

# PHY321: Two-body scattering and Variational calculus

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## Aims and Overarching Motivation

Monday.

- Two-body scattering and [Rutherford scattering](#).

**Reading suggestion:** Taylor sections 14.1-14.6 and Lecture notes

Wednesday.

- Lagrangian formalism, top down approach first and derivation of the Euler-Lagrange equations
- Principle of Least Action, watch [Feynman Lecture](#).

**Reading suggestion:** Taylor sections 6.1-6.3

Friday.

- [Variational Calculus](#)

**Reading suggestion:** Taylor sections 6.3-6.4

## Scattering and Cross Sections

Scattering experiments don't measure entire trajectories. For elastic collisions, they measure the distribution of final scattering angles at best. Most experiments use targets thin enough so that the number of scatterings is typically zero or one. The cross section,  $\sigma$ , describes the cross-sectional area for particles to scatter with an individual target atom or nucleus. Cross section measurements form the basis for MANY fields of physics. BThe cross section, and the differential

cross section, encapsulates everything measurable for a collision where all that is measured is the final state, e.g. the outgoing particle had momentum  $\mathbf{p}_f$ . y studying cross sections, one can infer information about the potential interaction between the two particles. Inferring, or constraining, the potential from the cross section is a classic *inverse* problem. Collisions are either elastic or inelastic. Elastic collisions are those for which the two bodies are in the same internal state before and after the collision. If the collision excites one of the participants into a higher state, or transforms the particles into different species, or creates additional particles, the collision is inelastic. Here, we consider only elastic collisions.

## Scattering: Coulomb forces

For Coulomb forces, the cross section is infinite because the range of the Coulomb force is infinite, but for interactions such as the strong interaction in nuclear or particle physics, there is no long-range force and cross-sections are finite. Even for Coulomb forces, the part of the cross section that corresponds to a specific scattering angle,  $d\sigma/d\Omega$ , which is a function of the scattering angle  $\theta_s$  is still finite.

If a particle travels through a thin target, the chance the particle scatters is  $P_{\text{scatt}} = \sigma dN/dA$ , where  $dN/dA$  is the number of scattering centers per area the particle encounters. If the density of the target is  $\rho$  particles per volume, and if the thickness of the target is  $t$ , the areal density (number of target scatterers per area) is  $dN/dA = \rho t$ . Because one wishes to quantify the collisions independently of the target, experimentalists measure scattering probabilities, then divide by the areal density to obtain cross-sections,

$$\sigma = \frac{P_{\text{scatt}}}{dN/dA}. \quad (1)$$

## Scattering, more details

Instead of merely stating that a particle collided, one can measure the probability the particle scattered by a given angle. The scattering angle  $\theta_s$  is defined so that at zero the particle is unscattered and at  $\theta_s = \pi$  the particle is scattered directly backward. Scattering angles are often described in the center-of-mass frame, but that is a detail we will neglect for this first discussion, where we will consider the scattering of particles moving classically under the influence of fixed potentials  $U(\mathbf{r})$ . Because the distribution of scattering angles can be measured, one expresses the differential cross section,

$$\frac{d^2\sigma}{d\cos\theta_s d\phi}. \quad (2)$$

Usually, the literature expresses differential cross sections as

$$d\sigma/d\Omega = \frac{d\sigma}{d\cos\theta d\phi} = \frac{1}{2\pi} \frac{d\sigma}{d\cos\theta}, \quad (3)$$

where the last equivalency is true when the scattering does not depend on the azimuthal angle  $\phi$ , as is the case for spherically symmetric potentials.

The differential solid angle  $d\Omega$  can be thought of as the area subtended by a measurement,  $dA_d$ , divided by  $r^2$ , where  $r$  is the distance to the detector,

$$dA_d = r^2 d\Omega. \quad (4)$$

With this definition  $d\sigma/d\Omega$  is independent of the distance from which one places the detector, or the size of the detector (as long as it is small).

## Differential scattering cross sections

Differential scattering cross sections are calculated by assuming a random distribution of impact parameters  $b$ . These represent the distance in the  $xy$  plane for particles moving in the  $z$  direction relative to the scattering center. An impact parameter  $b = 0$  refers to being aimed directly at the target's center. The impact parameter describes the transverse distance from the  $z = 0$  axis for the trajectory when it is still far away from the scattering center and has not yet passed it. The differential cross section can be expressed in terms of the impact parameter,

$$d\sigma = 2\pi b db, \quad (5)$$

which is the area of a thin ring of radius  $b$  and thickness  $db$ . In classical physics, one can calculate the trajectory given the incoming kinetic energy  $E$  and the impact parameter if one knows the mass and potential.

