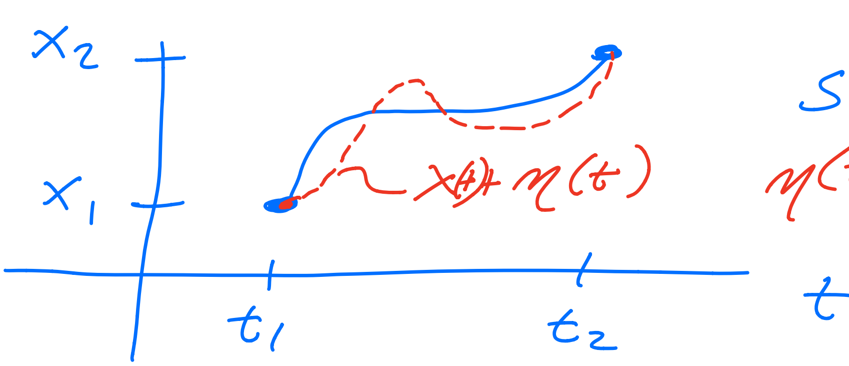
PHY 321 APRIL 9

- Principle of least action
- Variational calculus
- Lagrangian
- Euler - Lagrange equations

Variational calculus deals with finding min or max of a quantity that can be expressed as an integral,

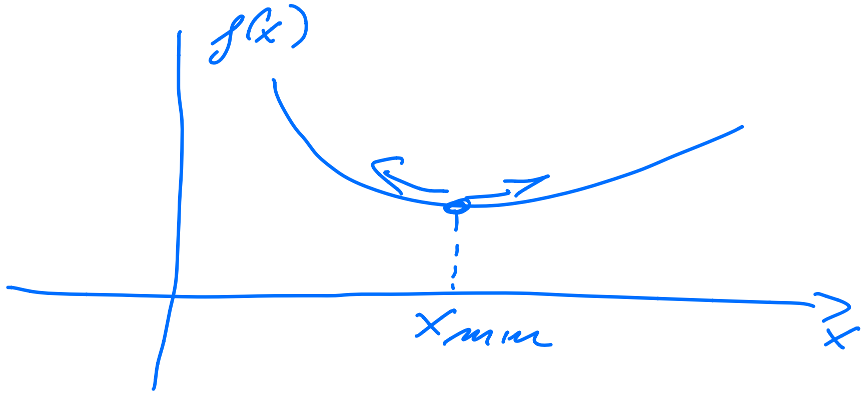
$$S = \int_{t_1}^{t_2} \mathcal{L}\left(x, \frac{dx}{dt}, t\right) dt$$

$$\left(\int_{x_1}^{x_2} f\left(y, \frac{dy}{dx}, x\right) dx \right)$$



S = action

$\eta(t_1) = \eta(t_2) = 0$



$x \to x + \varepsilon$

$f(x) \to f(x+\varepsilon)$

$$f(x+\varepsilon) = f(x) + \varepsilon \frac{df}{dx} + \frac{\varepsilon^2}{2!}\frac{d^2 f}{dx^2} + O(\varepsilon^3)$$

$$\Delta f = f(x+\varepsilon) - f(x) = \varepsilon \frac{df}{dx} + \frac{\varepsilon^2}{2!}\frac{d^2 f}{dx^2} + O(\varepsilon^3)$$

$$\frac{df}{dx} = 0$$

$$\Delta f \cong \frac{\varepsilon^2}{2!}\frac{d^2 f}{dx^2}$$

change to 2nd order in ε

$\Delta f = f(x+\varepsilon) - f(x)$ to

<u>first order in ε.</u>

$\delta S = \underline{S[x+\eta] - S[x]}$

to first order in η.

$$\mathcal{L}(x, v, t) = \mathcal{L}\left(x, \frac{dx}{dt}, t\right)$$

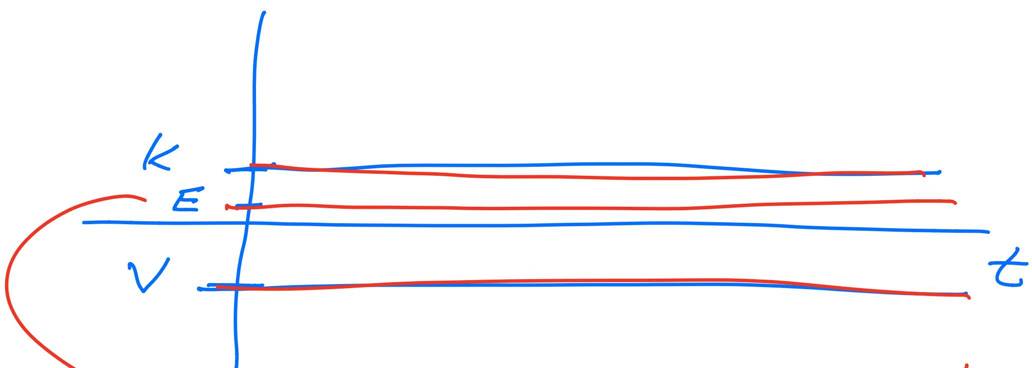
v

$\mathcal{L} = \frac{1}{2}mv^2 - V(x)$

why not $\mathcal{L} = \frac{1}{2}mv^2 + V(x)$?

