KINEMATICS

- relationship between joint positions and end-effector position and orientation

Rotation matrix

Representations of orientation

Homogeneous transformations

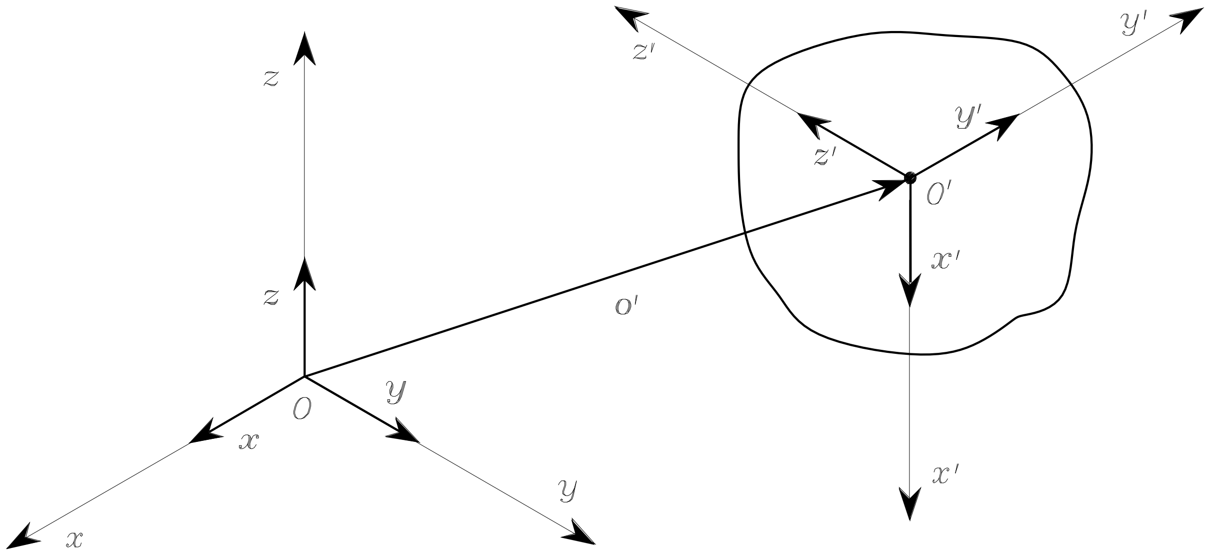
Direct kinematics

Joint space and operational space

Kinematic calibration

Inverse kinematics problem

POSE OF A RIGID BODY



- Position

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$

- Orientation

$$\boldsymbol{x}' = x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z}$$
$$\boldsymbol{y}' = y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z}$$
$$\boldsymbol{z}' = z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}$$

ROTATION MATRIX

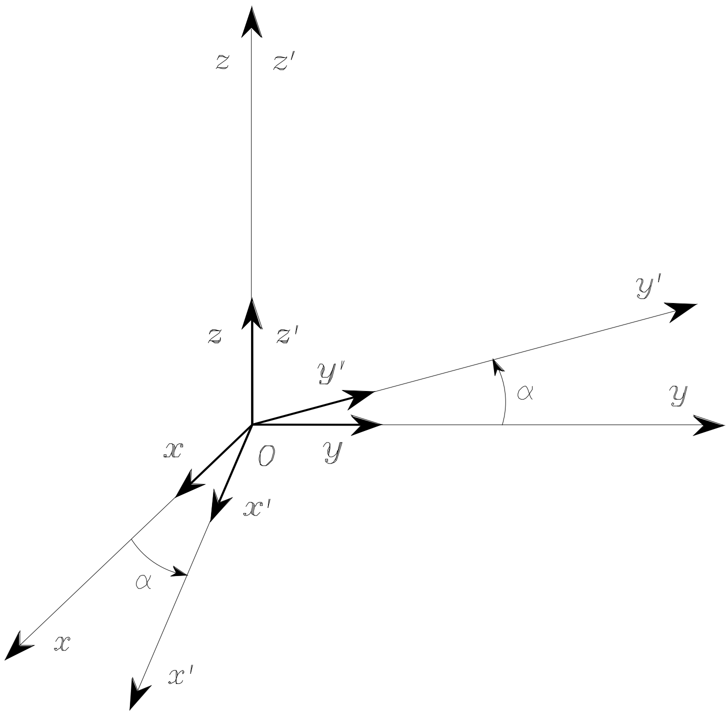
$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$$

$$\boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

Elementary rotations

- rotation of α about z



$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

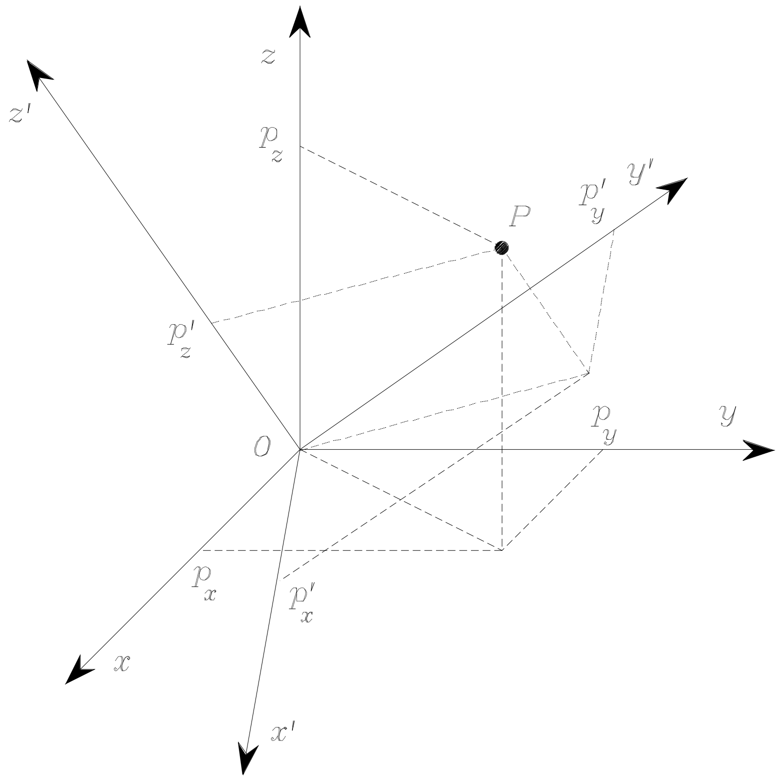
- rotation of β about y

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- rotation of γ about x

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Representation of a vector

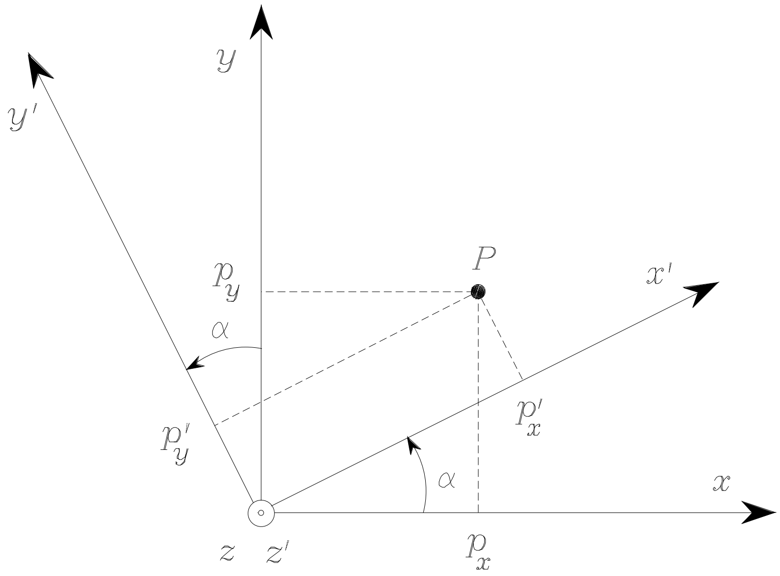


$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{p} &= \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}' \\ &= \boldsymbol{R}\boldsymbol{p}' \end{aligned}$$

$$\boldsymbol{p}' = \boldsymbol{R}^T \boldsymbol{p}$$

- Example



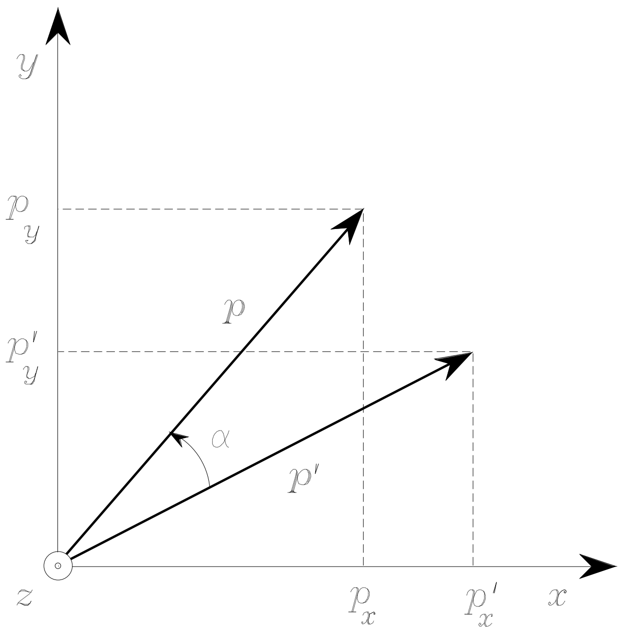
$$
\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z
\end{aligned}
$$

Rotation of a vector

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'$$

$$\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{p}'^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p}'$$

- Example



$$\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z
\end{aligned}$$

$$\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$$

- Rotation matrix
  - ⋆ it describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame
  - ⋆ it represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin)
  - ⋆ it is the operator that allows the rotation of a vector in the same coordinate frame

COMPOSITION OF ROTATION MATRICES

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

$$\boldsymbol{R}_i^j = (\boldsymbol{R}_j^i)^{-1} = (\boldsymbol{R}_j^i)^T$$

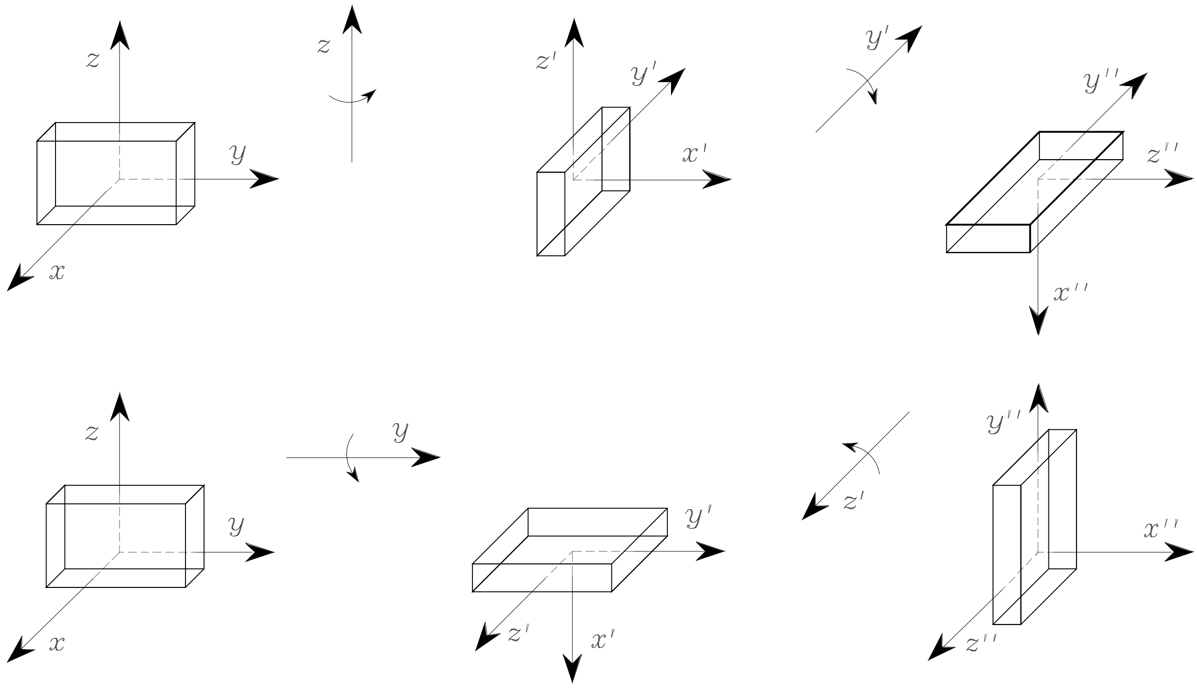
- *Current frame* rotation

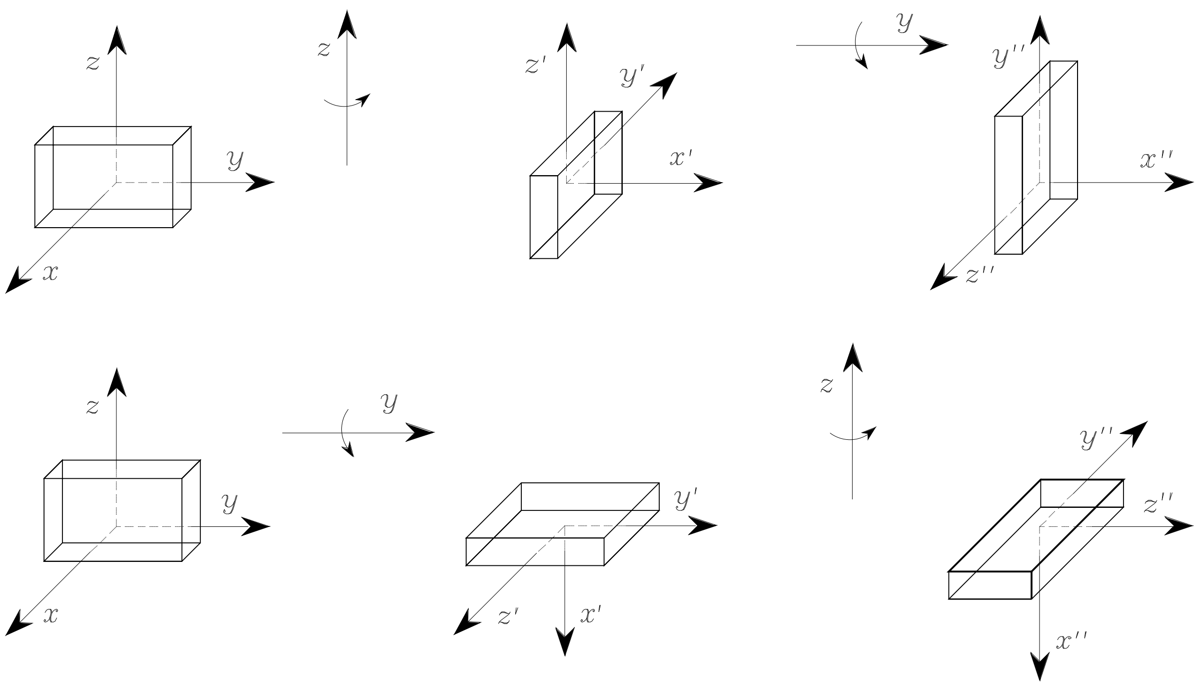
$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$

