MATH W53
MODULE 1 SECTION 1/2

Finding the equation of a tangent to a curve.
Calculate the slope at the given point with $y=g(t)$ & $x=f(t)$ so slope $= \frac{g'(t)}{f'(t)} = \frac{dy}{dx}$
Then use the line formula $y-y_0 = \frac{dy}{dx}(x-x_0)$ to find the equation of the tangent line.
Ex: Find the eq of the tangent of $x=1+\sqrt{t}$, $y=e^{t^2}$ @ $(2,e)$

$\left.\begin{aligned} \frac{dx}{dt} &= \frac{1}{2\sqrt{t}} \to @\, t=1 \to \frac{1}{2} \\ \frac{dy}{dt} &= 2te^{t^2} \to @\, t=1 \to 2e \end{aligned}\right]$ slope $=4e$

We get $y-e = 4e(x-2) \Rightarrow y = 4ex - 7e$

or

$y=e^{t^2} \to \ln(y) = t^2 \to t = \sqrt{\ln(y)}$
$x = 1+\sqrt{t} \to x = 1+(\ln(y))^{1/4} \to \ln y = (x-1)^4$
$y = e^{(x-1)^4} \Rightarrow \frac{dy}{dx} = 4(x-1)^3 e^{(x-1)^4}$ @ $x=2 \Rightarrow 4e$

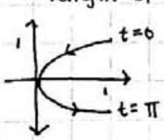
Determining concavity of a parametrized curve
If $\frac{d^2y}{dx^2} > 0$ then it is convex; if $\frac{d^2y}{dx^2} < 0$ then it is concave.
Find $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}$.
Ex: For which values of t is the curve $x=t^2+1$ & $y=e^t-1$ concaved?

We know $\left.\begin{aligned} \frac{dy}{dx} &= \frac{e^t}{2t}. \text{ Thus } \frac{d}{dt}\left(\frac{e^t}{2t}\right) = \frac{2te^t - 2e^t}{4t^2} \\ \frac{dx}{dt} &= 2t \end{aligned}\right] \frac{d^2y}{dx^2} = \frac{2te^t - 2e^t}{8t^3} = \frac{e^t(t-1)}{4t^3}$

when t = positive $e^t(t-1) < 0 \to t < 1$. Notice $\frac{e^t(t-1)}{4t^3} = \frac{e^t}{4t^2}\frac{(t-1)}{t} \to 1 - \frac{1}{t} < 0$ so $0 < t < 1$

Finding the area enclosed by a curve.
With $x=f(t)$ & $y=g(t)$ the equation is $\int_a^b g(t)f'(t)dt$ where t goes from a to b.
Be careful parametrizing t! Curves under the x-axis will negate area.
Ex: Find the area enclosed by $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$.
This is an ellipse — use symmetry.
$\int_a^b b\sin\theta(-a\sin\theta)\,d\theta$ where $0 \le \theta \le 2\pi$ or $-2\int_0^\pi -ab\sin^2\theta\, d\theta = 2ab\left(\frac{1}{2}\right)\int_0^\pi 1 - \cos(2\theta)d\theta$
$= ab\left(\theta - \frac{1}{2}\sin(2\theta)\Big|_0^\pi\right) = ab\pi$

Distance traveled on a curve vs. length of a curve
Length $= \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt$ (integrating length of velocity vector / speed)
Distance = length × number of times traveled on the length.
Ex: Compare the distance traveled by a particle w/ position $x=\cos^2 t$ & $y=\cos t$ where $0 \le t \le 4\pi$ w/ the length of the curve



We know $x=y^2$ so the curve ranges $0 \le t \le \pi$
The distance is 4 × this measure
$L = 2\int_0^{\pi/2} \sqrt{(-2\sin t\cos t)^2 + (-\sin t)^2}\,dt = 2\int_0^{\pi/2} \sin t\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \ln(\sqrt{5}+2)^{\frac{1}{2}}$
$D = 4\sqrt{5} + 4\ln(\sqrt{5}+2)^{\frac{1}{2}}$

Finding the area of surface of revolution of a parametrized curve.
Area (surface revolved around x axis) $= \int_\alpha^\beta 2\pi y\sqrt{x'(t)^2 + y'(t)^2}\,dt$ (careful when revolving closed circles!)
Ex: Area of $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le \theta \le \frac{\pi}{2}$ revolved around the x axis.
We get $\int_\alpha^\beta 2\pi(a\sin^3\theta)\sqrt{(-a(3\cos^2\theta)\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\,d\theta$
$= 2\pi a\int_0^{\pi/2} \sin^3\theta\sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\,d\theta = 2\pi a\int_0^{\pi/2} 3a\sin^4\theta\cos\theta\, d\theta$
$= 6\pi a^2\int_0^{\pi/2} u^4 du$ where $u = \sin\theta$ & $du = \cos\theta d\theta$
$= 6\pi a^2\left[\frac{1}{5}\sin^5\theta\Big|_0^{\pi/2}\right] = \frac{6}{5}\pi a^2$

Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$
Slope $= \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$
Area under the curve $= \int_\alpha^\beta \frac{f(\theta)^2}{2}d\theta$
Length $= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

MATH W53
MODULE 1 SECTION 3/4

Determining the vector & parallel equation of a line in 3D space (including symmetric eq).

We use the equation for vector equations as: $\vec{r} = \vec{r_0} + t\vec{v}$ where $\vec{r_0}$ describes a point on the line $\langle x_0, y_0, z_0 \rangle$
$\vec{v}$ is a parallel vector $\langle a, b, c \rangle$

This can be converted to parametric equations: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

Symmetric equations solve for t in above & are in the form $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

Ex 1: Find the three equations for a line through point $(0, 14, -10)$ & parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.

We have point $(0, 14, -10)$ & parallel vector $\langle 2, -3, 9 \rangle$.

Thus $\vec{r} = (0, 14, -10) + t\langle 2, -3, 9 \rangle$

$\vec{r} = \langle 2t, 14 - 3t, -10 + 9t \rangle$

P.eq: $x = 2t$, $y = 14 - 3t$, $z = -10 + 9t$

S.eq: $\frac{x}{2} = \frac{14 - y}{3} = \frac{z + 10}{9}$

Ex 2: Find the three equations for a line through points $(-8, 1, 4)$ & $(3, -2, 4)$.

Direction vector $= \langle 3 - (-8), -2 - 1, 4 - 4 \rangle$
$= \langle 11, -3, 0 \rangle$

Using random point, we get

$\vec{r} = (-8, 1, 4) + t\langle 11, -3, 0 \rangle$

$\vec{r} = \langle -8 + 11t, 1 - 3t, 4 \rangle$

P.eq.: $x = -8 + 11t$, $y = 1 - 3t$, $z = 4$

S.eq.: $\frac{x + 8}{11} = \frac{1 - y}{3}$ $z = 4$

can also use: $\vec{r}(t) = (1 - t)\vec{r_0} + t\vec{r_1}$ 0

Parallel, Intersecting vs. Skewed Lines.

Lines are parallel if their direction vectors are parallel.
——— intersecting if they are not parallel & there is a value of t & s equal on all variables
——— skewed if they are not parallel nor intersecting.

Ex: Determine if lines L_1 & L_2 are parallel, skewed or intersect.

$L_1: \frac{x - 2}{2} = \frac{y - 3}{-2} = \frac{z}{7} \Rightarrow x = 2t + 2$, $y = 3 + -2t$, $z = 7t$

$L_2 \quad x = \frac{y - 1}{-1} = \frac{z - 2}{3} \qquad x = s$, $y = 1 - s$, $z = 3s + 2$

We set the eq. by their t & s values so we rearrange above & solve for below.

$2t + 2 = s$ (1), $3 - 2t = 1 - s$ (2), $7t = 3s + 2$ (3)

Substituting (1) in (2): $3 - 2t = 1 - (2t + 2) \Rightarrow 3 = -1$

Because the system of equations does not work & their direction vectors are not parallel, L_1 & L_2 are skewed.

Finding the equation of a plane.

The equation of a plane is given by $ax + by + cz = ax_0 + by_0 + cz_0$ where $\langle a, b, c \rangle = \vec{n}$ (the normal, or perpendicular, vector) & (x_0, y_0, z_0) is a point on the plane

Ex 1: Find the eq of a plane through point $(3, -2, 8)$ & parallel to plane $z = x + y$.

We have $\vec{n} = \langle 1, 1, -1 \rangle$ & point $(3, -2, 8)$

Thus $1(x) + 1(y) - 1(z) = 1(3) + 1(-2) + -1(8)$

$x + y - z = -7 \Rightarrow x + y + 7 = z$

Ex 2: Find the eq of a plane through the points $(0, 1, 1)$, $(1, 0, 1)$, & $(1, 1, 0)$

We find vector $\vec{a} = \langle 1, -1, 0 \rangle$
$\vec{b} = \langle 0, -1, 1 \rangle$

To find $\vec{n}$, we can $\vec{a} \times \vec{b} = \langle 1, 1, 1 \rangle$

Thus $x + y + z = 0 + 1 + 1 \Rightarrow x + y + z = 2$

Finding a plane equidistant from two points.

Use the distance formula & equate the two expressions

Ex: Find an eq. for a plane w/points all equidistant from $(2, 5, 5)$ & $(-6, 3, 1)$

We have $\sqrt{(x-2)^2 + (y-5)^2 + (z-5)^2} = \sqrt{(x+2+4)^2 + (y-3)^2 + (z-1)^2}$

$(x-2)^2 + (y-5)^2 + (z-5)^2 = (x+6)^2 + (y-3)^2 + (z-1)^2$

$x^2 - 4x + 4 + y^2 - 10y + 25 + z^2 - 10z + 25 = x^2 + 12x + 36 + y^2 - 6y + 9 + z^2 - 2z + 1$

$16x + 4y + 8z = 8$

$4x + y + 2z = 2 \Rightarrow 4x + y + 2z - 2 = 0$

Distance from a point to a line or a plane.

