thus we conclude $d(x, x_0) < \delta \implies \tilde{d}(Tx, Tx_0) < \varepsilon$. ☐

1.2.3 Separability

The set theory we have built up in Section 1.2.1 leads us to the important concept of dense subsets and separable spaces which will be important throughout the remainder of our studies:

Definition 1.9 (Dense set, separable space): A subset M of a metric space X is said to be **dense** in X if $\bar{M} = X$. X is said to be **separable** if it has a *countable* subset which is dense in X.

Consequently, if M is dense in X, then every ball in X (with nonzero radius), no matter how small, will contain a point of M.

Example 1.10 (Weierstrass Approximation Theorem): The set of polynomial functions defined on the interval $[a, b] \subset \mathbb{R}$ is dense in the metric space $C[a, b]$.[1]

Example 1.11: $\mathbb{R}$ is separable.

Proof. $\mathbb{Q}$ is a countable, dense subset of $\mathbb{R}$. □

Example 1.12: $\mathbb{C}$ is separable.

Example 1.13: ℓ^∞ is **not** separable. ℓ^P is separable for $1 \leq p < \infty$.

Proof. We prove that ℓ^p is separable.
Let us define the set

$$M_n = \{(\eta_1, \dots, \eta_n, 0, 0, \dots) \mid \eta_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n\}$$

which consists of those sequences whose first n terms are rational and remaining terms are zero. Each of these sequences is in ℓ^p since it only has finitely many nonzero terms. Note that M_n is countable since it is equal to the Cartesian product of countable sets.
From here the set

$$M = \bigcup_{n \in \mathbb{N}} M_n$$

is countable because it is a countable union of countable sets.
We claim that M is dense in ℓ^p. To prove this, take an arbitrary element $x = (\xi_1, \xi_2, \dots) \in \ell^p$. Then for each $\varepsilon > 0$, there must exist some N_ε such that $\sum_{n > N_\varepsilon} |\xi_i|^p < \frac{\varepsilon^p}{2}$ (by definition of the

[1] My favorite proof of the Weierstrass Approximation Theorem is that which relies upon Fejer's Theorem.

space ℓ^p). And since $\mathbb{Q}$ is dense in the Euclidean metric space $\mathbb{R}$, for each ξ_n, $1 \leq n \leq N_\varepsilon$ there exists an η_n such that $|\xi_n - \eta_n| < \frac{\varepsilon^p}{2^{n+1}}$.
Let us define $w = (\eta_1, \ldots, \eta_n, 0, 0 \ldots,) \in M$. Then we have that

$$\begin{aligned} d(x,w) = \left(\sum_{n=1}^{\infty} |\xi_n - \eta_n|^p \right)^{1/p} &= \left(\sum_{n=1}^{N_\varepsilon} |\xi_n - \eta_n|^p + \sum_{n > N_\varepsilon} |\xi_n|^p \right)^{1/p} \\ &\leq \left(\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} \right)^{1/p} = \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily, we deduce that M is indeed dense in ℓ^p for each $1 \leq p < \infty$, and so ℓ^p is separable. $\square$

Exercise 1.8: Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \tilde{B}_r(x)$, where $x(t) = \sin(t)$ and $y(t) = \cos(t)$.
First, we comment that

$$y \in \tilde{B}_r(x) \iff \max_{t \in [0,2\pi]} |\sin(t) - \cos(t)| \leq r.$$

And so we must maximize the function $f(x) = |\sin(t) - \cos(t)|$ over the interval $[0, 2\pi]$. To do so, we find the extreme points of the function $\tilde{f}(t) = \sin(t) - \cos(t)$:

$$\begin{aligned} \tilde{f}'(t) = 0 &\iff \frac{d}{dt}(\sin(t) - \cos(t)) = 0 \\ &\implies \cos(t) + \sin(t) = 0 \implies t \in \left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}. \end{aligned}$$

$t = \frac{3\pi}{4}$ is a maximizer of $\tilde{f}$, whereas $t = \frac{7\pi}{4}$ is a minimizer of $\tilde{f}$. Plugging each of these values for t into the original objective f, we get that $\max_{t \in [0,2\pi]} |\sin(t) - \cos(t)| = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) = \sqrt{2}$. We conclude that the smallest value of r such that $y \in \tilde{B}_r(x)$ is $r = \sqrt{2}$.

Exercise 1.9: If x_0 is a limit point of a set $A \subset (X, d)$, show that any neighborhood of x_0 contains infinitely many points of A. A **neighborhood** is defined to be any set which contains $B_r(x_0)$ for some $r > 0$.

Proof. Let M be an arbitrary neighborhood of x_0. Suppose towards contradiction that M contains only finitely many points of A. Let $a_1, \ldots, a_N$ be the points of A contained in M which are distinct from x_0. Since M is a neighborhood of x_0, then $B_r(x_0) \subset M$ for some $r > 0$. Let $r^* = \min\{d(x_0, a_1), \ldots, d(x_0, a_N), r\}$. Then $B_{r^*}(x_0) \subset M$ and $B_{r^*}(x_0)$ does not contain any points of A distinct from x_0. This, though, contradicts our assumption that x_0 is a limit point of the set A. Note that we needed M to be a neighborhood of x_0 since this implies $r > 0$. $\square$

Exercise 1.10: A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M (as defined in Exercise 1.7). To prove this, show that $x \in \bar{A}$ if and only if $D(x, A) = 0$; here, A is any nonempty subset of X.

Proof. $\Longrightarrow$ First let us suppose that $x \in \bar{A}$. If $x \in A$, then $D(x, A) = 0$. Otherwise, x must be a limit point of the set A. Accordingly, there exists a sequence of points $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in A$ such that $a_n \in B_{n^{-1}}(x) \iff d(x, a_n) < n^{-1}$. Therefore, we have the sequence $\{d(x, a_n)\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} d(x, a_n) = 0$. This implies $D(x_0, A) = 0$.
$\Longleftarrow$ Otherwise, suppose $D(x, A) = 0$. By definition of the infimum, for every $\varepsilon > 0$, there exists an $a_\varepsilon \in A$ such that $d(x, a_\varepsilon) < D(x, A) + \varepsilon = \varepsilon$. However, this then implies $a_\varepsilon \in B_\varepsilon(x)$ for every $\varepsilon > 0$. We conclude that x must be a limit point of A, and so $x \in \bar{A}$. $\square$

Exercise 1.11 (The function space $B(A)$): Let X consist of all functions that are bounded on a set A. And define the metric function

$$d(x, y) = \sup_{t\in A} |x(t) - y(t)| \quad x, y \in B(A).$$

(X, d) defines the metric space $B(A)$.
Show that $B[a, b]$, $a < b$, is not separable.

Proof. Let us consider the set of functions

$$F = \bigcup_{x\in[a,b]\cap(\mathbb{R}\setminus\mathbb{Q})} \mathbb{1}_x,$$

where

$$\mathbb{1}_x(t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{otherwise} \end{cases}.$$

We make two comments regarding this set F. First, $F \subset B[a, b]$ since for each $f \in B[a, b]$, $\sup_{t\in[a,b]} |f(t)| = 1$. Also, our set F contains uncountably many functions since the set $[a, b]\cap(\mathbb{R}\setminus\mathbb{Q})$ is uncountable.
Now that we have defined this set F, let us consider $\mathbb{1}_x, \mathbb{1}_y \in F$ for $x \neq y$. In particular, we have $d(\mathbb{1}_x, \mathbb{1}_y) = |\mathbb{1}_x(x) - \mathbb{1}_y(x)| = 1$. And so if we let each function $\mathbb{1}_x \in F$ be the center of an open ball $B_r(\mathbb{1}_x)$ with radius $r \leq 1$, we have uncountably many balls that are pairwise disjoint. That is, we can cover the set F by an uncountable collection of disjoint, open balls. Accordingly, if M is any dense set in $B[a, b]$, each of the uncountably many balls must contain an element of M. Therefore, there can exist no countable, dense subset of $B[a, b]$. $\square$

Exercise 1.12: Show that the image of an open set under continuous mapping need not be open. Let $X = Y = (\mathbb{R}, d)$ be the Euclidean metric space and define the map $T : X \longrightarrow Y$ according to

$Tx := 0$ for every $x \in X$. Clearly T is continuous because for every $x_0 \in X$ and for every $\varepsilon > 0$, $d(x, x_0) < 1 \implies d(Tx, Tx_0) = d(0, 0) = 0 < \varepsilon$. However, the image of the open set $X = \mathbb{R}$ under T is $\{0\}$, a closed set in $(\mathbb{R}, d)$.

1.3 Convergence, Cauchy Sequences, and Completeness

Definition 1.10: A sequence (x_n) in a metric space $X = (X, d)$ is said to **converge** or to **be convergent** if there exists an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

If this is the case, x is called the **limit** of (x_n) and we write

$$\lim_{n\to\infty} x_n = x \qquad \text{or} \qquad x_n \longrightarrow x.$$

We say the sequence (x_n) **converges to** x or **has the limit** x. If (x_n) is not convergent, it is said to be **divergent**.

Note 1.2: One must be careful about the stipulation that $x \in X$. In particular, suppose our metric space satisfies $X = (a, b)$ with d the Euclidean metric. Then the sequence $(x_n) \subset X$ defined $x_n = a + n^{-1},\ n \in \mathbb{N}$ is divergent. If we were to let $X = \mathbb{R}$, though, then $x_n \longrightarrow a$ for any $a \in \mathbb{R}$.

