SOLOW GROWTH MODEL

Production Function: $Y(t) = F(K(t), A(t)L(t))$

Assumptions: Freeze Frame Approach

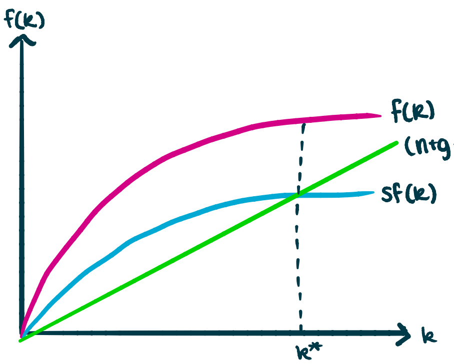
1. Constant returns to scale
   $F(cK, cAL) = cF(K, AL)$, $\forall c \geq 0$
2. Positive diminishing marginal returns
   $F_K > 0$, $F_{KK} < 0$ $\forall$ $AL > 0$
3. Inada Conditions (IC)
   $\lim_{K \to 0} F_K(K, AL) = \infty$ $\quad \forall$ $AL > 0$
   $\lim_{K \to \infty} F_K(K, AL) = 0$
4. Analogous assumptions (2) + (3) wrt AL
5. Intensive form: $y = f(k) = F(k, 1) = \frac{F(K, AL)}{AL}$
   $f'(k) > 0$, $f''(k) < 0$, $f(0) = 0$
   IC: $\lim_{k \to 0} f'(k) = \infty$, $\lim_{k \to \infty} f'(k) = 0$

Assumptions: How K, L, A evolve over time

1. Time is continuous
2. $A(0), L(0), K(0)$ given + strictly positive
3. L + A grow at constant rates (n + g)
   $L(t) = L(0)e^{nt}$, $A(t) = A(0)e^{gt}$ $\Rightarrow$ $\left.\begin{array}{l} \dot{L}(t) = nL(t) \\ \dot{A}(t) = gA(t) \end{array}\right]$
   L + K grow exponentially
4. K grows at a constant rate
   $\dot{K}(t) = sY(t) - \delta K(t)$
   s = savings (exogenous), $0 \leq s \leq 1$
   δ = depreciation
   $n + g + \delta > 0$
   only consump. + invest./savings

Equation Index

$Y(t) = F(K(t), A(t)L(t))$
$\dot{L}(t) = nL(t)$
$\dot{A}(t) = gA(t)$
$\dot{K}(t) = sY(t) - \delta K(t)$
$0 \leq s \leq 1$, $n + g + \delta > 0$
$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t)$
$f'(k^*_{GR}) = (n+g+\delta)k^*$
$c^* = f(k^*) - (n+g+\delta)k^*$
$f'(k^*_{GR}) = (n+g+\delta)k^*$
$\frac{\partial y^*}{\partial s}\left(\frac{s}{y^*}\right) = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$
$\alpha(k^*) = \frac{f'(k^*)k^*}{f(k^*)}$
$k(t) \simeq k^* + e^{-\lambda t}[k(0) - k^*]$
$y(t) \simeq y^* + e^{-\lambda t}[y(0) - y^*]$
$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t)$



Let $\dot{x}(t)$ = derivative of x wrt time.

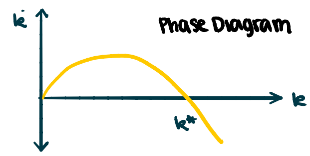
$$\dot{k}(t) = \underbrace{sf(k(t))}_{\text{actual investment per unit of AL}} - \underbrace{(n+g+\delta)k(t)}_{\text{breakeven investment}}$$

breakeven investment = amt of investment to keep k(t) the same

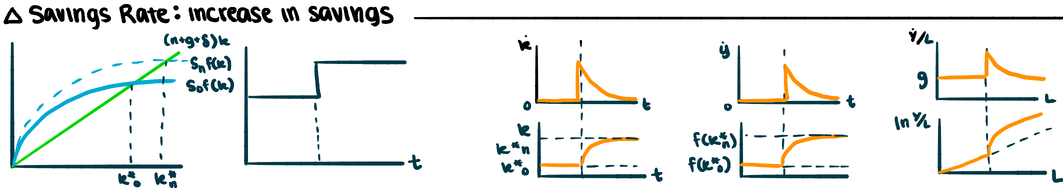
When $k = k^*$ then the economy converges to a balanced growth path (bgp).

var	growth rate
k	0
y	0
L	n
A	g
AL	n+g

var	growth rate
K	n+g
Y	n+g
K/L	g
Y/L	g



Δ Savings Rate: increase in savings



Δing savings has a level effect but not a growth effect

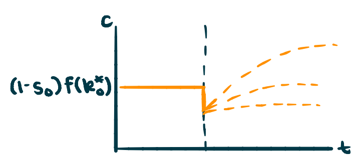
Consumption $= f(k) - sf(k) = f(k) - (n+g+\delta)k$
Δ in c depends on $f'(k)$

