March 26 (Rudin: Ch 1; B/S: Ch 1.7, 1.8)

Fields: set of #'s or things w/2 operations $(F, +, \cdot)$

→ Axioms:
1. Associativity
2. Communativity
3. Identities (0 for + + 1 for $\cdot$)
4. Inverses: $\forall a \ \exists (-a)$ s.t. $a + (-a) = 0$ } not equiv. $1 \neq 0$
   If $a \neq 0$, $\exists a^{-1}$ s.t. $a \cdot a^{-1} = 1$
5. Distributive Law: $a(b+c) = ab + ac$

→ $\mathbb{Z}$ is not a field b/c no multiplicative inverse, but ex. includes $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$

Complex numbers: an ordered pair of real numbers (a,b); write as $z = (a,b)$ or $z = a + ib$

→ If $w = (c,d)$ then $z + w := (a+c, b+d)$
$zw := (ac - bd, ad + bc)$

→ Define $i = (0,1)$, $1 = (1,0)$, so $i^2 = (-1,0) = -1$

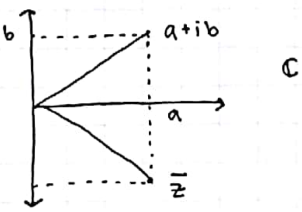
$z^{-1} = \frac{a - ib}{a^2 + b^2}$

$\bar{z} := a - ib$ (conjugate)

$|\bar{z}| = \sqrt{a^2+b^2} = (z \cdot \bar{z})^{\frac{1}{2}}$ (absolute value)

real: $Re(z) = a$

imaginary: $Im(z) = b$



$a = r\cos\theta$ $\quad z = |z|(\cos\theta + i\sin\theta)$ + $e^{i\theta} := \cos\theta + i\sin\theta$

$b = r\sin\theta$ $\quad = |z| e^{i\theta}$

$r = |z|$ $\quad$ Adding Angles: $e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta + \alpha)}$

$Arg(z) = \theta$ $\quad (e^{i\theta})^n = e^{in\theta} \ \forall n \in \mathbb{N}$

ex: $z = 1 - i$
$Arg(z) = -\frac{\pi}{4}$
$|z| = \sqrt{2}$
$z = \sqrt{2} e^{-i\frac{\pi}{4}}$

Theorem: z, w complex numbers then
- a. $|z| \geq 0$ + $|z| = 0 \Leftrightarrow z = 0$
- b. $|\bar{z}| = |z|$
- c. $|z \cdot w| = |z| \cdot |w|$
- d. $|Re(z)| \leq |z|$
- e. $|z + w| \leq |z| + |w|$ (triangle inequality)

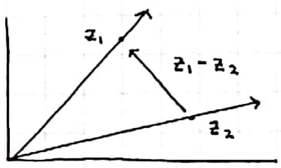
$z + \bar{z} = 2Re(z)$
$z - \bar{z} = 2i\,Im(z)$

Proof c: $|zw|^2 = (z \cdot w)(\overline{z \cdot w})$
$= z \cdot w \cdot \bar{z} \cdot \bar{w} = z \cdot \bar{z} \cdot w \cdot \bar{w} = |z|^2 |w|^2$
take the sq. root of both sides

Proof e: $z = a + ib$ $\quad \sqrt{(a+b)^2 + (c+d)^2} \leq \sqrt{(a^2+b^2)} + \sqrt{(c^2+d^2)}$
$w = c + id$
$|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\bar{z} + \bar{w})$
$= z \cdot \bar{z} + w \cdot \bar{w} + z \cdot \bar{w} + \bar{z} \cdot w$
$= |z|^2 + |w|^2 + 2Re(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}|$
$= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$

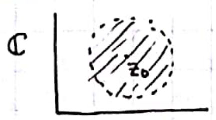
March 28 (B/S: Ch 0.2, 1.8)

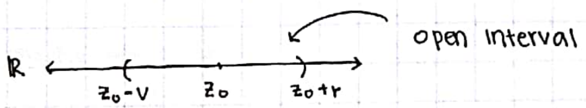
Proposition: Abs. value in $\mathbb{R}, \mathbb{C}$ satisfy
1. $|z| \geq 0$ + $|z| = 0 \Leftrightarrow z = 0$
2. $|z + w| \leq |z| + |w|$ (tri. ineq.)
3. $|z \cdot w| = |z||w|$



Distance: If $z_1, z_2 \in \mathbb{C}$ then distance b/t z_1, z_2 is equal to $|z_1 - z_2|$

An open ball of radius r centered @ a point z_0 is given by $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$

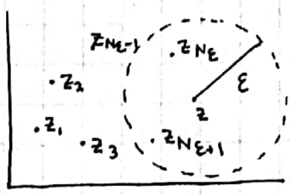


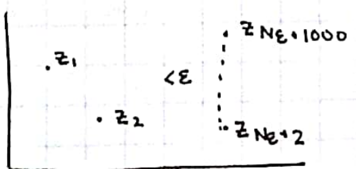


A sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in \mathbb{C}$ is bounded if $\exists r > 0$ s.t. $\forall n$: $z_n \in B_r(0)$

is convergent / converging to z if $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. if $n \geq N_\varepsilon$ then $|z_n - z| < \varepsilon$

is Cauchy if $\forall \varepsilon > 0 \ \exists N_\varepsilon > 0$ so that if $n, m \geq N_\varepsilon$ then $|z_n - z_m| < \varepsilon$





Theorem (Completeness of $\mathbb{R}, \mathbb{C}$): A seq. $(z_n)_{n\in\mathbb{N}}$ is convergent $\Leftrightarrow$ $(z_n)_{n\in\mathbb{N}}$ is Cauchy

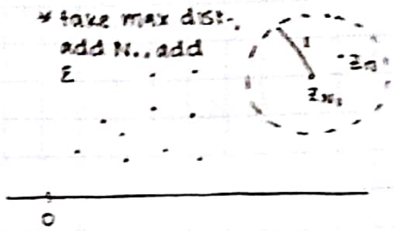
Exercises: 1) Prove $z_n = 2i + (-1)^n$ is bounded

2) Prove $w_n = \frac{3+4i}{n}$ is convergent

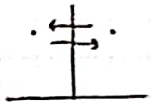
3) If $(z_n)_{n\in\mathbb{N}}$ is convergent $\Rightarrow$ $(z_n)_{n\in\mathbb{N}}$ is bounded

4) ________ Cauchy $\Rightarrow$ ________ bounded

* take max dist., add N_1, add ε



1)



$|z_n| = \sqrt{5}$ for all n.

So $|z_n| < 3$. Let $r = 3$.

$\Rightarrow z_n \in B_3(0)$

4) Let $\varepsilon = 1$. $\exists N_1$ s.t. if $n, m \geq N_1 \Rightarrow |z_n - z_m| < 1$.

Take $r = \max\{|z_1|+1, |z_2|+1, \dots, |z_{N_1-1}|+1, |z_{N_1}|+2\}$. Show $z_k \in B_r(0)$ for every $k \in \mathbb{N}$

Case 1: $k < N_1$; $|z_k| < |z_k| + 1 \leq r$

Case 2: $k \geq N_1$; $|z_k| \leq |z_k - z_{N_1}| + |z_{N_1}| < 1 + |z_{N_1}| \leq r$

March 30

Construction of real numbers: Set Theory Axioms $\Rightarrow \mathbb{Z} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{R} \Rightarrow \mathbb{C}$

construction using equiv relations

A **relation** is a subset S of $X \times X$, written as aRb or $a \sim b$ which means $(a,b) \in S$ or $a, b \in X$

$\rightarrow X \times X = \{(a,b) \mid a \in X, b \in X\}$

An **equivalence relation** is a relation satisfying

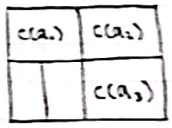
1. **Reflexive**: $\forall a \in X$, $a \sim a$
2. **Symmetric**: $\forall a, \forall b$ if $a \sim b \Rightarrow b \sim a$
3. **Transitive**: $\forall a, b, c \in X$, if $a \sim b$ + $b \sim c$ then $a \sim c$.

Ex: 1) $X = \mathbb{N}$, $S_1 = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ divides } b\}$] not symm so not an equiv. relation

2) $X = \mathbb{C}$, $a \sim b$ if $|a| = |b|$

3) $X = \mathbb{Z}$, $a \sim b$ if $a - b$ is a multiple of 3] prove equiv. relation

If $a \in X$, an **equivalent class** of a is $c(a) := \{b \in X \mid a \sim b\}$

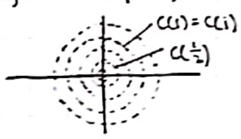


Ex: (Exe 2) $c(1) = \{a, b \in \mathbb{C} \mid |b| = |1| = 1\} = c(i)$

(Exe 3) $c(2) = \{b \in \mathbb{Z} \mid 2 - b \text{ is a multiple of } 3\} = \begin{Bmatrix} 2, 5, 8, 11, \dots \\ -1, -4, -7, -9, \dots \end{Bmatrix}$

No overlap of equiv. classes (prove disjoint or equal)

$\rightarrow$ In ex 2, partition in circles



$\rightarrow$ In ex 3, only 3 equiv. classes

mult of 3: $\{3, 6, 9, \dots\}$

remainder 1: $\{-2, 1, 4, \dots\}$

remainder 2: $\{-1, 2, 5, \dots\}$

Construction of $\mathbb{Q}$ from $\mathbb{Z}$ ($\mathbb{Z} \times \mathbb{Z}$)

$X = \{(a,b) \mid (a,b) \in \mathbb{Z} \times \mathbb{Z} \text{ or } a, b \in \mathbb{Z}, b \neq 0\}$ relation $(a,b) \sim (c,d)$ if $ad = bc$ * $\frac{a}{b} = \frac{c}{d}$

$\rightarrow$ construct $\mathbb{Z}$ w/ Naive Set Theory by Paul Halmos

Ex: $\sim$ is an equiv. relation

$\mathbb{Q} := \{c((a,b)) \mid (a,b) \in X\} = \frac{X}{\sim}$ $\quad [\frac{2}{3} := c((2,3)) = \{(2,3), (4,6), \dots\}]$

$\rightarrow$ Addition of rationals: $c((a,b)) + c((c,d)) := c((ad+bc, bd))$

$\rightarrow$ Multiplication: $c((a,b))\, c((c,d)) := c((ac, bd))$

Exercise: Prove $\mathbb{Q}$ is a field

$\mathbb{Q}$ is **incomplete**: a Cauchy seq $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{Q}$ is not necessarily convergent to a rational number

$a_1 = 3$ $\quad a_4 = 3.141$

$a_2 = 3.1$ $\quad a_5 = 3.1417$ $\qquad$ * a_n is Cauchy but $\lim_{n\to\infty} a_n = \pi \notin \mathbb{Q}$

$a_3 = 3.14$

Another ex: $b_1 = 1$, $b_{n+1} = \frac{1}{2}(b_n + \frac{2}{b_n})$ $\rightarrow$ Prove $\lim_{n\to\infty} b_n = \sqrt{2}$ or $\lim_{n\to\infty} b_n^2 = 2$

Real Numbers

$\rightarrow$ Talk about Cauchy w/o talking about limits

$C = \left\{ (a_n)_{n\in\mathbb{N}} \;\middle|\; \begin{matrix} a_n \in \mathbb{Q} \\ (a_n) \text{ is a Cauchy seq} \end{matrix} \right\}$

$\forall r > 0 + r \in \mathbb{Q}\ \exists N;\ n, m \geq N \Rightarrow |a_n - a_m| < r$

Relation $(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}}$ if $\lim_{n\to\infty} a_n - b_n = 0$

$\rightarrow \mathbb{R} = \frac{C}{\sim}$: set of eq. classes of the relation of C

$\rightarrow$ two Cauchy is the same if $\lim$ is same

April 2 (B/S: Ch 1.5)

Construction of $\mathbb{R}$ from $\mathbb{Q}$: $C = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{Q},\ (a_n)_{n\in\mathbb{N}}$ is Cauchy$\}$

→ **Relation**: $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in C$ $\quad (a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}}$ if $\lim_{n\to\infty} a_n - b_n = 0$

$\mathbb{R} := \{$set of eq. classes of $C\} = \frac{C}{\sim}$

→ Notation: $[a_n] = C((a_n)_{n\in\mathbb{N}})$ ← eq. classes of sequences

all seq. related by this relation

Addition / Multiplication

Lemma: If $(a_n), (b_n)$ are Cauchy seq, then $(a_n + b_n)_{n\in\mathbb{N}}$ + (a_nb_n) are Cauchy seq.

→ Def: $(a_n), (b_n) \in C$; $a = [a_n]$, $b = [b_n] \in \mathbb{R}$ (real #)

$[a_n] + [b_n] := [a_n + b_n]$

$[a_n][b_n] := [a_nb_n]$

Theorem: $(\mathbb{R}, +, \cdot)$ is a field

* Prove on own

→ Ex: $1_{\mathbb{R}} = [c_n]$ $\quad c_n = 1, n \geq 1$

$0_{\mathbb{R}} = [d_n]$ $\quad d_n = 0, n \geq 1$

→ Check if addition + multiplication are well-defined

$(a_n) \sim (c_n)$, $(b_n) \sim (d_n)$ $\Rightarrow (a_n + b_n) \sim (c_n + d_n)$ + $(a_nb_n) \sim (c_nd_n)$

Addition: $\lim_{n\to\infty} a_n - c_n = 0$

$\lim_{n\to\infty} b_n - d_n = 0$

$\lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} (c_n + d_n) = 0$

$\lim_{n\to\infty} (a_n - c_n) - \lim_{n\to\infty} (b_n - d_n) = 0$

* $\frac{a}{b} < \frac{c}{d} \Leftrightarrow ad \lessgtr cb$

Multiplication: Show Cauchy ⇒ bounded first

$\{a_n\}, \{b_n\}$ are bounded $\exists M$ s.t. $|a_n| \leq M$, $|b_n| \leq M$

Take N_1, N_2 s.t. $n, m \geq N_1 \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2M}$

$n, m \geq N_2 \Rightarrow |b_n - b_m| < \frac{\varepsilon}{2M}$

prod of lim = lim of prod.

$|a_nb_n - a_mb_m| = |a_nb_n - a_mb_m + a_nb_m - a_nb_m|$

$\leq |a_n||b_n - b_m| + |b_m||a_n - a_m| < M\left(\frac{\varepsilon}{2M}\right) \cdot M\left(\frac{\varepsilon}{2M}\right) = \varepsilon$

Order in $\mathbb{R}$: notation $\underset{\mathbb{R}}{<}$

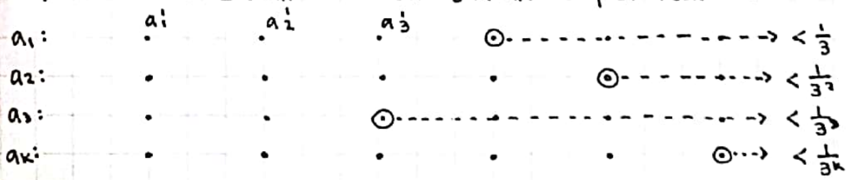
→ Def. $a = [a_n]$, $b = [b_n] \in \mathbb{R}$, we say that $a \underset{\mathbb{R}}{<} b$ if there exists N s.t. if $n \geq N \Rightarrow a_n \underset{\mathbb{Q}}{<} b_n$

→ HW: If the # $(c) = [c_k]$ + $\lim_{k\to\infty} c_k \neq 0$ (diff than zero) $\Rightarrow c > 0$ or $c < 0$

→ $\mathbb{R}$ satisfies axiom of an ordered field

Theorem: Every Cauchy seq of real numbers is convergent

seq $a^m = [a^m_n]_{n\in\mathbb{N}}$ $\quad \{a^m\}_{m\in\mathbb{N}}$: seq of real #'s



all Cauchy but how to take lim?

↳ b/c Cauchy, dist. b/t points eventually $< \varepsilon$

↳ use these seq.

Take N_k s.t. if $n, m \geq N_k$ then $|a^k_m - a^k_n| < \frac{1}{3^k}$

↳ blue point: $b_k := a^k_{N_k}$

Cauchy seq of rational #'s = real #'s = lim(seq)

Lemma 1: $(b_k)_{k\in\mathbb{N}}$ is Cauchy → why?

$\lim_{m\to\infty} [a^m] = [b_k]$

April 4 (B/S: Ch 1.4; 2.5; Rudin: Ch 1 last three pages)

Euclidean Space (in n dimensions): $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ $\quad$ * $\mathbb{C}^n$ defined similarly

$\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$

$\vec{y} = (y_1, y_2, y_3, \ldots, y_n)$

$\alpha \in \mathbb{R}$

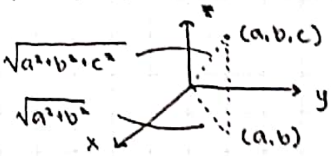
1. Addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
2. **Dot product / Standard inner product**: $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ $\quad$ vector → #
3. **Scalar Multiplication**: $\alpha\vec{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

* In $\mathbb{C}$, $\vec{x} \cdot \vec{y} = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$ (not commutative – use conjugate)

Norm: $\|x\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $\quad$ distance from $\vec{x}$ to $\vec{0}$ in $\mathbb{R}^n$

→ $\vec{0} = (0, 0, 0, \ldots, 0)$

→ Pythag. Theorem in $\mathbb{R}^2$ extrapolated



Think of x = slope

y = intercept

Properties

1. $|\vec{x}| \geq 0$ + $|\vec{x}| = 0 \Rightarrow \vec{0}$
2. **Cauchy-Schwartz Inequality**: $|\vec{x} \cdot \vec{y}| \leq |x| \cdot |y|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

absolute value of real #

Proof of Cauchy-Schwartz Inequality

Def. $f(\alpha) := \|\alpha\vec{x} + \vec{y}\|^2 = (\alpha x + y)(\alpha x + y)$

(func of $\mathbb{R}$; $\alpha \in \mathbb{R}$) $= \alpha^2\|x\|^2 + 2\alpha x\cdot y + \|y\|^2$

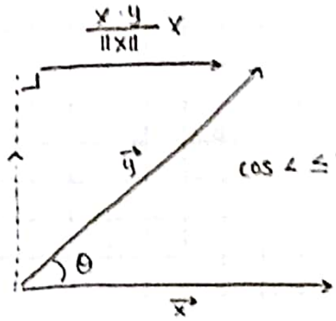
$f'(\alpha) = 2\alpha\|x\|^2 + 2x\cdot y$

Find min, set $f'(\alpha) = 0$; $\alpha = -\frac{x\cdot y}{\|x\|^2}$

Plug back in: $f(\alpha) = \left(-\frac{x\cdot y}{\|x\|}\right)^2\|x\|^2 - 2\left(\frac{x\cdot y}{\|x\|^2}\right)(x\cdot y) + \|y\|^2 > 0$ norm is always positive

$= -\frac{(x\cdot y)^2}{\|x\|^2} + \|y\|^2 \geq 0$

$\|y\|^2 \geq \frac{(x\cdot y)^2}{\|x\|^2} \Rightarrow \|x\|^2\|y\|^2 \geq (x\cdot y)^2$. ▨



Interpretation: $|x_1y_1 + y_2x_2 + x_3y_3 + \cdots + x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$ take sq root

$\cos\theta = \frac{\left\|\frac{x\cdot y}{\|x\|^2}x\right\|}{\|y\|}$

positive b/c $\vec{x} \parallel \frac{x\cdot y}{\|x\|}x$

$= \frac{\left\|\frac{x\cdot y}{\|x\|^2}\cdot\|x\|\right\|}{\|y\|} = \frac{|x\cdot y|}{\|x\|\cdot\|y\|} = \frac{x\cdot y}{\|x\|\cdot\|y\|}$

Lines in $\mathbb{R}^n$: $\vec{x}, \vec{y}$

$\ell = \{\alpha\vec{x} + \vec{y} \mid \alpha \in \mathbb{R}\}$: line passing through $\vec{y}$ w/direction $\vec{x}$

Corollary: $\vec{x}, \vec{y}$ are perpendicular iff $\vec{x}\cdot\vec{y} = 0$

April 6 (B/S: Ch. 03; Rudin: Ch 2)

Countability + Cardinality: A, B sets + $f: A \to B$

1. $f: A \to B$ is **injective** iff $\forall x, \forall y$ $f(x) = f(y) \Rightarrow x = y$
2. f is **surjective** if $\forall y \in B\ \exists x \in A$ $f(x) = y$
3. **Bijective** = injective + surjective

def. A, B have same **cardinality** ($|A| = |B|$) if $\exists$ bijection $f: A \to B$

def. A is **finite** if $\exists n \in \mathbb{N}$ + bijective $f: A \to \{1, 2, 3, \ldots, n\}$

A is **infinite** if A is not finite

def. A is **countable** if 1) A is finite or

2) $\exists$ bijection $f: A \to \mathbb{N}$ (infinite countable)

Ex: $|\mathbb{N}| = |\{1, 3, 5, 7, \ldots\}|$ Proof: $f: \mathbb{N} \to \mathbb{N}$, $f(n) = 2n - 1$ (f is a bijection)

Notation. $E_1, E_2, \ldots, E_n, \ldots$

$\bigcup_{i=1}^{\infty} E_i = \{x \mid x \in E_i \text{ for some } i\}$

$\bigcap_{i=1}^{\infty} E_i = \{x \mid x \in E_i \text{ for all } i\}$

$\{E_\alpha\}_{\alpha\in A}$ → family of sets indexed by A

$\bigcup_{\alpha\in A} E_\alpha = \{x \mid x \in E_\alpha \text{ for some } \alpha \in A\}$

Union of Countable Sets

Theorem: If $E_1, E_2, \ldots, E_n, \ldots$ are countable then $\bigcup_{i=1}^{\infty} E_i$ is countable

Countable union of countable sets is countable

def. A is **uncountable** if A is not countable

Theorem (Schroder-Bernstein): If there is $h: A \to B$, $g: B \to A$ surjective or injective functions then $\exists$ bijection $f: A \to B$

→ same cardinality

Theorem: If S is the set of all infinite seq. consisting of 0's + 1's then S is uncountable

→ Ex: element of S: 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, ...