Prior to introducing the first lemma regarding convergent sequences, we first define a bounded subset of a metric space:

Definition 1.11 (Bounded set, diameter): Let (X, d) be a metric space and $M \subseteq X$ a (nonempty) subset of X. Then M is a **bounded set** if

$$\delta(M) = \sup_{x,y\in M} d(x, y) < \infty.$$

For a bounded set M, $\delta(M)$ is called the **diameter** of M.

Lemma 1.2 (Boundedness, uniqueness of convergent sequences): Let $X = (X, d)$ be a metric space. Then

1. A convergent sequence in X is bounded and its limit is unique.
2. If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, the $d(x_n, y_n) \longrightarrow 0$.

Proof. We first prove statement (1). Let (x_n) be a convergent sequence in X with $x_n \longrightarrow x$. Then there exists a $N \in \mathbb{N}$ such that $n > N \implies d(x_n, x) < 1$. Let $k = \max\{d(x_1, x), \ldots, d(x_N, x), 1\}$. With this k, we can say that $d(x_n, x) \leq k$ for every $n \in \mathbb{N}$. Therefore, for every $n, m \in \mathbb{N}$, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq 2k$ by the triangle inequality. We conclude $\sup_{n,m\in\mathbb{N}} d(x_n, x_m) \leq 2k$, and so (x_n) is bounded.
For the second part of proof, suppose towards contradiction that we have both $x_n \longrightarrow x$ as well as $x_n \longrightarrow y$ but $x \neq y$. By the triangle inequality we know that

$$\begin{aligned} &d(x, y) \leq d(x, x_n) + d(x_n, y), \quad \forall n \in \mathbb{N} \\ \implies &d(x, y) \leq \lim_{n\to\infty} d(x_n, x) + \lim_{n\to\infty} d(x_n, y) = 0. \end{aligned}$$

This is a contradiction, since by the axioms of (X, d) a metric space we know $d(x, y) \leq 0 \implies d(x, y) = 0 \implies x = y$.
Next, we prove statement (2), which is a simple consequence of the triangle inequality. In particular,

$$\begin{aligned} &d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \\ \implies &d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\ \implies &d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y). \end{aligned}$$

And so, together, these two inequalities imply

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore, we conclude $d(x_n, y_n) \longrightarrow d(x, y)$ (in the Euclidean metric space) as $n \longrightarrow \infty$. □

Now that we have defined convergent sequences in a metric space (X, d), we expand our consideration to a more general class of sequences.

Definition 1.12 (Cauchy criterion, Cauchy sequence): A sequence (x_n) in a metric space $X = (X, d)$ is said to be **Cauchy** if for every $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \qquad \text{for every } m, n > N_\varepsilon.$$

The previous statement is called the **Cauchy criterion**.

Definition 1.13 (Completeness): A metric space $X = (X, d)$ is **complete** if every Cauchy sequence in X converges.

Example 1.14: The real line $(\mathbb{R}, d)$ and the complex plane $(\mathbb{C}, d)$ are complete metric spaces.

Example 1.15: The metric space $((a, b), d)$, where d is the Euclidean metric, is not complete. To understand why this is the case, see Note 1.2.

Example 1.16: The rational line $(\mathbb{Q}, d)$ (d is the Euclidean metric) is not complete.[2]

Although, as we have seen, $(x_n) \subset (X, d)$ is Cauchy $\not\Longrightarrow$ (x_n) is convergent, the opposite implication is, in fact, true:

[2]For an interesting example of a Cauchy sequence in $\mathbb{Q}$ that is divergent, see Morales, J. V. (2009). Math Bite: $\mathbb{Q}$ Is Not Complete. Mathematics Magazine, 82(4), 293-294.

Theorem 1.4: Every convergent sequence in a metric space (X, d) is a Cauchy sequence.

Proof. Let $(x_n) \subset X$ be an arbitrary convergent sequence with $x_n \longrightarrow x$. And take $\varepsilon > 0$ to be arbitrary. Since (x_n) converges, then there exists some N_ϵ such that $n > N_\epsilon \implies d(x_n, x) < \varepsilon$. Having established this fact, then for any $n, m > N_\epsilon$ it must be true that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from the Definition 1.12, the sequence (x_n) is Cauchy. □

Theorem 1.5: Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure. Then:

1. $x \in \bar{M}$ is and only if there is a sequence $(x_n) \subset M$ such that $x_n \longrightarrow x$.
2. M is closed if and only if $(x_n) \subset M$, $x_n \longrightarrow x$ implies that $x \in M$.

Proof. We begin with a proof of (1).
$\implies$ Suppose $x \in \bar{M}$. Then there are two cases to consider. First, if $x \in M$, then let $x_n := x$ be the constant sequence. Clearly we have $x_n \longrightarrow x$ since $d(x_n, x) = 0$ for every $n \in \mathbb{N}$. Otherwise, it must be the case that x is a limit point of M. Therefore, for every $n \in \mathbb{N}$, there exists an $x_n \in M$ such that $x_n \in B_{n^{-1}}(x) \iff d(x, x_n) < n^{-1}$. This sequence of points $(x_n) \subset M$ satisfies $x_n \longrightarrow x$ since $\lim_{n\to\infty} n^{-1} = 0$. And so we deduce that there is a sequence $(x_n) \subset M$ such that $x_n \longrightarrow x$.
$\impliedby$ Conversely, suppose that there exists a sequence $(x_n) \subset M$ such that $x_n \longrightarrow x$. If $x_n = x$ for any $n \in \mathbb{N}$, then $x \in M \implies x \in \bar{M}$. Otherwise, we have that for every $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies d(x_n, x) < \varepsilon \iff x_n \in B_\varepsilon(x)$. But because $x_n \in M$ with $x_n \neq x$ for every $n \in \mathbb{N}$, this implies that x must be a limit point of M. By definition of the closure, we deduce $x \in \bar{M}$.
Now we proceed to prove statement (2).
$\implies$ Let us first assume M is closed. We know from (1) that $(x_n) \subset M$ and $x_n \longrightarrow x$ implies $x \in \bar{M}$. But M closed implies $\bar{M} = M$, meaning $(x_n) \subset M$ and $x_n \longrightarrow x$ implies $x \in M$.
$\impliedby$ Conversely, assume $(x_n) \subset M$, $x_n \longrightarrow x$ implies $x \in M$. And let x be an arbitrary point in $\bar{M}$. Then from (1) we have that there exists a sequence $(x_n) \subset M$ with $x_n \longrightarrow x$. By our assumption, this implies $x \in M$. Hence we have shown $\bar{M} = M$, and so M is closed. □

Theorem 1.5 lends itself to the following result, which is important for understanding complete metric spaces:

Theorem 1.6: A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Proof. $\implies$ Suppose $M \subset X$ is complete. Let us consider an arbitrary $x \in \bar{M}$. Then by Theorem 1.5, there exists a sequence $(x_n) \subset M$ such that $x_n \longrightarrow x$. By assumption, though, M is complete, and the limit x of the convergent sequence (x_n) is unique (Lemma 1.2), meaning that it must be true that $x \in M$. Thus $\bar{M} = M$, and so M is closed in X.
$\impliedby$ Conversely, suppose that M is closed in X. Take (x_n) to be an arbitrary Cauchy sequence in M. Since X is a complete metric space, it must be the case that $x_n \longrightarrow x$ for some $x \in X$. By Theorem, 1.5, this implies that $x \in \bar{M}$. By assumption, though, $\bar{M} = M$, and so we have shown that each Cauchy sequence in M converges to a limit in M. □

The final theorem in this section is of great importance when studying continuous maps. Specifically, it relates the concepts of continuity and convergence.

Theorem 1.7 (Continuous mapping theorem): A mapping $T : X \longrightarrow Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point x_0 if and only if

$$x_n \longrightarrow x_0 \quad \Longrightarrow \quad Tx_n \longrightarrow Tx_0.$$

Proof. $\Longrightarrow$ First suppose that T is continuous at x_0. And let $(x_n) \subset X$ be an arbitrary convergent sequence in X that satisfies $x_n \longrightarrow x_0$. By assumption, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x_0) < \delta \implies \tilde{d}(Tx, Tx_0) < \varepsilon$. But since $x_n \longrightarrow x_0$, then there exists a $N_\delta \in \mathbb{N}$ such that $n > N_\delta \implies d(x_n, x_0) < \delta$. Therefore, it is true that for all $n > N_\delta$, $\tilde{d}(Tx_n, Tx_0) < \varepsilon$. We conclude that $Tx_n \longrightarrow Tx_0$.
$\Longleftarrow$ Suppose $x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0$. And in the direction of contradiction, let us suppose that T is not continuous at x_0. This means that there exists some $\varepsilon' > 0$ such that for every $\delta > 0$, there exists a $x_\delta \in X$ with $d(x_\delta, x_0) < \delta$ but $\tilde{d}(Tx_\delta, Tx_0) \geq \varepsilon'$. Therefore, we can form the sequence $(x_n) \subset X$ with $d(x_n, x_0) < n^{-1}$ and $d(Tx_n, Tx_0) \geq \varepsilon'$. Patently, since $\lim_{n\to\infty} n^{-1} = 0$, then $x_n \longrightarrow x_0$. However, $d(Tx_n, Tx_0) \geq \varepsilon'$ for every $n \in \mathbb{N}$ implies $\liminf_{n\to\infty} d(Tx_n, Tx_0) \geq \varepsilon' > 0$. This is a contradiction, though, to our assumption that $Tx_n \longrightarrow Tx_0$. We conclude that the map $T : X \longrightarrow Y$ must be continuous at x_0. □

Exercise 1.13: If a sequence (x_n) in a metric space X is convergent and has limit x, show that every subsequence (x_{n_k}) of (x_n) is convergent and has the same limit x.