- *Fixed frame* rotation

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_2^1 \boldsymbol{R}_1^0$$

- Example

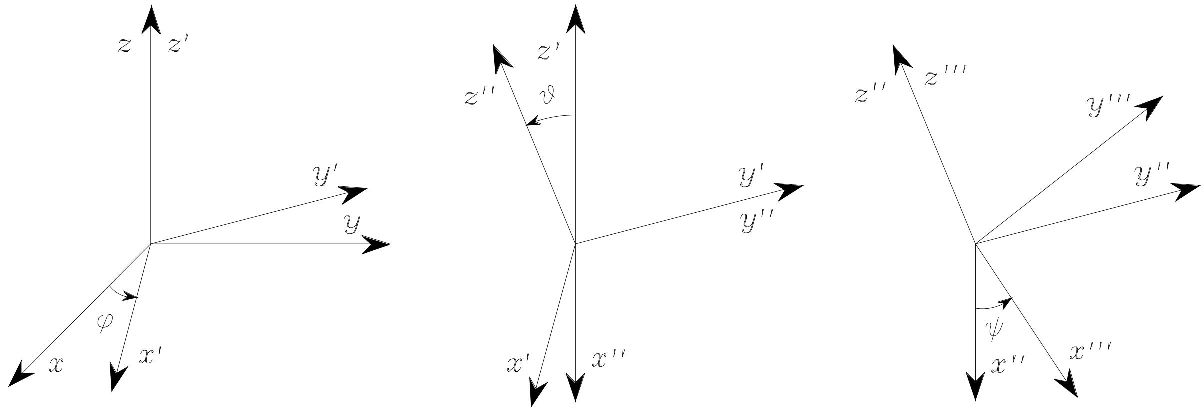




EULER ANGLES

- rotation matrix
  - ⋆ 9 parameters with 6 constraints
- minimal representation of orientation
  - ⋆ 3 independent parameters

ZYZ angles



$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

- Inverse problem
  - ⋆ Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

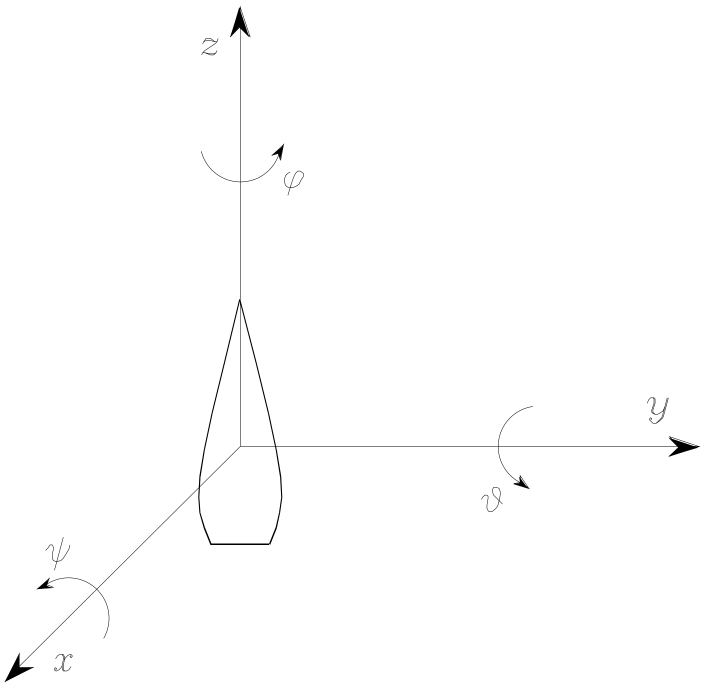
the three ZYZ angles are ($\vartheta \in (0, \pi)$)

$$\begin{aligned} \varphi &= \mathrm{Atan2}(r_{23}, r_{13}) \\ \vartheta &= \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \\ \psi &= \mathrm{Atan2}(r_{32}, -r_{31}) \end{aligned}$$

or ($\vartheta \in (-\pi, 0)$)

$$\begin{aligned} \varphi &= \mathrm{Atan2}(-r_{23}, -r_{13}) \\ \vartheta &= \mathrm{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \\ \psi &= \mathrm{Atan2}(-r_{32}, r_{31}) \end{aligned}$$

RPY angles



$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

- Inverse problem
  - ⋆ Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

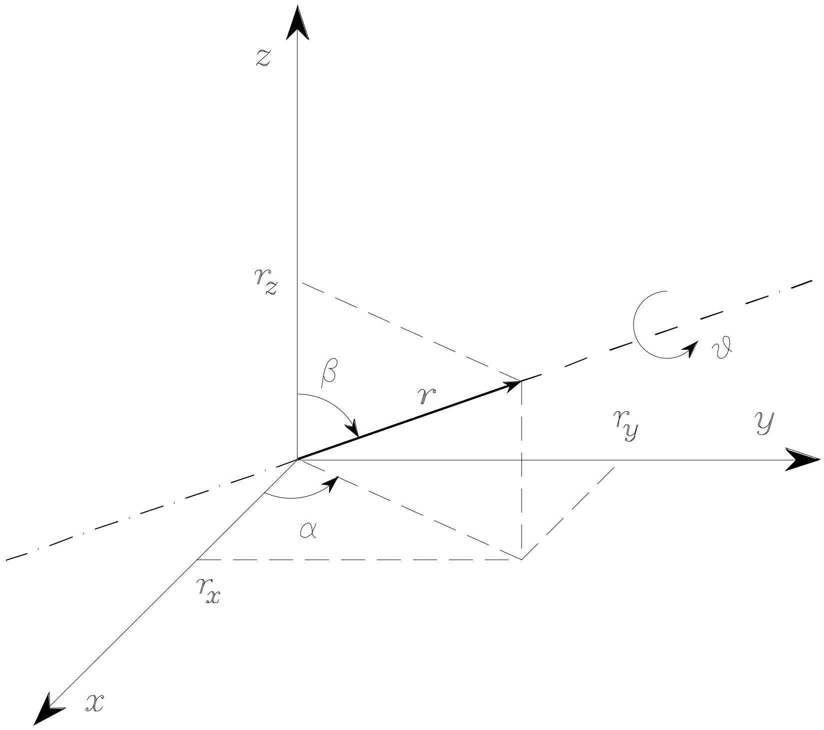
the three RPY angles are ($\vartheta \in (-\pi/2, \pi/2)$)

$$\begin{aligned} \varphi &= \mathrm{Atan2}(r_{21}, r_{11}) \\ \vartheta &= \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right) \\ \psi &= \mathrm{Atan2}(r_{32}, r_{33}) \end{aligned}$$

or ($\vartheta \in (\pi/2, 3\pi/2)$)

$$\begin{aligned} \varphi &= \mathrm{Atan2}(-r_{21}, -r_{11}) \\ \vartheta &= \mathrm{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right) \\ \psi &= \mathrm{Atan2}(-r_{32}, -r_{33}) \end{aligned}$$

ANGLE AND AXIS



$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z$$

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_y r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{bmatrix}$$

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}(-\vartheta, -\boldsymbol{r})$$

- Inverse problem
  - ⋆ Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

the angle and axis of rotation are ($\sin\vartheta \neq 0$)

$$\vartheta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

with

$$r_x^2 + r_y^2 + r_z^2 = 1$$

UNIT QUATERNION

- 4-parameter representation $\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\}$

$$\eta = \cos\frac{\vartheta}{2}$$
$$\boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\,\boldsymbol{r}$$

$$\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$$

$\star$ $(\vartheta, \boldsymbol{r})$ and $(-\vartheta, -\boldsymbol{r})$ give the same quaternion

$$\boldsymbol{R}(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2+\epsilon_x^2)-1 & 2(\epsilon_x\epsilon_y-\eta\epsilon_z) & 2(\epsilon_x\epsilon_z+\eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y+\eta\epsilon_z) & 2(\eta^2+\epsilon_y^2)-1 & 2(\epsilon_y\epsilon_z-\eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z-\eta\epsilon_y) & 2(\epsilon_y\epsilon_z+\eta\epsilon_x) & 2(\eta^2+\epsilon_z^2)-1 \end{bmatrix}$$

- Inverse problem
  - ⋆ Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

the quaternion is ($\eta \geq 0$)

$$\eta = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$\boldsymbol{\epsilon} = \frac{1}{2} \begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

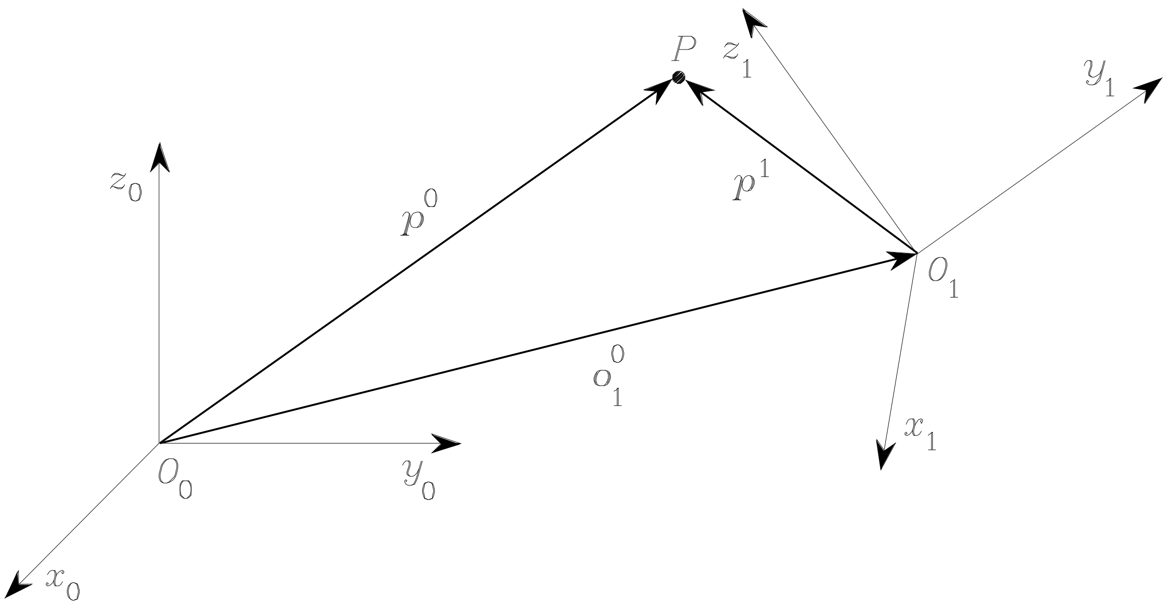
- quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$

$$\mathcal{Q}^{-1} = \{\eta, -\boldsymbol{\epsilon}\}$$

- quaternion product

$$\mathcal{Q}_1 * \mathcal{Q}_2 = \{\eta_1\eta_2 - \boldsymbol{\epsilon}_1^T\boldsymbol{\epsilon}_2, \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2\}$$

HOMOGENEOUS TRANSFORMATIONS



- Coordinate transformation (*translation* + *rotation*)

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- Inverse transformation

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$

- Homogeneous representation

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

- Homogeneous transformation matrix

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

- Coordinate transformation

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1$$

- Inverse transformation

$$\tilde{\boldsymbol{p}}^1 = \boldsymbol{A}_0^1 \tilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \tilde{\boldsymbol{p}}^0$$

ove

$$\boldsymbol{A}_0^1 = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

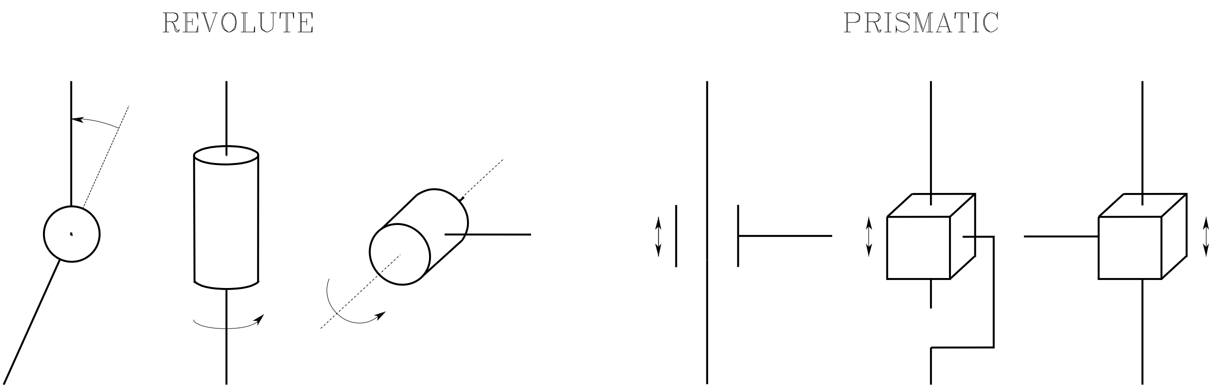
$$\boldsymbol{A}^{-1} \neq \boldsymbol{A}^T$$

- Sequence of coordinate transformations

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \dots \boldsymbol{A}_n^{n-1} \tilde{\boldsymbol{p}}^n$$

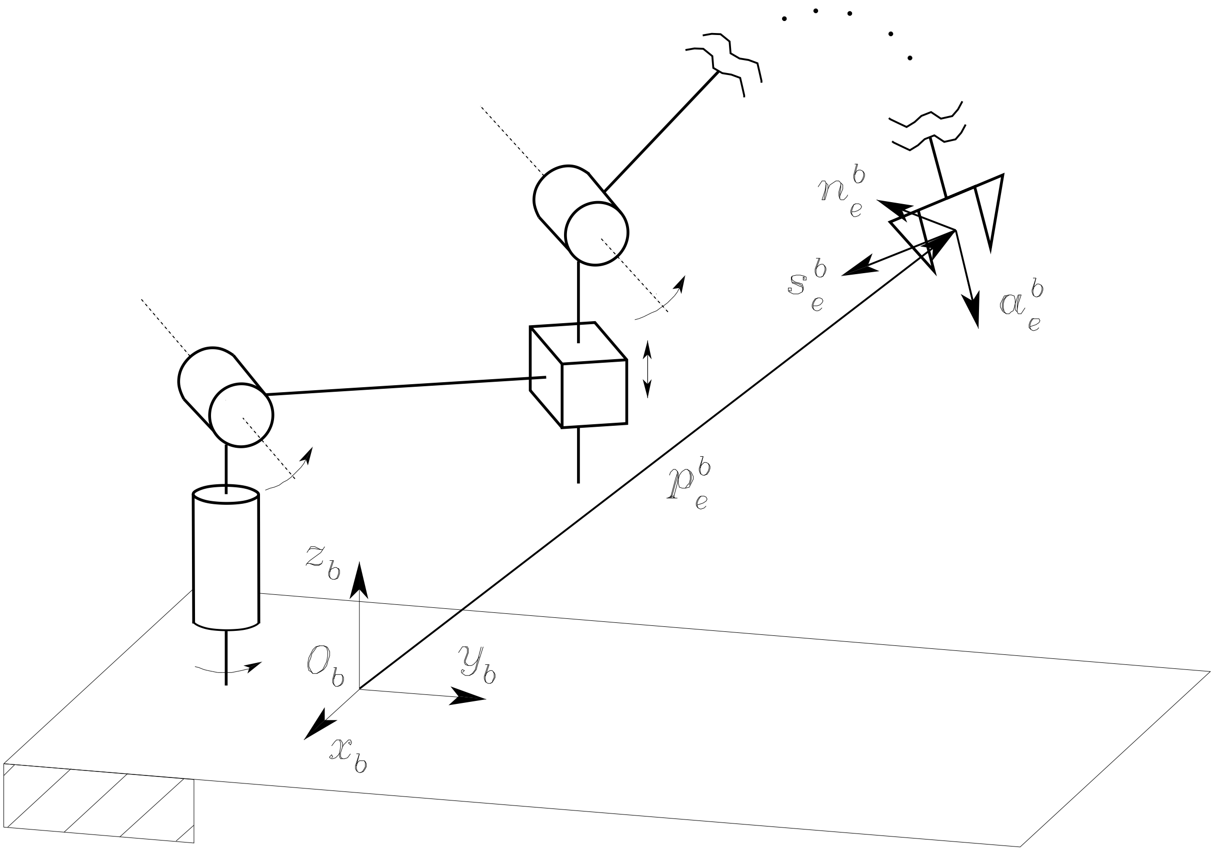
DIRECT KINEMATICS

- Manipulator
  - ⋆ series of *links* connected by means of *joints*



- Kinematic chain (from base to end-effector)
  - ⋆ open (only one sequence)
  - ⋆ closed (loop)
- Degree of freedom
  - ⋆ associated with a joint articulation = *joint variable*