Proof by Contradiction: Assume countably many elements in S

Cantor's Diagonal Argument

Bijection:
$1 \to a_1^1\ a_2^1\ a_3^1 \cdots a_n^1 \cdots$
$2 \to a_1^2\ a_2^2\ a_3^2 \cdots a_n^2 \cdots$
$3 \to a_1^3\ a_2^3\ a_3^3 \cdots a_n^3 \cdots$
$\vdots$
$n \to a_1^n\ a_2^n\ a_3^n \cdots a_n^n \cdots$

Take the following seq $b_i = \begin{cases} 1 & \text{if } a_i^i = 0 \\ 0 & \text{if } a_i^i = 1 \end{cases}$

$(b_i) \in S$

$(b_i) \notin (a_i^k)$ b/c $b_k \neq a_k^k$

HW1: 7c) $\lim e^{i\theta} = L$

$\lim e^{(i+1)\theta} = L \Rightarrow \lim e^{i\theta}\lim e^{\theta} = Le^i \Rightarrow e^i = 1$

$L(e^i - 1) = 0 \Rightarrow L = 0$ but L cannot be 0 b/c $|e^{i\theta}| = 1 \neq 0$

April 9 (B/S: Ch 3.1-3.3; Rudin: Ch. 2.2.3)

Metric Space: (X, d)

a set ↩ ↪ $d: X \times X \to \mathbb{R}$ (metric or distance function)

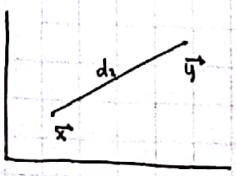
* elements of X are called **points**

The function d satisfies the following:

1. $\forall x\ \forall y$ $d(x,y) \geq 0$ + $d(x,y) = 0$ iff $x = y$
2. $\forall x\ \forall y$ $d(x,y) = d(y,x)$
3. $\forall x\ \forall y\ \forall z$ $d(x,z) \leq d(x,y) + d(y,z)$ triangular inequality

Ex1: **Euclidean Space** $(\mathbb{R}^n, d_2)$ $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n)$

↪ **L^2 metric**



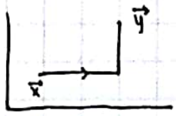
$d_2(x,y) = \|\vec{x} - \vec{y}\|$

$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots}$

Ex 2: L^1 metric (taxicab metric): $(\mathbb{R}^n, d_1)$ $\quad \vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $n = 2$

$\hookrightarrow (d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|)$

$= |x_1 - y_1| + |x_2 - y_2|$



ex: $\vec{x} = (1,3)$, $\vec{y} = (2,5)$

$d_\infty(x,y) = 2$

$d_p(x,y) = 1 + 2$

$d_2(x,y) = \sqrt{1^2 + 2^2}$

Ex 3: Discrete metric is defined by $d_{disc}(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

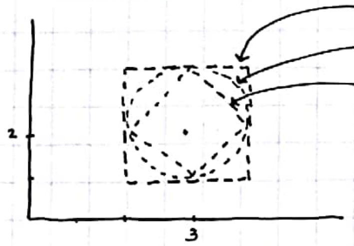
Ex 4: L^p metric: $(\mathbb{R}^n, d_p)$, $p \geq 1$ where $d_p(\vec{x}, \vec{y}) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$

L^∞ metric: $d_\infty(\vec{x}, \vec{y}) = \max(1 \leq i \leq n \mid x_i - y_i)$

Topology of Metric Spaces (X, d)

The open ball centered @ x w/radius r is defined by $B_r(x) = \{y \in X \mid d(x,y) < r\}$

— closed ball — $\bar{B_r}(x) = \{y \in X \mid d(x,y) \leq r\}$

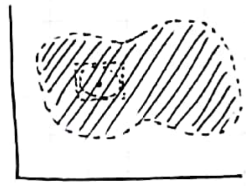


d_∞ metric: $B_1(p) = \{x\ (x_1, x_2) \in \mathbb{R}^2 \mid \max\{|x-2|, |y-3| < 1\}\}$

d_2 metric: $B_1(p)$

d_1 metric: $B_2(p) = \{(x,y) \in \mathbb{R}^2 \mid |x-2| + |y-3| < 1\}$

Ex: $(\mathbb{R}^2, d_2)$ → standard metric



Open set: $E \subseteq X$ is said to be open if $\forall x \in E$, there is $r > 0$ so that $B_r(x) \subseteq E$

Closed set: $E \subseteq X$ is closed if E^c $(X \setminus E)$ is open (e.g. point, segment)

* For $(\mathbb{R}^n, d_p)$ metric, open in one is open for all but not in general

Exercise: Show an open ball is an open set.

(X, d_{disc}) Show that any subset $Y \subseteq X$ is both open + closed

→ d(bt any 2 pts) $= 1$: so ball is either a whole space or one point

April 11 (B/S: Ch 3.3 ; Rudin: Ch 2.2)

Basic Properties of Open + Closed Sets:

Theorem: 1) $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets then $\bigcup_{\alpha \in A} U_\alpha$ is open

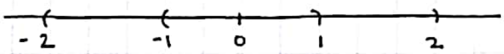
2) If $U_1, U_2, \dots, U_n$ are open sets, then $\bigcap_{i=1}^{n} U_i$ is open

Theorem: 1) If $\{C_\alpha\}_{\alpha \in A}$ is a collection of closed sets then $\bigcap_{\alpha \in A} C_i$ is closed

2) If $C_1, C_2, \dots, C_n$ are closed sets, then $\bigcup_{i=1}^{n} C_i$ is closed

Ex: $X = \mathbb{R}$, $U_n = B_{\frac{1}{n}}(0) = (-\frac{1}{n}, \frac{1}{n})$ are open sets (like an interval)

$\bigcap_{n=1}^{\infty} U_n = \{0\}$ → not open



* $C_n = U_n^c$

$= (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$

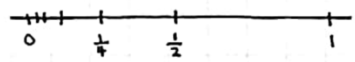
$\bigcup_{n=1}^{\infty} C_n = (-\infty, 0) \cup (0, \infty) \Rightarrow$ not closed

Proof 2: Take $x \in \bigcap_{i=1}^{n} U_i \Rightarrow x \in U_i$ for all $1 \leq i \leq n$

$\exists\ r_i$ s.t. $B_{r_i}(x) \subseteq U_i$

Take $r = \min_{1 \leq i \leq n}\{r_i\} > 0$ (b/c finite) $\Rightarrow B_r(x) \subseteq B_{r_i}(x) \subseteq U_i$

Ex: $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$



S is not closed b/c $0 \in S^c$ but all $B_r(0)$ for $\forall r > 0$ contains points of S

Problem: there are points accumulating @ 0

Accumulation points: $E \subseteq X$

def. a point $p \in X$ is an accumulation point of E if $\forall r > 0$, $B_r(p) \cap E$ contains points of E different from p itself; $(B_r(p) \setminus \{p\}) \cap E \neq \emptyset$, $\forall r > 0$

Theorem: A set E is closed iff E contains all its accumulation points

* Remark: p can be in E or E^c

Ex: $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ is a closed set

Proof: $(\Rightarrow)$ Assume E is closed. Take accumulation point x of E

If $x \in E$ we are done; if not $x \in E^c$ which is open

$\Rightarrow \exists r\ B_r(x) \subseteq E^c \Rightarrow B_r(x) \cap E = \emptyset \Rightarrow x$ is not a.p.

* Exercise: Find ∞ union of closed sets is open.

$(\Leftarrow)$ If E contains all a.p. Show E^c is open

Pf by Contr. Assume E^c is not open

$\Rightarrow \exists\ x \in E^c$ s.t. $\forall r > 0$ $B_r(x)$ not contained in E^c

$\Rightarrow B_r(x) \cap E \neq \emptyset \Rightarrow B_r(x) \setminus \{x\} \cap E \neq \emptyset$

$\Rightarrow x$ is an a.p. of E $\Rightarrow x \in E$ + $x \in E^c$.

Closed sets in $\mathbb{R}$

1. Finite collection of points
2. Closed intervals
3. $\mathbb{R}, \Phi, \mathbb{Z}$

Miderm I Review

Fields: set of #'s w/2 operations

→ Axioms: 1. Associativity
2. Commutativity
3. Identities
4. Inverses
5. Distributive

Open ball of radius r centered at a point z_0 is given by:
$B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$.

Complex numbers: ordered pair of real #'s; $z = (a,b)$, $z = a + bi$

$z + w := (a+c, b+d)$ $\bar{z} := a - bi$ (conjugate)

$zw := (ac - bd, ad + bc)$ $|z| = \sqrt{a^2 + b^2} = (z \cdot \bar{z})^{\frac{1}{2}}$

$e^{i\theta} := \cos\theta + i\sin\theta$ distance $= |z_1 - z_2|$

Theorem: 1. $|z| \geq 0$ + $|z| = 0 \Leftrightarrow z = 0$
2. $|\bar{z}| = |z|$
3. $|z \cdot w| = |z| \cdot |w|$
4. $|Re(z)| \leq |z|$
5. $|z + w| \leq |z| + |w|$ (triangle inequality)

A seq is **bounded** if $\exists\ r > 0$ s.t. $\forall n: z_n \in B_r(0)$

convergent to z if $\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon$ s.t. if $n \geq N_\varepsilon$ then $|z_n - z| < \varepsilon$

Cauchy if $\forall\ \varepsilon > 0\ \exists\ N_\varepsilon > 0$ s.t. if $n, m \geq N_\varepsilon$ then $|z_n - z_m| < \varepsilon$

Completeness of $\mathbb{R}, \mathbb{C}$: A seq $(z_n)_{n \in \mathbb{N}}$ is convergent $\Leftrightarrow$ $(z_n)_{n \in \mathbb{N}}$ is Cauchy

Equivalence relation is a relation satisfying
1. **Reflexive**: $\forall a \in X$, $a \sim a$
2. **Symmetric**: $\forall a, b$ if $a \sim b \Rightarrow b \sim a$
3. **Transitive** $\forall a, b, c \in X$, if $a \sim b$ + $b \sim c$ then $a \sim c$

Equivalence class of $a \in X$ is $c(a) := \{b \in X \mid a \sim b\}$

→ no overlap; disjoint or equal

→ $\mathbb{Q} := \{c(a,b) \mid (a,b) \in X\} = \frac{X}{\sim}$

Euclidean Space: $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n \mid x_i \in \mathbb{R}\}$

Dot/Inner Product: $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

Norm: $\|\vec{x}\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Properties of Euclidean Space:
1. $|\vec{x}| \geq 0$ + if $|\vec{x}| = 0 \Rightarrow \vec{0}$
2. **Cauchy-Schwarz Inequality**: $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Thm: $\vec{x}$ + $\vec{y}$ are $\perp \Leftrightarrow \vec{x} \cdot \vec{y} = 0$

Metric Space (X, d) where the metric func. $d: X \times X \to \mathbb{R}$ satisfies
1. $\forall x, y$ $d(x,y) \geq 0$ + $d(x,y) = 0 \Leftrightarrow x = y$
2. —— $d(x,y) = d(y,x)$
3. $\forall x, y, z$ $d(x,z) \leq d(x,y) + d(y,z)$

Discrete metric: $d_{disc}(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

L^p metric $(\mathbb{R}^n, d_p)$, $p \geq 1$: $d_p(\vec{x}, \vec{y}) = (\sum_{i=1}^n |x_i - y_i|^p)^{\frac{1}{p}}$

→ **L^1 (taxicab metric)**: $d_1(\vec{x}, \vec{y}) = \sum_{i=1}^n |x_i - y_i|$

→ **L^2 metric**: $d_2(\vec{x}, \vec{y}) = (\sum_{i=1}^n |x_i - y_i|^2)^{\frac{1}{2}} = \|\vec{x} - \vec{y}\|$

→ **L^∞ metric**: $d_\infty(\vec{x}, \vec{y}) = \max(1 \leq i \leq n \mid x_i - y_i)$

Open ball centered @ x w/radius r is $B_r(x) = \{y \in X \mid d(x,y) < r\}$

Closed ball is $\bar{B}_r(x) = \{y \in X \mid d(x,y) \leq r\}$

Open set $E \subseteq X$: if $\forall x \in E$, there is $r > 0$ s.t. $B_r(x) \subseteq E$

Closed set $E \subseteq X$: E^c $(X \setminus E)$ is open

A point $p \in X$ is an **accumulation point** of E if $\forall\ r > 0$, $B_r(p) \cap E$ contains points of E different from p itself

→ $(B_r(p) \setminus \{p\}) \cap E \neq \emptyset, \forall r > 0$

Thm: A set E is closed $\Leftrightarrow$ E contains all its accumulation points

Negation of limits: If $a_k \nrightarrow L$, $\exists\ \varepsilon > 0$ s.t. $\forall\ N_\varepsilon$, $\exists\ n \geq N_\varepsilon$ s.t. $|a_n - L| \geq \varepsilon$.

For finding ε + N_ε: Solve for ε in $|a_k - L| < \varepsilon$.

→ use n to find N_ε.

Construction of $\mathbb{R}$ from $\mathbb{Q}$

$C = \{(a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{Q}, (a_n)_{n \in \mathbb{N}}$ is Cauchy$\}$

$\forall (a_n), (b_n)_{n \in \mathbb{N}}$, $(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$ if $\lim_{n \to \infty} a_n - b_n = 0$

$\mathbb{R} := \{$set of equivalence classes of C$\} = \frac{C}{\sim}$

Thm: If $(a_n), (b_n)$ are Cauchy seq, then $(a_n + b_n)_{n \in \mathbb{N}}$ + $(a_n b_n)$ are also Cauchy seq.

$[a_n] + [b_n] := [a_n + b_n]$; $[a_n][b_n] := [a_n b_n]$

Order in $\mathbb{R}$: $a = [a_n]$, $b = [b_n] \in \mathbb{R}$, $a \underset{\mathbb{R}}{\leq} b$ if $\exists$ s.t. if $n \geq N \Rightarrow a_n \underset{\mathbb{Q}}{\leq} b_n$

Thm: every Cauchy seq of reals is convergent

A, B have the same **cardinality** ($|A| = |B|$) if $\exists$ bijection $f: A \to B$

→ **Bijection** = injective + surjective

→ f is **injective** $\Leftrightarrow$ $\forall x, y$ $f(x) = f(y) \Rightarrow x = y$

→ f is **surjective** if $\forall y \in B$ $\exists x \in A$ s.t. $f(x) = y$

A is **finite** if $\exists n \in \mathbb{N}$, bijection $f: A \to \{1, 2, 3, \ldots, n\}$

infinite if A is not finite

A is **countable** if
1. A is finite or
2. $\exists$ bijection $f: A \to \mathbb{N}$ (**infinitely countable**)

A is **uncountable** if A is not countable

Schroder-Bernstein Thm: If there is $h: A \to B$ + $g: B \to A$ for surjection/injection then $\exists$ bijection $f: A \to B$

Thm: If S is the set of all infinite seq. consisting of 0's + 1's then S is uncountable (Cantor's Diagonal Arg)

Thm: 1. $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets then $\bigcup_{\alpha \in A} U_\alpha$ is open
2. $\{C_\alpha\}_{\alpha \in A}$ is a collection of closed sets then $\bigcap_{\alpha \in A} C_\alpha$ is closed

Thm: 1. If $U_1, U_2, \ldots, U_n$ are open sets then $\bigcap_{i=1}^n$ is open
2. If $C_1, C_2, \ldots, C_n$ are closed sets then $\bigcup_{i=1}^n C_i$ is closed

April 13 (B/S: Ch 3.3; Rudin: Ch 2.2)

[HW: every point = open set = $\cup$ of open set = open
def of limit: $\forall \varepsilon>0 \ \exists N$ s.t. $n \geq N \rightarrow |a_n - L| < \varepsilon$
neg of limit: $\exists \varepsilon > 0 \ \forall N \ \exists n \geq N \Rightarrow |a_n - L| \geq \varepsilon$
$\exists$ subseq. $n_1 < n_2 < \cdots < n_k = \{a_n\}$ s.t. $|a_{n_1} - L| \geq \varepsilon$]

(X,d) metric spaces + subsets

→ $Y \subseteq X$ is a subset where Y has a metric given by the metric on X so $(Y,d) \subseteq (X,d)$

→ Ex 1: $Y = [0,1]$. $\mathbb{R}$ w/standard metric $(d(x,y) = |x-y|)$

In Y, $B_r(0) = \{y \in X \mid d(0,y) < r\}$

Case 1: $0 \leq r \leq 1$, $B_r(0) = [0,r) = B_r^*(0) \cap Y$

Case 2: $r > 1$. $B_r(0) = [0,1]$

→ Ex 2: $Y = \mathbb{Q}$, $X = \mathbb{R}$

$B_r(p) = (p-r, r+p) \cap \mathbb{Q} = B_r^*(p) \cap \mathbb{Q}$

Note that $[-\sqrt{2}, \sqrt{2}]$ is not an open set in $\mathbb{R}$

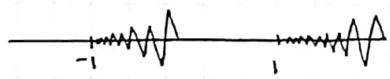
$[-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$ is an open + closed set (contains all acc. points)

$(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is the same as above.