Point to Plane: $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Point to Line: $\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$ where $\vec{a}$ is on the line
$\vec{b}$ is b/t the point & the line

<u>Ex 1</u>: Find the distance between point $(4,1,-2)$ & line $x=1+t$, $y=3-2t$, $z=4-3t$

@ $t=0$, we have point $P=(1,3,4)$

@ $t=1$, we have point $Q=(2,1,1)$

$\vec{a} = \overrightarrow{PQ} = \langle 1,-2,-3 \rangle$

$\vec{b} = \overrightarrow{P \text{ to point}} = \langle 3,-2,-6 \rangle$

Thus we get $D = \dfrac{|\langle 1,-2,-3\rangle \times \langle 3,-2,-6 \rangle|}{\sqrt{1^2+(-2)^2+(-3)^2}}$

$= \dfrac{|\langle 12-6, -9+6, -2+6 \rangle|}{\sqrt{14}}$

$= \dfrac{\sqrt{6^2+(-3)^2+(+4)^2}}{\sqrt{14}} = \dfrac{\sqrt{61}}{\sqrt{14}}$

<u>Ex 2</u>: Find the distance from point $(-6,3,5)$ to plane $x-2y-4z=8$.

we have $ax+by+cz+d = x-2y-4z-8$

↳ $a=1$; $b=-2$; $c=-4$; $d=-8$

↳ $x_1=-6$; $y_1=3$; $z_1=5$

$D = \dfrac{|-6+(-6)+(-20)+(-8)|}{\sqrt{1^2+(-2)^2+(-4)^2}}$

$= \dfrac{|-40|}{\sqrt{21}} = \dfrac{40}{\sqrt{21}} = \dfrac{40\sqrt{21}}{21}$

MATH W53
MODULE 1 SECTION 5

Finding vector functions to represent the curve of intersection of two surfaces.

Given two surfaces, solve for each dimension w/respect to x & set $x=t$ (or whatever convenient).
Alternatively, you can look for trig identities & substitute the variables for those.

Ex 1: Find the vector function that represents the curve of intersection of paraboloid $z=4x^2+y^2$ & parabolic cylinder $y=x^2$.
Let $x=t$ so $y=t^2$
Substitute for z $\Rightarrow z=4t^2+(t^2)^2=4t^2+t^4$
Thus we get $\vec{r}(t)=\langle t, t^2, 4t^2+t^4\rangle$

Ex 2: ··· semiellipsoid $x^2+y^2+4z^2=4$, $y\geq 0$, & cylinder $x^2+z^2=1$.
$\cos^2 t+\sin^2 t=1 \Rightarrow x=\cos t$ & $z=\sin t$
Substitute in $y=\sqrt{4-4z^2-x^2} \Rightarrow y=\sqrt{4-4\sin^2 t-\cos^2 t}$
$y=\sqrt{(\sin^2 t+\cos^2 t)(4-1-3\sin^2 t)}=\sqrt{3-3\sin^2 t}=\sqrt{3}|\cos t|$
$\vec{r}(t)=\langle\cos t, \sqrt{3}|\cos t|, \sin t\rangle$

Determining the parametric eq for a tangent line to a curve at a point.

Given the curve, solve for t which will yield the given point.
Use that t to input in $\vec{r}'(t)$ to get the normal vector ($\perp$) of the curve.
Then use $\vec{r}(t)=\vec{a}+t\vec{b}$ where $\vec{a}$ = point & $\vec{b}$ = direction vector.

Ex: We have $x=e^{-t}\cos t$, $e^{-t}\sin t$, $z=e^{-t}$ @ $(1,0,1) \Rightarrow t=0$
$\vec{r}'(t)=\langle -e^{-t}\cos t-e^{-t}\sin t, -e^{-t}\sin t+e^{-t}\cos t, -e^{-t}\rangle \Rightarrow \vec{r}'(0)=\langle -1,1,-1\rangle$
$\vec{r}(t)=(1,0,1)+t\langle -1,1,-1\rangle \Rightarrow \vec{r}(t)=\langle 1-t, t, 1-t\rangle \Rightarrow x=1-t,\ y=t,\ z=1-t$

Finding the integral of $\vec{r}'(t)$ given $\vec{r}'(t)$ & $\vec{r}$(t= some given value).

Integrate $\vec{r}'(t)$ with $\int_a^b \vec{r}'(t)dt=\langle\int x'(t)dt, \int y'(t)dt, \int z'(t)dt\rangle$
To find the constants for x(t), y(t), & z(t), use the given $\vec{r}$ at a point to solve.

Ex: Find $\vec{r}(t)$ if $\vec{r}'(t)=2t\,i+3t^2 j+\sqrt{t}\,k$ & $\vec{r}(1)=i+j$
We know $x'(t)=2t$, $y'(t)=3t^2$, & $z=\sqrt{t}$
Thus $\int\vec{r}'(t)dt=\langle t^2+C_1, t^3+C_2, \frac{2}{3}t^{3/2}+C_3\rangle$ Given $t=1$ we get $x=1, y=1, z=0$
$x(1)=(1)^2+C_1=1$ & $y(1)=(1)^3+C_2=1$ & $z(1)=\frac{2}{3}(1)^{3/2}+C_3=0$
$C_1=0$, $C_2=0$, $C_3=-\frac{2}{3}$
Therefore we have $\vec{r}(t)=\langle t^2, t^3, \frac{2}{3}t^{3/2}-\frac{2}{3}\rangle$

MATH W53
MODULE 2 SECTION 1

Proving that the limit of $f(x,y)$ exists / does not exist.

To prove a limit exists, it is best to use definitions & theorems to show this. (polar sub, definitions, etc.)

To disprove a limit exists, we can find an instance that is not equivalent by setting y as a function of x.

__Ex 1__: Find the limit of $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Multiply conjugate $\lim_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 = \sqrt{1}+1 = 2$

__Ex 2__: Find $\lim_{(x,y)\to(1,0)} \dfrac{xy-y}{(x-1)^2+y^2}$

Approach x & y axis $\to (1,0) = 0$

Sub $y = m(x-1)$ to get

$\lim_{(x,y)\to(1,0)} \dfrac{xm(x-1)-m(x-1)}{(x-1)^2 + [m(x-1)]^2}$

$\lim_{(x,y)\to(1,0)} \dfrac{m(x-1)(x-1)}{(x-1)^2(1+m^2)} = \dfrac{m}{1+m^2}$

Does not exist!

__Ex 3__: Find $\lim_{(x,y)\to(3,2)} x^2y^3 - 4y^2$

Plug in: $(3)^2(2)^3 - 4(2)^2 = 9\cdot 8 - 16 = 56$

Methods of finding partial derivatives.

1) By definition, we use $\lim_{h\to0} (f(x+h,y)-f(x,y))/h = \frac{\partial f}{\partial x}$ & $\lim_{h\to0} (f(x,y+h)-f(x,y))/h = \frac{\partial f}{\partial y}$ when we are given the explicit formula $z=f(x,y)$
2) When given eq. in the form of $F(x,y,z)$, we use implicit differentiation.

__Ex 1__: Use the def of partial derivatives to find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ of $xy^2 - x^3y$

$\frac{\partial z}{\partial x} = (x+h)y^2 - (x+h)^3y - xy^2 + x^3y/h = xy^2 + y^2h - x^3y + 3x^2yh + 3xh^2y + yh^3 - xy^2 + x^3y/h$

$= hy^2 - 3x^2hy - 3xh^2y - h^3y/h = y^2 - 3x^2y - 3xhy - h^3y$

$\lim_{h\to0} (y^2 - 3x^2y - 3xhy - h^3y) = y^2 - 3x^2y$

__Ex 2__: Use implicit differentiation on $x^2 - y^2 + z^2 - 2z = 4$.

$\frac{\partial z}{\partial x}: 2x + 2z\left(\frac{\partial z}{\partial x}\right) - 2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x}(1-z) = x \Rightarrow \frac{\partial z}{\partial x} = \frac{x}{1-z}$

$\frac{\partial z}{\partial y}: -2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0 \Rightarrow -z\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = -y \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{1-z}$

Finding the equation of a tangent plane to a given surface at a point.

First, find the partial derivatives of z w/respect to x & y & find the value of z_x & z_y at the point

Input into equation $z - z_0 = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

__Ex__: Find the tangent plane to $z = x\sin(x+y)$ @ point $(-1,1,0)$

$z_x = \sin(x+y) + x\cos(x+y) \Rightarrow @ (-1,1,0) = 0 + 1(-1) = -1$

$z_y = x\cos(x,y) \Rightarrow @ (-1,1,0) = -1$

Thus we have $z - 0 = -1(x+1) - 1(y-1) \Rightarrow z = -x - 1 - y + 1 \Rightarrow x + y + z = 0$

__Ex__: We have $f(2,5) = 6$, $f_x = 1$, $f_y = -1$ @ $(2,5)$. Use linear approx to est. $f(2.2, 4.9)$

We have $z - 6 = 1(2.2-2) - 1(4.9-5) \Rightarrow z = 0.2 + 0.1 + 6 = 6.3$

MATH W53
MODULE 2 SECTION 3/4

Use the chain rule to find the partial derivative.

Given a function of x & y [f(x,y)] & the function x(t) & y(t), we may use the chain rule to find the derivative of f with respect to t with $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. This can be expanded as well.

EX: $w = \ln\sqrt{x^2+y^2+z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{1}{(\sqrt{x^2+y^2+z^2})^2}(x)(\cos t) + \frac{1}{(\sqrt{x^2+y^2+z^2})^2}(y)(-\sin t) + \frac{1}{(\sqrt{x^2+y^2+z^2})^2}(z)\sec^2 t$$

$$= (x^2+y^2+z^2)^{-1}(x\cos t - y\sin t + z\sec^2 t)$$

$$= (\sin^2 t + \cos^2 t + \tan^2 t)^{-1}(\sin t\cos t - \sin t\cos t + \tan t\sec^2 t)$$

$$= (1+\tan^2 t)^{-1}(\tan t\sec^2 t) = (\sec^2 t)^{-1}(\tan t\sec^2 t) = \tan t$$

(Alternatively, we could plug in x, y, & z from the onset & have $w = \ln\sqrt{1+\tan^2 t} = \ln(\sec(t))$)

EX: Let $R(s,t) = G(u(s,t), v(s,t))$ where G, u, & v are differentiable, $u(1,2)=5$, $u_s(1,2)=4$, $u_t(1,2)=-3$, $v(1,2)=7$, $v_s(1,2)=2$, $v_t(1,2)=6$, $G_u(5,7)=9$, $G_v(5,7)=-2$.

$$R_s = \frac{\partial R}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial s} = 9(4) + (-2)(2) = 32$$

$$R_t = \frac{\partial G}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial t} = 9(-3) + (-2)(6) = -39$$

Finding the implicit partial differentiation of a function.

