MATHEMATICAL TOOLS FOR ECONOMICS: PROOFS

Thm 1. (Deduction) Prove that $f(x)=x^2$ is continuous at $x=5$.

Proof. We know that $f(x)$ continuous means: $\forall \varepsilon>0\ \exists \delta>0$ s.t. $|x-5|<\delta \Rightarrow |f(x)-f(5)|<\varepsilon$.

Let $\varepsilon>0$ be given.

Let $\delta=\min\{1, \frac{\varepsilon}{11}\}>0 \Rightarrow \delta \le \frac{\varepsilon}{11}$

→ Suppose $|x-5|<\delta$. Since $\delta\le 1$, $4<x<6$ so $9<x+5<11$ & $|x+5|<11$.

Then $|f(x)-f(5)| = |x^2-25| = |(x+5)(x-5)| = |x+5||x-5|$

$< 11\cdot\delta \le 11\cdot\frac{\varepsilon}{11} = \varepsilon$

Thus $\forall \varepsilon>0\ \exists \delta>0$ s.t. $|x-5|<\delta \Rightarrow |f(x)-f(5)|<\varepsilon$.

So f is continuous at $x=5$. ▨

Thm 2. (Contraposition) $(P\Rightarrow Q) \Leftrightarrow (\neg P \vee Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

Proof. ($\Rightarrow$) Suppose $P\Rightarrow Q$ is true. Then $\neg P \vee Q$ is true.

Therefore either $\neg P$ is true or $\neg Q$ is false. So $\neg(\neg Q)\vee(\neg P)$. Thus $\neg Q \Rightarrow \neg P$.

($\Leftarrow$) Suppose $\neg Q\Rightarrow \neg P$ is true. Then $Q\vee\neg P$ is true. So $P\Rightarrow Q$ is true. ▨

Thm 3. (Induction) $\forall n\in \mathbb{N}_0=\{0,1,2,3,\dots\}$ (i.e. $\mathbb{N}\cup\{0\}$) then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof. (Base Step). Let $n=0$.

LHS: $\sum_{k=1}^{0} k = \text{empty sum} = 0$

RHS: $\frac{0\cdot 1}{2}=0$ } LHS = RHS

(Induc. Step). We will show $\sum_{k}^{n+1} k = \frac{(n+1)(n+1+1)}{2}$

LHS: $\sum_k^{n+1} k = \sum_k^n k + (n+1) = \frac{n(n+1)}{2} + n+1$

$= \frac{n(n+1)+2n+2}{2} = \frac{(n+1)(n+2)}{2}$

RHS: $\frac{(n+1)(n+1+1)}{2} = \frac{(n+1)(n+2)}{2}$ } LHS = RHS

By induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n\in\mathbb{N}_0$. ▨

Thm 4. (Contradiction) There is no rational number q s.t. $q^2=2$.

Proof. Suppose $q^2=2$ where $q\in\mathbb{Q}$. Then we can write $q=\frac{m}{n}$ for some integers $m,n\in\mathbb{Z}$.

We can also assume m & n have have no common factor.

Thus $2=q^2=\frac{m^2}{n^2}$. Therefore $m^2=2n^2$ so m^2 is even.

→ Claim: m is even.

If m is odd then $m=2p+1$ for some $p\in\mathbb{Z}$.

Then $m^2=(2p+1)^2=4p^2+4p+1=2(2p^2+2p)+1$ which is odd.

This is a contradiction to $m^2=2n^2$.

So m is even → $m=2r$ for some $r\in\mathbb{Z}$.

So $m^2=(2r)^2=4r^2=2n^2 \rightarrow n^2=2r^2$. Thus n^2 is even & n is even (argument above)

Therefore $n=2s$ for some $s\in\mathbb{Z}$.

Thus m & n have a common factor. Contradiction. ▨

Thm 5. (Partition) Let R be an equivalence relation on X. Then $\forall x\in X$, $x\in[x]$. Given $x,y\in X$, either $[x]=[y]$ (same set) or $[x]\cap[y]=\varnothing$ (disjoint).

Proof. If $x\in X$ then xRx bc R is reflexive so $x\in[x]$.

1) Suppose $x,y\in X$. If $[x]\cap[y]=\varnothing$ then we are done.

2) Suppose $[x]\cap[y]\neq\varnothing$.

Choose $z\in[x]\cap[y]$. Then $z\in[x]$ so xRz. By symmetry, zRx.

Also $z\in[y]$ so yRz. By symmetry, zRy.

Let $w\in[x] \rightarrow xRw$.

Since zRx & R is transitive, zRw. By symmetry, wRz.

—— zRy ——, wRy. ——, yRw. So $w\in[y]$.

Thus $w\in[y] \rightarrow [x]\subseteq[y]$.

3) Similarly $[y]\subseteq[x] \rightarrow [x]=[y]$. ▨

Thm 6. The set of rational numbers $\mathbb{Q}$ is countable.

Proof. $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$
$= \{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \}$

Go on diagonal & omit repeats.

$f: \mathbb{N} \to \mathbb{Q}$ is 1-1 & onto (bijection)

$$\begin{array}{c|cccc} & & m & & \\ n & 0 & 1 & -1 & \cdots \\ \hline 1 & 0 & 1 & -1 & \\ 2 & 0 & \frac{1}{2} & -\frac{1}{2} & \\ 3 & 0 & \frac{1}{3} & -\frac{1}{3} & \\ \vdots & & & & \end{array}$$

$\boxtimes$

Thm 7. (Cantor's Diagonalization). $2^{\mathbb{N}}$, the set of all subsets of $\mathbb{N}$, is not countable.

Proof. Suppose $2^{\mathbb{N}}$ is countable. Then $\exists$ bijection $f: \mathbb{N} \to 2^{\mathbb{N}}$.
Let $A_m = f(m)$. We create an infinite matrix whose $(m,n)^{th}$ entry is 1
if $n \in A_m$ & 0 otherwise (indicator func. of set A_m)

	1	2	3	4	... $\mathbb{N}$
$A_1 = \emptyset$	0	0	0	0	
$A_2 = \{1\}$	1	0	0	0	
$A_3 = \{1, 2, 3\}$	1	1	1	0	
$A_4 = \mathbb{N}$	1	1	1	1	
$A_5 = 2\mathbb{N}$	0	1	0	1	

Goal: Find one more set not enumerated

Change all 0's to 1's & vice versa on diagonal

Formal. Let $t_{mn} = \begin{cases} 1 & \text{if } n \in A_m \\ 0 & \text{if } n \notin A_m \end{cases}$

Let $A = \{ m \in \mathbb{N} : t_{mm} = 0 \}$
Then $m \in A \Leftrightarrow t_{mm} = 0 \Leftrightarrow m \notin A_m$
$1 \in A \Leftrightarrow 1 \notin A_1$ so $A \neq A_1$
$2 \in A \Leftrightarrow 2 \notin A_2$ so $A \neq A_2$
$\vdots$
$m \in A \Leftrightarrow m \notin A_m$ so $A \neq A_m$

Therefore $A \neq f(m)$ for any m, so f is not onto. Contradiction

$\boxtimes$

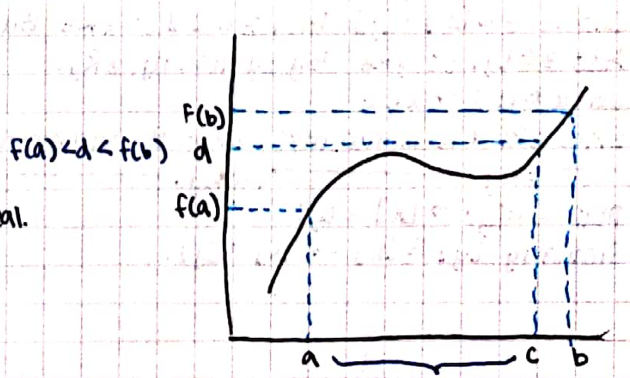
Thm 8. The Supremum Property & the Completeness Axiom are equivalent.

Proof. ($\Leftarrow$) Assume Completeness Axiom.
Let $X \subseteq \mathbb{R}$ be a nonempty set bounded above.
Let U be the set of all upper bounds for X. Since X is bounded above, $U \neq \emptyset$.
If $x \in X$ & $u \in U$, $x \leq u$ since u is an upper bound for X.
So $x \leq u$ $\forall x \in X$, $u \in U$.
By the Completeness Axiom, $\exists \alpha \in \mathbb{R}$ s.t. $x \leq \alpha \leq u$ $\forall x \in X$, $u \in U$.
$\rightarrow \alpha$ is an upper bound for X & it is $\leq$ every other upper bound for X.
So it is the least upper bound for X.
So $\sup X = \alpha \in \mathbb{R}$.
The case which X is bounded below is similar.
Thus the Supremum Property holds.

($\Rightarrow$) Assume the Supremum Property. Suppose $L, H \subseteq \mathbb{R}$, $L \neq \emptyset \neq H$ & $l \leq h$ $\forall l \in L$, $h \in H$.
Since $L \neq \emptyset$ & L is bounded above by any element of H, $\alpha = \sup L$ exists & is real.
By def of supremum, α is an upper bound for L so $l \leq \alpha$ $\forall l \in L$.
Suppose $h \in H$. Then h is an upper bound for L so by the def of supremum, $\alpha \leq h$.
Therefore $l \leq \alpha \leq h$ $\forall l \in L$, $h \in H$.
So Completeness Axiom holds. $\boxtimes$

Thm 9. (Intermediate Value Theorem) Suppose $f: [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is continuous, & $f(a) < d < f(b)$.
Then there exists $c \in (a,b)$ s.t. $f(c) = d$.

Proof. Use the Supremum Property.
Let $B = \{ x \in [a,b] : f(x) < d \}$
$\rightarrow a \in B$ so $B \neq \emptyset$
$\rightarrow B \subseteq [a,b]$ so B is bounded above
$\rightarrow$ By Supremum Prop., $\sup B$ exists & is real.
Let $c = \sup B$.
$\rightarrow$ Since $a \in B$, $c \geq a$; $B \subseteq [a,b]$ so $c \leq b$.
Therefore $c \in [a,b]$
We claim that $f(c) = d$



$B = \{ x \in [a,b] : f(x) < d \}$
$c = \sup B$

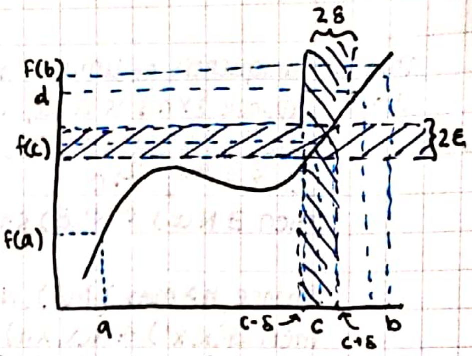
① Suppose $f(c) < d$.

Since $f(b) > d$, $c \neq b$, so $c < b$.

Let $\varepsilon = \frac{d - f(c)}{2} > 0$. Since f is continuous at c, $\exists\, \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\Rightarrow f(x) < f(c) + \varepsilon = f(c) + \frac{d - f(c)}{2} = \frac{f(c) + d}{2} < \frac{d + d}{2} = d$$



So $(c, c+\delta) \subseteq B$, so $c \neq \sup B$. Contradiction.

$f(c) < d \Rightarrow \exists\, \delta > 0$ s.t. for $x \in (c-\delta, c+\delta)$, $f(x) < d$ $\Rightarrow c \neq \sup B$

② Suppose $f(c) > d$.

Since $f(a) < d$, $a \neq c$ so $c > a$.

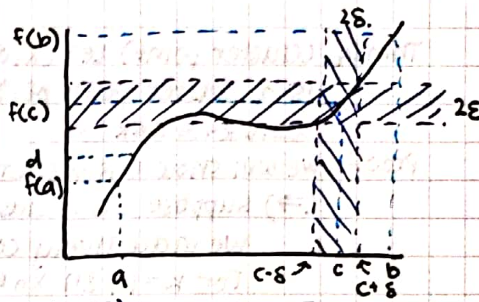
Let $\varepsilon = \frac{f(c) - d}{2} > 0$.

$f(c) > d \Rightarrow \exists\, \delta > 0$ s.t. $f(x) > d \ \forall x \in (c-\delta, c+\delta)$

Since f is continuous at c, $\exists\, \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\Rightarrow f(x) > f(c) - \varepsilon = f(c) - \frac{f(c) - d}{2} = \frac{f(c) + d}{2} > \frac{d + d}{2} = d$$



So $(c - \delta, c + \delta) \cap B = \emptyset$. So either

→ $\exists\, x \in B$ w/ $x \geq c + \delta$ so c is not an upper bound for B

→ $c - \delta$ is an upper bound for B → c is not the least upper bound for B

$(c-\delta, c+\delta) \cap B = \emptyset \Rightarrow$ either $\exists\, y \in [c+\delta, b] \cap B$ or $B \subseteq [a, c-\delta)$. $c \neq \sup B$

Thus $c \neq \sup B$. Contradiction.

Since $f(c) \not< d$, $f(c) \not> d$, & the order is complete, $f(c) = d$.

Since $f(a) < d$ & $f(b) > d$, $a \neq c \neq b$ so $c \in (a, b)$. ▨

Thm 10. (Archimedean Property). $\forall x, y \in \mathbb{R}, y > 0\ \exists\, n \in \mathbb{N}$ s.t. $ny = (y + \cdots + y) > x$

Proof. Contradiction using supremum property.

Separate exercise.

Thm 11. Let $(V, \|\cdot\|)$ be a normed vector space. Let $d: V \times V \to \mathbb{R}_+$ but defined by $d(v, w) = \|v - w\|$. Then (V, d) is a metric space.

Proof. Properties of a metric

1. Let $v, w \in V$. Then by def, $d(v, w) = \|v - w\| \geq 0$ &

$$d(v, w) = 0 \Leftrightarrow \|v - w\| = 0$$
$$\Leftrightarrow v - w = 0 \Leftrightarrow \text{vector additive identity}$$
$$\Leftrightarrow (v + (-w)) + w = w \Leftrightarrow v + ((-w) + w) = w$$
$$\Leftrightarrow v + 0 = w \Leftrightarrow v = w.$$

2. Symmetry: note $\forall x \in X$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$ so $0 \cdot x = 0$.

Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1)x$ so $(-1) \cdot x = -x$.

Then let $v, w \in V$ so

$$d(v, w) = \|v - w\| = |-1|\, \|v - w\| = \|(-1)(v + (-w))\|$$
$$= \|(-1)v + (-1)(-w)\| = \|-v + w\| = \|w + (-v)\|$$
$$= \|w - v\| = d(w, v).$$

3. Tri ineq: Let $u, w, v \in V$

$$d(u, w) = \|u - w\| = \|u + (-v + v) - w\|$$
$$= \|(u - v) + (v - w)\|$$
$$\leq \|u - v\| + \|v - w\| = d(u, v) + d(v, w).$$

Thus d is a metric on V. ▨

Thm 12. (Cauchy-Schwartz Inequality). If $v, w \in \mathbb{R}^n$ then

$$\left(\sum_{i=1}^n v_i w_i\right)^2 \leq \left(\sum_{i=1}^n v_i^2\right)\left(\sum_{i=1}^n w_i^2\right)$$

$$|\langle v, w \rangle|^2 = |v \cdot w|^2 \leq |v|^2 |w|^2 = \|v\|^2 \|w\|^2$$

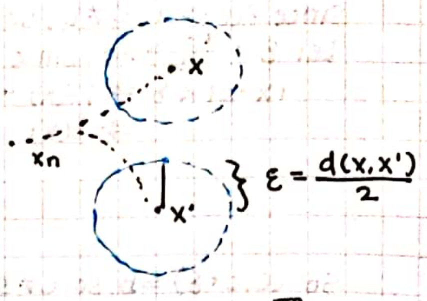
$$|\langle v, w \rangle| = |v \cdot w| \leq |v|\,|w| = \|v\|\,\|w\|$$

Show the triangle inequality of $\|\cdot\|_2$ in $\mathbb{R}^n$ follows from the C-S inequality.

