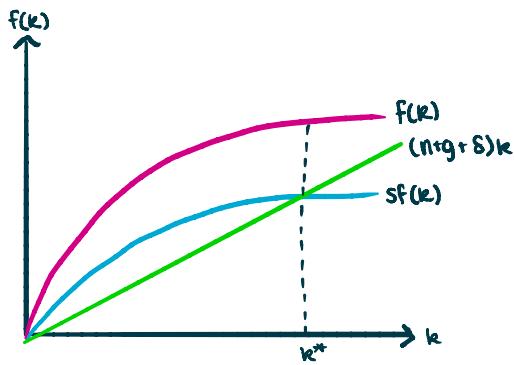


# SOLOW GROWTH MODEL

Production Function:  $Y(t) = F(K(t), A(t)L(t))$

Assumptions: Freeze Frame Approach

1. Constant returns to scale  
 $F(cK, cAL) = cF(K, AL)$ ,  $\forall c \geq 0$
2. Positive diminishing marginal returns  
 $F_K > 0$ ,  $F_{KK} < 0$  &  $AL > 0$
3. Inada Conditions (IC)  
 $\lim_{K \rightarrow 0} F_K(K, AL) = \infty$  &  $AL > 0$   
 $\lim_{K \rightarrow \infty} F_K(K, AL) = 0$
4. Analogous assumptions (2) + (3) wrt AL
5. Intensive form:  $y = f(k) = F(k, 1) = \frac{F(k, AL)}{AL}$   
 $f'(k) > 0$ ,  $f''(k) < 0$ ,  $f(0) = 0$   
IC:  $\lim_{k \rightarrow 0} f'(k) = \infty$ ,  $\lim_{k \rightarrow \infty} f'(k) = 0$



Assumptions: How K, L, A evolve over time

1. Time is continuous
2.  $A(0), L(0), K(0)$  given & strictly positive
3.  $L + A$  grow at constant rates ( $n + g$ )  
 $L(t) = L(0)e^{nt}$   $\Rightarrow L(t) = nL(t)$   
 $A(t) = A(0)e^{gt}$   $\Rightarrow A(t) = gA(t)$
4.  $K$  grows at a constant rate  
 $\dot{K}(t) = sY(t) - \delta K(t)$   
 $s = \text{savings (exogenous)}, 0 \leq s \leq 1$   
 $\delta = \text{depreciation}$   
 $n + g + \delta > 0$   
only consump. + invest. / savings

Let  $\dot{x}(t) = \text{derivative of } x \text{ wrt time.}$

$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t)$$

actual investment per unit of AL      break-even investment per unit of AL

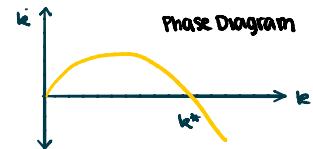
Equation Index

$$\begin{aligned} Y(t) &= F(K(t), A(t)L(t)) \\ L(t) &= nL(t) \\ A(t) &= gA(t) \\ K(t) &= sY(t) - \delta K(t) \\ 0 \leq s \leq 1, n+g+\delta > 0 \\ k(t) &= sf(k(t)) - (n+g+\delta)k(t) \\ f'(k^*) &= (n+g+\delta)k^* \\ C^* &= f(k^*) - (n+g+\delta)k^* \\ f'(k^*) &= (n+g+\delta)k^* \\ \frac{\partial y^*}{\partial s} \left(\frac{s}{y^*}\right) &= \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)} \\ d(k^*) &= \frac{f'(k^*)}{f(k^*)}k^* \\ k(t) &\approx k^* + e^{-\lambda t}[k(0) - k^*] \\ y(t) &\approx y^* + e^{-\lambda t}[y(0) - y^*] \\ \frac{y(t)}{Y(t)} \cdot \frac{L(t)}{L(E)} &= \alpha_k(t) \left[ \frac{\dot{k}(t)}{k(t)} - \frac{\dot{L}(t)}{L(t)} \right] + R(t) \end{aligned}$$

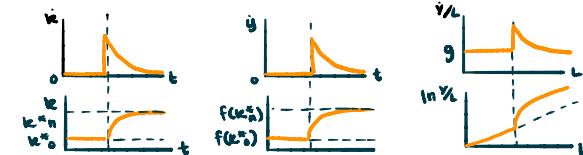
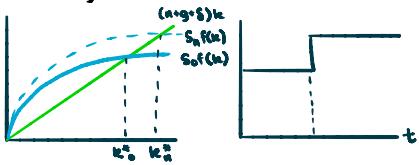
When  $k = k^*$  then the economy converges to a balanced growth path (bfp).

