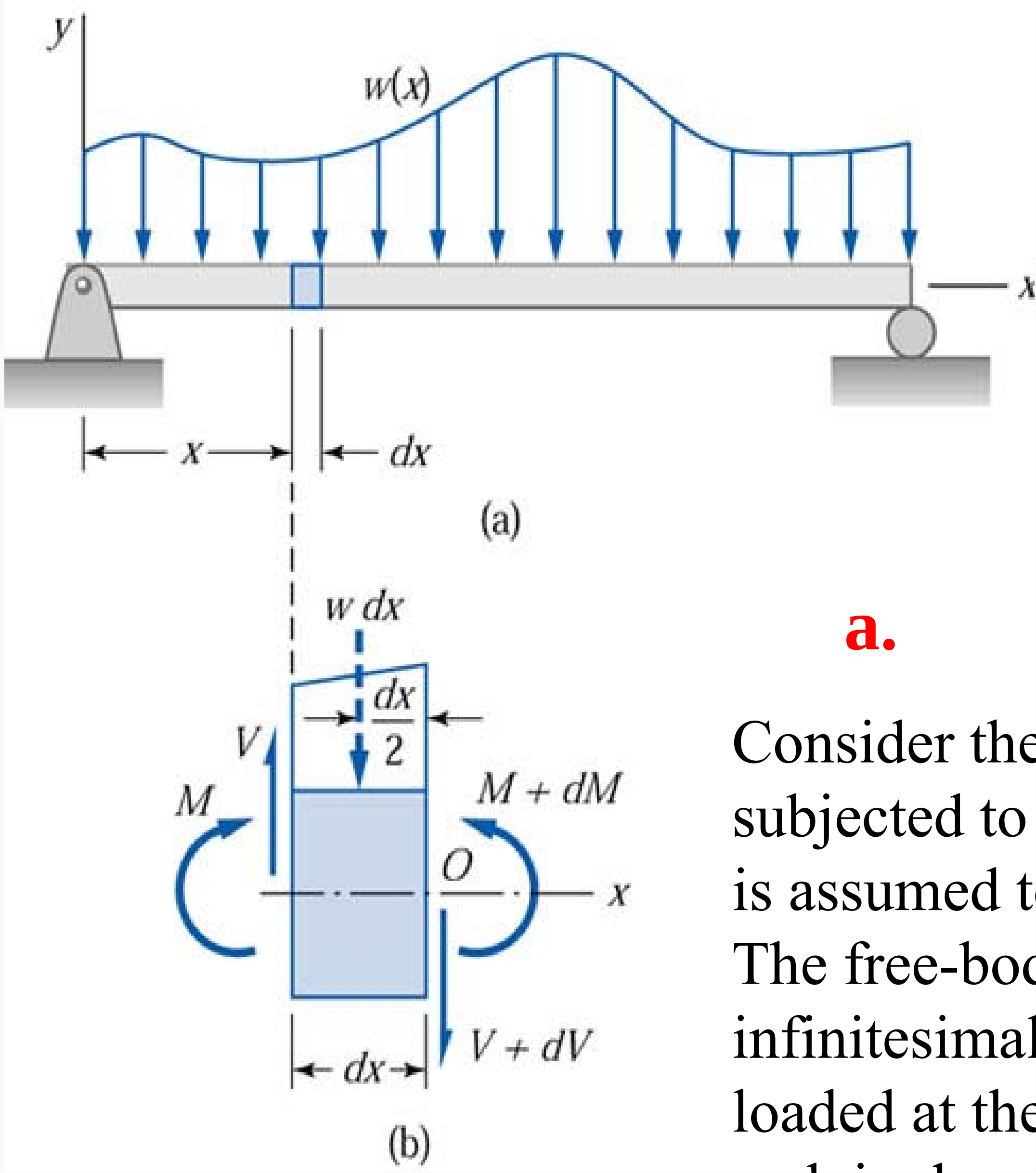
4.4 *Area Method for Drawing Shear- Moment Diagrams*

- Useful relationships between the loading, shear force, and bending moment can be derived from the equilibrium equations.
- These relationships enable us to plot the shear force diagram directly from the load diagram, and then construct the bending moment diagram from the shear force diagram. This technique, called the ***area method***, allows us to draw the shear force and bending moment diagrams without having to derive the equations for V and M.
- First consider beam subjected to distributed loading and then discuss concentrated forces and couples.



4.4 Figures (a) Simply supported beam carrying distributed loading; (b) free-body diagram of an infinitesimal beam segment.

a. *Distributed loading*

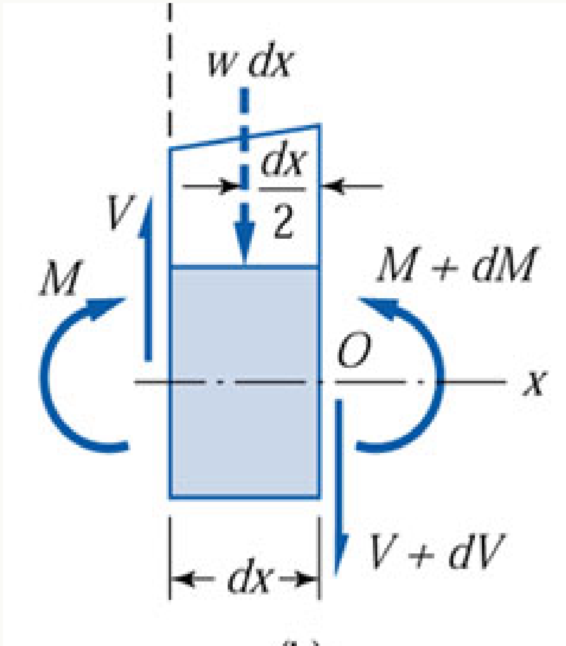
Consider the beam in Fig. 4.4 (a) that is subjected to a the distributed load w (x) is assumed to be a continuous function. The free-body diagram of an infinitesimal element of the beam, loaded at the distance x from the left end, is shown in Fig. 4.4 (b)

The force equation of equilibrium is

$$\Sigma F_y = 0 + \uparrow \quad V - w\,dx - (V + dV) = 0$$

From which we get

$$\boxed{w = -\frac{dV}{dx}} \qquad (4.1)$$



(b)

The moment equation of equilibrium yields

$$\Sigma M_0 = 0 + \circlearrowleft \quad -M - Vdx + (M + dM) + wdx(dx/2) = 0$$

After canceling M and dividing by *dx*, we get

$$-V + \frac{dM}{dx} + \frac{w\cancel{dx}}{\cancel{2}} = 0 \qquad \boxed{V = \frac{dM}{dx}} \qquad (4.2)$$

Equations (4.1) and (4.2) are called the ***differential equations of equilibrium*** for beams.

