Lecture Notes
PHY 321 - Classical Mechanics I

www.pa.msu.edu/courses/phy321

Instructor: Scott Pratt, prattsc@pa.msu.edu

These notes are **NOT** meant to be a substitute for the book. The text Classical Dynamics of Particles and Systems by Thornton and Marion is required. The course will follow the order of material in the text. These notes are only meant to outline what was covered in class and to provide something students can print and annotate rather than taking full lecture notes.

1 Gravity and Central Forces

1.1 Gravity

The gravitational potential energy and forces involving two masses a and b are

$$\begin{aligned} U_{ab} &= -\frac{Gm_a m_b}{|\vec{r}_a - \vec{r}_b|}, \\ F_{ba} &= -\frac{Gm_a m_b}{|\vec{r}_a - \vec{r}_b|^2}\hat{r}_{ab}, \\ \hat{r}_{ab} &= \frac{\vec{r}_b - \vec{r}_a}{|\vec{r}_a - \vec{r}_b|}. \end{aligned} \tag{1.1}$$

Here $G = 6.67 \times 10^{-11}$ Nm^2/kg^2, and F_{ba} is the force on b due to a. By construction, the force on b due to a and the force on a due to b are equal and opposite. The net potential energy for a large number of masses would be

$$U = \sum_{a<b} U_{ab}. \tag{1.2}$$

Just like electrodynamics, one can define "fields", which are the force and potential energy due to a small additional mass m. The gravitational field related to the force has dimensions of force per mass, or acceleration, and can be labeled $\vec{g}(\vec{r})$. The potential energy per mass has dimensions of energy per mass.

Since the field $\vec{g}$ obeys the same inverse square law for a point mass as the electric field does for a point charge, the gravitational field also satisfies a version of Gauss's law,

$$\oint d\vec{A} \cdot \vec{g} = -4\pi G M_{\text{inside}}. \tag{1.3}$$

Here, M_{inside} is the net mass inside a closed area. This can be understood by realizing that if a vector due to a point measure falls as $1/r^2$ and points outward, that it behaves the same as the flux of fluid spraying uniformly outward from a point source. If one imagines a point source spraying R gallons of paint per second at a steady rate, the number of gallons of paint per second caught by a closed surface would also be R. If the flux, number of gallons per area per second is $\vec{J}$, then integrating $d\vec{A} \cdot \vec{J}$ around the closed surface will be R regardless of the shape of the surface because for a steady state the rate at which paint is expelled from the point source would equal the rate at which it is deposited on the surface.

Example I:
Consider Earth to have its mass M uniformly distributed in a sphere of radius R. Find the magnitude of the gravitational acceleration as a function of the radius r in terms of the acceleration of gravity at the surface $g(R)$.
Solution: Take the ratio of Eq. (1.3) for two radii, R and $r < R$,

$$\begin{aligned} \frac{4\pi r^2 g(r)}{4\pi R^2 g(R)} &= \frac{4\pi G M_{\text{inside r}}}{4\pi G_{\text{inside R}}} \\ &= \frac{r^3}{R^3} \\ g(r) &= g(R)\frac{r}{R}\,. \end{aligned}$$

The potential energy per mass is similar conceptually to the voltage, or electric potential energy per charge that was studied in electromagnetism. If $V \equiv U/m$, $\vec{g} = -\nabla V$.

1.2 Tidal Forces

Consider a spherical object of radius r a distance D from another body of mass M. The magnitude of the force due to M on an object of mass δm on surface of the planet can be calculated by performing a Taylor expansion about the center of the spherical object.

$$F = -\frac{GM\delta m}{D^2} + 2\frac{GM\delta m}{D^3}\Delta D + \cdots \tag{1.4}$$

If the z direction points toward the large object, ΔD can be referred to as z. In the accelerating frame of an observer at the center of the object,

$$\delta m\frac{d^2z}{dt^2} = F - \delta ma' + \text{other forces acting on } \delta m, \tag{1.5}$$

where a' is the acceleration of the observer. Since $\delta ma'$ equals the gravitational force on δm if it were located at the object's center, one can write

$$m\frac{d^2z}{dt^2} = 2\frac{GM\delta m}{D^3}z + \text{other forces acting on } \delta m. \tag{1.6}$$