Given function z of x, y defined implicitly by $F(x,y,z)=0$ & F is a differentiable func., we have $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}$ & $\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}$

EX: Find $\partial y/\partial x$ of $e^y\sin x = x + xy$

$F(x,y) = x + xy - e^y\sin x = 0$

$\frac{\partial y}{\partial x} = 1 + y - e^y\cos x / x - e^y\sin x \neq 0$ (-1)

$= e^y\cos x - 1 - y / x - e^y\sin x$

EX: Find $\frac{\partial z}{\partial x}$ of $x^2 - y^2 + z^2 - 2z = 4$

$F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4$

$\frac{\partial z}{\partial x} = -\frac{2x}{2z-2} = \frac{-x}{z-1}\frac{(2)}{(2)}$

$= \frac{x}{1-z}$

Finding the directional derivative of f at point P in direction $\vec{u}$.

First, find the gradient of f given by $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle (x,y,z)$

Then, find the unit vector. If given a regular vector, divide the values by $\sqrt{x^2+y^2+z^2}$.

To get the directional derivative, find the dot product of ∇f & the unit vector evaluated at P.

The maximum directional derivative occurs at $|\nabla f|$

EX: Find the directional derivative of $f(x,y,z) = x^2y + y^2z$ at (1,2,3) in direction $\vec{v} = \langle 2,-1,2 \rangle$

$\nabla f = \langle 2xy, x^2+2yz, y^2 \rangle \Rightarrow \nabla f(1,2,3) = \langle 4, 13, 4 \rangle$

$\vec{v}$ in unit vectors is $\vec{u} = \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle$

$D_u f(1,2,3) = \langle 4, 13, 4 \rangle \cdot \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle$

$= 4(\frac{2}{3}) - 13(\frac{1}{3}) + 4(\frac{2}{3}) = 1$

EX: Find the max rate of change of $f(x,y) = \sin(xy)$ at (1,0) in the direction it occurs.

$\nabla f = \langle y\cos(xy), x\cos(xy) \rangle$

$\nabla f(1,0) = \langle 0, 1 \rangle$

$\max = |\nabla f| = \sqrt{0^2+1^2}$

max = 1 in direction ⟨0,1⟩

Solving for the equations of the tangent plane & normal line to a given surface.

Because ∇f is perpendicular to tangent vector $\vec{r}'(t)$, we find the tangent plane via $\nabla f \cdot \vec{r}'(t) = 0$

Thus we get $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

The normal line is given by $\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$

EX: Find the tangent plane & normal line of $x+y+z = e^{xyz}$ at (0,0,1)

$\nabla f = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$

$\nabla f(0,0,1) = \langle 1,1,1 \rangle$

Tangent Plane: $(x-0) + (y-0) + (z-1) = 0$

$x + y + z = 1$

Normal Line: $x = y = z - 1$

EX: $g(x,y) = x^2 + y^2 - 4x$. Find $\nabla g(1,2)$ to find tangent line to level curve $g(x,y) = 1$ @ (1,2)

$\nabla g = \langle 2x-4, 2y \rangle$

$\nabla g(1,2) = \langle -2, 4 \rangle$

direction vector: $\nabla g(1,2) = 0$ so $\vec{v} = \langle 4,2 \rangle$

eq of tangent line: $\vec{r} = (1,2) + t(4,2)$

$x = 1 + 4t$ & $y = 2 + 2t \Rightarrow y = 2 + 2(\frac{x-1}{4})$

$y = \frac{1}{2}x + \frac{3}{2}$

MATH W53
MODULE 2 SECTION 5

Find critical points using partial derivatives

Set the function as $f(x,y)$. Find f_x & f_y & set them equal to 0. Make sure all values are within the domain.

If the region is defined, check boundary points to see if they are local/absolute min/max. Check corners.

To determine a min, max, or saddle point, calculate D as (cross product)

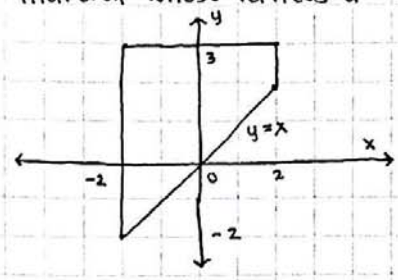
$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

$D>0$, $f_{xx}(a,b)>0$: local min

$D>0$, $f_{xx}(a,b)<0$: local max

$D<0$: saddle point

Ex: Find the abs. max & min values of f on the set D: $f(x,y) = x^3 - 3x - y^3 + 12y$ where D is a quadrilateral whose vertices are $(-2,3)$, $(2,3)$, $(2,2)$, & $(-2,-2)$



- $f_x = 3x^2 - 3 \rightarrow f_x = 0 \rightarrow x = \pm 1$
  $f_y = -3y^2 + 12 \rightarrow f_y = 0 \rightarrow y = \pm 2$
  $\left.\right\}$ $\underline{f(1,2) = 14}$ $\underline{f(-1,2) = 18}$ $f(-1,-2) = -14$ $f(1,-2) = -18$
  * $f(-1,-2)$ & $f(1,-2)$ is not in the domain
- $x = 2$ & $2 \le y \le 3$
  $f(2,y) = 8 - 6 - y^3 + 12y = 2 - y^3 + 12y$
  $f_y = -3y^2 + 12 \rightarrow f_y = 0 \rightarrow y = \pm 2$ $(2,-2)$ is not in the domain
  $\underline{f(2,2) = 8 - 6 + 8 - 24 = -14}$
- $-2 \le x \le 2$ & $y = 3$
  $f(x,3) = x^3 - 3x + 9 \rightarrow f_x = 3x^2 - 3 \rightarrow f_x = 0 \rightarrow x = \pm 1$
  $\underline{f(1,3) = 7}$ & $\underline{f(-1,3) = 11}$
- $x = -2$, $-2 \le y \le 3$
  $f(-2,y) = -2 - y^3 + 12y \rightarrow f_y = -3y^2 + 12 \rightarrow y = \pm 2$
  $\underline{f(-2,2) = 14}$ Check $(-2,3) \rightarrow \underline{f(-2,3) = -18}$

$x = y \rightarrow f(x,x)$

$f(x,x) = 9x \rightarrow f_x = 9$

corner: $(-2,-2)$

$\underline{f(-2,-2) = 18}$

Abs. Min: -18 @ $(-2,3)$

Abs. Max: 18 @ $(-1,2)$ & $(-2,-2)$

MATH W53
MODULE 3 SECTION 2

Find the area of the region using polar coordinates.

Find the area of integration of r & θ. You can find r by direct solving or with the equation $r=\sqrt{x^2+y^2}$. Find θ by (1) graphing the region, (2) setting upper = lower, or (3) by default 0 to 2π.

Ex: Find the area of the region inside the circle $(x-1)^2+y^2=1$ & outside the circle $x^2+y^2=1$.

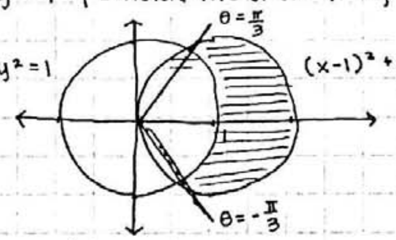
Convert: $(r\cos\theta-1)^2+r^2\sin^2\theta=1$

$r^2\cos^2\theta-2r\cos\theta+1+r^2\sin^2\theta=1$

$r^2=2r\cos\theta \rightarrow r=2\cos\theta$

We know $r=\sqrt{x^2+y^2}=1$

$1=2\cos\theta \rightarrow \frac{1}{2}=\cos\theta \rightarrow \theta=\pm\frac{\pi}{3}$



We get $\int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta=\int_{-\pi/3}^{\pi/3}\frac{r^2}{2}\Big|_{r=1}^{r=2\cos\theta}d\theta$

$=\int_{-\pi/3}^{\pi/3}\frac{4\cos^2\theta}{2}-\frac{1}{2}\,d\theta \leftarrow \cos^2\theta=\frac{1}{2}(1-\cos(2\theta))$

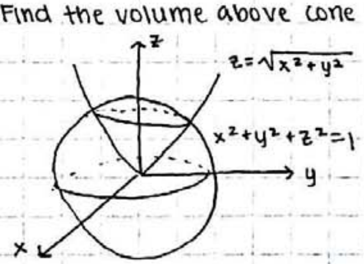
$=\int_{-\pi/3}^{\pi/3}\frac{1}{2}-\cos(2\theta)\,d\theta=\frac{\theta}{2}+\frac{1}{2}\sin(2\theta)\Big|_{\theta=-\pi/3}^{\theta=\pi/3}$

$=\frac{\pi}{6}+\frac{\sqrt{3}}{4}+\frac{\pi}{6}+\frac{\sqrt{3}}{4}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

Find the volume of the region using polar coordinates

The equation is $V=\iint_{shadow}$(upper height − lower height)$r\,dA$. Find the region of integration of θ & r w/steps above, looking at the shadow. Integrate z_2-z_1.