The following five theorems relating the load, the shear force, and the bending moment diagrams follow from these equations.

1. The load intensity w at any section of a beam is equal to the negative of the slope of the shear force diagram at the section.

Proof — follows directly from Eq. (4.1).

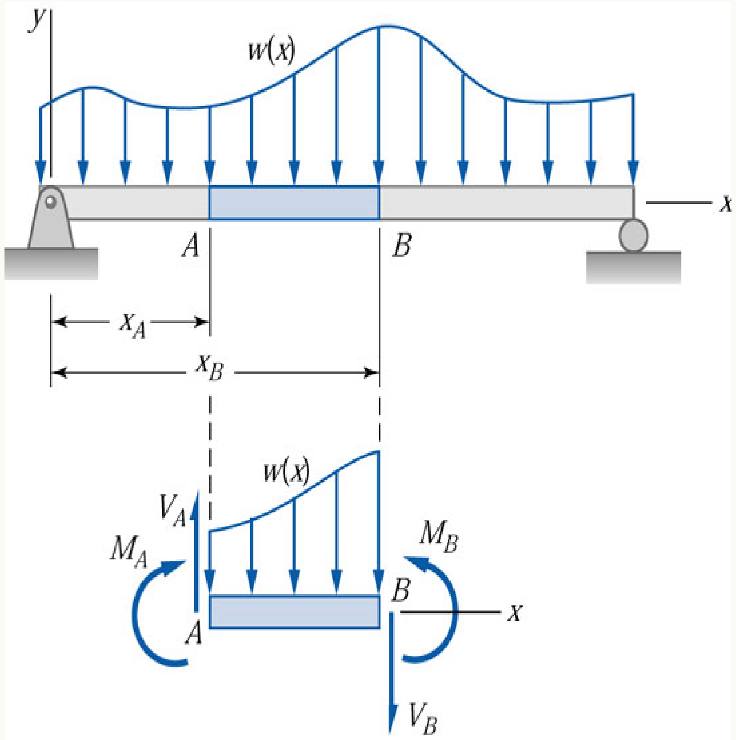
$$\boxed{w = -\frac{dV}{dx}}$$

2. The shear force V at any section is equal to the slope of the bending moment diagram at that section.

Proof — follows directly from Eq. (4.2).

$$\boxed{V = \frac{dM}{dx}}$$

3. The difference between the shear forces at two sections of a beam is equal to the negative of the area under the load diagram between those two sections.



Proof —integrating Eq. (4.1) between sections *A* and *B* in Fig. 4.5, we obtain

$$w = -\frac{dV}{dx} \qquad \boxed{\int_{xA}^{xB} \frac{dV}{dx}\, dx = V_B - V_A = -\int_{xA}^{xB} w\, dx}$$

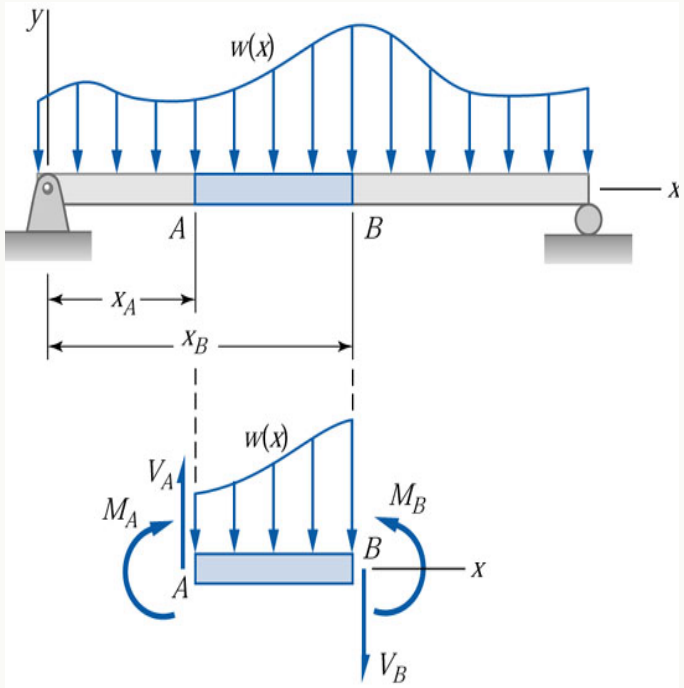
$$V_B - V_A = -\text{area of } w\text{-diagram}\Big]_A^B$$

$$\boxed{V_B = V_A - \text{area of } w\text{ -diagram}\Big]_A^B \qquad (4.3)}$$

Note that the signs in Eq. (4.3) are correct only if $x_B > x_A$.

4. The difference between the bending moments at two sections of a beam is equal to the area of the shear force diagram between these two sections.

Proof —integrating Eq. (4.2) between sections *A* and *B* in (see Fig. 4.5),



$$V = \frac{dM}{dx}$$

$$\boxed{\int_{XA}^{XB} \frac{dM}{dx} dx = M_B - M_A = \int_{XA}^{XB} V dx}$$

$M_B - M_A$ = area of V-diagram$]_A^B$ Q.E.D

$$\boxed{M_B = M_A + \text{area of } V\text{-diagram}]_A^B \qquad (4.4)}$$

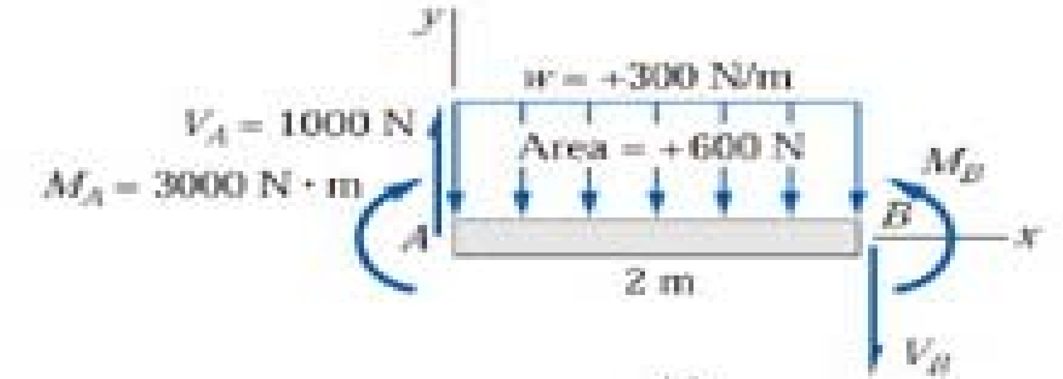
The signs in Eq. (4.4) are correct only if $x_B > x_A$.