Proof. Separate exercise.

Thm 13: (Uniqueness of Limits) In a metric space (X,d), if $x_n \to x$ & $x_n \to x'$ then $x = x'$.

Proof. Suppose $\{x_n\}$ is a seq. in X, $x_n \to x$, $x_n \to x'$, & $x \neq x'$.
Since $x \neq x'$, $d(x,x') > 0$.
Let $\varepsilon = \frac{d(x,x')}{2} > 0$.
Then $\exists N(\varepsilon)$ & $N'(\varepsilon)$ s.t. $n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$ $(x_n \to x)$.
$n > N'(\varepsilon) \Rightarrow d(x_n, x') < \varepsilon$ $(x_n \to x')$.
Choose $n > \max\{N(\varepsilon), N'(\varepsilon)\}$
Then $d(x,x') \leq d(x,x_n) + d(x_n, x')$
$< \varepsilon + \varepsilon = 2\varepsilon = d(x,x')$.
But $d(x,x') < d(x,x')$ is a contradiction. ▨



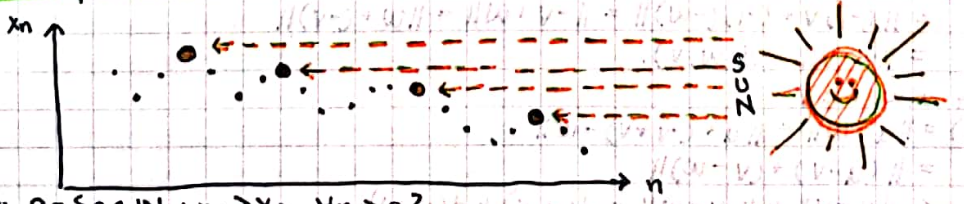
Thm 14. (Cluster point) Let (X,d) be a metric space, $c \in X$ & $\{x_n\}$ a sequence in X. Then c is a cluster point of $\{x_n\}$ if & only if there is a subsequence $\{x_{n_k}\}$ s.t. $\lim_{k\to\infty} x_{n_k} = c$.

Proof. We will show c is a cluster point of $\{x_n\} \Leftrightarrow \exists \{x_{n_k}\}$ s.t. $\lim_{n\to\infty} x_{n_k} = c$.
($\Rightarrow$) Suppose c is a cluster ______.
We inductively construct a subseq that converges to c.
For $k=1$, $\{n : x_n \in B_1(c)\}$ is infinite, so nonempty.
Let $n_1 = \min\{n : x_n \in B_1(c)\}$ (first seq in $B_1(c)$).
Suppose we chose $n_1 < n_2 < \cdots < n_k$ s.t. $x_{n_j} \in B_{\frac{1}{j}}(c)$ for $j = 1, \ldots, k$.
$\{n : x_n \in B_{\frac{1}{k+1}}(c)\}$ is infinite so it contains at least one element larger than n_k.
Let $n_{k+1} = \min\{n : n > n_k, x_n \in B_{\frac{1}{k+1}}(c)\}$
Thus we have chosen $n_1 < n_2 < \cdots < n_k < n_{k+1}$ s.t. $x_{n_j} \in B_{\frac{1}{j}}(c)$ for $j = 1, \ldots, k, k+1$
Thus by induc., we obtain subseq. $\{x_{n_k}\}$ s.t. $x_{n_k} \in B_{\frac{1}{k}}(c)$.
Given any $\varepsilon > 0$ by the Archimedean property, $\exists N(\varepsilon) > \frac{1}{\varepsilon}$
$k > N(\varepsilon) \Rightarrow x_{n_k} \in B_{\frac{1}{k}}(c)$
$\Rightarrow x_{n_k} \in B_\varepsilon(c)$
$x_{n_k} \to c$ as $k \to \infty$.
($\Leftarrow$) Conversely, suppose that there is a subseq $\{x_{n_k}\}$ converging to c.
Given any $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_\varepsilon(c)$.
Therefore, $\{n : x_n \in B_\varepsilon(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \ldots\}$
Since $n_{K+1} < n_{K+2} < n_{K+3} < \cdots$, this set is infinite, so c is a cluster point of $\{x_n\}$. ▨

Thm 15. (Increasing/Decreasing Sequences) Let $\{x_n\}$ be an increasing (decreasing) seq of real numbers. Then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ ($\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$). In particular, the limit exists.

Proof. Separate exercise & try w/unbounded case

Thm 16. (Rising Sun Lemma) Every seq of real numbers contains an increasing or decreasing subseq or both!



Proof. Let $S = \{s \in \mathbb{N} : x_s > x_n, \forall n > s\}$
Either S is infinite or S is finite
→ If S is infinite, let $n_1 = \min S$
$n_2 = \min(S \setminus \{n_1\})$
$n_3 = \min(S \setminus \{n_1, n_2\})$
$\vdots$
$n_{k+1} = \min(S \setminus \{n_1, n_2, \cdots n_k\})$
Then $n_1 < n_2 < n_3 < \cdots$; $x_{n_1} > x_{n_2}$ since $n_1 \in S$ & $n_2 > n_1$
$\vdots$
$x_{n_k} > x_{n_{k+1}}$ since $n_k \in S$ & $n_{k+1} > n_k$.

So $\{x_{n_k}\}$ is a strictly decreasing subseq. of $\{x_n\}$

→ If S is finite & nonempty, let $n_1 = \max S + 1$; if $S = \emptyset$, let $n_1 = 1$.

Then $n_1 \notin S$ so $\exists\, n_2 > n_1$ s.t. $x_{n_2} \geq x_{n_1}$

$n_2 \notin S$ so $\exists\, n_3 > n_2$ s.t. $x_{n_3} \geq x_{n_2}$

$\vdots$

$n_k \notin S$ so $\exists\, n_{k+1} > n_k$ s.t. $x_{n_{k+1}} \geq x_{n_k}$.

∴ So $\{x_{n_k}\}$ is a (weakly) increasing subseq of $\{x_n\}$. ▨

Thm 17. (Bolzano-Weierstrass) Every bounded seq of real numbers contains a converging subsequence.

Proof. Let $\{x_n\}$ be a bounded seq of real numbers.

By the Rising Sun Lemma, find an incr. or decr. subseq $\{x_{n_k}\}$.

→ If $\{x_{n_k}\}$ is incr., then $\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbb{N}\} \leq \sup\{x_n : n \in \mathbb{N}\} < \infty$

Thm 18. (Openness & Closeness). In any metric space (X,d), $\emptyset$ & X are both open & closed.

Proof. (use $\emptyset$ case)

→ Note $\forall x \in \emptyset\ \exists\, \varepsilon > 0\ B_\varepsilon(x) \subseteq \emptyset$ is vacuously true since $\nexists\, x \in \emptyset$. Open.

→ Note $B_\varepsilon(x) = \{z \in X : d(z,x) < \varepsilon\}$ is trivially contained in X.

Since X is open, $\emptyset$ is closed. ▨

Thm 19. (1) The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.
(2) The intersection of a finite collection of open sets is open

Proof. (1) Suppose $\{A_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets.

$x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow \exists\, \lambda_0 \in \Lambda$ s.t. $x \in A_{\lambda_0}$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$

So $\bigcup_{\lambda \in \Lambda} A_\lambda$ is open.

(2) Suppose $A_1, \dots, A_n \subseteq X$ are open sets. If $x \in \bigcap_{i=1}^n A_i$ then

$x \in A_1,\ x \in A_2, \dots, x \in A_n$ so

$\exists\, \varepsilon_1 > 0, \dots, \varepsilon_n > 0$ s.t. $B_{\varepsilon_1}(x) \subseteq A_1, \dots, B_{\varepsilon_n}(x) \subseteq A_n$.

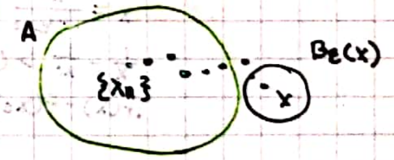
Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} > 0$

Then $B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \dots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$.

So $B_\varepsilon(x) \subseteq \bigcap_{i=1}^n A_i$ which proves $\bigcap_{i=1}^n A_i$ is open. □

Thm 20. A set A in a metric space (X,d) is closed $\Leftrightarrow$ $\{x_n\} \subset A$, $x_n \to x \in X \Rightarrow x \in A$

Proof. ($\Rightarrow$) Suppose A is closed. Then $X \setminus A$ is open.



Consider a convergent seq $x_n \to x \in X$ w/ $x_n \in A$, $\forall n$.

So $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq X \setminus A$.

Since $x_n \to x$, $\exists\ N(\varepsilon)$ s.t. $n > N(\varepsilon) \Rightarrow x_n \in B_\varepsilon(x)$

$\Rightarrow x_n \in X \setminus A \Rightarrow x_n \notin A$

∴ Contradiction. Thus $x_n \subset A$, $x_n \to x \in X \Rightarrow x \in A$.

($\Leftarrow$) Suppose $\{x_n\} \subset A$, $x_n \to x \in X \Rightarrow x \in A$.

Show $X \setminus A$ is open. Suppose not ($X \setminus A$ is not open).

Then $\exists\ x \in X \setminus A$ s.t. $\forall\, \varepsilon > 0$, $B_\varepsilon(x) \not\subseteq X \setminus A$.

→ $\exists\ y \in B_\varepsilon(x)$ s.t. $y \notin X \setminus A$. Then $y \in A$.

Hence $B_\varepsilon(x) \cap A \neq \emptyset\quad \forall \varepsilon > 0$

Construct a seq $\{x_n\}$.

→ For each n, choose $x_n \in B_{\frac{1}{n}}(x) \cap A$.

Given $\varepsilon > 0$, $\exists\ N(\varepsilon)$ s.t. $N(\varepsilon) > \frac{1}{\varepsilon}$ by Arch. Prop.

Thus $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, so $x_n \to x$.

Then $\{x_n\} \subseteq A$, $x_n \to x$ so $x \in A$.

Contradiction. Thus $X \setminus A$ is open.

A is closed. □

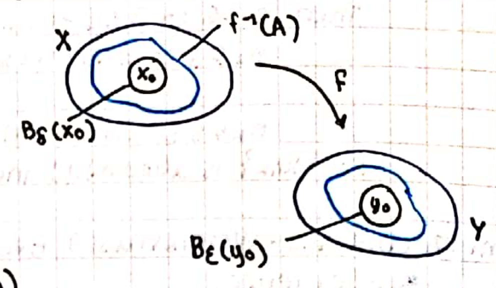
Thm 21. Let (X,d) & (Y,ρ) be metric spaces & $f: X \to Y$.
f is continuous $\Leftrightarrow$ $f^{-1}(A)$ is open in X $\forall A \subseteq Y$ s.t. A is open in Y.

Proof. ($\Rightarrow$) Suppose f is continuous.

Given $A \subseteq Y$, A open, we must show $f^{-1}(A)$ is open in X.
Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$.
Since A is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(y_0) \subseteq A$.
Since f is contin, $\exists\, \delta > 0$ s.t.

$$x \in B_\delta(x_0) \Rightarrow d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$
$$\Rightarrow f(x) \in B_\varepsilon(y_0)$$
$$\Rightarrow f(x) \in A \Rightarrow x \in f^{-1}(A)$$

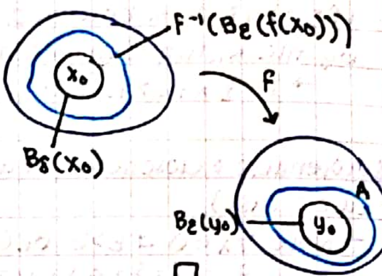
So $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open.



($\Leftarrow$) Suppose $f^{-1}(A)$ is open in X $\forall A \subseteq Y$ s.t. A is open in Y.
Let $x_0 \in X$, $\varepsilon > 0$ & $A = B_\varepsilon(f(x_0))$
A is an open ball $\Rightarrow$ A is an open set $\Rightarrow$ $f^{-1}(A)$ is open in X
$x_0 \in f^{-1}(A)$ so $\exists\, \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(A)$

$$d(x, x_0) < \delta \Rightarrow x \in B_\delta(x_0) \Rightarrow x \in f^{-1}(A)$$
$$\Rightarrow f(x) \in A \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Thus f is contin at x_0, $x_0 \in X$. $\square$



Thm 22. Let (X, d_X), (Y, d_Y), & (Z, d_Z) be metric spaces. If $f: X \to Y$ & $g: Y \to Z$ are contin, then $g \circ f: X \to Z$ is contin.

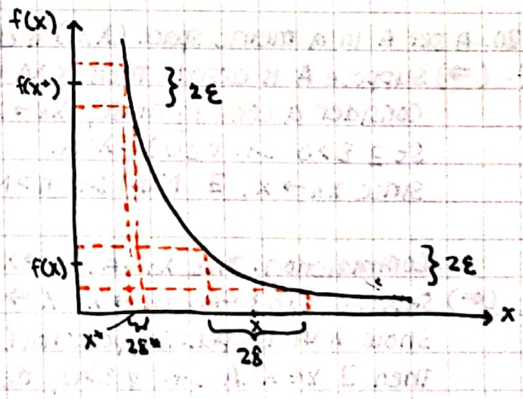
Proof. Suppose $A \subseteq Z$ is open.

Since g is contin, $g^{-1}(A)$ is open in Y.
Since f is contin, $f^{-1}(g^{-1}(A))$ is open in X.
We claim $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$
Observe $x \in f^{-1}(g^{-1}(A)) \Leftrightarrow f(x) \in g^{-1}(A) \Leftrightarrow g(f(x)) \in A$
$\Leftrightarrow (g \circ f)(x) \in A \Leftrightarrow x \in (g \circ f)^{-1}(A)$
So $(g \circ f)^{-1}(A)$ is open in X $\to$ $g \circ f$ is continuous. $\square$

Thm 23. (Uniform Continuity) We have continuous func.
(1) $f(x) = \frac{1}{x}$, $x \in (0,1]$ & (2) $f(x) = \sqrt{x}$, $x \in [0,1]$
Prove whether they are uniformly continuous.

Proof. (1) Fix $\varepsilon > 0$ & $x_0 \in (0,1]$. If $\frac{x_0}{1+\varepsilon x_0} = x$, then

$$1 + \varepsilon x_0 > 1$$
$$x = \frac{x_0}{1+\varepsilon x_0} < x_0$$
$$\frac{1}{x} - \frac{1}{x_0} > 0$$
$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right|$$
$$= \frac{1}{x} - \frac{1}{x_0}$$
$$= \frac{1+\varepsilon x_0}{x_0} - \frac{1}{x_0}$$
$$= \frac{\varepsilon x_0}{x_0} = \varepsilon$$



Thus $\delta(x_0, \varepsilon)$ must be chosen small enough s.t.

$$\left|\frac{x_0}{1+\varepsilon x_0} - x_0\right| \geq \delta(x_0, \varepsilon)$$
$$\delta(x_0, \varepsilon) \leq x_0 - \frac{x_0}{1+\varepsilon x_0} = \frac{\varepsilon (x_0)^2}{1+\varepsilon x_0} < \varepsilon (x_0)^2$$

Which converges to 0 as $x_0 \to 0$. So there is no $\delta(\varepsilon)$ will work for all $x_0 \in (0,1]$

(2). Given $\varepsilon > 0$; Let $\delta = \varepsilon^2$.
Given any $x_0 \in [0,1]$, $|x - x_0| < \delta$ implies by FTC

$$|f(x) - f(x_0)| = \left|\int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt\right|$$
$$\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt = \sqrt{|x - x_0|}$$
$$< \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

Thus f is uniformly contin on $[0,1]$.
(even though $f'(x) \to \infty$ as $x \to 0$) $\square$

Thm 24. (Extreme Value Theorem) Let $a,b \in \mathbb{R}$ with $a \le b$ & let $f:[a,b] \to \mathbb{R}$ be a continuous func. Then f assumes its min & max on $[a,b]$. That is if
$M = \sup_{t \in [a,b]} f(t)$ & $m = \inf_{t \in [a,b]} f(t)$
then $\exists\, t_m, t_M \in [a,b]$ s.t. $f(t_M) = M$ & $f(t_m) = m$. f is bounded above & below.