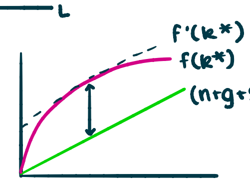
$$\frac{\partial c}{\partial s} = \left[f'(k^*(s,n,g,\delta)) - (n+g+\delta)\right]\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

"Golden Rule" capital stock is the k^* that gives the max of c^* s.t. $f'(k^*_{GR}) = (n+g+\delta)k^*$

$$\frac{\partial y^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)}$$

$$\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$$ = elasticity of output wrt capital at k^* where $\alpha(k^*) = \frac{f'(k^*)k^*}{f(k^*)}$





Speed of Convergence:

$$-\frac{\partial \dot{k}(k)}{\partial k} = \lambda = [1 - \alpha_K(k^*)](n+g+\delta)$$

$k(t) \simeq k^* + e^{-\lambda t}[k(0) - k^*]$
$y(t) \simeq y^* + e^{-\lambda t}[y(0) - y^*]$

Growth Accounting

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)}\dot{K}(t) + \frac{\partial Y(t)}{\partial L(t)}\dot{L}(t) + \frac{\partial Y(t)}{\partial A(t)}\dot{A}(t)$$

$$= \underbrace{\alpha_K(t)}_{\text{elasticity of output wrt capital at t}}\frac{\dot{K}(t)}{K(t)} + \underbrace{\alpha_L(t)}_{\text{elasticity of output wrt labor at t}}\frac{\dot{L}(t)}{L(t)} + \underbrace{R(t)}_{\text{Solow residual}}$$

growth rate of output per worker:

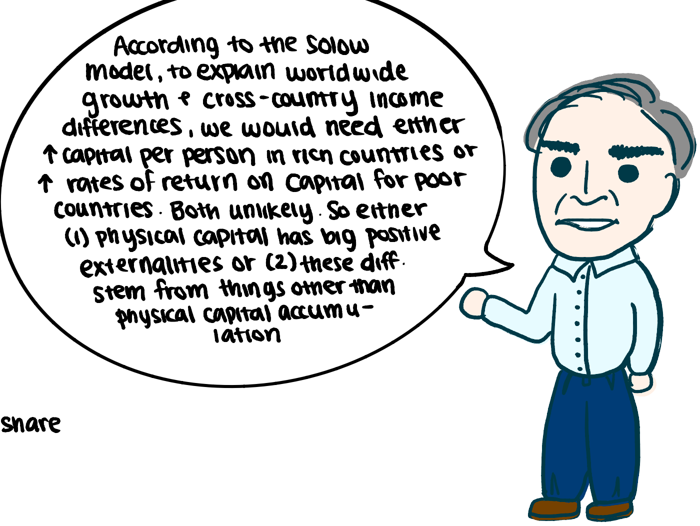
$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t)$$

Rognlie Reading

Picketty: "Central contradiction of capitalism: r > g"

Rognlie Arguments:

1. elasticity of substitution < 1; share inc to capital ↓
   → hard to get big Δ's: ↑ capital accumulation = ↑ capital share
2. endogeneity: mechanical ↑ price driving ↑ share of output
3. evidence on housing: may drive ↑ in K/Y



Kristy Kim 2019

RAMSEY-CASS-KOOPMANS MODEL

Assumptions of the Model (fixed pop/hh ; normalize to one hh)

Firms:

1. Many firms & all competitive
2. A is a public good & growing at rate g
3. Each firm has access to production func. $F(K, AL)$
4. max profits which accrue to hh.

Households:

1. Size of hh grows at rate n
2. Each member of hh supplies 1 unit of L at every point in time
3. owns capital stock ($k(0)$ is given)
   - → rents its capital to firms
   - → no depreciation ($\delta=0$) $\Rightarrow \dot{K}(t) = Y(t) - C(t)L(t)$
4. Size $L(0)e^{nt}$ (supplies labor inelastically)
5. Paths of r & w given ($R(t) = \int_{\tau=0}^{t} r(\tau)\,d\tau$)
5. Objective Function: $U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) L(t)\,dt$

discounted at rate ρ · utility · consumption per person · hh size growing at rate n

→ $u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}$, $\theta > 0$ & $\rho - n - (1-\theta)g > 0$

↳ coeff. of relative risk aversion (CRRA), $\theta = \frac{-u''(C)C}{u'(C)}$

- → small θ = allow consumption to vary over time
- → near zero θ = large swings in consump for diff. in ρ & r
- → $\theta \to 1$: $u(C(t)) = \ln C$
- → elasticity of sub. of C across periods is $1/\theta$