Here the other forces could represent the forces acting on δm from the spherical object such as the gravitational force of the contact force with the surface. If θ is the angle w.r.t. the z axis, the effective force acting on δm is

$$F_{\text{eff}} = 2\frac{GM\delta m}{D^3}r\cos\theta + \text{other forces acting on } \delta m. \tag{1.7}$$

This first force is the "tidal" force. It pulls objects outward from the center of the object. If the object were covered with water, it would distort the objects shape so that the shape would be elliptical, stretched out along the axis pointing toward the large mass M. The force is always along (either parallel or antiparallel to) the $\hat{r}$ direction.

Example IIa:
Consider the Earth to be a sphere of radius R covered with water, with the gravitational acceleration at the surface noted by g. Now assume that a distant body provides an additional constant gravitational acceleration $\vec{a}$ pointed along the z axis. Find the distortion of the radius as a function of θ. Ignore planetary rotation and assume $a << g$.
Solution: Since Earth would then accelerate with a, the field a would seem invisible in the accelerating frame.
Example IIb:
Now consider that the field is no longer constant, but that instead $a = -kz$ with $|kR| << g$.
Solution: The surface of the planet needs to be at constant potential (if the planet is not accelerating). Otherwise water would move to a point of lower potential. Thus

$$V(R) + gh + \frac{1}{2}kr^2\cos^2\theta = \text{Constant}$$
$$V(R) - Rg + gh + \frac{1}{2}kR^2\cos^2\theta + kRh\cos^2\theta + \frac{1}{2}kh^2\cos^2\theta = \text{Constant}.$$

Here, the potential due to the external field is $(1/2)kz^2$ so that $-\nabla U = -kz$. One now needs to solve for $h(\theta)$. Absorbing all the constant terms from both sides of the equation into one constant C, and since both h and kR are small, we can through away terms of order h^2 or kRh. This gives

$$\begin{aligned} gh + \frac{1}{2}kR^2\cos^2\theta &= C, \\ h &= \frac{C}{g} + \frac{1}{2g}kR^2\cos^2\theta, \\ h &= \frac{1}{2g}kR^2(\cos^2\theta - 1/3). \end{aligned}$$

The term with the factor of $1/3$ replaced the constant and was chosen so that the average height of the water would be zero.

1.3 Deriving elliptical orbits

Kepler's laws state that a gravitational orbit should be an ellipse with the source of the gravitational field at one focus. Deriving this is surprisingly messy. To do this, we first derive the equations of motion in terms of r and θ rather than in terms of x and y. The equations of motion give

$$\begin{aligned} \frac{d}{dt}r^2 &= \frac{d}{dt}(x^2+y^2) = 2x\dot{x} + 2y\dot{y}, \\ \dot{r} &= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}, \\ \ddot{r} &= \frac{x}{r}\ddot{x} + \frac{y}{r}\ddot{y} + \frac{\dot{x}^2+\dot{y}^2}{r} - \frac{\dot{r}^2}{r} \\ &= \frac{F_x\cos\theta + F_y\sin\theta}{m} + \frac{\dot{r}^2 + r^2\dot{\theta}^2}{r} - \frac{\dot{r}^2}{r} \\ &= \frac{F}{m} + \frac{r^2\dot{\theta}^2}{r} \\ &= \frac{F}{m} + \frac{L^2}{m^2r^3}. \end{aligned} \tag{1.8}$$

This derivation used the fact that the force was radial, $F = F_r = F_x\cos\theta + F_y\sin\theta$, and that angular momentum is $L = mrv_\theta = mr^2\dot{\theta}$. The term $L^2/mr^3 = mv^2/r$ behaves appears like an additional force. Sometimes this is referred to as a centrifugal force, but it is not a force, but merely the consequence of considering the motion in a rotating (and therefore accelerating) frame.