Proof. Let $M = \sup\{f(t) : t \in [a,b]\}$
If M is finite then for each $n \in \mathbb{N}$, we may choose $t_n \in [a,b]$ s.t. $M \ge f(t_n) \ge M - \frac{1}{n}$
→ By construction $f(t_n) \to M$
→ If we could not make the choice, $M - \frac{1}{n}$ would be an upper bound & M would not be the supremum.
If M is infinite, choose t_n s.t. $f(t_n) \ge n$
→ By Bolzano-Weierstrass, $\{t_n\}$ contains a convergent subseq. $\{t_{n_k}\}$ w/ $\lim_{k\to\infty} t_{n_k} = t_0 \in [a,b]$
→ Since f is contin. $f(t_0) = \lim_{t \to t_0} f(t) = \lim_{k \to \infty} f(t_{n_k}) = M$
↳ so M is finite & $f(t_0) = M = \sup\{f(t) : t \in [a,b]\}$
So f attains its max & is bounded above. Argument for min is similar. □

Thm 25. (Intermediate Value Theorem) Suppose $f:[a,b] \to \mathbb{R}$ is contin & $f(a) < d < f(b)$. Then $\exists\, c \in (a,b)$ s.t. $f(c) = d$.

Proof. Let $B = \{t \in [a,b] : f(t) < d\}$ where $a \in B$ so $B \ne \emptyset$.
By the Supremum Prop, $\sup B$ exists & is real so let $c = \sup B$
→ Since $a \in B$, $c \ge a$; $B \subseteq [a,b]$ so $c \le b$; Thus, $c \in [a,b]$. We claim $f(c) = d$
→ Let $t_n = \min\{c + \frac{1}{n}, b\} \ge c$
↳ Either $t_n > c \to t_n \notin B$ or $t_n = c \to t_n = b$ so $f(t_n) > d$ & $t_n \notin B$.
↳ Either case, $f(t_n) \ge d$
↳ Since f is contin at c, $f(c) = \lim_{n \to \infty} f(t_n) \ge d$ ①
Since $c = \sup B$, we may find $s_n \in B$ s.t. $c \ge s_n \ge c - \frac{1}{n}$ $\forall n \in \mathbb{N}$
→ Since $s_n \in B$, $f(s_n) < d$. Since f is contin at c, $f(c) = \lim_{n \to \infty} f(s_n) \le d$.
→ Since $d \le f(c) \le d$, $f(c) = d$.
Since $f(a) < d$ & $f(b) > d$, $a \ne c \ne b$ so $c \in (a,b)$ □

Thm 26. Let $a,b \in \mathbb{R}$ with $a<b$ & let $f:(a,b) \to \mathbb{R}$ be monotonically increasing. Then the one-sided limits $f(t^+) = \lim_{u \to t^+} f(u)$ & $f(t^-) = \lim_{u \to t^-} f(u)$ exist & are real num $\forall t \in (a,b)$

Proof. Analogous to proof that bounded monotone seq. converges.

Thm 27. Let $a,b \in \mathbb{R}$ w/ $a<b$ & let $f:(a,b) \to \mathbb{R}$ be monotonically increasing. Then $D = \{t \in (a,b) : f \text{ is discontinuous at } t\}$ is finite (possibly empty) or countable.
(Result shows a monotonic func. is contin. "almost everywhere")

Proof. If $t \in D$ then $f(t^-) < f(t^+)$.
$\mathbb{Q}$ is dense in $\mathbb{R} \Rightarrow x,y \in \mathbb{R}$ & $x<y$ then $\exists r \in \mathbb{Q}$ s.t. $x<r<y$.
$\forall t \in D$ we may choose $r(t) \in \mathbb{Q}$ s.t. $f(t^-) < r(t) < f(t^+)$.
This defines a func $r: D \to \mathbb{Q}$.
Notice that $s>t \Rightarrow f(s^-) \ge f(t^+)$ so $s>t$, $s,t \in D \Rightarrow r(s) > f(s^-) \ge f(t^+) > r(t)$ so $r(t) \ne r(s)$.
Therefore r is one-to-one so it is a bijection from D to a subset of $\mathbb{Q}$
Thus D is finite or countable. □

Thm 28. Every convergent seq in a metric space is Cauchy.

Proof. Let $x_n \to x$.
For every $\varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$.
Then $m,n > N \Rightarrow d(x_n, x_m) = d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

Thm 29. $\mathbb{R}$ is complete w/ the usual metric (so E^1 is a Banach space)

Proof. Suppose $\{x_n\}$ is a Cauchy seq in $\mathbb{R}$. Fix $\varepsilon > 0$.
Find $N(\frac{\varepsilon}{2})$ s.t. $n,m > N(\frac{\varepsilon}{2}) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$
Let $\alpha_n = \sup\{x_k : k \ge n\}$ & $\beta_n = \inf\{x_k : k \ge n\}$
Fix $m > N(\frac{\varepsilon}{2})$, then $k \ge m \Rightarrow k > N(\frac{\varepsilon}{2}) \Rightarrow x_k < x_m + \frac{\varepsilon}{2}$
$\Rightarrow \alpha_m = \sup\{x_k : k \ge m\} \le x_m + \frac{\varepsilon}{2}$

Since $a_m < \infty$, $\limsup x_n = \lim_{n\to\infty} a_n \le a_m \le x_m + \frac{\varepsilon}{2}$ since seq $\{a_n\}$ is decreasing.
Similarly, $\liminf x_n \ge x_m - \frac{\varepsilon}{2}$.
Therefore $x_m - \frac{\varepsilon}{2} \le \liminf x_n = \limsup x_n \le x_m + \frac{\varepsilon}{2}$

$$0 \le \lim_{n\to\infty} \sup x_n - \lim_{n\to\infty} \inf x_n \le \varepsilon.$$ □

Since ε is arbitrary, $\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \inf x_n \in \mathbb{R}$.
So $\lim_{n\to\infty} x_n$ exists & is real. Thus $\{x_n\}$ is convergent.

Thm 30. E^n is complete $\forall n \in \mathbb{N}$.
Proof. In dIF textbook.

Thm 31. Suppose (X,d) is a complete metric space & $Y \subseteq X$.
Then $(Y,d) = (Y, d|_Y)$ is complete $\Leftrightarrow$ Y is a closed subset of X.

Proof. ($\Rightarrow$) Suppose (Y,d) is complete. We need to show Y is closed.
Consider a seq. $\{y_n\} \subseteq Y$ s.t. $y_n \xrightarrow{(X,d)} x \in X$.
→ then $\{y_n\}$ is Cauchy in X, hence Cauchy in Y
Since Y is complete, $y_n \xrightarrow{(Y,d)} y$ for some $y \in Y$. Therefore $y_n \xrightarrow{(X,d)} y$.
By uniqueness of limits, $y = x$ so $x \in Y$.
Thus Y is closed.
($\Leftarrow$) Suppose Y is closed. We need to show Y is complete.
Let $\{y_n\}$ be a Cauchy seq in Y.
→ then $\{y_n\}$ is Cauchy in X, hence convergent so $y_n \xrightarrow{(X,d)} x$ for some $x \in X$.
Since Y is closed, $x \in Y$ so $y_n \xrightarrow{(Y,d)} x \in Y$. □
Thus Y is complete.

Thm 32. Every contraction is uniformly continuous.
Proof. Let $\delta = \frac{\varepsilon}{\beta}$ & $\varepsilon > 0$.
Then $\forall x,y$ s.t. $d(x,y) < \delta$, $d(T(x),T(y)) \le \beta d(x,y) < \beta\delta = \varepsilon$. □

Thm 33. (Contraction Mapping Theorem) Let (X,d) be a nonempty complete metric space & $T: X \to X$ a contraction with modulus $\beta < 1$. Then
1. T has a unique fixed point x^*.
2. For every $x_0 \in X$, the seq. defined by
$x_1 = T(x_0)$
$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$ ...
$x_{n+1} = T(x_n) = T^n(x_0)$ converges to x^*.

Proof. Define the seq $\{x_n\}$ as above by first fixing $x_0 \in X$ & then letting $x_n = T(x_{n-1}) = T^n(x_0)$ for $n = 1, 2, \cdots$, where $T^n = T^n = T \circ T \circ \cdots \circ T$ is the n-fold iteration of T.

② We first show that $\{x_n\}$ is Cauchy & hence converges to a limit x.
Then
$$d(x_{n+1}, x) = d(T(x_n), T(x_{n-1})) \le \beta d(x_n, x_{n-1}) \le \beta^2 d(x_{n-1}, x_{n-2}) \le \cdots \le \beta^n d(x_1, x_0)$$
Then for any $n > m$,
$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\le (\beta^{n-1} + \beta^{n-2} + \cdots + \beta^m)\, d(x_1, x_0) \\ &= d(x_1, x_0) \textstyle\sum_{\ell=m}^{n-1} \beta^\ell \\ &< d(x_1, x_0) \textstyle\sum_{\ell=m}^{\infty} \beta^\ell \\ &= \frac{\beta^m}{1-\beta} d(x_1, x_0) \end{aligned}$$
Fix $\varepsilon > 0$. Since $\frac{\beta^m}{1-\beta} \to 0$ as $m \to \infty$, choose $N(\varepsilon)$ s.t. $\forall m > N(\varepsilon)$, $\frac{\beta^m}{1-\beta} < \frac{\varepsilon}{d(x_1,x_0)}$.
Then for $n, m > N(\varepsilon)$, $d(x_n, x_m) \le \frac{\beta^m}{1-\beta} d(x_1, x_0) < \varepsilon$.
Therefore $\{x_n\}$ is Cauchy. Since (X,d) is complete, $x_n \to x^*$ for some $x^* \in X$.

① We show that x^* is a fixed point of T.
$$\begin{aligned} T(x^*) &= T(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} T(x_n) \text{ since } T \text{ is contin.} \\ &= \lim_{n\to\infty} x_{n+1} = x^* \end{aligned}$$
So x^* is a fixed point of T.

Finally we show $\exists$ at most one fixed point.
Suppose $x^* \neq y^*$ are both fixed points of T, so $T(x^*) = x^*$ & $T(y^*) = y^*$.
Then $d(x^*, y^*) = d(T(x^*), T(y^*))$

$$\leq \beta d(x^*, y^*)$$

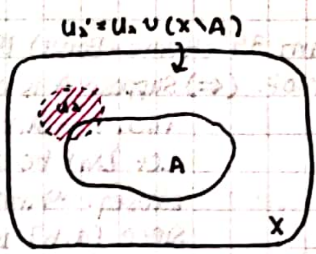
$$(1-\beta) d(x^*, y^*) \leq 0$$

$$d(x^*, y^*) \leq 0$$ $\leftarrow$ but $d(x,y)$ must be positive $\forall x, y$.

So $d(x^*, y^*) = 0$ which implies $x^* = y^*$. $\square$

Thm 34. Every closed subset A of a compact metric space (X, d) is compact.

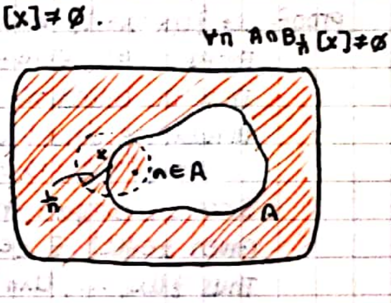
Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A



Let $U'_\lambda = U_\lambda \cup (X \setminus A)$, which is an open cover of X, $\forall \lambda \in \Lambda$.
Since A is closed, $X \setminus A$ is open. Since U_λ is open, so is U'_λ.
Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$
→ If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$.
→ If $x \in X \setminus A$ then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$
Therefore $X \subseteq \bigcup_{\lambda \in \Lambda} U'_\lambda$ so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of X.
Since X is compact, $\exists \lambda_1, \dots, \lambda_n \in \Lambda$ s.t. $X \subseteq U'_{\lambda_1} \cup \dots \cup U'_{\lambda_n}$
Then $a \in A \Rightarrow a \in X$
$\Rightarrow a \in U'_{\lambda_i}$ for some i $\Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \Rightarrow a \in U_{\lambda_i}$.
So $A \subseteq U_{\lambda_1} \cup \dots \cup U_{\lambda_n}$. Thus A is compact. $\square$

Thm 35. If A is a compact subset of the metric space (X, d) then A is closed.

Proof. By contradiction, suppose A is not closed. Then $X \setminus A$ is not open.
We can find point $x \in X \setminus A$ s.t. $\forall \varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$ & hence $A \cap B_\varepsilon[x] \neq \emptyset$.



For $n \in \mathbb{N}$ let $U_n = X \setminus B_{\frac{1}{n}}[x]$ which is open.
And $\bigcup_{n \in \mathbb{N}} U_n = X \setminus \{x\} \supseteq A$ since $x \notin A$
Therefore $\{U_n : n \in \mathbb{N}\}$ is an open cover for A.
Since A is compact, there is a finite subcover $\{U_{n_1}, \dots, U_{n_k}\}$.
Let $n = \max\{n_1, \dots, n_k\}$ then $U_n = X \setminus B_{\frac{1}{n}}[x]$

$$\supseteq X \setminus B_{\frac{1}{n_j}}[x] \quad (j = 1, \dots, k)$$

$$\Rightarrow U_n \supseteq \bigcup_{j=1}^{k} U_{n_j} \supseteq A$$

But $A \cap B_{\frac{1}{n}}[x] \neq \emptyset$ so $A \not\subseteq X \setminus B_{\frac{1}{n}}[x] = U_n$. Contradiction. A is closed. $\square$.

Thm 36. A set A in a metric space (X, d) is compact $\Leftrightarrow$ A is sequentially compact.

Proof. ($\Rightarrow$) Suppose A is compact
By contradiction, we will assume A is not seq compact.
We can find a seq $\{x_n\}$ of elements of A s.t. no subseq. converges to any element of A
Recall a is a cluster point of the seq $\{x_n\}$ means that $\forall \varepsilon > 0$ $\{n : x_n \in B_\varepsilon(a)\}$ is infinite
→ equivalent to $\exists$ subseq $\{x_{n_k}\}$ converging to a
Thus no element $a \in A$ can be a cluster point for $\{x_n\}$.
Hence $\forall a \in A$ $\exists \varepsilon_a > 0$ s.t. $\{n : x_n \in B_{\varepsilon_a}(a)\}$ is finite
Then $\{B_{\varepsilon_a}(a) : a \in A\}$ is an open cover of A
→ since A is compact, there is a finite subcover $\{B_{\varepsilon_{a_1}}(a_1), \dots, B_{\varepsilon_{a_m}}(a_m)\}$
Then $N = \{n : x_n \in A\}$

$$\subseteq \{n : x_n \in (B_{\varepsilon_{a_1}}(a_1) \cup \dots \cup B_{\varepsilon_{a_m}}(a_m))\}$$

$$= \{n : x_n \in B_{\varepsilon_{a_1}}(a_1)\} \cup \dots \cup \{n : x_n \in B_{\varepsilon_{a_m}}(a_m)\}$$

So N is contained in a finite union of sets which is finite. contradiction.
($\Leftarrow$) For converse, see dIF. $\square$

Thm 37. Let A be a subset of metric space (X, d). Then A is compact $\Leftrightarrow$ A is complete & totally bounded.