5. If the load W diagram is a polynomial of degree *n*, then shear force V diagram is polynomial of degree (*n* +1), and the bending moment M diagram is polynomial of degree (*n* +2).

Proof — followings directly from the integration of Eqs. (4.1) and (4.2).

$$\boxed{w = -\frac{dV}{dx}} \qquad \boxed{V = \frac{dM}{dx}}$$

- Consider the beam segment shown in Fig. 4.6 (a), which is 2 m long and is subjected to a uniformly distributed load w = 300 N /m. Figure 4.6 (a) shows the shear force and the bending moment at the left end are V_A= +1000 N and M_A = +3000 N· m.



(a)

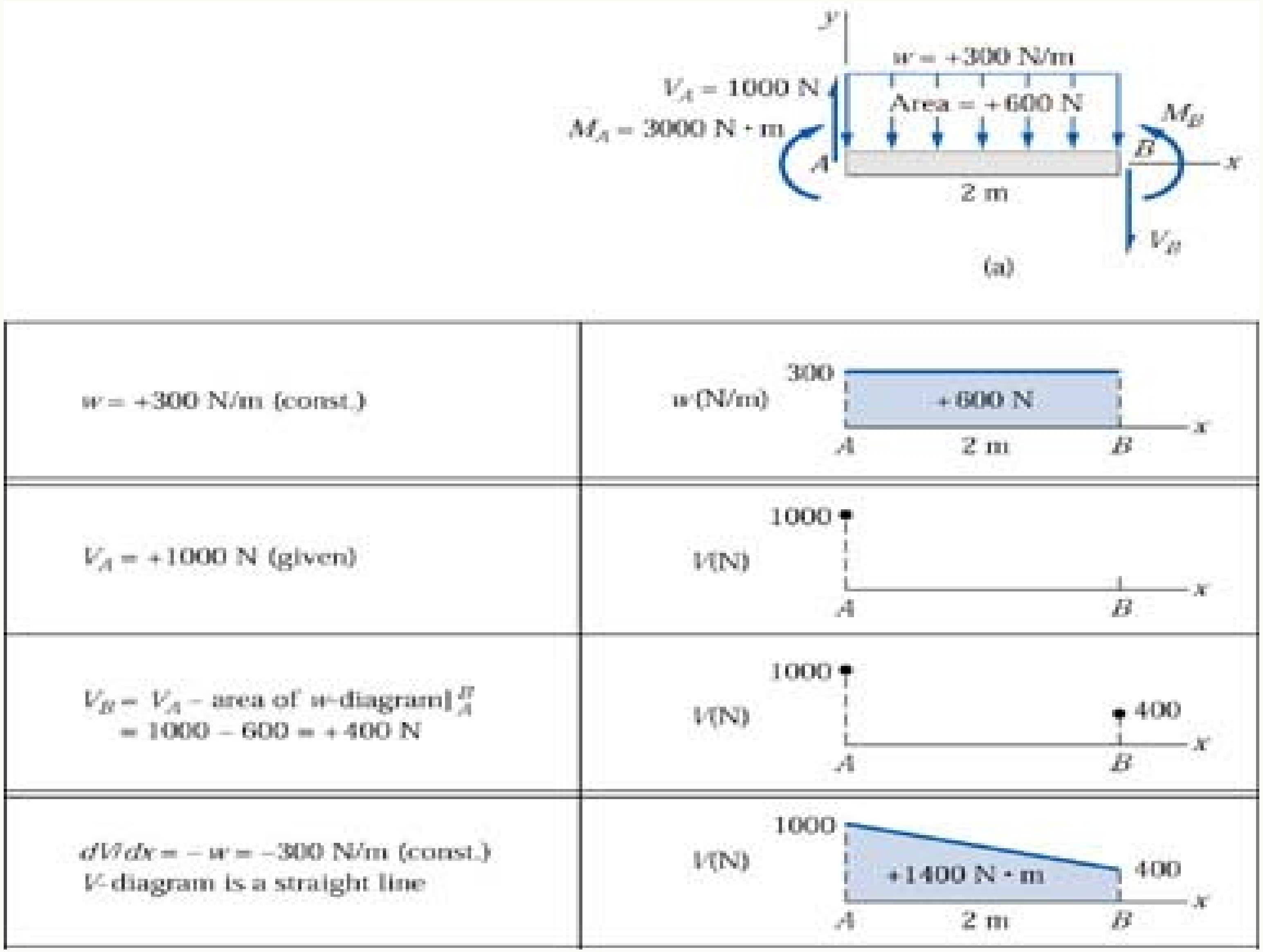


Figure 4.6 (a) Free-body diagram of a beam segment carrying uniform loading;