Now, we switch to the particular case of an attractive inverse square force, $F = -\alpha/r^2$, and show that the trajectory, $r(\theta)$, is an ellipse. To do this we transform derivatives w.r.t. time to derivatives w.r.t. θ using the chain rule,

$$\begin{aligned} \dot{r} &= \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}, \\ \ddot{r} &= \frac{d^2r}{d\theta^2}\left(\frac{L}{mr^2}\right)^2 - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\left(\frac{L}{mr^2}\right)^2 \end{aligned} \tag{1.9}$$

Equating the two expressions for $\ddot{r}$ in Eq. (1.8) and Eq. (1.9) provides a differential equation,

$$\frac{d^2r}{d\theta^2}\left(\frac{L}{mr^2}\right)^2 - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\left(\frac{L}{mr^2}\right)^2 = \frac{F}{m} + \frac{L^2}{m^2r^3}, \tag{1.10}$$

that when solved yields the trajectory. Up to this point the expressions work for any radial force, not just forces that fall as $1/r^2$.

The trick to simplifying this differential equation for the inverse square problems is to make a substitution, $u \equiv 1/r$, and rewrite the differential equation for $u(\theta)$.

$$\begin{aligned} r &= 1/u, \qquad (1.11)\\ \frac{dr}{d\theta} &= \frac{-1}{u^2}\frac{du}{d\theta},\\ \frac{d^2r}{d\theta^2} &= \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\theta^2}. \end{aligned}$$

Plugging these expressions into Eq. (1.10) gives an expression in terms of u, $du/d\theta$, and $d^2u/d\theta^2$. After some tedious algebra,

$$\frac{d^2u}{d\theta^2} = -u - \frac{Fm}{L^2u^2}. \qquad (1.12)$$

For the attractive inverse square law force, $F = -\alpha u^2$,

$$\frac{d^2u}{d\theta^2} = -u + \frac{m\alpha}{L^2}. \qquad (1.13)$$

The solution has two arbitrary constants,

$$\begin{aligned} u &= \frac{m\alpha}{L^2} + A\cos(\theta - \theta_0), \qquad (1.14)\\ r &= \frac{1}{(m\alpha/L^2) + A\cos(\theta - \theta_0)}. \end{aligned}$$

The radius will be at a minimum when $\theta = \theta_0$ and at a maximum when $\theta = \theta_0 + \pi$. The constant A represents the eccentricity of the orbit. When $A = 0$ the radius is a constant $r = L^2/m\alpha$. If one solved the expression $mv^2/r = -\alpha/r^2$ for a circular orbit, using the substitution $v = L/mr$, one would reproduce the expression above with $A = 0$.

The form describing the elliptical trajectory in Eq. (1.14) can be identified as an ellipse with one focus being the center of the ellipse by considering the definition of an ellipse as being the points such that the sum of the two distances between the two foci are a constant. Making that distance $2D$, the distance between the two foci as $2a$ and putting one focus at the origin,

$$\begin{aligned} 2D &= r + \sqrt{(r\cos\theta - 2a)^2 + r^2\sin^2\theta}, \qquad (1.15)\\ 4D^2 + r^2 - 4Dr &= r^2 + 4a^2 - 4ar\cos\theta,\\ r &= \frac{D^2 - a^2}{D + a\cos\theta} = \frac{1}{D/(D^2 - a^2) - a\cos\theta/(D^2 - a^2)}. \end{aligned}$$

By inspection, this is the same form as Eq. (1.14).

The total energy of a particle is

$$\begin{aligned} E &= U(r) + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mv_\theta^2 \qquad (1.16)\\ &= U(r) + \frac{L^2}{2mr^2} + \frac{1}{2}m\dot{r}^2. \end{aligned}$$

The second term then contributes to the energy like an additional repulsive potential. The term is sometimes referred to as the centrifugal potential.