Proof. ($\Rightarrow$) compact implies totally bounded.
Suppose $\{x_n\}$ is a cauchy seq in A. Since A is compact, A is seq. compact.
→ Hence $\{x_n\}$ has a convergent subseq $x_{n_k} \to a \in A$.
Since $\{x_n\}$ is Cauchy, $x_n \to a$.
A is complete.

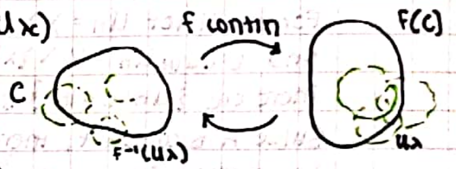
($\Leftarrow$) Suppose A is complete & totally bounded.
Let $\{x_n\}$ be a seq A.
Bc A is totally bounded, we can extract a Cauchy subseq $\{x_{n_k}\}$
Bc A is complete, $x_{n_k} \to a$ for some $a \in A$ which shows A is seq compact.
Hence A is compact. $\square$

Thm 38. Let A be a subset of a complete metric space (X, d). A is compact $\Leftrightarrow$ A is closed & bounded
Proof. Exercise.

Thm 39. (Heine-Borel) If $A \subseteq E^1$, then A is compact $\Leftrightarrow$ A is closed & bounded.
Proof. ($\Leftarrow$) Suppose A is closed, bounded subset of $\mathbb{R}$
Then $A \subseteq [a,b]$ for some interval $[a,b]$
Let $\{x_n\}$ be a seq of elements of $[a,b]$. By Bolzano-Weierstrass, $\{x_n\}$ contains a convergent subseq. w/limit $x \in \mathbb{R}$.
Since $[a,b]$ is closed, $x \in [a,b]$.
Thus $[a,b]$ is seq compact, hence compact.
($\Rightarrow$) A is compact $\to$ A is closed & totally bounded. $\square$

Thm 40. (Heine-Borel) If $A \subseteq E^n$, then A is compact $\Leftrightarrow$ A is closed & bounded.
Proof. See dIF.

Thm 41. Let (X, d) & (Y, ρ) be metric spaces. If $X \to Y$ is cont. & C is compact subset of (X,d) then $f(C)$ is compact in (Y, ρ). i.e. a contin image of a compact set is compact.
Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f(C)$
For each $c \in C$, $f(c) \in f(C)$ so $f(c) \in U_{\lambda_c}$ for some $\lambda_c \in \Lambda$, that is, $c \in f^{-1}(U_{\lambda_c})$
Thus the collection $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is a cover of C.
In addition, since f is contin, each set $f^{-1}(U_\lambda)$ is open in C
$\to \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of C



Since C is compact, there is a finite subcover $\{f^{-1}(U_{\lambda_1}), \dots, f^{-1}(U_{\lambda_n})\}$ of C.
Given $x \in f(C)$, $\exists\, c \in C$ s.t. $f(c) = x$ & $c \in f^{-1}(U_{\lambda_i})$ for some i, so $x \in U_{\lambda_i}$.
Thus $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$ is a finite subcover of $f(C)$ so $f(C)$ is compact. $\square$

Thm 42. (Extreme Value Theorem) Let C be a compact set in a metric space (X,d), & suppose $f: C \to \mathbb{R}$ is contin. Then f is bounded on C & attains its min & max on C.
Proof. Fix $\varepsilon > 0$. Given $c \in C$, find $\delta(c) > 0$ s.t. $x \in C$, $d(x,c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$
Let $U_c = B_{\delta(c)}(c)$ $\leftarrow$ open balls so U_c is an open set
Then $\{U_c : c \in C\}$ is an open cover of C.
Since C is compact, there is a finite subcover $\{U_{c_1}, \dots, U_{c_n}\}$
Let $\delta = \min\{\delta(c_1), \dots, \delta(c_n)\} > 0$
Given $x, y \in C$ with $d(x,y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \dots, n\}$ so $d(x, c_i) < \delta(c_i)$

$$d(y, c_i) \le d(y,x) + d(x, c_i)$$
$$< \delta + \delta(c_i) \le \delta(c_i) + \delta(c_i) = 2\delta(c_i)$$
$$\rho(f(x), f(y)) \le \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So f is uniformly contin. $\square$

switch proofs

Thm 43. Let (X, d) & (Y, ρ) be metric spaces, C a compact subset of X, & $f: C \to Y$ a contin func. Then f is uniformly contin. on C.
Proof. Since C is compact & f is contin, $f(C) \subset \mathbb{R}$ is compact $\to$ closed & bounded.
Let $M = \sup f(C)$, $M < \infty$.
Then $\forall m > 0$ $\exists\, y_m \in f(C)$ s.t. $M - \frac{1}{m} \le y_m \le M$
So $y_m \to M$ & $\{y_m\} \subseteq f(C)$.
Since $f(C)$ is closed, $M \in f(C)$
$\to \exists\, c \in C$ s.t. $f(c) = M = \sup f(C)$
So f attains its max at c.
Proof of min is similar. $\square$ A

Thm 44. A set $S \subseteq E'$ of real numbers is connected $\Leftrightarrow$ S is an interval (i.e. if $x, y \in S$ + $z \in (x,y)$, then $z \in S$.)

Proof. ($\Rightarrow$) Use contrapositive: if S is not an interval then it is not connected

If S is not an interval, find $x, y \in S$ + $z \notin S$ s.t. $x < z < y$

Let $A = S \cap (-\infty, z)$, $B = S \cap (z, \infty)$

Then $\bar{A} \cap B \subseteq \overline{(-\infty, z)} \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset$

$A \cap \bar{B} \subseteq (-\infty, z) \cap \overline{(z, \infty)} = (-\infty, z) \cap [z, \infty) = \emptyset$

$A \cup B = (S \cap (-\infty, z)) \cup (S \cap (z, \infty)) = S \setminus \{z\} = S$

$x \in A$, so $A \neq \emptyset$ + $x \in B$ so $B \neq \emptyset$.

So S is not connected $\leftarrow$ S is not an interval

Thus if S is connected, then S is an interval.

($\Leftarrow$) Use IVT proof (see diff). $\square$

Thm 45. Let X be a metric space + $f: X \to Y$ be contin. If C is a connected subset of X, then $f(C)$ is a connected subset of Y.

Proof. Use contrapositive: if $f(C)$ is not connected, then C is not connected.

Suppose $f(C)$ is not connected.

Then $\exists$ P, Q s.t. $P \neq \emptyset \neq Q$, $f(C) = P \cup Q$, + $\bar{P} \cap Q = P \cap \bar{Q} = \emptyset$.

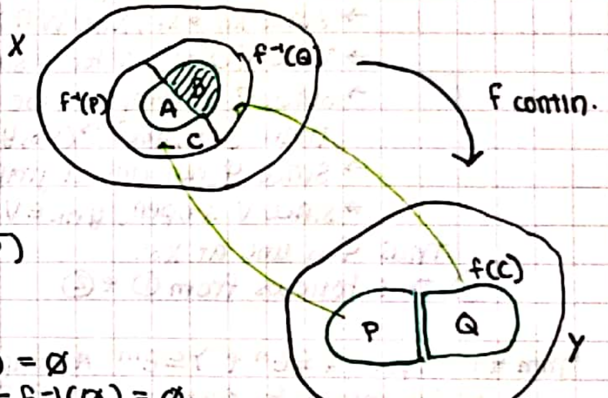
Let $A = f^{-1}(P) \cap C$ + $B = f^{-1}(Q) \cap C$

Then $A \cup B = (f^{-1}(P) \cap C) \cup (f^{-1}(Q) \cap C)$

$= (f^{-1}(P) \cup f^{-1}(Q)) \cap C$

$= f^{-1}(P \cup Q) \cap C$

$= f^{-1}(f(C)) \cap C = C$



Also $A = f^{-1}(P) \cap C \neq \emptyset$ + $B = f^{-1}(Q) \cap C \neq \emptyset$.

Then note $A = f^{-1}(P) \cap C \subseteq f^{-1}(P) \subseteq f^{-1}(\bar{P})$

Since f is contin, $f^{-1}(\bar{P})$ is closed so $\bar{A} \subseteq f^{-1}(\bar{P})$

Similarly, $B = f^{-1}(Q) \cap C \subseteq f^{-1}(Q) \subseteq f^{-1}(\bar{Q})$

$\rightarrow$ + $f^{-1}(\bar{Q})$ is closed so $\bar{A} \subseteq f^{-1}(\bar{P})$

Then $\bar{A} \cap B \subseteq f^{-1}(\bar{P}) \cap f^{-1}(Q) = f^{-1}(\bar{P} \cap Q) = f^{-1}(\emptyset) = \emptyset$

Similarly, $A \cap \bar{B} \subseteq f^{-1}(P) \cap f^{-1}(\bar{Q}) = f^{-1}(P \cap \bar{Q}) = f^{-1}(\emptyset) = \emptyset$

So C is not connected b/c $f(C)$ is not connected.

Thus C connected $\rightarrow$ $f(C)$ connected. $\square$

Thm 46. (IVT) If $f: [a,b] \to \mathbb{R}$ is contin + $f(a) < d < f(b)$, then $\exists$ $c \in (a,b)$ s.t. $f(c) = d$.

Proof. Since $[a,b]$ is an interval, it is connected.

So $f([a,b])$ is connected, hence $f([a,b])$ is an interval.

$\rightarrow$ $f(a) \in f([a,b])$ + $f(b) \in f([a,b])$ + $d \in [f(a), f(b)]$

Since $f([a,b])$ is an interval, $d \in f([a,b])$,

$\rightarrow$ $\exists$ $c \in [a,b]$ s.t. $f(c) = d$

Since $f(a) < d < f(b)$, $c \neq a$ + $c \neq b$ so $c \in (a,b)$. $\square$

Thm 47. Let $X \subseteq E^n$, $Y \subseteq E^m$ + $f: X \to Y$. Let $\varphi: X \to 2^Y$ be defined by $\varphi(x) = \{f(x)\}$ $\forall x \in X$. Then φ is uhc $\Leftrightarrow$ f is contin.

Proof. Suppose φ is uhc. We consider the metric spaces (X, d) + (Y, d) where d is the Euclidean metric. Fix V open in Y.

Then $f^{-1}(V) = \{x \in X : f(x) \in V\}$

$= \{x \in X : \varphi(x) \subseteq V\}$

Thus f is contin $\Leftrightarrow$ $f^{-1}(V)$ is open in X for each open V in Y

$\Leftrightarrow$ $\{x \in X : \varphi(x) \subseteq V\}$ is open in X for each open V in Y.

$\Leftrightarrow$ φ is uhc. $\square$

Thm 48. Suppose $X \subseteq E^n$ + $Y \subseteq E^m$ + $\varphi: X \to 2^Y$

1.) If φ is closed-value + uhc $\Rightarrow$ φ has closed graph

2.) If φ has closed graph + there is an open set W w/ $x_0 \in W$ + a compact set Z s.t. $x \in W \cap X \Rightarrow \varphi(x) \subseteq Z$, $\Rightarrow$ φ is uhc at x_0.

3.) If Y is compact then φ has a closed graph $\Leftrightarrow$ φ is closed-valued + uhc.

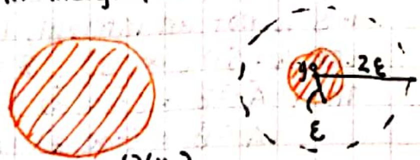
Proof: ① Suppose φ is closed-valued & uhc.

If φ does not have a closed graph, we can find seq $(x_n, y_n) \to (x_0, y_0)$ where (x_n, y_n) lies in the graph of φ (so $y_n \in \varphi(x_n)$) but (x_0, y_0) does not lie in the graph of φ (so $y_0 \notin \varphi(x_0)$).

Since φ is closed-valued, $\varphi(x_0)$ is closed.

Since $y_0 \notin \varphi(x_0)$, $\exists\, \varepsilon > 0$ s.t. $\varphi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$

$\varphi(x_0) \subseteq E^m \setminus B_\varepsilon[y_0]$



$\varphi(x_0)$

Let $V = E^m \setminus B_\varepsilon[y_0]$.

Then V is open & $\varphi(x_0) \subseteq V$.

Since φ is uhc, there is an open set U w/ $x_0 \in U$ s.t. $x \in U \cap X \Rightarrow \varphi(x) \subseteq V$.

Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \in U$ for n sufficiently large.

So $y_n \in \varphi(x_n) \subseteq V$.

Thus for n sufficiently large, $\|y_n - y_0\| \geq \varepsilon$ which implies $y_n \not\to y_0$ & $(x_n, y_n) \not\to (x_0, y_0)$.

Contradiction. Thus φ is closed-graph.

② Suppose φ has closed graph & there is an open set W w/ $x_0 \in W$ & a compact set Z s.t. $x \in W \cap X \Rightarrow \varphi(x) \subseteq Z$.

Since φ has closed graph, it is closed-valued.

Let V be any open set s.t. $V \supseteq \varphi(x_0)$.

We need to show $\exists$ an open set U with $x_0 \in U$ s.t. $x \in U \cap X \Rightarrow \varphi(x) \subseteq V$.

If not, we can find seq $x_n \to x_0$ & $y_n \in \varphi(x_n)$ s.t. $y_n \notin V\ \forall n$.

- → Since $x_n \to x_0$, $x_n \in W \cap X\ \forall n$ sufficiently large
- → Thus $\varphi(x_n) \subseteq Z$ for n sufficiently large
- → Since Z is compact, we can find a convergent subseq. $y_{n_k} \to y'$. Then $(x_{n_k}, y_{n_k}) \to (x_0, y')$
- → Since φ has closed graph, $y' \in \varphi(x_0)$ so $y' \in V$
- → Since V is open, $y_{n_k} \in V\ \forall k$ sufficiently large. Contradiction.

Thus φ is uhc at x_0.

③ This follows from ① & ②. $\square$

Thm 49. Suppose $X \subseteq E^n$ & $Y \subseteq E^m$. A compact-valued correspondence $\varphi: X \to 2^Y$ is uhc at $x_0 \in X$ if & only if for every seq $\{x_n\} \subseteq X$ w/ $x_n \to x_0$ & every seq $\{y_n\}$ s.t. $y_n \in \varphi(x_n)$ for each n, there is a convergent subseq $\{y_{n_k}\}$ s.t. $\lim y_{n_k} \in \varphi(x_0)$.

Proof. See dIF.

Thm 50. A correspondence $\varphi: X \to 2^Y$ is lhc at $x_0 \in X \Leftrightarrow$ for every seq $\{x_n\} \subseteq X$ w/ $x_n \to x_0$ & every $y_0 \in \varphi(x_0)$, $\exists$ a companion seq $\{y_n\}$ w/ $y_n \in \varphi(x_n)$ for each n s.t. $y_n \to y_0$.

Proof. See dIF.

Thm 51. Let V be a Hamel basis for X. Then every vector $x \in X$ has a unique representation as a linear combo of a finite number of elements of V (w/all coef. nonzero).

