PHY 321 MARCH 24

Elliptical orbits & Kepler's laws!

$$\mu \frac{d^2 r}{dt^2} = F(r) + \frac{L^2}{\mu r^3} \qquad \frac{d\phi}{dt} = \frac{L}{\mu r^2}$$

$$F(r) = -\frac{dV(r)}{dr}$$

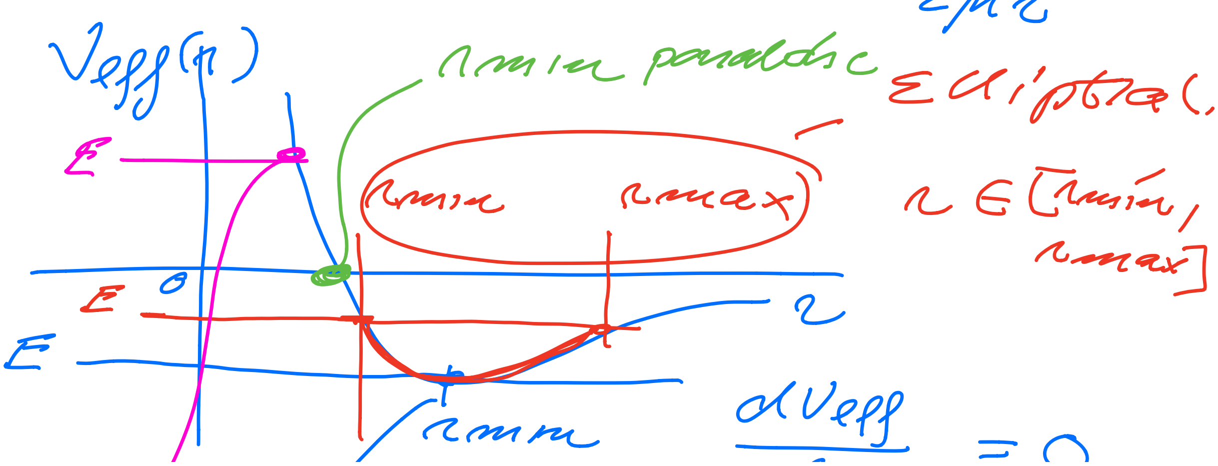
$$V(r) = -\alpha/r \qquad \alpha = G m_1 m_2$$

$$F(r) = -\alpha/r^2$$

$$\mu \ddot{r} = -\alpha/r^2 + \frac{L^2}{\mu r^3}$$

$$(-kr)$$

$$V_{eff}(r) = -\alpha/r + \frac{L^2}{2\mu r^2}$$



$$\frac{dV_{eff}}{dr} = 0$$

dr

hyperbolic

circular motion.

$$\frac{d^2 r}{dt^2} = 0$$

$$\boxed{\mu \ddot{r} = -\alpha/r^2 + \frac{L^2}{\mu r^3}}$$

Trick : $u = \frac{1}{r}$

chain rule and rewrite of

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi}$$

$$\dot{\phi} = \frac{L}{\mu r^2} \qquad = \frac{L}{\mu r^2}\frac{d}{d\phi}$$

$$= \frac{L u^2}{\mu}\frac{d}{d\phi}$$

$$\frac{d^2 r}{dt^2} = ? \qquad \frac{dr}{dt} = \frac{L u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right)$$

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) =$$

$$\frac{L u^2}{\mu}\frac{d}{d\phi}\left[-\frac{L}{\mu}\frac{du}{d\phi}\right]$$

$$= -\frac{L^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}$$

u^3

$$\mu \cdot \frac{d^2 r}{dt^2} = -\alpha/r^2 + \frac{L^2}{\mu r^3}$$

$r = \frac{1}{u}$

$$\boxed{\frac{d^2 u}{d\phi^2} = -u + \frac{\mu\alpha}{L^2}}$$

$\alpha = 0$

$$\frac{d^2 u}{d\phi^2} = -u$$

$$u = A\cos(\phi - \delta)$$

$$\frac{d^2 u}{d\phi^2} = -u + \frac{\mu\alpha}{L^2}$$

Define $\quad u = w(\phi) + \frac{\mu\alpha}{L^2}$

$$w(\phi) = \underline{u(\phi)} - \frac{\mu\alpha}{L^2}$$

$$\frac{d^2 w}{d\phi^2} = -w$$

$$w(\phi) = A\cos(\phi - \delta)$$

$$u(\phi) = A\cos(\phi - \delta) + \frac{\mu\alpha}{L^2}$$

$$\delta = 0$$

$$\varepsilon = \frac{AL^2}{\alpha\mu} \quad (\text{Dimless const})$$

$$u(\phi) = \frac{\mu\alpha}{L^2}(1 + \varepsilon\cos\phi)$$

$u = \frac{1}{r}$

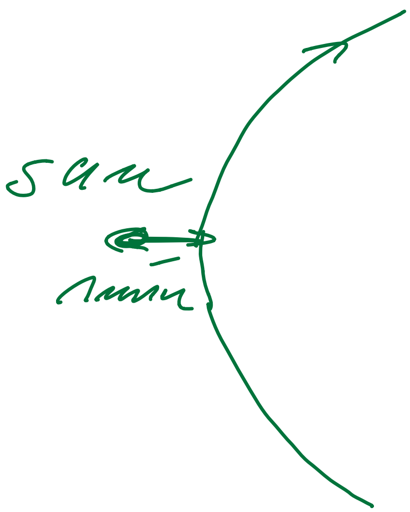
eccentricity of an ellipse

Eccentricity	Energy	orbit
bound motion: $\varepsilon = 0$	$E < 0$	circle
$0 < \varepsilon < 1$	$E < 0$	ellipse
$\varepsilon = 1$	$E = 0$	parabola.