Ex: Find the volume above cone $z=\sqrt{x^2+y^2}$ & below sphere $x^2+y^2+z^2=1$



We know $z=r$ & $x^2+y^2=r^2$

Thus, we have $r^2+r^2=1 \rightarrow 2r^2=1 \rightarrow r^2=\frac{1}{2} \rightarrow r=\frac{1}{\sqrt{2}}$

We know $0\le\theta\le 2\pi$ mag factor: r

$\int_0^{2\pi}\int_0^{\sqrt{1/2}} r(\sqrt{1-r^2}-r)\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{1/2}} r\sqrt{1-r^2}-r^2\,dr\,d\theta$

$\frac{d}{dr}(1-r^2)^{3/2}=-\frac{3}{2}(1-r^2)^{\frac{1}{2}}(2r)=-3r(1-r^2)^{\frac{1}{2}}$

$=\int_0^{2\pi}-\frac{1}{3}(1-r^2)^{3/2}-\frac{1}{3}r^3\Big|_{r=0}^{r=\sqrt{1/2}}d\theta$

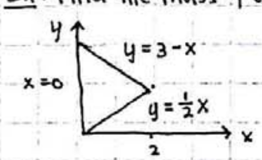
$=\int_0^{2\pi}-\frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right)-\frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right)+\frac{1}{3}\,d\theta$

$=\frac{1}{3}\theta-\frac{1}{3\sqrt{2}}\theta\Big|_{\theta=0}^{\theta=2\pi}=\frac{2\pi}{3}-\frac{2\pi}{3\sqrt{2}}$

Finding the mass & center of mass of a lamina in region D w/density function ρ

mass $=\iint_D \rho(x,y)\,dA=\int_a^b\int_c^d \rho(x,y)\,dy\,dx$; The center of mass is where $\bar{x}=\iint_D x\rho(x,y)\,dA\left(\frac{1}{m}\right)$, $\bar{y}=\frac{1}{m}\iint_D y\rho(x,y)\,dA$

Ex: Find the mass & center of mass where D is enclosed by (0,0), (2,1) & (0,3) & $\rho(x,y)=x+y$



$m=\int_0^2\int_{1/2x}^{3-x}(x+y)\,dy\,dx=\int_0^2 xy+\frac{1}{2}y^2\Big|_{y=1/2x}^{y=3-x}dx$

$=\int_0^2 3x-x^2+\frac{1}{2}(3-x)^2-\frac{x^2}{2}-\frac{1}{8}x^2\,dx=\int_0^2-\frac{13}{8}x^2+3x+\frac{1}{2}(9-6x+x^2)\,dx$

$=\int_0^2-\frac{9}{8}x^2+\frac{9}{2}\,dx=-\frac{9}{24}x^3+\frac{9}{2}x\Big|_0^2=-3+9=6$

$\bar{x}=\frac{1}{6}\int_0^2\int_{1/2x}^{3-x} x^2+yx\,dy\,dx=\frac{1}{6}\int_0^2 x^2y+\frac{y^2x}{2}\Big|_{y=1/2x}^{y=3-x}dx$

$=\frac{1}{6}\int_0^2(3x^2-x^3)+\frac{9x-6x^2+x^3}{2}-\frac{x^3}{2}-\frac{x^3}{8}\,dx \leftarrow \frac{9x}{2}-\frac{9x^3}{8}$

$=\frac{1}{6}\int_0^2\frac{9}{8}(4x-x^3)\,dx=\frac{3}{16}\left(2x^2-\frac{1}{4}x^4\Big|_{x=0}^{x=2}\right)=\frac{12}{16}=\frac{3}{4}$

$\bar{y}=\frac{1}{6}\int_0^2\int_{1/2x}^{3-x} xy+y^2\,dy\,dx=\frac{1}{6}\int_0^2\frac{1}{2}xy^2+\frac{1}{3}y^3\Big|_{1/2x}^{3-x}dx$ $\quad\frac{9x-6x^2+x^3}{2}+\frac{-x^3+9x^2-27x+27}{3}-\frac{1}{8}x^3+\frac{1}{24}x^3$

$=\frac{1}{6}\int_0^2 9-\frac{9x}{2}\,dx=\frac{1}{6}\left(18-\frac{36}{4}\right)=\frac{9}{6}=\frac{3}{2}$ $\quad(\bar{x},\bar{y})=\left(\frac{3}{4},\frac{3}{2}\right)$ $\quad m=\frac{1}{6}$

Finding the area of the surface.

The equation is as follows: Area(surface) $=\iint_D\sqrt{1+f_x^2+f_y^2}\,dA$. Change the eq. of the surface in terms of z & then find the limits of integration.

Ex: Find the area of the sphere $x^2+y^2+z^2=4z$ inside paraboloid $z=x^2+y^2$.

Δ surface in terms of z: $x^2+y^2+z^2=4z \Rightarrow x^2+y^2+(z-2)^2=4 \Rightarrow z=2\pm\sqrt{4-x^2-y^2}$

$f_x=\frac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2x)=-x(4-x^2-y^2)^{-\frac{1}{2}} \Rightarrow f_y=-y(4-x^2-y^2)^{-\frac{1}{2}}$

$\sqrt{1+f_x^2+f_y^2}=\sqrt{\frac{x^2+y^2+(4-x^2-y^2)}{4-x^2-y^2}}=\sqrt{\frac{4}{4-x^2-y^2}}=\sqrt{\frac{4}{4-r}}$ where $r=\sqrt{3}$ (b/c $z=3$)

We get $\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2}{\sqrt{4-r}}r\,dr\,d\theta=\ldots$ u sub $\cdots=4\pi$

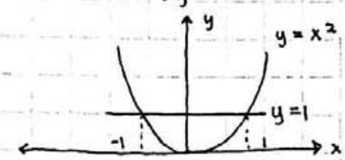
MATH W53
MODULE 3 SECTION 3/4/5

Using triple integrals to find the volume of a solid.

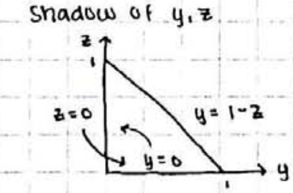
The volume of a solid is $V=\iiint_D 1\,dV$. Iterated integrals can be placed in any of the six integrals. To find the limits of integration, find the range of one variable, then fix that & find the range of the other. Solve the last variable by fixing the previous two.

Ex: Find the volume enclosed by cylinder $y=x^2$, $z=0$, & $y+z=1$.

Shadow of x, y



Shadow of y, z



We get $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$

$\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-x^2} dz\,dx\,dy$

$\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz \quad = \frac{8}{15}$

$\int_0^1\int_0^{1-y}\int_{-\sqrt{1-z}}^{\sqrt{1-z}} dx\,dz\,dy$

$\int_{-1}^{1}\int_0^{1-x^2}\int_0^{1-z} dy\,dz\,dx$

$\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1} dy\,dx\,dz$

Finding the volume in cylindrical coordinates.

Volume $=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{\phi_1(\theta)}^{\phi_2(\theta)} r\,dz\,dr\,d\theta$

Recall $x=r\cos\theta$, $y=r\sin\theta$, $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$

Ex: Evaluate $\iiint_E z\,dV$ where E is enclosed by the paraboloid $z=x^2+y^2$ & plane $z=4$

We know $z=r^2$ so $r^2\le z\le 4$. The shadow of x, y plane is $x^2+y^2=4 \Rightarrow r=2$ & $0\le\theta\le 2\pi$

Thus we get $\int_0^{2\pi}\int_0^2\int_{r^2}^4 rz\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{2}z^2 r\Big|_{z=r^2}^{z=4} dr\,d\theta = \int_0^{2\pi}\int_0^2 8r-\frac{r^5}{2}\,dr\,d\theta$

$=\int_0^{2\pi} 4r^2-\frac{r^6}{12}\Big|_{r=0}^{r=2} d\theta = 2\pi\left(16-\frac{16}{3}\right)=\frac{64\pi}{3}$

Evaluating triple integrals in spherical coordinates.

Volume $=\int_c^d\int_\alpha^\beta\int_a^b \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

We have $z=\rho\cos\phi$, $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $\rho=\sqrt{x^2+y^2+z^2}$, $r=\rho\sin\phi$

where ϕ goes from 0 (positive z axis) to π (negative z axis)

Ex: Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$ in spherical coordinates.

We get $z=\sqrt{x^2+y^2}=\sqrt{2-x^2-y^2} \Rightarrow z^2=x^2+y^2$ & $x^2+y^2+z^2=2$

$\rightarrow \rho=\sqrt{x^2+y^2+z^2}=\sqrt{2}$

We know $y\ge 0$ & $x^2+y^2=1$ & $x\ge 0$ so $0\le\theta\le\frac{\pi}{2}$

If $z^2=x^2+y^2 \Rightarrow \rho^2\cos^2\phi=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta \Rightarrow \rho^2\cos^2\phi=\rho^2\sin^2\phi \Rightarrow \phi=\frac{\pi}{4}$

We get $\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^2\sin\phi\,(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,d\rho\,d\theta\,d\phi$

$=\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^4\sin^3\phi\,\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi = \cdots = \frac{4\sqrt{2}}{15}-\frac{1}{3}$

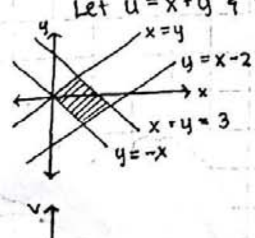
Using transformations to evaluate integrals.

1. Choose a transformation which makes the equations nicer; understand the change in geometry
2. Compute the magnification factor; Jacobian: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$
3. Evaluate integral $\iint_R f\,dA=\iint_S (f\circ T)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$ where T is a differentiable bijection from S

* May have to change to polar/spherical/cylindrical coordinates — if so, don't forget their magnification factor!

