Math 22B Solutions
Homework 1
Spring 2008

Section 1.1

22. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution Let V = volume, t = time, and S = surface area. Then $\frac{dV}{dt} = -kS$, for $k > 0$. Since the volume of the raindrop is given by $V = \dfrac{4}{3}\pi r^3$, where r is its radius, and its surface area is given by $S = 4\pi r^2$, we can solve for r and substitute to get $S = 4\pi \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V^{\frac{2}{3}}$. Thus, $\frac{dV}{dt} = -cV^{\frac{2}{3}}$, for some constant $c > 0$.

23. The temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings. The ambient temperature is 70^o F, the rate constant is $0.05\ (min)^{-1}$. Write a differential equation for the temperature of the object at any time.

Solution Let T be the temperature of the object, and t be time. Then we have $\frac{dT}{dt} = c(T-70) = -0.05(T-70)$ degrees F/min. Notice that the coefficient c is negative because the object is cooling.

Section 1.2

3 Consider the differential equation $\frac{dy}{dt} = -ay + b$, where both a and b are positive numbers.

(a) Solve the differential equation

Solution Since $\frac{dy}{dt} = -ay + b \Rightarrow \frac{dy}{-ay+b} = dt$. Now we integrate both sides and solve for y to get

$$\int \frac{dy}{-ay+b} = \int dt$$

$$\frac{-1}{a}\ln|-ay+b| = t + c$$

$$\ln|-ay+b|^{\frac{-1}{a}} = e^{t+c} = c'e^{t}$$
$$-ay+b = c'e^{-at}$$
$$\Rightarrow y(t) = \frac{b-ce^{-at}}{a}$$

(b) see figure 1

(c)

(i) As a increases the equilibrium solution gets smaller. The convergence rate increases as well.

(ii) As b increases, the equilibrium solution gets larger. The convergence rate stays the same.

(iii) As a and b both increase, the equilibrium solution stays the same, but the convergence rate increases for all solutions.

8.

(a) Find the rate constant r if the population doubles in 30 days.

Solution The general solution is $p(t) = p_0e^{rt}$ where time is measured in months, so 30 days is 1 month for $t = 1$. We plug in constants to get $2p_0 = p_0e^{r} * 1$. If we solve for r, we get $r = \ln(2)$ per month.

(b) Find r if the population doubles in N days.

Solution Again, time is in months, so $t = \frac{N}{30}$ months. We solve for r as in part (a) to get $2p_0 = p_0e^{r\frac{N}{30}} \Leftrightarrow r = \ln(2)\frac{30}{N}$ per month.

17.

(a) $R\frac{dQ}{dt} + \frac{Q}{C} = V$. In order to separate variables we need $R\frac{dQ}{dt} = V - \frac{Q}{C} \Leftrightarrow \frac{dQ}{V-\frac{Q}{C}} = \frac{dt}{R}$. Integrating both sides we get

$$\int \frac{dQ}{V-\frac{Q}{C}} = \int \frac{dt}{R}$$

$$-c\ln V - \frac{Q}{C} = \frac{t}{R} + C$$
$$\ln V - \frac{Q}{C} = \frac{t}{-cR} + c$$
$$V - \frac{Q}{C} = ce^{\frac{-t}{cR}}$$

$$Q = CV - Cce^{\frac{-t}{cR}}$$
$$Q(0) = 0 = CV - Cce^{\frac{-t}{cR}}$$
$$Q(t) = CV(1 - e^{\frac{-t}{cR}}$$

See figure 2 for more reference.

Section 1.3

Determine the order and state whether it is linear or nonlinear.

1. $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sint$

Solution Since the highest derivative is second order, the equation is second order. Moreover, the equation is linear since it has the form

$$a_n(t)y^{(n)} + ... + a_0(t)y = g(t) \qquad (1)$$

3. $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$

Solution The highest derivative is fourth order, so the equation is fourth order. This equation also has the form (1), so it is also linear.

4. $\frac{dy}{dt} + ty^2 = 0$

Solution There is only one derivative, and it is of order 1, so the equation is first order. The equation is non-linear because it has the term y^2.

5. $\frac{d^2y}{dt^2} + sin(t + y) = sint$

Solution The equation is second order because the highest order derivative is of order two. Since $sin(t + y)$ is a term, the equation is not linear.

Verify that each is a solution of the differential equation

9. $ty' - y = t^2$, $y = 3t + t^2$

Solution We differentiate y to obtain $y' = 3 + 2t$. Substituting in to the differential equation gives

$$ty' - y = t(3t + 2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2, \text{ as desired.}$$

Determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

15. $y' + 2y = 0$

Solution We substitute $y = e^{rt}$ and $y' = re^{rt}$ into the equation to get $re^{rt} + 2e^{rt} = 0$. This implies that $re^{rt} = -2e^{rt}$, which gives us that $r = -2$.

16. $y'' - y = 0$

Solution We substitute $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$ into the equation to get $r^2e^{rt} - e^{rt} = 0$. Then we have $(r^2 - 1)e^{rt} = 0$, which implies that $r^2 - 1 = 0$ since e^{rt} is never zero. Thus, $r = \pm 1$.

29. See textbook page 25 for question.

Solution

(a) see figure (3)

(b) We have that $F = ma$, but for the tangential direction. Let $F_\theta = ma_theta$. The only force acting in the tangential direction is the weight, so $F_\theta = -mgsin\theta$. The acceleration a_theta is the linear acceleration along the path $a_theta = L\frac{d^2\theta}{dt^2}$, as given in the problem. So $-mgsin\theta = mL\frac{d^2\theta}{dt^2}$. See figure 4 for more reference.

(c) From part (b), we have $-mgsin\theta = mL\frac{d^2\theta}{dt^2}$, so $L\frac{d^2\theta}{dt^2} + gsin\theta = 0$, which implies that $\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\theta = 0$.

30. See book pgs. 25-26 for problem.

(a) The kinetic energy of mass m is given by $T = \frac{1}{2}mv^2$, where v is the velocity. A particle in motion on a circle of radius L has speed $L\frac{d\theta}{dt}$, where θ is angular position, $\frac{d\theta}{dt}$ is angular speed. So $T = \frac{1}{2}m(L\frac{d\theta}{dt})^2$.

(b) Potential energy is given by mgh, where h is the height and g is the gravity. If we take $v = 0$, the lowest point of the swing is $h = L$. If the pendulum is not at the lowest point of the swing, $h = L(1 - cos\theta)$. So $V = mgL(1 - cos\theta)$ See figure 5 for more reference.

(c) $E = T + V \;\Rightarrow\; E = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mgL(1 - cos\theta)$. So then

$$\frac{dE}{dt} = mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgLsin\theta\frac{d\theta}{dt}$$

by the Chain Rule. Then we set $\frac{dT}{dt} = 0$ to get

$$0 = mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mglsin\theta\frac{d\theta}{dt}$$

Dividing both sides by $mL\frac{d\theta}{dt}$ we get

$$0 = L\frac{d^2\theta}{dt^2} + gsin\theta, \text{ or } 0 = \frac{d^2\theta}{dt^2} + \frac{1}{L}gsin\theta$$