var	growth rate
$k$	0
$y$	0
$L$	$n$
$A$	$g$
$AL$	$n+g$

var	growth rate
$K$	$n+g$
$Y$	$n+g$
$\frac{K}{L}$	$g$
$\frac{Y}{L}$	$g$



$\Delta$  Savings Rate: Increase in savings



$$\text{Consumption} = f(k) - sf(k) = f(k) - (n+g+\delta)k$$

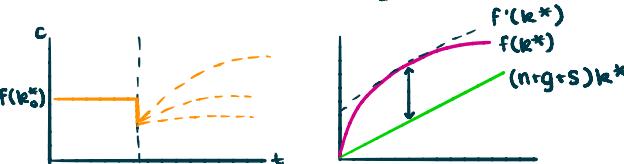
$\Delta$  in C depends on  $f'(k)$

$$\frac{\partial C}{\partial s} = [f'(k^*(s, n, g, s)) - (n+g+\delta)] \frac{\partial k^*(s, n, g, \delta)}{\partial s}$$

"Golden Rule" capital stock is the  $k^*$  that gives the max of  $C^*$  s.t.  $f(k^*) = (n+g+\delta)k^*$

$$\frac{\partial y^*}{\partial s} = \frac{f'(k^*)}{(n+g+\delta) - sf'(k^*)}$$

$$\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)} = \text{elasticity of output wrt capital at } k^*$$



Speed of Convergence:

$$-\frac{\partial \dot{k}(t)}{\partial k} = \lambda = [1 - \alpha_k(k^*)](n+g+\delta)$$

$$k(t) \approx k^* + e^{-\lambda t}[k(0) - k^*]$$

$$y(t) \approx y^* + e^{-\lambda t}[y(0) - y^*]$$

Growth Accounting

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)} \dot{K}(t) + \frac{\partial Y(t)}{\partial L(t)} \dot{L}(t) + \frac{\partial Y(t)}{\partial A(t)} \dot{A}(t)$$

$$= \alpha_K(t) \frac{\dot{K}(t)}{K(t)} + \alpha_L(t) \frac{\dot{L}(t)}{L(t)} + R(t)$$

elasticity of output wrt capital at t      elasticity of output wrt labor at t      Solow residual

growth rate of output per worker:

$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t) \left[ \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} \right] + R(t)$$

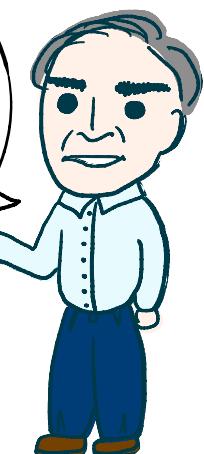
Rognlie Reading

Picketty: "Central contradiction of capitalism:  $r > g$ "

Rognlie Arguments:

1. elasticity of substitution  $< 1$ ; share inc to capital +  
→ hard to get big  $\Delta$ 's: ↑ capital accumulation = ↑ capital share
2. endogeneity: mechanical ↑ price driving ↑ share of output
3. evidence on housing: may drive ↑ in  $k/y$

According to the Solow model, to explain worldwide growth & cross-country income differences, we would need either ↑ capital per person in rich countries or ↑ rates of return on Capital for poor countries. Both unlikely. So either (1) physical capital has big positive externalities or (2) these diff. stem from things other than physical capital accumulation