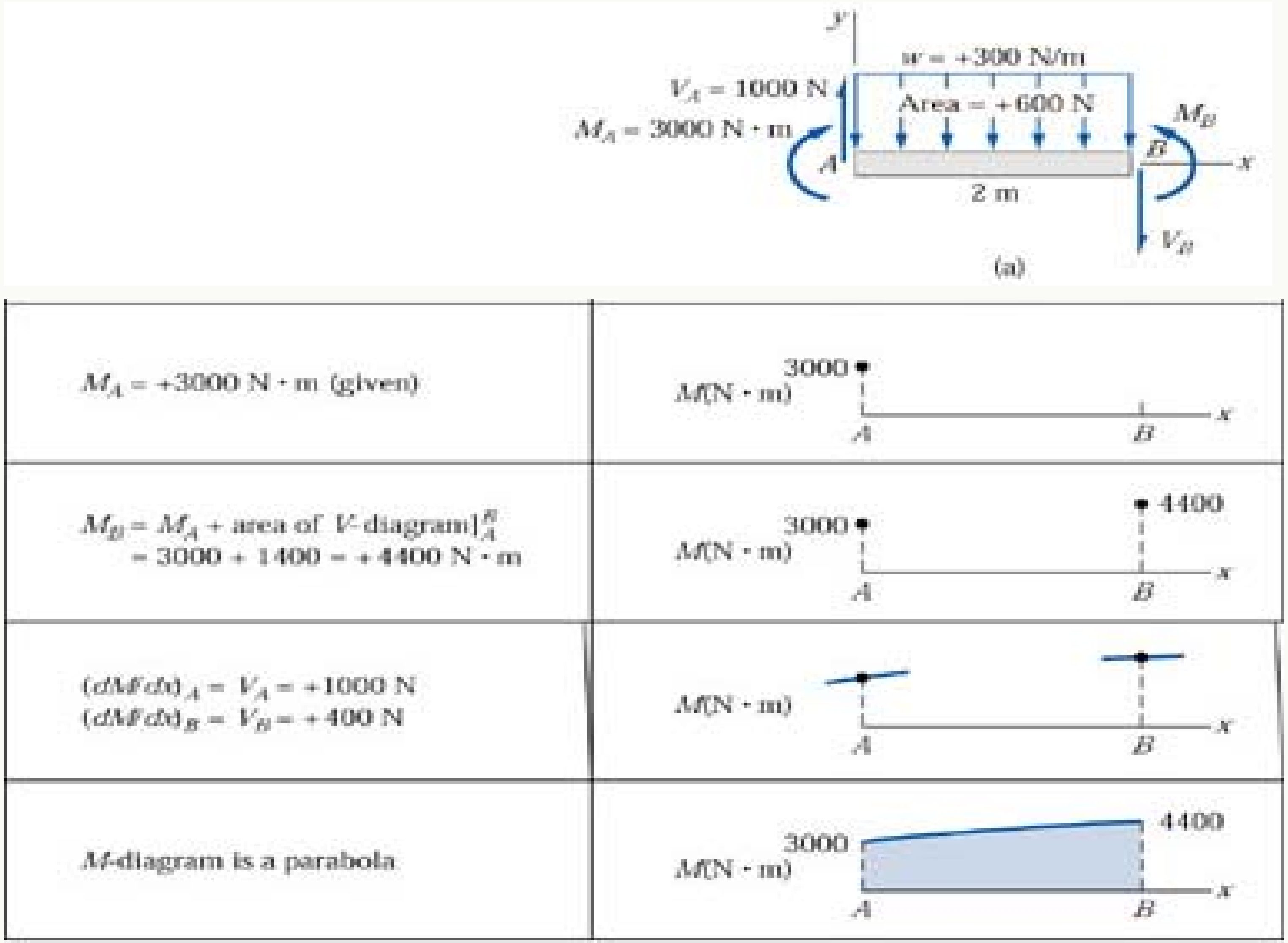


Figure 4.6(b) constructing shear force and bending moment diagrams for the beam segment.

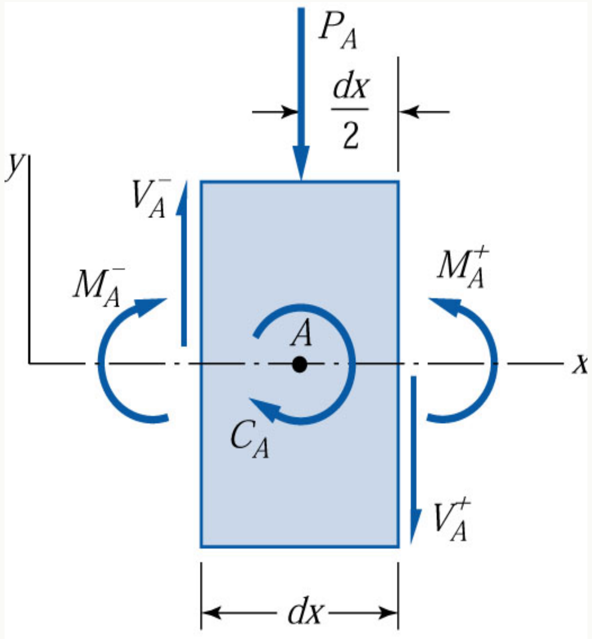


Figure 4.7 Free-body diagram of an infinitesimal beam element carrying a concentrated force P_A and a concentrated couple C_A.

b. *Concentrated forces and couples*.

- The force equilibrium equation

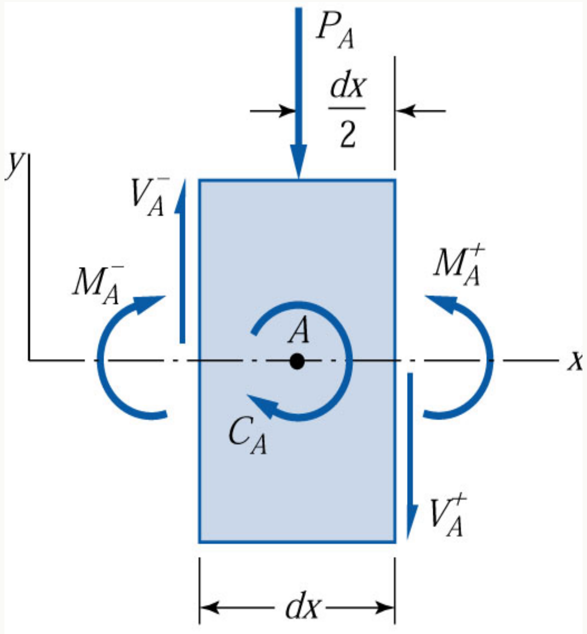
$$\sum F_y = 0 + \uparrow \qquad V_A^- - P_A - V_A^+ = 0 \qquad \boxed{V_A^+ = V_A^- - P_A \qquad (4.5)}$$

Equation (4.5) indicates that a positive concentrated force causes a negative jump discontinuity in the shear force diagram at A (a concentrated couple does not affect the shear force diagram).

The moment equilibrium equation yields

$$\sum M_A = 0 \; +\circlearrowleft \qquad M_A^+ - M_A^- - C_A - V_A^+ \frac{dx}{2} - V_A^- \frac{dx}{2} = 0$$

$$\boxed{M_A^+ = M_A^- + C_A}$$



Thus, a positive concentrated couple causes a positive jump in the bending moment diagram.

Figure 4.7 Free-body diagram of an infinitesimal beam element carrying a concentrated force P_A and a concentrated couple C_A.

c. Summary

The area method is useful only if the area under the load and shear force diagrams can be easily computed.