↳ $\rho - n - (1-\theta)g > 0$ ensures lifetime utility does not diverge

Behavior of Firms

real interest rate at t: $r(t) = f'(k(t))$

real wage at t: $W(t) = A(t)[f(k(t)) - k(t)f'(k(t))]$

wage per AL: $w(t) = f(k(t)) - k(t)f'(k(t))$

Behavior of Households

one unit at t_0 saved yields $e^{R(t)}$ at t

→ effect of continuously compounding interest over $[0,t]$

HH Lifetime Budget Constraint:

PV of consumption $\leq D(0)$ + PV of lifetime labor inc.

hh wealth at t (in eq. $D(t) = K(t)$)

PV of lifetime consumption $= \int_{t=0}^{\infty} e^{-R(t)} C(t) L(t)\,dt$

$\leq D(0) + \int_{t=0}^{\infty} e^{-R(t)} L(t) W(t)\,dt$

→ can rewrite as

No Ponzi Game: $\lim_{s\to\infty} e^{-R(s)} D(s) \geq 0$

HH Maximization Problem:

$$\max_{\{c(t)\}} U = B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} \quad \text{s.t.} \begin{cases} \dot{d}(t) = (r(t)-n-g)d(t) + w(t) - c(t) \\ \lim_{t\to\infty} e^{-(R(t)-(n+g)t)} d(t) \geq 0 \\ R(t) = \int_{\tau=0}^{t} r(\tau)\,d\tau \end{cases}$$

where we have $d(t) = \frac{D(t)}{A(t)L(t)}$, $c(t) = \frac{C(t)}{A(t)}$, $B = A(0)^{1-\theta}L(0)$

$\beta = \rho - n - (1-\theta)g\ (>0)$, $w(t) = \frac{W(t)}{A(t)L(t)}$

We can find Euler's Equation: $\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{r(t) - n - g - \beta}{\theta}$

which states consumption rises if the rate of return > future consump. discount rate

↳ otherwise, hh arrange consump to ↑ lifetime utility w/o changing present value of lifetime spending

Hamiltonian

Present Value: $H(k_t, c_t, t) = e^{-\rho t}\underbrace{[v(k_t, c_t, t) + e^{\rho t}\mu_t\, g(k_t, c_t, t)]}_{\text{Current Value, } \tilde{H}}$

discount rate · control · flow utility at t · shadow price, q · costate · $\frac{d\,\text{state}(t)}{dt}$ (i.e. $\dot{k}_t$)

Conditions for Optimality:

1. 1^{st} oo: $\frac{\partial H}{\partial\,\text{control}(t)} = \frac{\partial H}{\partial c_t} = 0\ \forall t$ (cost v. saving at t)
2. 2^{nd} oo: internal consistency (valuing k consistently over time)
   - → PV: $\frac{\partial H}{\partial\,\text{state}(t)} + \dot{\mu}_t = 0 = \frac{\partial H}{\partial k_t} + \dot{\mu}_t$
   - → CV: $\frac{\partial \tilde{H}}{\partial k_t} + \dot{q}_t - \rho q_t = 0$
3. Transversality Condition (TVC): don't do globally stupid stuff "no money left on the table"
   - → PV: $\lim_{t\to\infty} \mu_t k_t \leq 0$
   - → CV: $\lim_{t\to\infty} e^{-\rho t} q_t k_t \leq 0$
4. No Ponzi Game (NPG): can't do impossible stuff
   $\lim_{t\to\infty} e^{-[R(t)+(n+g)t]} k(t) \geq 0$

Social Planner Problem: $\max_{\{c(t)\}} B\int_{-\infty}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta}\,dt$

where $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$, $k(0)$ given, $k(t) \not< 0\ \forall t$

$H(k(t), c(t)) = \frac{Bc(t)^{1-\theta}}{1-\theta} + \mu(t)[f(k(t)) - c(t) - (n+g)k(t)]$

FOC:

1. $\frac{\partial H}{\partial c} = 0 \Rightarrow Bc(t)^{-\theta} - \mu(t) = 0 \Rightarrow \mu(t) = Bc(t)^{-\theta}$
2. $\frac{\partial H}{\partial k} + \dot{q}_t - \rho q_t = 0 \Rightarrow \mu(t)[f'(k(t)) - (n+g)] - \beta\mu(t) + \dot{\mu}(t) = 0$

TVC & Feasibility:

3. $\lim_{t\to\infty} e^{-\rho t} q_t k_t \leq 0 \Rightarrow \lim_{t\to\infty} e^{-\beta t}\mu(t)k(t) \leq 0$
4. $k(0)$ given & $k(t) \not< 0$ (no NPG in closed economy)

From ① $\frac{\dot{\mu}(t)}{\mu(t)} = -\theta\frac{\dot{c}(t)}{c(t)} \Rightarrow \frac{\dot{c}(t)}{c(t)} = \frac{-1}{\theta}\frac{\dot{\mu}(t)}{\mu(t)}$

② $\frac{\dot{\mu}(t)}{\mu(t)} = \beta - [f'(k(t)) - (n+g)]$

Combine to get $\frac{\dot{c}(t)}{c(t)} = -\frac{1}{\theta}(\beta - f'(k(t)) + (n+g))$

