Diff. Eq. App.(110.302) Midterm 1 Solutions

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Problem 1.[$3 \times 15 = 45$ points] Solve the following differential equations. (Hint: Identify the types of the equations first.)

(1.1) $y' = \frac{1}{2}y - \frac{x}{2}y^3$

Solution. This is a Bernoulli equation with $n = 3$. We first check that $y = 0$ is a solution. If y is not the zero function, we can let $u = u(x) = y^{1-n} = y^{-2}$.

$$u' = -2y^{-3}y' = -2y^{-3}(\frac{1}{2}y - \frac{x}{2}y^3) = -y^{-2} + x = -u + x.$$

This means $u = u(x)$ is a solution to the first order linear equation $u' + u = x$. The integrating facto is $v = e^{\int 1dx} = e^x$. Multiplying by the integrating factor, we have $(uv)' = xe^x$. So

$$y^{-2} = u(x) = v^{-1} \int xe^x dx = e^{-x}(xe^x - e^x + c) = x - 1 + ce^{-x}$$

So $y = \pm 1/\sqrt{x - 1 + ce^{-x}}$, where c is an arbitrary constant, or $y = 0$.

Remark *The zero solution is not required. But if one noticed this solution, 1 extra point was assigned.*

(1.2) $y' = \frac{y^3}{x^3 + xy^2}$

Solution. This is a homogeneous equation. Let $u = y/x$, then

$$y' = \frac{(y/x)^3}{1 + (y/x)^2} = \frac{u^3}{1 + u^2}.$$

On the other hand, $y = ux$ implies that $y' = u'x + u$. So we have

$$u'x + u = \frac{u^3}{1 + u^2} \Longleftrightarrow x\frac{du}{dx} = -\frac{u}{1 + u^2}.$$

The last equation is separable and we have (if $u \neq 0$)

$$(\frac{1}{u} + u)du = -\frac{dx}{x} \Longrightarrow \ln|u| + \frac{u^2}{2} = -\ln|x| + c.$$

Notice that $u = 0 \iff y = 0$ is also a solution.

Remark *The zero solution is not required. But if one noticed this solution, 1 extra point was assigned. You may simplify your answer further, but not required.*

(1.3) $y' = -\frac{2y^2x+2y}{3yx^2+4x}$ **Solution.** This is not homogeneous or Bernoulli or linear. The only remaining equation we learnt is exact equation. Rewrite the differential equation as

$$M + Ny' = (2y^2x + 2y) + (3yx^2 + 4x)y' = 0.$$

Compute $M_y - N_x = (4yx + 2) - (6yx + 4) = -2yx - 2$, and hence $\frac{M_y - N_x}{M} = \frac{-1}{y}$ is a function on y. So let the integrating factor u to be a function on y, i.e., $u = u(y)$. (Assume you do not remember the differential equation for u.) We want $uM + uNy' = 0$ to be exact, that means

$$(uM)_y = (uN)_x \iff u_yM + uM_y = uN_x \implies \frac{du}{dy} = u\frac{N_x - M_y}{M} = \frac{-u}{y}.$$

So we can choose $u = y$.

Now we want to find $\phi(x, y)$ such that $\phi_x = uM = 2y^3x + 2y^2$ and that $\phi_y = uN = 3y^2x^2 + 4xy$. From $\phi_x = 2y^3x + 2y^2$ we see

$$\phi(x, y) = \int 2y^3x + 2y^2 dx = y^3x^2 + 2y^2x + g(y).$$

In order to get $g(y)$, we take y-derivative to the above equation to obtain

$$3y^2x^2 + 4xy = \phi_y = 3y^2x^2 + 4yx + g_y.$$

So $g_y = 0$ which means $g(y)$ is a constant function $g(y) = c$. So $\phi(x, y) = y^3x^2 + 2y^2x + c$ and the solution to the original differential equation is

$$y^3x^2 + 2y^2x = c.$$

Problem 2.[10 points] Find the equilibrium solutions and classify each one as stable or unstable.

$$dy/dt = y^3 - y.$$

Solution. From $y^3 - y = 0$ we have $y = -1, 0, 1$. These are the equilibrium solutions. $d(y^3 - y)/dy = 3y^2 - 1$. At the equilibrium $y = -1$, the derivative is $2 > 0$, so $y = -1$ is not stable. For the same reason, $y = 1$ is unstable. At the equilibrium $y = 0$, the derivative is $-1 < 0$, so $y = 0$ is stable.