$$\boxed{w = -\frac{dV}{dx}} \qquad (4.1)$$

$$\boxed{V = \frac{dM}{dx}} \qquad (4.2)$$

$$\boxed{V_B = V_A - \text{area of w-diagram}]_A^B} \qquad (4.3)$$

$$\boxed{M_B = M_A + \text{area of w-diagram}]_A^B} \qquad (4.4)$$

$$\boxed{V_A^+ = V_A^- - P_A} \qquad (4.5)$$

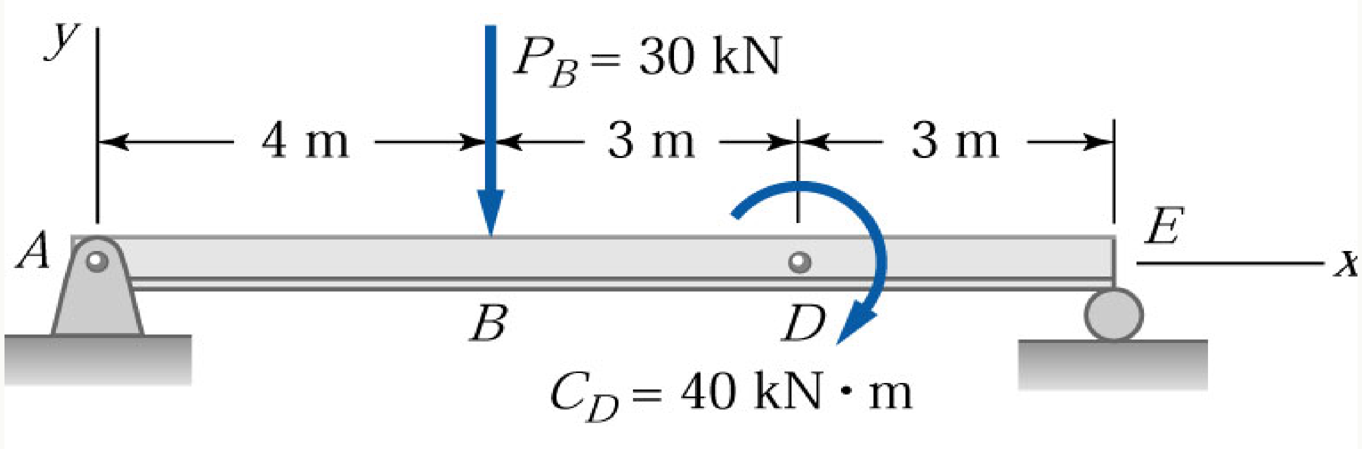
$$\boxed{M_A^+ = M_A^- + C_A} \qquad (4.6)$$

Procedure for the Area Method

- Compute the support reactions force the free-body diagrams (FBD) of the entire beam.
- Draw the load diagram of the beam (which is essentially a FBD) showing the values of the loads, including the support reactions. Use the sign conventions in Fig. 4.3 to determine the correct sign of each load.
- Working from left to right, construct the *V*-and *M*-diagram for each segment of the beam using Eqs. (4.1)-(4.6).
- When reach the right end of the beam, check to see whether the computed values of *V*-and *M* are consistent with the end conditions. If they are not, you made an error in the computations.

Sample Problem 4.4

The simply supported beam in Fig. (a) supports 30-kN concentrated force at *B* and a 40-kN·m couple at *D*. Sketch the shear force and beading moment diagrams by the area method. Neglect the weight of the beam.

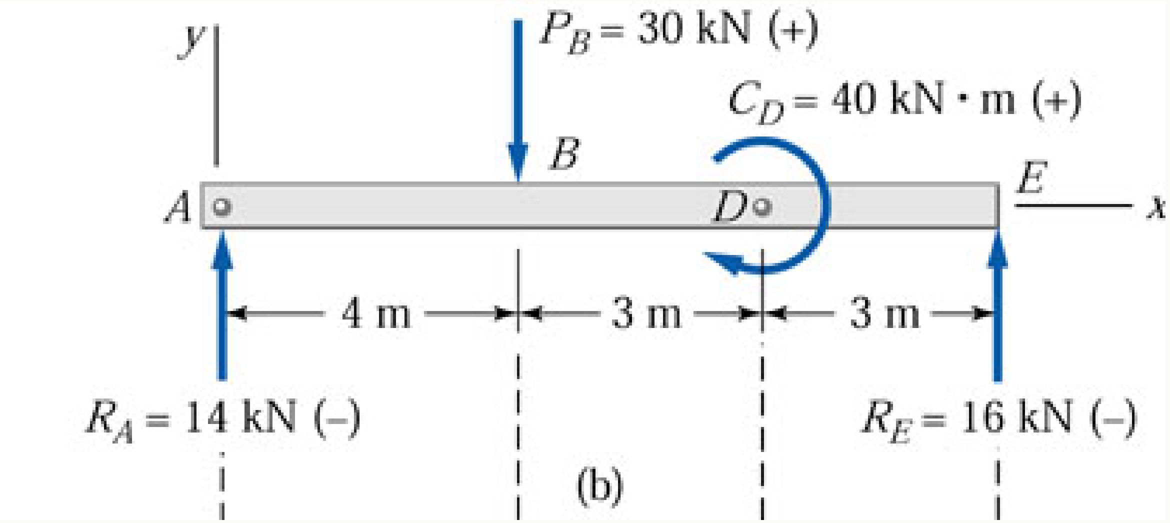


(a)

Solution

Load Diagram

The load diagram for the beam is shown in Fig. (b). The reactions at *A* and *E* are found from equilibrium analysis. Indicating its sign as established by the sign conventions in Fig. 4.3.



(b)

Shear Force Diagram

There are concentrated forces at *A*,*B*, noting that V_A - = 0 because no load is applied to the left of A

