PHY 321 MARCH 8

Harmonic oscillations

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega t)$$

$$\omega_0 = \sqrt{k/m} \qquad \tau = \omega_0 \cdot t$$

$$\tilde{\omega} = \frac{\omega}{\omega_0} \qquad \tilde{F}_0 = \frac{F_0}{m\omega_0^2}$$

$$2\gamma = \frac{b}{2m\omega_0}$$

$$\frac{d^2x}{d\tau^2} + 2\gamma \frac{dx}{d\tau} + \underline{x} = \tilde{F}_0 \cos(\tilde{\omega}\tau)$$

$$x(\tau) = x_h(\tau) + x_p(\tau)$$

$$x_h(\tau) = A_1 e^{-\tau(\gamma + \sqrt{\gamma^2-1})} + A_2 e^{-\tau(\gamma - \sqrt{\gamma^2-1})}$$

$(\tilde{F}_0 = 0)$

A_1 and A_2 given by initial conditions, infinity of A_1 and A_2

$$x_p(\tau) = D \cos(\tilde{\omega}\tau - \delta)$$

$$D = \frac{F_0}{\sqrt{(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2}}$$

$$\delta = \tan^{-1}\left(\frac{2\gamma\tilde{\omega}^2}{1-\tilde{\omega}^2}\right)$$

<u>Example: $F_0 = 0$</u>

$m = 2\,kg \qquad b = 20\,NS/m$

$K = 32\,N/m = 32\,kg\,m/s^2/m$

$\omega_0 = \sqrt{K/m} = 4\,s^{-1}$

$\gamma = \frac{b}{2m\omega_0} = \frac{5}{4}$

initial values $\quad x_0 = 0.125\,m$

$\qquad\qquad\qquad v_0 = -2.0\,m/s$

$$X = A_1 e^{-\underbrace{(\gamma+\sqrt{\gamma^2-1})}_{\beta_1}\tau} + A_2 e^{-\underbrace{(\gamma-\sqrt{\gamma^2-1})}_{\beta_2}\tau}$$

$\beta_1 = 2 \quad \wedge \quad \beta_2 = 1/2$

$x_0 = x(t=0) = x(\tau=0) = A_1 + A_2$

$v_0 = v(\tau=0) = \left.\frac{dx}{d\tau}\right|_{\tau=0}$

$\quad = -2\,m/s = -\beta_1 A_1 - \beta_2 A_2$

$A_1 = 17/24 \qquad A_2 = -7/6$

$$\underline{x(\tau) = 17/24\, e^{-2\tau} - 7/6\, e^{-1/2\tau}}$$

Particular solution:

$$\frac{d^2x}{d\tau^2} + 2\gamma \frac{dx}{d\tau} + x = \tilde{F}_0 \cos(\tilde{\omega}\tau)$$

we can guess a solution

$$\underline{x_p(\tau)} = D\cos(\tilde{\omega}\tau - \delta)$$

insert in $\qquad \sin(\tilde{\omega}\tau - \delta)$

$$D\Big(-\tilde{\omega}^2\cos(\tilde{\omega}\tau - \delta) - 2\gamma\tilde{\omega}\,\underline{\sin(\tilde{\omega}\tau - \delta)}$$

$$+ \cos(\tilde{\omega}\tau - \delta)\Big) = \tilde{F}_0\cos(\tilde{\omega}\tau)$$

Trigonometric relations:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$D\Big\{\Big(-\tilde{\omega}^2\cos\delta + 2\gamma\tilde{\omega}\sin\delta + \cos\delta\Big)\underline{\cos(\tilde{\omega}\tau)}$$

$$+\left(-\tilde{\omega}^2 \sin\delta - 2\gamma\tilde{\omega}\cos\delta + \sin\delta\right)\underline{\sin(\tilde{\omega}\tau)}\Big\} = \underline{\tilde{F}_0\cos(\tilde{\omega}\tau)}$$

The $\sin(\tilde{\omega}\tau)$ and $\cos(\tilde{\omega}\tau)$ need to satisfy the following equations if the expressions are to hold for all times:

$$D\left\{-\tilde{\omega}^2\underline{\cos\delta} + 2\gamma\tilde{\omega}\underline{\sin\delta} + \underline{\cos\delta}\right\} = \tilde{F}_0$$

and

$$-\tilde{\omega}^2\sin\delta - 2\gamma\tilde{\omega}\underline{\cos\delta} + \sin\delta = 0$$