Proof. Suppose (x_n) is an arbitrary convergent sequence in X satisfying $x_n \longrightarrow x$. And suppose that $(x_{n_k})_{k\in\mathbb{N}}$ is an arbitrary subsequence of (x_n).[3] Because (x_n) is convergent, then for each $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies d(x_n, x) < \varepsilon$. But this then means that there exists some $K_\varepsilon \in \mathbb{N}$ such that $k > K_\varepsilon \implies n_k > N_\varepsilon \implies d(x_{n_k}, x) < \varepsilon$. Therefore, we conclude that the subsequence (x_{n_k}) converges with limit x. □

Exercise 1.14: If (x_n) is Cauchy and has a convergent subsequence, say $x_{n_k} \longrightarrow x$, show that (x_n) is convergent with the limit x.

Proof. Suppose that (x_n) is a Cauchy sequence in metric space (X, d) with convergent subsequence $x_{n_k} \longrightarrow x$. We will show that the sequence (x_n) itself converges to the limit x.
To begin, fix $\varepsilon > 0$ to be arbitrary. By our assumption that (x_n) is Cauchy, there exists a $N_\varepsilon \in \mathbb{N}$ such that $n, m > N_\varepsilon \implies d(x_n, x_m) < \frac{\varepsilon}{2}$. Additionally, since the subsequence (x_{n_k}) is convergent, then there exists some $K_\varepsilon \in \mathbb{N}$ such that $k > K_\varepsilon \implies d(x_{n_k}, x) < \frac{\varepsilon}{2}$.
Using these two positive integers N_ε and K_ε, let us define $M_\varepsilon := \max\{N_\varepsilon, n_{K_\varepsilon}\}$. Note that because

[3] Notice that we index a subsequence (x_{n_k}) of (x_n) by k rather than by n. Particularly, $n_1, n_2, \ldots, n_k, \ldots$ defines a sequence of natural numbers such that $n_1 < n_2 < \ldots < n_k < \ldots$.

$M_\varepsilon \in \mathbb{N}$, then there also exists a $j \in \mathbb{N}$ such that $n_j > M_\varepsilon$. Therefore, we have that for all $n > M_\varepsilon$,

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_j}) + d(x_{n_j}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The inequality on the second line follows from (1) the Cauchy criterion since $n, n_j > M_\varepsilon \geq N_\varepsilon$ as well as from (2) the convergence of the subsequence (x_{n_k}) since $n_j > M_\varepsilon \geq n_{K_\varepsilon} \implies j > K_\varepsilon$. Because we let $\varepsilon > 0$ be arbitrary, this proves that $x_n \longrightarrow x$. □

Exercise 1.15: Show that every Cauchy sequence is bounded.

Proof. Let $X = (X, d)$ be a metric space and $(x_n) \subset X$ an arbitrary Cauchy sequence. Then by the Cauchy criterion (taking $\varepsilon = 1$), there exists an $N \in \mathbb{N}$ such that $n, m > N \implies d(x_n, d_m) < 1$. Using this positive integer N, let us define $k := \max\{d(x_1, x_{N+1}), \ldots, d(x_N, x_{N+1}), 1\}$. Then for every $n \in \mathbb{N}$, it must be the case that $d(x_n, x_{N+1}) \leq k$. By appealing to the triangle inequality, we deduce that for each $n, m \in \mathbb{N}$,

$$d(x_n, x_m) \leq d(x_n, x_{N+1}) + d(x_{N+1}, x_m) \leq 2k.$$

We conclude that the Cauchy sequence (x_n) is bounded. □

Exercise 1.16: Is boundedness of a sequence sufficient for the sequence to be Cauchy? How about convergent?

This is not the case. As a counterexample, consider the Euclidean metric space $(\mathbb{R}, d)$ and the sequence (x_n) defined $x_n := (-1)^n$. For each $n, m \in \mathbb{N}$ we have

$$d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |x_m| = 2,$$

and so the sequence is bounded.

We will show that the sequence is not convergent, and thus it is not Cauchy. Consider an arbitrary $x \in \mathbb{R}$ and define $k := \max\{|x - 1|, |x + 1|\}$, which is the maximum of the distances of x to each of -1 and 1. Evidently, it must be the case that $k > 0$. Then for each $\varepsilon < k$ and for every $N_\varepsilon \in \mathbb{N}$, there exists a $n > N_\varepsilon$ with $d(x_n, x) = k > \varepsilon$. Therefore, it cannot be the case that $x_n \longrightarrow x$. But we chose $x \in \mathbb{R}$ to be arbitrary, and so the sequence (x_n) is divergent.

Exercise 1.17: If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.

Proof. Let $X = (X, d)$ be an arbitrary metric space and (x_n), (y_n) two Cauchy sequences. Take $\varepsilon > 0$ to be arbitrary. Then by the Cauchy criterion, there exists a $N_\varepsilon^{(1)} \in \mathbb{N}$ such that $n, m >$

$N_\varepsilon^{(1)} \implies d(x_n, x_m) < \frac{\varepsilon}{2}$ as well as a $N_\varepsilon^{(2)} \in \mathbb{N}$ such that $n, m > N_\varepsilon^{(2)} \implies d(y_n, y_m) < \frac{\varepsilon}{2}$. Using these $N_\varepsilon^{(1)}$ and $N_\varepsilon^{(e)}$, define $N = \max\{N_\varepsilon^{(1)}, N_\varepsilon^{(2)}\}$. Then by the triangle inequality inequality, we have that

$$\begin{aligned} n, m > N &\implies d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ &\iff d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n) \\ &\iff d(x_n, y_n) - d(x_m, y_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} n, m > N &\implies d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \\ &\iff d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m) \\ &\iff d(x_m, y_m) - d(x_n, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Altogether, we have shown

$$n, m > N \implies |d(x_n, y_n) - d(x_m, y_m)| < \varepsilon.$$

Because $\varepsilon > 0$ was taken to be arbitrary, this proves that (a_n) defined $a_n := d(x_n, y_n)$ is a Cauchy sequence in the Euclidean metric space. And since the real line is complete, we conclude that (a_n) converges. □

Exercise 1.18: If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,

$$a \cdot d_1(x, y) \leq d_2(x, y) \leq b \cdot d_1(x, y)$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Proof. Start by assuming $(x_n) \subset X$ is a Cauchy sequence in (X, d_1). Then for each $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $n, m > N_\varepsilon \implies d_1(x_n, x_m) < \frac{\varepsilon}{b}$. Note that $b, \varepsilon > 0 \implies \frac{\varepsilon}{b} > 0$. But by the properties of d_1 and d_2, we also know that $n, m > N_\varepsilon$ implies

$$d_2(x_n, x_m) \leq b \cdot d_1(x_n, x_m) < b \cdot \frac{\varepsilon}{b} = \varepsilon.$$

Thus, the sequence (x_n) is also a Cauchy sequence in the metric space (X, d_2).
Now, suppose that $(x_n) \subset X$ is a Cauchy sequence in (X, d_2). Accordingly, for each $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $n, m > N_\varepsilon \implies d_2(x_n, x_m) < a \cdot \varepsilon$. Just as in the previous argument, $a, \varepsilon > 0 \implies a \cdot \varepsilon$. Therefore, we have that for every $n, m > N_\varepsilon$,

$$d_1(x_n, x_m) \leq \frac{1}{a} \cdot d_2(x_n, x_m) < \frac{1}{a} \cdot a \cdot \varepsilon = \varepsilon.$$

And so we conclude that (x_n) must also be a Cauchy sequence in (X, d_1).
Since we have shown that every Cauchy in (X, d_1) is a Cauchy sequence in (X, d_2) and vice versa, then the Cauchy sequences in (X, d_1) and (X, d_2) are the same. □

1.4 Examples of Complete Metric Spaces

Prior to diving into examples of complete metric spaces, we present a general overview of how one would prove that an arbitrary metric space (X, d) is complete.

Note 1.3: Suppose we have a metric space $X = (X, d)$ and want to show that X is complete. To do so, we consider an arbitrary Cauchy sequence $(x_n) \subset X$. Using this Cauchy sequence, we

1. Find a candidate x for the limit of (x_n)
2. Prove that this candidate x is in the metric space X
3. Prove $x_n \longrightarrow x$ in the metric space X

Since we have shown that an arbitrary Cauchy sequence in X is convergent, then we conclude X is complete.

Let us see this proof technique in action:

Example 1.17: The Euclidean metric space $(\mathbb{R}^n, d)$ and unitary metric space $(\mathbb{C}^n, d)$ are complete.

Example 1.18: The sequence space ℓ^∞ is complete.