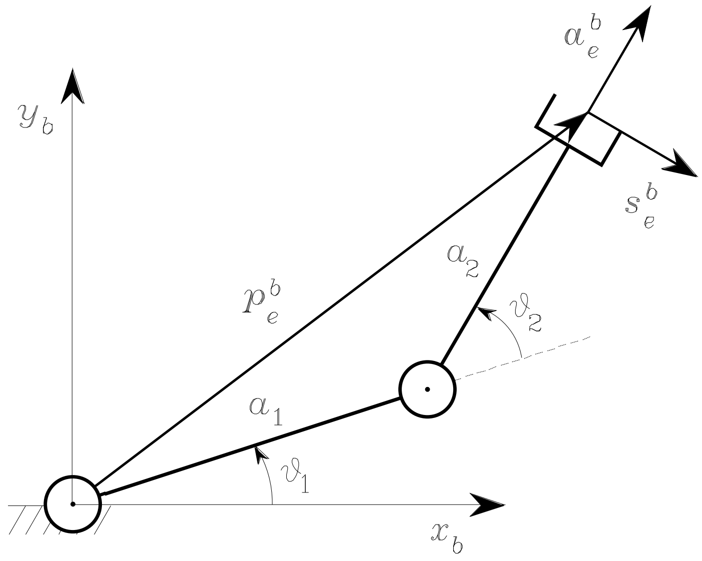
Base frame and end-effector frame



- Direct kinematics equation

$$
\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b(\boldsymbol{q}) & \boldsymbol{s}_e^b(\boldsymbol{q}) & \boldsymbol{a}_e^b(\boldsymbol{q}) & \boldsymbol{p}_e^b(\boldsymbol{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

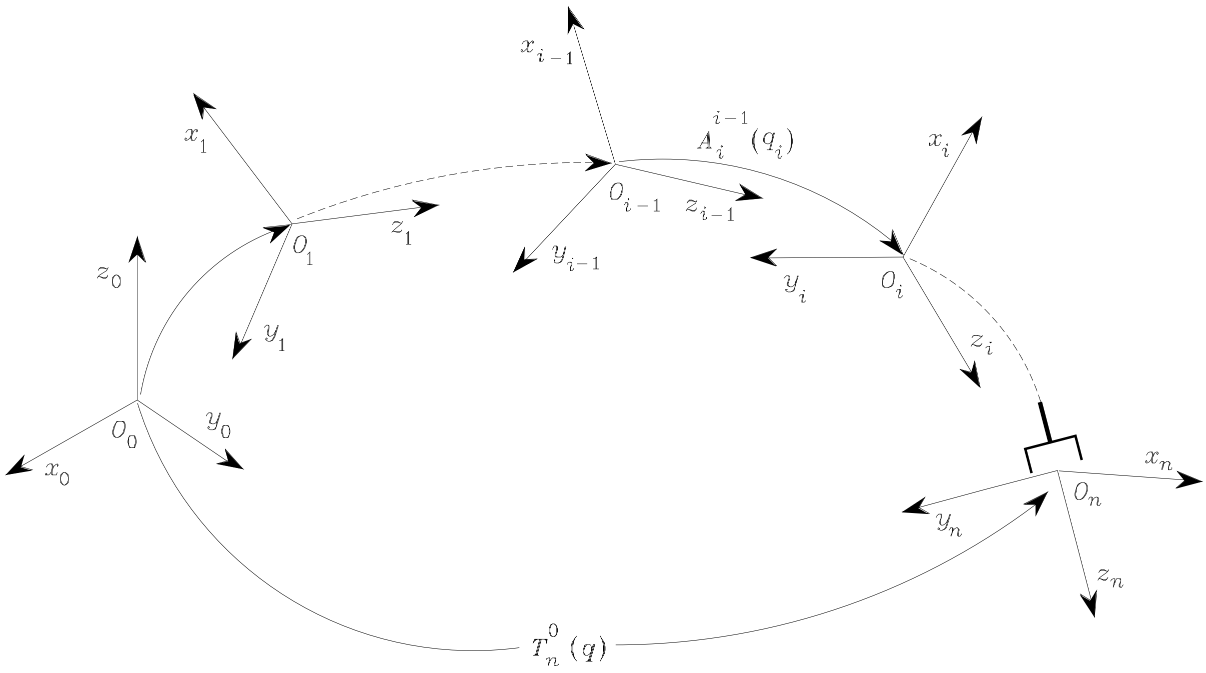
Two-link planar arm



$$
\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b & \boldsymbol{s}_e^b & \boldsymbol{a}_e^b & \boldsymbol{p}_e^b \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & s_{12} & c_{12} & a_1 c_1 + a_2 c_{12} \\ 0 & -c_{12} & s_{12} & a_1 s_1 + a_2 s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

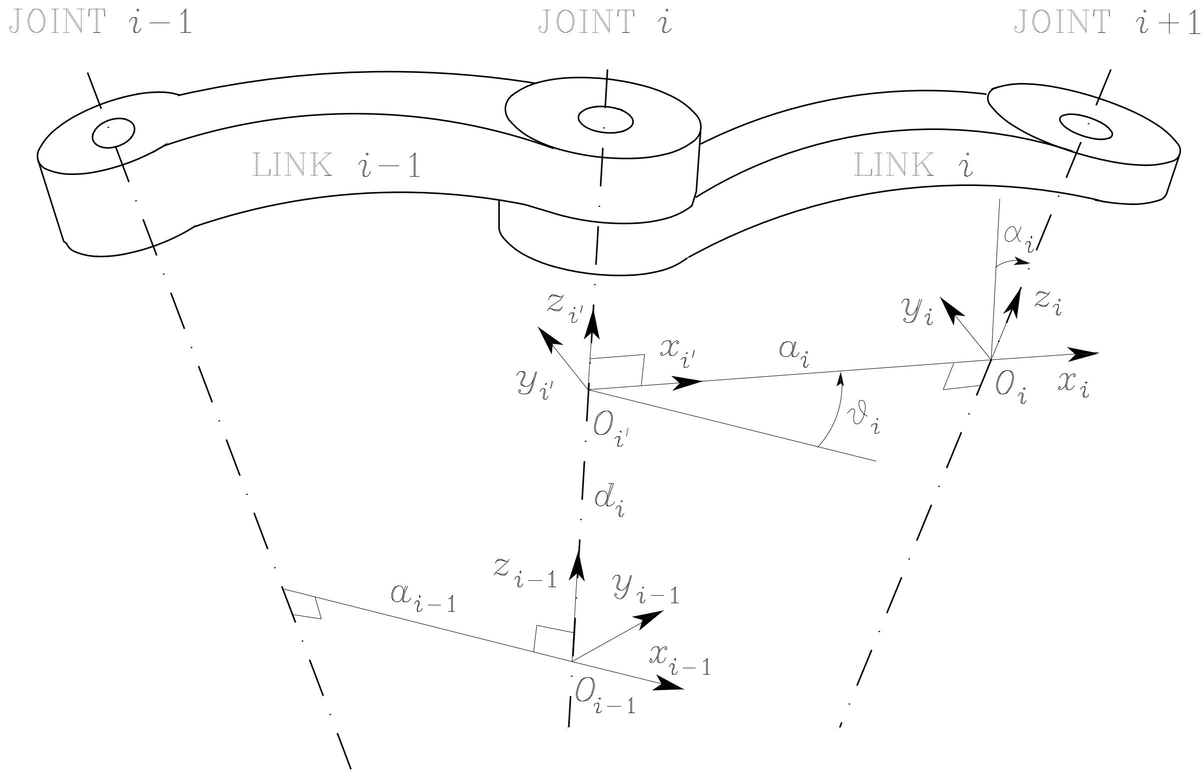
Open chain



$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0(q_1)\boldsymbol{A}_2^1(q_2)\dots\boldsymbol{A}_n^{n-1}(q_n)$$

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b\boldsymbol{T}_n^0(\boldsymbol{q})\boldsymbol{T}_e^n$$

Denavit–Hartenberg convention



- choose axis z_i along axis of Joint $i+1$

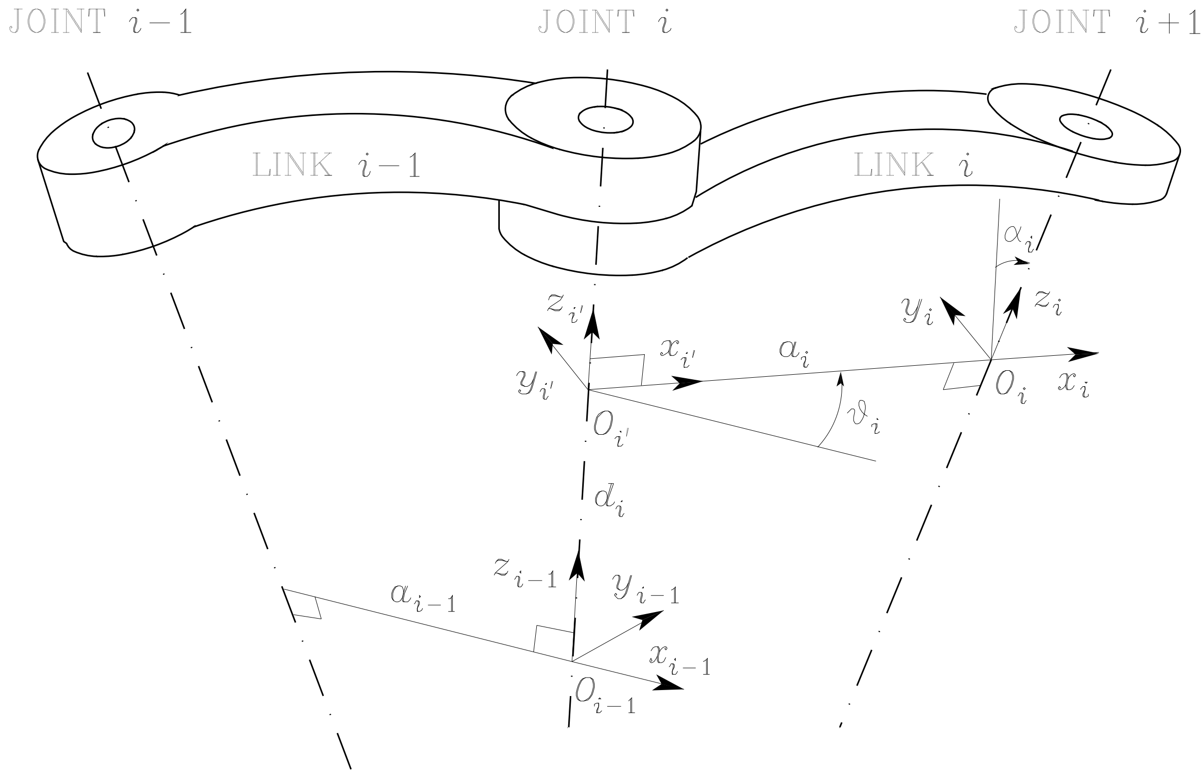
- locate O_i at the intersection of axis z_i with the common normal to axes z_{i-1} and z_i, and O_i' at intersection of common normal with axis z_{i-1}

- choose axis x_i along common the normal to axes z_{i-1} and z_i with positive direction from Joint i to Joint $i+1$

- choose axis y_i so as to complete right-handed frame

- Nonunique definition of link frame:
  - ⋆ for Frame 0, only the direction of axis z_0 is specified: then O_0 and x_0 can be chosen arbitrarily
  - ⋆ for Frame n, since there is no Joint $n+1$, z_n is not uniquely defined while x_n has to be normal to axis z_{n-1}); typically Joint n is revolute and thus z_n can be aligned with z_{n-1}
  - ⋆ when two consecutive axes are parallel, the common normal between them is not uniquely defined
  - ⋆ when two consecutive axes intersect, the positive direction of x_i is arbitrary
  - ⋆ when Joint i is prismatic, only the direction of z_{i-1} is specified

Denavit–Hartenberg parameters



a_i distance between O_i and O_i';

d_i coordinate of O_i' along z_{i-1};

α_i angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise

ϑ_i angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise

- a_i and α_i are always constant
- if Joint i is *revolute* the variable is ϑ_i
- if Joint i is *prismatic* the variable is d_i

- Coordinate transformation

$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Procedure

1. Find and number consecutively the joint axes; set the directions of axes $z_0, \ldots, z_{n-1}$

2. Choose Frame 0 by locating the origin on axis z_0; axes x_0 and y_0 are chosen so as to obtain a right-handed frame. If feasible, it is worth choosing Frame 0 to coincide with the base frame

Execute steps from **3** to **5** for $i = 1, \ldots, n-1$:

3. Locate the origin O_i at the intersection of z_i with the common normal to axes z_{i-1} and z_i. If axes z_{i-1} and z_i are parallel and Joint i is revolute, then locate O_i so that $d_i = 0$; if Joint i is prismatic, locate O_i at a reference position for the joint range, e.g., a mechanical limit

4. Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from Joint i to Joint $i+1$

5. Choose axis y_i so as to obtain a right-handed frame

To complete:

6. Choose Frame n; if Joint n is revolute, then align z_n with z_{n-1}, otherwise, if Joint n is prismatic, then choose z_n arbitrarily. Axis x_n is set according to step **4**

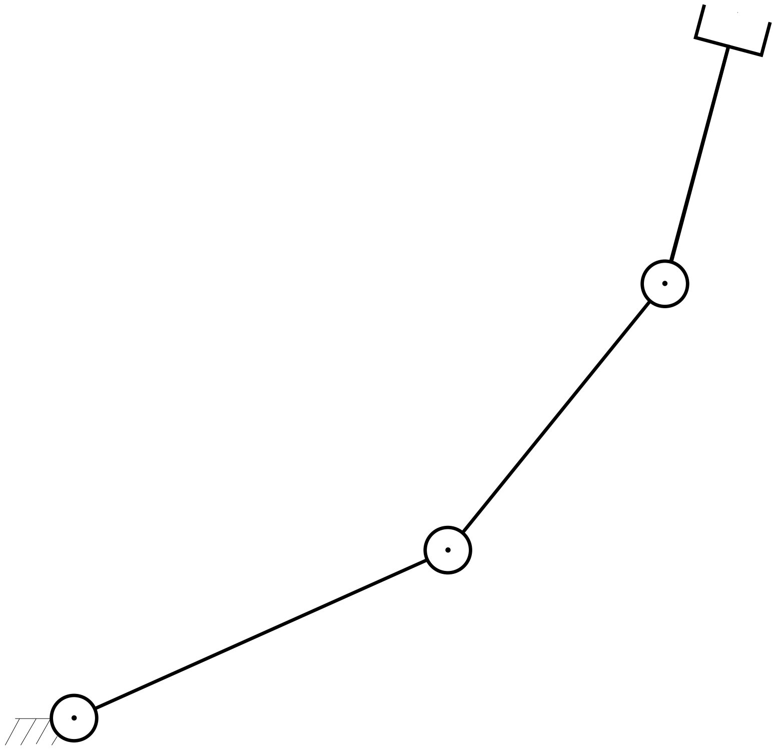
7. For $i = 1, \ldots, n$, form the table of parameters $a_i, d_i, \alpha_i, \vartheta_i$