Ex: Evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$ where R is enclosed by $x-y=0$, $x-y=2$, $x+y=0$ & $x+y=3$

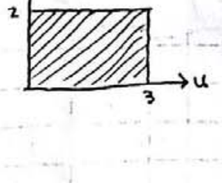
Let $u=x+y$ & $v=x-y$ $\Rightarrow x=v+y \rightarrow u=v+2y \Rightarrow y=\frac{u-v}{2}$ & $x=\frac{u+v}{2}$



$x=y: u-v=u+v \Rightarrow v=0$; $y=x-2: u-v=u+v-4 \Rightarrow v=2 \Rightarrow 0\le v\le 2$

$y=-x: u-v=-u-v \Rightarrow u=0$; $u+v+u-v=6 \Rightarrow u=3 \Rightarrow 0\le u\le 3$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{matrix}\right| = \left|-\frac{1}{4}-\frac{1}{4}\right| = \frac{1}{2}$



We get $\frac{1}{2}\int_0^3\int_0^2 ue^{uv}\,dv\,du = \frac{1}{2}\int_0^3 e^{uv}\Big|_{v=0}^{v=2} du$

$=\frac{1}{2}\int_0^3 (e^{2u}-1)\,du = \frac{1}{2}\left(\frac{1}{2}e^{2u}-u\Big|_{u=0}^{u=3}\right)$

$=\frac{1}{2}\left(\frac{1}{2}e^6-3-\frac{1}{2}\right)=\frac{1}{2}\left(\frac{1}{2}e^6-\frac{7}{2}\right)$

$=\frac{1}{4}e^6-\frac{7}{4}$

MATH W53
MODULE 4 SECTION 1/2/3

Evaluating the line integral.

Integration w.r.t. arc length: $\int_C f\,ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$

x or y: $\int_C f\,dx = \int_a^b f(\vec{r}(t))\,x'(t)dt$; $\int_a^b f(\vec{r}(t))y'(t)dt = \int_C f\,dy$; $\int_a^b f(\vec{r}(t))z'(t) = \int_C f\,dz$

Line Integrals of Vector Fields: $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt = \int_C (P\,dx + Q\,dy + R\,dz)$

* none depend on parametrization ; #2 & 3 are negative if you go backwards

* Remember: $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$

<u>EX:</u> Evaluate $\int_C x^2dx + y^2dy$, where C is the arc of a circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by a line segment from $(0,2)$ to $(4,3)$

We have $x = \cos t\,(2) = 2\cos t$, $y = \sin t\,(2) = 2\sin t$ & $C_1: 0 \le t \le \frac{\pi}{2}$ & $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$

$C_1: \int_0^{\pi/2} (4\cos^2 t)(-2\sin t) + (4\sin^2 t)(2\cos t)\,dt = 8\int_0^{\pi/2} \sin^2 t\cos t - \sin t\cos^2 t\,dt$

–integration by parts– $= 8\left(\frac{1}{3}\sin^3 t + \frac{1}{3}\cos^3 t \Big|_0^{\pi/2}\right) = 0$

For C_2, we have $\vec{r}(t) = \langle 0,2\rangle\cdot(1-t) + t\cdot\langle 4,3\rangle = \langle 4t, 2+t\rangle \Rightarrow \vec{r}\,'(t) = \langle 4,1\rangle$

$\left.\begin{array}{l} x = 4t \\ y = 2+t \end{array}\right]$ $0 \le t \le 1 \Rightarrow \int_0^1 16t^2(4) + (2+t)^2\,dt = \int_0^1 64t^2 + 4 + 4t + t^2\,dt = \int_0^1 65t^2 + 4 + 4t\,dt$

$= \frac{65}{3}t^3 + 4t + 2t^2\Big|_0^1 = \frac{65}{3} + 4 + 2 = \frac{83}{3}$

<u>EX:</u> Evaluate $\int_C y^2z\,ds$ where C goes from $(3,1,2)$ to $(1,2,5)$

We have $\vec{r}(t) = (1-t)\langle 3,1,2\rangle + t\langle 1,2,5\rangle = \langle 3-2t, 1+t, 2+3t\rangle \Rightarrow 0 \le t \le 1$

Thus $\vec{r}\,'(t) = \langle -2, 1, 3\rangle$ & $f(\vec{r}(t)) = (1+t)^2(2+3t) = (1+2t+t^2)(2+3t) = 2+4t+2t^2+3t+6t^2+3t^3$

We get $\int_0^1 (2+7t+8t^2+3t^3)\sqrt{14}\,dt = \sqrt{14}\left(2t + \frac{7}{2}t^2 + \frac{8}{3}t^3 + \frac{9}{4}t^4\Big|_0^1\right) = \sqrt{14}\left(2 + \frac{7}{2} + \frac{8}{3} + \frac{3}{4}\right) = \frac{107\sqrt{14}}{12}$

<u>EX:</u> Find the work done by $\vec{F}(x,y) = x\vec{i} + (y+2)\vec{j}$ in moving an object along an arch of the cycloid $\vec{r}(t) = (t-\sin t)\vec{i} + (1-\cos t)\vec{j}$, $0 \le t \le 2\pi$

We have $\vec{r}\,'(t) = \langle 1-\cos t, \sin t\rangle$ & $\vec{F}(\vec{r}(t)) = \langle t-\sin t, 3-\cos t\rangle$

$\int_0^{2\pi} (1-\cos t)(t-\sin t) + \sin t(3-\cos t)\,dt = \int_0^{2\pi} t - t\cos t + 2\sin t\,dt$ –integration by parts–

$= t\sin t + \cos t + \frac{t^2}{2} - 2\cos t\Big|_0^{2\pi} = 2\pi^2$

Using the Fundamental Theorem of Line Integrals to evaluate line integrals.

FTLI: If $\vec{F} = \nabla f$ & C is a parametrized curve from A to B then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$ (path independent)

If A=B (closed curve) then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = 0$.

<u>EX:</u> Evaluate $\vec{F}(x,y,z) = yz\vec{i} + xz\vec{j} + (xy+2z)\vec{k}$ where C goes from $(1,0,-2)$ to $(4,6,3)$

Check if conservative: $P_y = z = Q_x \Rightarrow$ conservative & path independent

$f = xyz + z^2 + k$ (constant)

$\vec{r}(t) = (1-t)\langle 1,0,-2\rangle + t(\langle 4,6,3\rangle) = \langle 1+3t, 6t, -2+5t\rangle \Rightarrow 0 \le t \le 1$

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}_1(t)) - f(\vec{r}_0(t)) = (4)(6)(3) + 9 - (1)(0)(-2) - 4 = 77$

* If $\vec{F}$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (Clairaut's Theorem)

Using Green's Theorem to evaluate line integrals.

If C is a simple, closed curve in $\mathbb{R}^2$ & D is the region it bounds, then

$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$ & $\oint \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$ (where $\oint$ is the positive orientation)

<u>EX:</u> Evaluate $\int_C y^3dx - x^3dy$ where C is the circle $x^2+y^2=4$, positively oriented.

$\frac{\partial P}{\partial y} = 3y^2$ & $\frac{\partial Q}{\partial x} = -3x^2$, $r = 2 = \sqrt{x^2+y^2}$

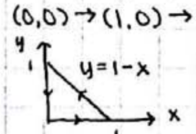
$\iint_D -3x^2 - 3y^2\,dA = \iint -3(x^2+y^2)dA$

Convert to polar coordinates

$-3\int_0^{2\pi}\int_0^2 r^2(r)\,dr\,d\theta = -3\int_0^{2\pi} \frac{1}{4}r^4\Big|_{r=0}^{r=2} d\theta$

$= -3\int_0^{2\pi} 4\,d\theta = -3(4)(2\pi) = -24\pi$

<u>EX:</u> Find the work done by force $\vec{F}(x,y) = \langle x(x+y), xy^2\rangle$ in amoving particle $(0,0) \to (1,0) \to (0,1) \to (0,0)$



positively oriented

$\frac{\partial P}{\partial y} = x$ & $\frac{\partial Q}{\partial x} = y^2 \Rightarrow \int_0^1\int_0^{1-x} y^2 - x\,dy\,dx$

$\int_0^1 \frac{1}{3}y^3 - xy\Big|_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{3}(1-3x+3x^2-x^3) - x + x^2\,dx$

$= \frac{1}{3}\left(\frac{1}{4}\right) - \frac{1}{6} = -\frac{1}{12}$

MATH W53
MODULE 4 SECTION 4/5

Calculating & using properties of curl $\vec{F}$ & divergence $\vec{F}$.

Curl $\vec{F}$ measures the local rotation of $\vec{F}$ (direction & magnitude)

$\rightarrow$ $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$

* If $\vec{F} = \langle P, Q, R \rangle$ is conservative, $\text{curl } \vec{F} = \nabla \times \vec{F} = 0$

Divergence of $\vec{F}$ measures expansion & compression (<0 = incr. pressure)

* $\vec{F}$ is conservative if curl $\vec{F} = 0$ & R is simply connected

$\rightarrow$ $\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$\text{Function} \xrightarrow{\text{gradient}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{divergence}} \text{Function}$$

(gradient then curl $=0$; curl then divergence $=0$)

$\text{div}(\text{curl } \vec{F}) = 0$

Laplacian: $\Delta f = \nabla^2 f = \text{div}(\text{grad } f) = f_{xx} + f_{yy} + f_{zz}$

Alternate statement of Green's Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \cdot \vec{n}\, dA$

Ex: Is $\vec{F} = \langle z\cos y, xz\sin y, x\cos y \rangle$ conservative?

$\text{curl } \vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos y & xz\sin y & x\cos y \end{vmatrix}$

$= \langle -x\sin y - x\sin y, \cos y - \cos y, z\sin y + z\sin y \rangle$

$\neq 0 \Rightarrow$ not conservative

Ex: Is there a vector field on $\mathbb{R}^3$ such that $\text{curl } \vec{G} = \langle x\sin y, \cos y, z - xy \rangle$?

$\text{div}(\text{curl } \vec{F}) = 0$

$\text{div}(\text{curl } \vec{G}) = \langle \sin y, -\sin y, 1 \rangle \neq 0 \Rightarrow$ no $\vec{G}$

Find the equation of a tangent plane to given parametric surface at a specified point.