Proof. Suppose (x_n) is an arbitrary Cauchy sequence in ℓ^∞. Let $x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots)$ be the nth element of our Cauchy sequence. Then by the Cauchy criterion, we know that for each $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that $n, m > N_\varepsilon \implies d(x_n, x_m) < \varepsilon$. By the definition of metric d, we also know that

$$\begin{aligned} d(x_n, x_m) < \varepsilon &\iff \sup_{i \in \mathbb{N}} |\xi_i^{(n)} - \xi_i^{(m)}| < \varepsilon && (n, m > N_\varepsilon) \\ &\implies |\xi_i^{(n)} - \xi_i^{(m)}| < \varepsilon \quad \forall i \in \mathbb{N}. && (n, m > N_\varepsilon) \qquad (1) \end{aligned}$$

That is, $n, m > N_\varepsilon \implies |\xi_i^{(n)} - \xi_i^{(m)}| < \varepsilon$ for each $i \in \mathbb{N}$. Therefore, $\left(\xi_i^{(n)}\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Euclidean metric space $(\mathbb{R}, d)$ for each $i \in \mathbb{N}$. And since $(\mathbb{R}, d)$ is complete (see Example 1.14), then this means $\xi_i^{(n)} \longrightarrow \xi_i$ for each $i \in \mathbb{N}$, where the limit ξ_i is in $\mathbb{R}$.
From here, we have a candidate for the limit of our Cauchy sequence (x_n), namely $x = (\xi_1, \xi_2, \ldots)$. By the continuous mapping theorem (Theorem 1.7), we know

$$\xi_i^{(m)} \longrightarrow \xi_i \implies |\xi_i^{(n)} - \xi_i^{(m)}| \longrightarrow |\xi_i^{(n)} - \xi_i|.$$

And from (1) $n, m > N_\varepsilon \implies |\xi_i^{(n)} - \xi_i^{(m)}| < \varepsilon$, which, together with the previous statement, means

$$n > N_\varepsilon \implies |\xi_i^{(n)} - \xi_i| \leq \varepsilon. \qquad (2)$$

Now, by our assumption that $(x_n) \subset \ell^\infty$, then for each $n \in \mathbb{N}$ we have $|\xi_i^{(n)}| \leq k_n, \forall i \in \mathbb{N}$ for some $k_n \in \mathbb{R}_+$. That is, the terms in each sequence x_n are bounded by some constant k_n. Consequently, by the triangle inequality in the Euclidean metric space,

$$|\xi_i| \leq |\xi_i - \xi_i^{(n)}| + |\xi_i^{(n)}| \leq \varepsilon + k_n$$

for each $n \in \mathbb{N}$ satisfying $n > N_\varepsilon$. Because the right-hand side of the inequality does not depend on the particular term $i \in \mathbb{N}$ of the sequence x, we have proven

$$\sup_{i \in \mathbb{N}} |\xi_i| \leq \varepsilon + k_n \implies x \in \ell^\infty.$$

Finally, to show that $x_n \longrightarrow x$, recall our statement (2) that $n > N_\varepsilon \implies |\xi_i^{(n)} - \xi_i| \leq \varepsilon$ for each $i \in \mathbb{N}$. Accordingly, we have $n > N_\varepsilon \implies d(x_n, x) = \sup_{i \in \mathbb{N}} |\xi_i^{(n)} - \xi_i| \leq \varepsilon$. Because $\varepsilon > 0$ was chosen arbitrarily, this proves that $x_n \longrightarrow x$.
We conclude that an arbitrary Cauchy sequence (x_n) in ℓ^∞ converges, and so ℓ^∞ is complete. □

Example 1.19: The space c consists of all convergent sequences $x = (\xi_i)$ of complex numbers, with the metric induced from the space ℓ^∞. c is complete.

Proof. c is a closed subset of ℓ^∞, which implies by Theorem 1.6 that c is complete. □

Example 1.20: The sequence space ℓ^p is complete for p is fixed and $1 \leq p < \infty$.

Example 1.21: The function space $C[a, b]$ is complete; $[a, b]$ is any given closed interval on $\mathbb{R}$.

Proof. Let (x_n) be any Cauchy sequence in $C[a, b]$. Then for every $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that $n, m > N_\varepsilon \implies d(x_n, x_m) = \max_{t \in [a,b]} |x_n(t) - x_m(t)| < \varepsilon$. This implies that for each fixed $t_0 \in [a, b]$,

$$n, m > N_\varepsilon \implies |x_n(t_0) - x_m(t_0)| \leq \max_{t \in [a,b]} |x_n(t) - x_m(t)| < \varepsilon, \tag{3}$$

and so $(x_n(t_0))$ is a Cauchy sequence in the Euclidean space $(\mathbb{R}, d)$. Because $(\mathbb{R}, d)$ is a complete metric space (Theorem 1.14), then this implies that $x_n(t_0) \longrightarrow x(t_0)$ for each $t_0 \in [a, b]$. Using these individual limits $x(t_0)$, $t_0 \in [a, b]$, we define the function $x(t)$.
And so it remains to show that the candidate limit x is indeed in $C[a, b]$ and $x_n \longrightarrow x$. In order to do so, we use the continuous mapping theorem to say that for each $t_0 \in [a, b]$,

$$x_m(t_0) \longrightarrow x(t_0) \implies |x_n(t_0) - x_m(t_0)| \longrightarrow |x_n(t_0) - x(t_0)|.$$

Therefore, from (3) we get that for each $t_0 \in [a, b]$

$$n > N_\varepsilon \implies |x_n(t_0) - x(t_0)| \leq \varepsilon. \tag{4}$$

Equipped with this result, we claim the proposed limit x is continuous on the interval $[a, b]$. In particular, let us consider an arbitrary $t_0 \in [a, b]$. And take $\tilde{\varepsilon} > 0$ to be arbitrary. Then by (4), there exists a $M_{\tilde{\varepsilon}} \in \mathbb{N}$ such that $n > M_{\tilde{\varepsilon}} \implies |x_n(t) - x(t)| \leq \frac{\tilde{\varepsilon}}{3}$ for all $t \in [a, b]$. Using this value $M_{\tilde{\varepsilon}}$, choose some $k > M_{\tilde{\varepsilon}}$. Since x_k is continuous, then there exists a $\delta > 0$ such that $|t - t_0| < \delta \implies |x_k(t) - x_k(t_0)| < \frac{\tilde{\varepsilon}}{3}$. Altogether, we have

$$\begin{aligned}|t - t_0| < \delta \implies |x(t) - x(t_0)| &\leq |x(t) - x_k(t)| + |x_k(t) - x_k(t_0)| + |x_k(t_0) - x(t_0)| \\ &\leq \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} = \tilde{\varepsilon}.\end{aligned}$$

This proves that x is continuous on $[a, b]$, meaning $x \in C[a, b]$.
Finally to prove $x_n \longrightarrow x$, we invoke (4) to say

$$n > N_\varepsilon \implies d(x_n, x) = \max_{t \in [a,b]} |x_n(t) - x(t)| \leq \varepsilon.$$

While this may have seemed obvious when we initially stated (4), we could not consider $d(x_n, x)$ since we did not a priori know that $x \in C[a, b]$. Because $\varepsilon > 0$ was chosen to be arbitrary, we conclude that $x_n \longrightarrow x$.
(x_n) was an arbitrary Cauchy sequence, and so we have proven the metric space $C[a, b]$ is complete.
□

Although we assumed our functions $x \in C[a, b]$ are real-valued, this does not need to be the case. That is, we only considered the real $C[a, b]$. The complex $C[a, b]$, consisting of those complex-valued, continuous functions defined on the interval $[a, b] \subset \mathbb{R}$, is also a complete metric space. Note that for the complex $C[a, b]$, the metric d is defined in terms of the complex modulus function rather than the absolute value. The proof is nearly identical to that we presented in Example 1.21.

Theorem 1.8 (Uniform convergence): Convergence $x_n \longrightarrow x$ in the space $C[a, b]$ is **uniform convergence**. That is, (x_m) converges uniformly on $[a, b]$ to x.

Hence, the metric on $C[a, b]$ describes uniform convergence on $[a, b]$ and, for this reason, is called the **uniform metric**.

Now that we have given some examples of complete metric spaces, let us turn our attention to examples of incomplete metric spaces:

Example 1.22: Consider the subset of $C[a, b]$, denoted $\mathcal{P}$, consisting of all polynomials defined on $[a, b]$. Then the metric space $(\mathcal{P}, d)$, where d is the metric induced on $\mathcal{P}$ by $C[a, b]$, is not complete.

Proof. Let x be an arbitrary continuous function on $[a, b]$ that is not a polynomial (i.e. $x \in C[a, b] \setminus \mathcal{P}$). By the Weierstrass Approximation Theorem (Example 1.10) $\mathcal{P}$ is dense in $C[a, b]$, meaning there exists a sequence $(x_n) \subset P$ such that $x_n \longrightarrow x$. By Theorem 1.4, (x_n) a convergent sequence in $C[a,b]$ implies (x_n) is a Cauchy sequence in $C[a,b]$. And because d is the metric induced on $\mathcal{P}$ by $C[a,b]$, then (x_n) is also a Cauchy sequence in $\mathcal{P}$. Therefore, we have given an instance of a Cauchy sequence (x_n) in $(\mathcal{P},d)$ that is divergent. We conclude that $(\mathcal{P},d)$ is an incomplete metric space. □

Example 1.23: Let X be the set of all continuous, real-valued functions on $[0,1]$ and let

$$d(x,y) := \int_0^1 |x(t)-y(t)|dt.$$

The metric space (X,d) is incomplete.