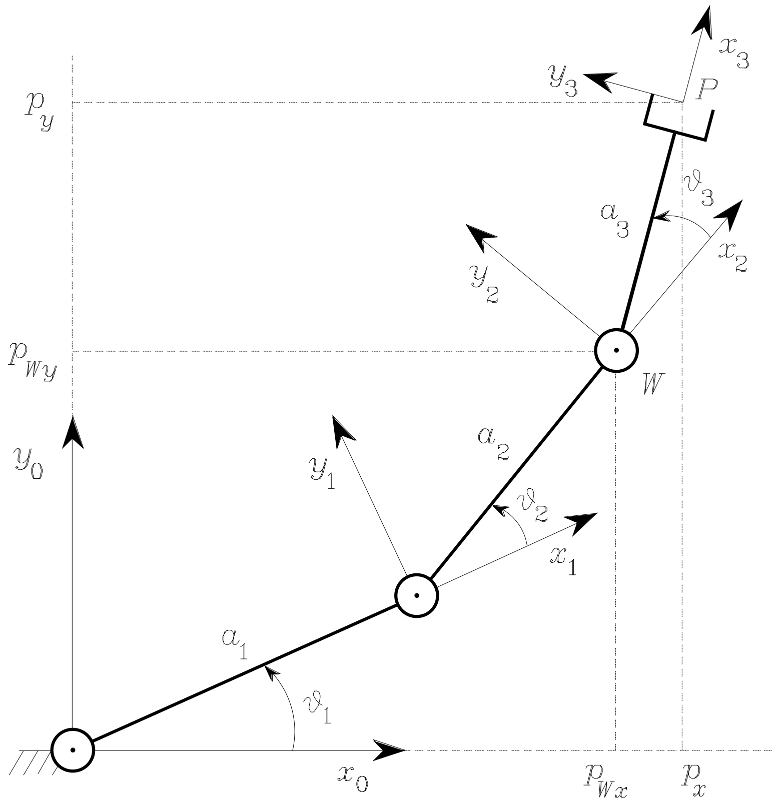
8. On the basis of the parameters in **7**, compute the homogeneous transformation matrices $\boldsymbol{A}_i^{i-1}(q_i)$ for $i = 1, \ldots, n$

9. Compute the homogeneous transformation $\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \ldots \boldsymbol{A}_n^{n-1}$ that yields the position and orientation of Frame n with respect to Frame 0

10. Given $\boldsymbol{T}_0^b$ and $\boldsymbol{T}_e^n$, compute the direct kinematics function as $\boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b \boldsymbol{T}_n^0 \boldsymbol{T}_e^n$ that yields the position and orientation of the end-effector frame with respect to the base frame

Three–link planar arm



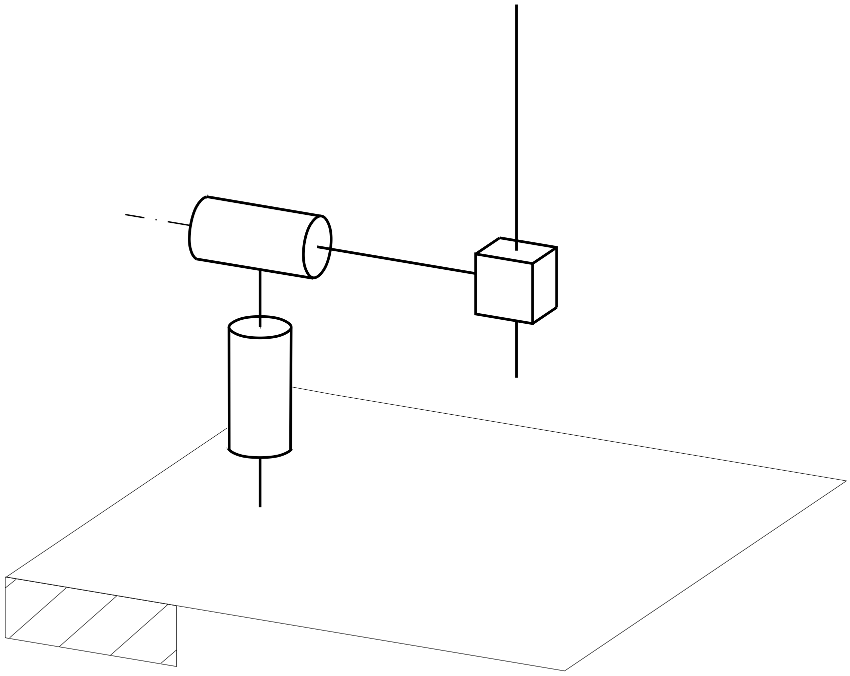


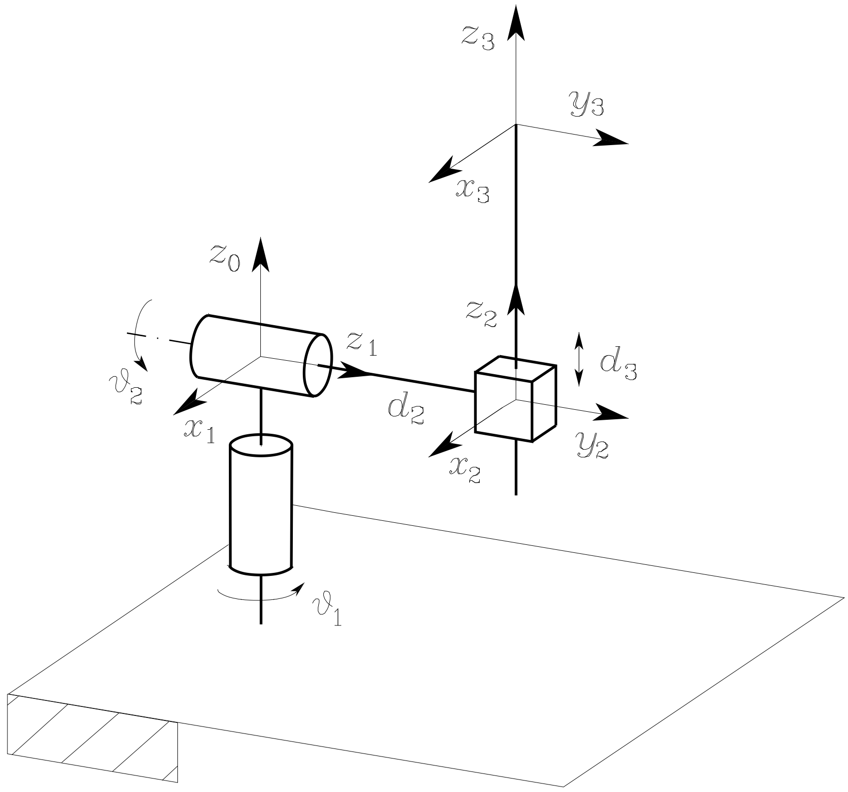
Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3$$

$$\begin{aligned} \boldsymbol{T}_3^0 &= \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 \\ &= \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Spherical arm





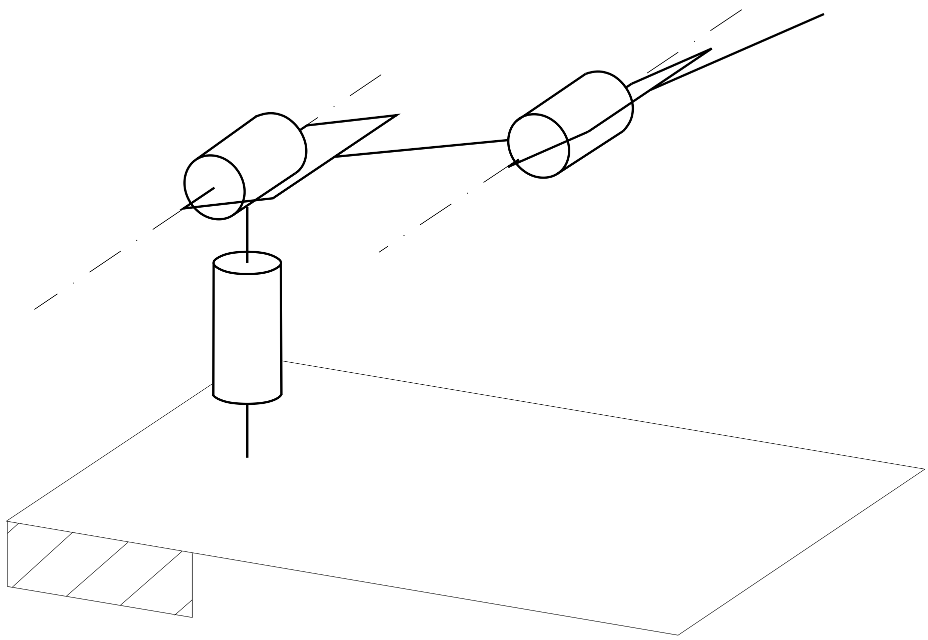
Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0

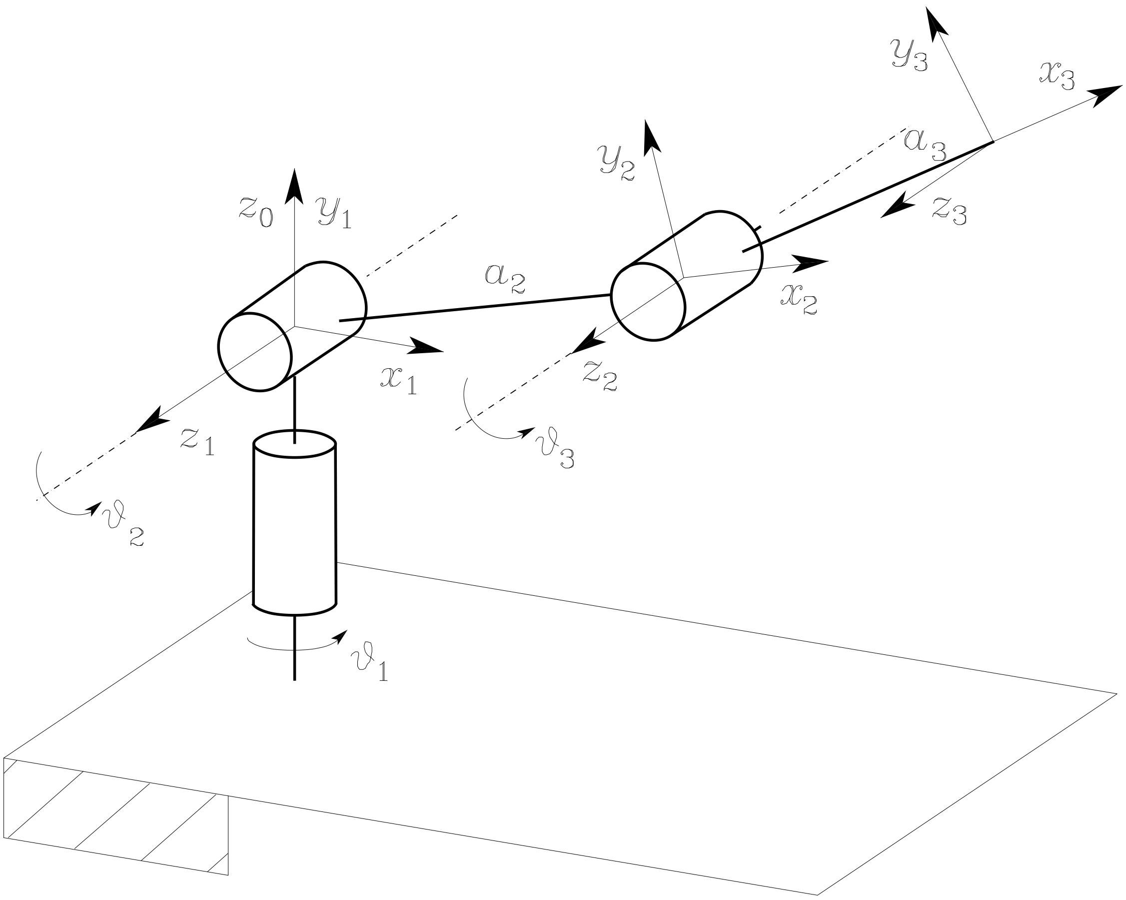
$$\boldsymbol{A}_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{T}_3^0 &= \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 \\ &= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Anthropomorphic arm





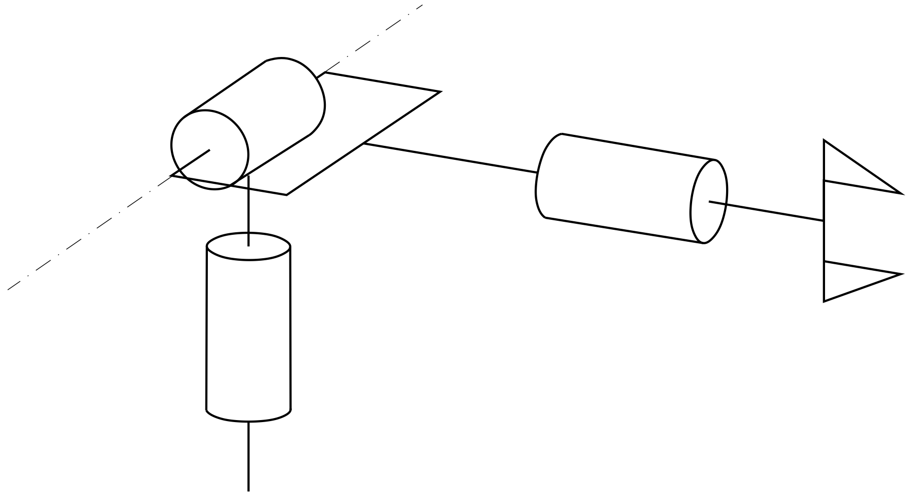
Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