## More on Differential Cross Sections

From the trajectory, one then finds the scattering angle  $\theta_s(b)$ . The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{d\sigma}{d\cos\theta_s} = b \frac{db}{d\cos\theta_s} = \frac{b}{(d/db)\cos\theta_s(b)}. \quad (6)$$

Typically, one would calculate  $\cos\theta_s$  and  $(d/db)\cos\theta_s$  as functions of  $b$ . This is sufficient to plot the differential cross section as a function of  $\theta_s$ .

The total cross section is

$$\sigma_{\text{tot}} = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int d\cos\theta_s \frac{d\sigma}{d\Omega}. \quad (7)$$

Even if the total cross section is infinite, e.g. Coulomb forces, one can still have a finite differential cross section as we will see later on.

## Rutherford Scattering

This refers to the calculation of  $d\sigma/d\Omega$  due to an inverse square force,  $F_{12} = \pm\alpha/r^2$  for repulsive/attractive interaction. Rutherford compared the scattering of  $\alpha$  particles ( ${}^4\text{He}$  nuclei) off of a nucleus and found the scattering angle at which the formula began to fail. This corresponded to the impact parameter for which the trajectories would strike the nucleus. This provided the first measure of the size of the atomic nucleus. At the time, the distribution of the positive charge (the protons) was considered to be just as spread out amongst the atomic volume as the electrons. After Rutherford's experiment, it was clear that the radius of the nucleus tended to be roughly 4 orders of magnitude smaller than that of the atom, which is less than the size of a football relative to Spartan Stadium.

## Rutherford Scattering, more details

In order to calculate differential cross section, we must find how the impact parameter is related to the scattering angle. This requires analysis of the trajectory. We consider our previous expression for the trajectory where we derived the elliptic form for the trajectory, For that case we considered an attractive force with the particle's energy being negative, i.e. it was bound. However, the same form will work for positive energy, and repulsive forces can be considered by simple flipping the sign of  $\alpha$ . For positive energies, the trajectories will be hyperbolas, rather than ellipses, with the asymptotes of the trajectories representing the directions of the incoming and outgoing tracks.

## Rutherford Scattering, final trajectories

We have

$$r = \frac{1}{\frac{m\alpha}{L^2} + A \cos \theta}. \quad (8)$$

Once  $A$  is large enough, which will happen when the energy is positive, the denominator will become negative for a range of  $\theta$ . This is because the scattered particle will never reach certain angles. The asymptotic angles  $\theta'$  are those for which the denominator goes to zero,

$$\cos \theta' = -\frac{m\alpha}{AL^2}. \quad (9)$$

## Rutherford Scattering, Closest Approach

The trajectory's point of closest approach is at  $\theta = 0$  and the two angles  $\theta'$ , which have this value of  $\cos \theta'$ , are the angles of the incoming and outgoing particles. From Fig (to come), one can see that the scattering angle  $\theta_s$  is given by,

$$\begin{aligned}
2\theta' - \pi &= \theta_s, & \theta' &= \frac{\pi}{2} + \frac{\theta_s}{2}, \\
\sin(\theta_s/2) &= -\cos \theta' \\
&= \frac{m\alpha}{AL^2}.
\end{aligned} \tag{10}$$

Now that we have  $\theta_s$  in terms of  $m, \alpha, L$  and  $A$ , we wish to re-express  $L$  and  $A$  in terms of the impact parameter  $b$  and the energy  $E$ . This will set us up to calculate the differential cross section, which requires knowing  $db/d\theta_s$ . It is easy to write the angular momentum as

$$L^2 = p_0^2 b^2 = 2mEb^2. \tag{11}$$

### Rutherford Scattering, getting there

Finding  $A$  is more complicated. To accomplish this we realize that the point of closest approach occurs at  $\theta = 0$ , so from Eq. (8)

$$\begin{aligned}
\frac{1}{r_{\min}} &= \frac{m\alpha}{L^2} + A, \\
A &= \frac{1}{r_{\min}} - \frac{m\alpha}{L^2}.
\end{aligned} \tag{12}$$

Next,  $r_{\min}$  can be found in terms of the energy because at the point of closest approach the kinetic energy is due purely to the motion perpendicular to  $\hat{r}$  and

$$E = -\frac{\alpha}{r_{\min}} + \frac{L^2}{2mr_{\min}^2}. \tag{13}$$

### Rutherford Scattering, More Manipulations

One can solve the quadratic equation for  $1/r_{\min}$ ,

$$\frac{1}{r_{\min}} = \frac{m\alpha}{L^2} + \sqrt{(m\alpha/L^2)^2 + 2mE/L^2}. \tag{14}$$