$$V_A^+ = V_A^- - R_A = 0 - (-14) = (+14kN)$$

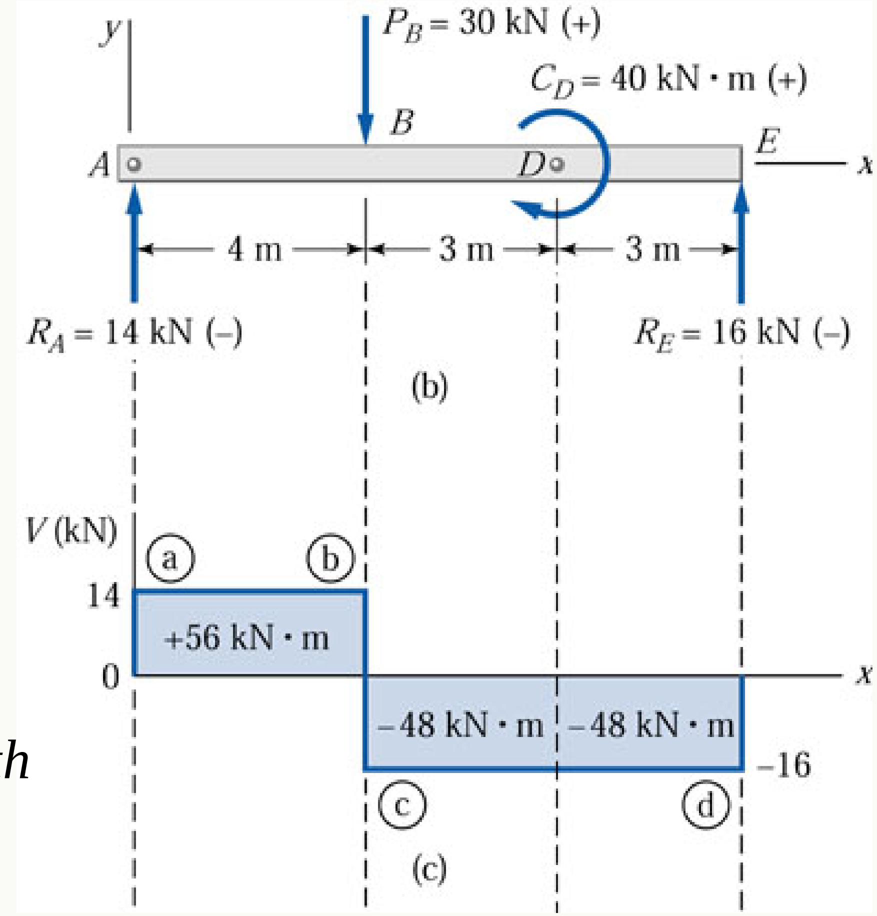
Plot point. a

$V_B^- = V_A^-$ - area of *w*-diagram$]_A^B$

=14-0 = 14 kN *Plot point.* b

Because $w = -dV/dx = 0$ between *A* and *B*,

Connect a and b *with a horizontal straight line*



$$V_B^+ = V_B^- - P_B = 14 - (+30) = -16kN$$ *Plot point.* c

$V_E^- = V_B^+$ -area of *w*-diagram$]_B^E$ = -16-0 = -16 kN *Plot point.* d

Because $w = -dV/dx = 0$ between *B* and *E*

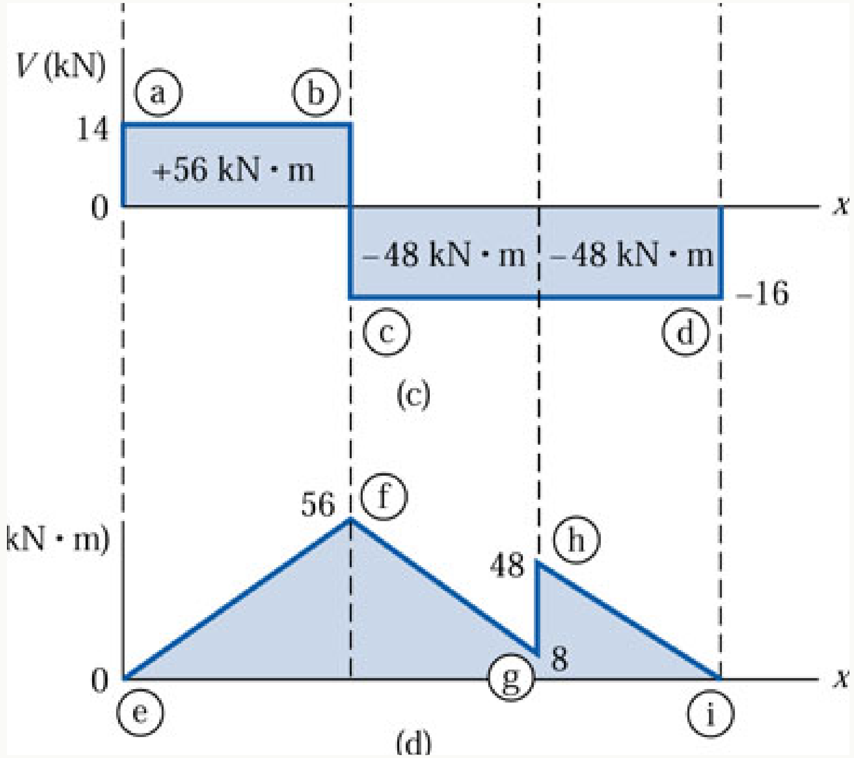
Connect c and d *with a horizontal line*

$$V_E^+ = V_E^- - R_E = -16 - (-16) = 0$$

Check !

Bending Moment Diagram

- The applied couple is cause a jump in the bending moment diagram at D.
- The areas are either positive or negative, depending on the sign of the shear force in Fig. (c). $M_A = 0$ (there is no couple applied at A). point (e)



$M_B = M_A +$ area of V-diagram$]_A^B = 0 + (+56) = 56$ kN· m , point (f)

The shear force bewteen A and B is constant and positive. The slope of the M-diagram between these two sections is also constant and positive. (recall that $V = dM/dx = 0$), connect (e) and (f) with straight line.

$M_D^- = M_B +$ area of V-diagram$]_B^D = 56 + (-48) = 8$ kN· m , point (g)