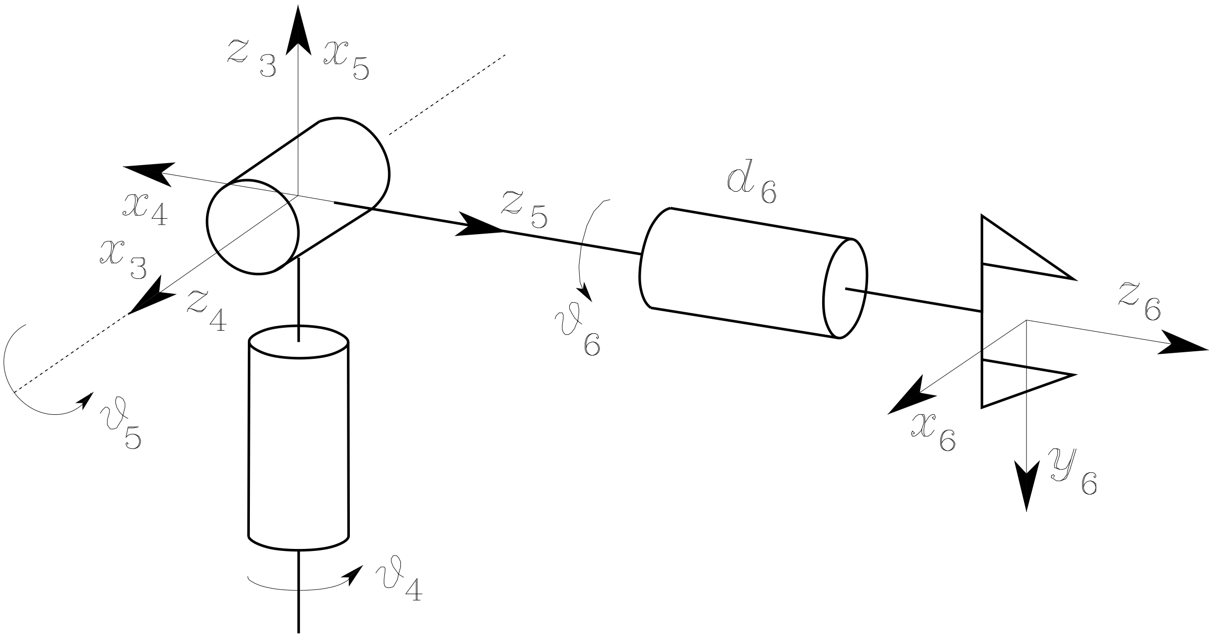
$$\boldsymbol{A}_1^0 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3$$

$$\begin{aligned} \boldsymbol{T}_3^0 &= \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 \\ &= \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Spherical wrist





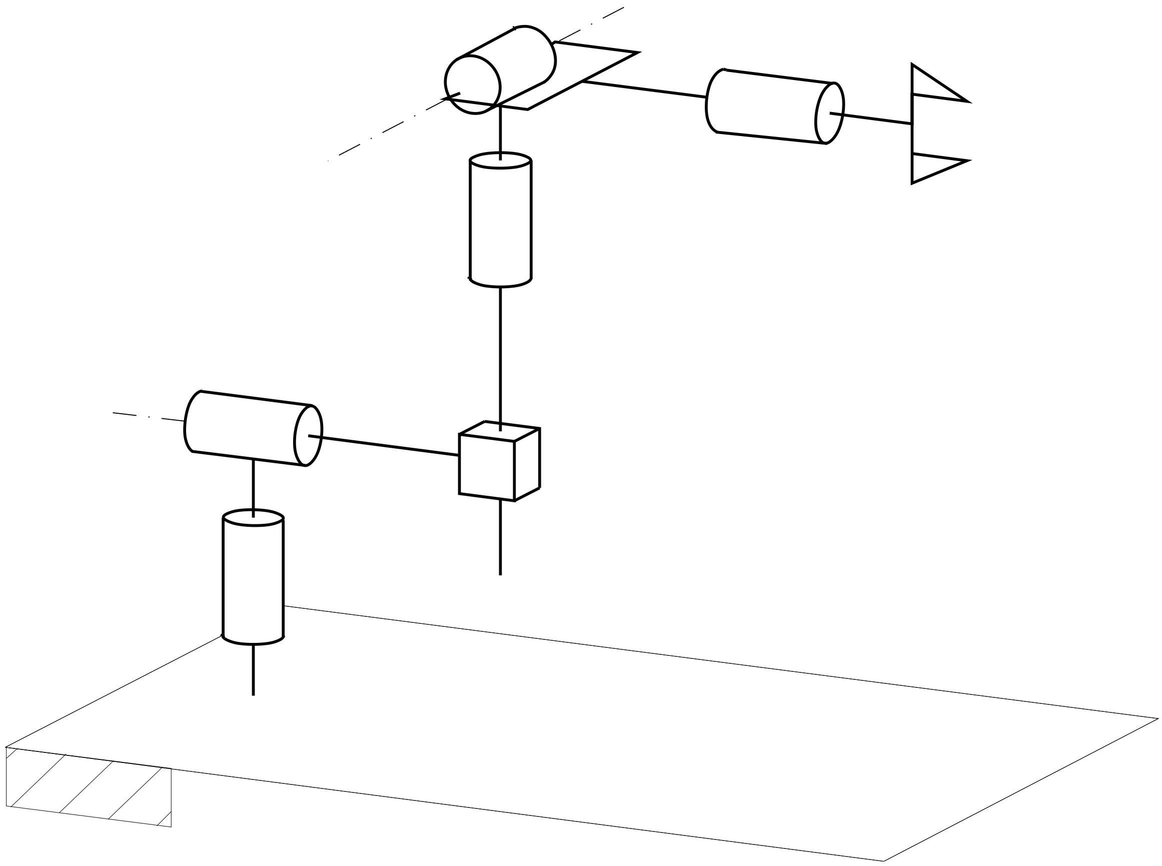
Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

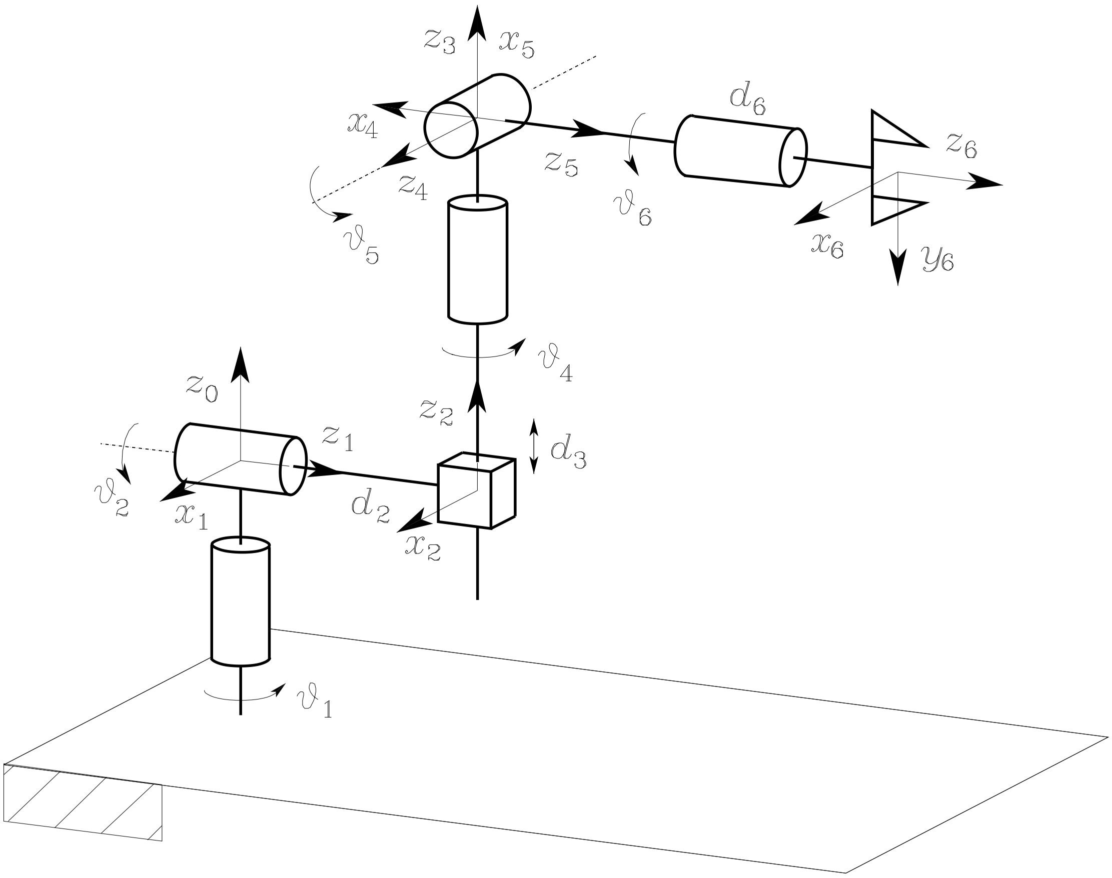
$$\boldsymbol{A}_4^3 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_5^4 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_6^5 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{T}_6^3 &= \boldsymbol{A}_4^3 \boldsymbol{A}_5^4 \boldsymbol{A}_6^5 \\ &= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Stanford manipulator





$$
\boldsymbol{T}_6^0 = \boldsymbol{T}_3^0 \boldsymbol{T}_6^3 = \begin{bmatrix} \boldsymbol{n}^0 & \boldsymbol{s}^0 & \boldsymbol{a}^0 & \boldsymbol{p}^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

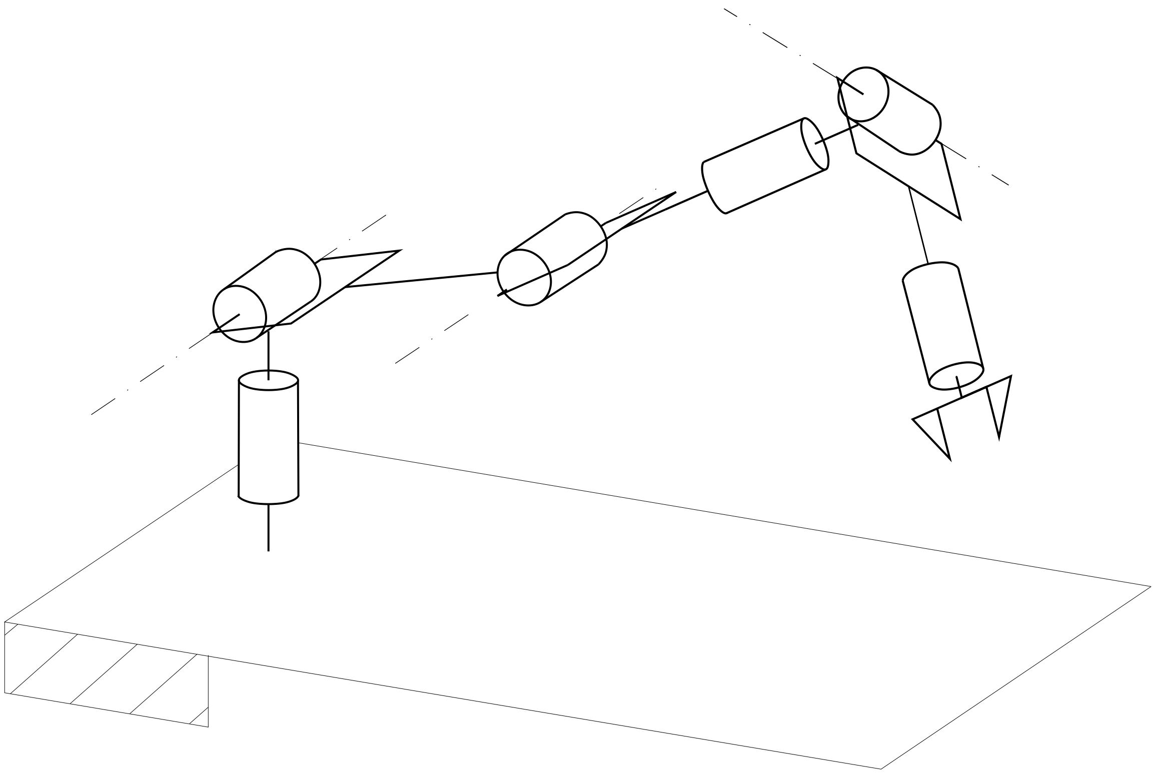
$$\boldsymbol{p}^0 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + \big(c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5\big) d_6 \\ s_1 s_2 d_3 + c_1 d_2 + \big(s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5\big) d_6 \\ c_2 d_3 + (-s_2 c_4 s_5 + c_2 c_5) d_6 \end{bmatrix}$$

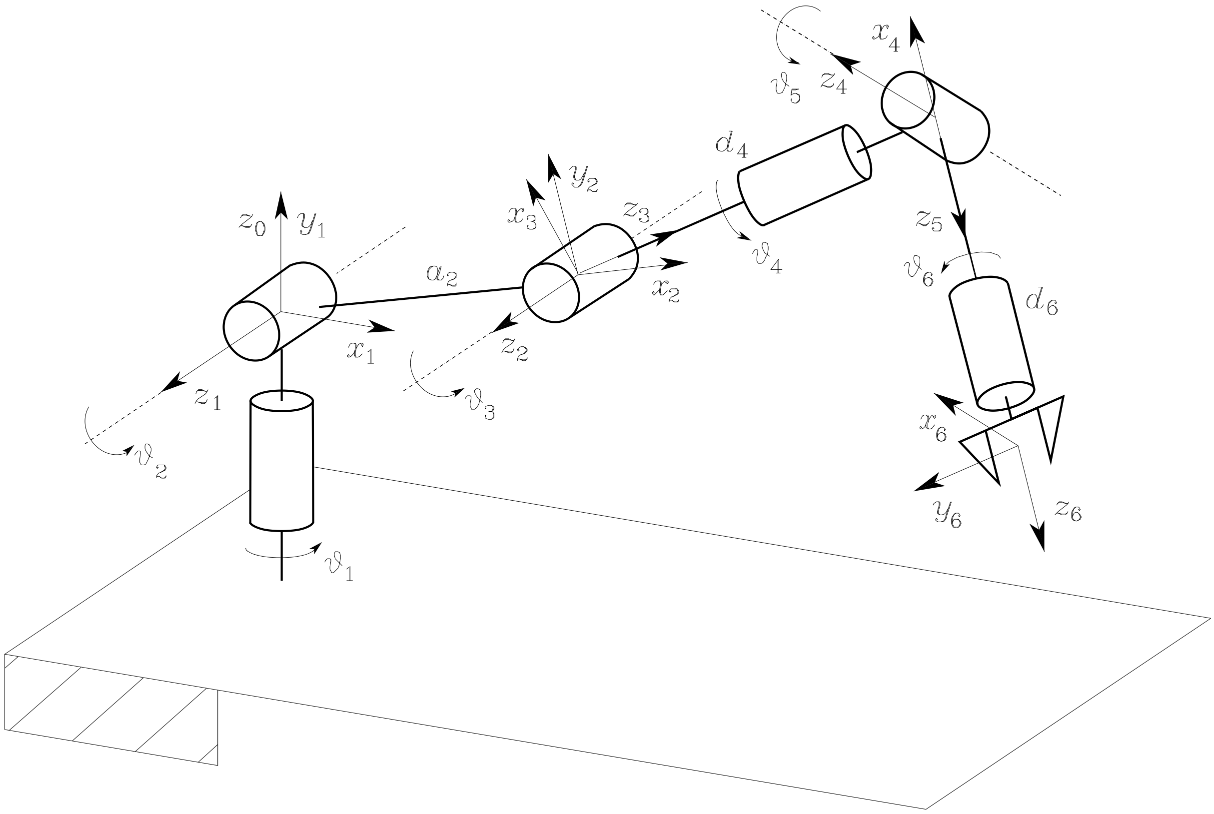
$$\boldsymbol{n}^0 = \begin{bmatrix} c_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) - s_1(s_4 c_5 c_6 + c_4 s_6) \\ s_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) + c_1(s_4 c_5 c_6 + c_4 s_6) \\ -s_2(c_4 c_5 c_6 - s_4 s_6) - c_2 s_5 c_6 \end{bmatrix}$$

$$\boldsymbol{s}^0 = \begin{bmatrix} c_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) - s_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) + c_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_2(c_4 c_5 s_6 + s_4 c_6) + c_2 s_5 s_6 \end{bmatrix}$$

$$\boldsymbol{a}^0 = \begin{bmatrix} c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5 \\ s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