Problem 3.[$3 \times 5 = 15$ points] Solve the following differential equations.

(3.1) $y'' - 4y' + 3y = 0$.

Solution. The characteristic equation $T^2 - 4T + 3 = 0$ has distinct real roots $T = 1$ and $T = 3$. So the solutions are

$$y(t) = c_1 e^t + c_2 e^{3t}$$

where c_1 and c_2 are arbitrary constants.

(3.2) $y'' - 4y' + 4y = 0$.

Solution. The characteristic equation $T^2 - 4T + 4 = 0$ has only one real root $T = 2$, repeated twice. So the solutions are

$$y(t) = (c_1 + c_2 t)e^{2t}$$

where c_1 and c_2 are arbitrary constants.

(3.3) $y'' - 4y' + 5y = 0$.

Solution. The characteristic equation $T^2 - 4T + 5 = 0$ has complex roots $T = 2 \pm i$. So the solutions are

$$y(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$$

where c_1 and c_2 are arbitrary constants.

Problem 4.[20 points] Solve the differential equation

$$t^2y'' - ty' + y = t^2.$$

Solution. This is a non-homogeneous equation. The first step is to solve the corresponding homogeneous equation $t^2y'' - ty' + y = 0$, which is an Euler equation. Let $u = \ln t$, either by the formula if remember or by direct computation we see that, as a function on u, the differential equation for $y = y(u)$ becomes $\frac{d^2y}{du^2} - 2\frac{dy}{du} + y = 0$. The characteristic equation has only one repeated root $T = 1$. So the solutions (to the homogeneous equation, not the original equation) are

$$y = (c_1 + c_2u)e^u = (c_1 + c_2 \ln t)t.$$

The second step, find a particular solution to the non-homogeneous equation. If use the method of undetermined coefficients, we try polynomials of degree at most 2 (Why?). By computation we found $y = t^2$ is a solution. So the solutions of the differential equation are

$$y = (c_1 + c_2u)e^u = (c_1 + c_2 \ln t)t + t^2$$

where c_1 and c_2 are arbitrary constants.

Remark. In the second step, if you want to use the method of variation of parameters. $W = W(t, t\ln t) = t$. So one solution is

$$y = -t \cdot \int \frac{t\ln t \cdot 1}{t} dt + t\ln t \int \frac{t \cdot 1}{t} dt \qquad (\text{why } g(t) = 1 \text{ not } t^2?)$$

$$= -t(t\ln t - t + c_1) + t\ln t(t + c_2) = -c_1t + c_2t\ln t + t^2.$$

Problem 5.[10 points] Let $A(x)$ and $B(x)$ be functions on $\mathbb{R}$ which have derivatives of any order. Does there always exist a second order linear homogeneous differential equation $y'' + p(x)y' + q(x)y = 0$ which satisfies the following condition?

- $p(x)$ and $q(x)$ are continuous functions on $\mathbb{R}$;
- $A(x)$ and $B(x)$ are solutions of the differential equation.

If you think it exists, justify yourself by a proof. Otherwise, provide a counterexample.

Solution. No, such a desired equation does not exist in general.

Method 1. If A and B are solutions, then $W(A, B) = c \cdot e^{-\int p(x)dx}$ is either the zero function or nowhere zero **on** $\mathbb{R}$. We can easily construct A and B so that the Wronskian is not of this type. For example, $A = 1$ and $B = x^2$. Then $W(1, x^2) = 2x$ is not the zero function but it is zero when $x = 0$.

Method 2. Let A and B be tow functions such that $A(0) = B(0) = s$ and $A'(0) = B'(0) = t$ but $A \neq B$. Any such differential equation $y'' + p(x)y' + q(x)y = 0$ must have only one solution $y(x)$ with the initial values $y(0) = s$ and $y'(0) = t$. So A and B can not both be solutions to such a differential equation. (Warning: of course they could be solutions of a common $y''+p(x)y'+q(x)y = 0$. The point is that both p and q are required to be continuous over $\mathbb{R}$).

Method 3. (Due to Tim Tran) Let $A(x) = x^2$, if A is a solution to $y'' + p(x)y' + q(x)y = 0$, then we have

$$2 + 2xp + qx^2 = 0.$$

If p and q are defined at $x = 0$, then evaluate $x = 0$ into the equation we have $2 + 0 + 0 = 0$, contradiction.