Remember: accumulation points (= limit points in Rudin): $E \subseteq X$, $p \in E$ is an acc. point if $\forall r > 0$ or $(B_r(p) \setminus \{p\}) \cap E$

Notation: E' denotes the set of acc. points of E

Ex: $S = \{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}\}$ so $S' = \{-1, 1\}$



Closure of a set: $E \subseteq X$ (notation: $\bar{E} : E \cup E'$)

Ex: $\bar{S} = S \cup S' = \{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{-1, 1\}$

Theorem: (X,d) metric space, $E \subseteq X$
1) $\bar{E}$ is a closed set
2) $E = \bar{E}$ iff E is closed
3) If F is closed + $E \subseteq F$, then $\bar{E} \subseteq F$

* $\bar{E}$ is the smallest closed set containing E
* $\bar{E}$ is the intersection of all closed sets containing E

Proof: 1) Prove $\bar{E}^c$ is open ($\bar{E}^c = X - \bar{E}$)

Let $p \in X - \bar{E}$ then $p \notin E$ + $p \notin E'$

$\Rightarrow \exists r > 0$, $B_r(p) \setminus \{p\} \cap E = \emptyset$

$\Rightarrow B_r(p) \cap E = \emptyset$ + $B_r(p) \cap \bar{E} = \emptyset$

If $B_r(p) \cap \bar{E} \neq \emptyset$ then $B_r(p) \cap E \neq \emptyset$ if $q \in E'$.
Any open set containing q contains points of E
So $B_r(p) \cap E \neq \emptyset$

$B_r(p) \subseteq X - \bar{E}$ $\Rightarrow X \setminus \bar{E}$ is open

2) If E is closed $\Leftrightarrow$ contains all acc. points $\Leftrightarrow E' \subseteq E \Leftrightarrow E = E \cup E' = \bar{E}$

3) If $E \subseteq F$ + F is closed so $E' \subseteq F' \subseteq$ (by theorem from last lecture) F

$E \cup E' \subseteq F' \Rightarrow \bar{E} \subseteq F$

$B_s(q) \cap E \neq \emptyset$ b/c q is an accumulation point

HW: look up isolated points + boundary points (B/S: Ch. 3.3)

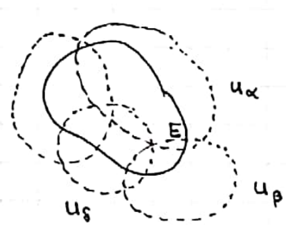
April 16

An **open cover** of E is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets s + $E \subseteq \bigcup_{\alpha \in A} U_\alpha$

E is **compact** if any open cover $\{U_\alpha\}_{\alpha \in A}$ E has a finite subcover

→ $\exists \alpha_1, \alpha_2, \ldots \alpha_n$ s.t. $E \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \cup \cdots \cup U_{\alpha_n} = \bigcup_{i=1}^{n} U_{\alpha_i}$

$\{V_\alpha\}_{\alpha \in A}$, $\{U_{\alpha'}\}_{\alpha \in A'}$ cover E then $\{V_\alpha\}$ is a subcover if $\forall \alpha$ $V_\alpha = U_{\alpha'}$ for some α'



Theorem: A finite collection of points is compact.

Proof: $S = \{p_1, p_2, \ldots, p_n\}$. If $\{U_\alpha\}$ is an open cover of S then $\exists U_{\alpha_i}$ s.t. $p_i \in U_{\alpha_i}$

Exercises: 1) $(0,1)$ is not compact

2) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ is compact

3) $\mathbb{Z}$ is not compact.

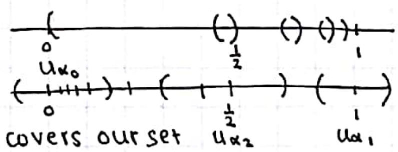
* To prove not compact: find an ∞ collection of open sets w/ no finite subcover

Rough Pfs: 1) $\{(1 - \frac{1}{2^i} - \varepsilon, 1 - \frac{1}{2^{i+1}})\} \cup \{0, \frac{1}{2}\}$, $\varepsilon = \frac{1}{8}$

2) If $\{U_\alpha\}_{\alpha \in A}$ covers $\{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$

$\Rightarrow \exists U_{\alpha_1}$ s.t. $0 \in U_{\alpha_1} \Rightarrow \exists m \in \mathbb{N}$ $(-\frac{1}{m}, \frac{1}{m}) \subseteq U_{\alpha_0}$

$\Rightarrow$ for each $1 \leq i \leq m$, $\frac{1}{i} \in U_{\alpha_i} \Rightarrow U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}$ covers our set



3) exercise.

Thm: Compact sets are closed sets.

Proof: E compact, show that E^c is open

Take $p \in E^c$, if $q \in E$ + $r = \frac{1}{3}(d(p,q))$, $B_r(p) \cap B_r(q) = \emptyset$

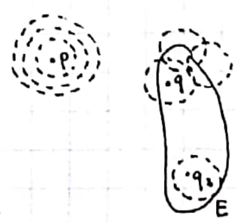
$\Rightarrow (B_{r_q}(q))_{q \in E}$ is an open cover of E

$\Rightarrow$ Since E is compact, $B_{r_{q_1}}(q_1), B_{r_{q_n}}(q_n)$ covers E.

If we let $r^* = \min\{r_{q_1}, r_{q_2}, \ldots, r_{q_n}\}$ then $B_{r^*}(p) \cap E = \emptyset \Rightarrow B_{r^*}(p) \subseteq E^c$

$B_{r^*}(p) \cap B_{r_i}(q_i) \subseteq B_{r_i}(p) \cap B_{r_i}(q_i) = \emptyset \Rightarrow B_{r^*}(p) \subseteq (B_{r_i}(q_i))$ $\forall i$

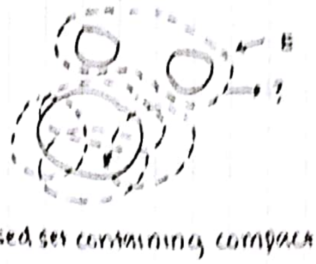
(take min of disjoint ball)



can take balls so $p + q$ are disjoint

April 20

E is **compact** ($E \subseteq X$) if every open cover $\{U_\alpha\}_{\alpha \in A}$ of E has a finite subcover



Theorem: $E \subseteq X$ compact + $F \subseteq E$ is closed → F is compact

Proof: Take an open cover $\{U_\alpha\}$ of F.

Take $\{U_\alpha\} \cup \{X - F\}$, this is an open cover of E. ← assuming $F \subsetneq E$

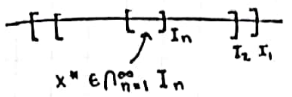
Then it has a finite subcover $U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots, U_{\alpha_n}, X - F$

← still cover F

If $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} \neq \emptyset$ for all possible indices $i_1, i_2, i_3, \dots, i_n$

Exercise: Show thm is false if K_i are not compact.

~~Nested Interval Theorem~~: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n \dots$, collection of intervals in $\mathbb{R}$.



$\Rightarrow \bigcap_{n=1}^{\infty} I_n$ is non-empty

$[a,b]$ (is compact) $\subseteq \mathbb{R}$

Nested Interval Theorem

Exercise: False for open intervals (ex: $I_n = (0, \frac{1}{n})$)

Proof: $I_n = [a_n, b_n]$, $a_1 \le a_2 \le a_3 \le \dots \le a_n \le \dots$, $\dots b_n \le \dots \le b_3 \le b_2 \le b_1$

Take seq $\{a_n\}$ → $\{a_n\}$ is increasing, bounded above then $x^* = \lim_{n\to\infty} a_n$ exists.

$\Rightarrow a_n \le x^* \ \forall n$, $x^* \le b_m \ \forall m$ so $x^* \le \dots \le b_n \le \dots \le b_2 \le b_1$

$\Rightarrow x^* \in I_n$ for all n ; $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Finite Intersection Property: $K_1, K_2, \dots, K_n, \dots$ (collection of compact sets)

If $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} \neq \emptyset$ for all possible indices $i_1, i_2, \dots, i_n$

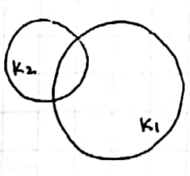
$\Rightarrow \bigcap_{i=1}^{\infty} K_i \neq \emptyset$

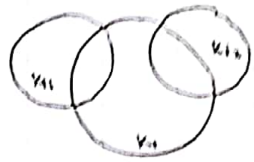
Exercise: Show thm is false if K_i are not compact.

Proof by Contradiction: Assume $\bigcap_{i=1}^{\infty} K_i$ is empty. (know K_i are closed sets)

Take $\{X - K_i\}_{i \in \mathbb{N}}$ which is an open cover of K_1

1. $X - K_i$ are open sets
2. Covers K_n b/c $\bigcap_{i=1}^{\infty} K_i$ empty



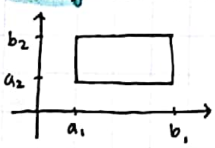


Then there is a finite subcover $X \setminus K_{i_1}, X \setminus K_{i_2}, \dots, X \setminus K_{i_n}$ covering K_1

$K_1 \cap K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} = \emptyset$

contradicts finite intersection is non-empty

A **k-cell** in $\mathbb{R}^k$ is a subset of the form $I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$ for some a_i, b_i such that $a_i < b_i$



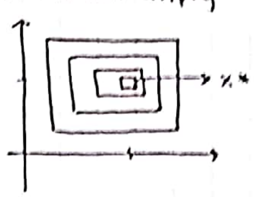
$k = 2$ (2-cell) $= \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid a_i \le x_i \le b_i\}$

Nested K-cell Theorem: If $I_1, I_2, \dots, I_n$ are k-cells s.t. $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \Rightarrow \bigcap_{i=1}^{\infty} I_i$ is non-empty

Theorem: Every k-cell in $\mathbb{R}^k$ is compact in standard topology

Heine Borel Theorem: $E \subseteq \mathbb{R}^k$ then following statements are equivalent:

1. E is compact
2. E is bounded & closed
3. Any infinite subset of E has a limit point in E



A set $E \subseteq \mathbb{R}^k$ is bounded if there is $x \in \mathbb{R}^k$ + $r > 0$ s.t. $B_r(x) \supseteq E$

April 23 : (Rudin: Ch 2)

Recap:

Compact set: $E \subseteq X$ (metric space $\mathbb{R}^k$), E is compact if any open cover $\{U_\alpha\}$ admits a finite subcover

k-cell: $I = \{(x_1, \dots, x_k) \mid \forall i \ a_i \le x_i \le b_i\}$

Theorem: k-cells are compact subsets of $\mathbb{R}^k$

(no finite subcover)

Proof by Contradiction (case k=2): Assume $\exists \ I = [a_1, b_1] \times [a_2, b_2]$ + $\{U_\alpha\}_{\alpha \in A}$ infinite open cover

→ Divide I into 4 2-cells

→ Observe: $\{U_\alpha\}$ also covers those 2-cells + therefore at least one of those 2-cells does not admit a finite subcover; let's call it I_2.

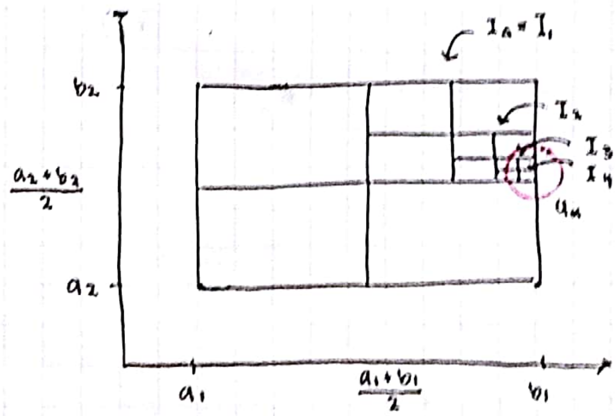
→ using the same reason, we construct $I_2 \subseteq I_1$, $I_3 \subseteq I_2$, $I_{n+1} \subseteq I_n$ (converging to a point)

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n$

→ Nested k-cell theorem: $\bigcap_{i=1}^{\infty} I_i \neq \emptyset$

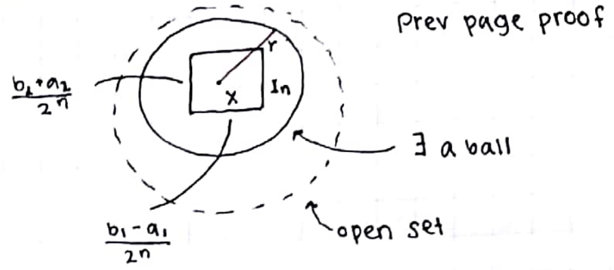
$\Rightarrow x^* \in \bigcap_{i=1}^{\infty} I_i \Rightarrow x^* \in I \Rightarrow \exists \ a \in A$ s.t. $x^* \in U_\alpha$

→ Then $\exists$ n s.t. $I_n \subseteq U_\alpha \Rightarrow I_n$ is covered by only one set of $\{U_\alpha\}_{\alpha \in A}$ ⇄



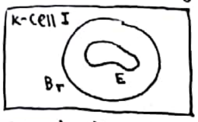
Heine-Borel Theorem: The following are equivalent ($E \subseteq \mathbb{R}^k$):
a. E is closed + bounded
b. E is compact
c. Any infinite subset $S \subseteq E$ has an accumulation / limit point in E.

Prev page proof



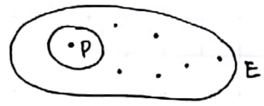
Exercise: 1. Show $[0,1] \cup [2,3] \subseteq \mathbb{R}$ is compact
→ union of closed sets = closed, bounded
2. Show that $\{\frac{1}{n^2}\}$ is not compact
→ Not closed b/c no limit point
3. Show that $E \subseteq \mathbb{R}^n$ is compact $\Rightarrow$ E is bounded (no H-B theorem)
→ Take $r_1, r_2, \dots, r_n$ will cover

Prove $a \Rightarrow b$ (by graph)



closed, bounded $\Rightarrow$ in a ball
ball $\Rightarrow$ k-cell, I is compact
closed subset of compact sets are compact (Thm 5.A)

Prove $b \Rightarrow c$ (by contradiction):
→ $\exists\, S \underset{\text{infinite}}{\subseteq} E$ w/o acc. points in E
$\forall p \in E \;\exists\; B_r(p) \cap S$ contains at most one point of S
$\{B_{r_p}(p)\}_{p \in E}$ is an open cover of E
Then $\exists\, p_1, p_2, \dots, p_n$ s.t. $B_{r_{p_1}}(p_1), \dots, B_{r_{p_n}}(p_n)$ cover E
$\Rightarrow$ S has at most n points. Contradiction.



Prove $c \Rightarrow a$ (exercise)
→ E is finite (do yourself)
→ E is infinite: $E' \subseteq E \Rightarrow E$ is closed; if not bounded, must find some points going to infinity

April 25

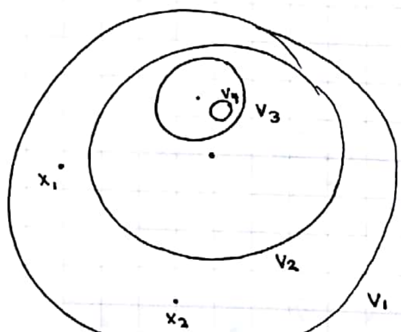
$E \subseteq X$ is **perfect** if $E' = E$ ($\forall$ point $p \in E$ is an acc. point)
→ ex: $X = \mathbb{R}$, $E = [a,b]$ if $a \neq b$ (not a single point)
→ ex not perfect: $X = \mathbb{R}$, $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

* If E is perfect, E is closed

Theorem: If $p \subseteq \mathbb{R}^k$, p is perfect, then p is uncountable

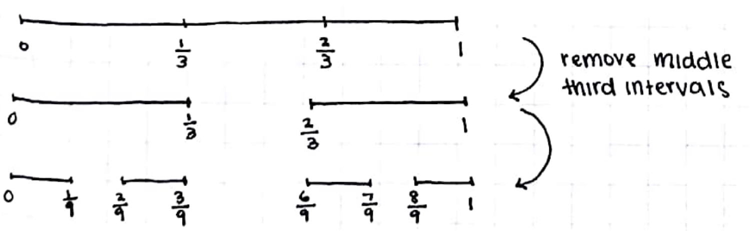
* Corollary: $\mathbb{R}$ is uncountable

Proof: (By Contradiction) Assume p is countable
Let $x_1, x_2, \dots, x_n, \dots$ be all points of p
Let $V_1 = B_1(x_1)$. So V_2 will satisfy the following
1. $x_1 \notin \overline{V_2}$
2. $\overline{V_2} \subseteq V_1$
3. $V_2 \cap p \neq \emptyset$



We construct inductively open balls V_n s.t.
1. $x_{n-1} \notin \overline{V_n}$
2. $\overline{V_n} \subseteq V_{n-1}$
3. $V_n \cap p \neq \emptyset$ ← compact
Define $k_n = \overline{V_n} \cap p$ ← closed ← nested
Observe $k_1 \supseteq k_2 \supseteq \dots \supseteq k_n \supseteq \dots$ So k_n is compact + non-empty $\Rightarrow \bigcap_{n=1}^{\infty} k_n \neq \emptyset$ (Finite Intersection Prop).
If $x^* \in \bigcap_{n=1}^{\infty} k_n$, $x^* \in p \Rightarrow x^* = x_\ell$ for some $\ell \Rightarrow x^* \in k_{\ell+1} \Rightarrow x^* \notin k_{\ell+1}$

Cantor Sets ($C \subseteq \mathbb{R}$)



$E_0 = [0,1]$
$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
$\vdots$
E_n
$C = \bigcap_{n=0}^{\infty} E_n$, $E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots \supseteq E_n$

Properties: 1. C is compact (exercise)
2. C is perfect (———)
3. C is uncountable (perfect = uncountable)
4. C has measure zero
→ def. $\forall\, \varepsilon > 0\; \exists$ intervals $I_1, \dots, I_n$ s.t. $C \subseteq \bigcup I_n$
$\sum_{k=1}^{n} \text{length}(I_k) < n$

Consists of 2^n intervals of size 3^{-n}; dimension (c) = $\log(2) / \log(3)$

Proof 1) E_0 has open cover b/c $[0,1]$ is closed + bounded (nested inside)

Expansion in base b ($b \in \mathbb{N}, b > 1$):

Every $x \in \mathbb{R}$ can be expressed as $x = \sum_{k=0}^{m} a_k b^k + \sum_{k=1}^{\infty} c_k b^{-k}$ $\qquad a_k, c_k \in \mathbb{N}$, $0 \leq a_k, c_k \leq b-1$

→ if $b = 10$, $x = 31{,}25$ $\quad 3 \cdot 10^1 + 1 \cdot 10^0 + 2 \cdot 10^{-1} + 5 \cdot 10^{-2}$

$b = 2$ (binary), $x = 20$ $\quad 20 = 16 + 4 = 2^4 + 2^2$ (10100 in binary)

Exercise: Find $\frac{1}{3}$ in binary.

Fact: Cantor sets consists of all $x \in [0,1]$ whose ternary expansion (base 3) consists of numbers w/digits zero or two

$\forall x \in C$, $x = \sum_{k=1}^{\infty} c_k 3^{-k}$, $c_k = 0, 2$

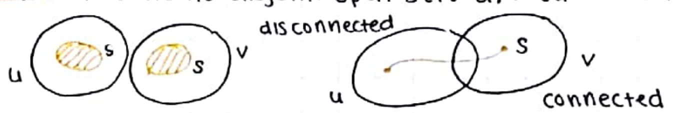
$\frac{1}{3} = \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \dots$

April 27

X, d metric space, $S \subseteq X$: a set is **connected** if there are no disjoint open sets U, V ($U \cap V = \emptyset$) s.t.:

1. $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$
2. $S = (S \cap U) \cup (S \cap V)$

$\Big\}$ ★



→ Q: Connected sets in $\mathbb{R}$?