Divide by $\cos\delta$

$$\underline{\tan\delta} = \frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}$$

$$\underline{\delta} = \tan^{-1}\left[\frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}\right]$$

$$\sin^2\delta + \cos^2\delta = \underline{1}$$

$$\sin\delta = \frac{\tan\delta}{\sqrt{\tan^2\delta + \underline{1}}}$$

$$\sin\delta = \frac{2\gamma\tilde{\omega}}{\sqrt{(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2}}$$

$$\cos\delta = \frac{1-\tilde{\omega}^2}{\sqrt{(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2}}$$

$$D = \frac{\tilde{F}_0}{\sqrt{(\underline{1-\tilde{\omega}^2})^2 + 4\tilde{\omega}^2\gamma^2}}$$

$$\delta = \tan^{-1}\left[\frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}\right]$$

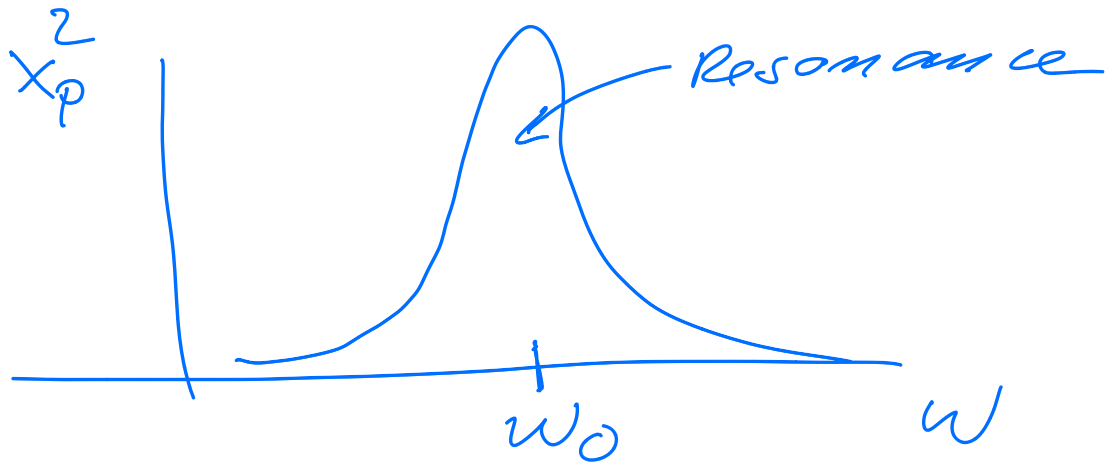
1) after a time $\tilde{\tau}_1$, the system was governed by $x_p(\tau) =$

$$D\cos(\tilde{\omega}\tau - \delta)$$

$$x_h(\tau) = A_1 e^{-\tau(\gamma + \sqrt{\gamma^2 - 1})} + A_2 e^{-\tau(\gamma - \sqrt{\gamma^2 - 1})}$$

2) $\omega = \omega_0 \Rightarrow \tilde{\omega} = 1$

$$D = \frac{\tilde{F}_0}{\sqrt{4\gamma^2}} = \frac{\tilde{F}_0}{2\gamma}$$



Denominator: Max value of D when denominator has its minimum.

Keep ω_0 fixed ($\omega_0 = \sqrt{k/m}$) and vary ω

$$\frac{d}{d\tilde{\omega}}\left[(1-\tilde{\omega}^2)^2 + 4\tilde{\omega}^2\gamma^2\right] = 0$$

$2(-2\tilde{\omega})(1-\tilde{\omega}^2)$

$$\tilde{\omega} = \frac{\omega}{\omega_0}$$

$$\Rightarrow \quad -4\tilde{\omega}(1-\tilde{\omega}^2) + 8\omega\gamma^2 = 0$$

$$\boxed{\tilde{\omega} = \sqrt{1-2\gamma^2}}$$

$$\gamma = \frac{b}{2m\omega_0}$$

$\omega = \omega_0$ is almost <u>true</u> TRUE

since $\gamma \ll 1$