$\varepsilon = 1$ $\quad E = 0$

$\varepsilon > 1$ $\quad E > 0$ $\quad$ hyperbola

unbound motion.



$$u(\phi) = \frac{\alpha\mu}{L^2}(1 + \varepsilon\cos\phi)$$

$$c = \frac{L^2}{\alpha\mu}$$

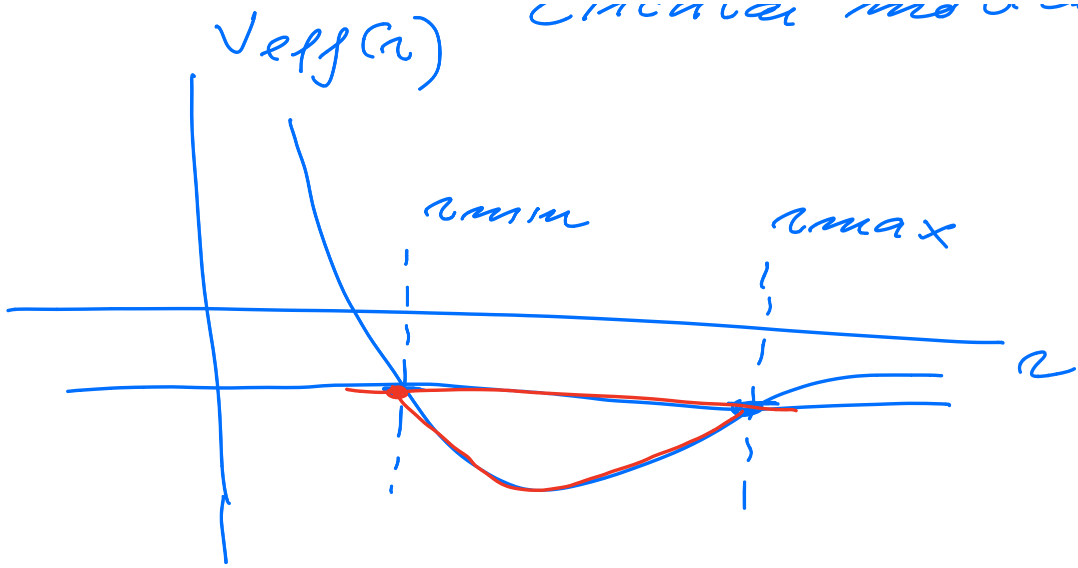
$$r = \frac{1}{u} \Rightarrow$$

$$r(\phi) = \frac{c}{1 + \varepsilon\cos\phi}$$

$\varepsilon = 0 \Rightarrow$ r is constant $\Rightarrow$

circular motion

Elliptical motion,



$$\frac{dr}{d\phi} = 0 \Rightarrow \sin\phi = 0$$

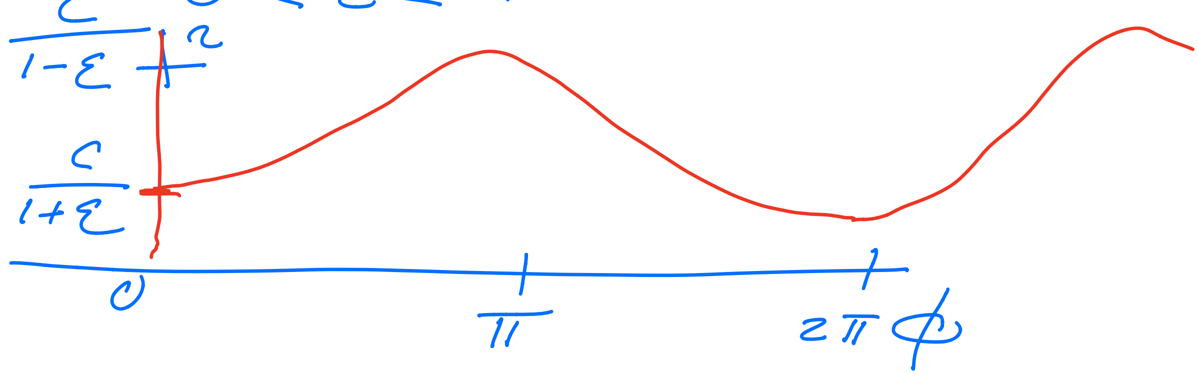
$$\Rightarrow \phi = 0 \ \vee \ \phi = \pi$$

$\phi = 0$

$$r(\phi=0) = \frac{c}{1+\varepsilon} = R_{min}$$

$$r(\phi=\pi) = \frac{c}{1-\varepsilon} = R_{max}$$

$$0 < \varepsilon < 1$$



period for $r = 2\pi$

The motion is an ellipse;

$$\underline{x} = r\cos\phi \qquad y = r\sin\phi$$

$$\underline{r}(1+\varepsilon\underline{\cos\phi}) = c$$

$$= r + \varepsilon x = c$$

square both sides

$$r^2 = c^2 + \varepsilon^2 x^2 - 2x\varepsilon c$$

$$= x^2 + y^2$$

$$x^2(1-\varepsilon^2) + 2c\varepsilon x + y^2 = c^2$$

$$\downarrow$$

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = \underline{1}$$