Exercises 6

More exercises are available in *Elementary Differential Equations*. If you have a problem to solve any of them, feel free to come to office hour.

1 Problem 1

Use Euler's formula to write the given expression in the form $a + ib$:

1. $\exp(1 + 2i)$
2. $e^{i\pi}$
3. 2^{1-i}

1.1 Solution

Euler's formula is:

$$e^{it} = \cos t + i \sin t. \tag{1}$$

1.
$$\begin{aligned} \exp 1 + 2i &= e^1 \cdot e^{2i}, \\ &= e^1 \left(\cos 2 + i \sin 2\right). \end{aligned} \tag{2}$$

2.
$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi, \\ &= -1. \end{aligned} \tag{3}$$

3.
$$\begin{aligned} 2^{1-i} &= 2 \cdot 2^{-i}, \\ &= 2e^{\ln 2^{-i}}, \\ &= 2e^{-i \ln 2}, \\ &= 2\left(\cos\left(\ln 2\right) - i \sin\left(\ln 2\right)\right). \end{aligned} \tag{4}$$

2 Problem 2

Find the general solution of the given differential equation:

1. $y'' - 2y' + 2y = 0$
2. $y'' + 2y' - 8y = 0$
3. $y'' + 6y' + 13y = 0$
4. $y'' + 2y' + 1.25y = 0$
5. $y'' + y' + 1.25y = 0$

2.1 Solution

1. First, we need to write the characteristic equation:

$$\lambda^2 - 2\lambda + 2 = 0. \tag{5}$$

The solutions of this equation are $\lambda = 1 + i$ and $\lambda = 1 - i$. Thus, the general solution is:

$$y = c_1 e^t \cos t + c_2 e^t \sin t. \tag{6}$$

2. First, we need to write the characteristic equation:

$$\lambda^2 + 2\lambda - 8 = 0. \tag{7}$$

The solutions of this equation are $\lambda = -4$ and $\lambda = 2$. Thus, the general solution is:

$$y = c_1 e^{-4t} + c_2 e^{2t}. \tag{8}$$

3. First, we need to write the characteristic equation:

$$\lambda^2 + 6\lambda + 13 = 0 \tag{9}$$

The solutions of this equation are $\lambda = -3 + 2i$ and $\lambda = -3 - 2i$. Thus, the general solution is:

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t). \tag{10}$$

4. First, we need to write the characteristic equation:

$$\lambda^2 + 2\lambda + 1.25 = 0. \tag{11}$$

The solutions of this equation are $\lambda = -1 - \frac{i}{2}$ and $\lambda = -1 + \frac{i}{2}$. Thus, the general solution is:

$$y = c_1 e^{-t} \cos\left(\frac{t}{2}\right) + c_2 e^{-t} \sin\left(\frac{t}{2}\right). \tag{12}$$

5. First, we need to write the characteristic equation:

$$\lambda^2 + \lambda + 1.25 = 0. \tag{13}$$

The solutions of this equation are $\lambda = -\frac{1}{2} + i$ and $\lambda = -\frac{1}{2} - i$. Thus, the general solution is:

$$y = c_1 e^{-\frac{t}{2}} \cos t + c_2 e^{-\frac{t}{2}} \sin t. \tag{14}$$

3 Problem 3

Find the solution of the given initial value problem. Sketch the graph of the solution of the solution and describe its behavior for increasing t.

1. $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$
2. $y'' - 2y' + 5y = 0$, $y\left(\frac{\pi}{2}\right) = 0$, $y'\left(\frac{\pi}{2}\right) = 2$
3. $y'' + y' + 1.25y = 0$, $y(0) = 3$, $y'(0) = 1$

3.1 Solution

1. First, we need to write the characteristic equation:
$$\lambda^2 + 4 = 0. \tag{15}$$
The solutions of this equation are $\lambda = -2i$ and $\lambda = 2i$. Thus, the general solution is:
$$y = c_1 \cos(2t) + c_2 \sin(2t). \tag{16}$$
To use the I.C., we need y':
$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t). \tag{17}$$
Thus, we have the system of equations:
$$c_1 = 0, \tag{18}$$
$$2c_2 = 1. \tag{19}$$
The final solution is:
$$y = \frac{1}{2}\sin(2t). \tag{20}$$
When t increases, we have a steady oscillation.