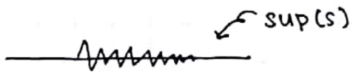
A: $\mathbb{R}$, any continuous interval $[a,b]$ (a,b) etc., single point

Least upper bound property: If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ is bounded above, then S has least upper bound ($\sup(S)$)

→ x is an **upper bound** of S if $x \geq s$ for all $s \in S$

→ **least upper bound**: x upper bound + $x \leq y$ for all y upper bounds

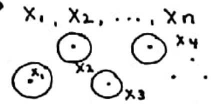
↳ not necessarily true for rationals



Exercise: 1. $\mathbb{R} \setminus \{0\}$ is not connected, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$

2. A finite collection of $n \geq 2$ points is not connected in any metric space.

→ $x_1, x_2, \dots, x_n$

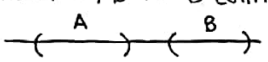


Take $r = \min_{i \neq j} (d(x_i, x_j))$ $\qquad U = B_{r/2}(x_1)$

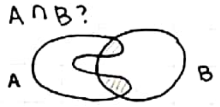
$V = \bigcup_{i=2}^{n} B_{r/2}(x_i)$

3. If A, B are connected, is $A \cup B$ connected? $A \cap B$?

→ No. Ex:



or



4. Prove $[0,1]$ is connected (HARD)

5. Prove the Cantor Set C is disconnected in $\mathbb{R}$

Theorem: A set $S \subseteq \mathbb{R}$ is connected iff S satisfies

1. $\forall x, y \in S$, if $x \leq z \leq y \Rightarrow z \in S$ ☺
2. If a set containing 1 satisfies ☺ then S is an interval (exe. prove)

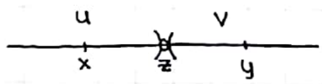
Proof 1: ⇒ (by contrapositive)

If S is connected + if $x, y \in S$ + $x < z < y$ + $z \notin S$

Let $U = (-\infty, z)$ + $V = (z, \infty)$

Then $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$, + $S = (S \cap U) \cup (S \cap V)$

Then S is not connected.



⇐ (by Contradiction)

Suppose S is disconnected; $\exists\, U, V$ satisfying ★

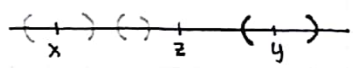
Take $x \in S \cap U$, $y \in S \cap V$. Assume $x < y$.

$T = \{z \in [x,y] \text{ s.t. } z \in S \cap U\}$

Take sup of (), let $z = \sup(T)$

$z \in S \cap U$

$z \in S \cap V$ because of ☺, then if 2 intervals overlap = bigger # than sup.



April 30 (B/S: Ch 3.4, 3.5; Rudin: Ch 4)

HW: $A, B \rightarrow \bar{A} \cap B = A \cap \bar{B} = \emptyset$

A might not be closed or open but $\bar{A}$ & $\bar{B}$ are closed so $(\bar{A})^c$, $(\bar{B})^c$ are open

Two metric spaces (X, d_X), (Y, d_Y) function: $f: X \to Y$

def. $(x_n)_{n \in \mathbb{N}}$ seq in X, we say $\lim_{n \to \infty} x_n = x$ if $\forall \varepsilon > 0\ \exists N$ s.t. if $n \geq N$ then $d(x_n, x) < \varepsilon$.

Exercise 1) $(x_n) \in \mathbb{R}^k$, $x_n = (x_n^1, x_n^2, \dots, x_n^k)$, $x = (x^1, x^2, \dots, x^k)$

$\lim_{n \to \infty} x_n = x \Leftrightarrow \lim_{n \to \infty} x_n^i = x^i$, $\forall_{1 \leq i \leq n}\, i$

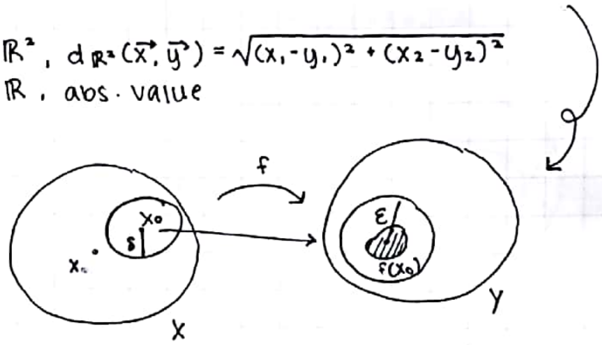
2) (closed set) $C \subseteq X$ is closed iff $\forall (x_n)$ seq in X s.t. 1. $x_n \in C$, 2. $\lim_{n \to \infty} x_n = x$ $\Rightarrow x \in C$

f is said to be continuous at $x = x_0 \in X$ if for any $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.
$\to f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$
$\to$ f is continuous if f is continuous at x_0 $\forall x_0 \in X$

Exercise: $f, g, h : \mathbb{R}^2 \to \mathbb{R}$
$f(x,y) = x$
$g(x,y) = x + y$ $\quad \to |(x+y) - (a+b)| < \varepsilon$
$h(x,y) = x^2 + y$ $\quad \to |x^2 + y - (a^2+b)| < \varepsilon$

$X = \mathbb{R}^2$, $d_{\mathbb{R}^2}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
$Y = \mathbb{R}$, abs. value

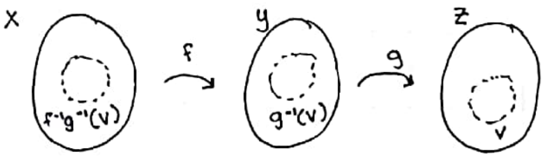
Prove f, g, h are continuous.
① f is continuous
$\vec{x}_0 = (a,b)$, $\vec{x} = (x,y)$, $f(\vec{x}) = x$, $f(\vec{x}_0) = a$
$\forall \varepsilon > 0$ $\exists \delta > 0$ if $\sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |x - a| < \varepsilon$
Then $\delta = \varepsilon \Rightarrow |x - a| \le \sqrt{(x-a)^2 + (y-b)^2}$
$\Rightarrow$ if $\sqrt{(x-a)^2 + (y-b)^2} < \varepsilon \Rightarrow |x - a| < \varepsilon$
② ③ Use thm below



Theorem (Rudin 4.9): If f, g are continuous real/complex func. $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$ (or $\mathbb{C}$)
then $f+g$, $f \cdot g$, $\frac{f}{g}$ are contin., $g \ne 0$ $\forall x \in X$

Theorem: $f: X \to Y$ is continuous $\Leftrightarrow$ $\forall$ open set $V \subseteq Y$, $f^{-1}(V)$ is an open set.

Corollary: $f: X \to Y$, $g: Y \to Z$ continuous $\Rightarrow f \circ g$ is continuous.
$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

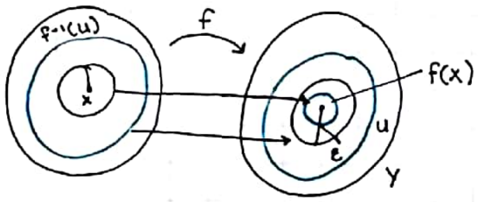


$(g \circ f)^{-1}(V)$ is open

May 2 (B/S: Ch. 4.5, 4.6; Rudin: Ch. 4.1, 4.2)
$f: X \to Y$, X, Y metric spaces

Theorem: ($f: X \to Y$) continuous $\Leftrightarrow$ $\forall$ U open set in Y, $f^{-1}(U)$ is open.
* $f^{-1}(u) = \{x \in X \mid f(x) \in u\}$

Proof: ($\Rightarrow$) Take $x \in f^{-1}(u)$.
Show $\exists$ δ s.t. $B_\delta(x) \subseteq f^{-1}(u)$.
$\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subseteq u$.
By def of continuity $\exists$ $\delta > 0$ s.t.
$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq u \Rightarrow B_\delta(x) \subseteq f^{-1}(u)$
($\Leftarrow$) Exercise.



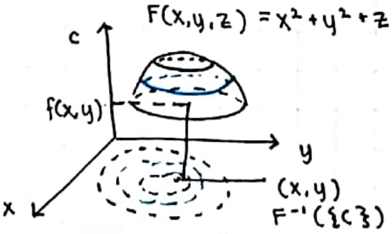
Corollary: $C \subseteq Y$ is closed, $f^{-1}(C)$ is closed (where f is contin)
Proof: C is closed $\Rightarrow Y \setminus C$ is open.
Then $f^{-1}(Y \setminus C)$ is open; $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$
* f^{-1} not necessarily a func.

Exercise: If D is closed/open in X, is $f(D)$ closed/open?

HW: $F(x,y,z) = x^2 + \frac{y^2}{4} + z^2$ $\to$ sum of a contin. func. $\to F: \mathbb{R}^3 \to \mathbb{R}$
$C = (-\infty, 1] \subseteq \mathbb{R}$ $\quad F^{-1}(C) = \{(x,y,z) \mid F(x,y,z) \le 1\}$
closed so $\quad$ is closed $\quad x^2 + \frac{y^2}{4} + z^2 \le 1$

$F(x,y) = x - y$, $(x,y) \in \mathbb{R}^2$, $x^2 \le y \le x$
$G(x,y) = y - x^2$ $\quad \hookrightarrow x - y \ge 0$ + $y - x^2 \ge 0$
closed
$F(x,y) \ge 0$ + $G(x,y) \ge 0$ (where F, G contin) $\Rightarrow F^{-1}([0,\infty)) \cap G^{-1}([0,\infty)) \Rightarrow$ closed

Corollary: $F: \mathbb{R}^k \to \mathbb{R}$, $c \in \mathbb{R}$, F contin $\Rightarrow F^{-1}(\{c\})$ is closed
$F(x,y,z) = x^2 + y^2 + z^2$ $\quad F^{-1}(\{1\}) = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\} \to$ closed



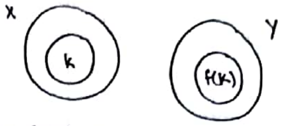
level curves are closed

Theorem: $f: X \to \mathbb{R}^k$ contin, then $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$, $f_i: X \to \mathbb{R}$
f is contin $\Leftrightarrow$ f_i are contin
* ex: $F: \mathbb{R}^2 \to \mathbb{R}$ $\quad F(x,y) = (x^2 - y, x^2 \cdot x^3)$ (f_1, f_2)

Proof: ($\Rightarrow$) F contin so $\lim_{x \to p} F(x) = F(p)$
$\forall \varepsilon > 0$ $\exists \delta > 0$ if $d(x,p) < \delta \Rightarrow \|F(x) - F(p)\| < \varepsilon$
$\|F(x) - F(p)\| = \sqrt{(f_1(x) - f_1(p))^2 + (f_2(x) - f_2(p))^2 + \dots + (f_k(x) - f_k(p))^2}$
$\ge \sqrt{(f_i(x) - f_i(p))^2} = |f_i(x) - f_i(p)| < \varepsilon$
So f_i is contin.

May 4

HW typo: $f(x,y) = \frac{(xy)^2}{x^2+y^2}$ if $(x,y) \neq 0$
0 if $(x,y) = 0$

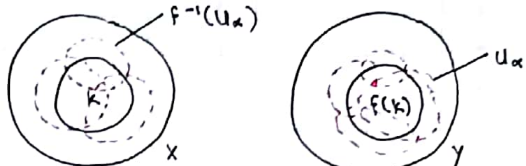


$f: X \to Y$ continuous
$k \subseteq X$
$f(k) = \{y \in Y \mid \exists x \in k, f(x) = y\}$

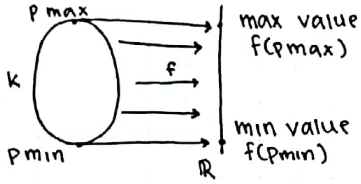
Compact Sets + Continuous Functions:

Theorem: $f: X \to Y$ continuous + $k \subseteq X$ compact then $f(k)$ compact
→ False for open + closed sets

Proof: (use def of compactness) Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $f(k)$
⇒ $\{f^{-1}(U_\alpha)\}$ covers k b/c its continuous; $f^{-1}(U_\alpha)$ is open
⇒ ∃ finite subcover $f^{-1}(U_1) \dots f^{-1}(U_m)$
⇒ $f(k)$ is covered by $f(f^{-1}(U_1)) \dots f(f^{-1}(U_m))$
$f(f^{-1}(U_i)) \subseteq U_i$ $\quad \subseteq U_1 \quad \dots \quad \subseteq U_m$

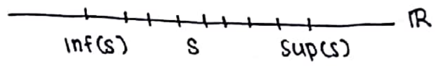


Corollary: $f: X \to \mathbb{R}$, $k \subseteq X$ compact then ∃ $p_{max}, p_{min} \in k$ s.t. $f(p_{min}) \leq f(k) \leq f(p_{max})$ $\forall k \in K$



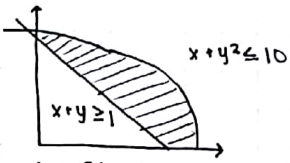
Proof: Take $S = f(k)$ compact in $\mathbb{R}$ → closed + bounded
(Exercise) If $S \subseteq \mathbb{R}$ closed + bounded, then $\sup(S), \inf(S) \in S$.
So $\inf(S), \sup(S) \in S$ ⇒ ∃ $p_{min} \in k$ $\quad f(p_{min}) = \inf(S)$
$p_{max} \in k$ $\quad f(p_{max}) = \sup(S)$

$\inf(S) \leq f(k) \leq \sup(S)$



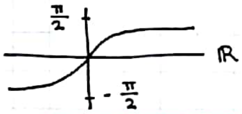
Exercise 1: $f(x,y) = x^2y + x$ under constraints $x + y \geq 1$
$x + y^2 \leq 10$

Show f has a max + min.



$F(x,y) = x + y$ $\quad G(x,y) = x + y^2$
$F^{-1}([1,\infty))$ $\quad G^{-1}(x,y) = ((-\infty, 10])$
↳ closed $\quad$ ↳ closed
closed + bounded = compact.

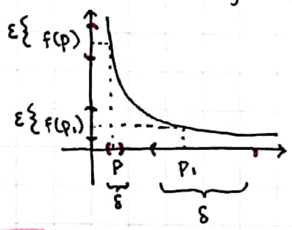
Exercise 2: If $k \subseteq X$ closed/open, $f(x)$ might fail to be closed/open.



$f(x) = \arctan$
$f(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$

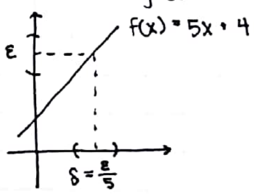
all linear func works!

f is **uniformly continuous** if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$
→ δ does not depend on a point, only ε
→ Ex: not uniformly continuous



$f: (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$
* δ depends on point
If restricted to closed interval, then it works

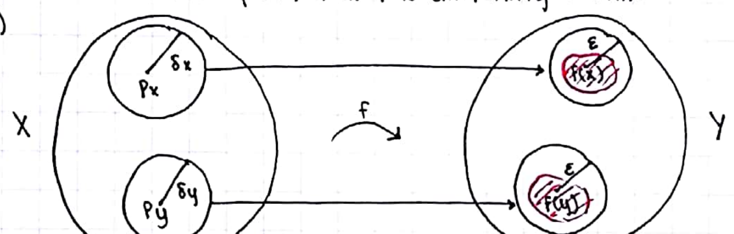
→ Ex: uniformly contin



* does not depend on point

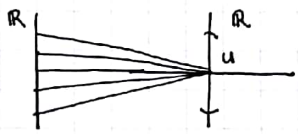
Theorem: $f: X \to Y$ is contin + X is compact, then f is uniformly contin.

Proof: (Graphical)



May 7

HW: $f: X \to \mathbb{R}^k$
$\underset{\in \mathbb{R}^k}{f(x)} \cdot \underset{\in \mathbb{R}^k}{g(x)}$ (dot product of vectors)

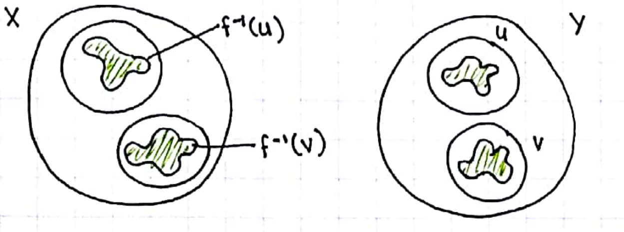


$f: \mathbb{R} \to \mathbb{R}$ $\quad f(\underbrace{f^{-1}(u)}_{\mathbb{R}}) = \{c\}$
$f(x) = c$
(use Rudin 4.-.10)

Continuous Functions + Connected Set, $f: X \to Y$ contin. func.

Theorem: $(f: X \to Y)$ contin, $k \subseteq X$ connected, then $f(k)$ is connected

Proof:



proof contin: Assume $f(K)$ is disconnected

$\exists\, U, V$ open sets $U \cap f(K) \neq \emptyset$, $V \cap f(K) \neq \emptyset$, $U \cap V = \emptyset$, $f(K) \subseteq U \cup V$

$f(K) = (f(K) \cap U) \cup (f(K) \cap V) \Rightarrow f^{-1}(U), f^{-1}(V)$ will disconnect K

$f^{-1}(U) \cap K \neq \emptyset$, $f^{-1}(V) \cap K \neq \emptyset$, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ } proof by contrapositive

[Ex: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$]

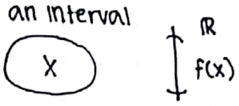
Corollary (Intermediate Value Theorem): $f: X \to \mathbb{R}$ contin, X connected, + $\exists\, p, q$ s.t. $f(p) < 0$ + $f(q) > 0 \Rightarrow$
$\exists\, r \in X$ $f(r) = 0$

→ Exercise: Replace 0 w/any $c \in \mathbb{R}$

→ Exercise: $f(x,y) = \cos(x + y^2) + y$, $f: \mathbb{R}^2 \to \mathbb{R}$. Show $\exists\, (x,y)$ s.t. $f(x,y) = 0$

Corollary: $f: X \to \mathbb{R}$ contin, X connected then $f(x)$ is an interval

proof: $f(X)$ is connected (from thm)
connected sets of $\mathbb{R}$ are intervals



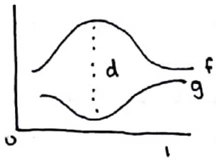
Spaces of Functions (B/S: 3.3.5, 3.4.5 - 3.4.23, 3.5.25, 3.6.11)

• $B(X, \mathbb{R}) = \{f$, bounded func: $f: X \to \mathbb{R}\}$
metric space — could be $\mathbb{C}$
$f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$

L^∞ metric $B(X, \mathbb{R})$: $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$

* bounded: $\exists\, c \in \mathbb{R}$ s.t. $\forall x \in X$, $|f(x)| < c$

Ex: [0, 1]



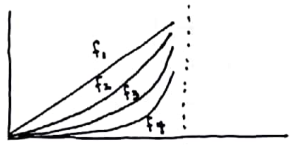
* Exercise: d is a metric

Convergence of func: $f_n: X \to \mathbb{R}$, $f: X \to \mathbb{R}$
↳ seq of func. ↳ limit

$f_n \to f$ **converges pointwise** if $\forall x \in X$ $\lim_{n \to \infty} f_n(x) = f(x)$

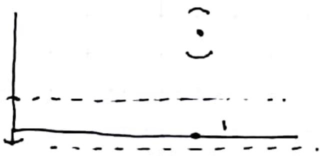
$f_n \to f$ **uniformly converges** if $\forall \varepsilon > 0\ \exists N$ s.t. if $n \geq N$, $\forall x \in X$ $|f_n(x) - f(x)| < \varepsilon$
↳ N does not depend on x

Ex: $X = [0, 1]$



Ex: $f_n: [0,1] \to \mathbb{R}$
$f_n(x) = x^n$

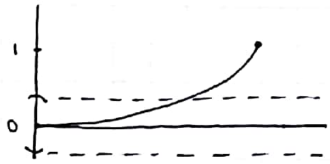
$\lim_{n \to \infty} x^n = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$



$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$

$f_n \to f$ converges pointwise but not uniformly

If f_n restricted to $[0, \frac{1}{2}] \to \mathbb{R}$, $f_n(x) = x^n$ then
$f_n \to f$ uniformly



$\varepsilon = \frac{1}{4}$

May 9

- Last time: $f: X \to \mathbb{R}$, $f_n \to f$

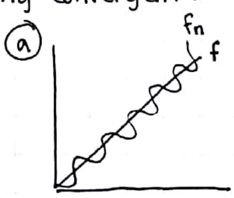
pointwise: $\forall x \in X$ $\lim_{n \to \infty} f_n(x) \to f(x)$; graph not exactly converging

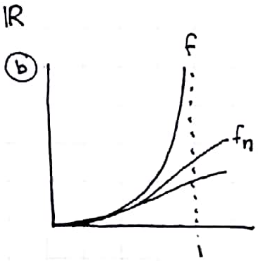
uniform: $\forall \varepsilon > 0\ \exists N_\varepsilon$ if $n \geq N_\varepsilon \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in X$

uniform $\Rightarrow$ pointwise

Exercise 1: Find lim of the following seq; uniformly convergent? $f_n: \mathbb{R} \to \mathbb{R}$

a) $f_n(x) = x + \frac{1}{n}\sin(nx)$
→ $\lim_{n \to \infty} f_n(x) \to f(x) = x$
bounded independently of x





b) $f_n(x) = 1 + x + x^2 + \cdots + x^n$, $x \in (-1, 1)$
→ $\lim_{n \to \infty} f_n(x) \to f(x) = \frac{1}{1-x}$

$f_n(x) = \dfrac{(1 + x + x^2 + \cdots + x^n)(1-x)}{(1-x)} = \dfrac{1 - x^{n-1}}{1-x}$

polynomials have no asymptotes
pointwise but not uniform

c) $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

$f_n \to f$ uniformly $\Leftrightarrow \lim_{n \to \infty} f_n = f$ in $B(X, \mathbb{R}) \Leftrightarrow \lim_{n \to \infty} d_{B(X,\mathbb{R})}(f_n, f) = 0$ (Thm)

→ $B(X, \mathbb{R}) = \{f: X \to \mathbb{R} \mid f$ is bounded$\}$
$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$, $f, g \in B(X, \mathbb{R})$

A seq $(x_n)_{n \in \mathbb{N}}$ $x_n \in X$ is Cauchy if $\forall \varepsilon > 0\ \exists N$ if $n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon$

X is **complete** if any Cauchy seq is convergent (to a point in X)

→ $\forall (x_n)$ Cauchy seq $\Rightarrow \lim_{n \to \infty} x_n = x^*$ for some $x^* \in X$

*any cauchy converges to a constant

→ Ex: $\mathbb{R}, \mathbb{C}, \mathbb{R}^k, \mathbb{C}^k$, discrete metric space, $[0,1]$, $B(X, \mathbb{R})$

→ Ex of not: $\mathbb{Q}$, $\mathbb{Q}^k$, $(0,1)$

Theorem: $B(X,\mathbb{R})$ is complete (metric space).