Example III:

Consider a particle of mass m in a 2-dimensional harmonic oscillator with potential

$$U = \frac{1}{2}kr^2.$$

If the orbit has angular momentum L, find
a) the radius and angular velocity of the circular orbit
b) the angular frequency of small radial perturbations
Solution:
a) Consider the effective potential. The radius of a circular orbit is at the minimum of the potential (where the effective force is zero).

$$\begin{aligned}
U_{\rm eff} &= \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}, \\
0 &= kr_{\rm min} - \frac{L^2}{mr_{\rm min}^3}, \\
r_{\rm min} &= \left(\frac{L^2}{mk}\right)^{1/4}, \\
\dot{\theta} &= \frac{L}{mr_{\rm min}^2} = \sqrt{k/m}.
\end{aligned}$$

b) Now consider small vibrations about $r_{\rm min}$. The effective spring constant is the curvature of the effective potential.

$$\begin{aligned}
k_{\rm eff} &= \left.\frac{d^2}{dr^2}U(r)\right|_{r=r_{\rm min}} = k + \frac{3L^2}{mr_{\rm min}^4} \\
&= 4k, \\
\omega &= \sqrt{k_{\rm eff}/m} = 2\sqrt{k/m} = 2\dot{\theta}.
\end{aligned}$$

Unlike the inverse-square force, the harmonic oscillator has two minima and two maxima. The orbits are elliptical, but the center of the ellipse coincides with $r = 0$.
The solution is also simple to write down exactly in Cartesian coordinates. The x and y equations of motion separate,

$$\begin{aligned}
\ddot{x} &= -kx, \\
\ddot{y} &= -ky.
\end{aligned}$$

So the general solution can be expressed as

$$\begin{aligned}
x &= A\cos\omega_0 t + B\sin\omega_0 t, \\
y &= C\cos\omega_0 t + D\sin\omega_0 t.
\end{aligned}$$

With some work using double angle formulas, one can calculate

$$\begin{aligned}
r^2 &= x^2 + y^2 = \alpha + \beta \cos 2\omega_0 t + \gamma \sin 2\omega_0 t, \\
\alpha &= \frac{A^2 + B^2 + C^2}{2}, \quad \beta = \frac{A^2 - B^2 + C^2 - D^2}{2}, \quad \gamma = AB + CD, \\
r^2 &= \alpha + (\beta^2 + \gamma^2)^{1/2} \cos(2\omega_0 t - \delta), \quad \delta = \arctan(\gamma/\beta).
\end{aligned}$$

1.4 Stability of Orbits

The effective force can be extracted from the effective potential, U_{eff}. Beginning from the equations of motion, Eq. (1.8), for r,

$$\begin{aligned}
m\ddot{r} &= F + \frac{L^2}{mr^3} \\
&= F_{\text{eff}} \\
&= -\partial_r U_{\text{eff}}, \\
U_{\text{eff}} &= -\partial_r \left[U(r) - (L^2/2mr^2) \right].
\end{aligned} \tag{1.17}$$

For a circular orbit, the radius must be fixed as a function of time, so one must be at a maximum or a minimum of the effective potential. However, if one is at a maximum of the effective potential the radius will be unstable. For the attractive Coulomb force the effective potential will be dominated by the $-\alpha/r$ term for large r since the centrifugal part falls off more quickly, $\sim 1/r^2$. A low r the centrifugal piece wins and the effective potential is repulsive. Thus, the potential must have a minimum somewhere with negative potential. The circular orbits are then stable to perturbation.

If one considers a potential that falls as $1/r^3$, the situation is reversed. The repulsive centrifugal piece dominates at large r and the attractive Coulomb piece wins out at small r. The circular orbit is then at a maximum of the effective potential and the orbits are unstable. It is the clear that for potentials that fall as r^n, that one must have $n > -2$ for the orbits to be stable.