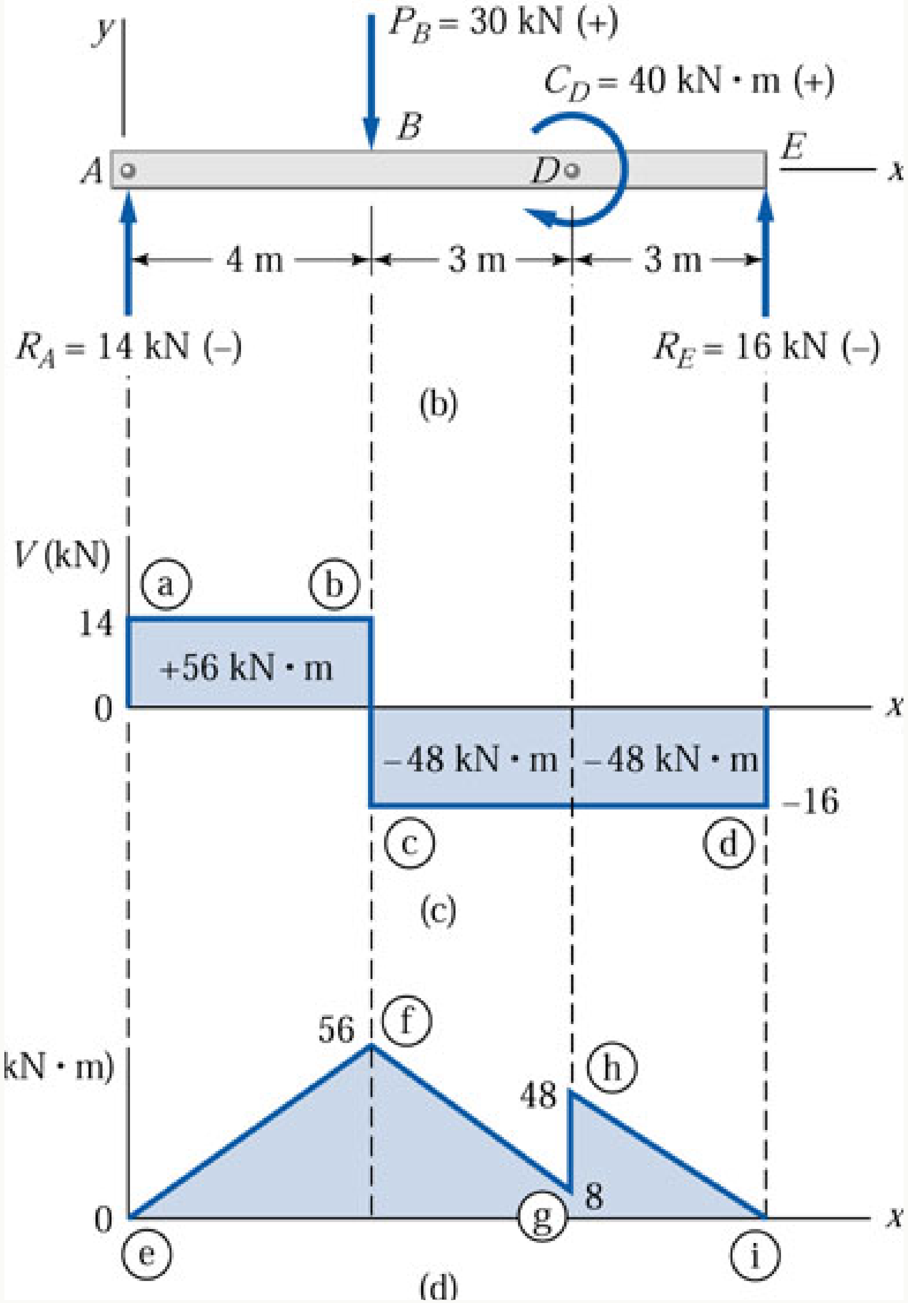
The slope of the V diagram between B and D is negative and constant, the M-diagram has a constant, negative slope in this segment,), connect (f) and (g) with straight line.

$$M_D^{+} = M_D^{-} + C_D$$
$$= 8 + (+40) = 48 \text{ kN}\cdot\text{m}$$

Point (h), note that $M_E = 0$ (these is no couple applied at *E*).

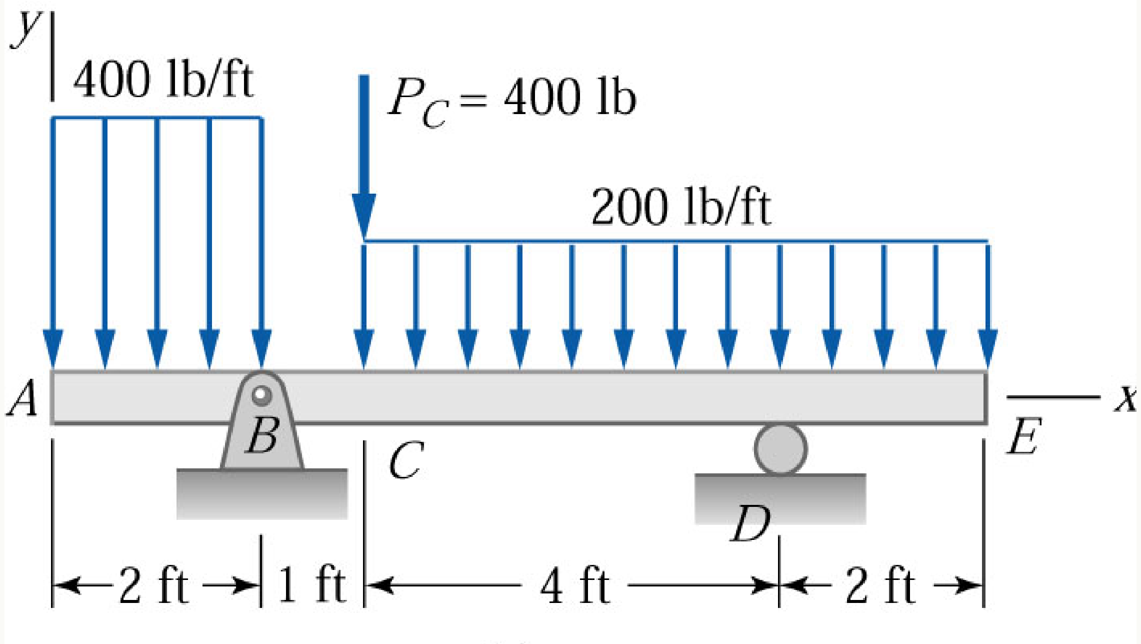
$$M_E = M_D^{+} + \text{area of } V\text{-diagram}]_D^E$$
$$= 48 + (-48) = 0 \quad \textit{Check !}$$

The shear force between D and E is negative and constant, which means that the slope of the M -diagram for this segment is also constant and negative, connect (h) and (i) with straight line.