2. First, we need to write the characteristic equation:
$$\lambda^2 - 2\lambda + 5 = 0. \tag{21}$$
The solutions of this equation are $\lambda = 1 + 2i$ and $\lambda = 1 - 2i$. Thus, the general solution is:
$$y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t). \tag{22}$$
To use the I.C., we need y':
$$y' = c_1 e^t \left(\cos(2t) - \sin(2t)\right) + c_2 e^t \left(\sin(2t) + \cos(2t)\right). \tag{23}$$
Thus, we have the system of equations:
$$-c_1 e^{\pi/2} = 0, \tag{24}$$
$$-c_2 e^{\pi/2} = 1. \tag{25}$$
The final solution is:
$$y = -e^{t-pi/2}\sin(2t). \tag{26}$$
When t increases, we have a growing oscillation.

3. First, we need to write the characteristic equation:
$$\lambda^2 + \lambda + 1.25y = 0. \tag{27}$$
The solutions of this equation are $\lambda = -\frac{1}{2} - i$ and $\lambda = -\frac{1}{2} + i$. Thus, the general solution is:
$$y = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t. \tag{28}$$
To use the I.C., we need y':
$$y' = c_1 e^{-t/2}\left(-\frac{1}{2}\cos t - \sin t\right) + c_2 e^{-t/2}\left(-\frac{1}{2}\sin t + \cos t\right). \tag{29}$$
Thus, we have the system of equations:
$$c_1 = 3, \tag{30}$$
$$-c_1\frac{1}{2} + c_2 = 1. \tag{31}$$
The final solution is:
$$y = 3e^{-t/2}\cos t + \frac{5}{2}e^{-t/2}\sin t. \tag{32}$$
When t increases, we have a decaying oscillation.

4 Problem 4

Show that $W\left(e^{\lambda t}\cos\left(\mu t\right), e^{\lambda t}\sin\left(\mu t\right)\right) = \mu e^{2\lambda t}$.

4.1 Solution

We need to compute:

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \tag{33}$$

Here, we have:

$$\begin{align} y_1 &= e^{\lambda t}\cos\left(\mu t\right), \tag{34}\\ y_1' &= e^{\lambda t}\left(\cos\left(\mu t\right) - \mu\sin\left(\mu t\right)\right), \tag{35}\\ y_2 &= 2^{\lambda t}\sin\left(\mu t\right), \tag{36}\\ y_2' &= e^{\lambda t}\left(\sin\left(\mu t\right) + \mu\cos\left(\mu t\right)\right). \tag{37}\end{align}$$

Thus, W is given by:

$$\begin{aligned} W &= e^{2\lambda t}\left(\sin(\mu t)\cos(\mu t) + \mu\left(\cos(\mu t)\right)^2\right) - e^{2\lambda t}\left(\sin(\mu t)\cos(\mu t) - \mu\left(\cos(\mu t)\right)^2\right), \\ &= e^{2\lambda t}\mu\left(\left(\cos(2t)\right)^2 + \left(\sin(2t)\right)^2\right), \\ &= \mu e^{2\lambda t}. \end{aligned} \tag{38}$$

5 Problem 5

An equation of the form:

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \tag{39}$$

where $t > 0$, and α and β are real constants, is called an Euler equation.

1. Let $x = \ln t$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
2. Use the results of part (1) to transform equation (39) into:

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{40}$$

Observe equation (40) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of equation (40), then $y_1\left(\ln t\right)$ and $y_2\left(\ln t\right)$ form a fundamental set of solutions of equation (39).