We can plug the expression for  $r_{\min}$  into the expression for  $A$ , Eq. (12),

$$A = \sqrt{(m\alpha/L^2)^2 + 2mE/L^2} = \sqrt{(\alpha^2/(4E^2b^4) + 1/b^2)} \tag{15}$$

### Rutherford Scattering, final expression

Finally, we insert the expression for  $A$  into that for the scattering angle, Eq. (10),

$$\begin{aligned}
\sin(\theta_s/2) &= \frac{m\alpha}{AL^2} \\
&= \frac{a}{\sqrt{a^2 + b^2}}, \quad a \equiv \frac{\alpha}{2E}
\end{aligned} \tag{16}$$

## Rutherford Scattering, the Differential Cross Section

The differential cross section can now be found by differentiating the expression for  $\theta_s$  with  $b$ ,

$$\begin{aligned}
\frac{1}{2} \cos(\theta_s/2) d\theta_s &= \frac{ab \, db}{(a^2 + b^2)^{3/2}} = \frac{bdb}{a^2} \sin^3(\theta_s/2), \\
d\sigma &= 2\pi b db = \frac{\pi a^2}{\sin^3(\theta_s/2)} \cos(\theta_s/2) d\theta_s \\
&= \frac{\pi a^2}{2 \sin^4(\theta_s/2)} \sin \theta_s d\theta_s \\
\frac{d\sigma}{d \cos \theta_s} &= \frac{\pi a^2}{2 \sin^4(\theta_s/2)}, \\
\frac{d\sigma}{d\Omega} &= \frac{a^2}{4 \sin^4(\theta_s/2)}.
\end{aligned} \tag{17}$$

where  $a = \alpha/2E$ . This is the Rutherford formula for the differential cross section. It diverges as  $\theta_s \rightarrow 0$  because scatterings with arbitrarily large impact parameters still scatter to arbitrarily small scattering angles. The expression for  $d\sigma/d\Omega$  is the same whether the interaction is positive or negative.

## Rutherford Scattering, Example

Consider a particle of mass  $m$  and charge  $z$  with kinetic energy  $E$  (Let it be the center-of-mass energy) incident on a heavy nucleus of mass  $M$  and charge  $Z$  and radius  $R$ . We want to find the angle at which the Rutherford scattering formula breaks down.

Let  $\alpha = Zze^2/(4\pi\epsilon_0)$ . The scattering angle in Eq. (16) is

$$\sin(\theta_s/2) = \frac{a}{\sqrt{a^2 + b^2}}, \quad a \equiv \frac{\alpha}{2E}.$$

The impact parameter  $b$  for which the point of closest approach equals  $R$  can be found by using angular momentum conservation,

$$\begin{aligned}
p_0 b &= b\sqrt{2mE} = Rp_f = R\sqrt{2m(E - \alpha/R)}, \\
b &= R \frac{\sqrt{2m(E - \alpha/R)}}{\sqrt{2mE}} \\
&= R\sqrt{1 - \frac{\alpha}{ER}}.
\end{aligned}$$

## Rutherford Scattering, Example, wrapping up

Putting these together

$$\theta_s = 2 \sin^{-1} \left\{ \frac{a}{\sqrt{a^2 + R^2(1 - \alpha/(RE))}} \right\}, \quad a = \frac{\alpha}{2E}.$$

It was from this departure of the experimentally measured  $d\sigma/d\Omega$  from the Rutherford formula that allowed Rutherford to infer the radius of the gold nucleus,  $R$ .

## Variational Calculus

The calculus of variations involves problems where the quantity to be minimized or maximized is an integral.

The usual minimization problem one faces involves taking a function  $\mathcal{L}(x)$ , then finding the single value  $x$  for which  $\mathcal{L}$  is either a maximum or minimum. In multivariate calculus one also learns to solve problems where you minimize for multiple variables,  $\mathcal{L}(x_1, x_2, \dots, x_n)$ , and finding the points  $(x_1 \dots y_n)$  in an  $n$ -dimensional space that maximize or minimize the function. Here, we consider what seems to be a much more ambitious problem. Imagine you have a function  $\mathcal{L}(x(t), \dot{x}(t), t)$ , and you wish to find the extrema for an infinite number of values of  $x$ , i.e.  $x$  at each point  $t$ . The function  $\mathcal{L}$  will not only depend on  $x$  at each point  $t$ , but also on the slope at each point, plus an additional dependence on  $t$ . Note we are NOT finding an optimum value of  $t$ , we are finding the set of optimum values of  $x$  at each point  $t$ , or equivalently, finding the function  $x(t)$ .