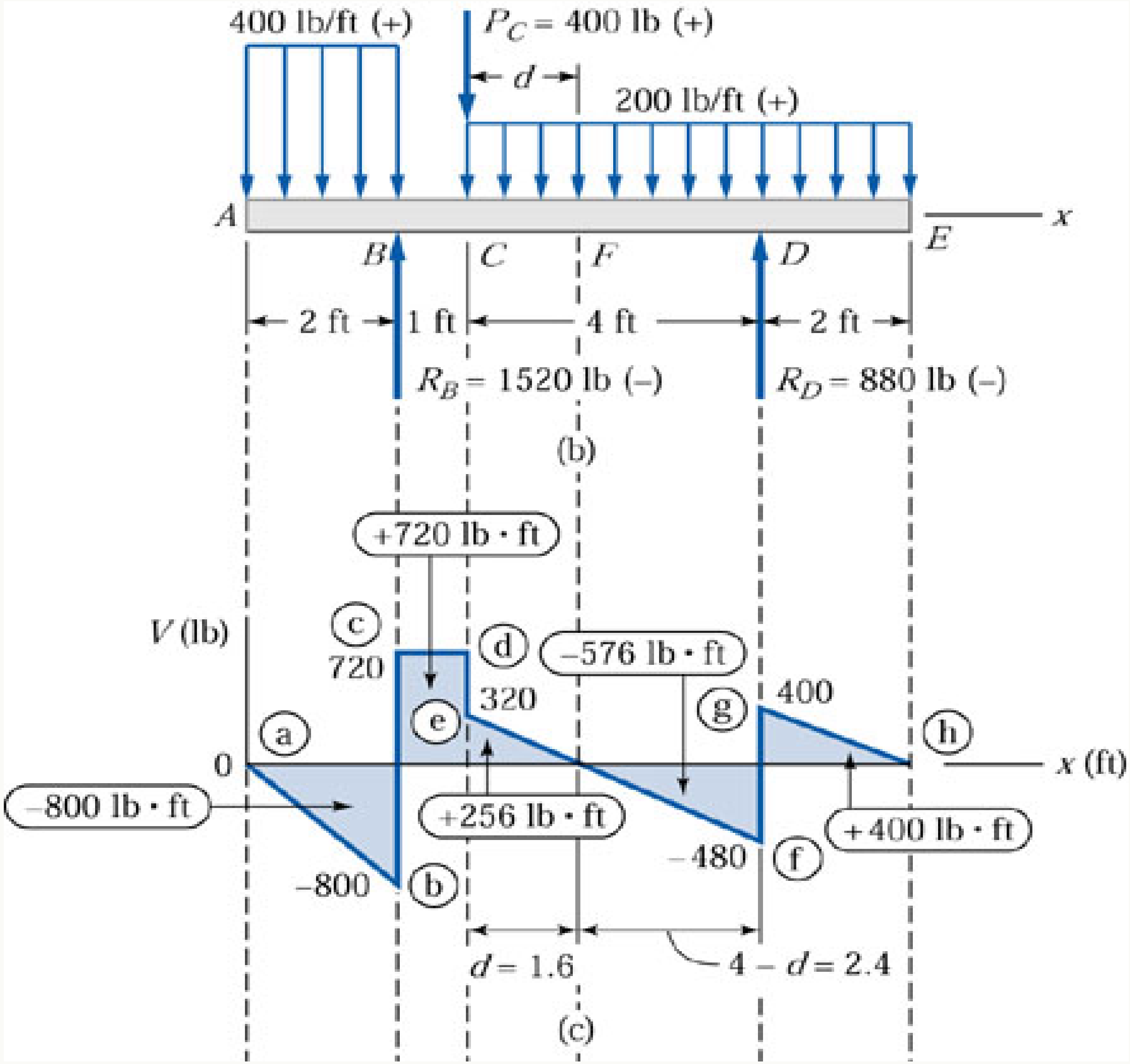


Sample Problem 4.5

The overhanging beam in Fig. (a) carries two uniformly distributed loads and a concentrated load Using the area method. Draw the shear force and bending moment diagrams for the beam.



(a)

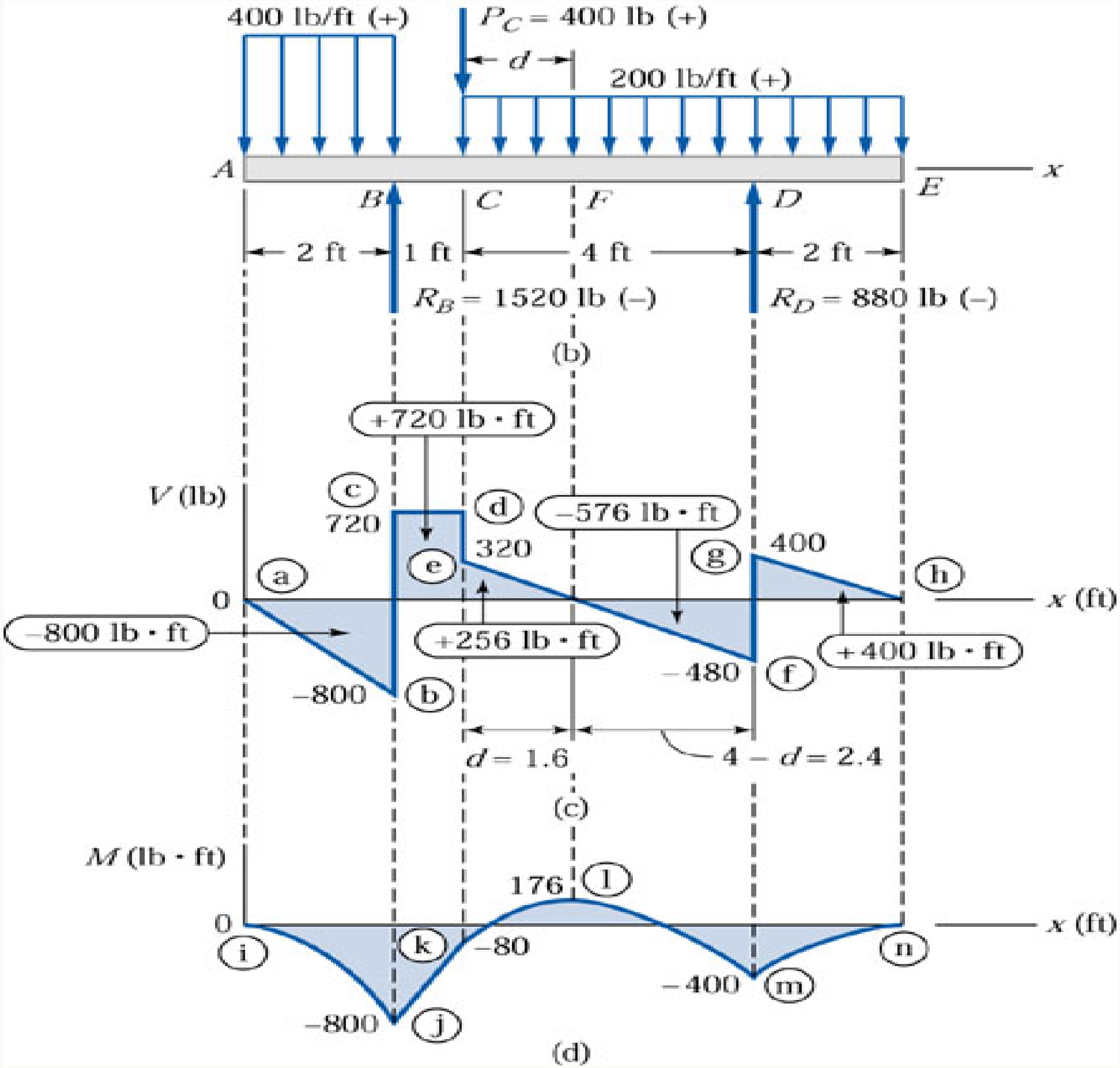


Load Diagram

The load diagram for the beam is given in Fig. (b)

Shear Force Diagram

The steps required to construct the shear force diagram in Fig. (c) are now detailed.



Bending Moment Diagram

The slope of the *M*-diagram is discontinuous at j and m.