$= -\frac{1}{\theta}(\rho + \theta g - f'(k))$

$= \frac{f'(k(t)) - \rho - \theta g}{\theta}$

= so planner's solution is the same as competitive equilibrium, just all hh's at the same time

Constructing the Phase Diagram

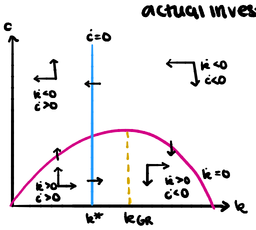
general equilibrium of the model: $r(t) = \frac{\partial F(K(t), A(t)L(t))}{\partial K(t)}$

$= A(t)L(t) f\left(\frac{K(t)}{A(t)L(t)}\right) = f'(k(t))$

$W(t) = f(k(t)) - k(t)f'(k(t))$, $d(t) = k(t)$

We know $\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - g\theta}{\theta} = \frac{f'(k) - n - g - \beta}{\theta}\ \forall t$

$\dot{k}(t) = \underbrace{f(k(t)) - c(t)}_{\text{actual investment}} - \underbrace{(n+g)k(t)}_{\text{breakeven}}$



Why is $k^* < k_{GR}$?

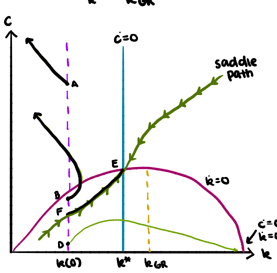
k^*: $f'(k^*) = \rho + \theta g$

k_{GR}: $f'(k_{GR}) = n + g$ $\Rightarrow k^* < k_{GR}$

$\Rightarrow \rho + \theta g > n + g$

$\rho - n - (1-\theta)g > 0 \Rightarrow \beta = 0$

violates TVC



At D, $k\uparrow$ & $c\downarrow$ & capital accum. without consumption

F satisfies all conditions

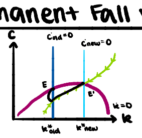
E is at BGP

→ y, k, c constant

savings rate from output invested $1 - c/y = 1 - c^*/f(k^*)$

→ $Y/L = g$ (same as Solow: $Y/L = Y/AL \cdot A$)

Permanent Fall in ρ



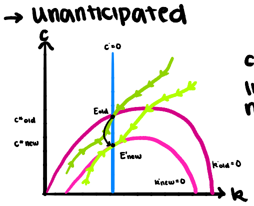
jumps down to new saddle path

→ people are more patient – consume less today & more in the future

Permanent Incr. in Gov't Purchases

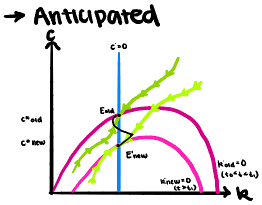
$\dot{k}_t = f(k_t) - c_t - (n+g)k_t - G_t$ $\quad \frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - \rho - \theta g}{\theta}$ $\quad$ * like lump sum taxes: $G_t = T_t$

→ Unanticipated

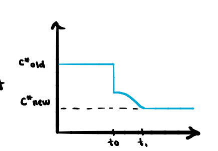


c jumps immed. @ new BGP

→ Anticipated



jump down in new dyn. until you hit the new saddle path ⇒



DIAMOND OVERLAPPING GENERATIONS MODEL

Assumptions: Demographics

1. Time is discrete ($t = 0, 1, 2, \dots$)
2. People live for two periods
3. L_t individuals born in period t
4. $L_{t+1} = (1+n)L_t$: constant exog. growth, $n > -1$
5. Supply 1 unit of labor when young, 0 units when old
6. Only consume savings + interest when old (in period 2)
7. Constant relative risk aversion utility: $U_t = U(C_{1,t}, C_{2,t+1}) = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}$

   where $\theta > 0$ + $\rho > -1$ (positive $C_{2,t+1}$)

Assumptions: Tech + Markets

1. many firms w $Y_t = F(K_t, A_t L_t)$
   → constant returns to scale; Inada conditions
2. $A_{t+1} = (1+g)A_t$, $g > -1$
3. No depreciation
4. markets are competitive
   → no profits; factors earn marg. prod.
   → $r_t = f'(k_t)$
   → $w_t = f(k_t) - k_t f'(k_t)$
5. $A(0) > 0$
   $K(0) > 0$ owned equally by old hh@ $t=0$
6. $k_{t+1} = L_t [w_t A_t - C_{1,t}]$
   → old supply capital, young supply labor
   → get MPK but not b/t $t=1$ + $t=2$

Household Optimization: $\max_{(C_{1,t}, C_{2,t+1})} \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}$ s.t. $C_{1,t} + \frac{1}{1+r_{t+1}} C_{2,t+1} \leq A_t W_t$