$= E$

hw 6



$E = K + V =$ constant

$$S = \int_{t_1=0}^{t_2} E\,dt = E \cdot t_2$$

$$S(x+\eta) = \int_{t_1}^{t_2} \left\{ \frac{1}{2}m\left[\frac{d}{dt}(x+\eta)\right]^2 - V(x+\eta) \right\} dt$$

$$\left(\frac{d}{dt}(x+\eta)\right)^2 = \left(\frac{dx}{dt} + \frac{d\eta}{dt}\right)^2$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + 2\frac{dx}{dt}\frac{d\eta}{dt}$$

$(\; dt \;) \quad (\; dt \;) \quad dt \; dt$

$$\underline{V(x)} = V(x+\eta) = V(x) + \eta \frac{dV}{dx} + \underbrace{O(\eta^2)}_{\text{ignore}}$$

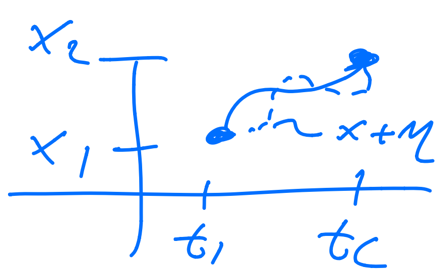
$$\underline{S(x+\eta)} - \underline{S(x)} = \delta S$$

$$= \int_{t_1}^{t_2} \left\{ \cancel{\frac{1}{2}} \left[\cancel{2} \underbrace{\frac{dx}{dt} \frac{d\eta}{dt}} \right] m - \eta \frac{dV}{dx} \right\} dt$$

$$\frac{d}{dt}\left(\underline{\frac{dx}{dt}} \eta \right) = \frac{d\eta}{dt} \frac{dx}{dt} + \underline{\frac{d^2x}{dt^2}} \eta$$

$$\frac{dx}{dt} \frac{d\eta}{dt} = \frac{d}{dt}\left(\frac{dx}{dt} \eta \right) - \frac{d^2x}{dt^2} \eta$$

$$\int_{t_1}^{t_2} \underline{\frac{dx}{dt} \frac{d\eta}{dt}} \, dt = \int_{t_1}^{t_2} \underline{\frac{d}{dt}\left[\frac{dx}{dt} \eta \right]} - \int_{t_1}^{t_2} \underline{\frac{d^2x}{dt^2}} \eta \, dt$$



$$\int^{t_2} \frac{d}{dt} \left[\frac{dx}{dt} \eta \right] dt = \frac{dx}{dt} \eta \Big|^{t_2} = 0$$

t_1 $\qquad$ t_1

$$\delta S = \int_{t_1}^{t_2} \underbrace{\left[- m \frac{d^2 x}{dt^2} - \frac{dV}{dx} \right]}_{} \eta \, dt$$

$$\delta S[x] = 0$$

$$+ m \frac{d^2 x}{dt^2} = - \frac{dV}{dx} = F(x)$$

Newton's law, equation of motion from

$$\mathcal{L} = K - V$$

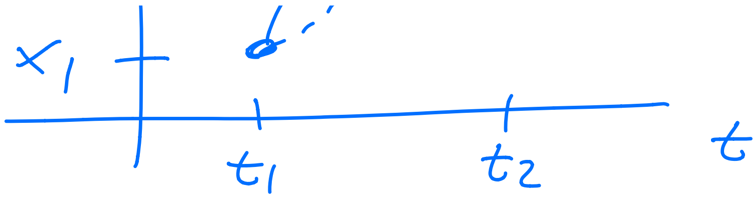
Somewhat more general:
Euler-Lagrange equations

$\mathcal{L}(x, v, t) \qquad v = \frac{dx}{dt}$

$x(t) + \delta x(t)$

$v(t) + \delta v(t)$

$\delta x(t_2) = \delta x(t_1)$
$= 0$



$$\delta v(t_2) = \delta v(t_1) = 0$$

$$\mathcal{L}(x+\delta x, v+\delta v, t)$$
$$= \mathcal{L}(x, v, t) +$$
$$\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial v}\delta v$$
$$+ \cdots$$

$$\delta v = \frac{d}{dt}\delta x$$

$$\delta S = S(x+\delta x, v+\delta v, t)$$
$$- S(x, v, t)$$

$$= \int_{t_1}^{t_2} \left(\underline{\frac{\partial \mathcal{L}}{\partial x}\delta x} + \underbrace{\frac{\partial \mathcal{L}}{\partial v}\delta v}_{\frac{\partial \mathcal{L}}{\partial v}\frac{d}{dt}\delta x} \right) dt$$

integrate by parts

$$= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} \right] \delta x \, dt$$

$$+ \frac{\partial \mathcal{L}}{\partial v} \delta x \Big|_{t_1}^{t_2}$$

$$= 0$$

$\Rightarrow$ in order $\delta S = 0$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = 0$$

Euler-Lagrange equations