Proof: If (f_n) is a Cauchy seq in $B(X,f)$ $\forall \varepsilon>0, \exists N$ s.t.

if $n,m \geq N$ $\sup_{x\in X} |f_n(x)-f_m(x)| < \varepsilon$ $\Rightarrow$ Cauchy seq of a real #

$\forall x \in X$ $|f_n(x)-f_m(x)|<\varepsilon$ so $\lim_{n\to\infty} f_n(x)$ exists

Then define $f^*(x) = \lim_{n\to\infty} f_n(x)$

f^* is bounded; $f_n \to f^*$ (exercise)

$\lim_{n\to\infty} d(f_n, f^*) = 0$

$d(f_n,f_m) = \sup_{x\in X}|f_n(x)-f_m(x)| < \varepsilon \Rightarrow |f_n(x)-f_m(x)|<\varepsilon, \forall x$ $|f_n(x)-f^*(x)| \leq \varepsilon < 2\varepsilon$

$d(f_n, f^*) \leq \varepsilon$

May 11

HW: Rudin Thm 4.9 for dot product

Last time: $B(X,\mathbb{R}) = \{f: X\to\mathbb{R} \mid f \text{ bounded}\}$ $f,g \in B(X,\mathbb{R})$

$d(f,g) = \sup_{x\in X}|f(x)-g(x)|$

Theorem: $B(X,\mathbb{R})$ is complete

Def: proper subset of space of bounded func. $BC(X,\mathbb{R}) = \{f: X\to\mathbb{R} \mid f \text{ contin, bounded}\}$

infinite dimensional vector space

Exercise: 1) (Z,d) complete metric space, (Y,d) is complete iff $Y\subseteq Z$ + Y is closed in Z

2) $Z=\mathbb{R}$ complete $\Rightarrow Y=[0,1]$ (example)

Remark: A seq $(f_n)_{n\in\mathbb{N}} \in B(X,\mathbb{R})$ converges to $f\in B(X,\mathbb{R}) \Leftrightarrow f_n \to f$ uniformly

Theorem: $BC(X,\mathbb{R})$ is closed in $B(X,\mathbb{R})$ + therefore $BC(X,\mathbb{R})$ is complete.

Proof: Suppose $f_n \to f$ in $B(X,\mathbb{R})$, $f_n \in BC(X,\mathbb{R})$, f is bounded a priori

We show f is contin at $p\in X$

Let $\varepsilon>0$, there is N_ε $n\geq N_\varepsilon \Rightarrow \forall x\in X$ $|f_n(x)-f(x)|<\frac{\varepsilon}{3}$

$\rightarrow$ know f_n converges uniformly b/c of space

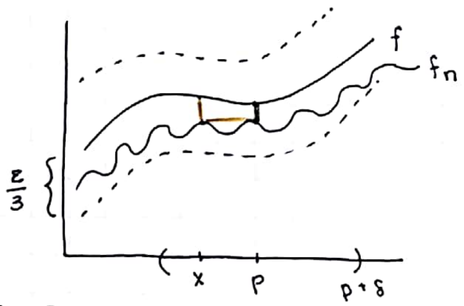
f_N is contin, $\exists \delta>0$ if $d(x,p)<\delta \Rightarrow d(f_n(x), f_n(p)) < \frac{\varepsilon}{3}$

$\Rightarrow$ if $d(x,p)<\delta \Rightarrow |f_N(p)-f(p)| < \frac{\varepsilon}{3}$

$|f_N(x)-f(x)| < \frac{\varepsilon}{3}$

$|f_N(x)-f_N(p)| < \frac{\varepsilon}{3}$

$\Rightarrow |f(x)-f(p)| \leq |f(x)-f_N(x)| + |f_N(x)-f_N(p)| + |f_N(p)-f(p)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$



Dense Set: $S\subseteq X$, X metric space

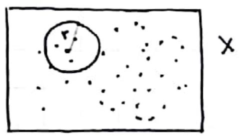
S is dense in X if $\forall x\in X$ $\forall r>0$, $B_r(x)\cap S \neq \emptyset$

Exercise: Show $X=\mathbb{R}$, $S=\mathbb{Q}$ is dense (b/t 2 $\mathbb{R}$ is $\mathbb{Q}$).

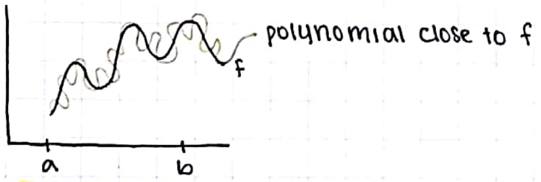
$X=\mathbb{R}^k$, $S=\mathbb{Q}^k$

$S=\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$

$\mathbb{Z}$ not dense in $\mathbb{R}$.

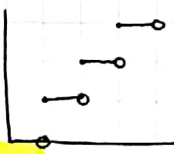


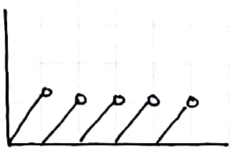
Stone-Weierstrass Theorem: the set of polynomials is dense in $BC([a,b],\mathbb{R})$



$x\in\mathbb{R}$: floor, integer part $\lfloor x\rfloor :=$ largest integer smaller than x

fractional part $\{x\} = x - \lfloor x \rfloor$, $0\leq\{x\}<1$





$[3.1415]=3$

$\{3.12\}=0.12$

$\{-0.6\}=0.4$

Theorem: α is irrational $\Rightarrow S=\{\{n\alpha\}, n\in\mathbb{Z}\}$ is dense in $[0,1]$

Proof: Divide $[0,1]$ in n pieces

Look at $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots, \{(n+1)\alpha\}$

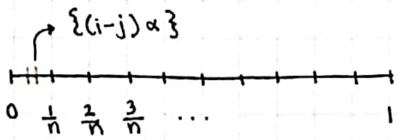
$\exists \{i\alpha\}, \{j\alpha\}$ s.t. $|\{i\alpha\}-\{j\alpha\}|<\frac{1}{n}$

cannot be endpoints b/c not $\mathbb{Q}$

Suppose $\alpha>0$, $i>j$ $|\{(i-j)\alpha\}| < \frac{1}{n}$

$\Rightarrow \exists k$ s.t. $\{k(i-j)\alpha\} \in [\frac{\ell}{n}, \frac{\ell+1}{n}]$ $\forall \ell$

$\hookrightarrow$ multiply small pieces to cover



Midterm II Review

Closure: $\bar{E} = E \cup E'$

Thm: 1) $\bar{E}$ is a closed set

2) $E = \bar{E}$ if E is closed

3) If F is closed + $E \subseteq F$, then $\bar{E} \subseteq F$

Isolated point: $\exists\, r > 0$ s.t. $B_r(x_0) \cap A = \{x_0\}$

Boundary point: $\forall\, r > 0$, $B_r(x_0) \cap A \neq \emptyset$ & $B_r(x_0) \cap A^c \neq \emptyset$

Interior point: A° = union of all open sets in A ($A^\circ \subseteq A$; $A^\circ = A$ if open)

Limit point: $\forall\, r > 0$, $(B_r(x_0) \setminus \{x_0\}) \cap A \neq \emptyset$

* for $A \subseteq X$, $x_0 \in A$

Open cover of E is a collection of open sets $\{U_\alpha\}_{\alpha \in A}$ s.t. $E \subseteq \bigcup_{\alpha \in A} U_\alpha$

E is compact if any open cover $\{U_\alpha\}_{\alpha \in A}$ has a finite subcover

Thm: a finite collection of points is compact

Thm: compact sets are closed

Thm: If $E \subseteq X$ is compact + $F \subseteq E$ is closed $\Rightarrow$ F is compact

Finite Intersection Property: For a collection of compact sets $K_1, K_2, \dots K_n, \dots$: If $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} \neq \emptyset$ for all possible indices $i_1, i_2, \dots, i_n$ $\Rightarrow \bigcap_{i=1}^{\infty} K_i \neq \emptyset$

Nested Interval Theorem: For a collection of intervals in $\mathbb{R}$: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n \supseteq \dots \Rightarrow \bigcap_{n=1}^{\infty} I_n$ is non-empty

k-cell in $\mathbb{R}^k$ is a subset of the form $I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$ for some a_i, b_i s.t. $a_i < b_i$

Nested k-cell Theorem: If $I_1, I_2, \dots, I_n$ are k-cells s.t. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n \Rightarrow \bigcap_{i=1}^{\infty}$ is non-empty

Thm: Every k-cell in $\mathbb{R}^k$ is compact in a standard topology

Heine-Borel Theorem: $E \subseteq \mathbb{R}^k$ then the following are equivalent

1. E is compact
2. E is bounded + closed
3. Any infinite subset of E has a limit point in E

Perfect: $E \subseteq X$ if $E = E'$

Thm: If $P \subseteq \mathbb{R}^k$, P is perfect, then P is uncountable

Properties of Cantor Sets:

1. C is compact
2. C is perfect
3. C is uncountable
4. C has measure zero

f is continuous at $x = x_0 \in X$ if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. if $d_X(x, x_0) < \delta$ + $d_Y(f(x), f(x_0)) < \varepsilon$

$\rightarrow f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$

Thm: If f, g are contin $\mathbb{R}/\mathbb{C}$ func. then $f+g$, $f \cdot g$, $\frac{f}{g}$ are contin

Thm: $f: X \to Y$ contin $\Leftrightarrow$ $\forall$ open set $V \subseteq Y$, $f^{-1}(V)$ is an open set

Cor: $C \subseteq Y$ closed, $f^{-1}(C)$ is closed

Cor: $f: X \to Y$, $g: Y \to Z$ contin $\Rightarrow$ $f \circ g$ is contin.

Cor: $F: \mathbb{R}^k \to \mathbb{R}$, $c \in \mathbb{R}$, F contin $\Rightarrow$ $F^{-1}(\{c\})$ is closed

Thm: $f: X \to \mathbb{R}^k$ contin, then $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$, $f_i: X \to \mathbb{R}$

$\rightarrow f$ contin $\Leftrightarrow f_i$ contin

$S \subseteq X$ is a connected set if there are no disjointed, open sets U, V ($U \cap V = \emptyset$) s.t.

1. $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$
2. $S = (S \cap U) \cup (S \cap V)$

x is an upper bound of S if $x \geq s\ \forall s \in S$

least upper bound if x is an u.b., $x \leq y$ for all $y \in$ u.b.

Least Upper Bound Property: If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ is bounded above, then S has a least upper bound, $\sup(S)$

Thm: A set $S \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ $\forall x, y \in S$ if $x \leq z \leq y \Rightarrow z \in S$

Compact + Continuity

Thm: $f: X \to Y$ contin + $K \subseteq X$ compact then $f(K)$ compact

Cor: $f: X \to \mathbb{R}$ contin + $K \subseteq X$ compact then $\exists\ p_{max}, p_{min} \in K$ s.t. $f(p_{min}) \leq f(K) \leq f(p_{max})$, $\forall\, k \in K$

uniformly continuous: $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. if $d_X(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

$\rightarrow$ δ does not depend on point, only ε

Thm: $f: X \to Y$ contin + X compact $\Rightarrow$ f is uniformly contin

Connectedness + Continuity

Thm: $f: X \to Y$ contin, $K \subseteq X$ connected $\Rightarrow$ $f(K)$ connected

Intermediate Value Thm: $f: X \to \mathbb{R}$ contin, X connected, + $\exists\ p, q$ s.t. $f(p) < 0$ + $f(q) > 0$ $\Rightarrow \exists\, r \in X$, $f(r) = 0$.

$f_n \to f$ converges pointwise if $\forall\, x \in X$ $\lim_{n \to \infty} f_n(x) = f(x)$

$f_n \to f$ uniformly converges if $\forall\, \varepsilon > 0\ \exists\, N$ s.t. if $n \geq N$, $\forall\, x \in X$ $|f_n(x) - f(x)| < \varepsilon$

$\rightarrow$ N does not depend on x; uniform $\Rightarrow$ pointwise

X is complete if any Cauchy seq is convergent to a point in X

$\rightarrow \forall\ x_n$ Cauchy seq $\Rightarrow \lim_{n \to \infty} x_n = x^*$ for some $x^* \in X$

Thm: $B(X, \mathbb{R})$ is complete

$B(X, \mathbb{R}) = \{f \text{ bounded}: f: X \to \mathbb{R}\}$

Thm: $f_n \to f$ uniformly $\Leftrightarrow \lim_{n \to \infty} f_n = f$ in $B(X, \mathbb{R})$

$\Leftrightarrow \lim_{n \to \infty} d_{B(X,\mathbb{R})}(f_n, f) = 0$

May 14 (Rudin Ch 4)

$f: X \to Y$, today $- f: \mathbb{R} \to \mathbb{R}$

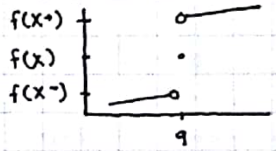
f is **discontinuous** at x if $\lim_{x \to x_0} f(x) \neq f(x_0)$ [or $\lim_{x\to x_0} f(x)$ does not exist]

$f: (a,b) \to \mathbb{R}$ $\quad$ notation $f(x^+)$

right hand limit: $\lim_{x \to a^+} f(x) = L$ $\forall \varepsilon > 0$ $\exists \delta > 0$ if $0 < x - a < \delta$ then $|f(x) - L| < \varepsilon$

left hand limit: $\lim_{x \to a^-} f(x) = L$ ——— $0 < a - x < \delta$ ———

suppose f is discontinuous at $a \in \mathbb{R}$, f has a discontinuity of the **first kind** if both $f(x^+)$, $f(x^-)$ exists.



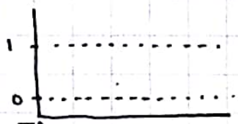
→ otherwise f has a discontinuity of the **second kind**

→ no third

Ex 1: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Second kind – @ any point, $f(x^+)$, $f(x^-)$ DNE

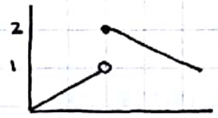
$\lim_{x \to a^+} f(x)$ DNE



Ex 2: $f(x) = \begin{cases} x & \text{if } x < 1 \\ 3 - x & \text{if } x \geq 1 \end{cases}$

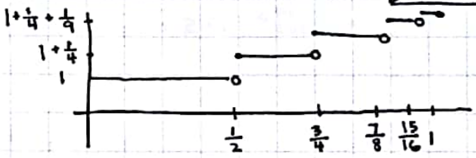
first kind: $f(1^+) = 2$

$f(1^-) = 1$



Ex 3: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$g(x) = \begin{cases} \sum_{k=1}^{n} \frac{1}{k^2}, & 1 - \frac{1}{2^{n-1}} \leq x < 1 - \frac{1}{2^n} \\ \frac{\pi^2}{6} & , x = 1 \end{cases}$



countably many discontinuities

$f: (a,b) \to \mathbb{R}$

f is **monotonic** if f is increasing $\quad \forall x, y \in (a,b)$ if $x < y \Rightarrow f(x) \leq f(y)$

or decreasing ——— $x < y \Rightarrow f(x) \geq f(y)$

Exercise: If f is incr. + decr., it is a constant func.

Theorem: If $f: (a,b) \to \mathbb{R}$ is monotonic then every discontinuity is of the first kind, or $f(x^+)$ + $f(x^-)$ exists at every $x \in (a,b)$

Proof: Assume f increasing. Take $S = \{f(y) \mid a < y < x\}$

Show $f(x^-)$ exist.

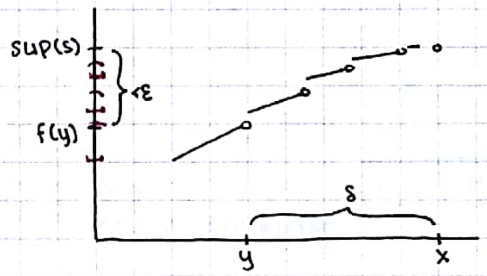
S is non-empty + bounded above b/c $f(x) \geq s$ $\forall s \in S$

$\Rightarrow \sup(S)$ exists

Given any $\varepsilon > 0$ $\exists$ $s \in S$ s.t. $s > \sup(S) - \varepsilon$.

Then $\exists$ $y < x$ s.t. $f(y) > \sup(S) - \varepsilon$

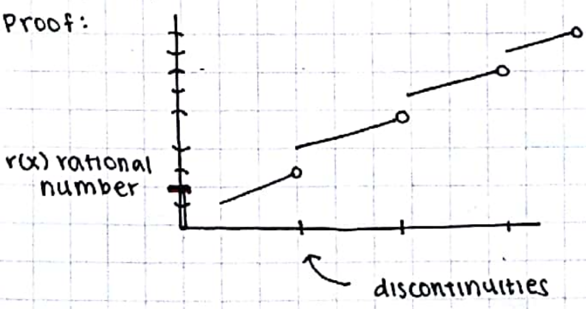
Take $\delta = x - y \Rightarrow$ if $0 < x - z < \delta \Rightarrow |f(x) - f(z)| < \varepsilon$



$f: (a,b) \to \mathbb{R}$

Theorem: If f is monotonic then the set of discontinuities of f is countable

Proof:



Assume f is increasing

Let E be the set of discontin. of f.