→ Relationship b/t real rate of return + discount rate determines whether consumption is increasing or decreasing over time: $\frac{C_{2,t+1}}{C_{1,t}} = \left[\frac{1+r_{t+1}}{1+\rho}\right]^{1/\theta}$

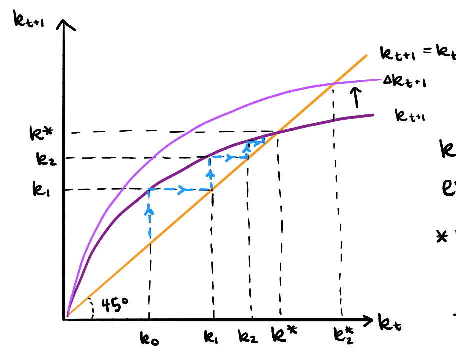
→ Fraction of income saved: $s(r_{t+1}) = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}}$
↳ $\theta\downarrow$, $r\uparrow \Rightarrow s\uparrow$; θ=log ⇒ no response

→ Interest rates determine fraction of consumption in period 1
$C_{1,t} = (1 - s(r_{t+1}))A_t w_t$

Dynamics of the Economy

Fundamental Eq: $k_{t+1} = \frac{1}{(1+n)(1+g)} \underbrace{s(f'(k_{t+1}))}_{t+1 \text{ savings}} \underbrace{(f(k_t) - k_t f'(k_t))}_{t \text{ wage}}$

Special Case: Log Utility + Cobb-Douglas Production

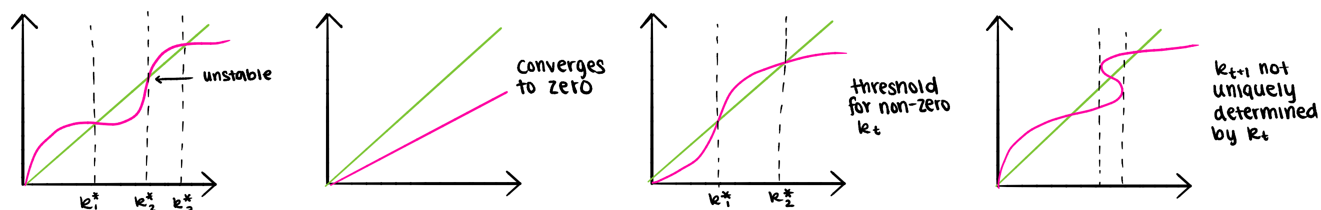


$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)k_t^\alpha$

ex: permanent fall in ρ

* note we can solve for k^*, speed of convergence, show $k^* > k^{GR}$ is possible in this special case

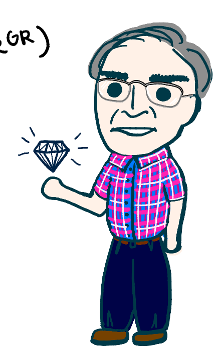
General Cases: Various Possibilities



*eventually go below 45° line

Main Results

1. "Solow messages" still hold
   a. growth of A is the only possible source of long-run growth of Y/L
      → since $k_{t+1} < f(k_t)/[(1+n)(1+g)]$, k settles + cannot grow w/o bound
      → Y/L grows at g
   b. If c.c. inc. diff. are due to diff in k, we get the same quantitative issues
2. Even though the economy is perfectly competitive (etc), equilibrium can be Pareto inefficient
   a. Specifically, economy can converge to a bgp with $k^* > k^{GR}$
      → ∞ agents give planner a way of providing for the consumption of the old that's not available to market
      → known as dynamic inefficiency
3. If there is dynamic inefficiency, gov't can use debts + markets to solve
   ex: suppose on bgp, $k^* > k^{GR}$ then the gov't can
   a. levy a tax on the young in some period to lower their after tax income to what their labor income would be if $k = k^{GR}$
   b. issue debt of amt = (what the young want to save) − (savings that would cause $k_{t+1} = k^{GR}$)
   c. give stuff to the old
   d. roll over debt forever (Ponzi scheme since ∃ ∞ agents)
4. Weird stuff can happen
5. The model can be extended in many ways
6. Anticipation does not matter
   a. what matters in next period is savings + labor unit (not how much you make)



Kristy Kim, 2019

EGGERTSSON-MEHROTA-ROBBINS MODEL

Assumptions

1. OLG where individuals live 3 periods
2. $N_t > 0$ people born in period t
3. each individual born at t has endowment incomes : $0,\ Y^m_{t+1} > 0,\ Y^o_{t+2} \geq 0$
4. individ born at t can borrow B_t but cannot exceed $D_t/(1+r_t)$ where
   - → D_t is exogenous $t > 0$
   - → r_t is the real interest rate from t to $t+1$
   - → $B_t(1+r_t) \leq D_t$
   - → borrowing constraint always binds
5. utility of individ. born at t from $C^m_{t+1} + C^o_{t+2}$ is $U_t = \ln C^m_{t+1} + \beta \ln C^o_{t+2},\ \beta > 0$
   - → note: no utility in young