Proof. Let $x \in X$. Since V spans X we can write $x = \sum_{s \in S_1} \alpha_s v_s$.

→ S_1 is finite, $\alpha_s \in F$, $\alpha_s \neq 0$, & $v_s \in V$ for each $s \in S_1$.

Now suppose $x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$

→ S_2 is finite, $s \in F$, $\beta_s \neq 0$, & $v_s \in V$ for each $s \in S_2$

Let $S = S_1 \cup S_2$ & define $\alpha_s = 0$ for $s \in S_2 \setminus S_1$

$\beta_s = 0$ for $s \in S_1 \setminus S_2$

$$\begin{aligned} \text{Then } 0 = x - x &= \textstyle\sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s \\ &= \textstyle\sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s \\ &= \textstyle\sum_{s \in S} (\alpha_s - \beta_s) v_s \end{aligned}$$

Since V is linearly independent, we must have $\alpha_s - \beta_s = 0$ so $\alpha_s = \beta_s$ for all $s \in S$.

$s \in S_1 \Leftrightarrow \alpha_s \neq 0 \Leftrightarrow \beta_s \neq 0 \Leftrightarrow s \in S_2$.

So $S_1 = S_2$ & $\alpha_s = \beta_s$ for $s \in S_1 = S_2$, so the representation is unique. $\square$

Thm 51. Every vector space has a Hamel basis.

Proof. Use Axiom of Choice. $\square$

Thm 52. If X is a vector space & $V \subseteq X$ is linearly independent, then $\exists$ linearly independent set $W \subseteq X$ s.t. $V \subseteq W \subseteq \text{span } W = X$.

Proof. Exercise.

Thm 53. Any two Hamel bases of a vector space X have the same cardinality (are num. equivalent).

Proof. Exchange Lemma.

Suppose $V = \{v_\lambda : \lambda \in \Lambda\}$ & $W = \{w_\gamma : \gamma \in \Gamma\}$ are Hamel bases of X.

Remove one vector v_{λ_0} from V so it no longer spans

→ If it still did, then v_{λ_0} would be a linear combo of other elements V (& not lin. indep.)

If $w_\gamma \in \text{span}(V \setminus \{v_{\lambda_0}\})$ for every $\gamma \in \Gamma$ then since W spans, $V \setminus \{v_{\lambda_0}\}$ would also span. Contrad.

We can choose $\gamma_0 \in \Gamma$ s.t. $w_{\gamma_0} \notin \text{span}(V \setminus \{v_{\lambda_0}\})$

Because $w_{\gamma_0} \in \text{span } V$, we can write $w_{\gamma_0} = \sum_{i=0}^{n} \alpha_i v_{\lambda_i}$

→ where α_0, the coef of v_{λ_0}, is not zero

↳ If so, we would have $w_{\gamma_0} \in \text{span}(V \setminus \{v_{\lambda_0}\})$

Since $\alpha_0 \neq 0$, we can solve for v_{λ_0} as a linear combo of w_{γ_0} & $v_{\lambda_1}, \dots, v_{\lambda_n}$ so

$\text{span}((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\}) \ni v_{\lambda_0} \quad \supseteq \text{span } V = X$

So $((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\})$ spans X.

Since $w_{\gamma_0} \notin \text{span}(V \setminus \{v_{\lambda_0}\})$ then $((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\})$ is lin. independent

→ It is a basis of X

Repeat this process to exchange every element of V w/ element of W

→ when V is infinite, process is called transfinite induction.

We obtain a bijection $V \to W$ s.t. V & W are num equivalent. □

Thm 54. Suppose $\dim X = n \in \mathbb{N}$. If $V \subseteq X$ & $|V| > n$ then V is linearly dependent.

Proof. If not, so V is linearly indep., then there is a basis W (V extended) for X that contains V.

But $|W| \geq |V| > n = \dim X$, a contradiction. □

Thm 55. Suppose $\dim X = n \in \mathbb{N}$ & $V \subseteq X$, $|V| = n$.

1) If V is linearly indep. then V spans X so V is a Hamel basis

2) If V spans X, then V is linearly indep.

Proof. ① If V does not span X then there is a basis W for X that contains V as a proper subset.

Then $|W| > |V| = n = \dim X$, a contradiction.

② If V is not lin indep, then there is a proper subset V' of V that is linearly indep & for which $\text{span } V' = \text{span } V = X$. But then $|V'| < |V| = n = \dim X$. Contradiction. □

Thm 56: $L(X,Y)$ is a vector space over F.

Proof. Define linear combos in $L(X,Y)$ as follows:

For $T_1, T_2 \in L(X,Y)$ & $\alpha, \beta \in F$, define $\alpha T_1 + \beta T_2$ by $(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$.

We need to show $\alpha T_1 + \beta T_2 \in L(X,Y)$.

$$\begin{aligned}(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) &= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2)\\ &= \alpha(\gamma T_1(x_1) + \delta T(x_2)) + \beta(\gamma T_2(x_1) + \delta T_2(x_2))\\ &= \gamma(\alpha T_1(x_1) + \beta T_2(x_1)) + \delta(\alpha T_1(x_2) + \beta T_2(x_2))\\ &= \gamma(\alpha T_1 + \beta T_2)(x_1) + \delta(\alpha T_1 + \beta T_2)(x_2)\end{aligned}$$

So $\alpha T_1 + \beta T_2 \in L(X,Y)$.

Then check vector space axioms. □

Thm 57. (Rank-Nullity Thm) Let X be a finite-dim vector space & $T \in L(X,Y)$. Then $\text{Im } T$ & $\ker T$ are vector subspaces of Y & X respectively & $\dim X = \dim \ker T + \text{Rank } T$.

Proof. First show $\text{Im } T$ & $\ker T$ are vector subspaces of Y & X, respectively.

Then let $V = \{v_1, \dots, v_k\}$ be a basis for $\ker T$

→ note: $\ker T \subseteq X$ so $\dim \ker T \leq \dim X = n$

If $\ker T = \{0\}$, take $k=0$ so $V = \emptyset$.

Extend V to a basis W for X w/ $W = \{v_1, \dots, v_k, w_1, \dots, w_r\}$

Then $\{T(w_1), \dots, T(w_r)\}$ is a basis for $\text{Im } T$.

By def, $\dim \ker T = k$ & $\dim \text{Im } T = r$

Since W is a basis for X, $k + r = |W| = \dim X \to \dim X = \dim \ker T + \text{Rank } T$. □

Thm 58. $T \in L(X,Y)$ is 1-1 $\Leftrightarrow \ker T = \{0\}$

Proof. ($\Rightarrow$) Suppose T is 1-1 & $x \in \ker T$. Then $T(x) = 0$.

Since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$

Since T is 1-1, $x = 0$, so $\ker T = \{0\}$

($\Leftarrow$) Suppose $\ker T = \{0\}$. Suppose $T(x_1) = T(x_2)$.

Then $T(x_1 - x_2) = T(x_1) - T(x_2) = 0$

Thus $x_1 - x_2 \in \ker T \rightarrow x_1 - x_2 = 0 \rightarrow x_1 = x_2$. Thus T is 1-1. $\square$

Thm 59. If $T \in L(X,Y)$ is invertible, then $T^{-1} \in L(Y,X)$, i.e. T^{-1} is linear

Proof. Suppose $\alpha, \beta \in F$ & $v, w \in Y$. Since T is invertible, $\exists$ unique $v', w' \in X$ s.t.

$T(v') = v \qquad T^{-1}(v) = v'$

$T(w') = w \qquad T^{-1}(w) = w'$

Then $T^{-1}(\alpha v + \beta w) = T^{-1}(\alpha T(v') + \beta T(w'))$

$= T^{-1}(T(\alpha v' + \beta w'))$

$= \alpha v' + \beta w'$

$= \alpha T^{-1}(v) + \beta T^{-1}(w)$ so $T^{-1} \in L(Y,X)$. $\square$

Thm 60. Let X, Y be 2 vector spaces over field F & let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T \in L(X,Y)$ is completely determined by its values on V:

1) Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X,Y)$ s.t. $T(v_\lambda) = y_\lambda \ \forall \lambda \in \Lambda$

2) If $S, T \in L(X,Y)$ & $S(v_\lambda) = T(v_\lambda) \ \forall \lambda \in \Lambda$ then $S = T$

Proof. ① If $x \in X$, x has a unique rep of the form $x = \sum_{i=1}^n \alpha_i v_{\lambda i}$ w/ $\alpha_i \neq 0 \ \forall i = 1, \dots, n$

$\rightarrow$ Recall if $x = 0$ then $n = 0$

Define $T(x) = \sum_{i=1}^n \alpha_i y_{\lambda i} = T(v_{\lambda i})$

Then $T(x) \in Y$.

② Suppose $S(v_\lambda) = T(v_\lambda) \ \forall \lambda \in \Lambda$. Given $x \in X$, $S(x) = S(\sum_{i=1}^n \alpha_i v_{\lambda i})$

$= \sum_{i=1}^n \alpha_i S(v_{\lambda i}) = \sum_{i=1}^n \alpha_i T(v_{\lambda i})$

$= T(\sum_{i=1}^n \alpha_i v_{\lambda i}) = T(x)$

So $S = T$. $\square$

Thm 61. Two vector spaces X, Y over the same field are isomorphic $\Leftrightarrow \dim X = \dim Y$

Proof. ($\Rightarrow$) Suppose X, Y are isomorphic, & let $T \in L(X,Y)$ be an isomorphism.

Let $U = \{u_\lambda : \lambda \in \Lambda\}$ be a basis of X.

Let $v_\lambda = T(u_\lambda)$, $V = \{v_\lambda : \lambda \in \Lambda\}$

Since T is 1-1, U & V have the same cardinality

If $y \in Y$, $\exists x \in X$ s.t. $y = T(x) = T(\sum_{i=1}^n \alpha_{\lambda i} u_{\lambda i})$

$= \sum_{i=1}^n \alpha_{\lambda i} T(u_{\lambda i}) = \sum_{i=1}^n \alpha_{\lambda i} u_{\lambda i}$

$\rightarrow$ shows V spans Y

To see V is linearly indep. suppose $0 = \sum_{i=1}^m \beta_i v_{\lambda i}$

$= \sum_{i=1}^m \beta_i T(u_{\lambda i}) = T(\sum_{i=1}^m \beta_i u_{\lambda i})$

Since T is 1-1, $\ker T = \{0\}$ so $\sum_{i=1}^m \beta_i u_{\lambda i} = 0$.

Since U is a basis for X, we have $\beta_1 = \dots = \beta_m = 0$, so V is linearly indep

Thus V is a basis of Y; since U & V are num equiv, $\dim X = \dim Y$.

($\Leftarrow$) Suppose $\dim X = \dim Y$.

Let $U = \{u_\lambda : \lambda \in \Lambda\}$ & $V = \{v_\lambda : \lambda \in \Lambda\}$ be bases of X & Y

$\rightarrow$ Note we can use same index set Λ for both b/c $\dim X = \dim Y$

By thm above (60), $\exists$ unique $T \in L(X,Y)$ s.t. $T(u_\lambda) = v_\lambda \ \forall \lambda \in \Lambda$

If $T(x) = 0$ then $0 = T(x) = T(\sum_{i=1}^n \alpha_i u_{\lambda i}) = \sum_{i=1}^n \alpha_i T(u_{\lambda i}) = \sum_{i=1}^n \alpha_i v_{\lambda i}$

$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$ since V is a basis

$\Rightarrow x = 0 \Rightarrow \ker T = \{0\} \Rightarrow T$ is 1-1

$\Big\}$ T is 1-1

If $y \in Y$ write $y = \sum_{i=1}^m \beta_i v_{\lambda i}$. Let $x = \sum_{i=1}^m \beta_i u_{\lambda i}$.

Then $T(x) = T(\sum_{i=1}^m \beta_i u_{\lambda i}) = \sum_{i=1}^m \beta_i T(u_{\lambda i})$

$= \sum_{i=1}^m \beta_i T(u_{\lambda i}) = \sum_{i=1}^m \beta_i v_{\lambda i} = y$

$\Big\}$ T is onto

So T is onto

Hence T is isomorphism & X, Y are isomorphic

Thm 62. If X is a vector space w/ $\dim X = n$ for some $n \in \mathbb{N}$ & W is a vector subspace of X then $\dim(X/W) = \dim X - \dim W$.

proof. Sketch. Begin w/ a basis $\{w_1, \ldots, w_c\}$ for W, & a basis $\{[x_1], \ldots, [x_k]\}$ for X/W.
Show that $\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$ is a basis for X.
$\rightarrow |c| = \dim W$; $|k| = \dim X/W$; $\dim X = k + c$

Thm 63. Let X & Y be vector spaces over the same field F & $T \in L(X,Y)$. Then $\operatorname{Im} T$ is isomorphic to $X/\ker T$.

Proof. ① Notice that if X is finite-dimensional, then

$$\begin{aligned}\dim(X/\ker T) &= \dim X - \dim\ker T \quad \text{(from above thm 62)}\\ &= \operatorname{Rank} T \quad \text{(by Rank-Nullity Thm)}\\ &= \dim \operatorname{Im} T\end{aligned}$$

So $X/\ker T$ is isomorphic to $\operatorname{Im} T$.
We will prove this is true in general & the isomorphism is natural.

② Define $\tilde{T}: X/\ker T \rightarrow \operatorname{Im} T$ by $\tilde{T}([x]) = T(x)$
We need to check if this is well-defined
$\rightarrow [x] = [x'] \Rightarrow x \sim x' \Rightarrow x - x' \in \ker T$
$\Rightarrow T(x - x') = 0 = T(x) - T(x')$
So $\tilde{T}$ is well-defined
We will show $\tilde{T}$ is an isomorphism
$\rightarrow \tilde{T}([x]) = \tilde{T}([y]) \Rightarrow T(x) = T(y) \Rightarrow T(x-y) = 0$
$\Rightarrow x - y \in \ker T \Rightarrow x \sim y \Rightarrow [x] = [y]$
So $\tilde{T}$ is one-to-one
$\rightarrow y \in \operatorname{Im} T \Rightarrow \exists x \in X$ s.t. $T(x) = y \Rightarrow \tilde{T}([x]) = y$
so $\tilde{T}$ is onto
$\tilde{T}$ is an isomorphism. $\square$

Thm 64. Let X & Y be vector spaces over the same field F, w/ $\dim X = n$, $\dim Y = m$. Then $L(X,Y)$, the space of linear transformations from X to Y, is isomorphic to $F_{m \times n}$, the vector space $m \times n$ matrices over F. If $V = \{v_1, \ldots, v_n\}$ is a basis for X & $W = \{w_1, \ldots, w_m\}$ is a basis for Y then $Mtx_{W,V} \in L(L(X,Y), F_{m\times n})$ & $Mtx_{W,V}$ is an isomorphism from $L(X,Y)$ to $F_{m\times n}$.

Thm 65. Let X, Y, Z be finite-dimensional vector spaces over the same field F w/ bases U, V, W respectively. Let $S \in L(X,Y)$ & $T \in L(Y,Z)$. Then $Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$ (i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations)

Thm 66. Suppose that X is a finite-dimensional vector space.
1) If $T \in L(X,X)$ then any two matrix representations of T are similar. That is, if U, W are any two bases of X, then $Mtx_W(T)$ & $Mtx_U(T)$ are similar.
2) Conversely, two similar matrices represent the same linear transformation T, relative to suitable bases. That is, given similar matrices A, B with $A = P^{-1}BP$ & any basis U, there is a basis W & $T \in L(X,X)$ s.t. $B = Mtx_U(T)$, $A = Mtx_W(T)$, $P = Mtx_{U,W}(Id)$, $P^{-1} = Mtx_{W,U}(Id)$

Thm 67. Let X be a finite-dimensional vector space & U a basis. Then λ is an eigenvalue of $T \Leftrightarrow \lambda$ is an eigenvalue of $Mtx_U(T)$. v is an eigenvector of T corresponding to $\lambda \Leftrightarrow crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to λ.