# DIAMOND OVERLAPPING GENERATIONS MODEL

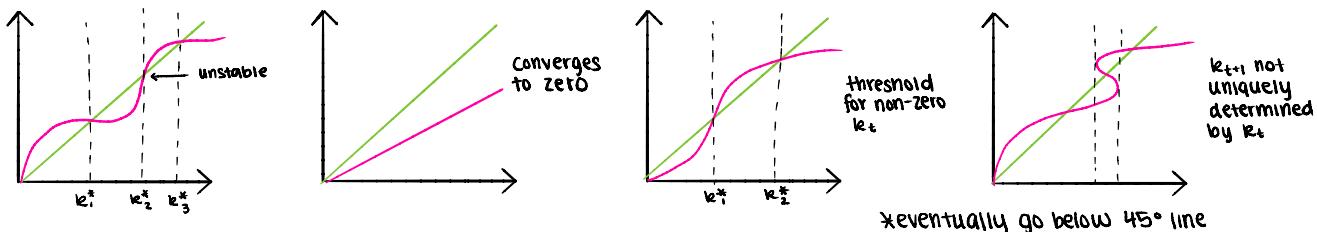
## Assumptions: Demographics

1. Time is discrete ( $t=0, 1, 2, \dots$ )
2. People live for two periods
3.  $L_t$  individuals born in period  $t$
4.  $L_{t+1} = (1+n)L_t$ : constant exog. growth,  $n > 1$
5. Supply 1 unit of labor when young, 0 units when old
6. Only consume savings + interest when old (in period 2)
7. Constant relative risk aversion utility:  
 $U_t = U(C_{1,t}, C_{2,t+1}) = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2,t+1}^{1-\theta}}{1-\theta}$   
 where  $\theta > 0$  +  $\rho > 1$  (positive  $C_{2,t+1}$ )

## Assumptions: Tech + Markets

1. Many firms w/  $Y_t = F(K_t, A_t L_t)$   
 → constant returns to scale; Inada conditions
2.  $A_{t+1} = (1+g)A_t$ ,  $g > -1$
3. No depreciation
4. Markets are competitive  
 → no profits; factors earn marg. prod.  
 $\rightarrow r_t = f'(K_t)$   
 $\rightarrow w_t = f(K_t) - k_r f'(K_t)$
5.  $A(0) > 0$   
 $K(0) > 0$  owned equally by old mn@  $t=0$
6.  $k_{t+1} = L_t [W_t + A_t - C_{1,t}]$   
 → old supply capital, young supply labor  
 → get MPK but not b/t  $t=1$  +  $t=2$

## General Cases: Various Possibilities



## Main Results

1. "Solow messages" still hold
  - growth of  $A$  is the only possible source of long-run growth of  $Y/L$   
 $\rightarrow$  Since  $k_{t+1} < f(k_t)/[(1+n)(1+g)]$ ,  $k_t$  settles + cannot grow w/o bound  
 $\rightarrow Y/L$  grows at  $g$
  - If c.c. inc. diff. are due to diff in  $k$ , we get the same quantitative issues
2. Even though the economy is perfectly competitive (etc), equilibrium can be Pareto inefficient
  - Specifically, economy can converge to a bgp with  $k^* > k^{GR}$   
 $\rightarrow \infty$  agents give planner a way of providing for the consumption of the old that's not available to market  
 $\rightarrow$  known as dynamic inefficiency
3. If there is dynamic inefficiency, gov't can use debts + markets to solve
 

Ex: suppose on bgp,  $k^* > k^{GR}$  then the gov't can

  - levy a tax on the young in some period to lower their after tax income to what their labor income would be if  $k = k^{GR}$
  - issue debt of amt = (what the young want to save) - (savings that would cause  $k_{t+1} = k^{GR}$ )
  - give stuff to the old
  - roll over debt forever (Ponzi scheme since  $\exists \infty$  agents)
4. Weird stuff can happen
5. The model can be extended in many ways
6. Anticipation does not matter
  - what matters in next period is savings + labor unit (not how much you make)



# EGGERTSSON - MEHROTA - ROBBINS MODEL

Assumptions

1. OLG where individuals live 3 periods
2.  $N_t > 0$  people born in period  $t$
3. each individual born at  $t$  has endowment incomes:  $0, Y_{t+1}^m > 0, Y_{t+2}^o \geq 0$
4. individ. born at  $t$  can borrow  $D_t$  but cannot exceed  $D_t/(1+r_t)$  where
  - $D_t$  is exogenous  $t > 0$
  - $r_t$  is the real interest rate from  $t$  to  $t+1$
  - $B_t(1+r_t) \leq D_t$
  - borrowing constraint always binds
5. utility of individ. born at  $t$  from  $C_{t+1}^m + C_{t+2}^o$  is  $U_t = \ln C_{t+1}^m + \beta \ln C_{t+2}^o, \beta > 0$ 
  - note: no utility in young

Consumption at period  $t$

$$\textcircled{1} \quad C_t^Y = \frac{D_t}{1+r_t}$$

$$\textcircled{2} \quad C_t^m = \frac{1}{1+\beta} [y_t^m + \frac{1}{1+r_t} Y_{t+1}^o - D_{t+1}]$$

