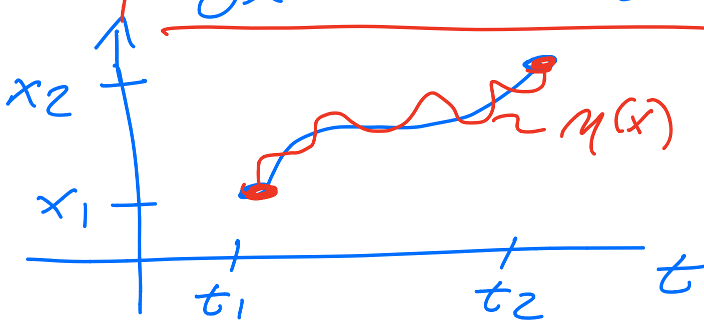
PHY 321 APRIL 12

action $S = \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\, dt$

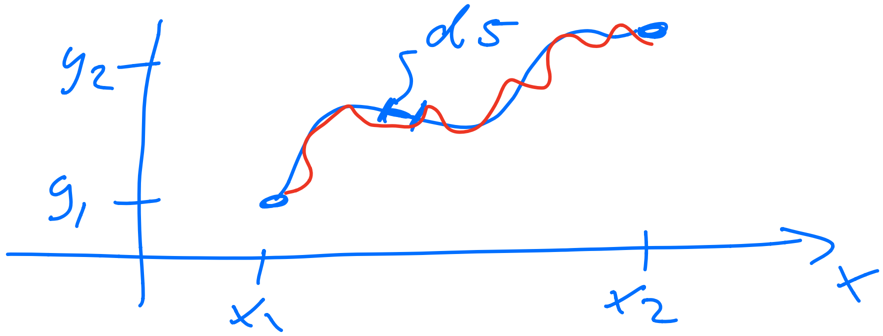
$\mathcal{L}(\underline{x, v,} t) = \frac{1}{2} m v^2 - V(x)$

Euler-Lagrange eqs:

$$\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = 0}$$



Example



$$L = \int_{x_1}^{x_2} ds$$

$$ds = \sqrt{dx^2 + \underline{dy}^2}$$

$$dy = \frac{dy}{dx}\,dx = y'\,dx$$

$$ds = \sqrt{1 + y'^2}\,dx$$

$$L = \int_{x_1}^{x_2} \underline{\sqrt{1+y'^2}}\,dx$$

$$= \int_{x_1}^{x_2} f(x, y, y')\,dx$$

Euler-Lagrange

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \qquad \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \implies$$

$$\frac{\partial f}{\partial y'} = c \quad \text{(constant)}$$

$$f = (1 + y'^2)^{1/2} \implies$$

$$c^2 = 1 + y'^2 \Rightarrow$$

$$y' = D$$

$$\frac{dy}{dx} = D \Rightarrow$$

$y(x) = Dx + B$, shortest path is a straight line

Example 2

$$\mathcal{L}(x, v, t) \qquad \frac{dx}{dt} = v = \dot{x}$$

$$\mathcal{L}(x, v, t) = \frac{1}{2} m v^2 - V(x)$$

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\, dt$$

S - stationary quantity.

$$\frac{d\mathcal{L}}{dt} = 0 =$$

$$\frac{\partial \mathcal{L}}{\partial} \frac{dv}{} + \frac{\partial \mathcal{L}}{\partial} \frac{dx}{}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial v}\frac{d^2x}{dt^2} + \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}\right]\frac{dx}{dt}$$

$$= \frac{d}{dt}\left[\frac{dx}{dt}\frac{\partial \mathcal{L}}{\partial v}\right]$$

$$= \frac{d}{dt}\left[v\frac{\partial \mathcal{L}}{\partial v}\right]$$

$$\frac{d}{dt}\left[v\frac{\partial \mathcal{L}}{\partial v} - \mathcal{L}\right] = 0$$

conserved E

define $\mathcal{L} = \frac{1}{2}mv^2 - V(x)$

$$\frac{d}{dt}\left[mv^2 - \frac{1}{2}mv^2 + V(x)\right]$$

$$= 0$$

$$\frac{d}{dt} E = 0$$

$x \to q$ general position

$v \to \dot{q}$ general velocity

($x, y \to r, \phi$ in polar coordinates)

$\mathcal{L}(x, v, t) \to \mathcal{L}(q, \dot{q}, t)$

can have more than one dim

$$\mathcal{L} = \sum_{i=1}^{n} \mathcal{L}(q_i, \dot{q}_i, t)$$

Euler-Lagrange

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$$

Example 3

$$x, y \Rightarrow r, \phi$$

$$\mathcal{L}(x, \dot{x}, y, \dot{y}, t) \Rightarrow \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r, \phi)$$

$\underline{r}$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$m r \dot{\phi}^2 \underbrace{- \frac{dV}{dr}}_{F_r} = \frac{d}{dt}(m\dot{r})$$

$$m\ddot{r} = F_r + r m \dot{\phi}^2$$

$$\boxed{\ddot{r} = F_r/m + r\dot{\phi}^2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$-\frac{dV}{d\phi} = \frac{d}{dt}(m r^2 \dot{\phi})$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$F_\phi = -\frac{dV}{d\phi} = -\frac{1}{r}\frac{\partial V}{\partial \phi}$$

$$\underbrace{rF_\phi}_{\text{Torque}} = \frac{d}{dt}\left(mr^2\dot{\phi}\right)$$

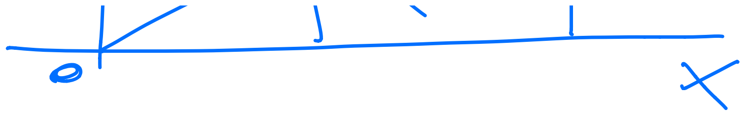
Variational calculus & constrained motion

Example 1:

minimize $f(x_1, x_2) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$

subject to $x_1 + x_2 = 5$

Example 2



minimize $\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$
$- mgy$

constraint :

$$y = x \tan\alpha$$