Proof. By the Commutative Diagram Theorem, $T(v) = \lambda v \Leftrightarrow crd_U(T(v)) = crd_U(\lambda v)$
$\Leftrightarrow Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v))$ $\square$

Thm 68. Let X be an n-dimensional vector space, $T \in L(X,X)$, U any basis of X & $A = Mtx_U(T)$. Then the following are equivalent:
1. A can be diagonalized
2. There is a basis W for X consisting of eigenvectors of T
3. ——— V for $\mathbb{R}^n$ ——— A

Thm 69. Let X be a vector space & $T \in L(X,X)$.

1. If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T w/corresponding eigenvectors $v_1, \dots, v_m$, then $\{v_1, \dots, v_m\}$ is linearly independent
2. If $\dim X = n$ & T has n distinct eigenvalues, then X has a basis consisting of eigenvectors of T; consequently if U is any basis of X, then $Mtx_U(T)$ is diagonalizable.

Thm 70. A real $n \times n$ matrix A is unitary $\Leftrightarrow$ the columns of A are orthonormal.

Proof. Let v_j denote the j^{th} col of A.

$$A^T = A^{-1} \Leftrightarrow A^T A = I$$
$$\Leftrightarrow v_i \cdot v_j = \delta_{ij} \;\forall i,j$$
$$\Leftrightarrow \{v_1, \dots, v_n\} \text{ is orthonormal.}$$
$\square$

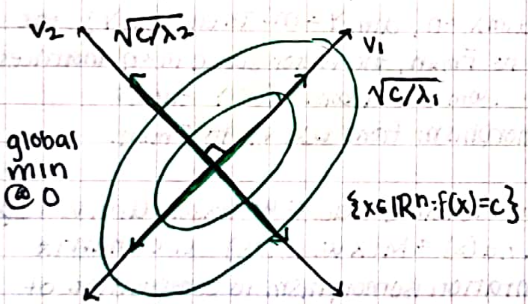
Thm 71. Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ & W be the std basis of $\mathbb{R}^n$. Suppose that $Mtx_W(T)$ is symmetric. Then the eigenvectors of T are all real & there is an orthonormal basis $V = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of T so that $Mtx_W(T)$ is diagonalizable:

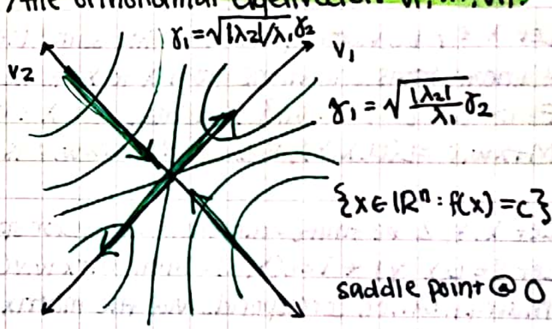
$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id)$$

where $Mtx_V T$ is diagonaliz & the change of basis matrices $Mtx_{V,W}(id)$ & $Mtx_{W,V}(id)$ are unitary.

Thm 72. Consider the quadratic form $f(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j$.

1. f has a global min at $0 \Leftrightarrow \forall i$ the level sets of f are ellipsoids w/principal axes aligned w/orthonormal eigenvectors $v_1, \dots, v_n$.
2. f has global max at $0 \Leftrightarrow \lambda_i \geq 0 \;\forall i$ the level sets of f are ellipsoid w/principal axes aligned w/orthonormal eigenvectors $v_1, \dots, v_n$
3. If $\lambda_i < 0$ for some i & $\lambda_j > 0$ for some j, then f has a saddle point at 0; the level sets of f are hyperboloids w/principal axes aligned w/the orthonormal eigenvectors $v_1, \dots, v_n$.





Thm 73. Let X, Y be normed vector spaces & $T \in L(X,Y)$. Then

T is continuous at some point $x_0 \in X \Leftrightarrow T$ is continuous at every $x \in X$
$\Leftrightarrow T$ is uniformly continuous on X
$\Leftrightarrow T$ is bounded
$\Leftrightarrow T$ is Lipschitz

Proof. ① & ② Suppose T is continuous at x_0. Fix $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$

Now suppose x is any element of X. If $\|y - x\| < \delta$, let $z = y - x + x_0$, so $\|z - x_0\| = \|y - x\| < \delta$.

$$\|T(y) - T(x)\| = \|T(y-x)\|$$
$$= \|T(y - x + x_0 - x_0)\|$$
$$= \|T(z) - T(x_0)\| < \varepsilon$$

which proves T is continuous at every x & uniformly continuous.

③ We claim that T is bounded $\Leftrightarrow$ continuous at 0. Suppose T is not bounded.

($\Rightarrow$) Then $\exists \{x_n\}$ s.t. $\|T(x_n)\| > n\|x_n\| \;\forall n$

Note $x_n \neq 0$. Let $\varepsilon = 1$, fix $\delta > 0$ & choose n s.t. $\frac{1}{n} < \delta$.

Let $x'_n = \frac{x_n}{n\|x_n\|} = \frac{1}{n} \frac{x_n}{\|x_n\|}$

So $\|x'_n\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} < \delta$

$$\|T(x'_n) - T(0)\| = \|T(x'_n)\| = \frac{1}{n\|x_n\|}\|T(x_n)\|$$
$$> \frac{n\|x_n\|}{n\|x_n\|} = 1 = \varepsilon$$

$\forall \delta$, T is not contin at 0. Thus T contin. at $0 \to T$ is bounded.

($\Leftarrow$) Suppose T is bounded.

Find M s.t. $\|T(x)\| \leq M\|x\| \ \forall x \in X$.

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$ then $\|x - 0\| < \delta \Rightarrow \|x\| < \delta$

$\Rightarrow \|T(x) - T(0)\| = \|T(x)\| < M\delta$

$\Rightarrow \|T(x) - T(0)\| < \varepsilon$

So T is continuous at 0.

We have shown: contin at point $x_0 \to$ uniform continuity $\to$ continuity at every point $\to$ T contin at $0 \to T$ bounded $\to$ & backwards

④ Suppose T is bound w/constant $M > 0$.

Then $\|T(x) - T(y)\| = \|T(x-y)\|$

$\leq M\|x - y\|$

So T is Lipschitz w/constant M.

Conversely if T is Lipschitz w/constant M then T is bounded w/constant M

All statements are equivalent. $\square$

Basics of Linear Algebra.

→ linear combination of $x_1, \ldots, x_n \in X$ (where X is a vector space over field F) is a vector of the form $y = \sum_{i=1}^{n} \alpha_i x_i$ where $\alpha_1, \ldots, \alpha_n \in F$ is the coefficient

→ hamel basis, or basis, of a vector space X is a linearly independent set of vectors in X that spans X.

↳ span of V ($\subseteq X$) is the set of all linear comb. of elements of V

↳ a set $V \subseteq X$ is linearly dependent if $\exists\, v_1, \ldots, v_n \in V$ & $\alpha_1, \ldots, \alpha_n \in F$ not all zero s.t. $\sum_{i=1}^{n} \alpha_i v_i = 0$

→ Thus, $\sum_{i=1}^{n} \alpha_i v_i = 0,\ v_i \in V\ \forall i \Rightarrow \alpha_i = 0\ \forall i \Leftrightarrow$ linearly independent

→ dimension of vector space X ($\dim X$) is the cardinality of any basis of X

↳ finite-dimensional: if $\dim X = n$ for some $n \in \mathbb{N}$

→ $T: X \to Y$ is a linear transformation if $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)\ \forall x_1, x_2 \in X$; $\alpha_1, \alpha_2 \in F$

↳ image of $T \to \operatorname{Im} T = T(X)$

↳ kernel of $T \to \ker T = \{x \in X : T(x) = 0\}$

↳ rank of $T \to \operatorname{Rank} T = \dim(\operatorname{Im} T)$

→ Rank-Nullity Thm: $\dim X = \dim \ker T + \operatorname{Rank} T$ where X is finite-dim & $T \in L(X, Y)$

→ $T \in L(X, Y)$ is invertible if $\exists\, S: Y \to X$ s.t. $S(T(x)) = x\ \forall x \in X$ & $T(S(y)) = y\ \forall y \in Y$.

↳ Two vector spaces X, Y over field F are isomorphic if $\exists$ invertible $T \in L(X, Y)$

↳ $T \in L(X, Y)$ is an isomorphism if it is invertible (1-1 & onto)

→ quotient vector space X/W is the set of equivalence classes where $x \sim y \Leftrightarrow x - y \in W$

↳ vectors in X/W are sets of vectors in X: $[x] = \{x + w : w \in W\}$ for $x \in X$

↳ $\dim(X/W) = \dim X - \dim W$

Thm 74. Suppose $X \subseteq \mathbb{R}^n$ is open & $f: X \to \mathbb{R}^m$ is differentiable at $x \in X$. Then $\frac{\partial f^i}{\partial x_j}$ exists at x for $1 \le i \le m$, $1 \le j \le n$ & $Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$ i.e. the Jacobian at x is the matrix of partial deriv. at x

Proof. f is differentiable at $x \Leftrightarrow E_f(h) = o(h)$ as $h \to 0$

$Df(x) = (a_{ij})$ is the matrix rep of $T_x = df_x$.

Let $\{e_1, \ldots, e_n\}$ be the std basis of $\mathbb{R}^n$. Look in direction e_j (note $|\gamma e_j| = |\gamma|$)

$$o(\gamma) = f(x+\gamma e_j) - (f(x) + T_x(\gamma e_j))$$
$$= f(x+\gamma e_j) - \left[f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \gamma \\ \vdots \end{pmatrix} \right] \leftarrow j^{th} \text{ spot}$$
$$= f(x+\gamma e_j) - \left[f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right]$$

For $i = 1, \ldots, m$, let f^i denote the i^{th} component of the func f:

$$\forall i : f^i(x+\gamma e_j) - (f^i(x) + \gamma a_{ij}) = o(\gamma)$$
$$\text{so } a_{ij} = \frac{\partial f^i}{\partial x_j}(x). \qquad \square$$

Thm 75. If all first-order partial deriv. $\frac{\partial f^i}{\partial x_j}$ $(1 \le i \le m, 1 \le j \le n)$ exists & are contin at x then f is differentiable at x.

Thm 76. (Chain Rule) Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be open, $f: X \to Y$, $g: Y \to \mathbb{R}^p$. Let $x_0 \in X$ & $F = g \circ f$. If f is differentiable at x_0 & g is differentiable at $f(x_0)$ then $F = g \circ f$ is differentiable at x_0 &

$dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$ (composition of linear transformations)

$DF(x_0) = Dg(f(x_0))\, Df(x_0)$ (matrix multiplication)

Thm 77. (Mean Value Theorem) (univariate case) Let $a, b \in \mathbb{R}$. Suppose $f: [a,b] \to \mathbb{R}$ is contin on $[a,b]$ & differentiable on (a,b). Then $\exists\, c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$. That is, s.t. $f(b) - f(a) = f'(c)(b-a)$.

Proof. Consider the func. $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$.

Then $g(a) = 0 = g(b)$.

Note for $x \in (a,b)$, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ so it suffices to find $c \in (a,b)$ s.t. $g'(c) = 0$.

→ Case I: If $g(x) = 0$ $\forall x \in [a,b]$, any arbitrary $c \in (a,b) \to g'(c) = 0$

→ Case II: suppose $g(x) > 0$ for some $x \in [a,b]$. Since g is contin on $[a,b]$, it attains its max at some point $c \in (a,b)$. Since g is differentiable at c & c is an interior point of domain g, we have $g'(c) = 0$.

→ Case III: Similiar argument for $g(x) < 0$. $\square$

Thm 78. (Mean Value Theorem) Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable on an open set $X \subseteq \mathbb{R}^n$, $x, y \in X$ & $\ell(x,y) \subseteq X$. Then $\exists\, z \in \ell(x,y)$ s.t. $f(y) - f(x) = Df(z)(y-x)$

Thm 79. Suppose $X \subseteq \mathbb{R}^n$ is open & $f: X \to \mathbb{R}^m$ is differentiable. If $x, y \in X$ & $\ell(x,y) \subseteq X$, then $\exists\, z \in \ell(x,y)$ s.t. $|f(y) - f(x)| \le |df_z(y-x)| \le \|df_z\|\, |y-x|$

Thm 80. (Taylor's Thm in $\mathbb{R}$) Let $f: I \to \mathbb{R}$ be n-times differentiable where $I \subseteq \mathbb{R}$ is an open interval. If $x, x+h \in I$ then $f(x+h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$ where $f^{(k)}$ is the k^{th} derivative of f & $E_n = \frac{f^{(n)}(x+\lambda h)h^n}{n!}$ for some $\lambda \in (0,1)$.

(Alternate Taylor's Thm in $\mathbb{R}^1$) Let $f: I \to \mathbb{R}$ be n times differentiable where $I \subseteq \mathbb{R}$ is an open interval & $x \in I$. Then $f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + o(h^n)$ as $h \to 0$.

If f is $(n+1)$ times continuously differentiable

→ i.e. all derivatives up to order $n+1$ exist & are continuous

then $f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + O(h^{n+1})$ as $h \to 0$.

Thm 81. Suppose $X \subseteq \mathbb{R}^n$ is open & $f: X \to \mathbb{R}^m$. Then f is continuously differentiable on $X \Leftrightarrow f$ is C^1.

Thm 82. Suppose $X \subseteq \mathbb{R}^n$ is open & $x \in X$. If $f: X \to \mathbb{R}^m$ is differentiable then $f(x+h) = f(x) + Df(x)h + o(h)$ as $h \to 0$

Thm 83. Suppose $X \subseteq \mathbb{R}^n$ is open & $x \in X$. If $f: X \to \mathbb{R}^m$ is C^2 then $f(x+h) = f(x) + Df(x)h + O(|h|^2)$ as $h \to 0$.