## Variational Calculus, introducing the action

One treats the function  $x(t)$  as being unknown while minimizing the action

$$S = \int_{t_1}^{t_2} dt \mathcal{L}(x(t), \dot{x}(t), t).$$

Thus, we are minimizing  $S$  with respect to an infinite number of values of  $x(t_i)$  at points  $t_i$ . As an additional criteria, we will assume that  $x(t_1)$  and  $x(t_2)$  are fixed, and that that we will only consider variations of  $x$  between the boundaries. The dependence on the derivative,  $\dot{x} = dx/dt$ , is crucial because otherwise the

solution would involve simply finding the one value of  $x$  that minimized  $\mathcal{L}$ , and  $x(t)$  would equal a constant if there were no explicit  $t$  dependence. Furthermore,  $x$  wouldn't need to be continuous at the boundary.

## Variational Calculus, general Action

In the general case we have an integral of the type

$$S[q] = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t) dt,$$

where  $S$  is the quantity which is sought minimized or maximized. The problem is that although  $\mathcal{L}$  is a function of the general variables  $q(t), \dot{q}(t), t$  (note our change of variables), the exact dependence of  $q$  on  $t$  is not known. This means again that even though the integral has fixed limits  $t_1$  and  $t_2$ , the path of integration is not known. In our case the unknown quantities are the positions and general velocities of a given number of objects and we wish to choose an integration path which makes the functional  $S[q]$  stationary. This means that we want to find minima, or maxima or saddle points. In physics we search normally for minima. Our task is therefore to find the minimum of  $S[q]$  so that its variation  $\delta S$  is zero subject to specific constraints. The constraints can be treated via the technique of Lagrangian multipliers as we will see below.

## Variational Calculus, Optimal Path

We assume the existence of an optimum path, that is a path for which  $S[q]$  is stationary. There are infinitely many such paths. The difference between two paths  $\delta q$  is called the variation of  $q$ .

We call the variation  $\eta(t)$  and it is scaled by a factor  $\alpha$ . The function  $\eta(t)$  is arbitrary except for

$$\eta(t_1) = \eta(t_2) = 0,$$

and we assume that we can model the change in  $q$  as

$$q(t, \alpha) = q(t) + \alpha \eta(t),$$

and

$$\delta q = q(t, \alpha) - q(t, 0) = \alpha \eta(t).$$

## Variational Calculus, Condition for an Extreme Value

We choose  $q(t, \alpha = 0)$  as the unknown path that will minimize  $S$ . The value  $q(t, \alpha \neq 0)$  describes a neighbouring path.

We have



$$S[q(\alpha)] = \int_{t_1}^{t_2} \mathcal{L}(q(t, \alpha), \dot{q}(t, \alpha), t) dt.$$

The condition for an extreme of

$$S[q(\alpha)] = \int_{t_1}^{t_2} \mathcal{L}(q(t, \alpha), \dot{q}(t, \alpha), t) dt,$$

is

$$\left[ \frac{\partial S[q(\alpha)]}{\partial t} \right]_{\alpha=0} = 0.$$

### Variational Calculus. $\alpha$ Dependence

The  $\alpha$  dependence is contained in  $q(t, \alpha)$  and  $\dot{q}(t, \alpha)$  meaning that

$$\left[ \frac{\partial E[q(\alpha)]}{\partial \alpha} \right] = \int_{t_1}^{t_2} \left( \frac{\partial \downarrow}{\partial q} \frac{\partial q}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \alpha} \right) dt.$$

We have defined

$$\frac{\partial q(x, \alpha)}{\partial \alpha} = \eta(x)$$

and thereby

$$\frac{\partial \dot{q}(t, \alpha)}{\partial \alpha} = \frac{d(\eta(t))}{dt}.$$

### INtegrating by Parts

Using

$$\frac{\partial q(t, \alpha)}{\partial \alpha} = \eta(t),$$

and

$$\frac{\partial \dot{q}(t, \alpha)}{\partial \alpha} = \frac{d(\eta(t))}{dt},$$

in the integral gives

$$\left[ \frac{\partial S[q(\alpha)]}{\partial \alpha} \right] = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial q} \eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d(\eta(t))}{dt} \right) dt.$$

Integrating the second term by parts

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d(\eta(t))}{dt} dt = \eta(t) \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_1}^{t_2} - \int_a^b \eta(t) \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{q}} dt,$$

and since the first term disappears due to  $\eta(a) = \eta(b) = 0$ , we obtain

$$\left[ \frac{\partial S[q(\alpha)]}{\partial \alpha} \right] = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \eta(t) dt = 0.$$

## Euler-Lagrange Equations

The latter can be written as

$$\left[ \frac{\partial S[q(\alpha)]}{\partial \alpha} \right]_{\alpha=0} = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q(t) dt = \delta S = 0.$$

The condition for a stationary value is thus a partial differential equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0,$$

known as the **Euler-Lagrange** equation.