Proof. Consider the sequence of continuous functions $(x_n) \subset X$ defined as follows

$$x_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ nt - \frac{n}{2} & \frac{1}{2} < t \le \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} < t \le 1 \end{cases}.$$

We can compute the distance between x_n and x_m, $n > m$ as

$$d(x_n, x_m) = \int_0^1 |x_n(t) - x_m(t)|dt = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right).$$

Notice that this integral is simply equal to the area of the triangle with base of length $\frac{1}{m} - \frac{1}{n}$ and height of length 1.
Now for each $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$. And so by our previous computation, for each $n, m > N_\varepsilon$, $n > m$ we have

$$d(x_n, x_m) = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) < \frac{1}{2} \cdot \frac{1}{m} < \frac{1}{2} \cdot \frac{1}{N_\varepsilon} < \varepsilon.$$

We deduce that (x_n) is a Cauchy sequence in the metric space (X,d).
However, we claim that (x_n) is a divergent sequence. Assume towards contradiction that (x_n) converges to a limit x in X. Then by splitting the integral over the regions $\left[0, \frac{1}{2}\right]$, $\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right)$, and $\left[\frac{1}{2} + \frac{1}{n}, 1\right]$, we compute the distance between x_n and x as

$$d(x_n, x) = \int_0^{\frac{1}{2}} |x(t)|dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |x(t) - x_n(t)|dt + \int_{\frac{1}{2}+\frac{1}{n}}^1 |1 - x(t)|dt, \quad n \in \mathbb{N}.$$

And so by the Dominated Convergence Theorem

$$d(x_n, x) \longrightarrow 0 \implies \int_0^{\frac{1}{2}} |x(t)|dt + \int_{\frac{1}{2}}^1 |1 - x(t)|dt = 0.$$

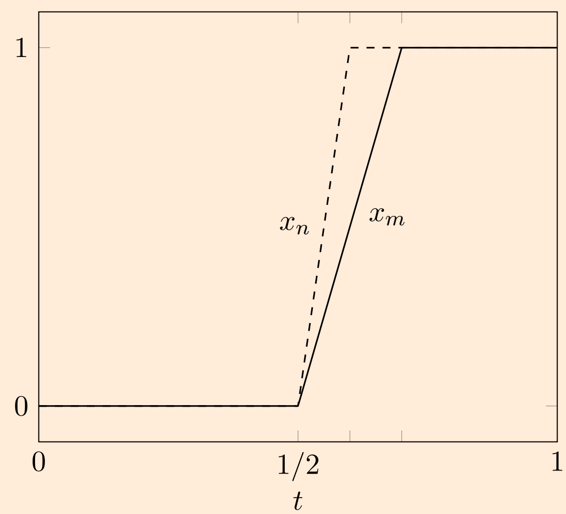
However, each of the integrands is strictly nonnegative, which means

$$\int_0^{\frac{1}{2}} |x(t)|dt = \int_{\frac{1}{2}}^1 |1 - x(t)|dt = 0.$$

And since $x \in X$, then x is continuous on $[0,1]$. This implies

$$x(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} < t < 1 \end{cases}.$$

Clearly, though, this violates the continuity of x on $[0,1]$ as $\lim_{t \to \frac{1}{2}^-} x(t) \neq \lim_{t \to \frac{1}{2}^+} x(t)$. We conclude that the Cauchy sequence (x_n) in the metric space (X,d) does not converge. Thus, (X,d) is incomplete. $\square$



Two functions x_n and x_m in our Cauchy sequence $(x_n) \subset X$

Exercise 1.19: Let $M \subset \ell^\infty$ be the subspace consisting of all sequences $x = (\xi_n)$ with at most finitely nonzero terms. Find a Cauchy sequence in M which does not converge in M, so that M is incomplete.

Consider the sequence $(x_n) \subset M$ defined $x_n := (1, 1/2, \dots, 1/n, 0, 0, \dots)$, $n \in \mathbb{N}$. Notice that for $x_n = (\xi_i^{(n)})$, $|\xi_i^{(n)}| \le 1$ and $\xi_i^{(n)} = 0, \forall i > n$, meaning that $x_n \in M$. This is a Cauchy sequence because for each $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$. And so for $n, m > N_\varepsilon$, $n \neq m$ we have

$$d(x_n, x_m) = \sup_{i \in \mathbb{N}} |\xi_i^{(n)} - \xi_i^{(m)}| = \sup_{\substack{i \in \mathbb{N} \\ \min\{n,m\} < i \le \max\{n,m\}}} i^{-1} = (\min\{n,m\} + 1)^{-1} < \varepsilon.$$

Thus, the sequence (x_n) is indeed Cauchy in M.

However, (x_n) does not converge in M. To see why this is the case, we show $x_n \longrightarrow x$ for $x \in \ell^\infty \backslash M$. But by the uniqueness of the limit, this would imply that (x_n) cannot converge in M. In particular, we claim $x_n \longrightarrow x$ where $x = (1, 1/2, \dots, 1/n, \dots)$. Clearly $x = (\xi_i)$ satisfies $|\xi_i| \le 1$, and so $x \in \ell^\infty$; but we also know $|\xi_i| > 0, \forall i \in \mathbb{N}$, so $x \notin M$. For an arbitrary $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$. And so for all $n > N_\varepsilon$

$$d(x_n, x) = \sup_{i\in\mathbb{N}} |\xi_i^{(n)} - \xi_i| = \sup_{\substack{i\in\mathbb{N}\\ i>n}} i^{-1} = (n+1)^{-1} < \varepsilon.$$

This proves that $x_n \longrightarrow x$ in ℓ^∞. We have given an example of a Cauchy sequence in M that is divergent, and so we conclude that M is incomplete.

Exercise 1.20: Show that M in Exercise 1.19 is incomplete by using Theorem 1.6.

Proof. We wish to show that M is incomplete by showing that the set M is not closed in the metric space ℓ^∞. To do so, it suffices to find some $x \in \bar{M}$ such that $x \notin M$. An example of such an x is $x = (1, 1/2, \ldots, 1/n, \ldots) \in X \setminus M$ from the previous problem. For every $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$. Then for $x_n = (1, 1/2, \ldots, 1/n, 0, \ldots) \in M$ with $n > N_\varepsilon$ we have

$$d(x, x_n) = (n+1)^{-1} < \varepsilon \iff x_n \in B_\varepsilon(x).$$

Because $\varepsilon > 0$ was taken to be arbitrary, this proves that x is a limit point of M. Thus $\bar{M} \neq M$, which implies by Theorem 1.6 that M is incomplete. $\square$

Exercise 1.21: Show that the subspace $Y \subset C[a, b]$ consisting of all $x \in C[a, b]$ such that $x(a) = x(b)$ is complete.

Proof. Let us consider an arbitrary element $y \in \bar{Y}$. By Theorem 1.5, there exists a sequence $(y_n) \subset Y$ such that $y_n \longrightarrow y$. Suppose towards contradiction that $y(a) \neq y(b)$. Then because $y_n \longrightarrow y$, for each $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \implies d(y_n, y) = \sup_{t\in[a,b]} |y_n(t) - y(t)| < \varepsilon.$$

But this also implies

$$n > N_\varepsilon \implies |y_n(a) - y(a)| < \varepsilon, |y_n(b) - y(b)| < \varepsilon.$$

Therefore, the sequence $(y_n(a))$ satisfies $y_n(a) \longrightarrow y(a)$ and $(y_n(b))$ satisfies $y_n(b) \longrightarrow y(b)$ in the Euclidean metric space $(\mathbb{R}, d)$. But because $(y_n) \subset Y$, we know $y_n(a) = y_n(b), \forall n \in \mathbb{N}$. And so we have proven both $y_n(a) \longrightarrow y(a)$ and $y_n(a) \longrightarrow y(b)$ for $y(a) \neq y(b)$. This, though, contradicts the uniqueness of the limit of a convergent sequence (Theorem 1.2).
Thus $y \in Y$, which implies that $\bar{Y} = Y$ since we chose y to be arbitrary. Finally by Theorem 1.6, Y closed in $C[a, b]$ implies Y is a complete metric space. $\square$

Exercise 1.22: Show that a discrete metric space is complete.