Consumption at period t

① $C^y_t = \frac{D_t}{1+r_t}$

② $C^m_t = \frac{1}{1+\beta}\left[Y^m_t + \frac{1}{1+r_t} Y^o_{t+1} - D_{t-1}\right]$

→ maximize log utility w/ $Y^m_t + \frac{1}{1+r_t} Y^o_{t+1} - D_{t-1}$

③ $C^o_t = Y^o_t + \frac{D_{t-1} N_{t-1}}{N_{t-2}}$

Equilibrium in Loan Markets at Period t

Loan Demand = Loan Supply

$$\frac{N_t D_t}{1+r_t} = N_{t-1}(Y^m_t - D_{t-1} - C^m_t)$$

Plug in ②

*all terms are exogenous

$$1 + r_t = \frac{1}{Y^m_t - D_{t-1}}\left[\frac{1+\beta}{\beta}(1+n_t)D_t + \frac{1}{\beta} Y^o_{t+1}\right] \quad \text{where } 1+n_t = \frac{N_t}{N_{t-1}}$$

Example

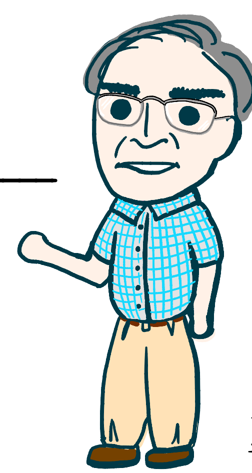
Let $\beta = 1,\ n = 0,\ Y^m_{t+1} = 1,\ Y^o_{t+2} = 0,\ D_t = D_{t-1} = D < 1$

Then we have $1 + r_t = \frac{1}{1-D}(2D) \Rightarrow r_t = \frac{3D-1}{1-D}$

- → As $D\uparrow$, people borrow a lot $\Rightarrow r\uparrow$
- → As $D\downarrow$, many want to borrow $\Rightarrow r\downarrow$
- → If we ↑ borrowing in the past, ↓ loan supply today

Motivation of EMR

- → Unlike Ramsey, we show 3 periods to allow for borrowing
  - ↳ Also allow for flexible price rates (low or negative)
- → Want to show secular stagnation (equilibrium real rate < 0) for an extended time or permanent



Kristy Kim, 2019

ENDOGENOUS GROWTH MODEL

Assumptions

1. Fraction a_L of L used in R&D] exogenous & constant
2. Fraction a_K of K used in R&D
3. $Y(t) = [(1-a_K)K(t)]^{\alpha}[A(t)(1-a_L)L(t)]^{1-\alpha}$
   $\rightarrow 0<a_L<1,\ 0<a_K<1,\ 0\leq\alpha<1$
4. $\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma}A(t)^{\theta}$
   $\rightarrow B>0$ (shift parameter), $\beta\geq 0,\ \gamma\geq 0$
5. $\dot{K}(t) = sY(t),\ s>0,\ s<1,\ \delta=0$
6. $\dot{L}(t) = nL(t),\ n\geq 0$
7. $K(0), L(0), A(0)$ all given & >0
8. Continuous time

Model Without Capital

$Y(t) = A(t)(1-a_L)L(t),\quad 0<a_L<1$

$\dot{A}(t) = B[a_L L(t)]^{\gamma}A(t)^{\theta},\quad B>0,\ \gamma\geq 0$

$\frac{Y(t)}{L(t)} = (1-a_L)A(t)$

Focus: $\frac{\dot{A}(t)}{A(t)} = g_A(t)$

$g_A(t) = B[a_L L(t)]^{\gamma}A(t)^{\theta-1}$

$\frac{\dot{g}_A(t)}{g_A(t)} = \gamma n + (\theta-1)g_A(t)$

Case 1: $\theta<1$

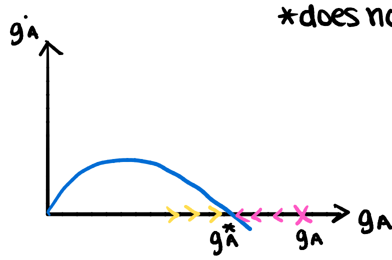
If $\dot{g}_A = 0$ then $g_A>0$

g_A^* satisfies $\gamma n g_A^* + (\theta-1)g_A^{*2} = 0$

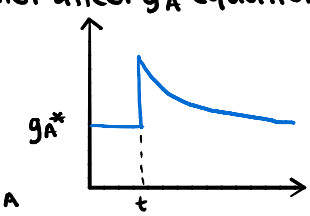
$\Rightarrow g_A^* = \frac{\gamma}{1-\theta}n$

ex: start at eq., permanent ↑ a_L

$\Rightarrow$ jumps $\rightarrow$ dynamics back to g_A^*



*does not affect $\dot{g}_A$ equation, but g_A

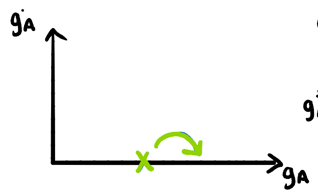


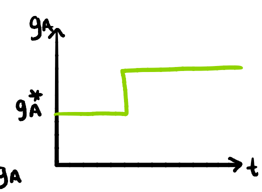
Case 2: $\theta=1, n=0$

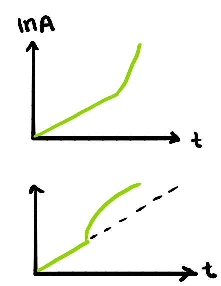
We have $g_A(t) = B[a_L \bar{L}]^{\gamma}$