If $\vec{r}_u$ & $\vec{r}_v$ are linearly independent ($\vec{r}_u \times \vec{r}_v \neq 0$) & normal vector is $\vec{n} = \vec{r}_u \times \vec{r}_v$ the the equation of the tangent plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

Ex: Find the equation of the tangent plane to $\vec{r} = \langle u\cos v, u\sin v, v \rangle$ where $u = 1$ & $v = \frac{\pi}{3}$

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$ $\quad$ $\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

@ $u = 1$ & $v = \frac{\pi}{3}$, $\vec{n} = \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \rangle$ & $\vec{r}_0 = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \rangle$

we get $\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \rangle \cdot \langle x - \frac{1}{2}, y - \frac{\sqrt{3}}{2}, z - \frac{\pi}{3} \rangle = \frac{\sqrt{3}}{2}(x - \frac{1}{2}) - \frac{1}{2}(y - \frac{\sqrt{3}}{2}) + z - \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{4} - \frac{1}{2}y + \frac{\sqrt{3}}{4} + z - \frac{\pi}{3} \Rightarrow \frac{x\sqrt{3}}{2} - \frac{y}{2} + z = \frac{\pi}{3}$

Find the area of a surface.

Solve using the equation $\iint_D |\vec{r}_u \times \vec{r}_v|\, dA$ or $\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$.

Ex: Find the area of the surface $x = u^2, y = uv, z = \frac{1}{2}v^2$, $0 \leq u \leq 1, 0 \leq v \leq 2$

* Also see ex 61 on pg 1123 (PS 4.5)

$\vec{r}_u = \langle 2u, v, 0 \rangle$ $\quad$ $\vec{r}_u \times \vec{r}_v = \langle v^2, -2uv, 2u^2 \rangle$

$\vec{r}_v = \langle 0, u, v \rangle$ $\quad$ $|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = (v^2 + 2u^2)$

$\text{Area} = \int_0^1 \int_0^2 v^2 + 2u^2\, dv\, du = \int_0^1 \frac{1}{3}v^3 + 2u^2 v \Big|_0^2 du = \int_0^1 \frac{8}{3} + 4u^2\, du = \frac{8}{3}u + \frac{4}{3}u^3 \Big|_0^1 = 4$

Evaluating surface integrals.

We have $\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$ (integration wrt surface area) $\quad$ * $|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + f_x^2 + f_y^2}$

We have $\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA = \iint_D (-Pg_x - Qg_y + R)\, dA$ where $\vec{r} = \langle x, y, g(x,y) \rangle$

$\rightarrow$ measures flux across S using normal vector to S

$\rightarrow$ depends on orientation (neg/pos) but not parametrization

Ex: Evaluate $\iint_S x^2 yz\, dS$ where S is $z = 1 + 2x + 3y$ above the rectangle $[0,3] \times [0,2]$.

$z = g(x,y) \Rightarrow g_x = 2, g_y = 3 \Rightarrow \sqrt{1 + 4 + 9} = \sqrt{14}$

$\iint_S x^2 y(1 + 2x + 3y)\sqrt{14}\, dS = \int_0^3 \int_0^2 \sqrt{14}(x^2 y + \cdots$

$2x^3 y + 3x^2 y^2)\, dy\, dx$

$= \sqrt{14} \int_0^3 \frac{1}{2}x^2 y^2 + x^3 y^2 + x^2 y^3 \Big|_0^2 dx$

$= \sqrt{14} \int_0^3 2x^2 + 4x^3 + 8x^2\, dx$

$= \sqrt{14}\left(\frac{2}{3}x^3 + x^4 + \frac{8}{3}x^3 \Big|_0^3\right) = 171\sqrt{14}$

Ex: Evaluate $\vec{F} = \langle 0, y, -z \rangle$ where S is $y = x^2 + z^2$, $0 \leq y \leq 1$ & $x^2 + z^2 \leq 1, y = 1$

Let $x = r\cos\theta, y = r^2, z = r\sin\theta$

$\vec{r}_r = \langle \cos\theta, 2r, \sin\theta \rangle$ & $\vec{r}_\theta = \langle -r\sin\theta, 0, r\cos\theta \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos\theta, -r, 2r^2\sin\theta \rangle$ @ $\frac{\pi}{2}$ points out

$\vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta) = -r^3 - 2r^3\sin^2\theta$

$-\int_0^{2\pi} \int_0^1 r^3 + 2r^3\sin^2\theta\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4}r^4 + \frac{1}{2}r^4\sin^2\theta \Big|_0^1 d\theta$

$= -\int_0^{2\pi} \frac{1}{4} + \frac{1}{2}\sin^2\theta\, d\theta = -\int_0^{2\pi} \frac{1}{4} + \frac{1 - \cos 2\theta}{4}\, d\theta$

$= -\left(\frac{2}{4}\theta - \frac{\sin(2\theta)}{8}\right)\Big|_0^{2\pi} = -\pi$

$\text{DISK} = \pi \Rightarrow \pi - \pi = 0$

MATH W53
MODULE 4 SECTION 6/7

Using Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$.

Stokes' Theorem states $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS = \iint_D -P\left(\frac{\partial y}{\partial x}\right) - Q\left(\frac{\partial y}{\partial y}\right) + R\,dA$

→ $\vec{n}$ is the normal vector (for planes, it is $\langle a, b, 1\rangle$)

→ begin by finding the curl $\vec{F}$ & then identify your surface to get the normal vector

EX: Use Stokes' to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle 1, x+yz, xy-\sqrt{z}\rangle$ & C is the boundary of plane $3x+2y+z=1$ in the first octant

curl $\vec{F} = \langle x-y, -y, 1\rangle$ & $\vec{n} = \langle 3,2,1\rangle \Rightarrow \iint_S 3(x-y) - 2(y) + 1\,dS = \iint_S 3x - 5y + 1\,dS$

We get $\int_0^{1/3}\int_0^{\frac{1}{2}-\frac{3}{2}x} 3x - 5y + 1\,dy\,dx = \cdots = \frac{1}{24}$

EX: Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ of $\vec{F}(x,y,z) = \langle x^2\sin z, y^2, xy\rangle$ on $z = 1-x^2-y^2$ (xy plane, oriented upward)

We know $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$\vec{r}(t) = \langle \cos t, \sin t, 0\rangle$ & $\vec{r}\,'(t) = \langle -\sin t, \cos t, 0\rangle \Rightarrow \vec{F}(\vec{r}(t)) = \langle 0, \sin^2 t, \sin t\cos t\rangle$

We get $\int_0^{2\pi} \sin^2 t\cos t\,dt$ ← we let $u = \sin t$ & $du = \cos t\,dt$

$\int_{t=0}^{t=2\pi} u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\sin^3 t\Big|_0^{2\pi} = 0$

Using the Divergence Theorem to calculate $\iint_S \vec{F}\cdot d\vec{S}$ (flux $\vec{F}$ across S).

Divergence Theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV$. where $\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

EX: Use the Divergence Theorem to find $\vec{F} = |\vec{r}|^2\vec{r}$ where $\vec{r} = \langle x, y, z\rangle$ & S is a sphere with radius R & center origin

$|\vec{r}|^2 = (\sqrt{x^2+y^2+z^2})^2 = x^2+y^2+z^2 \Rightarrow |\vec{r}|^2\vec{r} = \langle x^3+y^2x+z^2x,\ yx^2+y^3+yz^2,\ zx^2+zy^2+z^3\rangle$

$\text{div}\,\vec{F} = 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2 = 5(x^2+y^2+z^2)$

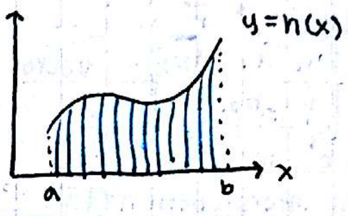
We know $0\le\theta\le 2\pi$, $0\le\phi\le\pi$, $0\le R\le 2$ & $\rho^2 = x^2+y^2+z^2$

$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 5\rho^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = 2\pi\int_0^R 5\rho^4\,d\rho\int_0^{\pi}\sin\phi\,d\phi$

$= 2\pi\left(\rho^5\Big|_0^R\right)\left(-\cos\phi\Big|_0^{\pi}\right) = (2\pi)(R^5)(2) = 4\pi R^5$

FINAL REVIEW

Parametrized Curves: we let $x = f(t)$ & $y = g(t)$



Area $= \int_a^b h(x)\,dx$

$dx = f'(t)\,dt$

$h(x) = g(t)$

How to Graph (1) Plot the points

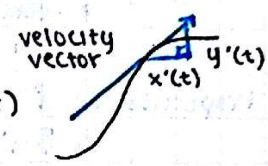
(2) Eliminate parameter t

(3) Use symmetry

slope $= \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$

area without vertical tangents $= \pm \int_\alpha^\beta g(t) f'(t)\,dt$

→ be wary of curves which go beneath the x axis (it may cancel out)



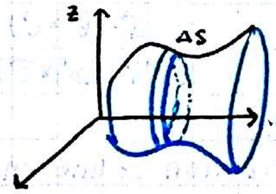
length $= \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt$

→ does not depend on parametrization (as long as you do not cover the same part twice)

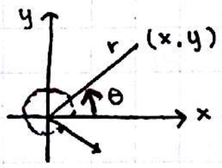
area of surface of revolution $= \int_\alpha^\beta 2\pi y \sqrt{x'(t)^2 + y'(t)^2}\,dt$

→ where $0 \le \alpha, \beta \le$

→ careful on parameters (i.e. one revolution of unit circle is from $0 \le \theta \le \pi$)



Polar Coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

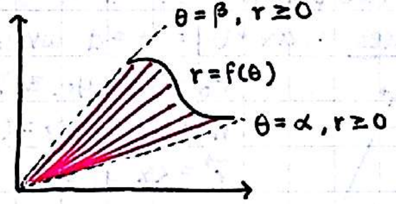
$\tan\theta = \frac{y}{x}$

How to Graph (1) Plot the points

(2) Convert to Cartesian coordinates

Given $x = f(\theta)\cos\theta$ & $y = f(\theta)\sin\theta$ where $r = f(\theta)$

slope $= \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{d/d\theta\,[f(\theta)\sin\theta]}{d/d\theta\,[f(\theta)\cos\theta]}$



area $= \int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\,d\theta$

length of polar curve $= \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$

3-D Space & Vectors

Distance between (x_1, y_1, z_1) & (x_2, y_2, z_2) $= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Properties of Vectors (1) Addition: $\vec{a} + \vec{b} = \langle a_1+b_1, a_2+b_2, \dots \rangle$