By thm, $f(x^-) < f(x^+)$ if $x \in E$

Then let $r: E \to \mathbb{Q}$ by choosing a rational number: $f(x^-) < r(x) < f(x^+)$

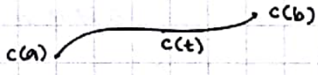
$r(x)$ exists b/c $\mathbb{Q}$ is dense

Exercise: r is injective

Therefore it is countable

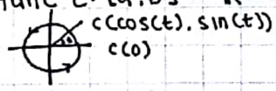
May 18 (B/S: Ch. 3.3.43 – 49, Ch. 3.6.23 – 24)

A **parametrized curve** in $\mathbb{R}^k$ is a contin. func $c: [a,b] \to \mathbb{R}^k$

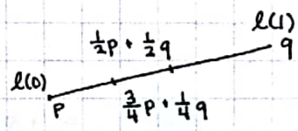


ex: $c(t) = (\cos(t), \sin(t))$

$c: [0, 2\pi] \to \mathbb{R}$



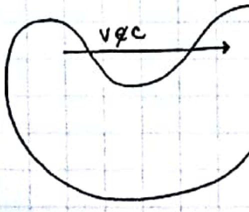
Line segment b/t points p, q in $\mathbb{R}^k$: $q \cdot t + (1-t) \cdot p = \ell(t)$



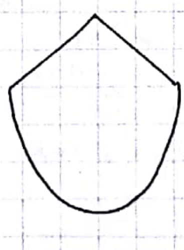
$C \subseteq \mathbb{R}^k$ is **convex** if $\forall p, q \in C$, $\forall 0 \leq t \leq 1$, $tq + (1-t)p \in C$

same for C

ex:



not convex



Convex

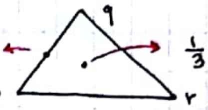
Given $\{p_1, p_2, \ldots, p_m\}$ in $\mathbb{R}^k$, x is a convex combination of S if $\exists\, t_1, t_2, \ldots, t_m \geq 0$ + $\sum_{i=1}^{n} t_i = 1$ s.t. $x = t_1p_1 + t_2p_2 + \cdots + t_mp_m$

ex: x is a convex combo of $\{p, q\}$ ← strict

$x = t_1 p + t_2 p$ $\quad t_1, t_2 \geq 0$

$x = t_1 p + (1-t_1) q$ $\quad t_1 + t_2 = 1$

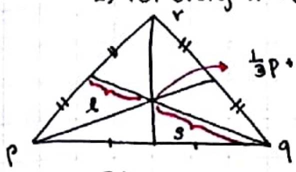
$\frac{1}{2}p + \frac{1}{2}q$ or ←



→ $\frac{1}{3}p + \frac{1}{3}q + \frac{1}{3}r$

Exercise: 1) Given non-collinear $p, q, r \in \mathbb{R}^2$, the set of convex combo of $\{p, q, r\}$ coincide w/ the triangle.

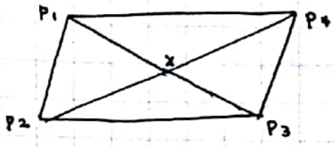
2) For every $x \in t$, $\exists$ unique $t_1, t_2, t_3 \geq 0$, $t_1 + t_2 + t_3 = 1$, $x = t_1p_1 + t_2p_2 + t_3p_3$.



$\frac{1}{3}p + \frac{1}{3}q + \frac{1}{3}r$

s, l = mediums

* show $s = 2l$.

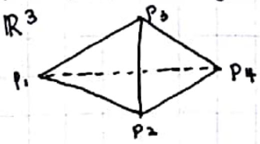


$x = \frac{1}{2}(p_1 + p_3)$
$= \frac{1}{2}(p_2 + p_4)$

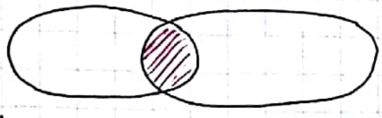
Remark: in general, $t_1, t_2, \ldots t_m$ are not unique

Given $A \subseteq \mathbb{R}^k$, convex hull of A (Hull(A)) is the intersection of all convex sets containing A (intersection of convex sets is convex; smallest convex set containing A)

$\mathbb{R}^3$



ex: If $C_1, C_2, \ldots, C_m$ are convex sets $\Rightarrow \bigcap_{i=1}^{\infty} C$ is convex



Hull(A)

Exercise: 1) If $A \subseteq \mathbb{R}^k$ then $\text{Hull}(A) = \left\{ p \in \mathbb{R}^k \;\middle|\; \begin{array}{l} \exists\, p_1, p_2, \ldots, p_m \in A \\ t_1, t_2, \ldots, t_m \geq 0, \sum_{i=1}^{m} t_i = 1 \end{array} \right\}$ s.t. $p = t_1p_1 + t_2p_2 + \cdots + t_mp_m$.

2) If A is convex $\Rightarrow \bar{A}$ is convex.

3) If A is convex $\Rightarrow A$ is connected.

Proof: 1) $\left\{ p \in \mathbb{R}^k \;\middle|\; p = \sum t_i p_i \;\; \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$ is convex

→ Take $p, q \in S$, then $p = \sum_{i=1}^{m} t_i p_i$ $\Rightarrow$ $tp + (1-t)q = \sum_{i=1}^{m} t t_i p_i$
$q = \sum_{i=1}^{n} t_i' p_i'$ $\quad = \sum_{i=1}^{n} (1-t) t_i' p_i'$

Show $\sum^m t t_i + \sum^n (1-t) t_i = 1$
$t \cdot 1 + (1-t) 1 = 1$

Prove $\text{Hull}(A) \subseteq S$ + $S \subseteq \text{Hull}(A)$. (by induction)

→ If convex sets contain points, then contains all combo.

→ intersection of convex sets

May 21 (B/S: Ch 3.7)

$f: X \to X$ is a contraction if $\exists\, \alpha \in [0,1)$ s.t. $\forall x, y: d(f(x), f(y)) \leq \alpha\, d(x, y)$ $\quad$ $\boxed{X, d \text{ metric space}}$

↖ rate of contraction

$f: X \to X$ then x_0 is a fixed point of f if $f(x_0) = x_0$

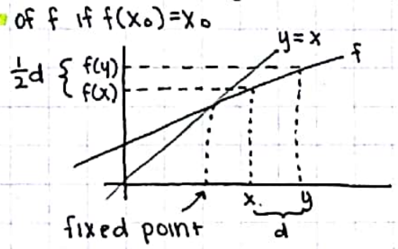
Ex1: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{2}x + 5$

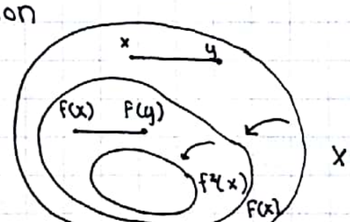
fixed point: $f(x) = x$

$\frac{1}{2}x + 5 = x$

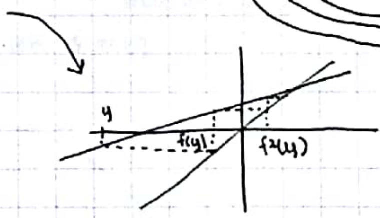
$x = 10$



contraction map, $\alpha = \frac{1}{2}$

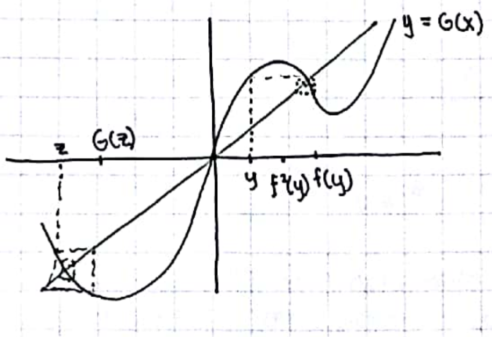


Ex 2: $F: \mathbb{R}^3 \to \mathbb{R}^3$
$F(x, y, z) = (\frac{1}{2}x, \frac{1}{3}y, \frac{1}{4}z)$] Exercise: Show F is a contraction.



General question: Given $G: X \to X$, how do you find a fixed point?

Answer: Try iterating G



Idea: many times $\{G^n(y)\}$ converges towards a fixed point.

space of function

Contraction Mapping Theorem: If $f: X \to X$ is a contraction + X is complete, then f has a unique fixed point

Proof: Take $y \in X$, define the seq $y_0 = y$, $y_1 = f(y)$, $y_2 = ff(y)$, $y_n = \underbrace{ff \cdots f}_{n\text{-times}}(y) = f^n(y)$

We show that $\lim_{n \to \infty} y_n$ exists + is a fixed point.

Suppose $p = \lim_{n\to\infty} y_n \Rightarrow f(p) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_{n+1} = p$

Proof: (cont.) Observation: $\forall n \geq 1,\ d(y_{n+1}, y_n) = d(f(y_n), f(y_{n-1}))$
$\leq \alpha\, d(y_n, y_{n-1})$ } apply n-times the fact that f is a contraction
$\leq \alpha^n d(y_1, y_0)$

$\Rightarrow$ If $m \geq n$, $d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \dots + d(y_{n+1}, y_n)$
$\leq \alpha^m d(y_1, y_0) + \alpha^{m-1} d(y_1, y_0) + \dots + \alpha^n d(y_1, y_0)$
$= \alpha^n \cdot d(y_1, y_0)(\alpha^{m-n} + \alpha^{m-n+1} + \dots + \alpha + 1)$
$= \alpha^n d(y_1, y_0)\left(\frac{1-\alpha^{m-n+1}}{1-\alpha}\right) \leq \underbrace{\alpha^n d(y_1, y_0)\frac{1}{1-\alpha}}_{< \varepsilon \text{ if } n \text{ is large enough};\ \lim_{n\to\infty} \alpha^n = 0}$ } multiply by $\frac{1-\alpha}{1-\alpha}$

$\Rightarrow \{y_n\}$ is Cauchy $\Rightarrow \{y_n\}$ is convergent

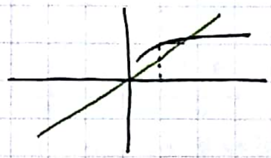
Exercise: Show f is cont. $\delta = \frac{\varepsilon}{\alpha}$.

Application: Newton's Theorem / Method: $x^2 = 2$

How to approximate $\sqrt{2}$?

$x = \frac{2}{x} \Rightarrow \frac{1}{2}(x + \frac{2}{x}) = x$

Take $f(x) = \frac{1}{2}(x + \frac{2}{x}) \Rightarrow f'(x) = \frac{1}{2} - \frac{1}{x^2} \Rightarrow f'(\sqrt{2}) = 0$



Babylonian method: to compute $\sqrt{2}$; $f(x) = \frac{1}{2}(x + \frac{2}{x}) \rightarrow$ has $\sqrt{2}$ as a fixed point

$x_0 = 1$

$x_1 = \frac{1}{2}(1+2) = \frac{3}{2} = 1.5$

$x_2 = \frac{1}{2}(\frac{3}{2} + \frac{4}{3}) = \frac{17}{12} = 1.4$

May 23 (B/S: Ch. 3.10.4)

Polynomial: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

$\alpha \in \mathbb{C}$ is a root of p if $p(\alpha) = 0$

Fact: If α is a root, then $p(x) = (x-\alpha)Q(x)$ ← degree = n−1

Corollary: A polynomial of degree n has at most n roots (proof by induction)

Fundamental Theorem of Algebra: If P is a polynomial (w/ coefficients in $\mathbb{C}$) of degree $n \geq 1$, then p has at least one root.

Corollary: Any complex polynomial can be factored as $P(x) = c(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots(x-\alpha_k)^{n_k}$ for some $\alpha_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$, $n_i \geq 1$ (can be double root)

Ex: 1) $P(x) = x^2 + 2 = (x + \sqrt{2}i)(x - \sqrt{2}i)$

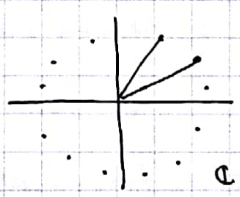
2) $Q(x) = x^n - 1 = (x - p_1)(x - p_2)\cdots(x - p_n)$

p_i: n-roots of unity

$p_i = e^{j\frac{2\pi i}{n}}$, $0 \leq j < n$

norm $|e_j| = 1$

$\theta = i\frac{2\pi}{n}$



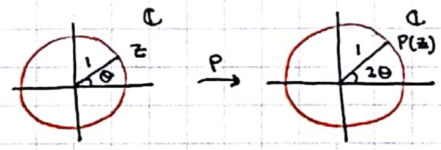
* convert to polar coord.

3) $P: \mathbb{C} \to \mathbb{C}$

$P(z) = z^2$

$P(r \cdot e^{i\theta}) = r^2 e^{i2\theta}$; if $r = 1$ then $P(e^{i\theta}) = e^{i2\theta}$

$P(z) = z^n \rightarrow P(e^{i\theta}) = e^{in\theta}$ (rotates n-times)



circle as θ varies

* moving 2x w/ r^2

Observation: $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$

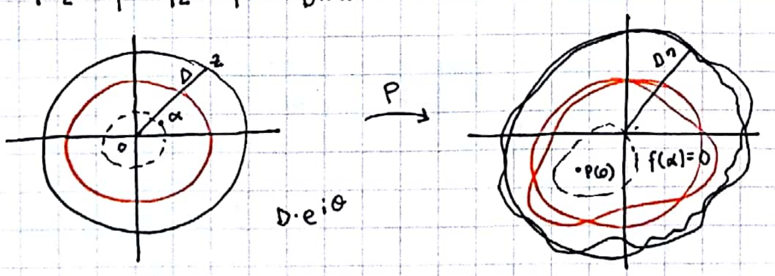
assume first coeff = 1 (can divide by first coeff if ≠ 1)

$= z^n \underbrace{\left(1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right)}_{* \ \sim 1 \text{ as } D \text{ is large}}$

If $|z| = \rho$, ρ is large, then * is very small

$\left|\frac{a_k}{z^{n-k}}\right| = \frac{|a_k|}{|z^{n-k}|} = \frac{|a_k|}{D^{n-k}}$

Proof:



$P(z) = z^n(1 + \dots)$

D is large

→ circle n-times w/ small wiggling b/c of small factor (small error)

once for n-many times

Look at $p(t \cdot e^{i\theta})$ as t goes from D to zero.

As it decreases to zero, picture is contin + at some point touches $P(\alpha) = 0$

Proof by Contradiction: Assume $P(z)$ has no roots.

$\Rightarrow P|z| \neq 0 \quad \forall z \in \mathbb{C}$

1) Prove $|P(z)|$ has min > 0; we can assume min occurs at $z = 0$ (translate if not)

$P(z) = a_0 + a_m z^m + a_{m+1}z^{m+1} + \dots + a_n z^n = \underbrace{a_0 + a_m z^m}_{\text{most important b/c small but } z^{m+1} \text{ is much smaller}} + z^{m+1}Q(z)$

* look @ z very close to zero

$z = t \cdot \left(\frac{-a_0}{a_m}\right)^{\frac{1}{m}}$ (t is small)

$P\left(t\left(-\frac{a_0}{a_m}\right)^{\frac{1}{m}}\right) = a_0 - t^m a_0 + t^{m+1} Q(*) = a_0(1-t^m) + t^{m+1} Q(*)$

$= |a_0|(1-t^m) + t^{m+1} |\cdot|$ bounded

can make smaller than $t^{m+1}|\cdot|$

$< |a_0|$ if t is small enough (see B/S for more details...)

May 25: (B/S: Ch. 3.8)

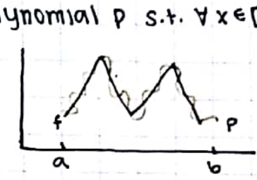
Stone-Weierstrass Theorem: $f:[a,b] \to \mathbb{R}$ contin. for any $\varepsilon > 0$ there is polynomial p s.t. $\forall x \in [a,b]$, $|P_\varepsilon(x) - f(x)| < \varepsilon$.

Remark 1: True for $f: C \to \mathbb{R}$, $C \subseteq \mathbb{R}^k$ compact

2: set of polynomials is dense in $BC([a,b], \mathbb{R})$

Idea: approximate f near $x \in [a,b]$ by a polynomial P_x

then use some combo of such P_x's to obtain P_ε



A function $f:[a,b] \to \mathbb{R}$ is **approximated** by polynomials if for any $\varepsilon > 0$ $\exists$ P poly. s.t. $\forall x \in [a,b]$ $|f(x) - P(x)| < \varepsilon$

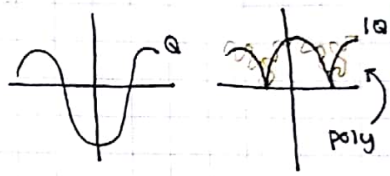
Proof Outline:

Step 1: $f(x) = |x|$ can be approx by polynomials

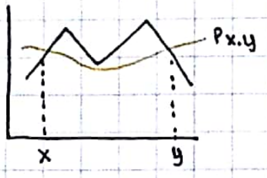
→ Corollary: For any poly. $Q(x) = |Q(x)|$ can be approx by poly

↳ sketch proof: $g_n(x)$ approx $|x|$ then $g_n(Q(x))$ approx $|Q(x)|$

→ If $f, g: [a,b] \to \mathbb{R}$, $\max\{f(x), g(x)\}$ $\min\{f(x), g(x)\}$ in terms of $f, g, |f-g|$



Step 2: Fix $\varepsilon > 0$, fix $x \in [a,b]$, take $y \in [a,b]$



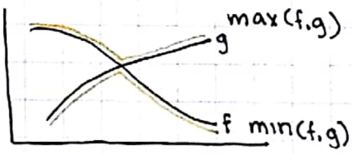
$P_{x,y}$ coincides w/ f @ x + y

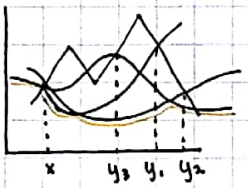
Now vary y

Exercise: If $x, y \in \mathbb{R}$

$\max\{x,y\} = \frac{1}{2}(x+y+|x-y|)$

$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|)$





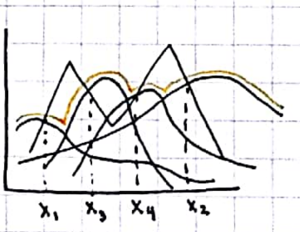
Take min

$g(x) = \min\{P_{xy_1}, P_{xy_2}, \dots, P_{xy_k}\}$

this func. can be approx by polynomials

$g(x)$ approx f near x + $g(x) \le f + \varepsilon$

Make x vary + look at $g(x)$



$\max\{g_{x_1}, g_{x_2}, \dots, g_{x_n}\}$ will approx f if one uses a lot of x_i s

Formal Proof: P_{xy} exists $\forall x, y \in [a,b]$ (exercise)

$f(x) = P_{xy}(x)$ $\quad f(y) = P_{xy}(y)$

$\exists$ $U_{x,y}$, $V_{x,y}$ open intervals containing $x + y$ $\quad |f(t) - P_{xy}(t)| < \frac{\varepsilon}{3}$ $\forall t \in V_{xy}$

$\{V_{x,y}\}_{y \in [a,b]}$ open cover of $[a,b]$ $\Rightarrow$ compact $\Rightarrow \exists$ $y_1, y_2, \dots, y_k$ s.t. $[a,b] \subseteq \bigcup_{j=1}^{k} V_{xy}$

Then define $g_x(t) = \min\{P_{xy_1}(t), \dots, P_{xy_k}(t)\} \le f(t) + \frac{\varepsilon}{3}$ $\forall t \in [a,b]$

$g(x)$ can be approx by poly.