Thm 84. Let $X \subseteq \mathbb{R}^n$ be open, $f: X \to \mathbb{R}$, $f \in C^2(X)$ & $x \in X$.
Then $f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^T(D^2f(x))h + o(|h|^2)$ as $h \to 0$
If $f \in C^3$, $f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^T(D^2f(x))h + O(|h|^3)$ as $h \to 0$

Thm 85. Suppose $X \subseteq \mathbb{R}^n$ is open & $x \in X$. If $f: X \to \mathbb{R}$ is C^2 then there is an orthonormal basis $\{v_1, \dots, v_n\}$ & corresponding eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ of $D^2f(x)$ s.t.
$f(x+h) = f(x + \gamma_1 v_1 + \dots + \gamma_n v_n) = f(x) + \sum_{i=1}^n (Df(x)v_i)\gamma_i + \frac{1}{2}\sum_{i=1}^n \lambda_i \gamma_i^2 + o(|\gamma|^2)$ where $\gamma_i = h \cdot v_i$.
1) If $f \in C^3$, we may strengthen $o(|\gamma|^2)$ to $O(|\gamma|^3)$.
2) If f has a local max or local min at x then $Df(x) = 0$
3) If $Df(x) = 0$ then $\lambda_1, \dots, \lambda_n > 0 \Rightarrow f$ has a local min at x
$\lambda_1, \dots, \lambda_n < 0 \Rightarrow$ ———— max ————
$\lambda_i < 0$ for some i, $\lambda_j > 0$ for some j $\Rightarrow f$ has a saddle at x
$\lambda_1, \dots, \lambda_n \geq 0$, $\lambda_i > 0$ for some i $\Rightarrow f$ has a local min or saddle at x
———— ≤ 0, $\lambda_i < 0$ for some i $\Rightarrow$ ———— max ————
$\lambda_1 = \dots = \lambda_n = 0$ $\Rightarrow$ no info

Proof. (Sketch) We know behavior of quadratic term is determined by signs of eigenvalues.
If $\lambda_i = 0$ for some i, then we know the quad. form from second partial deriv. is identically zero in the direction v_i & higher derivatives will determine the behavior of the func f in direction v_i
→ Ex: $f(x) = x^3$ then $f'(0) = 0$, $f''(0) = 0$
↳ we know f has a saddle at $x = 0$
↳ If $f(x) = x^4$ then $f'(0) = 0$ & $f''(0) = 0$ but f has a local & global min at $x = 0$ □

Thm 86. (Inverse Func Thm) Suppose $X \subseteq \mathbb{R}^n$ is open $f: X \to \mathbb{R}^n$ is C^1 on X, & $x_0 \in X$. If $\det D(f(x_0)) \neq 0$ (i.e. x_0 is a reg point of f) then there are open neighborhoods U of x_0 & V of $f(x_0)$ s.t.
$f: U \to V$ is 1-1 & onto
$f^{-1}: V \to U$ is C^1
$Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$
If in addition $f \in C^k$ then $f^{-1} \in C^k$.

Proof. (Sketch) Since $\det Df(x_0) \neq 0$ then $df_{x_0}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one & onto.
Find neighborhood U of x_0 sufficiently small s.t. Contraction mapping Thm implies 1-1 & onto.
→ Let $A = Df(x_0)$. Set $g(x) = x + A^{-1}(y - f(x))$. Note $y = f(x) \Leftrightarrow x$ is a fixed point of g.
Then show g is a contraction mapping on a suitably small neighborhood U around x_0
Then see formula for Df^{-1}, let id_U denote identity func. from U to U & I denote the $n \times n$ identity matrix
Then $Df^{-1}(f(x_0))Df(x_0) = D(f^{-1} \circ f)(x_0) = D(id_U)(x_0) = I$
$\Rightarrow Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$ □

Thm 87. (Implicit Func Thm) Suppose $X \subseteq \mathbb{R}^n$ & $A \subseteq \mathbb{R}^p$ are open & $f: X \times A \to \mathbb{R}^n$ is C^1. Suppose $f(x_0, a_0) = 0$ & $\det(D_x f(x_0, a_0)) \neq 0$ (i.e. x_0 is a reg point of $f(\cdot, a_0)$). Then there are open neighborhoods U of x_0 ($U \subseteq X$) & W of a_0 s.t. $\forall a \in W$ $\exists! x \in U$ s.t. $f(x, a) = 0$.
For each $a \in W$ let $g(a)$ be that unique x. Then $g: W \to X$ is C^1 & $Dg(a_0) = -[D_x f(x_0, a_0)]^{-1}[D_a f(x_0, a_0)]$
If $f \in C^k$, then $g \in C^k$.

Proof. Use Inverse Func Thm
→ Look at $F(x, a) = (f(x, a), a)$. So $F: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^p$
Show $DF(x_0, a_0)$ invertible $\Leftrightarrow D_x f(x_0, a_0)$ invertible
Then use Inverse Func Thm on F
Why is Dg formula correct?
$0 = Df(g(a), a)(a_0) = Dh(a_0)$ $\Leftarrow h(a) = f(g(a), a) = 0 \;\; \forall a \in W$
$= D_x f(x_0, a_0) Dg(a_0) + D_a f(x_0, a_0)$ } Chain Rule
$Dg(a_0) = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)$
Outline of proof that g is differentiable:
$f(x_0, a_0 + h) = f(x_0, a_0) + D_a f(x_0, a_0)h + o(h) = D_a f(x_0, a_0)h + o(h)$
Solve for Δx that brings f back to zero:
$0 = f(x_0 + \Delta x, a_0 + h)$
$= f(x_0, a_0 + h) + D_x f(x_0, a_0 + h)\Delta x + o(\Delta x)$

$= f(x_0,a_0) + D_a f(x_0,a_0)h + D_x f(x_0, a_0+h)\Delta x + o(\Delta x) + o(h)$

$= D_a f(x_0,a_0)h + D_x f(x_0, a_0+h)\Delta x + o(\Delta x) + o(h)$

$D_x f(x_0, a_0+h)\Delta x = -D_a f(x_0,a_0)h + o(\Delta x) + o(h)$

Because f is C^1 & the determinant is a contin func of the entries of the matrix, $\det D_x f(x_0, a_0+h) \neq 0$ for h sufficiently small so

$\Delta x = -[D_x f(x_0, a_0+h)]^{-1} D_a f(x_0,a_0)h + o(\Delta x) + o(h)$

$= -[D_x f(x_0,a_0) + o(1)]^{-1} D_a f(x_0,a_0)h + o(\Delta x) + o(h)$ since $f \in C^1$

$= -[D_x f(x_0,a_0)]^{-1} D_a f(x_0,a_0)h + o(\Delta x) + o(h)$ since $f \in C^1$

Then $|\Delta x + o(\Delta x)| = O(h)$

$\Rightarrow |\Delta x| = O(h)$

$\Rightarrow o(\Delta x) = o(h)$

$\Rightarrow \Delta x = -[D_x f(x_0,a_0)]^{-1} D_a f(x_0,a_0)h + o(h)$

By def of derivative, $Dg(a_0) = -[D_x f(x_0,a_0)]^{-1} D_a f(x_0,a_0)$. □

Thm 88. Suppose $X \subseteq \mathbb{R}^n$ & $A \subseteq \mathbb{R}^p$ are open & $f: X \times A \to \mathbb{R}^n$ is C^1. If 0 is a reg value of $f(\cdot, a_0)$ then the correspondence $a \mapsto \{x \in X : f(x,a) = 0\}$ is lower hemicontin at a_0.

Proof. If 0 is a reg value of $f(\cdot, a_0)$ then given any $x_0 \in \{x \in X : f(x, a_0) = 0\}$, we can find a local implicit func g

→ In other words, if a is sufficiently close to a_0 then $g(a) \in \{x \in X : f(x,a) = 0\}$.

→ The continuity of g then shows the correspondence $\{x \in X : f(x,a) = 0\}$ is lower hemi at a_0 □

Thm 89. (Sard's Thm) Let $X \subseteq \mathbb{R}^n$ be open & $f: X \to \mathbb{R}^m$ be C^r with $r \geq 1 + \max\{0, n-m\}$. Then the set of all critical values of f has Lebesgue measure zero.

Proof. Suppose $m = n$. Let C be the set of critical points of f, $f(C) = V$ the set of critical values.

Then $\mathrm{Vol}(V) = \mathrm{Vol}(f(C)) = \int_{f(C)} 1\, dy$

$\leq \int_C |\det Df(x)| dx$ (equality if f is 1-1 or linear)

$= \int_C 0\, dx = 0$

① Show $X = \bigcup_{j \in \mathbb{N}} X_j$ where each X_j is compact subset $[-j, j]^n$.

Let $C_j = C \cap X_j$. Fix j for now.

Since f is C^1, $x_k \to x \Rightarrow \det Df(x_k) \to \det Df(x)$

$\{x_k\} \subseteq C_j$, $x_k \to x \Rightarrow \det Df(x) = 0 \Rightarrow x \in C_j$ so C_j is closed, hence compact.

Since X is open & C_j is compact, $\exists\, \delta_1 > 0$ s.t. $B_{\delta_1}[C_j] = \bigcup_{x \in C_j} B_{\delta_1}[x] \subseteq X$.

→ $B_{\delta_1}[C_j]$ is bounded & using compactness of C_j, can show it's closed → compact

since $\det Df(x)$ is contin on $B_{\delta_1}[C_j]$, it is uniformly contin on $B_{\delta_1}[C_j]$

Then given $\varepsilon > 0$, we can find $\delta \leq \delta_1$ s.t. $B_\delta[C_j] \subseteq [-2j, 2j]^n$ &

$x \in B_\delta[C_j] \Rightarrow |\det Df(x)| \leq \frac{\varepsilon}{2 \cdot 4^n j^n}$

Then $f(C_j) \subseteq f(B_\delta[C_j])$

$\int_{f(B_\delta[C_j])} 1\, dy = \mathrm{Vol}(f(B_\delta[C_j])) \leq \int_{[-2j, 2j]^n} \frac{\varepsilon}{2 \cdot 4^n j^n} dx = \frac{\varepsilon}{2}$

Since f is C^1, show $f(C_j)$ can be covered by countable collection of rectangles of total vol $< \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $f(C_j)$ has Lebesgue measure zero.

Then $f(C) = f(\bigcup_{j \in \mathbb{N}} C_j) = \bigcup_{j \in \mathbb{N}} f(C_j)$ is a countable union of sets of Lebesgue measure zero. □

Thm 90. Let $X = [a,b]$ for $a, b \in \mathbb{R}$ w/ $a < b$ & let $f: X \to X$ contin. Then f has fixed point.

Proof. Let $g: [a,b] \to \mathbb{R}$ be given by $g(x) = f(x) - x$.

If either $f(a) = a$ or $f(b) = b$, we're done.

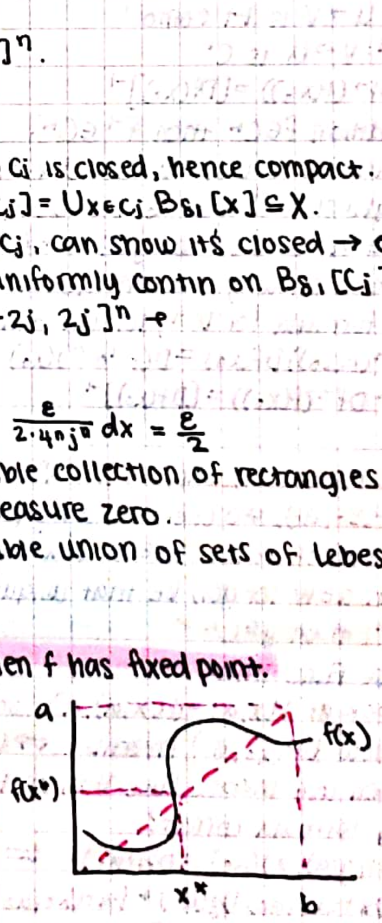
Assume $f(a) > a$ & $f(b) < b$.

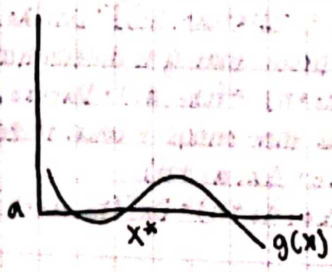
Then $g(a) = f(a) - a > 0$

$g(b) = f(b) - b < 0$

g is contin so by IVT $\exists\, x^* \in (a,b)$ s.t. $g(x^*) = 0$

→ s.t. $f(x^*) = x^*$

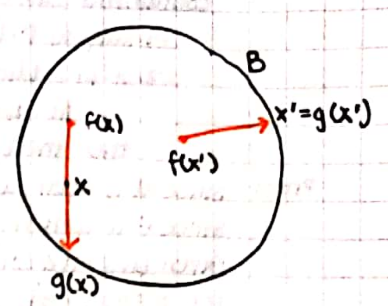




Thm 91. (Brouwer's Fixed Point Thm). Let $X \subseteq \mathbb{R}^n$ be nonempty, compact, & convex, & let $f: X \to X$ be contin. Then f has a fixed point

Proof. (Sketch) Consider the set X is the closed unit ball in $\mathbb{R}^n$

$\rightarrow X = B_1[0] = B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ w/o loss of generality.



Let $f: B \to B$ be a contin func.

∂B denotes boundary B so $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$

$\rightarrow \nexists$ contin func: $h: B \to \partial B$ s.t. $h(x') = x'$ $\forall x' \in \partial B$

Suppose f has no fixed points in B

$\rightarrow \forall x \in B$, $x \neq f(x)$; x & $f(x)$ are distinct points in B $\forall x$

$\rightarrow$ For each $x \in B$, construct line segment at $f(x)$ & going through x

Let $g(x)$ denote the intersection of this line seg w/ $\partial B = \{x \in B : \|x\| = 1\}$

$\rightarrow$ Note if $x' \in \partial B$ then $x' = g(x') \rightarrow g|_{\partial B} = id$

No such func. Contradiction. □

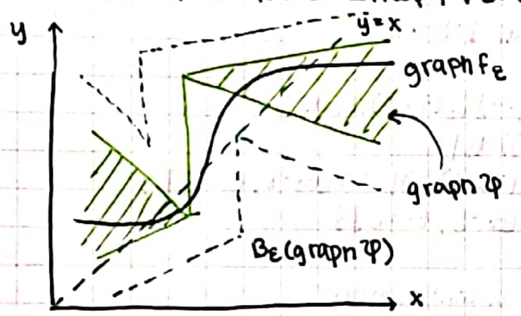
Thm 92. (Kakutani's Fixed Point Thm) Let $X \subset \mathbb{R}^n$ be a non-empty, compact, convex set & $\varphi: X \to 2^X$ be an upper hemi-contin corresp. w/ non-empty, convex, compact values. Then φ has a fixed point in X.

Proof. (Sketch) Look at weaker approx: Let $X \subset \mathbb{R}^n$ be non-empty, compact, convex set & let $\varphi: X \to 2^X$ be an uhc corresp. w/ nonempty, compact, convex values.

For $\forall \varepsilon > 0$, define ε ball about graph φ as

$$B_\varepsilon(\text{graph } \varphi) = \{z \in X \times X : d(z, \text{graph } \varphi) = \inf_{(x,y) \in \text{graph } \varphi} d(z,(x,y)) < \varepsilon\}$$

Since φ is a convex-valued corresp., $\forall \varepsilon > 0$ $\exists$ contin func $f_\varepsilon: X \to X$ s.t. graph $f_\varepsilon \subseteq B_\varepsilon(\text{graph } \varphi)$



Let $\varepsilon_n = \frac{1}{n} \to 0$

↳ we can find a seq of contin func $\{f_n\}$ s.t. graph $f_n \subseteq B_{\frac{1}{n}}(\text{graph } \varphi)$ for each n.

By Brouwer's FPT each func f_n has fixed point $\hat{x}_n \in X$ &

$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \varphi)$

for each n.

So for each n $\exists$ $(x_n, y_n) \in \text{graph } \varphi$ s.t.

$d(\hat{x}_n, x_n) < \frac{1}{n}$ & $d(\hat{x}_n, y_n) < \frac{1}{n}$

Since X is compact, $\{\hat{x}_n\}$ has convergent subseq $\{\hat{x}_{n_k}\}$ w/ $\hat{x}_{n_k} \to \hat{x} \in X$.

Then $x_{n_k} \to \hat{x}$ & $y_{n_k} \to \hat{x}$.

Since φ is uhc & closed-valued, it has closed graph so $(\hat{x}, \hat{x}) \in \text{graph } \varphi$

Thus $\hat{x} \in \varphi(\hat{x}) \rightarrow \hat{x}$ is a fixed point of φ □

Thm 93. (Separating Hyperplane Thm) Let $A, B \subseteq \mathbb{R}^n$ be non-empty, disjoint convex sets. Then $\exists$ a nonzero vector $p \in \mathbb{R}^n$ s.t. $p \cdot a \leq p \cdot b$ $\forall a \in A$, $b \in B$.