Proof. Let (X, d) be a discrete metric space, and suppose $(x_n) \subset X$ is an arbitrary Cauchy sequence in X.
We claim that there exists an $N \in \mathbb{N}$ such that $n, m > N \implies x_n = x_m$. By the Cauchy criterion, there exists an $N_{\varepsilon=1} \in \mathbb{N}$ such that $n, m > N_{\varepsilon=1} \implies d(x_n, x_m) < 1$. However, because X is a discrete metric space, $d(x_n, x_m) < 1 \implies d(x_n, x_m) = 0 \implies x_n = x_m$. Thus, $n, m > N = N_{\varepsilon=1} \implies x_n = x_m$.
Now that we have established this fact, we claim that $x_n \longrightarrow x$ where $x = x_{N+1} \in X$. To see why this is the case, let $\varepsilon > 0$ be arbitrary. For all $n > N$, the Cauchy sequence (x_n) satisfies

$$d(x_n, x) = d(x_n, x_{N+1}) = 0 < \varepsilon.$$

Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that indeed $x_n \longrightarrow x$. And since $(x_n) \subset X$ was taken to be an arbitrary Cauchy sequence, we have proven that every Cauchy sequence in X is convergent. Thus, (X, d) is a complete metric space. $\square$

Exercise 1.23: Show that in the space s defined in Exercise 1.4 we have $x_n \longrightarrow x$ if and only if $\xi_j^{(n)} \longrightarrow \xi_j$ for all $j \in \mathbb{N}$, where $x_n = (\xi_j^{(n)})$ and $x = (\xi_i)$.

Proof. $\implies$ Suppose $x_n \longrightarrow x$ and consider an arbitrary $j \in \mathbb{N}$. Then for each $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \implies d(x_n, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} < \frac{1}{2^j} \frac{\varepsilon}{1+\varepsilon}.$$

But since the individual terms of the infinite series are nonnegative, this also implies

$$\begin{aligned} &\frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \leq d(x_n, x) < \frac{1}{2^j} \frac{\varepsilon}{1+\varepsilon} \\ \iff &\frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} < \frac{\varepsilon}{1+\varepsilon} \\ \iff &|\xi_j^{(n)} - \xi_j| < \varepsilon. \end{aligned}$$

The last implication follows from the fact that the function $f(t) = t/(1+t)$ is strictly increasing on the interval $t > 0$. Because $\varepsilon > 0$ was taken to be arbitrary, this proves $\xi_j^{(n)} \longrightarrow \xi_j$ in the Euclidean metric space $(\mathbb{R}, d)$. And $j \in \mathbb{N}$ was also chosen arbitrarily, which implies that this result holds for all $j \in \mathbb{N}$.
$\impliedby$ Conversely, suppose $\xi_j^{(n)} \longrightarrow \xi_j$ for each $j \in \mathbb{N}$. Notice that for all $n \in \mathbb{N}$,

$$d(x_n, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Therefore, by the Dominated Convergence Theorem[4]

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\lim_{n\to\infty} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} \right).$$

Also, the continuous mapping theorem tells us that for each $j \in \mathbb{N}$,

$$\xi_j^{(n)} \longrightarrow \xi_j \implies \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \longrightarrow 0,$$

which implies

$$\lim_{n\to\infty} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} = 0.$$

Finally, we can compute the limit

$$\lim_{n\to\infty} d(x_n, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\lim_{n\to\infty} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} \right) = 0.$$

Hence, we have $x_n \longrightarrow x$. □

1.5 Completing Metric Spaces

We already know that the metric space $(\mathbb{Q}, d)$, where d is the Euclidean metric, can be **completed** by enlarging the rational line to the entire real line $\mathbb{R}$. This begs the question, though, of whether an arbitrary incomplete metric space can be completed in a similar manner.

Definition 1.14 (Isometric mapping, isometric spaces): Let $X = (X, d)$ and $\tilde{X} = (\tilde{X}, \tilde{d})$ be metric spaces. Then

1. A mapping $T : X \longrightarrow \tilde{X}$ is said to be **isometric** or an **isometry** if T preserves distances, that is, if for all $x, y \in X$

$$\tilde{d}(Tx, Ty) = d(x, y).$$

2. The space X is said to be **isometric** with the space $\tilde{X}$ if there exists a bijective isometry of X onto $\tilde{X}$. The spaces X and $\tilde{X}$ are then called **isometric spaces**.

To parse this definition, isometric spaces may differ at most by the "nature of their points", but these points are indistinguishable in terms of the respective metrics.

[4] This is because we can write $\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1+|\xi_i^{(n)} - \xi_i|} = \int f_n(i)\ d\mu(i)$, where μ is the counting measure on $\mathbb{N}$. And so if we can prove $|f_n(i)| \leq g(i)$ for g satisfying $\int g(i)\ d\mu(i) < \infty$, then we can invoke the Dominated Convergence Theorem to say $\lim_{n\to\infty} \int f_n(i)\ d\mu(i) = \int \lim_{n\to\infty} f_n(i)\ d\mu(i)$.

Theorem 1.9 (Completion): For a metric space $X = (X, d)$ there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W which isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique except for isometries, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric.

Proof. Just as is done by Kreyszig, we break down this proof into a number of smaller steps:

(a) Construct the metric space $\hat{X} = (\hat{X}, \hat{d})$:
Let (x_n) and (x'_n) be two Cauchy sequences in the metric space X. We define the equivalence relation $\sim$ such that

$$(x_n) \sim (x'_n) \iff \lim_{n \longrightarrow \infty} d(x_n, x'_n) = 0.$$

From Exercise 1.17, we know that the sequence (a_n) defined $a_n := d(x_n, x'_n)$ converges, and so limit $\lim_{n\longrightarrow\infty} d(x_n, x'_n)$ indeed exists. One can verify that $\sim$ indeed defines an equivalence relation on the set of Cauchy sequences of X.
Let $\hat{X}$ be defined to be the set of all equivalence classes under the the equivalence relation $\sim$. Then for any two elements $\hat{x}, \hat{y} \in \hat{X}$, we define the metric $\hat{d}$ on $\hat{X}$ such that

$$\hat{d}(\hat{x}, \hat{y}) := \lim_{n\to\infty} d(x_n, y_n),$$

where $(x_n) \in \hat{x}$ and $(y_n) \in \hat{y}$ are any two representatives of the equivalence classes $\hat{x}$ and $\hat{y}$, respectively. We must show that $\hat{d}$ does not depend on which representatives from the equivalence classes $\hat{x}$ and $\hat{y}$ we select.
To see why this is indeed the case, suppose $(x_n), (x'_n) \in \hat{x}$ and $(y_n), (y'_n) \in \hat{y}$. Let $l_1 = \lim_{n\to\infty} d(x_n, y_n)$ and $l_2 = \lim_{n\to\infty} d(x'_n, y'_n)$. And take $\varepsilon > 0$ to be arbitrary. Then we have

$$|l_1 - l_2| \leq |d(x_n, y_n) - l_1| + |d(x_n, y_n) - d(x'_n, y'_n)| + |d(x'_n, y'_n) - l_2|.$$

Clearly, we can find an $N \in \mathbb{N}$ sufficiently large such that $n > N$ implies $|d(x_n, y_n) - l_1| < \frac{\varepsilon}{3}$ and $|d(x'_n, y'_n) - l_2| < \frac{\varepsilon}{3}$. For the middle term, one can invoke the triangle inequality to show

$$\begin{aligned} &d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \\ \implies &d(x_n, y_n) - d(x'_n, y'_n) \leq d(x_n, x'_n) + d(y'_n, y_n) \end{aligned}$$

and similarly $d(x'_n, y'_n) - d(x_n, y_n) \leq d(x_n, x'_n) + d(y'_n, y_n)$. Together, these statements imply

$$|d(x'_n, y'_n) - d(x_n, y_n)| \leq d(x_n, x'_n) + d(y'_n, y_n).$$

But because $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$ we know it must the case that $\lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d(y_n, y'_n) = 0$. Therefore, we can also find an $M \in \mathbb{N}$ sufficiently large so that $n > M$ implies

$$|d(x'_n, y'_n) - d(x_n, y_n)| \leq d(x_n, x'_n) + d(y'_n, y_n) < \frac{\varepsilon}{3}.$$

Altogether, we have proven that for $n > \max\{N, M\}$, then

$$|l_1 - l_2| < \varepsilon.$$

Since $\varepsilon > 0$ was chosen to be arbitrary, we deduce

$$|l_1 - l_2| \leq 0 \implies |l_1 - l_2| = 0 \iff l_1 = l_2.$$

Now that we have shown that $\hat{d}$ is well-defined, it remains to prove that $\hat{d}$ is indeed a metric. From our previous note, we know that $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ exists as well as $d(x_n, y_n) \geq 0, \forall n \in \mathbb{N} \implies \hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n) \geq 0$. For the second axiom, let us first suppose that $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = 0$. By definition of our equivalence relation $\sim$, this means $(x_n) \sim (y_n) \implies \hat{x} = \hat{y}$. Conversely, $\tilde{d}(\hat{x},\hat{x}) = \lim_{n\to\infty} d(x_n, x_n) = 0$. For the third axiom, we have $\tilde{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \tilde{d}(\hat{y},\hat{x})$. Finally, for the triangle inequality, we know that since (X, d) is a metric space, then it holds that for every $n \in \mathbb{N}$,

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n),$$

where $(x_n) \in \hat{x}$, $(y_n) \in \hat{y}$, and $(z_n) \in \hat{z}$. Taking the limit on both sides of the inequality,

$$\hat{d}(\hat{x},\hat{z}) = \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) = \hat{d}(\hat{x},\hat{y}) + \hat{d}(\hat{y},\hat{z}).$$

And so we have proven $(\hat{X}, \hat{d})$ is indeed a metric space.