$g_x(x) = f(x)$ so $\exists$ neighborhood W_x open interval containing x s.t. $|g_x(t) - f(t)| < \frac{\varepsilon}{3}$ $\forall t \in W_x$

→ should be closed near x

$\{W_x\}_{x \in [a,b]}$ is an open cover of $[a,b]$

$\Rightarrow \exists$ $x_1, \dots, x_m$ s.t. $[a,b] \subseteq \bigcup_{i=1}^{m} W_{x_i}$

$f^*(t) = \max\{g_{x_1}(t), g_{x_2}(t), \dots, g_{x_m}(t)\}$

$\Rightarrow f^*(t) \le f(t) + \frac{\varepsilon}{3}$

$\Rightarrow f^*(t) \ge f(t) - \frac{\varepsilon}{3}$

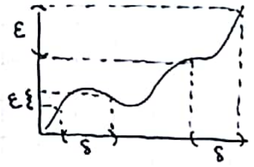
Negation of uniform continuity: $\exists\, \varepsilon>0$, $\exists\, n_1, n_2, \cdots, n_k$ for $k\to\infty$ (infinite subseq) s.t. $|f_n(x)-f(x)|\geq\varepsilon$ for some x

→ Take $\varepsilon=\frac{1}{3} \Rightarrow |f_n(x)-f(x)|\geq\varepsilon \Rightarrow$ Take $x=\frac{1}{4n} \Rightarrow |\frac{1}{4}-1|=\frac{3}{4}>\varepsilon$

Negation of limit: $\lim_{n\to\infty} a_n = L$: $\exists\, n_1<n_2<\cdots<n_k<\cdots$ $|a_{n_k}-L|\geq\varepsilon$

uniform continuity: $\forall\, \varepsilon>0\ \exists\, \delta>0\ \forall x,y\ \ d(x,y)<\delta \Rightarrow d(f(x),f(y))<\varepsilon$

↳ see if slope $\to\infty$



hw: $f(x)=x\log x$

$f'(x)=1+\log x$

$f: C\to\mathbb{R}$; $f(x)=\frac{1}{x}$, $x\in[1,2]$

Thm: If f is cont & C is compact, then f is uniformly cont.

Take $\forall n$ $f(x+\frac{1}{n})-f(x)$.

→ If $f(x+\frac{1}{n})-f(x)$ is unbounded $\forall n$ then f is not uniformly cont.

→ Ex: (use hw) $(x+\frac{1}{n})\log(x+\frac{1}{n}) - x\log(x)$

$$= \underbrace{x(\log(x+\tfrac{1}{n})-\log(x))}_{\geq 0 \text{ if } x\geq 0} + \underbrace{\tfrac{1}{n}\log(x+\tfrac{1}{n})}_{\to\infty \text{ if } x\to\infty}$$

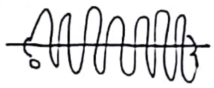
$\varepsilon=1$, $\delta\leq 1$, $h<\delta$: $\forall x$; $|f(x+h)-f(x)|<\delta$ (bounded if we fix h)

If $D\subseteq$ Cantor set & D is connected then D is a single point, $D\neq\emptyset$

↳ totally disconnected!

Exercise: Prove a set $C\subseteq\mathbb{R}$ (or any metric space) is compact iff every cont. func. is bounded $f:C\to\mathbb{R}$

↳ ex: $\sin(\frac{1}{x})$ not unif. cont.



FINAL REVIEW

Complex num: ordered pairs of real num (a,b); $z = a+ib \to Re(z)=a,\ Im(z)=b$

- $\to z+w := (a+c, b+d)$ $\quad zw := (ac-bd,\ ad+bc)$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$
- $\to$ conjugate: $\bar{z} = a-ib$ $\quad$ abs value: $|z| = \sqrt{a^2+b^2} = (z\bar{z})^{\frac{1}{2}}$ $\quad$ dist $= |z_1 - z_2|$
- $\to$ Thm: (1) $|z| \geq 0$ & $|z|=0$ iff $z=0$; (2) $|\bar{z}| = |z|$; (3) $|zw| = |z||w|$; (4) $|Re(z)| \leq |z|$;
  (5) Triangle inequality: $|z+w| \leq |z| + |w|$

HW1: Prove $x,y \in \mathbb{C}$ that $\overline{xy} = \bar{x}\bar{y}$.

Let $x = a+bi$, $y = c+di$, $a,b,c,d \in \mathbb{R}$

$xy = (ac-bd) + (ad+bc)i \to \overline{xy} = (ac-bd) - (ad+bc)i$

$\bar{x} = a-bi$, $\bar{y} = c-di \to \bar{x}\bar{y} = (ac-bd) - (ad+bc)i = \overline{xy}$

HW1: Prove for $x,y \in \mathbb{C}$ then $||x|-|y|| \leq |x-y|$.

By Δ ineq, $|x| = |(x-y)+y| \leq |x-y| + |y| \Rightarrow |x| - |y| \leq |x-y|$ ①

$|y| = |(y-x)+x| \leq |y-x| + |x| \Rightarrow |y| - |x| \leq |x-y| \Rightarrow |x|-|y| \geq -|x-y|$ ②

By ① & ②, we have $-|x-y| \leq |x|-|y| \leq |x-y|$.

Take the abs value to get $||x|-|y|| \leq |x-y|$.

HW1: Prove for $x,y \in \mathbb{C}$, $|x-y|^2 + |x+y|^2 = 2|x|^2 + 2|y|^2$.

Let $x = a+bi$, $y = c+di$, $a,b,c,d \in \mathbb{R}$.

$|x-y|^2 = |(a-c)+(b-d)i|^2 = (\sqrt{(a-c)^2+(b-d)^2})^2 = (a-c)^2 + (b-d)^2$.

$|x+y|^2 = |(a+c)+(b+d)i|^2 = (\sqrt{(a+c)^2+(b+d)^2})^2 = (a+c)^2 + (b+d)^2$.

$|x-y|^2 + |x+y|^2 = (a-c)^2 + (b-d)^2 + (a+c)^2 + (b+d)^2$

$= a^2 - 2ac + c^2 + b^2 - 2bd + d^2 + a^2 + 2ac + c^2 + b^2 + 2bd + d^2$

$= 2(a^2+b^2) + 2(c^2+d^2) = 2|x|^2 + 2|y|^2$.

Open ball: centered @ z_0, $B_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| < r\}$

Sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathbb{C}$ is **bounded** if $\exists r>0$ s.t. $\forall n: z_n \in B_r(0)$

convergent to z if $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. if $n \geq N_\varepsilon$ then $|z_n - z| < \varepsilon$

Cauchy if $\forall \varepsilon > 0\ \exists N_\varepsilon > 0$ s.t. if $n,m \geq N_\varepsilon$ then $|z_n - z_m| < \varepsilon$

If all Cauchy seq converge, it is **complete** ($\mathbb{R}$, $\mathbb{C}$, discrete metric)

$\to$ All convergent seq are cauchy

HW1: Prove if $(z_n)_{n\in\mathbb{N}} \in \mathbb{C}$ is convergent, then it is Cauchy.

Let z_n converge to L.

For $k \geq N$, $|z_k - L| < \frac{\varepsilon}{2}$, for some $N>0$.

For $m,n \geq N$, $|z_m - L| < \frac{\varepsilon}{2}$ & $|z_n - L| < \frac{\varepsilon}{2}$.

$|z_m - z_n| \leq |z_m - L| + |L - z_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

HW1: Prove if $(z_n)_{n\in\mathbb{N}} \in \mathbb{C}$ is convergent, then it is bounded.

Let $\varepsilon = 1$ & z_n converge to L.

Then $|z_n - z| < \varepsilon = 1$ for some $n \geq N_\varepsilon$, where $N_\varepsilon > 0$

Take $r = \max\{|z_1|+1, |z_2|+1, \dots, |z_n|+1, |L|+2\}$.

Then $\forall n \in \mathbb{N}$, $z_n \in B_r(0)$.

$\to$ Case 1: $n < N_\varepsilon \Rightarrow |z_n - L| \leq |z_n| + |L| < r$

Case 2: $n \geq N_\varepsilon \Rightarrow |z_n - L| \leq |z_n - z_{N_\varepsilon}| + |z_{N_\varepsilon} - L| < r$

HW1: If Cauchy seq $(a_k)_{k\in\mathbb{N}}$ does not converge to 0, all terms eventually have the same sign.

Suppose $(a_k)_{k\in\mathbb{N}}$ converges to a, $a \neq 0$.

Then $\forall \varepsilon > 0\ \exists N_\varepsilon > 0$ s.t. $\forall n, > N_\varepsilon$, $|a_n - a| < \varepsilon$.

Let $0 < \varepsilon = \frac{1}{2}\min\{x,y\}$ where $a = x+yi$.

Then $\forall n > N_\varepsilon$, $|a_n - a| < \varepsilon$, i.e. $\forall n > N_\varepsilon$, $a_n \in B_r(a)$, $r = \varepsilon$.

Let $r = \varepsilon$. $\forall n > N_\varepsilon$, a_n belongs to the same quadrant as a.

Negation of lim: if $a_k \not\to L$, $\exists \varepsilon > 0$ s.t. $\forall N_\varepsilon$, $\exists n \geq N_\varepsilon$ s.t. $|a_n - L| \geq \varepsilon$.

HW1: Show $z_n = \frac{ni+2n}{3ni+1}$ is bounded & convergent.

① We know $z_n = \frac{(3n^2+2n)+(n-6n^2)i}{9n^2+1}$

By def, $|z_n| = \sqrt{Re(z_n)^2 + Im(z_n)^2}$ $\quad \leftarrow Re(z_n)^2 = \frac{(3n^2+2n)^2}{(9n^2+1)^2} = \frac{9n^4+12n^3+4n^2}{(9n^2+1)^2}$

$= \left(\frac{5n^2(9n^2+1)}{(9n^2+1)^2}\right)^{\frac{1}{2}}$ $\quad Im(z_n)^2 = \frac{(n-6n^2)^2}{(9n^2+1)^2} = \frac{n^2-12n^3+36n^4}{(9n^2+1)^2}$

$= \left(\frac{5n^2}{9n^2+1}\right)^{\frac{1}{2}} < \left(\frac{5n^2}{9n^2}\right)^{\frac{1}{2}} < \frac{\sqrt{5}}{3}$

Let $r = 1 > \frac{\sqrt{5}}{3}$. Then $z_n \in B_1(0)$ & z_n is bounded.

② We know z_n converges to $\lim_{n\to\infty} Re(z_n) + \lim_{n\to\infty} Im(z_n) = \frac{1-2i}{3}$.

If convergent, $\forall \varepsilon > 0\ \exists N_\varepsilon > 0$ s.t. $|z_n - z| < \varepsilon$, $n \geq N_\varepsilon$.

$$|z_n - z| = \left|\frac{ni+2n}{3ni+1} - \frac{1-2i}{3}\right| = \left|\frac{3(ni+2n)-(3ni+1)(1-2i)}{3(3ni+1)}\right|$$

$$= \left|\frac{3ni+6n-(3ni-6ni^2+1-2i)}{3(3ni+1)}\right| = \left|\frac{1-2i}{3(3ni+1)}\right|$$

multiply by conjugate $\frac{3ni-1}{3ni-1}$.

$$= \left|\frac{(1-6n)-(3n+2)i}{3(9n^2+1)}\right|$$

We must find $\varepsilon > (Re(z)^2 + Im(z)^2)^{\frac{1}{2}} = \sqrt{\frac{5}{9(9n^2+1)}} > 0$.

Substitute N_ε for n:

$$\varepsilon^2 = \frac{5}{9(9N_\varepsilon^2+1)} \Rightarrow 9N_\varepsilon^2 + 1 = \frac{5}{9\varepsilon^2} \Rightarrow N_\varepsilon = \sqrt{\frac{5-9\varepsilon^2}{81\varepsilon^2}}$$

For $\varepsilon > \sqrt{\frac{5}{9(9N_\varepsilon^2+1)}}$, $|z_n - z| < \varepsilon$ for $n \geq \frac{\sqrt{5-9\varepsilon^2}}{9\varepsilon}$. Thus z_n is convergent.

Relation: a subset S of $X \times X$ (written aRb, $a \sim b$): $X \times X = \{(a,b) \mid a \in X, b \in X\}$

Equivalence Relation: is a relation satisfying 3 requirements

(1) Reflexive: $\forall a \in X$, $a \sim a$

(2) Symmetric: $\forall a, \forall b$ if $a \sim b \Rightarrow b \sim a$

(3) Transitivity: $\forall a, b, c \in X$ if $a \sim b$ & $b \sim c$ then $a \sim c$

Equivalence class: $c(a) := \{b \in X \mid a \sim b\}$

→ no overlap of classes; disjoint or equal

→ $\mathbb{Q} := \{c(a,b) \mid (a,b) \in X\} = \frac{X}{\sim}$

→ $\mathbb{R} := C = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{Q}, (a_n)_{n\in\mathbb{N}}$ is Cauchy$\}$: $a_n \sim b_n$ if $\lim_{n\to\infty} a_n - b_n = 0$

Lemma: If (a_n), (b_n) are Cauchy seq, then $(a_n + b_n)_{n\in\mathbb{N}}$ & $(a_n b_n)$ are Cauchy seq

Euclidean space (in n-dimensions) $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

→ Dot product: $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

→ Norm: $\|x\| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(1) $|\vec{x}| \geq 0$ & $|\vec{x}| = 0$ then $\vec{x} = \vec{0}$

(2) Cauchy-Schwarz Inequality: $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$

(3) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Corollary: $\vec{x}, \vec{y}$ are perpendicular iff $\vec{x} \cdot \vec{y} = 0$

Injective $f: A \to B$: $\forall x, y$ $f(x) = f(y) \Rightarrow x = y$

Surjective $f: A \to B$: $\forall y \in B\ \exists x \in A$, $f(x) = y$

Bijective: injective + surjective

A, B has the same cardinality ($|A| = |B|$) if $\exists$ bijection $f: A \to B$

X is countable if (1) X is finite: $\exists n \in \mathbb{N}$ s.t. bijection $f: X \to \{1, 2, 3, \ldots, n\}$

(2) infinite uncountable: $\exists$ bijection $f: X \to \mathbb{N}$

Thm: countable union of countable sets are countable

Schroder-Bernstein Thm: If there is $h: A \to B$, $g: B \to A$ surjective or injective funcs then $\exists$ bijection $f: A \to B$

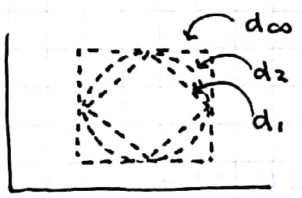
Cantor's Diagonalization Argument: If S is the set of all infinite seq consisting of 0's & 1's then S is countable

Metric space: (X, d) satisfies (1) $\forall x, y$, $d(x,y) \geq 0$ & $d(x,y) = 0$ iff $x = y$

(2) $\forall x, y$, $d(x,y) = d(y,x)$

(3) $\forall x, y, z$, $d(x,z) \leq d(x,y) + d(y,z)$ ← triangle inequality

→ discrete metric: $d_{disc}(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$



→ L^p metric: $(\mathbb{R}^n, d_p)$, $p \geq 1$: $d_p(\vec{x}, \vec{y}) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$

↳ L^1 (taxicab metric): $d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$

↳ L^2 metric: $d_2(\vec{x}, \vec{y}) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{\frac{1}{2}} = \|\vec{x} - \vec{y}\|$

↳ L^∞ metric: $d_\infty(\vec{x}, \vec{y}) = \max(1 \leq i \leq n \mid x_i - y_i)$

HW 2: Prove d_∞ defines a metric on $\mathbb{R}^n$

(1) Show $\forall x, y$ $d(x,y) \geq 0$ & $d(x,y) = 0 \Leftrightarrow x = y$

$|x_i - y_i| \geq 0\ \forall i = 1, \ldots, n \Rightarrow \max_{1 \leq i \leq n}(|x_i - y_i|) \geq 0$

If $d_\infty(x,y) = 0$, suppose $x \neq y$ then $\exists i$ s.t. $x_i \neq y_i$ $|x_i - y_i| > 0$. This contradicts $d_\infty(x,y) = 0$.

If $x = y$ then $x_i = y_i\ \forall i = 1, \ldots, n$ $|x_i - y_i| = 0\ \forall i$; $d_\infty(x,y) = 0$

(2) Show $\forall x, y$ $d(x,y) = d(y,x)$.

$|x_i - y_i| = |y_i - x_i|$ for all $i = 1, \ldots, n$; $d_\infty(x,y) = \max_{1 \leq i \leq n}(|x_i - y_i|) = d_\infty(y,x)$

(3) Show $\forall x, y, z$, $d(x,y) + d(y,z) \geq d(x,z)$

$|x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ $\forall i, \ldots, n$

$\max_{1 \leq i \leq n} |x_i - z_i| \leq \max_{1 \leq i \leq n} |x_i - y_i| + \max_{1 \leq i \leq n} |y_i - z_i|$

Thm: Every infinite subset of countable sets is countable

Open ball centered @ x: $B_r(x) = \{y \in X \mid d(x,y) < r\}$

Closed ball ———————— : $\overline{B_r}(x) = \{y \in X \mid d(x,y) \leq r\}$

Open set: $E \subseteq X$ is open if $\forall x \in E$, $\exists r > 0$ s.t. $B_r(x) \subseteq E$

Closed set: $E \subseteq X$ is closed if E^c is open or if E contains all its acc. points

Thm: If $\{E_\alpha\}$ a collection of sets E_α then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c)$

Thm: E is open $\Leftrightarrow$ E^c is closed; E is closed $\Leftrightarrow$ E^c is open; a set can be clopen

Thm: (1) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open

(2) ———————— $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed

(3) For any finite collection $G_1, \dots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open

(4) ———————— $F_1, \dots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Accumulation/limit points: a point $p \in E$, $\forall r > 0$ $(B_r(p) \setminus \{p\}) \cap E \neq \emptyset$

Interior point: $p \in E$ s.t. $\exists r$, $B_r(p) \subseteq E$; union of all open sets E°

Boundary point: $p \in E$, $\forall r > 0$ s.t. $B_r(p) \cap E \neq \emptyset$ & $B_r(p) \cap E^c \neq \emptyset$, $\forall r > 0$

Isolated point: $p \in E$, $\exists r > 0$ s.t. $B_r(p) \cap E = \{p\}$

Dense set: every point $p \in E$ is a limit point of E

Closure: $\overline{E} = E \cup E'$

(1) $E = \overline{E} \Leftrightarrow E$ is closed

(2) If F is closed & $E \subseteq F \Rightarrow \overline{E} \subseteq F$

Thm: If p is a limit point of E, then every ball around p contains infinitely many points of E

Cor: A finite point set has no limit points.

HW 2: open sets - any subset of discrete metric space, ~~single point~~, whole metric space, $\emptyset$

closed sets - ——————————————————————,

x-axis, closed ball, single point

HW 2: (1) Prove E' is closed.

Show $(E')' \subseteq E'$.

Let $p \in (E')'$. Then $\forall r_1 > 0$, $B_{r_1}(p)$ contains $q \in E'$ s.t. $p \neq q$.

Let $r_2 = \frac{1}{2}d(p,q)$ so $B_{r_2}(q) \subseteq B_{r_1}(p)$.

q is a limit point of E so $\exists$ some $x \in B_{r_2}(q) \in E$ s.t. $x \neq q$.

Then $x \in B_{r_2}(q) \subseteq B_{r_1}(p)$, $x \neq p$ since $p \notin B_{r_2}(q)$.

Any ball $B_{r_1}(p)$ contains $x \in E$. So p is a limit point of E.

(2) Prove E & $\overline{E}$ have the same limit points.

Show $E' = (\overline{E})'$.

ⓐ Show $E' \subseteq (\overline{E})'$

Let $p \in E'$. $\forall r > 0$, $q \in B_r(p)$ where $p \neq q$, $q \in E$.

$q \in \overline{E} = E \cup E'$. So $p \in (\overline{E})'$

ⓑ Show $(\overline{E})' \subseteq E'$

Let $p \in (\overline{E})'$. $\forall r_1 > 0$, $q \in B_{r_1}(p)$ where $p \neq q$, $q \in \overline{E}$.

If $q \in E$, $p \in E'$ then we are done.

If $q \in E'$, let $r_2 = \frac{1}{2}d(p,q) > 0$ s.t. $B_{r_2}(q) \subseteq B_{r_1}(p)$.

$\exists$ $x \in B_{r_2}(q)$ s.t. $x \in E$, $x \neq q \neq p$.