5.1 Solution

1. We need to use the chain rule:

$$\begin{align} \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}, \tag{41}\\ \frac{d^2y}{dt^2} &= \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} = \frac{d^2y}{dx^2}\frac{1}{t^2} - \frac{dy}{dx}\frac{1}{t^2}. \tag{42}\end{align}$$

2. We have successively:

$$\begin{align} \frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y = 0, \tag{43}\\ fracd^2ydx^2 + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{44}\end{align}$$

6 Problem 6

Find the general solution of the given differential equation:

1. $y'' - 2y' + y = 0$
2. $4y'' - 4y' - 3y = 0$
3. $y'' - 2y' + 10y = 0$
4. $4y'' + 17y' + 4y = 0$
5. $25y'' - 20y' + 4y = 0$

6.1 Solution

1. First, we need to write the characteristic equation:
$$\lambda^2 - 2\lambda + 1 = 0. \tag{45}$$
There is only one double root $\lambda = 1$. Thus, the general solution is given by:
$$y = c_1 te^t + c_2 e^t. \tag{46}$$
2. First, we need to write the characteristic equation:
$$4\lambda^2 - 4\lambda - 3 = 0. \tag{47}$$
The solutions of this equation are $\lambda = -1$ and $\lambda = \frac{3}{2}$. Thus, the general solution is:
$$y = c_1 e^{-t} + c_2 e^{-\frac{3t}{2}}. \tag{48}$$
3. First, we need to write the characteristic equation:
$$\lambda^2 - 2\lambda + 10 = 0. \tag{49}$$
The solutions of this equation are $\lambda = 1 - 3i$ and $\lambda = 1 + 3i$. Thus, the general solution is:
$$y = c_1 e^t \cos(3t) + c_2 e^t \sin(3t). \tag{50}$$
4. First, we need to write the characteristic equation:
$$4\lambda^2 + 17\lambda + 4 = 0 \tag{51}$$
The solutions of this equation are $\lambda = -\frac{1}{4}$ and $\lambda = -4$. Thus, the general solution is:
$$y = c_1 e^{-t/4} + c_2 e^{-4t}. \tag{52}$$
5. First, we need to write the characteristic equation:
$$25\lambda^2 - 20\lambda + 4 = 0. \tag{53}$$
There is only one double root $\lambda = \frac{2}{5}$. Thus, the general equation is:
$$y = c_1 te^{\frac{2t}{5}} + c_2 e^{\frac{2t}{5}}. \tag{54}$$

7 Problem 7

Solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t:

1. $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$.
2. $9y'' + 6y' + 82y = 0$, $y(0) = -1$, $y'(0) = 2$.

7.1 Solution

1. First, we need to write the characteristic equation:

$$9\lambda^2 - 12\lambda + 4 = 0. \tag{55}$$

There is only one double root $\lambda = \frac{2}{3}$. Thus, the general equation is:

$$y = c_1 t e^{\frac{2t}{3}} + c_2 e^{\frac{2t}{3}}. \tag{56}$$

To use the I.C., we need y':

$$y' = c_1 e^{\frac{2t}{3}} \left(1 + \frac{2t}{3}\right) + \frac{2c_2}{3} e^{\frac{2t}{3}}. \tag{57}$$

Thus, we have the system of equations:

$$c_2 = 2, \tag{58}$$

$$c_1 + \frac{4}{3} = -1. \tag{59}$$

We finally have:

$$y(t) = -\frac{7}{3} t e^{\frac{2t}{3}} + 2e^{\frac{2t}{3}}. \tag{60}$$

y goes to $-\infty$ when t goes ∞.