$\dot{g}_A(t) = \gamma n g_A(t)$

(linear growth model or AK model)



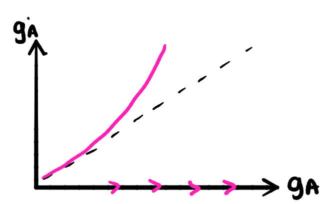




Case 3: $\theta>1$

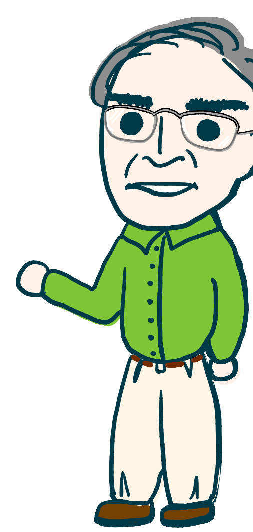
Production of new knowledge rises more than proportionally w/existing stock

*or if $\theta=1$ & $n>0$, g_A grows w/o bound



Key determinant of LR growth: returns to scale to produced factors

$\rightarrow$ increasing, decreasing, constant



Kristy Kim, 2019

(COUSIN) ROMER MODEL
ENDOGENOUS GROWTH MODEL

Assumptions
1. How ideas & labor combine to produce output at a point in time
   - → imperfect substitutes = downward sloping demand for ideas
   - → At time t, inputs available are $[0, A(t)]$
   - → Output at t: $Y=[\int_{i=0}^{A} L(i)^\phi di]^{1/\phi}$, $0<\phi<1$ * removed t from argument for now

     amt of labor used at t to produce 1 unit

     MP of $L(i) = \frac{1}{\phi}[\int_{i=0}^{A} L(i)^\phi di]^{\frac{1}{\phi}-1}\phi L(i)^{\phi-1}$ diminishing marginal product of L(i)
   - → suppose there are equal amts in all inputs s.t. $L(i)=\frac{L_Y}{A}$

     $Y=[A(\frac{L_Y}{A})^\phi]^{\frac{1}{\phi}} = A^{\frac{1-\phi}{\phi}}L_Y$ * CRS to L_Y given A; output is incr. in A given L_Y
2. Implications for demand for the use of an idea
   - → think of competitive output producer who can buy L(i) at price p(i)
   - → normalize p of final output to 1
   - → profit max: produce until $P\cdot MP(i)=P(i) \Rightarrow \Gamma L(i)^{\phi-1}=P(i) \Rightarrow L(i)=[\frac{P(i)}{\Gamma}]^{\frac{1}{\phi-1}} = B\cdot P(i)^{-\frac{1}{1-\phi}}$ constant   elasticity of demand
3. Market Structure
   - a. Given stock of ideas, how output is produced
     - → exclusive right to use idea held by monopolists who charge ↑MC (fixed P(i) per unit)
     - → output producers are competitive (zero π in eq)
     - → competitive labor market (going wage denoted w & workers produce their input)
     - → final goods are competitive
     - → Thm: $P(i)^* = \frac{\mu}{\mu-1}MC(i)$ where μ = abs. value of elasticity of demand

       $=\frac{1}{1-\phi}(\frac{1}{1-\phi}-1)^{-1}w(t) = \frac{1}{\phi}w(t)$
   - b. The production of new ideas
     - → free entry w/production func: $\dot A(t) = BL_A(t)A(t)$, $B>0$ * θ=1, δ=1 case
     - → MC of producing an idea: $\frac{w(t)}{BA(t)}$ (assume A(t) is used for free)
     - → PV of profits from creating an idea at t $\begin{cases} = \frac{w(t)}{BA(t)} & \text{if } L_A(t)>0 \\ \le \frac{w(t)}{BA(t)} & \text{if } L_A(t)=0\end{cases}$
4. Rest of the Model
   - → Population fixed at $\bar L=0$
   - → Labor market eq: $L_A(t)+L_Y(t)=\bar L$ $Y(t)=\bar L c(t)$
   - → Representative individuals have obj. func: $\int_{t=0}^\infty e^{-\rho t}\ln C(t)dt$, $\rho>0$
   - → Each person owns an equal share of patents on initial A(t) (given)