Then $x \in B_{r_2}(q) \subseteq B_{r_1}(p)$ so p is a limit point of $E \Rightarrow (\overline{E})' \subseteq E'$.

Open cover: a collection of open sets $S = \{U_\alpha\}_{\alpha \in A}$ s.t. set $E \subseteq \bigcup_{\alpha \in A} U_\alpha$

Compact: a set E which has a finite subcover for every open cover

Thm: A finite collection of points is compact

Thm: Compact sets are closed sets

Thm: $E \subseteq X$ compact & $F \subseteq E$ closed $\Rightarrow$ F is compact

Finite Intersection Thm: Let $K_1, K_2, \dots, K_n, \dots$ be a collection of compact sets. If $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_n} \neq \emptyset$ for all possible indices $i_1, i_2, \dots, i_n \Rightarrow \bigcap_{i=1}^\infty K_i \neq \emptyset$

Nested Interval Thm: If a collection of intervals in $\mathbb{R}$ s.t. $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \Rightarrow \bigcap_{n=1}^\infty I \neq \emptyset$

Nested K-cell Thm: If $I_1, \dots, I_n$ are k-cells s.t. $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \Rightarrow \bigcap_{i=1}^\infty I_i$

Thm: Every k-cell in $\mathbb{R}^k$ is compact

Heine-Borel Thm: the following are equivalent

(1) E is compact

(2) E is closed & bounded

(3) Any infinite subset of E has a limit point in E

Thm: For compact $E \subseteq X$, every infinite sequence in E has a subsequence that converges to a point in E

Thm: A closed/finite union of compact sets in compact
Thm: If F is closed & K is compact, $F \cap K$ is compact
Weierstrass Thm. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$
HW 3: Prove $C \cup D$ is compact if C, D are compact
Let $\mathcal{U}$ be an open cover of $C \cup D$ s.t. $C \subseteq \mathcal{U}$ & $D \subseteq \mathcal{U}$.
Since C is compact, $\exists$ finitely many subcovers $U_{11}, \dots, U_{1n}$ that cover C.
D ——— , ——— $U_{k1}, \dots, U_{kn}$ ——— D.
The union $(U_{11}, \dots, U_{1n}) \cup (U_{k1}, \dots, U_{kn})$ is a finite subcover of $C \cup D$.
Perfect set: the set E s.t. $E = E'$
Thm: If $P \subseteq \mathbb{R}^k$, P is perfect, then P is uncountable
Cor: Every interval $[a,b]$ s.t. $a<b$ is uncountable; the set of all real numbers is uncountable
Cantor set: $C = \bigcap_{n=1}^{\infty} I_n$ where $I_0 = [0,1]$ & subseqent n removes middle third
(1) C is compact
(2) C is perfect & uncountable
(3) C is totally disconnected
(4) C has measure zero, not dense $[0,1]$
→ ternary expansion consists of digits 0 or 2
Connected set: a) no disjoint, open sets U, V ($U \cap V = \emptyset$) s.t.
(1) $E \cap U \neq \emptyset$, $E \cap V \neq \emptyset$
(2) $E = (E \cap U) \cup (E \cap V)$
b) if not separated s.t. sets A, B
(1) $A \cap \overline{B} = \emptyset$
(2) $\overline{A} \cap B = \emptyset$

* closure of connected sets are connected; interiors of connected sets may not be connected

Thm: Subset of real line $\mathbb{R}$ is connected $\Leftrightarrow$ $x \in E$, $y \in E$ & $x<z<y$ then $z \in E$
Thm: A finite collection of $n \geq 2$ points is not connected in any metric space
Least Upper Bound Property: If $S \subseteq \mathbb{R}$ & $S \neq \emptyset$ is bounded above, then S has a least upper bound $\sup(S)$, $x \geq s$ $\forall s \in S$ & $x \leq y$ $\forall$ y upper bounds
HW 3: If C, D connected & $C \cap D \neq \emptyset$, then $C \cup D$ is connected.
Proof by contrad: Suppose $C \cup D$ is not connected. Let $S = C \cup D$.
So $\exists$ U, V open, disjoint sets ($U \cap V = \emptyset$) s.t. $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$, $S = (S \cap U) \cup (S \cap V)$.
Let $x \in C \cap D$. Let $x \in U$ (proof is symm. if V).
Then $x \in C$ & $x \in D$.
Let $y \in V$ then $y \in C$ or $y \in D$. Let $y \in C$ (proof symm if D).
Then $x \in U \cap C$ & $y \in V \cap C$ so $U \cap C, V \cap C \neq \emptyset$.
But $C \subseteq U \cup V$ which implies C is disconnected. Contradiction.
Limit: $\lim_{n \to \infty} x_n = x$ if $\forall \varepsilon > 0$ $\exists N$ s.t. if $n \geq N$ then $d(x_n, x) < \varepsilon$
Continuous func f: $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$ @ $x_0 \in X$
→ $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$; $\lim_{x \to x_0} f(x) = f(x_0)$
Thm: If f, g are cont real/complex func, then $f+g$, fg, $\frac{f}{g}$ are cont, $g \neq 0$ $\forall x \in X$
Cor: $f: X \to Y$, $g: Y \to Z$ cont. $\Rightarrow f \circ g$ cont.
Thm: $f: X \to Y$ cont $\Leftrightarrow$ $\forall$ open/closed set $V \subseteq Y$, $f^{-1}(V)$ is an open/closed set
Cor: $f: \mathbb{R}^k \to \mathbb{R}$, $c \in \mathbb{R}$, f cont $\Rightarrow f^{-1}(\{c\})$ is closed
Thm: $f: X \to \mathbb{R}^k$ cont, then $F(x) = (f_1(x), f_2(x), \dots f_k(x))$ $f_i: X \to \mathbb{R}$: f is cont $\Leftrightarrow$ f_i are cont.
Isolated points are continuous.
Thm: If $f: \mathbb{R} \to \mathbb{R}$ is a polynomial func in the usual metric, f is cont.
HW 4: If A is a bounded subset of X $\not\Rightarrow$ $f(A)$ is a bounded subset of X'
→ ex: $f(x) = \frac{1}{x}$, $A = (0,1]$, $f(A) = [1, \infty)$
If B is bounded in X', $\not\Rightarrow$ $f^{-1}(B)$ is bounded in X
→ ex: $f(x) = c$, $A = (-\infty, \infty)$, $f(A) = \{c\}$
If $x_0 \in A$ is isolated $\not\Rightarrow$ $f(x_0)$ is isolated
→ ex: $f(x) = \sin(x)$, $A = [0, 2\pi] \cup \{4\pi\}$, $f(A) = [-1, 1]$
If $f(x_0)$ is isolated in $f(A)$ $\not\Rightarrow$ x_0 is isolated in A
→ ex: $f(x) = c$, $A = (-\infty, \infty)$, $f(A) = \{c\}$

same for limit points

Thm: $f: X \to X'$ is cont. at x_0 $\Leftrightarrow$ $\forall$ seq $(x_n)_{n \in \mathbb{N}} \in X$ which converges to $x_0 \in X$, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$
→ See HW 4

HW 4: $f: X$ discrete $\to X'$ is cont (since all subset of discrete metric space is open)

HW 4: Define $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ (in lowest terms, } x \neq 0 \\ 0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q} \end{cases}$

f is cont at 0, $\mathbb{R} \setminus \mathbb{Q}$: $\varepsilon = \frac{2}{n} > \frac{1}{n}$, $\delta = \frac{1}{2} \min_{1 \le q \le n} \{ |x - \frac{p_i}{q_i}| \; i = 1, \dots, n \}$

f not cont at $\mathbb{Q} \setminus \{0\}$: $\varepsilon \le \frac{1}{q}$

HW 4: Prove $f(x,y) = \begin{cases} \frac{(xy)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is cont.

@ $(x,y) \neq (0,0)$ b/c polyn. func are cont & add. & mult. of cont func are cont.

@ $(x,y) = (0,0)$ (1) $f = \frac{1}{\frac{1}{x^2} + \frac{1}{y^2}} \Rightarrow \lim_{(x,y) \to (0,0)} f = 0 = f(0,0) = 0$

(2) $d((x,y), 0) = \sqrt{x^2+y^2} < \delta$

$d(f(x,y), f(0)) = f(x,y) < \varepsilon$

$x^2 + y^2 < \delta$ ①, $(x-y)^2 \ge 0 \Rightarrow x^2 + y^2 > xy$ ② $\Rightarrow \frac{1}{2} \ge \frac{xy}{x^2+y^2}$ ③

$\varepsilon < \frac{1}{2} \delta^2 \Rightarrow \varepsilon = \frac{1}{4} \delta^2$, $\delta = 2\sqrt{\varepsilon}$

Continuity & Compactness

Thm: $f: X \to Y$ cont & $K \subseteq X$ compact $\Rightarrow f(K)$ is compact

Thm: If f is a cont mapping of compact metric space $X \to \mathbb{R}^k$, then $f(x)$ is closed & bounded $\Rightarrow f$ is bounded

Cor: $f: X \to \mathbb{R}$, $K \subseteq X$ compact then $\exists$ $p_{max}, p_{min} \in K$ s.t. $f(p_{min}) \le f(k) \le f(p_{max})$ $\forall k \in K$

uniformly continuous: if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

→ δ does not depend on point, only ε

→ Thm: $f: X \to Y$ cont & X compact, then f is uniformly cont

→ see if slope of f tends to ∞ or $f(x + \frac{1}{n}) - f(x)$ is unbounded

Negation of uniform cont: $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \; d(x,y) < \delta \Rightarrow d(f(x), f(y)) \ge \varepsilon$

Thm: If f is cont 1-1 mapping of X compact $\to Y$ then $f^{-1}(f(x)) = x$, $x \in X$ is cont $Y \to X$

Thm: If E non-compact in $\mathbb{R}$ (1) $\exists$ cont f on E not bounded

(2) $\exists$ cont & bounded f on E w/ no max

(3) If E bounded, $\exists$ f cont. on E not uniformly cont.

Continuity & Connectedness

Thm: $(f: X \to Y)$ cont, $K \subseteq X$ connected then $f(K)$ connected.

Intermediate value theorem: $(f: X \to \mathbb{R}$ cont$)$, X connected & $\exists$ p, q s.t. $f(p) < 0$ & $f(q) > 0 \Rightarrow$ $\exists r \in X$ $f(r) = 0$

Cor: $(f: X \to \mathbb{R}$ cont$)$, X connected then $f(X)$ is an interval

$f_n \to f$ converges pointwise if $\forall x \in X$ $\lim_{n \to \infty} f_n(x) = f(x)$

$f_n \to f$ uniformly converges if $\forall$ $\varepsilon > 0 \; \exists N$ s.t. if $n \ge N$, $\forall x \in X$ $|f_n(x) - f(x)| < \varepsilon$

→ N does not depend on x; uniform $\Rightarrow$ pointwise

Ex: $f_n(x) = x + \frac{1}{n} \sin(nx)$ is uniformly conv.

$f_n(x) = 1 + x + \dots + x^n$ polynomial is pointwise but not convergent uniformly

Thm: $f_n \to f$ uniformly $\Leftrightarrow \lim_{n \to \infty} f_n = f$ in $B(X, \mathbb{R}) \Leftrightarrow \lim_{n \to \infty} d_{B(X,\mathbb{R})}(f_n, f) = 0$

HW: Prove $\sin(xy) + \cos(x+y) = x^2 + y$ has infinitely many solutions in x, y.

Let $f(x,y) = \sin(xy) + \cos(x+y)$, $g(x,y) = x^2 + y$ $\}$ $h(x,y) = f(x,y) - g(x,y)$, cont & X connected

Fix $y < -2$.

→ $\forall$ $x < \sqrt{|y| - 2}$, $g(x,y) < f(x,y)$ so $h(x,y) > 0$

→ $\forall$ $x > \sqrt{|y| + 2}$, $g(x,y) > f(x,y)$ so $h(x,y) < 0$

By IVT, $\forall$ $y \in (-\infty, -2)$ $\exists$ at least one solution for which $h(x,y) = 0$.

$B(X, \mathbb{R}) = \{ f$ bounded func; $f: X \to \mathbb{R} \}$

→ L^∞ metric : $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$

$BC(X, \mathbb{R}) = \{ f: X \to \mathbb{R} \mid f$ cont, bounded$\}$

Complete space: X is complete if any cauchy seq is convergent to a point in X

→ $\mathbb{C}$, $\mathbb{R}$, discrete, $B(X, \mathbb{R})$, $BC(X, \mathbb{R})$

Thm: Every bounded seq in $\mathbb{R}^n$ (or $\mathbb{C}^n$) w/usual metric has a convergent subseq.

Bolzano-Weierstrass Thm: If a set A is a bounded infinite subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, then A has an accumulation point

Stone-Weierstrass Thm: the set of polynomials is dense in $BC([a,b], \mathbb{R})$

floor, integer part: $\lfloor x \rfloor :=$ largest integer smaller than x

fractional part: $\{x\} = x - \lfloor x \rfloor$, $0 \le \{x\} < 1$

Thm: α is irrational $\Leftrightarrow S = \{ \{n\alpha\}, n \in \mathbb{Z} \}$ is dense in $[0,1]$

Discontinuous: f is discont at x if $\lim_{x\to x_0} f(x) \neq f(x_0)$ (or DNE)

→ first kind if $f(x^+)$ & $f(x^-)$ exists; otherwise of the second kind

↳ right hand limit: $\lim_{x\to q^+} f(x) = L$ $\forall \varepsilon > 0$ $\exists \delta > 0$ if $0 < x - a < \delta$ then $|f(x) - L| < \varepsilon$

↳ left hand limit: $\lim_{x\to q^-} f(x) = L$ ——— $0 < a - x < \delta$ ———

f is monotonic if f is increasing $\forall x,y \in (a,b)$ if $x < y \Rightarrow f(x) \leq f(y)$

or decreasing ——— $\Rightarrow f(x) \geq f(y)$

→ if f is increasing & decreasing, it is a constant func.

Thm: If $f:(a,b) \to \mathbb{R}$ is monotonic then every discontin. is of the first kind

→ $f(x^-)$ & $f(x^+)$ exists at every $x \in (a,b)$

Thm: If f is monotonic then the set of discontinuities is countable

→ $\exists$ infinitely many $x \in \mathbb{R}$ where f is cont.

Parametrized curve in $\mathbb{R}^k$ is a cont. func. $c: [a,b] \to \mathbb{R}^k$

Line segment b/t points p, q in $\mathbb{R}^k$: $\ell(t) = qt + (1-t)p$

$C \subseteq \mathbb{R}^k$ is convex if $\forall p, q \in C$, $\forall 0 \leq t \leq 1$, $tq + (1-t)p \in C$

Given $\{p_1, p_2, \dots, p_m\}$ in $\mathbb{R}^k$, x is a convex combination of S if $\exists$ $t_1, t_2, \dots, t_m \geq 0$ & $\sum_{i=1}^n t_i = 1$ s.t. $x = t_1p_1 + t_2p_2 + \dots + t_mp_m$.

HW5 → Proof by Induction: base case (n=1,2) vacuously or by def

$n = m+1$; true for $n = m$.

Let $T = \sum_{i=1}^m t_i$. Then $\sum_{i=1}^{m+1} t_i - \sum_{i=1}^m t_i = 1 - T = t_{m+1}$.

Convex combo: $t_1p_1 + t_2p_2 + \dots + t_{m+1}p_{m+1}$

$= \sum_{i=1}^m t_ip_i + t_{m+1}p_{m+1}$

$= T(\sum_{i=1}^m \frac{t_ip_i}{T}) + (1-T)p_{m+1}$

$= Tp' - (1-T)p_{m+1} \in$ convex set

$(\sum_{i=1}^m \frac{t_ip_i}{T}) \in$ set b/c $\sum_{i=1}^m \frac{t_i}{T} = 1$

Let $p' = \sum_{i=1}^m \frac{t_ip_i}{T} \in$ set.

Given $A \subseteq \mathbb{R}^k$, convex hull of A (Hull(A)) is the intersection of all convex sets containing A (smallest convex set containing A)

→ intersection of convex sets is convex

$f: X \to X$ is a contraction if $\exists$ $\alpha \in [0,1)$ s.t. $\forall x,y: d(f(x), f(y)) \leq \alpha d(x,y)$

$f: X \to X$ then x_0 is a fixed point of f if $f(x_0) = x_0$

Contraction Mapping Thm: If $f: X \to X$ is a contraction & X (space of func) is complete, then f has a unique fixed point

→ f is cont. using $\delta = \frac{\varepsilon}{\alpha}$

HW5: $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 100x^2 + 100$ has no fixed point

$f: (0,1) \to (0,1)$ where $f(x) = x^2$ does not have a fixed point

$f: [0,1] \to [0,1]$ has a fixed point (use IVT to show $f(x) = x$)

HW5: Show x_0 is the unique fixed point of f. (contraction map)

Proof by Contradiction: suppose $\exists > 1$ fixed point

Then $d(x_0, x_1) = d(f(x_0), f(x_1))$.

By contraction prop, $\exists$ $\alpha \in [0,1)$ s.t. $d(f(x), f(y)) \leq \alpha d(x,y)$ $\forall x, y$

So $d(x_0, x_1) \leq \alpha d(x_0, x_1) \Rightarrow 1 \leq \alpha$. Contradiction.

Thm: If C, D are convex, $C \cap D$ is convex

HW 5: Show a convex subset of $\mathbb{R}^n$ w/usual metric is connected

Proof by Contradiction: Suppose C convex is not connected

$\exists$ 2 open sets U, V s.t. $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$

$C = (U \cap C) \cup (V \cap C) \Rightarrow C \subseteq U \cap V$

Choose $x \in U \cap C$, $y \in V \cap L$.

Let $f: [0,1] \to \mathbb{R}^n$, $f(\ ) = tx + (1-t)y$ cont. Since $[0,1]$ is connected, $f([0,1])$ is connected.

→ since C is convex, $f([0,1]) \subseteq C$ then

(1) $x \in U \cap f([0,1]) \neq \emptyset$, $y \in V \cap f([0,1]) = \emptyset$

(2) $(U \cap f([0,1])) \cap (V \cap f([0,1])) \subseteq (U \cap C) \cap (U \cap V) = \emptyset$

(3) $f([0,1]) \subseteq U \cap V \Leftrightarrow f([0,1]) = (U \cap f([0,1])) \cup (V \cap f([0,1]))$

So $f([0,1])$ is disconnected. Contradiction.

Thm: A func $f: \mathbb{R}^k \to \mathbb{R}$ is convex if $\forall$ $0 \leq t \leq 1$ & $\forall x, y \in \mathbb{R}^k$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Polynomial: $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$

→ $\alpha \in \mathbb{C}$ is a root of p if $p(\alpha) = 0$

→ If α is a root, then $P(x) = (x - \alpha)Q(x)$ degree n-1

Cor: A polynomial of degree n has at most n roots

Fundamental Thm of Algebra: If P is a polynomial, w/coefficients in $\mathbb{C}$, of degree $n \geq 1$ then p has at least one root

Cor: any complex polynomial can be factored as $P(x) = c(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2} \dots (x-\alpha_k)^{n_k}$ for some $\alpha_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$, $n_i \geq 1$

Cor: $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0 = z^n(1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n})$

Stone - Weierstrass Thm : ($f:[a,b] \to \mathbb{R}$) cont $\forall \varepsilon > 0$ $\exists$ polynomial p s.t. $\forall x \in [a,b]$, $|P_\varepsilon(x) - f(x)| < \varepsilon$

→ True for $f: C \to \mathbb{R}$, $C \subseteq \mathbb{R}^k$ compact

→ set of polynomials is dense in $BC([a,b], \mathbb{R})$

A func. $f:[a,b] \to \mathbb{R}$ is approximated by polynomials if $\forall \varepsilon > 0$ $\exists$ polyn. p s.t. $\forall x \in [a,b]$ $|f(x) - P(x)| < \varepsilon$