Example IV:
Consider a potential $U(r) = \beta r$. For a particle of mass m with angular momentum L, find the angular frequency of a circular orbit. Then find the angular frequency for small radial perturbations.
Solution:
For the circular orbit you search for the position r_{min} where the effective potential is minimized,

$$\begin{aligned}
\partial_r \left\{ \beta r + \frac{L^2}{2mr^2} \right\} &= 0, \\
\beta &= \frac{L^2}{mr_{\text{min}}^3}, \\
r_{\text{min}} &= \left(\frac{L^2}{\beta m} \right)^{1/3}, \\
\dot{\theta} &= \frac{L}{mr_{\text{min}}^2} = \frac{\beta^{2/3}}{(mL)^{1/3}}
\end{aligned}$$

Now, we can find the angular frequency of small perturbations about the circular orbit. To do this we find the effective spring constant for the effective potential,

$$\begin{aligned} k_{\rm eff} &= \partial^2 \left. U_{\rm eff}\right|_{r_{\rm min}} \\ &= \frac{3L^2}{mr_{\rm min}^4}, \\ \omega &= \sqrt{\frac{k_{\rm eff}}{m}} \\ &= \frac{\beta^{2/3}}{(mL)^{1/3}}\sqrt{3}. \end{aligned}$$

For an ellipse, the two frequencies would have to be the same so that for each period the radius would return to the same value. In this case they differ by a factor of $\sqrt{3}$.

1.5 Scattering and Cross Sections

Scattering experiments don't measure entire trajectories. For elastic collisions, they measure the distribution of final scattering angles. Most experiments use targets thin enough so that the number of scatterings is typically zero or one. The cross section, σ, describes the cross-sectional area for particles to scatter. For Coulomb forces, this is infinite because the range of the Coulomb force is infinite, but for interactions such as those in nuclear or particle physics, there is no long-range force and cross-sections are finite. If a particle travels through a thin target, the chance the particle scatters is $\sigma dN/dA$, where dN/dA is the number of scattering centers per area the particle encounters. If the density of the target is ρ particles per volume, and if the thickness of the target is t, the areal density is $dN/dA = \rho t$. Since one wishes to quantify the collisions independently of the target, experimentalists measure scattering probabilities, then divide by the areal density to obtain cross-sections. Instead of merely stating that a particle collided, one can measure the probability the particle scattered by a given angle. The scattering angle θ_s is defined so that at zero the particle is unscattered and at $\theta_s = \pi$ the particle is scattered directly backward. Scattering angles are often described in the center-of-mass frame, but that is a detail we will neglect for this first discussion, where we will consider the scattering of particles moving classically under the influence of fixed potentials $U(\vec{r})$. Since the distribution of scattering angles can be measured, one expresses the differential cross section,

$$\frac{d\sigma}{d\cos\theta_s}. \tag{1.18}$$

Scattering cross sections are calculated by assuming a random distribution of impact parameters b. These represent the distance in the xy plane for particles moving in the z direction relative to the scattering center. An impact parameter $b = 0$ refers to a direct hit. The impact parameter describes the transverse distance from the $z = 0$ axis for the trajectory when it is still far away from the scattering center and has not yet passed it. The differential cross section can be expressed in terms of the impact parameter,

$$d\sigma = \pi b db, \tag{1.19}$$

which is the area of a thin ring of radius b and thickness db. In classical physics, one can calculate the trajectory given the incoming kinetic energy E and the impact parameter if one knows the mass and potential. From the trajectory, one then finds the scattering angle $\theta_s(b)$. The differential cross section is then

$$\frac{d\sigma}{d\cos\theta_s} = \pi b \frac{db}{d\cos\theta_s} = \frac{\pi b}{d/db\cos\theta_s(b)}. \tag{1.20}$$

Typically, one would calculate θ_s and $d/db\theta_s$ as functions of b. This is sufficient to plot the differential cross section as a function of θ_s.

The total cross section is

$$\sigma_{\text{tot}} = \int d\cos\theta_s \frac{d\sigma}{d\cos\theta_s}. \tag{1.21}$$

Even if the total cross section is infinite, e.g. Coulomb forces, one can still have a finite differential cross section as we will see later on.

Example V:

An asteroid of mass m and kinetic energy E approaches a planet of radius R and mass M. What is the cross section for the asteroid to impact the planet?