2. First, we need to write the characteristic equation:

$$9\lambda^2 + 6\lambda + 82 = 0. \tag{61}$$

The solutions of this equation are $\lambda = -\frac{1}{3} + 3i$ and $\lambda = -\frac{1}{3} - 3i$. Thus, the general equation is:

$$y = c_1 e^{-t/3} \cos(3t) + c_2 e^{-t/3} \sin(3t). \tag{62}$$

To use the I.C., we need y':

$$y' = -c_1 e^{-t/3} \left(\frac{1}{3}\cos(3t) + 3\sin(3t)\right) + c_2 e^{-t/3} \left(-\frac{1}{3}\sin(3t) + 3\cos(3t)\right). \tag{63}$$

Thus, we have the system of equations:

$$c_1 = -1, \tag{64}$$

$$\frac{1}{3} + 3c_2 = 2. \tag{65}$$

We finally have:

$$y = -e^{-t/3}\cos(3t) + \frac{5}{9} e^{-t/3} \sin(3t). \tag{66}$$

y goes to 0 when t goes ∞.

8 Problem 8

Use the method of reduction of order to find a second solution of the given differential equation:

1. $t^2y'' - 4ty' + 6y = 0$, $t > 0$, $y_1(t) = t^2$
2. $t^2y'' + 3ty' + y = 0$, $t > 0$, $y_1(t) = t^{-1}$

8.1 Solution

1. We are looking for a solution of the form:

$$y = u(t)t^2. \tag{67}$$

We will need:

$$\begin{aligned} y' &= u'(t)t^2 + 2tu(t), & (68) \\ y'' &= u''(t) + 2tu'(t) + 2u(t) + 2tu'(t) = u''(t) + 4tu'(t) + 2u(t). & (69) \end{aligned}$$

Using y, y', and y'', we get:

$$t^2\left(u''(t) + 4tu'(t) + 2u(t)\right) - 4t\left(u'(t)t^2 + 2tu(t)\right) + 6u(t)t^2 = 0, \tag{70}$$

$$u''t^2 + u'(4t^3 - 4t^3) + u(2t^2 - 8t^2 + 6t^2) = 0, \tag{71}$$

$$u''t^2 = 0, \tag{72}$$

$$u'' = 0, \tag{73}$$

$$u = c_1t + c_2. \tag{74}$$

Thus, we have:

$$\begin{aligned} y &= c_1t^3 + c_2t^2, & (75) \\ y_1 &= t^2, & (76) \\ y_2 &= t^3. & (77) \end{aligned}$$

2. We are looking for a solution of the form:

$$y = u(t)t^{-1}. \tag{78}$$

We will need:

$$\begin{aligned} y' &= u't^{-1} - ut^{-2}, & (79) \\ y'' &= u''t^{-1} - 2u't^{-2} + 2ut^{-3}. & (80) \end{aligned}$$

Using y, y', and y'', we get:

$$t^2\left(u''t^{-1} - 2u't^{-2} + 2ut^{-3}\right) + 3t\left(u't^{-1} - ut^{-2}\right) + u(t)t^{-1} = 0, \tag{81}$$

$$u''t + u'(-2 + 3) + \frac{u}{t}(2 - 3 + 1) = 0, \tag{82}$$

$$u''t + u' = 0. \tag{83}$$

Now, we introduce $y = u'$:

$$y't + y = 0, \tag{84}$$

$$y' = -\frac{y}{t}, \tag{85}$$

$$\ln y = -\ln t + c, \tag{86}$$

$$y = \frac{c_1}{t}. \tag{87}$$

Thus, we have:

$$u' = \frac{c_1}{t}, \tag{88}$$

$$u = c_1 \ln t + c_2. \tag{89}$$

We finally get:

$$y = c_1 \frac{\ln t}{t} + \frac{c_2}{t}, \qquad y_1 = t^{-1}, y_2 = \frac{\ln t}{t}. \tag{90}$$

9 Problem 9

If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

9.1 Solution

First, we need to write the characteristic equation:

$$a\lambda^2 + b\lambda + c = 0. \tag{91}$$

Now, we need to solve this equation:

$$\Delta = b^2 - 4ac, \tag{92}$$

Three cases are possible:

1. If $\Delta > 0$, we have:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{93}$$

   Since $b > \sqrt{b^2 - 4ac}$, λ_1 and λ_2 are always negative. Therefore, $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ approaches zero when t goes to ∞.