(2) Subtraction: $\vec{a} - \vec{b} = \langle a_1-b_1, a_2-b_2, \dots \rangle$

(3) Multiplication of Scalar: $c\vec{a} = \langle ca_1, ca_2, \dots \rangle$

(4) Length: $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \dots}$

(5) Triangle inequality: $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

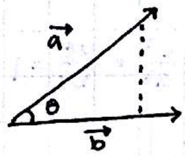
(6) Scalar Associative & Distributive: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

$c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$, $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$, $(cd)\vec{a} = c(d\vec{a})$

Dot Product: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 + \dots$

→ Geometric interpretation: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

→ If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$



→ Properties: (1) Communitive: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

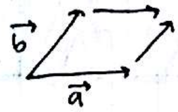
(2) Distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(3) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

(4) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

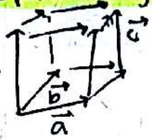
(5) Pythagorean Theorem: $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

Determinants : $\det(\vec{a} \,\&\, \vec{b}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \text{Area(parallelogram)}$



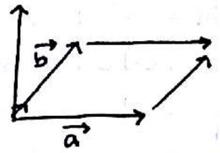
→ In 3-D: $\det = \pm$ Vol (Parallelopiped)

positive if satisfy "right hand rule"



Cross Product : $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ = vector ⊥ to $\vec{a}$ & $\vec{b}$

$\vec{a} \times \vec{b}$



→ If $\vec{a} \parallel \vec{b}$, then $\vec{a} \times \vec{b} = 0$

→ Geometric interpretation: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ = Area(parallelogram)

↳ $\vec{a}, \vec{b}$, & $\vec{a} \times \vec{b}$ satisfy the right hand rule

Properties (1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

(3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$) distributive

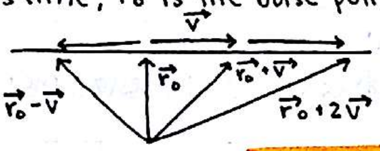
(5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ = Vol(parallelopiped of $\vec{a}, \vec{b}$, & $\vec{c}$)

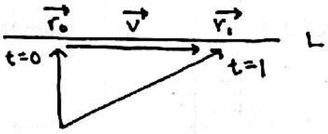
(6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Lines in 3D

Parametrization of lines in space: $\vec{r}(t) = \vec{r_0} + t\vec{v}$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

→ t is time, $\vec{r_0}$ is the base point, $\vec{v}$ is tangent vector





Line through two points: $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$

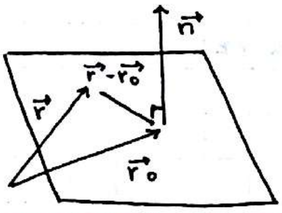
Planes & Surfaces : $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$

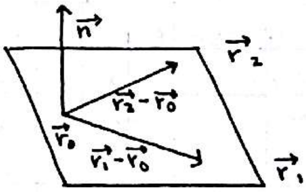
$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$\vec{n} = \langle a, b, c \rangle = (\vec{r_1} - \vec{r_0}) \times (\vec{r_2} - \vec{r_0})$

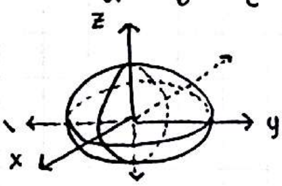
$\vec{r} = \langle x, y, z \rangle$

– Plane –

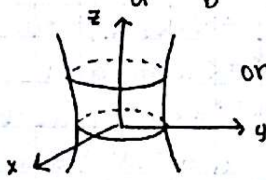




Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

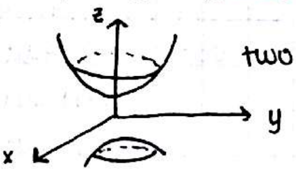


Hyperboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



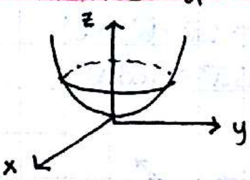
one sheet

$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

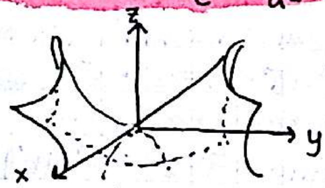


two sheets

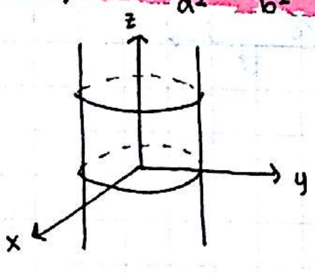
Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



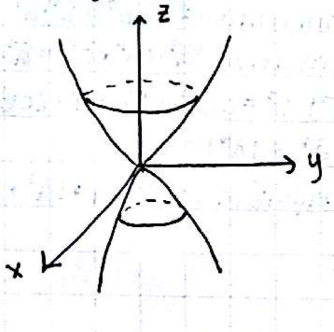
Hyperbolic Paraboloid: $\frac{z}{c} = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$



Quadric Cylinder: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Space Curves: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $\alpha \le t \le \beta$

→ velocity vector $= \langle x'(t), y'(t), z'(t) \rangle$

Tangent line to a curve $= \vec{r} + t\vec{r}'$ where $\vec{r}$ is the point & $\vec{r}'$ is the direction vector

Length $= \int_\alpha^\beta |\vec{r}'(t)|\,dt$

Product Rule of Vector-Valued Functions

(1) Vector & scalar: $\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$

(2) Dot Product: $\frac{d}{dt}(\vec{r}(t)\cdot\vec{s}(t)) = \vec{r}'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}'(t)$

(3) Cross Product: $\frac{d}{dt}(\vec{r}(t)\times\vec{s}(t)) = \vec{r}'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}'(t)$

Integration of Vector-Valued Functions: $\int_a^b \vec{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$

→ FTC: $\int_a^b \vec{r}'(t)\,dt = \vec{r}(b) - \vec{r}(a)$

→ Area (ribbon of vector) $= \frac{1}{2}\int_a^b |\vec{r}(t) \times \vec{r}'(t)|\,dt$

Functions of Several Variables

How to visualize a function of two variables (1) Graph it

(2) Level curves / set: $\{(x,y) \in D \mid f(x,y) = k\}$

Function of Two Variables: $\{(x,y,z) \mid (x,y) \in D,\ z = f(x,y)\}$

Function of Three Variables: $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = f(x,y,z)\}$

→ Level surfaces: $\{(x,y,z) \in D \mid f(x,y,z) = k\}$

Definition of a Limit: Let f be a function on domain D in $\mathbb{R}^2$. $\lim_{(x,y)\to(a,b)} f(x,y) = L$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $(x,y) \in D$ & $(x,y) \neq (a,b)$ & $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$ where ε = "error tolerance" & δ = "how close (x,y) needs to be to (a,b) to guarantee that $f(x,y)$ is within the error tolerance of L"

A function of two variables is continuous at (a,b) if $f(a,b)$ is defined & $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ [i.e. constants, coord. func., etc.]

Proving that the Limit Does not Exist

→ show discrepancies by approaching the limit in different ways (x & y axis, lines, etc.)

Linear Approximation:

Partial Derivatives: $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

Second Partial Derivatives: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$

Clairaut's Theorem: If f_{xy} & f_{yx} are defined in a neighborhood of (x,y) then $f_{xy}(x,y) = f_{yx}(x,y)$

Explicit Formula: $z = f(x,y)$ Implicit Formula: $F(x,y,z) = 0$

Tangent Planes: $z - z_0 = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$

→ tangent plane to $z = f(x,y)$ at $(x_0, y_0, z_0 = f(x_0,y_0))$ [← normal vector]

Linear Approximation: $f(x_0 + \Delta x) \approx y_0 + f'(x_0)\Delta x$

Differentiability: Let f be a function of two variables.

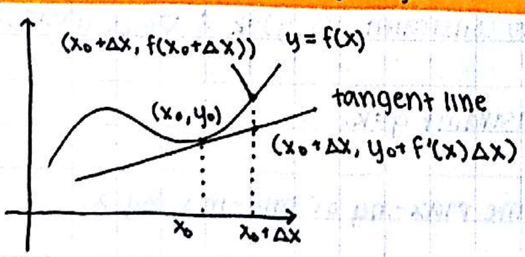
f is differentiable at (x_0,y_0) if the partial derivatives $f_x(x_0,y_0)$ & $f_y(x_0,y_0)$ are defined & there exists functions $\varepsilon_1, \varepsilon_2$ such that

$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1(\Delta x,\Delta y)\Delta x + \varepsilon_2(\Delta x,\Delta y)\Delta y$

where $\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(\Delta x,\Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(\Delta x,\Delta y) = 0$

"Graph F is well approx. by the tangent line"

Linear Approximation: $f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$



Chain Rule

Single variable: y is a func of x & x is a func. of t

$$\frac{\partial y}{\partial t} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \& \quad \frac{dy}{dt}(t) = \frac{dy}{dx}x(t)\frac{dx}{dt}(t)$$

Multi-variable: f is a differentiable func. of x & y; x & y are differentiable func. of t

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}(t)$$

Multi-variable Part 2: $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

General Rule: $\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{x_n}\frac{\partial x_n}{\partial t_i}$

Implicit Function Theorem: Suppose F is a differentiable function of x, y, z. Suppose $F(x_0, y_0, z_0) = 0$ & $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$. Then (1) you can solve $F(x,y,z) = 0$ for z as a function of x, y when (x, y) is close to (x_0, y_0) with z close to z_0

(2) z is a differentiable function of x, y

→ $\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$ & $\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}$

Directional Derivatives: Let f be a func. of x & y & $\vec{u} = \langle a, b \rangle$ be a unit vector, then

$D_{\vec{u}}f(x,y) = \frac{d}{dt}\big|_{t=0} f(x+at, y+bt)$

$D_{\vec{u}}f = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$ & $D_{\vec{u}}f = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}$

→ measures rate at which f changes as you move in any direction.