Solution:

Calculate the maximum impact parameter, b_{max}, for which the asteroid will hit the planet. The total cross section for impact is $\sigma_{\text{impact}} = \pi b_{\text{max}}^2$. The maximum cross-section can be found with the help of angular momentum conservation. The asteroid's incoming momentum is $p_0 = \sqrt{2mE}$ and the angular momentum is $L = p_0 b$. If the asteroid just grazes the planet, it is moving with zero radial kinetic energy at impact. Combining energy and angular momentum conservation and having p_f refer to the momentum of the asteroid at a distance R,

$$\begin{aligned}
\frac{p_f^2}{2m} &= E + \frac{GMm}{R}, \\
p_f R &= p_0 b_{\text{max}},
\end{aligned}$$

allows one to solve for b_{max},

$$\begin{aligned}
b_{\text{max}} &= R\frac{p_f}{p_0} \\
&= R\frac{\sqrt{2m(E+GMm/R)}}{\sqrt{2mE}} \\
\sigma_{\text{impact}} &= \pi R^2 \frac{E+GMm/R}{E}.
\end{aligned}$$

1.6 Center-of-mass Coordinates

Thus far, we have considered the trajectory as if the force is centered around a fixed point. When the two masses in question become similar, both masses circulate around the center of mass. One might think that solutions would become more complex when both particles move, but we will see here that one can transform the trajectories for $\vec{r}_1$ and $\vec{r}_2$ into the center-of-mass coordinates $\vec{R}_{\text{cm}}$ and the relative coordinate $\vec{r}$,

$$\begin{aligned}
\vec{R}_{\text{cm}} &\equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \\
\vec{r} &\equiv \vec{r}_1 - \vec{r}_2.
\end{aligned} \tag{1.22}$$

Here, we assume the two particles interact only with one another, so $\vec{F}_{12} = -\vec{F}_{21}$. The equations of motion then become

$$\begin{aligned}
\ddot{\vec{R}}_{\rm cm} &= \frac{1}{m_1+m_2}\left\{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2\right\} && (1.23)\\
&= \frac{1}{m_1+m_2}\left\{\vec{F}_{12} + \vec{F}_{21}\right\} = 0.\\
\ddot{\vec{r}} &= \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{\vec{F}_{12}}{m_1} + \frac{\vec{F}_{21}}{m_2}\right) && (1.24)\\
&= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}.
\end{aligned}$$

The first expression simply states that the center of mass coordinate $\vec{R}_{\rm cm}$ moves at a fixed velocity. The second expression can be rewritten in terms of the reduces mass μ.

$$\begin{aligned}
\mu\ddot{\vec{r}} &= \vec{F}_{12}, && (1.25)\\
\frac{1}{\mu} &= \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \frac{m_1 m_2}{m_1+m_2}. && (1.26)
\end{aligned}$$

Thus, one can treat the trajectory as a one-body problem where the reduced mass is μ, and a second trivial problem for the center of mass. The reduced mass is especially convenient when one is considering gravitational problems because then

$$\begin{aligned}
\mu\ddot{r} &= -\frac{Gm_1m_2}{r^2}\hat{r} && (1.27)\\
&= -\frac{GM\mu}{r^2}\hat{r}, \quad M \equiv m_1 + m_2.
\end{aligned}$$

The reduced mass then falls out and the trajectory depends only on the total mass M.
The kinetic energy and momenta also have analogues in center-of-mass coordinates. The total and relative momenta are

$$\begin{aligned}
\vec{P} &\equiv \vec{p}_1 + \vec{p}_2 = M\dot{\vec{R}}_{\rm cm}, && (1.28)\\
\vec{q} &\equiv \mu\dot{r}.
\end{aligned}$$

With these definitions, a little algebra shows that the kinetic energy becomes

$$\begin{aligned}
T &= \frac{1}{2}m_1|\vec{v}_1|^2 + \frac{1}{2}m_2|\vec{v}_2|^2 = \frac{1}{2}M|\dot{\vec{R}}_{\rm cm}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 && (1.29)\\
&= \frac{P^2}{2M} + \frac{q^2}{2\mu}.
\end{aligned}$$

The standard strategy is to transform into the center of mass frame, then treat the problem as one of a single particle of mass μ undergoing a force $\vec{F}_{12}$. Scattering angles can also be expressed in this frame.