Thm 94. Let $Y \subseteq \mathbb{R}^n$ be a nonempty convex set & $x \notin Y$. Then $\exists$ nonzero vector $p \in \mathbb{R}^n$ s.t. $p \cdot x \leq p \cdot y$ $\forall y \in Y$

Proof. Special case: Y compact to get conclusion: $\exists p \in \mathbb{R}^n$, $p \neq 0$ s.t. $p \cdot x < p \cdot y$ $\forall y \in Y$

Choose $y_0 \in Y$ s.t. $\|y_0 - x\| = \inf\{\|y - x\| : y \in Y\}$

$\rightarrow$ exists bc Y is compact, so dist func $g(y) = |y - x|$ assume min on Y

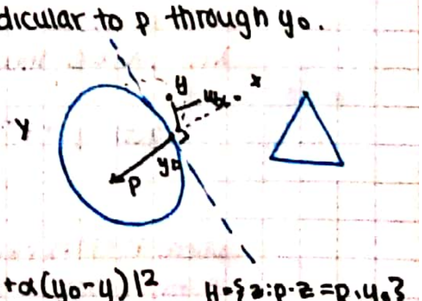
Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$.

Let $p = y_0 - x$. The set $H = \{z \in \mathbb{R}^n : p \cdot z = p \cdot y_0\}$ is the hyperplane perpendicular to p through y_0.

Then $p \cdot y_0 = (y_0 - x) \cdot y_0 = (y_0 - x) \cdot (y_0 - x + x)$

$= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x$

$= \|y_0 - x\|^2 + p \cdot x > p \cdot x$



We claim $y \in Y \Rightarrow p \cdot y \geq p \cdot y_0$.

$\rightarrow$ If not, $\exists$ $y \in Y$ s.t. $p \cdot y < p \cdot y_0$. Given $\alpha \in (0,1)$ let $w_\alpha = \alpha y + (1-\alpha) y_0$.

Since Y is convex, $w_\alpha \in Y$. Then for small α $|x - w_\alpha|^2 = |x - \alpha y - (1-\alpha) y_0|^2$

$= |x - y_0 + \alpha(y_0 - y)|^2 = |-p + \alpha(y_0 - y)|^2$

$= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2$

$< |p^2| = \|y_0 - x\|^2$

Thus for α suff close to zero, $|w_\alpha - x| < |y_0 - x|$

$\rightarrow y_0$ is not closest point to x in Y, contradiction. □

Thm 95. Consider the initial value problem $y'(t)=F(y(t),t)$, $y(t_0)=y_0$. Let U be an open set in $\mathbb{R}^n\times\mathbb{R}$ containing (y_0,t_0).

1. Suppose $F:U\to\mathbb{R}^n$ is contin. Then the initial value problem has a solution.
2. If in addition, F is Lipschitz in y on U (i.e. $\exists$ constant K s.t. $\forall (y,t),(\hat{y},t)\in U$)
$$\|F(y,t)-F(\hat{y},t)\|\le K\|y-\hat{y}\|$$
then there is an interval (a,b) containing t_0 s.t. solution is unique on (a,b).

Proof. Since U is open, we choose $r>0$ s.t. $R=\{(y,t):\|y-y_0\|\le r,\ \|t-t_0\|\le r\}\subseteq U$.
Since F is contin, we may find $M\ge 1\in\mathbb{R}$ s.t. $\|F(y,t)\|\le M\ \forall (y,t)\in\mathbb{R}$
Given Lipschitz condition, we can assume $|F(y,t)-F(\hat{y},t)|\le K|y-\hat{y}|\ \forall (y,t),(\hat{y},t)\in\mathbb{R}$
Let $\delta=\min\{\frac{1}{2K},\frac{r}{M}\}$.
We claim initial value problem has a unique solution on $(t_0-\delta,t_0+\delta)$.
Let C be the space of contin func from $[t_0-\delta,t_0+\delta]$ to $\mathbb{R}^n$, endowed w/ sup norm.
$$\|f\|_\infty=\sup\{|f(t)|:t\in[t_0-\delta,t_0+\delta]\}$$
Let $S=\{z\in C:(z(s),s)\in\mathbb{R}\ \forall s\in[t_0-\delta,t_0+\delta]\}$

$\to$ S is a closed subset of the complete metric space C so S is a complete metric space.

Consider the func $I:S\to C$ defined by $I(z)(t)=y_0+\int_{t_0}^{t}F(z(s),s)\,ds$.
$I(z)$ is defined & contin b/c F is bounded & contin on $\mathbb{R}$

$\to$ Observe if $(z(s),s)\in R\ \forall s\in[t_0-\delta,t_0+\delta]$ then
$$\begin{aligned}\|I(z)(t)-y_0\|&=\left\|\int_{t_0}^{t}F(z(s),s)\,ds\right\|\\&\le|t-t_0|\max\{\|F(y,s)\|:(y,s)\in R\}\\&\le\delta M\le r\end{aligned}$$
so $(I(z)(t),t)\in R\ \forall t\in[t_0-\delta,t_0+\delta]$. Thus $I:S\to S$.

Given two func $x,z\in S$ & $t\in[t_0-\delta,t_0+\delta]$,
$$\begin{aligned}|I(z)(t)-I(x)(t)|&=\left|y_0+\int_{t_0}^{t}F(z(s),s)\,ds-y_0-\int_{t_0}^{t}F(x(s),s)\,ds\right|\\&=\left|\int_{t_0}^{t}F(z(s),s)-F(x(s),s)\,ds\right|\\&\le|t-t_0|\sup\{|F(z(s),s)-F(x(s),s)|:s\in[t_0-\delta,t_0+\delta]\}\\&\le\delta K\sup\{|z(s)-x(s)|:s\in[t_0-\delta,t_0+\delta]\}\\&\le\tfrac{1}{2}\|z-x\|_\infty\end{aligned}$$
Thus $\|I(z)-I(x)\|_\infty\le\frac{1}{2}\|z-x\|_\infty$ so I is a contraction.

Since S is a complete metric space, I has unique fp $y\in S$.
Thus $\forall t\in[t_0-\delta,t_0+\delta]$ we have $y(t)=y_0+\int_{t_0}^{t}F(y(s),s)\,ds$.
F is contin so FTC implies $y'(t)=F(y(t),t)\ \forall t\in(t_0-\delta,t_0+\delta)$
Since we also have $y(t_0)=y_0+\int_{t_0}^{t_0}F(y(s),s)\,ds=y_0$, y (restricted to $(t_0-\delta,t_0+\delta)$) is a solution to the initial value problem.

$\to$ On the other hand, suppose $\hat{y}$ is any solution of the initial value problem on $(t_0-\delta,t_0+\delta)$
Check $(\hat{y}(s),s)\in R\ \forall s\in(t_0-\delta,t_0+\delta)$ so $\|F(\hat{y}(s),s)\|\le M$

$\hookrightarrow$ implies $\hat{y}$ has an extension to a contin func in S.

Since $\hat{y}$ is a solution to the initial value problem, FTC implies $I(\hat{y})=\hat{y}$
Since y is the unique fixed point of I, $\hat{y}=y$. $\square$

Thm 96. Consider the linear differential equation $y'=(y-y_s)'=M(y-y_s)$ where M is a real $n\times n$ matrix. Suppose M can be diagonalized over the complex field $\mathbb{C}$.
Let U be the std basis of $\mathbb{R}^n$ & $V=\{v_1,\dots,v_n\}$ be a basis of (complex) eigenvectors corresp. to eigenvalues $\lambda_1,\dots,\lambda_n\in\mathbb{C}$. Then the solution of the initial value problem is given by
$$y(t)=y_s+P^{-1}\begin{bmatrix}e^{\lambda_1(t-t_0)}&0&0&\cdots&0\\0&e^{\lambda_2(t-t_0)}&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&e^{\lambda_n(t-t_0)}\end{bmatrix}P(y(t_0)-y_s)$$
where $P=Mtx_{VU}(id)$ & general complex situation is obtained by allowing $y(t_0)$ to vary over $\mathbb{C}^n$; it has n complex degrees of freedom. The general real solution is obtained by allowing $y(t_0)$ to vary over $\mathbb{R}^n$; it has n real degrees of freedom. Every real solution is a linear comb of the real & imaginary parts of a complex solution. In particular,

1. If the real part of each eigenvalue is less than zero, all solutions converge to y_s.
2. If the real part of each eigenvalue is greater than zero, all solutions diverge from y_s & tend to infinity.

3. If the real parts of some eigenvalues are less than zero & the real parts of other eigenvalues are greater than zero, solutions follow roughly hyperbolic paths.

4. If the real parts of all eigenvalues are zero, all solutions follow closed cycles around y.

Proof. Let $P = (M\tau x)_{V,u}(id)$. Rewrite the diff. equation in terms of a new variable $z = Py$. the representation of the solution wrt the basis V of eigenvectors.

Let $z_s = Py_s$. Then $z - z_s = P(y - y_s)$

$$(z-z_s)' = z' = Py' = PM(y-y_s) = PMP^{-1}(z-z_s) = B(z-z_s)$$

where $B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Thus the ith component of $(z(t) - z_s)$ satisfies the differential equation: $(z(t)-z_s)_i' = \lambda_i (z(t)-z_s)_i$.

So $(z(t) - z_s)_i = e^{\lambda_i(t-t_0)}(z(t_0) - z_s)_i$

So $z(t) - z_s = \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{bmatrix} (z(t_0) - z_s)$

$$y(t) - y_s = P^{-1}(z(t) - z_s)$$

$$= P^{-1}\begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{bmatrix} (z(t_0) - z_s)$$

$$= P^{-1}\begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{bmatrix} P(y(t_0) - y_s)$$

$\square$

Thm 97. Consider the diff eq $y' = (y - y_s)' = M(y - y_s)$.

Suppose matrix M can be diagonalized over $\mathbb{C}$. Let the ~~eigenvect~~ values of M w/the correct multiplicity be $\underbrace{a_1+ib_1, a_1-ib_1, \ldots, a_m+ib_m, a_m-ib_m}_{2m \text{ complex eigenvalues}}, \underbrace{a_{m+1}, \ldots, a_{n-m}}_{n-2m \text{ real eigenvectors}}$.

Then for each fixed $i = 1, \ldots, n$, every real solution is of the form

$$(y(t)-y_s)_i = \sum_{j=1}^{m} e^{a_j(t-t_0)}(C_{ij}\cos b_j(t-t_0) + D_{ij}\sin b_j(t-t_0)) + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)}$$

Then n^2 parameters $\{C_{ij} : i=1,\ldots,n;\ j=1,\ldots,n-m\} \cup \{D_{ij} : i=1,\ldots,n;\ j=1,\ldots,m\}$ has n real degrees of freedom. The parameters are uniqlely determined from the n real initial conditions of an initial value problem.

Proof. Rewrite exp for the solution y as $(y(t)-y_s)_i = \sum_{j=1}^{n} \gamma_{ij} e^{\lambda_j(t-t_0)}$

Recall non-real eigenvalues occur in conjugate pairs, so suppose $\lambda_j = a+ib$, $\lambda_k = a-ib$ so the exp for $(y(t)-y_s)_i$ contains the pair of terms

$$\gamma_{ij} e^{\lambda_j(t-t_0)} + \gamma_{ik} e^{\lambda_k(t-t_0)} = \gamma_{ij} e^{a(t-t_0)}(\cos b(t-t_0) + i\sin b(t-t_0)) + \gamma_{ik} e^{a(t-t_0)}(\cos b(t-t_0) - i\sin b(t-t_0))$$
$$= e^{a(t-t_0)}((\gamma_{ij}+\gamma_{ik})\cos b(t-t_0) + i(\gamma_{ij}-\gamma_{ik})\sin b(t-t_0))$$
$$= e^{a(t-t_0)}(C_{ij}\cos b(t-t_0) + D_{ij}\sin b(t-t_0))$$

Since this must be real for all t, we must have $C_{ij} = \gamma_{ij} + \gamma_{ik} \in \mathbb{R}$ & $D_{ij} = i(\gamma_{ij} - \gamma_{ik}) \in \mathbb{R}$

$\rightarrow$ so γ_{ij} & γ_{ik} are complex conjugates

Thus if eigenvalues $\lambda_1, \ldots, \lambda_n$ are $a_1+ib_1, a_1-ib_1, a_2+ib_2, a_2-ib_2, \ldots, a_m+ib_m, a_m-ib_m, a_{m+1}, \ldots, a_{n-m}$.

Every real solution will be of the form

$$(y(t)-y_s)_i = \sum_{j=1}^{m} e^{a_j(t-t_0)}(C_{ij}\cos b_j(t-t_0) + D_{ij}\sin b_j(t-t_0)) + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)}$$

Diff eq satisfies Lipschitz condition

$\rightarrow$ Initial value problem has a unique solution determined by the n real initial conditions

Thus the general solution has exactly n real degrees of freedom in the n^2 coeff. $\square$

Thm 98. The general solution of the inhomogenous linear differential equation (1) is $y_n + y_p$, where y_n is the general solution of the homogeneous linear diff equation (2) & y_p is any particular solution of the inhomogenous linear diff. equation (1).

Proof. Fix any particular solution y_p of inhomogeneous eq (1). Suppose y_n is any solution of the corresponding homogeneous eq (2).

Let $y_i(t) = y_n(t) + y_p(t)$.

$$\begin{aligned} y_i'(t) &= y_n'(t) + y_p'(t) \\ &= M(t)y_n(t) + M(t)y_p(t) + H(t) \\ &= M(t)(y_n(t) + y_p(t)) + H(t) \\ &= M(t)y_i(t) + H(t) \end{aligned}$$

so y_i is solution of homogeneous eq (1).

Conversely suppose y_i is any solution of inhomogenous eq (1).

Let $y_n(t) = y_i(t) - y_p(t)$.

$$\begin{aligned} y_n'(t) &= y_i'(t) - y_p'(t) \\ &= M(t)y_i(t) + H(t) - M(t)y_p(t) - H(t) \\ &= M(t)(y_i(t) - y_p(t)) \\ &= M(t)y_n(t) \end{aligned}$$

so y_n is solution of homogenous eq (2) & $y_i = y_n + y_p$. □

Thm 99. Consider the inhomogenous linear diff. eq (1) & suppose that $M(t)$ is a constant matrix M, independent of t. A particular solution of the inhomogenous linear diff. eq (1), satisfying the initial condition $y_p(t_0) = y_0$ is given by $y_p(t) = e^{(t-t_0)M}y_0 + \int_{t_0}^{t} e^{(t-s)M}H(s)\,ds$.

Proof. We verify y_p solves (3):

$$\begin{aligned} y_p(t) &= e^{(t-t_0)M}y_0 + \int_{t_0}^{t} e^{(t-s)M}H(s)\,ds \\ &= e^{(t-t_0)M}y_0 + \int_{t_0}^{t} e^{(t-t_0)M}e^{-(s-t_0)M}H(s)\,ds \\ &= e^{(t-t_0)M}\left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M}H(s)\,ds\right) \end{aligned}$$

$$\begin{aligned} y_p'(t) &= Me^{(t-t_0)M}\left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M}H(s)\,ds\right) + e^{(t-t_0)M}\left(e^{-(t-t_0)M}H(t)\right) \\ &= My_p(t) + H(t) \end{aligned}$$

$$\begin{aligned} y_p(t_0) &= e^{(t_0-t_0)M}y_0 + \int_{t_0}^{t_0} e^{(s-t_0)M}H(s)\,ds \\ &= y_0. \end{aligned}$$

□