Gradient: $\text{grad } f(x,y,z) = \nabla f(x,y,z) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle (x,y,z)$

→ Relation w/ directional derivatives: $D_{\vec{u}}f = \vec{u} \cdot \nabla f$

→ Properties (1) If $\nabla f \neq 0$, then ∇f points in the direction in which $D_{\vec{u}}f$ is longest

(2) Vector version of the chain rule: $\frac{d}{dt}f(\vec{r}(t)) = \nabla f \cdot \vec{r}\,'(t)$

(3) If $\nabla f \neq 0$ then ∇f is perpendicular to the level sets $\{f = k\}$

(4) $\nabla(fg) = f\nabla g + g\nabla f$

Maxima & Minima

If (a,b) is a local extremum then at least one of the following holds:

(1) (a,b) is a critical point of f $[f_x = f_y = 0]$

(2) $f_x(a,b)$ & $f_y(a,b)$ are not both defined

(3) (a,b) is on the boundary of D

Second Derivative Test: Suppose (a,b) is a critical point of f. Assume f_{xx}, f_{xy}, f_{yy} are defined & continuous in a neighborhood of (a,b)

Define $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$

→ Local min: $D > 0$, $f_{xx}(a,b) > 0$

→ Local max: $D > 0$, $f_{xx}(a,b) < 0$

→ Inconclusive in all other cases

Extreme Value Theorem: Let f be a continuous function on $D \subset \mathbb{R}^2$. Assume D is closed & bounded. Then f has a global min & global max on D.

→ **bounded**: there exists $r > 0$ such that D is contained in disk of radius r centered at origin

→ **closed**: D contains all of its boundary points (i.e. no < or >)

Lagrange Multipliers: Suppose (x,y) min or max f subject to constraint $g(x,y) = k$ & $\nabla g(x,y) \neq 0$. Then $\nabla f(x,y) = \lambda \nabla g(x,y)$

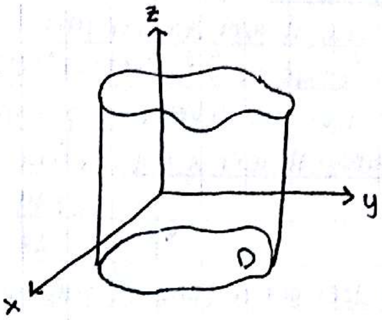
→ Let M(K) be the max or min value of f subject to constraint $g = k$.

Interpretation: $\frac{dM(k)}{dk} = \lambda$

Incr. constraint by 1 will approx incr. the max-ing or min-ing by λ.

Double Integrals

Interpretation: $\iint_R f\,dA$ = volume under the graph



→ $\iint_R f\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$

If f is continuous

→ computation: $\int_a^b \int_c^d f(x,y)\,dy\,dx$ (Fubini's Theorem)

$\int_c^d \int_a^b f(x,y)\,dx\,dy$

Type I Region: $D = \left\{(x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array}\right\}$ $\int_a^b \int_{g_1(x)}^{g_2(x)} f\,dy\,dx$

Type II Region: $D = \left\{(x,y) \,\middle|\, \begin{array}{l} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{array}\right\}$ $\int_c^d \int_{h_1(y)}^{h_2(y)} f\,dx\,dy$

Double Integral over Polar Rectangles: $\iint_R f\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} \text{Area}(R_{ij}) f(r_{ij}^*, \theta_{ij}^*)$

→ $\iint_R f\,dA = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\,dr\,d\theta$

Surface Area $= \iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA$

Triple Integrals

$\iiint_R f\,dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta x \Delta y \Delta z$

→ Fubini's Theorem: $\iiint_R f\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$

Type I Analogue: $\iiint_R f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

Center of Mass: Total Mass $= M = \iiint_E \rho\,dV$

$\bar{x} = \frac{1}{M}\iiint_E x\rho\,dV$ $\quad \bar{y} = \frac{1}{M}\iiint_E y\rho\,dV$ $\quad \bar{z} = \frac{1}{M}\iiint_E z\rho\,dV$

Triple Integrals in Cylindrical Coordinates: $\iiint_E f\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\phi_1(\theta)}^{\phi_2(\theta)} f(r,\theta,z)\, r\,dz\,dr\,d\theta$

in Spherical Coordinates: $\iiint_E f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$E = \left\{(\rho,\theta,\phi) \,\middle|\, \begin{array}{l} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ c \le \phi \le d \end{array}\right\}$

$x = \rho\sin\phi\cos\theta$ $\quad r = \rho\sin\phi$

$y = \rho\sin\phi\sin\theta$ $\quad \rho = \sqrt{x^2+y^2+z^2}$

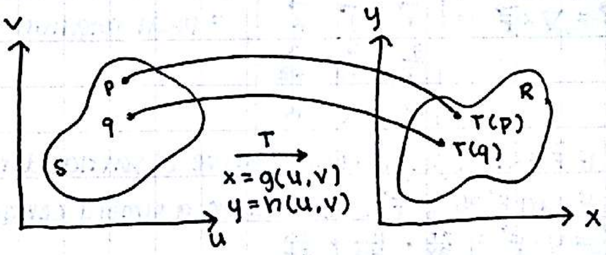
$z = \rho\cos\phi$ $\quad$ where $\rho \ge 0$ & $0 \le \phi \le \pi$

Change of Variables

→ Injective: if $p \ne q$ are different points in S then $T(p) \ne T(q)$

→ Subjective: if every point in R is T of some point in S

→ bijective / bijection / one-to-one correspondence: injective & surjective (one point in S corresponds in one point in R)



suppose $T: S \to R$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

Change of Variables of Double Integrals: $\iint_R f\,dA = \iint_S (f \circ T) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$

where T is a differentiable bijection from S (u,v) to R (x,y)

General Procedure
(1) Choose a transformation to make the equation nicer
(2) Understand the change in the region (geometry)
(3) Compute magnification factor
(4) Evaluate the integral

Vector Fields

Vector field on $\mathbb{R}^2$ is a function $\vec{F}$ associating to each point $(x,y) \in \mathbb{R}^2$; $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$

→ Conservative if $\vec{F} = \nabla f$ for some function f (a potential)

↳ $f(x,y) = f(0,y) + \int_0^x f_x(t,y)\,dt$

Line Integrals

Line Integral wrt Arc Length: $\int_C f\,ds$ where $f: C \to \mathbb{R}$ $= \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i^*))\Delta s_i$

→ $\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

→ does not depend on parametrization as long as you don't backtrack

Line Integral wrt x & y: $\int_C f\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$

$\int_C f\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$

$\int_C f\,ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$

→ does not depend on parametrization; negative if you go backwards

Line Integral of Vector Fields: $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

→ circulation of $\vec{F}$ along C (extent to which tangent vector & $\vec{F}$ point in same direction)

↳ if vector is ⊥ to tangent everywhere, $\int_C \vec{F}\cdot d\vec{r} = 0$

→ not depend on parametrization; negative in opposite orientation

Fundamental Theorem of Line Integrals: If $\vec{F} = \nabla f$ & C is a parametrized curve from A to B then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

→ If $\vec{F}$ is conservative, $\int_C \vec{F}\cdot d\vec{r}$ does not depend on C (only A & B)

If $\vec{F} (= \nabla f)$ is conservative & C is closed then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = 0$

Clairaut's Theorem: If $\vec{F} = \langle P, Q\rangle$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

A domain in D in $\mathbb{R}^2$ is **simple** if it does not intersect (except @ endpoints)

simply connected if it contains no holes

→ If D is simply connected & $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is a conservative vector field

Jordan Curve Theorem: A simple closed curve C in $\mathbb{R}^2$ is the boundary of a (unique) simply connected region D

Green's Theorem: Let C be a simple closed curve in $\mathbb{R}^2$ & D be the region it bounds & P, Q be differentiable functions on D. Then $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$ where $\oint$ is the positive orientation

Curl & Divergence

curl $\vec{F} = \nabla \times \vec{F}$ $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ = local rotation of $\vec{F}$ (direction & magnitude)

→ If $\vec{F} = \langle P, Q, R\rangle$ is conservative (& partial derivatives exist) then $\nabla \times \vec{F} = 0$

→ If curl $\vec{F} = 0$ & $\vec{F}$ is defined on a simply connected region, then $\vec{F}$ is conservative

div $\vec{F} = \nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

→ measures expansion & compression

Laplacian: $\nabla^2 f = \text{div}(\text{grad } f) = f_{xx} + f_{yy} + f_{zz}$

$$\text{Function} \xrightarrow{\text{gradient}} \text{Vector Field} \xrightarrow{\text{curl}} \text{Vector Field} \xrightarrow{\text{divergence}} \text{Function}$$

Alternate Statement of Green's Theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F}\cdot\vec{n})\,dA$

Parametrized Surface

parametrized curve: $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

velocity vector: $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$

normal vector: $\vec{n} = \vec{r}_u \times \vec{r}_v$ if $\vec{r}_u$ & $\vec{r}_v$ are linearly independent ($\vec{r}_u \times \vec{r}_v \neq 0$)

→ $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle$ & $\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$

Equation for tangent planes: $\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0$

Area of Parametrized Surface: $\iint_D |\vec{r}_u \times \vec{r}_v|\,dA$

Integration wrt Surface Area: $\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$

→ does not depend on parametrization as long as you do not backtrack

Flux across S: $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA = \iint_D (-Pg_x - Qg_y + R)\,dA$

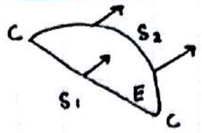
→ depends on orientation

Stokes' Theorem : $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$

Divergence Theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV$

→ $\iint_S (\text{curl}\,\vec{G})\cdot d\vec{S} = \iiint_E \text{div}(\text{curl}\,\vec{G})\,dV = 0$

→ $\iiint_E \text{div}(\vec{F})\,dV = \iint_{S_2}\vec{F}\cdot d\vec{S} - \iint_{S_1}\vec{F}\cdot d\vec{S}$



S is closed surface w/boundary E.