1.7 Rutherford Scattering

This refers to the calculation of $d\sigma/d\cos\theta$ due to an inverse square force, $F_{12} = \alpha/r^2$. Rutherford compared the scattering of α particles off of a nucleus and studied the angle at which the formula began to fail. This

corresponded to the impact parameter for which the trajectories would strike the nucleus. This provided the first measure of the size of the atomic nucleus. At the time, the distribution of the positive charge (the protons) was considered to be just as spread out amongst the atomic volume as the electrons. After Rutherford's experiment, it was clear that the radius of the nucleus tended to be 4 to 5 orders of magnitude smaller than that of the atom, or roughly the size of a football relative to Spartan Stadium.
We begin by considering our previous expression for the trajectory,

$$r = \frac{1}{\frac{m\alpha}{L^2} + A\cos\theta}. \tag{1.30}$$

Once A is large enough, which will happen when the energy is positive, the denominator will become negative for a range of θ. This is because the scattered particle will never reach certain angles. The asymptotic angles θ' are those for which the denominator goes to zero,

$$\cos\theta' = -\frac{m\alpha}{AL^2}. \tag{1.31}$$

The trajectory's point of closest approach is at $\theta = 0$ and the two angles θ' which have this value of $\cos\theta$ are the angles of the incoming and outgoing particles. From Fig. **??**, one can see that the scattering angle θ_s is given by,

$$2\theta' - \pi = \theta_s, \quad \theta' = \frac{\pi + \theta_s}{2}, \quad \cos\theta' = -\sin(\theta_s/2). \tag{1.32}$$

Now that we have θ_s in terms of m, α, L and A, we wish to re-express L and A in terms of the impact parameter b and the energy E. This will set us up to calculate the differential cross section, which requires knowing $db/d\theta_s$. It is easy to write the angular momentum as $L^2 = p_0^2 b^2 = 2mEb^2$. Finding A is more complicated. To accomplish this we realize that the point of closest approach occurs at $\theta = 0$,

$$\begin{aligned} \frac{1}{r_{\rm min}} &= \frac{m\alpha}{L^2} + A, \\ A &= \frac{1}{r_{\rm min}} - \frac{m\alpha}{L^2}. \end{aligned} \tag{1.33}$$

Next, $r_{\rm min}$ can be found in terms of the energy because at the point of closest approach the kinetic energy is due purely to the motion perpendicular to $\hat{r}$ and

$$E = -\frac{\alpha}{r_{\rm min}} + \frac{L^2}{2mr_{\rm min}^2}. \tag{1.34}$$

Once can solve the quadratic equation for $1/r_{\rm min}$,

$$\frac{1}{r_{\rm min}} = -\frac{m\alpha}{L^2} + \sqrt{(m\alpha/L^2)^2 + 2mE/L^2}. \tag{1.35}$$

We can plug the expression for $r_{\rm min}$ into the expression for A,

$$A = \sqrt{(m\alpha/L^2)^2 + 2mE/L^2} = \sqrt{(\alpha^2/(4E^2b^4) + 1/b^2} \tag{1.36}$$

Finally, we insert the expression for A into the that for the scattering angle,

$$\sin(\theta_s/2) = \frac{a}{\sqrt{a^2 + b^2}}, \quad a \equiv \frac{\alpha}{2E} \tag{1.37}$$

The differential cross section can now be found by differentiating the expression for θ_s with b,

$$\begin{aligned}
\frac{1}{2}\cos(\theta_s/2)d\theta_s &= \frac{abdb}{(a^2+b^2)^{3/2}} = \frac{bdb}{a^2}\sin^3(\theta_s/2), \\
d\sigma &= 2\pi bdb = \frac{\pi a^2}{\sin^3(\theta_s/2)}\cos(\theta_s/2)d\theta_s \\
\frac{d\sigma}{d\cos\theta_s} &= \frac{\pi a^2}{2\sin^4(\theta_s/2)},
\end{aligned} \tag{1.38}$$

where $a \equiv \alpha/2E$.