2. If $\Delta = 0$, we have:

$$\lambda = -\frac{b}{2a}. \tag{94}$$

   Thus, the general solution of the ODE is: $y = c_1 t e^{-\frac{bt}{2a}} + c_2 e^{-\frac{bt}{2a}}$. We know that the second term goes to zero as t goes to ∞. The first term gives:

$$\begin{aligned} \lim_{t\to\infty} c_1 t e^{-\frac{bt}{2a}} &= \lim_{t\to\infty} \frac{c_1 t}{e^{\frac{bt}{2a}}}, \\ &= \lim_{t\to\infty} \frac{c_1}{\frac{b}{2a} e^{\frac{bt}{2a}}}, \\ &= 0. \end{aligned} \tag{95}$$

3. If $\Delta < 0$, we have:

$$\lambda = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a}. \tag{96}$$

   Thus, the general solution of the ODE is: $y = c_1 e^{-\frac{bt}{2a}} \cos\left(\frac{\sqrt{|b^2-4ac|}}{2a} t\right) + c_2 e^{-\frac{bt}{2a}} \sin\left(\frac{\sqrt{|b^2-4ac|}}{2a} t\right)$. Therefore, the solution goes to zero when t goes to ∞.

10 Problem 10

Find the general solution of the given differential equation:

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - 2y' - 3y = -3te^{-t}$
3. $y'' + 9y = t^2e^{3t} + 6$
4. $2y'' + 3y' + y = t^2 + 3\sin t$
5. $y'' + y' + 4y = 2\sinh t$

10.1 Solution

1. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$\lambda^2 - 2\lambda - 3 = 0 \tag{97}$$
The solution of this equations are $\lambda = -1$ and $\lambda = 3$. Thus, the corresponding solution is:
$$y_c = c_1e^{-t} + c_2e^{3t}. \tag{98}$$
Now, we look for a particular solution of the form $y_p = ae^{2t}$. We will need:
$$y_p' = 2ae^{2t}, \tag{99}$$
$$y_p'' = 4ae^{2t}. \tag{100}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$4ae^{2t} - 4ae^{2t} - 3ae^{2t} = 3e^{2t}, \tag{101}$$
$$a = -1. \tag{102}$$
Thus, the general solution is:
$$y = c_1e^{-t} + c_2e^{3t} - e^{2t}. \tag{103}$$

2. First, we need to solve the corresponding homogeneous equation. The homogeneous equation is same as the previous example. Thus, the corresponding solution is:
$$y_c = c_1e^{-t} + c_2e^{3t}. \tag{104}$$
Now, we look for a particular solution of the form $y_p = (at^2+bt)e^{-t}$ ($(at+b)e^{-t}$ leads to a contradiction). We will need:
$$y_p' = e^{-t}(-at^2 + (2a-b)t + b), \tag{105}$$
$$y_p'' = e^{-t}\left(at^2 - (4a-b)t + 2(a-b)\right). \tag{106}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$e^{-t}\left(at^2 - (4a-b)t + 2(a-b)\right) - 2\left(e^{-t}(-at^2 + (2a-b)t + b)\right) - 3(at^2 + bt)e^{-t} = -3te^{-t}, \tag{107}$$
$$t(-4a + b - 4a + 2b - 3b) + 2a - 2b - 2b = -3t, \tag{108}$$
$$-8at + 2(a - 2b) = -3t. \tag{109}$$
Thus, we get the system:
$$-8a = -3, \qquad a - 2b = 0. \tag{110}$$
Thus, $a = \frac{3}{8}$ and $b = \frac{3}{16}$ and we finally have:
$$y = c_1e^{-t} + c_2e^{3t} + \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}. \tag{111}$$

3. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$\lambda^2 + 9 = 0. \tag{112}$$
The solutions of this equation are $\lambda = -3i$ and $\lambda = 3i$. Thus the corresponding solution is:
$$y_c = c_1 \cos(3t) + c_2 \sin(3t). \tag{113}$$
Now, we look for a particular solution of the form $y_p = \left(at^2 + bt + c\right) e^{3t} + f$. We will need:
$$y_p' = e^{3t}\left(3at^2 + 3bt + 3c + 2at + b\right), \tag{114}$$
$$y_p'' = e^{3t}\left(9at^2 + 9bt + 9c + 6at + 3b + 6at + 3b + 2a\right). \tag{115}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$\left(9at^2 + t\left(12a + 9b\right) + 2a + 6b + 9c\right) e^{3t} + 9e^{3t}\left(at^2 + bt + c\right) + 9f = t^2 e^{3t} + 6, \tag{116}$$
$$e^{3t}\left(18at^2 + t\left(12a + 18b\right) + 2a + 6b + 18c\right) + 9f = t^2 e^{3t} + 6, \tag{117}$$
$$18a = 1, \qquad 12a + 18b = 0, 2a + 6b + 18c = 0, \qquad 9f = 6, \tag{118}$$
$$a = \frac{1}{18}, \qquad b = -\frac{2}{54}, c = \frac{1}{162}, \qquad f = \frac{2}{3}. \tag{119}$$
Thus, we finally have:
$$y = c_1 \cos(3t) + c_2 \sin(3t) + e^{3t}\left(\frac{1}{18}t^2 - \frac{2}{54}t + \frac{1}{162}\right) + \frac{2}{3}. \tag{120}$$

4. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$2\lambda^2 + 3\lambda + 1 = 0. \tag{121}$$
The solutions of this equations are $\lambda = -\frac{1}{2}$ and $\lambda = -1$. Thus, the corresponding solution is:
$$y_c = c_1 e^{-\frac{t}{2}} + c_2 e^{-t}. \tag{122}$$
Now we are looking for a particular solution of the form $y_p = at^2 + bt + c + d\sin t + f\cos t$. We will need:
$$y_p' = 2at + b + d\cos t - f\sin t, \tag{123}$$
$$y_p'' = 2a - d\sin t - f\cos t. \tag{124}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$4a - 2d\sin t - 2f\cos t + 6at + 3b + 3d\cos t - 3f\sin t + at^2 + bt + c + d\sin t + f\cos t = t^2 + 3\sin t, \tag{125}$$
$$at^2 + t(6a + b) + 4a + 3b + c + \sin t\left(-d - 3f\right) + \cos t\left(-f + 3d\right) = t^2 + 3\sin t, \tag{126}$$
$$a = 1, \tag{127}$$
$$6a + b = 0, \tag{128}$$
$$4a + 3b + c = 0, \tag{129}$$
$$-d - 3f = 1, \tag{130}$$
$$-f + 3d = 0, \tag{131}$$

$$a = 1, \tag{132}$$
$$b = -6, \tag{133}$$
$$c = 14, \tag{134}$$
$$d = -\frac{1}{10}, \tag{135}$$
$$f = -\frac{3}{10}. \tag{136}$$

Thus, we finally have:

$$y = c_1 e^{-\frac{t}{2}} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{1}{10}\sin t - \frac{3}{10}\cos t \tag{137}$$

5. First, we rewrite the ODE:
$$y'' + y' + 4y = e^t - e^{-t}. \tag{138}$$
Now, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$\lambda^2 + \lambda + 4 = 0. \tag{139}$$
The solutions of this equation are $\lambda = \frac{-1+i\sqrt{5}}{2}$ $\lambda = \frac{-1-i\sqrt{5}}{2}$. Thus, the corresponding solution is:
$$y_c = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right). \tag{140}$$
Now, we are looking for a particular solution of the form $y_p = ae^t + be^{-t}$. We will need:
$$y_p' = ae^t - be^{-t}, \tag{141}$$
$$y_p'' = ae^t + be^{-t}. \tag{142}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$ae^t + be^{-t} + ae^t - be^{-t} + 4ae^t + 4be^{-t} = e^t - e^{-t}, \tag{143}$$
$$6ae^t + 4be^{-t} = e^t - e^{-t}, \tag{144}$$

$$6a = 1, \tag{145}$$
$$4b = -1. \tag{146}$$

$$a = \frac{1}{6}, \tag{147}$$
$$b = -\frac{1}{4}. \tag{148}$$
Thus, we finally have:
$$y = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}. \tag{149}$$