(b) There exists an isometry $T : X \longrightarrow W \subset \hat{X}$, where $\bar{W} = \hat{X}$:
Define $T : X \longrightarrow \hat{X}$ such that $Tx = \hat{x}$, where $\hat{x}$ is equivalence class which contains the constant sequence $(x, x, \ldots)$. Clearly $(x_n) = (x, x, \ldots) \in \hat{X}$ because $d(x_n, x_m) = d(x, x) = 0$ for every $n, m \in \mathbb{N}$. For our set W, we let $W = T(X)$, where $T(X)$ denotes the image of X under the map T.
Now we show that X is isometric with W by proving that $T : X \longrightarrow W$ is a bijective isometry. We see that T is an isometry because for $Tx = (x_n), Ty = (y_n)$, we have $\hat{d}(Tx, Ty) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y)$. T is injective because $Tx = Ty \iff \hat{d}(Tx, Ty) = 0 \iff d(x, y) = 0 \iff x = y$. And $T : X \longrightarrow W$ is surjective since we have defined $W = T(X)$.
Finally, we must prove that $\bar{W} = \hat{X}$, that is, W is a dense subset of the metric space $\hat{X}$. To prove that this, we show that every point in $\hat{X}$ is a limit point of W. In particular, consider an arbitrary $\hat{x} \in \hat{X}$. And let (x_n) be an arbitrary element of the equivalence class $\hat{x}$. Take $\varepsilon > 0$ to be arbitrary. Since (x_n) is a Cauchy sequence in X, there exists an $N \in \mathbb{N}$ such that

$$n > N \implies d(x_n, x_N) < \frac{\varepsilon}{2}.$$

Clearly, it is also the case that the constant sequence $(x_N, x_N, \ldots) \in W$. Let $\hat{x}_N$ denote the equivalence class containing this sequence. Then we have

$$\hat{d}(\hat{x}, \hat{x}_N) = \lim_{n\to\infty} d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Because $\varepsilon > 0$ was chosen arbitrarily, this proves that $\hat{x}$ is a limit point of X. And because $\hat{x}$ was taken to be arbitrary as well, we conclude $\bar{W} = \hat{X}$.

(c) $\hat{X}$ is complete:
Let $(\hat{x}_n)$ be an arbitrary Cauchy sequence in $\hat{X}$. Because W is dense in $\hat{X}$, then for every $n \in \mathbb{N}$, we can find a $\hat{z}_n \in W$ which satisfies

$$\hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}.$$

From here, we claim that the sequence $(\hat{z}_n)$ is Cauchy in $\hat{X}$. Take $\varepsilon > 0$ to be arbitrary. Then we can choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{3}$ and $n, m > N \implies \hat{d}(\hat{x}_n, \hat{x}_m) < \frac{\varepsilon}{3}$ (because $(\hat{x}_n)$ is a Cauchy sequence). By the triangle inequality, this gives us

$$n, m > N \implies \hat{d}(\hat{z}_n, \hat{z}_m) \leq \hat{d}(\hat{z}_n, \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{x}_m) + \hat{d}(\hat{x}_m, \hat{z}_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

And so because $\varepsilon > 0$ was chosen to be arbitrary, we deduce that $(\hat{z}_n)$ is indeed a Cauchy sequence.
Now, since $(\hat{z}_n) \subset W$ and $T : X \longrightarrow W$ is injective, then we can define the sequence $(z_n) \subset X$, where $z_n := T^{-1}\hat{z}_n$. But by our previous result that $(\hat{z}_n)$ is Cauchy and since T is an isometry, then the sequence (z_n) is Cauchy in the metric space X. That is, we have that (z_n) belongs to some equivalence class $\hat{x} \in \hat{X}$. This $\hat{x}$ is our candidate for the limit of $(\hat{x}_n)$.
It remains only to show that $\hat{x}_n \longrightarrow \hat{x}$ in the metric space $(\hat{X}, \hat{d})$. To start, let us take $\varepsilon > 0$ be arbitrary. And choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ as well as $n, m > N \implies d(z_n, z_m) < \frac{\varepsilon}{2}$, which we can do since (z_n) is a Cauchy sequence. Then we get that for $n > N$

$$\begin{aligned} \hat{d}(\hat{x}_n, \hat{x}) &\leq \hat{d}(\hat{x}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{x}) \\ &< \frac{1}{n} + \hat{d}(\hat{z}_n, \hat{x}) \\ &< \frac{\varepsilon}{2} + \hat{d}(\hat{z}_n, \hat{x}). \end{aligned}$$

And since $\hat{z}_n \in W$ then we have $(z_n, z_n, \dots) \in \hat{z}_n$, where $z_n = T^{-1}\hat{z}_n$ as we previously defined. By the definitions of $\hat{d}$ and $\hat{x}$, this implies

$$\begin{aligned} \hat{d}(\hat{x}_n, \hat{x}) &< \frac{\varepsilon}{2} + \lim_{m\to\infty} d(z_n, z_m) \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was taken to be arbitrary, we have proven that $\hat{x}_n \longrightarrow \hat{x}$. And as $(\hat{x}_n)$ was chosen to be an arbitrary Cauchy sequence in $\hat{X}$, then we conclude that the metric space $(\hat{X}, \hat{d})$ is complete.

(d) $\hat{X}$ is unique except for isometries:
Finally, we must prove that if $(\tilde{X}, \tilde{d})$ is another complete metric space with a dense subset $\tilde{W}$ that is isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric. Let $\tilde{x}$ and $\tilde{y}$ be arbitrary elements of $\tilde{X}$. By Theorem 1.5, there exists sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ in $\tilde{W}$ such that $\tilde{x}_n \longrightarrow \tilde{x}$ and $\tilde{y}_n \longrightarrow \tilde{y}$. By the triangle inequality, we have that

$$|\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{x}_n, \tilde{y}_n)| \leq \tilde{d}(\tilde{x}_n, \tilde{x}) + \tilde{d}(\tilde{y}_n, \tilde{y}) \longrightarrow 0,$$

which proves that

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\longrightarrow\infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n).$$

Since $\tilde{W}$ is isometric with X and X is isometric with $\hat{W}$, then $\tilde{W}$ is isometric with $\hat{W}$. And because $\bar{W} = \hat{X}$, then the distances on $\hat{X}$ and $\tilde{X}$ must be identical. Thus, $\hat{X}$ and $\tilde{X}$ must be isometric spaces.

$\square$

Exercise 1.24: If X_1 and X_2 are isometric and X_1 is complete, show that X_2 is complete.

Proof. Let (x_n) be an arbitrary Cauchy sequence in X_2. Since X_1 and X_2 are isometric, there exists a map $T : X_2 \longrightarrow X_1$ which is bijective and satisfies $d_2(x, y) = d_1(Tx, Ty)$. Accordingly, let us define the sequence (Tx_n) in X_1. Because (x_n) is a Cauchy sequence in X_2, then (Tx_n) is a Cauchy sequence in X_1. And so since X_1 is complete, this implies $Tx_n \longrightarrow x$ for some $x \in X_1$. We propose that $T^{-1}x \in X_2$ is the limit of the Cauchy sequence (x_n).
To see that this is the case, consider an arbitrary $\varepsilon > 0$. Because $Tx_n \longrightarrow x$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies d_1(Tx_n, x) < \varepsilon$. This fact allows us to deduce

$$n > N \implies d_2(x_n, T^{-1}x) = d_1(Tx_n, x) < \varepsilon.$$

Because $\varepsilon > 0$ was taken to be arbitrary, this proves that $x_n \longrightarrow T^{-1}x$. And since (x_n) was chosen to be an arbitrary Cauchy sequence in X_2, then X_2 must be a complete metric space. □

Exercise 1.25 (Homeomorphism): A **homeomorphism** is a continuous bijective map $T : X \longrightarrow Y$ whose inverse is continuous; the metric spaces X and Y are said to be **homeomorphic**. Show that

1. If X and Y are isometric, they are homeomorphic.

2. Illustrate with an example that a complete and an incomplete metric space may be homeomorphic.

Proof. We start with a proof of (1). Since the metric spaces (X, d) and $(Y, \tilde{d})$ are isometric, then there exists a bijective map $T : X \longrightarrow Y$ which satisfies $d(x_1, x_2) = \tilde{d}(Tx_1, Tx_2)$. And so it remains to show that T and T^{-1} are continuous.
Let us consider an arbitrary $x_0 \in X$. And take $\varepsilon > 0$ to be be arbitrary. Then we have that

$$d(x, x_0) < \varepsilon \implies \tilde{d}(Tx, Tx_0) < \varepsilon.$$

This proves that the map T is continuous. Similarly, we can take an arbitrary $y_0 \in Y$. Note that because X and Y are isometric, then the inverse of T is defined to be $y \mapsto T^{-1}y$, where T^{-1} is the inverse image of the point $y \in Y$. T^{-1} satisfies $\tilde{d}(y_1, y_2) = d(T^{-1}y_1, T^{-1}y_2)$. Consequently, we have

$$\tilde{d}(y, y_0) < \varepsilon \implies d(T^{-1}y, T^{-1}y_0) < \varepsilon.$$