Anthropomorphic arm with spherical wrist





Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	0	$\pi/2$	0	ϑ_3
4	0	$-\pi/2$	d_4	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

$$
\boldsymbol{A}_3^2 = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_4^3 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

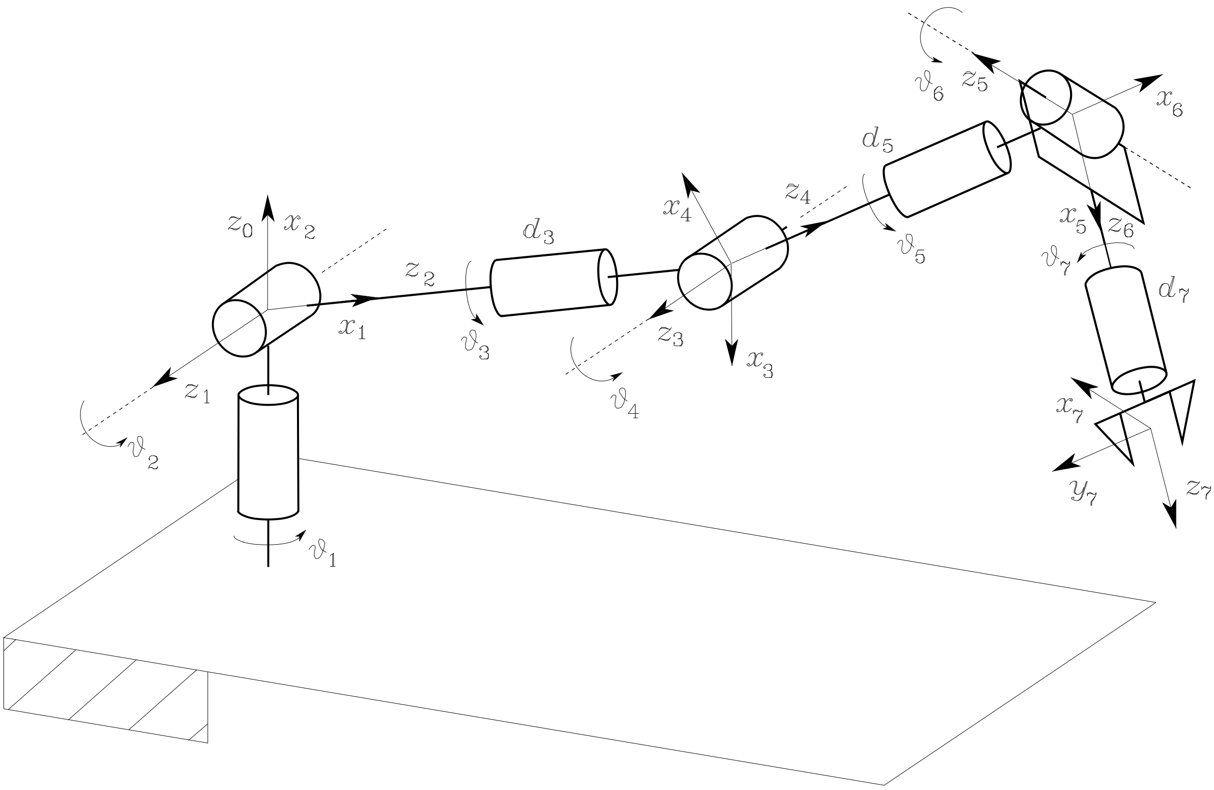
$$
\boldsymbol{p}^0 = \begin{bmatrix} a_2c_1c_2 + d_4c_1s_{23} + d_6\big(c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5\big) \\ a_2s_1c_2 + d_4s_1s_{23} + d_6\big(s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5\big) \\ a_2s_2 - d_4c_{23} + d_6(s_{23}c_4s_5 - c_{23}c_5) \end{bmatrix}
$$

$$
\boldsymbol{n}^0 = \begin{bmatrix} c_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) + s_1(s_4c_5c_6 + c_4s_6) \\ s_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) - c_1(s_4c_5c_6 + c_4s_6) \\ s_{23}(c_4c_5c_6 - s_4s_6) + c_{23}s_5c_6 \end{bmatrix}
$$

$$
\boldsymbol{s}^0 = \begin{bmatrix} c_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) + s_1(-s_4c_5s_6 + c_4c_6) \\ s_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) - c_1(-s_4c_5s_6 + c_4c_6) \\ -s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \end{bmatrix}
$$

$$
\boldsymbol{a}^0 = \begin{bmatrix} c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5 \\ s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5 \\ s_{23}c_4s_5 - c_{23}c_5 \end{bmatrix}
$$

DLR manipulator



Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	0	$\pi/2$	0	ϑ_2
3	0	$\pi/2$	d_3	ϑ_3
4	0	$\pi/2$	0	ϑ_4
5	0	$\pi/2$	d_5	ϑ_5
6	0	$\pi/2$	0	ϑ_6
7	0	0	d_7	ϑ_7

$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & 0 & s_i & 0 \\ s_i & 0 & -c_i & 0 \\ 0 & 1 & 0 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, \dots, 6$$

$$\boldsymbol{A}_7^6 = \begin{bmatrix} c_7 & -s_7 & 0 & 0 \\ s_7 & c_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{p}_7^0 = \begin{bmatrix} d_3 x_{d_3} + d_5 x_{d_5} + d_7 x_{d_7} \\ d_3 y_{d_3} + d_5 y_{d_5} + d_7 y_{d_7} \\ d_3 z_{d_3} + d_5 z_{d_5} + d_7 z_{d_7} \end{bmatrix}$$

$$\begin{aligned}
x_{d_3} &= c_1 s_2 \\
x_{d_5} &= c_1(c_2 c_3 s_4 - s_2 c_4) + s_1 s_3 s_4 \\
x_{d_7} &= c_1(c_2 k_1 + s_2 k_2) + s_1 k_3 \\
y_{d_3} &= s_1 s_2 \\
y_{d_5} &= s_1(c_2 c_3 s_4 - s_2 c_4) - c_1 s_3 s_4 \\
y_{d_7} &= s_1(c_2 k_1 + s_2 k_2) - c_1 k_3 \\
z_{d_3} &= -c_2 \\
z_{d_5} &= c_2 c_4 + s_2 c_3 s_4 \\
z_{d_7} &= s_2(c_3(c_4 c_5 s_6 - s_4 c_6) + s_3 s_5 s_6) - c_2 k_2
\end{aligned}$$

$$\begin{aligned}
k_1 &= c_3(c_4 c_5 s_6 - s_4 c_6) + s_3 s_5 s_6 \\
k_2 &= s_4 c_5 s_6 + c_4 c_6 \\
k_3 &= s_3(c_4 c_5 s_6 - s_4 c_6) - c_3 s_5 s_6
\end{aligned}$$

$$\boldsymbol{n}_7^0 = \begin{bmatrix} ((x_a c_5 + x_c s_5)c_6 + x_b s_6)c_7 + (x_a s_5 - x_c c_5)s_7 \\ ((y_a c_5 + y_c s_5)c_6 + y_b s_6)c_7 + (y_a s_5 - y_c c_5)s_7 \\ (z_a c_6 + z_c s_6)c_7 + z_b s_7 \end{bmatrix}$$

$$\boldsymbol{s}_7^0 = \begin{bmatrix} -((x_a c_5 + x_c s_5)c_6 + x_b s_6)s_7 + (x_a s_5 - x_c c_5)c_7 \\ -((y_a c_5 + y_c s_5)c_6 + y_b s_6)s_7 + (y_a s_5 - y_c c_5)c_7 \\ -(z_a c_6 + z_c s_6)s_7 + z_b c_7 \\ \end{bmatrix}$$

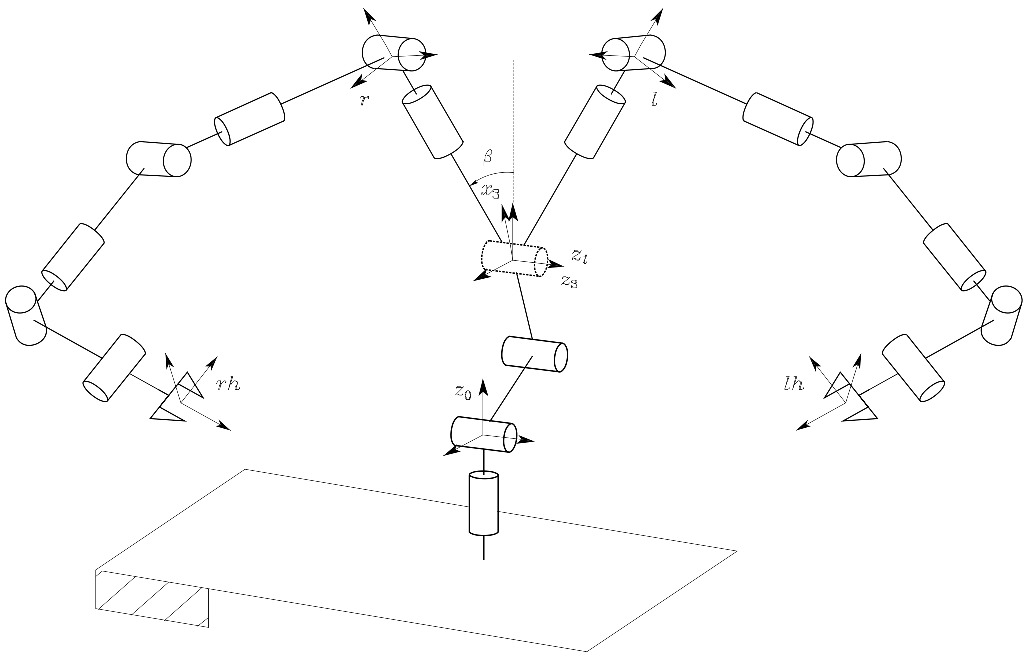
$$\boldsymbol{a}_7^0 = \begin{bmatrix} (x_a c_5 + x_c s_5)s_6 - x_b c_6 \\ (y_a c_5 + y_c s_5)s_6 - y_b c_6 \\ z_a s_6 - z_c c_6 \end{bmatrix}$$

$$\begin{aligned} x_a &= (c_1 c_2 c_3 + s_1 s_3)c_4 + c_1 s_2 s_4 \\ x_b &= (c_1 c_2 c_3 + s_1 s_3)s_4 - c_1 s_2 c_4 \\ x_c &= c_1 c_2 s_3 - s_1 c_3 \\ y_a &= (s_1 c_2 c_3 - c_1 s_3)c_4 + s_1 s_2 s_4 \\ y_b &= (s_1 c_2 c_3 - c_1 s_3)s_4 - s_1 s_2 c_4 \\ y_c &= s_1 c_2 s_3 + c_1 c_3 \\ z_a &= (s_2 c_3 c_4 - c_2 s_4)c_5 + s_2 s_3 s_5 \\ z_b &= (s_2 c_3 s_4 + c_2 c_4)s_5 - s_2 s_3 c_5 \\ z_c &= s_2 c_3 s_4 + c_2 c_4 \end{aligned}$$

$\star$ se $\alpha_7 = \pi/2$

$$\boldsymbol{A}_7^6 = \begin{bmatrix} c_7 & 0 & s_7 & a_7 c_7 \\ s_7 & 0 & -c_7 & a_7 s_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Humanoid manipulator



- Arms consisting of two DLR manipulators ($\alpha_7 = \pi/2$)
- Connecting device between the end-effector of the anthropomorphic torso and the base frames of the two manipulators
  - ⋆ permits keeping the 'chest' of the humanoid manipulator always orthogonal to the ground ($\vartheta_4 = -\vartheta_2 - \vartheta_3$)
- Direct kinematics

$$\boldsymbol{T}^0_{rh} = \boldsymbol{T}^0_3\, \boldsymbol{T}^3_t\, \boldsymbol{T}^t_r \boldsymbol{T}^r_{rh}$$
$$\boldsymbol{T}^0_{lh} = \boldsymbol{T}^0_3\, \boldsymbol{T}^3_t\, \boldsymbol{T}^t_l \boldsymbol{T}^l_{lh}$$

$$\boldsymbol{T}^3_t = \begin{bmatrix} c_{23} & s_{23} & 0 & 0 \\ -s_{23} & c_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ⋆ $\boldsymbol{T}_3^0$ like for anthropomorphic manipulator
- ⋆ $\boldsymbol{T}_r^t$ and $\boldsymbol{T}_l^t$ depend on β
- ⋆ $\boldsymbol{T}_{rh}^r$ and $\boldsymbol{T}_{lh}^l$ like for DLR manipulator

JOINT SPACE AND OPERATIONAL SPACE

- Joint space

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

  - ⋆ $q_i = \vartheta_i$ (revolute joint)
  - ⋆ $q_i = d_i$ (prismatic joint)

- Operational space

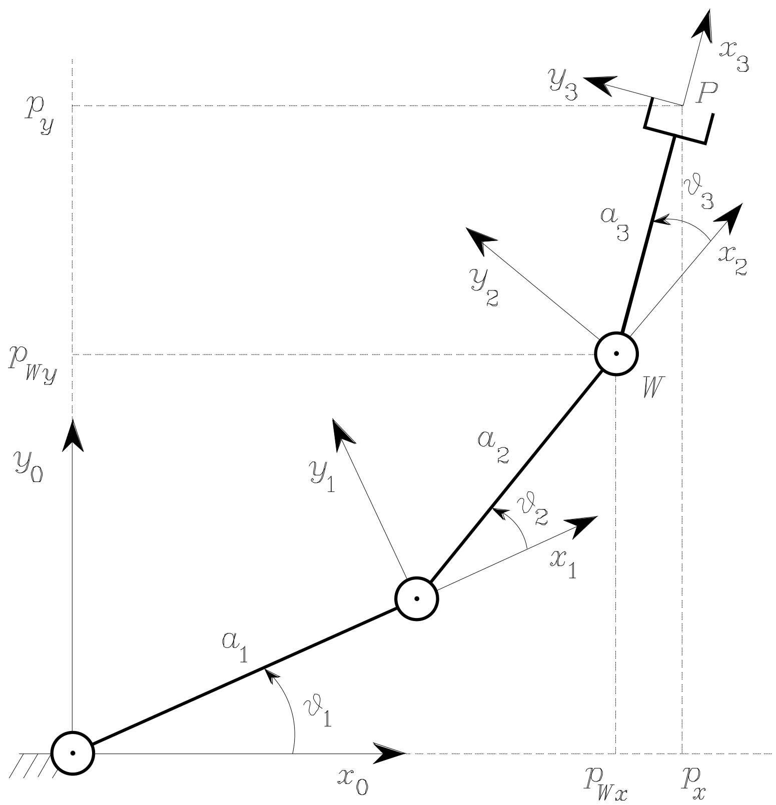
$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{\phi} \end{bmatrix}$$

  - ⋆ $\boldsymbol{p}$ (position)
  - ⋆ $\boldsymbol{\phi}$ (orientation)