11 Problem 11

Find the solution of the given initial value problem:

1. $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$
2. $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$
3. $y'' + 4y = 3\sin 2t$, $y(0) = 2$, $y'(0) = -1$

11.1 Solution

1. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$\lambda^2 + \lambda - 2 = 0. \tag{150}$$
The solutions of this equations are $\lambda = 1$ and $\lambda = -2$. Thus, the corresponding solution is:
$$y_c = c_1 e^t + c_2 e^{-2t}. \tag{151}$$
Now, we are looking for a particular solution of the form $y_p = at + b$. We will need:
$$y_p' = a, \tag{152}$$
$$y_p'' = 0. \tag{153}$$
Using y_p, y_p', and y_p'' in the nonhomogeneous equation, we get:
$$a - 2at - 2b = 2t, \tag{154}$$
$$a = -1, \tag{155}$$
$$b = -\frac{1}{2}. \tag{156}$$
Thus, we finally have:
$$y = c_1 e^t + c_2 e^{-2t} - t - \frac{1}{2}. \tag{157}$$
Now, we can find c_1 and c_2 using the I.C.:
$$c_1 + c_2 - \frac{1}{2} = 0, \tag{158}$$
$$c_1 - 2c_2 - 1 = 1, \tag{159}$$
$$c_1 = 1, \tag{160}$$
$$c_2 = -\frac{1}{2}. \tag{161}$$
We finally have:
$$y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}. \tag{162}$$

2. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:
$$\lambda^2 - 2\lambda + 1 = 0. \tag{163}$$
There is one double solution $\lambda = 1$. Thus, the corresponding solution is:
$$y_c = c_1 te^t + c_2 e^t. \tag{164}$$
Now, we are looking for a particular solution of the form $y_p = (at^3 + bt^2)e^t + c$. We will need:
$$y_p' = e^t(at^3 + (3a+b)t^2 + 2bt), \tag{165}$$
$$y_p'' = e^t(at^3 + (6a+b)t^2 + (6a+4b)t + 2b). \tag{166}$$
Using y_p, y_p', and y_p'' in the homogeneous equation, we get:
$$e^t(at^3 + (6a+b)t^2 + (6a+4b)t + 2b) - 2e^t(at^3 + (3a+b)t^2 + 2bt) + (at^3 + bt^2)e^t + c = te^t + 4, \tag{167}$$
$$e^t\,(6at + b) + c = te^t + 4, \tag{168}$$

$$a = \frac{1}{6}, \tag{169}$$
$$b = 0, \tag{170}$$
$$c = 4. \tag{171}$$

We get:

$$y = c_1 te^t + c_2 e^t + \frac{1}{6} t^3 e^t + 4. \tag{172}$$

Now, we can find c_1 and c_2 using the I.C.:

$$c_2 + 4 = 1, \tag{173}$$
$$c_1 + c_2 = 1, \tag{174}$$

$$c_2 = -3, \tag{175}$$
$$c_1 = 4. \tag{176}$$

We finally have:

$$y = 4te^t - 3e^t + \frac{1}{6} t^3 e6t + 4. \tag{177}$$

3. First, we need to solve the corresponding homogeneous equation. Thus, we start with the characteristic equation:

$$\lambda^2 + 4 = 0. \tag{178}$$

The solutions of this equation are $\lambda = -2i$ and $\lambda = 2i$. Thus, the corresponding solution is:

$$y_c = c_1 \cos(2t) + c_2 \sin(2t). \tag{179}$$

Now, we are looking for a particular solution of the form $y_p = at\cos(2t) + bt\sin(2t)$. We will need:

$$y_p' = a\cos(2t) - 2at\sin(2t) + b\sin(2t) + 2bt\cos(2t), \tag{180}$$
$$y_p'' = -4a\sin(2t) - 4at\cos(2t) + 4b\cos(2t) - 4bt\sin(2t). \tag{181}$$