And so we have proven that T^{-1} is also continuous.
Therefore, the metric spaces X and Y are homeomorphic.
Now for (2), let us consider the Euclidean metric space $X = (\mathbb{R}, d)$ as well as the metric space $Y = ((-1, 1), \tilde{d})$, where $\tilde{d}$ is the Euclidean metric. From Example 1.14, X is complete; but since $(-1, 1)$ is not a closed subset of $\mathbb{R}$, then Y is not complete. And so we must find a homeomorphism between the metric spaces X and Y.
Define the map $T : X \longrightarrow Y$ such that $Tx = \frac{x}{|x|+1}$. We claim that T is a homeomorphism. First, T is injective since $Tx_1 = Tx_2 \iff \frac{x_1}{|x_1|+1} = \frac{x_2}{|x_2|+1} \iff \frac{|x_2|+1}{|x_1|+1} = \frac{x_2}{x_1} \iff |x_2|+1 = k(|x_1|+1), x_2 =$

$kx_1, k > 0 \implies k|x_1|+1 = k(|x_1|+1) \iff k = 1 \implies x_2 = x_1$. And for the surjectivity of T, we have that $y \in (-1,1)$, $\frac{x}{|x|+1} = y \iff x = (|x|+1)y \iff x = |x|y + y$. Now, if $y \geq 0 \iff x \geq 0$, then $x = \frac{y}{1-y}$, and if $y < 0 \iff x < 0$, then $x = \frac{y}{1+y}$. Since for each $y \in (-1,1)$ we have found a $x \in X$ such that $Tx = y$, then the map T is surjective. Thus, T is a bijection.
Finally, we must show that T and T^{-1} are continuous. For T, notice that we can write $Tx = \frac{f(x)}{g(x)}$, where $f(x) = x$ and $g(x) = |x|+1$ are continuous functions mapping from X to itself. Then since $g(x) > 0$, we conclude that T, the quotient of two continuous functions, is also continuous. And for T^{-1}, we recall that

$$T^{-1}y = \begin{cases} \frac{y}{1-y} & y \geq 0 \\ \frac{y}{1+y} & y < 0 \end{cases} = \frac{y}{1-|y|}.$$

Thus, just as we had for T, T^{-1} is the quotient of two functions $f(y) = y$ and $g(y) = 1-|y|$ that are continuous maps from Y to itself. Also, have $g(y) > 0$. Therefore, we conclude that T^{-1} is continuous.
Altogether, we have shown that $T : X \longrightarrow Y$ is a homeomorphism, and so the complete metric space X and the incomplete metric space Y are homeomorphic. $\square$

Exercise 1.26: Recall from our proof of Theorem 1.9 that for an arbitrary metric space (X,d) and any two Cauchy sequences $(x_n), (x'_n)$ in X, we define the relation $\sim$ such that

$$(x_n) \sim (x'_n) \iff \lim_{n\longrightarrow\infty} d(x_n, x'_n) = 0.$$

Prove that $\sim$ defines an equivalence relation on the set of all Cauchy sequences of X.

Proof. In order to show that $\sim$ is an equivalence relation, we must show that it is symmetric, reflexive, and transitive:

(a) Symmetry:

$$(x_n) \sim (x'_n) \iff \lim_{n\longrightarrow\infty} d(x_n, x'_n) = 0 \iff \lim_{n\longrightarrow\infty} d(x'_n, x_n) = 0 \iff (x'_n) \sim (x_n)$$

(b) Reflexivity: For (x_n) an arbitrary Cauchy sequence in X, we have that $d(x_n, x_n) = 0$ for every $n \in \mathbb{N}$. This implies that

$$\lim_{n\longrightarrow\infty} d(x_n, x_n) = \lim_{n\longrightarrow\infty} 0 = 0.$$

And so indeed $(x_n) \sim (x_n)$.

(c) Transitivity: Suppose $(x_n), (y_n), (z_n)$ are Cauchy sequences in X that satisfy $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$. Let $\varepsilon > 0$ be arbitrary. Then by definition of $\sim$, we know that there exists a $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies d(x_n, y_n) < \frac{\varepsilon}{2}, d(y_n, z_n) < \frac{\varepsilon}{2}$. Accordingly,

$$n > N_\varepsilon \implies d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Because $\varepsilon > 0$ was taken to be arbitrary, this implies $\lim_{n\to\infty} d(x_n, z_n) = 0 \iff (x_n) \sim (z_n)$.

$\square$

Exercise 1.27 (Pseudometric): A **finite pseudometric** on a set X is a function $d : X \times X \longrightarrow \mathbb{R}_+$ which satisfies the following for all $x, y, z \in X$:

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

What is the difference between a metric and a pseudometric? Show that $d(x, y) = |\xi_1 - \eta_n|$ defines a pseudometric on the set of all ordered pairs of real numbers, where $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2)$.
The difference between a metric and a pseudometric is that for a pseudometric d we can have $x, y \in X$ with $x \neq y$ but also $d(x, y) = 0$.

Proof. Let X be the set of all ordered pairs of real numbers. We will now prove that d defines a pseudometric on X.
Since $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$, then $d(x, y) = |\xi_1 - \eta_1| \in \mathbb{R}_+$ by the properties of the absolute value function. For the first axiom, we have $d(x, x) = |\xi_1 - \xi_1| = 0$. Similarly, by the properties of the absolute value, we have $d(x, y) = |\xi_1 - \eta_1| = |\eta_1 - \xi_1| = d(y, x)$. Finally for $z = (\gamma_1, \gamma_2) \in X$, we have $d(x, z) = |\xi_1 - \gamma_1| \leq |\xi_1 - \eta_1| + |\eta_1 - \gamma_1| = d(x, y) + d(y, z)$. Notice that here we invoked the triangle inequality on the real line. Hence d, does indeed define a pseudometric on X.
Notice, though, that d does not define a metric on X. In fact, for any $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$ with $\xi_1 = \eta_1$ we have $d(x, y) = 0$. $\square$

Exercise 1.28: Does

$$d(x, y) = \int_a^b |x(t) - y(t)| dt$$

define a metric or pseudometric on X if X is

1. the set of all real-valued, continuous functions on $[a, b]$.
2. the set of all real-valued, Riemann integrable functions on $[a, b]$.

We know from Exercise 1.2 that d defines a metric on (1), the set of all real-valued, continuous functions on $[a, b]$. Now for (2), we claim that d defines a pseudometric on X but not a metric.
To see that d cannot define a metric on X, recall the **Riemann-Lebesgue lemma**, which states that a function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if f is bounded and the set $\mathcal{N}(f)$ of points at which f is discontinuous has Lebesgue measure zero. Accordingly, the function $f : [a, b] \longrightarrow \mathbb{R}$ defined

$$x(t) = \begin{cases} 0 & a \leq t < b \\ 1 & t = b \end{cases}$$

is Riemann integrable. Then for x_0 the zero function, we have

$$d(x, x_0) = \int_a^b \mathbb{1}_{t=b} dt = 0$$

but $x(b) \neq x_0(b) \implies x \neq x_0$. This implies that d cannot define a metric on X.
We can prove, though, that d defines a pseudometric on X. For the sake of not repeating ourselves, we do not include it here. The proof is identical to that in Exercise 1.2 except for the following pieces:

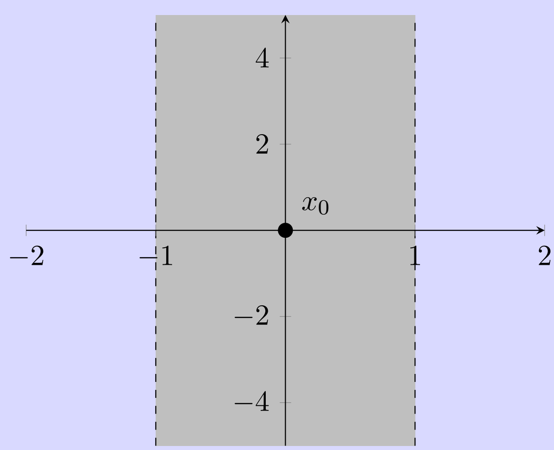
1. To prove $d(x, y) < \infty$, we use the result of the Riemann-Lebesgue lemma that $|x(t) - y(t)|$ is Riemann integrable on $[a, b]$ implies $|x(t) - y(t)| \leq c$ for all $t \in [a, b]$. Here $c \in \mathbb{R}_+$ is some constant.
2. We only need to prove $d(x, x) = 0$, not $d(x, y) = 0 \implies x = y$. As we showed above, the latter statement is not true.

Exercise 1.29: If (X, d) is a pseudometric space, we call a set

$$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}$$

an **open ball** with **center** x_0 and **radius** r. What are open balls of radius $r = 1$ in Exercise 1.27?
Let $x_0 = (\xi_1, \xi_2)$ be an arbitrary point in the pseudometric space (X, d). Then we have

$$\begin{aligned} x = (\eta_1, \eta_2) \in B_{r=1}(x_0) &\iff d(x, x_0) < 1 \iff |\eta_1 - \xi_1| < 1 \\ &\iff x \in \{(\eta_1, \eta_2) \mid |\eta_1 - \xi_1| < 1\}. \end{aligned}$$



The open unit ball $B_{r=1}(x_0)$ centered at $x_0 = (0, 0)$ in the pseudometric space (X, d). Note that $B_{r=1}(x_0) = B_{r=1}(x_0')$ for any $x_0' = (\xi_1', \xi_2') \in \mathbb{R}^2$ with $\xi_1' = 0$.

2 Normed Spaces, Banach Spaces

2.1 A Review of Vector Spaces