1.8 Exercises

1. Approximate Earth as a solid sphere of uniform density and radius $R = 6360$ km. Suppose you drill a tunnel from the north pole directly to another point on the surface described by a polar angle θ relative to the north pole. Drop a mass into the hole and let it slide through tunnel without friction. Find the frequency f with which the mass oscillates back and forth. Ignore Earth's rotation. Compare this to the frequency of a low-lying circular orbit.

2. Consider the gravitational field of the moon acting on the Earth.

   (a) Calculate the term k in the expansion

   $$g_{\rm moon} = g_0 + kz + \cdots,$$

   where z is measured relative to Earth's center and is measured along the axis connecting the Earth and moon. Give your answer in terms of the distance between the moon and the earth, R_m and the mass of the moon M_m.

   (b) Calculate the difference between the height of the oceans between the maximum and minimum tides. Express your answer in terms of the quantities above, plus Earth's radius, R_e. Then give you answer in meters.

3. Consider a particle in an attractive inverse-square potential, $U(r) = -\alpha/r$, where the point of closest approach is $r_{\rm min}$ and the total energy of the particle is E. Find the parameter A describing the trajectory in Eq. (1.14). Hint: Use the fact that at $r_{\rm min}$ there is no radial kinetic energy and $E = -\alpha/r_{\rm min} + L^2/2mr_{\rm min}^2$.

4. Consider the effective potential for an attractive inverse-square-law force, $F = -\alpha/r^2$. Consider a particle of mass m with angular momentum L.

   (a) Find the radius of a circular orbit by solving for the position of the minimum of the effective potential.

   (b) What is the angular frequency, $d\theta/dt$, of the orbit? Solve this by setting $F = m\omega^2 r$.

   (c) Find the effective spring constant for the particle at the minimum.

   (d) What is the angular frequency for small vibrations about the minimum? How does the compare with the answer to (b)?

5. Consider a particle of mass m and angular momentum L moving in a potential

$$U = \alpha \ln(r).$$

   (a) Find the radius of a circular orbit by solving for the position of the minimum of the effective potential.
   (b) What is the angular frequency, $d\theta/dt$, of the orbit? Solve this by setting $F = m\omega^2 r$.
   (c) Find the effective spring constant for the particle at the minimum.
   (d) What is the angular frequency for small vibrations about the minimum? How does the compare with the answer to (b)?

6. Consider a particle of mass m in an attractive potential, $U(r) = -\alpha/r$, with angular momentum L with just the right energy so that

$$A = m\alpha/L^2$$

for the trajectory described by Eq. (1.14). The trajectory can then be written as

$$r = \frac{2r_0}{1+\cos\theta}, \quad A \equiv L^2/m\alpha = 1/r_0.$$

   (a) Show that for this case the total energy $E = 0$.
   (b) Write this trajectory in a more recognizable parabolic form,

$$x = x_0 - \frac{y^2}{R}.$$

   I.e., express x_0 and R in terms of r_0.
   (c) What is the scattering angle for this trajectory?

7. Show that if one transforms to a reference frame where the total momentums if zero, $\vec{p}_1 = -\vec{p}_2$, that the relative momentum $\vec{q}$ corresponds to either $\vec{p}_1$ or $-\vec{p}_2$. This means that in this frame the magnitude of $\vec{q}$ is one half the magnitude of $\vec{p}_1 - \vec{p}_2$.

8. Given the center of mass coordinates $\vec{R}$ and $\vec{r}$ for particles of mass m_1 and m_2, find the coordinates $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$.

9. Consider two particles of identical mass scattering at an angle $\theta_{\rm cm}$ in the center of mass.

   (a) In a frame where one is the target (initially at rest) and one is the projectile, find the scattering angle in the lab frame, θ, in terms of $\theta_{\rm cm}$.
   (b) Express $d\sigma/d\cos\theta$ in terms of $d\sigma/d\cos\theta_{\rm cm}$. I.e., find the Jacobian for $d\cos\theta_{\rm cm}/d\cos\theta$.