PHY 321 APRIL 14

Constrained motion & Lagrangian formalism

Top-down

- constraint
- Lagrangian multipliers

Example 1

Minimize $f(x_1, x_2) =$

$$-3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$$

subject to

$$x_1 + x_2 = 5$$

$$\underline{x_2 = 5 - x_1}$$

constraint reduces the number of degrees of freedom $(x_1 x_2) \to x_1$

$$\boxed{-3x_1^2 - 6x_1(5-x_1) - 5(5-x_1)^2 + 7x_1 + 5(5-x_1)}$$

$$= -2x_1^2 + 22x_1 - 100 = g(x_1)$$

$$\frac{dg}{dx_1} = 0 \Rightarrow 4x_1 = 22 \Rightarrow x_1 = 11/2$$

$$\Rightarrow x_2 = -1/2$$

Def: $\mathcal{L}(x_1, x_2, \lambda) =$

$\uparrow$ Lagrange multiplier

$$f(x_1, x_2) + \cancel{\lambda}\, g(x_1, x_2)$$

$$g(x_1, x_2) = 0 = \cancel{x_1 + x_2 - 5}$$

Holonomic constraint

Euler-Lagrange

$$\left[\frac{\partial}{\partial x_1} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_1}\right] \mathcal{L}(x_1\, x_2\, \lambda) = 0$$

$$= -6x_1 - 6x_2 + 7 + \lambda = 0$$

$$x_1 + x_2 = \frac{1}{6}(7 + \lambda) = 5$$

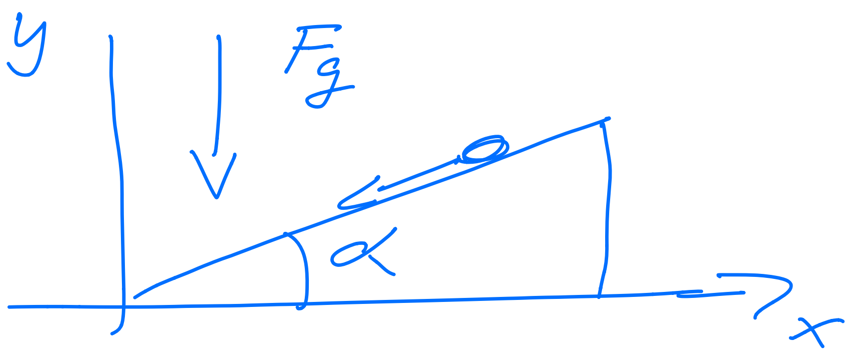
$$\boxed{\lambda = 23}$$

$$\left[\frac{\partial}{\partial x_2} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_2}\right] \mathcal{L}(x_1\, x_2\, \lambda) = 0$$

$$= -6x_1 - 10x_2 + 5 + \lambda = 0$$

$$\Rightarrow \boxed{x_2 = -1/2 \wedge x_1 = 11/2}$$

Example 2



$$y = \underline{x} \tan\alpha \Rightarrow$$

constraint: $g(x,y) = y - \underline{x}\tan\alpha = 0$

$$K = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$\mathcal{L} = \mathcal{L}(x, \dot{x}, y, \dot{y}, t) =$$

$$K - V + \lambda g(x,y)$$

$$\left[\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right]\mathcal{L} = 0$$

$$= -\lambda \tan\alpha - m\underline{\ddot{x}} = 0 \quad \underline{*}$$

$$\left[\frac{\partial}{\partial y} - \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\right]\mathcal{L} =$$

$$-\underline{m\ddot{y}} - mg + \lambda = 0 \quad **$$

$$\ddot{y} = \ddot{x}\tan\alpha$$

multiply * with $\tan\alpha$
and subtract 2nd eq (**)

$$-\lambda\tan^2\alpha - m\ddot{x}\tan\alpha$$
$$+ m\ddot{y} + mg - \lambda = 0$$

$$-\lambda\tan^2\alpha - \lambda - mg = 0$$

$$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$$

$$\lambda = mg\cos^2\alpha$$

$$\ddot{x} = -g\sin\alpha\cos\alpha$$
$$\ddot{y} = -g\sin^2\alpha$$