- Direct kinematics equation

$$\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q})$$

- Example



$$\boldsymbol{x} = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

Workspace

- *Reachable* workspace

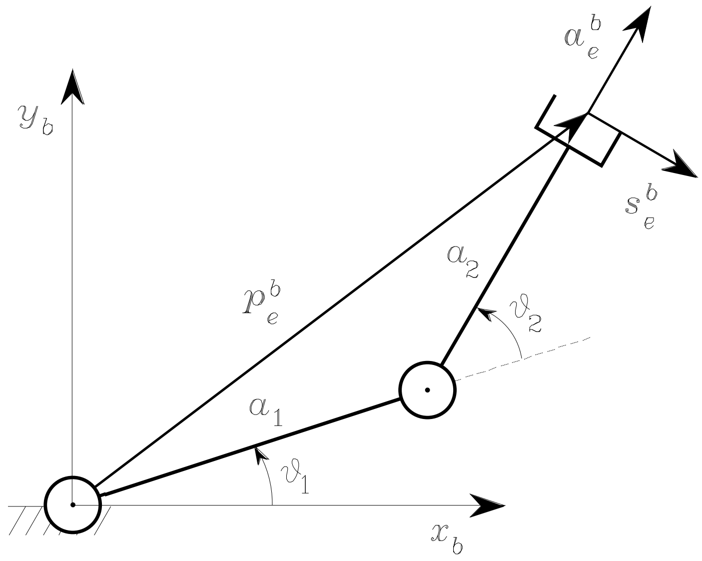
$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) \qquad q_{im} \leq q_i \leq q_{iM} \quad i = 1, \ldots, n$$

  ⋆ surface elements of planar, spherical, toroidal and cylindrical type

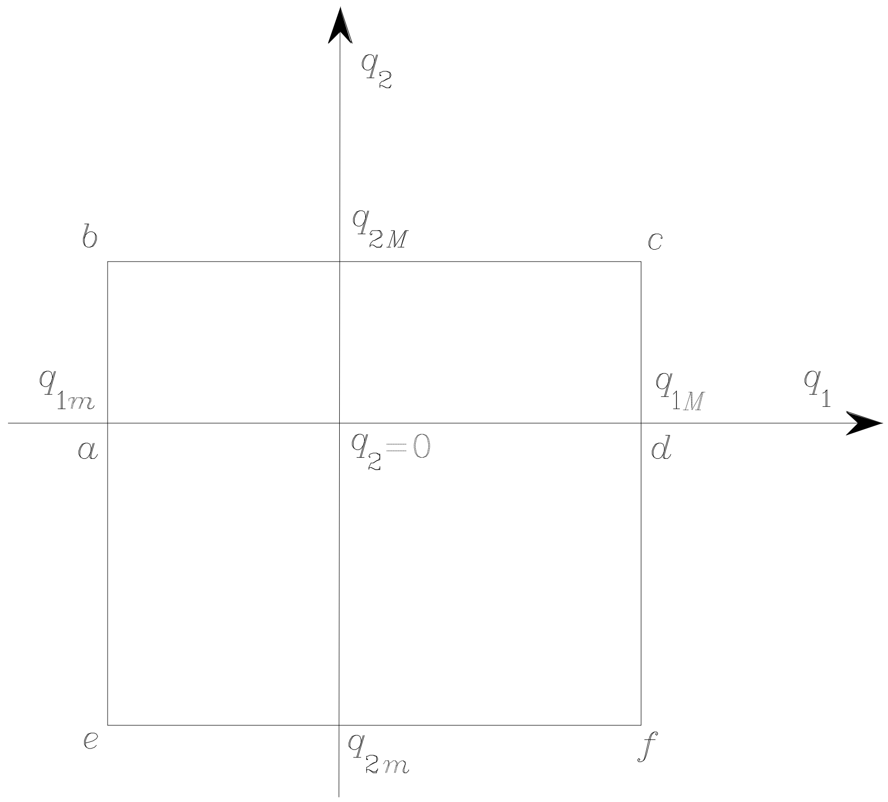
- *Dexterous* workspace

  ⋆ different orientations

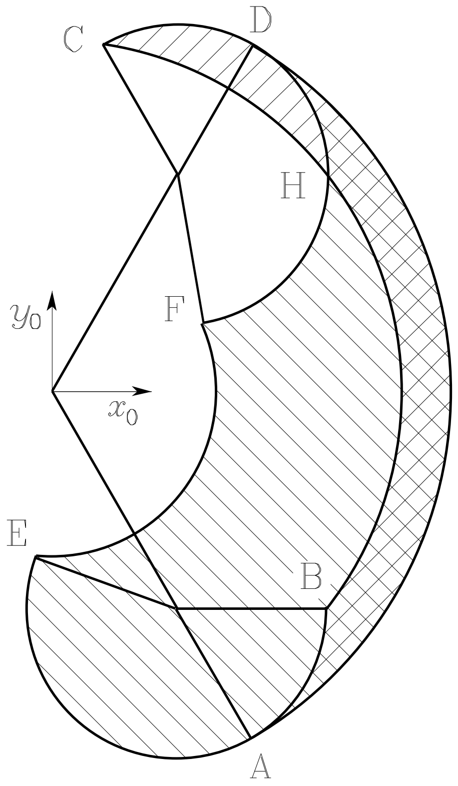
- Example



- ⋆ admissible configurations



⋆ workspace



- Accuracy
  - $\star$ deviation between the position reached in the assigned posture and the position computed via direct kinematics
  - $\star$ typical values: $(0.2, 1)$ mm
- Repeatability
  - $\star$ measure of manipulator's ability to return to a previously reached position
  - $\star$ typical values: $(0.02, 0.2)$ mm
- Kinematic redundancy
  - $\star$ $m < n$ (intrinsic)
  - $\star$ $r < m = n$ (functional)

KINEMATIC CALIBRATION

- Accurate estimates of DH parameters to improve manipulator accuracy

- Direct kinematics equation as a function of all parameters

$$\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{d}, \boldsymbol{\vartheta})$$

$\boldsymbol{x}_m$ measured pose

$\boldsymbol{x}_n$ nominal pose (fixed parameters + joint variables)

$$\begin{aligned}\Delta\boldsymbol{x} &= \frac{\partial\boldsymbol{k}}{\partial\boldsymbol{a}}\Delta\boldsymbol{a} + \frac{\partial\boldsymbol{k}}{\partial\boldsymbol{\alpha}}\Delta\boldsymbol{\alpha} + \frac{\partial\boldsymbol{k}}{\partial\boldsymbol{d}}\Delta\boldsymbol{d} + \frac{\partial\boldsymbol{k}}{\partial\boldsymbol{\vartheta}}\Delta\boldsymbol{\vartheta}\\ &= \boldsymbol{\Phi}(\boldsymbol{\zeta}_n)\Delta\boldsymbol{\zeta}\end{aligned}$$

- $\star$ l measurements ($lm \gg 4n$)

$$\Delta\bar{\boldsymbol{x}} = \begin{bmatrix} \Delta\boldsymbol{x}_1 \\ \vdots \\ \Delta\boldsymbol{x}_l \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1 \\ \vdots \\ \boldsymbol{\Phi}_l \end{bmatrix} \Delta\boldsymbol{\zeta} = \bar{\boldsymbol{\Phi}}\Delta\boldsymbol{\zeta}$$

- Solution

$$\Delta\boldsymbol{\zeta} = (\bar{\boldsymbol{\Phi}}^T\bar{\boldsymbol{\Phi}})^{-1}\bar{\boldsymbol{\Phi}}^T\Delta\bar{\boldsymbol{x}}$$

$$\boldsymbol{\zeta}' = \boldsymbol{\zeta}_n + \Delta\boldsymbol{\zeta}$$

...until $\Delta\boldsymbol{\zeta}$ converges

- $\star$ more accurate estimates of fixed parameters
- $\star$ corrections on transducers measurements

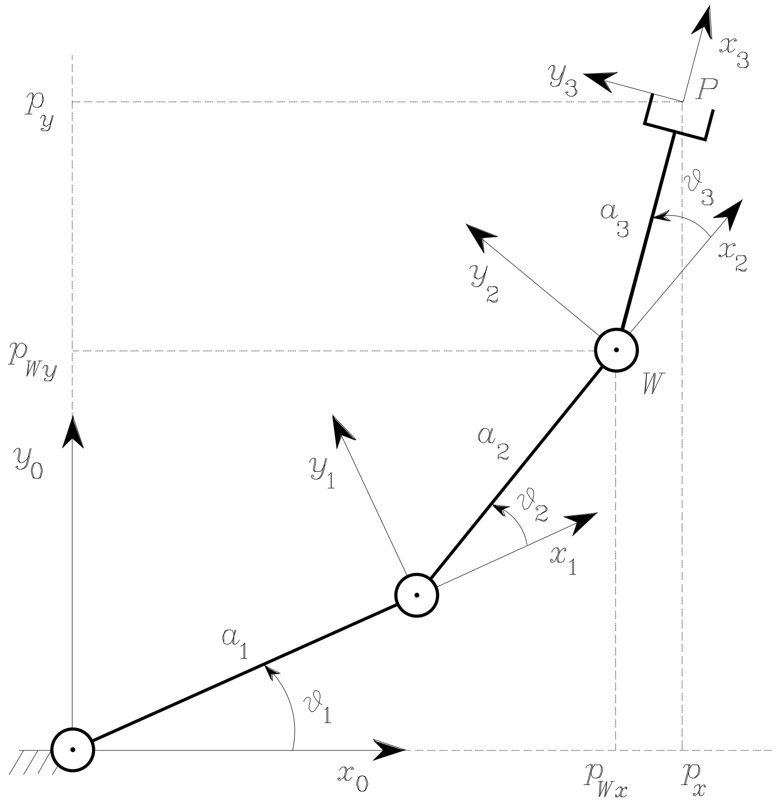
Start-up

- (*Home*) reference posture

INVERSE KINEMATICS PROBLEM

- Direct kinematics
  - $\boldsymbol{q} \quad \Longrightarrow \quad \boldsymbol{T}$
  - $\boldsymbol{q} \quad \Longrightarrow \quad \boldsymbol{x}$
- Inverse kinematics
  - $\boldsymbol{T} \quad \Longrightarrow \quad \boldsymbol{q}$
  - $\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{q}$
- Complexity
  - closed-form solution
  - multiple solutions
  - infinite solutions
  - no admissible solutions
- Intuition
  - algebraic
  - geometric
- Numerical techniques

Solution of three–link planar arm



- Algebraic solution

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

$$\begin{aligned} p_{Wx} &= p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12} \\ p_{Wy} &= p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12} \end{aligned}$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2}$$

$$s_2 = \pm\sqrt{1 - c_2^2}$$

$$\vartheta_2 = \mathrm{Atan2}(s_2, c_2)$$

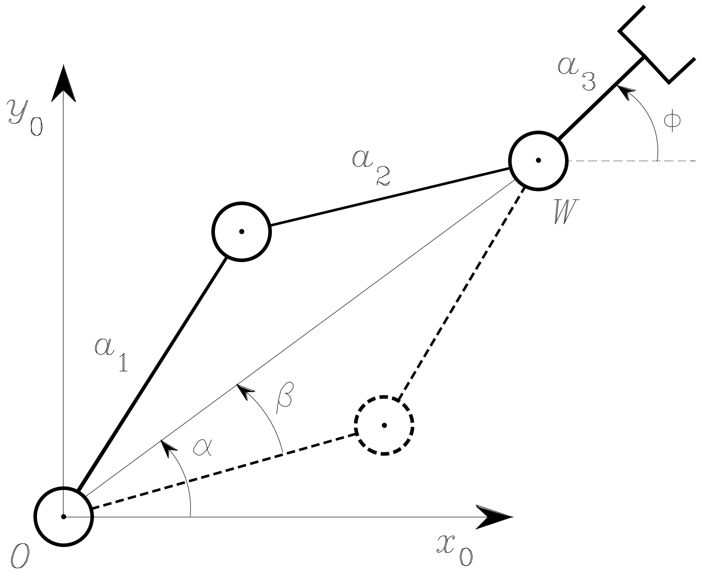
$$s_1 = \frac{(a_1 + a_2c_2)p_{Wy} - a_2s_2p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2c_2)p_{Wx} + a_2s_2p_{Wy}}{p_{Wy}^2 + p_{Wy}^2}$$

$$\vartheta_1 = \mathrm{Atan2}(s_1, c_1)$$

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2$$

- Geometric solution



$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2}.$$

$$\vartheta_2 = \cos^{-1}(c_2)$$

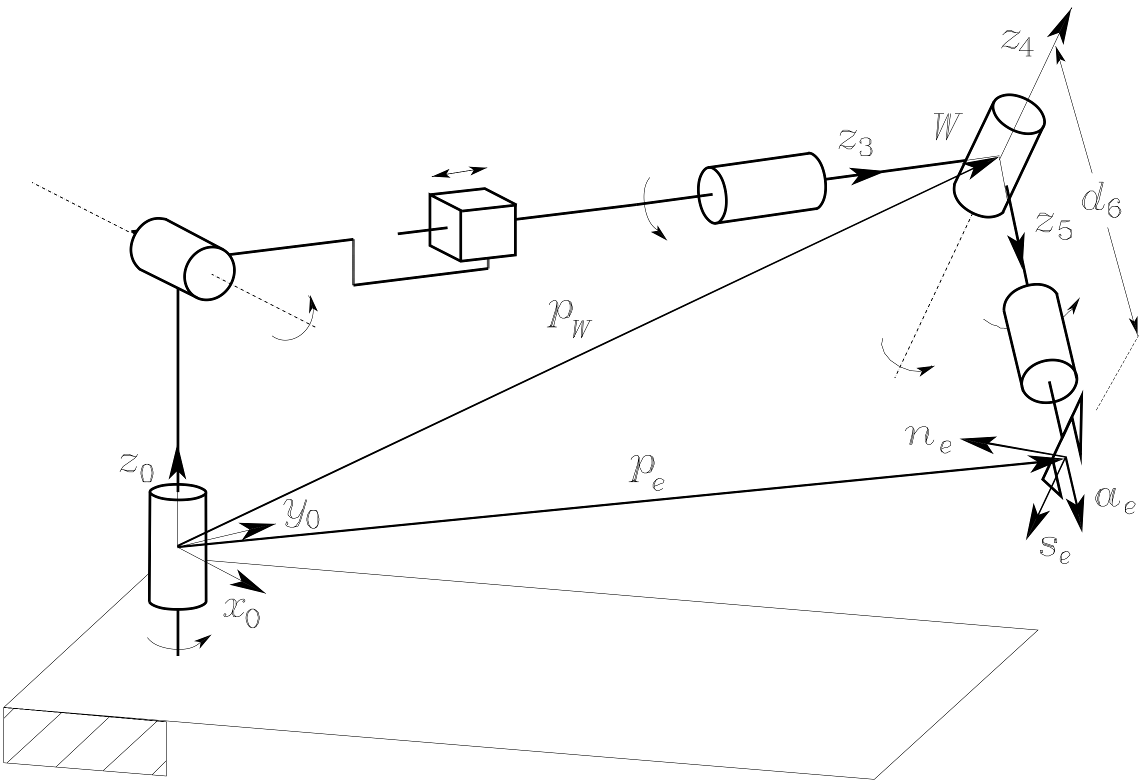
$$\alpha = \mathrm{Atan2}(p_{Wy}, p_{Wx})$$

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2c_2$$

$$\beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\vartheta_1 = \alpha \pm \beta$$

