PHY 321 April 16

$$\mathcal{L}(x, v, t) \to \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

Minimize

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt$$

Euler-Lagrange equations

$$\vec{q} = [q_1, q_2 \ldots q_N]$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

Constraints

- holonomic

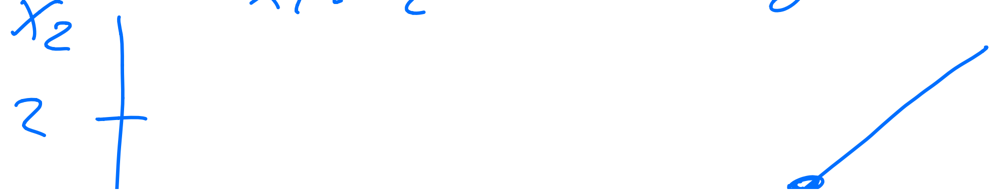
$$g(q_1, q_2 \cdots q_n) = 0$$

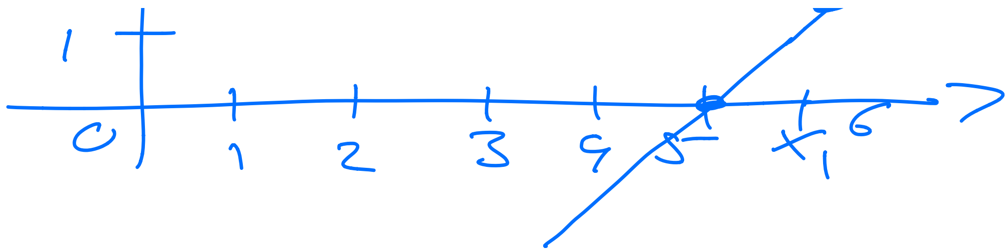
Example

$f(x_1, x_2)$ minimize/max

with a constraint

$$x_1 + x_2 - 5 = 0 = g(x_1, x_2)$$





Example

$$y - x\tan\alpha = 0 = g(x,y)$$

$$\mathcal{L} = K - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \underline{mgy}$$

Top-down

$$\mathcal{L}' = \mathcal{L} + \lambda g(x,y)$$

$$\frac{\partial \mathcal{L}'}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{x}} = 0$$

$$\frac{\partial \mathcal{L}'}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{y}} = 0$$

$\Rightarrow$

$$\ddot{x} = -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g\sin^2\alpha$$

$$x(t) = x_0 + \dot{x}_0 t - \frac{1}{2} g t^2 \sin\alpha\cos\alpha$$

$$y(t) = y_0 + \dot{y}_0 t - \frac{1}{2} g t^2 \sin^2\alpha$$

$$\lambda = mg\cos^2\alpha \quad ?$$

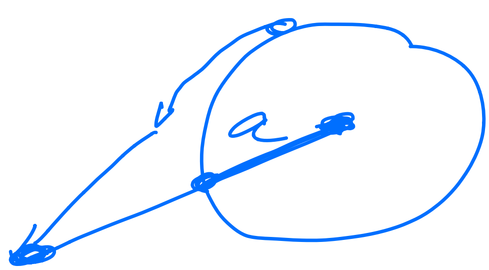
Force, normal force.

- non-holonomic

$$g(\vec{q}, \dot{\vec{q}}, t) = c$$

$$\mathcal{L}' = \underbrace{\mathcal{L}}_{K-V} + \lambda\left(g(\vec{q}, \dot{\vec{q}}, t) - c\right)$$

$r^2 - a^2 \geq c$



Lagrangian multipliers

Example: minimize

$f(x_1, x_2)$

subject to $g(x_1, x_2) = 0$

satisfied by $\tilde{x} = [\tilde{x}_1, \tilde{x}_2]$

necessary condition:

$$\Rightarrow df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

dx_1 and dx_2 are variations

i.e. points that are on the constraint

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) = 0$$

assume small

Taylor expand:

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) \simeq \underline{g(\tilde{x}_1, \tilde{x}_2)} + \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g(\tilde{x}_1 \tilde{x}_2)}{\partial x_2} dx_2$$

we require that

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0$$

at $(\tilde{x}_1, \tilde{x}_2)$

assume $\frac{\partial g}{\partial x_2} \neq 0$

$$dx_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\, dx_1$$

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial g/\partial x_1}{\partial g/\partial x_2}\frac{\partial f}{\partial x_2}\right] dx_1$$

it must be satisfied by all possible dx_1-values

$$\left[\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)} = 0$$

Define a parameter

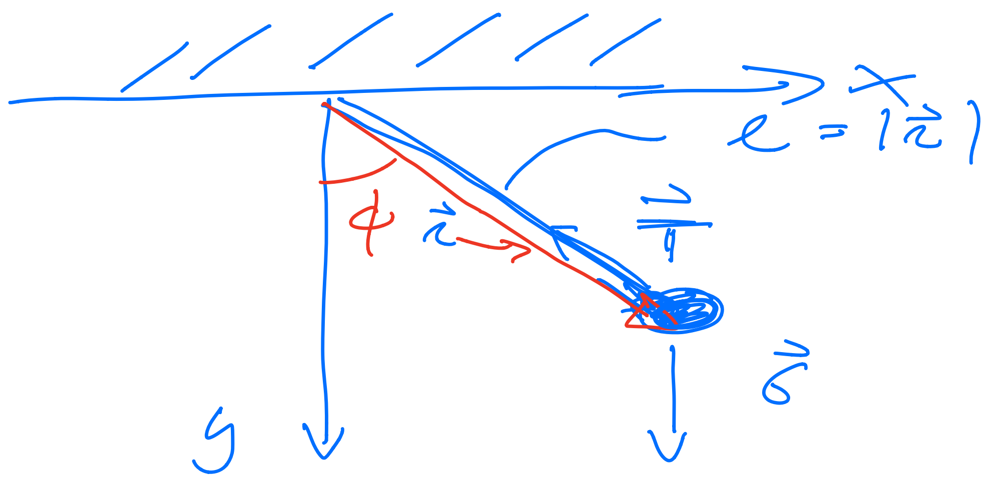
$$\lambda = \frac{\partial f}{\partial x_2} \Big/ \partial g/\partial x_2$$

$$df = 0 = \left[\frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1}\right] = 0$$

Lagrangian multiplier

$$\Rightarrow df' = d(f + \lambda g)$$

Example 3



constraint $\ell = |\vec{r}| = r$

$$K = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\phi}^2\right]$$

$$y = r\cos\phi \quad \wedge \quad x = r\sin\phi$$

$$V(y) = -mg\,\underbrace{r\cos\phi}_{y}$$

constraint $r - \ell = 0$

$$\boxed{\mathcal{L}' = \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\phi}^2\right] + mgr\cos\phi + \lambda(r-\ell)}$$

$$\left[\frac{\partial}{\partial r} - \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\right]\mathcal{L}' = 0$$

$$\frac{\partial L'}{\partial r} = m r \dot{\phi}^2 + m g \cos\phi + \lambda$$

$$\frac{\partial L'}{\partial \dot{r}} = m \dot{r} \qquad \frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} = m \ddot{r}$$

$$m \ddot{r} = m r \dot{\phi}^2 + m g \cos\phi + \lambda$$

$$\frac{\partial L'}{\partial \phi} = - m g r \sin\phi$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{\phi}} = m r^2 \ddot{\phi} \Rightarrow$$

$$m r^2 \ddot{\phi} = - m g r \sin\phi \Rightarrow$$

$$r = l \Rightarrow$$

$$\boxed{\ddot{\phi} = - g/l \sin\phi}$$

$$\lambda = m l \dot{\phi}^2 + m g \cos\phi = ?$$

Force