Using y_p, y_p', and y_p'' in the homogeneous equation, we get:

$$-4a\sin(2t) - 4at\cos(2t) + 4b\cos(2t) - 4bt\sin(2t) + 4at\cos(2t) + 3bt\sin(2t) = 3\sin(2t), \tag{182}$$
$$\sin(2t)\,(-4a - bt) + 4b\cos(2t) = 3\sin(2t), \tag{183}$$

$$a = -\frac{3}{4}, \tag{184}$$
$$b = 0. \tag{185}$$

We get:

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4} t\cos(2t), \tag{186}$$

Now, we can find c_1 and c_2 using the I.C.:

$$c_1 = 2, \tag{187}$$
$$2c_2 - \frac{3}{4} = -1, \tag{188}$$
$$\tag{189}$$

$$c_1 = 2, \tag{190}$$
$$c_2 = -\frac{1}{8}. \tag{191}$$

We finally have:

$$y = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3}{4} t\cos(2t). \tag{192}$$

12 Problem 12

Solve the differential equation:

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \dfrac{\pi}{2}, \\ 0, & t > \dfrac{\pi}{2} \end{cases} \tag{193}$$

with the initial condition $y(0) = 0$ and $y'(0) = 0$. Assume that y and y' are continuous at $t = \frac{\pi}{2}$.

12.1 Solution

First, we need to solve the corresponding homogeneous equation:

$$\lambda^2 + 2\lambda + 5 = 0, \tag{194}$$

The solutions of this equation are $\lambda = -1 + 2i$ and $\lambda = -1 - 2i$. Thus, the corresponding solution is:

$$y_c = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t). \tag{195}$$

When $0 \le t \le \pi$, $y_p = \frac{1}{5}$. Thus, we have:

$$y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + \frac{1}{5}. \tag{196}$$

Now, we can find c_1 and c_2 using the I.C.:

$$c_1 + \frac{1}{5} = 0, \tag{197}$$

$$-c_1 + 2c_2 = 0, \tag{198}$$

$$c_1 = -\frac{1}{5}, \tag{199}$$

$$c_2 = -\frac{1}{10}. \tag{200}$$

Thus, we have:

$$y = -\frac{1}{5} e^{-t} \cos(2t) - \frac{1}{10} e^{-t} \sin(2t) + \frac{1}{5}. \tag{201}$$

when $0 \le t \le \frac{\pi}{2}$.
Now, that we know the solution for $t \in \left[0, \frac{\pi}{2}\right]$, we can compute the solution for $t > \frac{\pi}{2}$. We assume that y and y' are continuous:

$$y\left(\frac{\pi}{2}\right) = \frac{1}{5} e^{-pi/2} + \frac{1}{5}, \tag{202}$$

$$y'\left(\frac{\pi}{2}\right) = 0. \tag{203}$$

These are the "initial" conditions for the second part of the problem. When $t > \frac{\pi}{2}$, the ODE is homogeneous:

$$y = d_1 \cos(2t) + d_2 \sin(2t). \tag{204}$$

We can find d_1 and d_2 using the "initial conditions":

$$-c_1 e^{-\pi/2} = \frac{1}{5} e^{-\pi/2} + \frac{1}{5}, \tag{205}$$

$$-c_1 e^{-\pi/2} + c_2 e^{-\pi/2} = 0, \tag{206}$$

$$c_1 = -\frac{1}{5}\left(1 + e^{\pi/2}\right), \tag{207}$$

$$c_2 = -\frac{1}{10}\left(1 + e^{\pi/2}\right). \tag{208}$$

Thus, we have:

$$y = -\frac{1}{5}\left(1 + e^{\pi/2}\right) e^{-t}\cos(2t) - \frac{1}{10}\left(1 + e^{\pi/2}\right) e^{-t}\sin(2t). \tag{209}$$

when $t > \frac{\pi}{2}$.