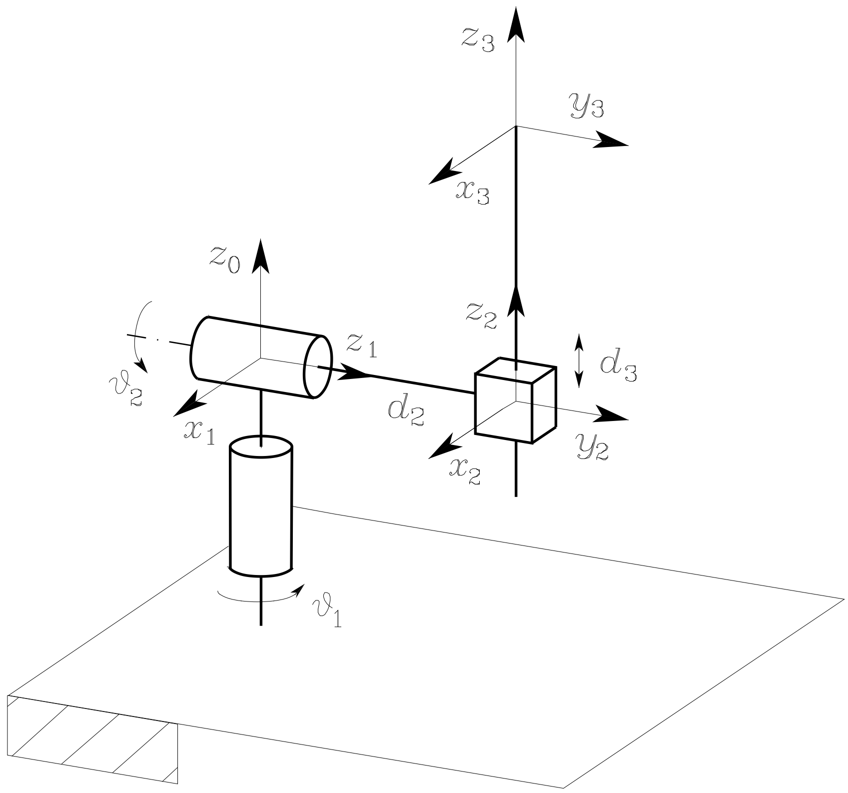
Solution of manipulators with spherical wrist



$$\boldsymbol{p}_W = \boldsymbol{p} - d_6\boldsymbol{a}$$

- Decoupled solution
  - ⋆ compute wrist position $\boldsymbol{p}_W(q_1, q_2, q_3)$
  - ⋆ solve inverse kinematics for (q_1, q_2, q_3)
  - ⋆ compute $\boldsymbol{R}_3^0(q_1, q_2, q_3)$
  - ⋆ compute $\boldsymbol{R}_6^3(\vartheta_4, \vartheta_5, \vartheta_6) = \boldsymbol{R}_3^{0T}\boldsymbol{R}$
  - ⋆ solve inverse kinematics for orientation $(\vartheta_4, \vartheta_5, \vartheta_6)$

Solution of spherical arm



$$(\boldsymbol{A}_1^0)^{-1}\boldsymbol{T}_3^0 = \boldsymbol{A}_2^1\boldsymbol{A}_3^2$$

$$\boldsymbol{p}_W^1 = \begin{bmatrix} p_{Wx}c_1 + p_{Wy}s_1 \\ -p_{Wz} \\ -p_{Wx}s_1 + p_{Wy}c_1 \end{bmatrix} = \begin{bmatrix} d_3 s_2 \\ -d_3 c_2 \\ d_2 \end{bmatrix}$$

$$c_1 = \frac{1-t^2}{1+t^2} \qquad s_1 = \frac{2t}{1+t^2}$$

$$(d_2 + p_{Wy})t^2 + 2p_{Wx}t + d_2 - p_{Wy} = 0$$

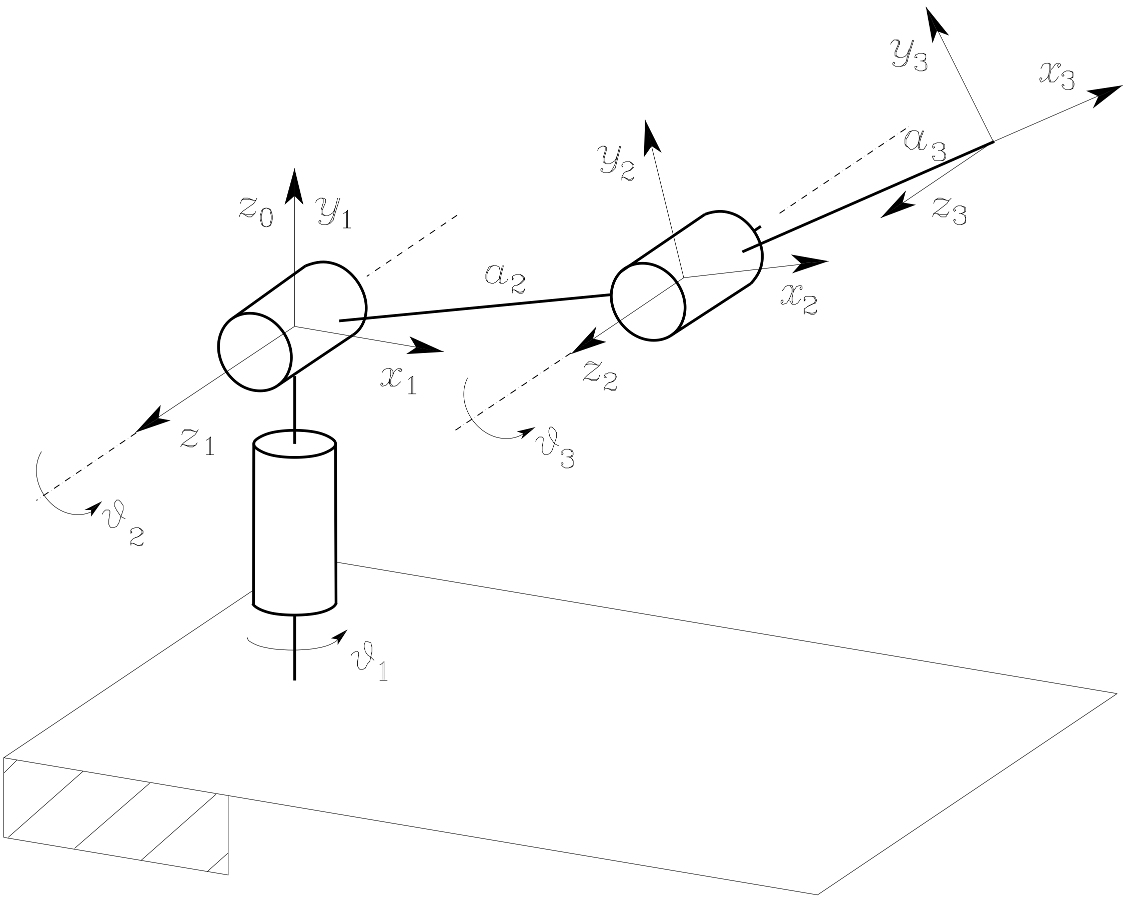
$$\vartheta_1 = 2\mathrm{Atan2}\Big(-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2},\ d_2 + p_{Wy}\Big)$$

$$\frac{p_{Wx}c_1 + p_{Wy}s_1}{-p_{Wz}} = \frac{d_3 s_2}{-d_3 c_2}$$

$$\vartheta_2 = \mathrm{Atan2}(p_{Wx}c_1 + p_{Wy}s_1, p_{Wz})$$

$$d_3 = \sqrt{(p_{Wx}c_1 + p_{Wy}s_1)^2 + p_{Wz}^2}$$

Solution of anthropomorphic arm



$$
\begin{aligned}
p_{Wx} &= c_1(a_2c_2 + a_3c_{23}) \\
p_{Wy} &= s_1(a_2c_2 + a_3c_{23}) \\
p_{Wz} &= a_2s_2 + a_3s_{23}
\end{aligned}
$$

$$
\begin{aligned}
c_3 &= \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3} \\
s_3 &= \pm\sqrt{1 - c_3^2}
\end{aligned}
$$

$$
\vartheta_3 = \mathrm{Atan2}(s_3, c_3)
$$

$$
\Downarrow
$$

$$\vartheta_{3,I} \in [-\pi, \pi]$$
$$\vartheta_{3,II} = -\vartheta_{3,I}$$

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3c_3) + p_{Wz}a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3}$$

$$\vartheta_2 = \mathrm{Atan2}(s_2, c_2)$$

$$\Downarrow$$

⋆ for $s_3^+ = \sqrt{1 - c_3^2}$:

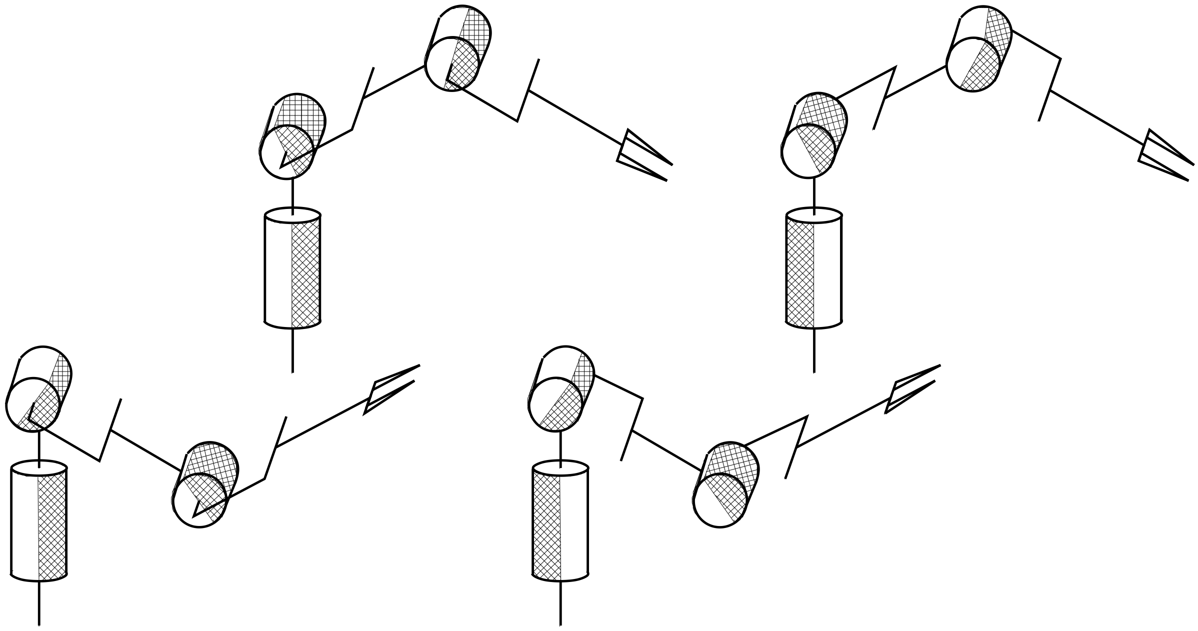
$$\vartheta_{2,I} = \mathrm{Atan2}\Big((a_2 + a_3c_3)p_{Wz} - a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},$$
$$(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big)$$
$$\vartheta_{2,II} = \mathrm{Atan2}\Big((a_2 + a_3c_3)p_{Wz} + a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},$$
$$-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big)$$

⋆ for $s_3^- = -\sqrt{1 - c_3^2}$:

$$\vartheta_{2,III} = \mathrm{Atan2}\Big((a_2 + a_3c_3)p_{Wz} - a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},$$
$$(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big)$$
$$\vartheta_{2,IV} = \mathrm{Atan2}\Big((a_2 + a_3c_3)p_{Wz} + a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},$$
$$-(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big)$$

$$
\begin{aligned}
\vartheta_{1,I} &= \mathrm{Atan2}(p_{Wy}, p_{Wx}) \\
\vartheta_{1,II} &= \mathrm{Atan2}(-p_{Wy}, -p_{Wx})
\end{aligned}
$$

- Four admissible configurations

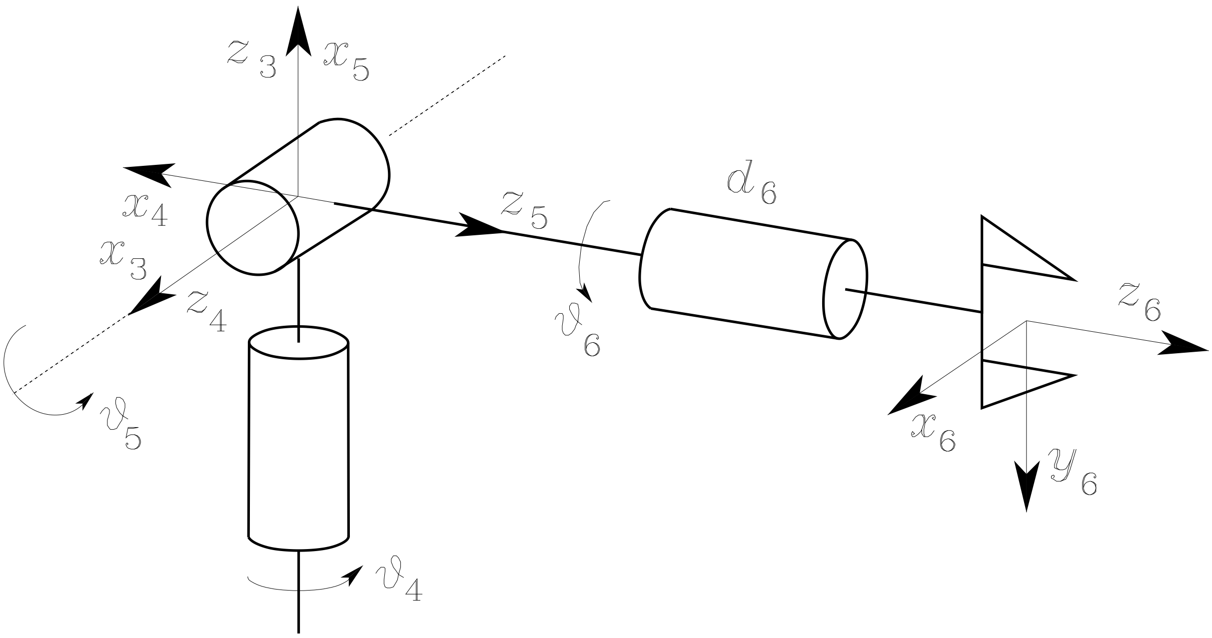


$$
(\vartheta_{1,I}, \vartheta_{2,I}, \vartheta_{3,I}) \quad (\vartheta_{1,I}, \vartheta_{2,III}, \vartheta_{3,II})
$$
$$
(\vartheta_{1,II}, \vartheta_{2,II}, \vartheta_{3,I}) \quad (\vartheta_{1,II}, \vartheta_{2,IV}, \vartheta_{3,II})
$$

⋆ infinite solutions (*kinematic singularity*)

$$
p_{Wx} = 0 \qquad p_{Wy} = 0
$$

Solution of spherical wrist



$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

$$\vartheta_4 = \mathrm{Atan2}(a_y^3, a_x^3)$$
$$\vartheta_5 = \mathrm{Atan2}\Big(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\Big)$$
$$\vartheta_6 = \mathrm{Atan2}(s_z^3, -n_z^3)$$

$$\vartheta_4 = \mathrm{Atan2}(-a_y^3, -a_x^3)$$
$$\vartheta_5 = \mathrm{Atan2}\Big(-\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\Big)$$
$$\vartheta_6 = \mathrm{Atan2}(-s_z^3, n_z^3)$$