Solving the Model
0. Look for an equilibrium with $L_A^*(t)$ constant
1. Preliminary result about symm. among inputs
   - → each monopolist faces same demand curve, elasticity of demand charges same p so $L(i,t)=\frac{L_Y(t)}{A(t)} = \frac{\bar L - L_A}{A(t)}$ * use equal amt of each input
2. For a given L_A, find:
   - a. Growth rate of economy: $\frac{\dot A(t)}{A(t)} = BL_A$
     - → Recall if all $L(i,t)=\frac{L_Y(t)}{A(t)}$, $Y(t)=A(t)^{\frac{1-\phi}{\phi}}L_Y(t) \Rightarrow g_Y = \frac{1-\phi}{\phi}g_A(t) = \frac{1-\phi}{\phi}BL_A$
   - b. Interest rate, r
     - → HH euler eq: $\frac{\dot c(t)}{c(t)} = r(t)-\rho$

       general eq: $c(t)=Y(t) \Rightarrow \frac{\dot c(t)}{c(t)} = \frac{\dot Y(t)}{Y(t)}$

       $r(t)=\rho+\frac{1-\phi}{\phi}BL_A$ * constant
   - c. Flow of π to a monopolist at given t from an input

     $(P-MC)Q = [\frac{1}{\phi}w(t) - w(t)]\frac{L_Y}{A(t)} \Rightarrow \frac{1-\phi}{\phi}w(t)\frac{\bar L - L_A}{A(t)}$
   - d. PV of π from creating an idea at t $=\int_{\tau=0}^\infty e^{-r\tau}\frac{1-\phi}{\phi}\frac{\bar L-L_A}{A(t+\tau)}w(t+\tau)d\tau$
     - → everything grows at constant rate (w grows at same rate as output)
3. Find L_A s.t. MC of producing an idea = PV of profits

   $\int_{\tau=0}^\infty e^{-r\tau}\frac{1-\phi}{\phi}w(t)e^{g_w\tau}\frac{\bar L-L_A}{A(t)e^{g_A\tau}}d\tau = \frac{1-\phi}{\phi}w(t)\frac{\bar L-L_A}{A(t)}e^{-r\tau}e^{g_w\tau}e^{-g_A\tau}$

   $\Rightarrow \frac{1-\phi}{\phi}\frac{w(t)}{A(t)}(\bar L-L_A)\frac{1}{\rho+BL_A} = \frac{w(t)}{BA(t)}$ if there is innovation

   $L_A = \max\{(1-\phi)\bar L - \frac{\phi\rho}{B}, 0\}$

   $\frac{\dot Y(t)}{Y(t)} = \max\{\frac{(1-\phi)^2}{\phi}B\bar L - (1-\phi)\rho, 0\}$

$\rho\uparrow \Rightarrow L_A\downarrow, \dot Y/Y\downarrow$
$\phi\uparrow \Rightarrow \dot Y/Y\downarrow$
$B\uparrow \Rightarrow \dot Y/Y\uparrow, L_A\uparrow$
$\bar L\uparrow \Rightarrow \dot Y/Y\uparrow$

Externalities for R&D
1. Consumer-surplus effect] cancels out
2. Business-stealing effect
3. R&D effects

Social Planner (max U of rep hh): $L_A^{opt} = \max\{\bar L - \frac{1-\phi}{\phi}\rho B, 0\}$

$L_A^{EQ} = (1-\phi)L_A^{opt}$

Readings
Hall & Jones

Approach 1: Are there permanent Δ's in growth rates? Are growth rates stationary?

Approach 2: Has growth been rising? If so, is it large?

Simple linear growth model:

$Y(t)=BK(t)$, $B>0$

$\frac{\dot Y(t)}{Y(t)} = \frac{\dot K(t)}{K(t)} = \frac{s(t)Y(t)-\delta K(t)}{Y(t)/B}$

$=Bs - \delta\frac{Bk}{Y} = Bs(t)+\delta$

Conclusion: Abandon linear & incr. returns to scale to produced factors model: focus more on models like "Case 1" "semi-endogenous"

Jones & Kim

Pareto distributions

Conditional on being above x_0

$1-CDF(\ln x | \ln x > \ln x_0)$

$=\exp\{-[\ln x - \ln x_0]\}$

$=\exp\{\ln[\frac{x}{x_0}]^{\xi}\} = (\frac{x}{x_0})^{\xi}$

1. High ξ = dies faster = more equal
2. If ξ low, mean is infinite
3. Pareto distrib. are fractal
4. But many things are just Pareto in nature

Toy model:

Entrepreneurs
- → initial inc: y_0
- → inc. conditional on staying an entr. grows at $\mu>0$
- → prob. δ per unit of time, revert to be a regular worker
- → # of entr. constant
- → only stationary distrib. of entr. inc. is Pareto $(Y/Y_0)^{-\xi}$
- → log inc: exponential above $\ln Y_0$, decay parameter δ/μ