→ maximize log utility  $\ln C_t^m + \frac{1}{1+r_t} Y_{t+1}^o - D_{t+1}$

$$\textcircled{3} \quad C_t^o = Y_t^o + \frac{D_{t+1} N_{t+1}}{N_{t+2}}$$

Equilibrium in Loan Markets at Period  $t$

Loan Demand = Loan Supply

$$\frac{N_t D_t}{1+r_t} = N_{t+1} (Y_t^m - D_{t+1} - C_t^m) \quad \text{Plug in } \textcircled{2} \quad * \text{all terms are exogenous}$$

$$1+r_t = \frac{1}{y_t^m - D_{t+1}} \left[ \frac{1+\beta}{\beta} (1+n_t) D_t + \frac{1}{\beta} Y_{t+1}^o \right] \quad \text{where } 1+n_t = \frac{N_t}{N_{t+1}}$$

Example

Let  $\beta=1, n=0, Y_{t+1}^m=1, Y_{t+2}^o=0, D_t=D_{t+1}=D < 1$

$$\text{Then we have } 1+r_t = \frac{1}{1-D} (2D) \Rightarrow r_t = \frac{3D-1}{1-D}$$

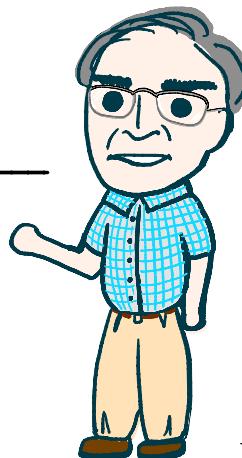
→ As  $D \uparrow$ , people borrow a lot  $\Rightarrow r \uparrow$

→ As  $D \downarrow$ , many want to borrow  $\Rightarrow r \downarrow$

→ If we  $\uparrow$  borrowing in the past,  $\downarrow$  loan supply today

Motivation of EMR

- Unlike Ramsey, we show 3 periods to allow for borrowing
  - ↳ Also allow for flexible price rates (low or negative)
- Want to show secular stagnation (equilibrium real rate  $< 0$ ) for an extended time or permanent



# ENDOGENOUS GROWTH MODEL

## Assumptions

1. Fraction  $\alpha_L$  of  $L$  used in R&D ] exogenous
2. Fraction  $\alpha_K$  of  $K$  used in R&D ]  $\gamma$  constant
3.  $Y(t) = [(1-\alpha_K)K(t)]^\alpha [A(t)(1-\alpha_L)L(t)]^{1-\alpha}$   
 $\rightarrow 0 < \alpha_L < 1, 0 < \alpha_K < 1, 0 \leq \alpha < 1$
4.  $A(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta$   
 $\rightarrow B > 0$  (shift parameter),  $\beta \geq 0, \gamma \geq 0$
5.  $K'(t) = S Y(t), S > 0, S < 1, \delta = 0$
6.  $L'(t) = nL(t), n \geq 0$
7.  $K(0), L(0), A(0)$  all given  $\neq 0$
8. Continuous time

## Model Without Capital

$$Y(t) = A(t)(1-\alpha_L)L(t), 0 < \alpha_L < 1$$

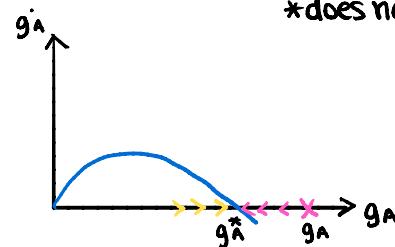
$$A'(t) = B[\alpha_L L(t)]^\gamma A(t)^\theta, B > 0, \gamma \geq 0$$

$$\frac{Y(t)}{L(t)} = (1-\alpha_L)A(t)$$

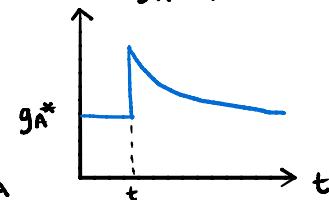
Focus:  $\frac{A'(t)}{A(t)} = g_A(t)$   
 $g_A(t) = B[\alpha_L L(t)]^\gamma A(t)^{\theta-1}$   
 $\frac{g_A'(t)}{g_A(t)} = \gamma n + (\theta-1)g_A(t)$

## Case 1: $\theta < 1$

If  $g_A' = 0$  then  $g_A > 0$   
 $g_A^*$  satisfies  $\gamma n g_A^* + (\theta-1)g_A^{*2} = 0$   
 $\Rightarrow g_A^* = \frac{\gamma}{1-\theta} n$   
ex: Start at eq., permanent  $\uparrow \alpha_L$   
 $\Rightarrow$  jumps  $\rightarrow$  dynamics back to  $g_A^*$

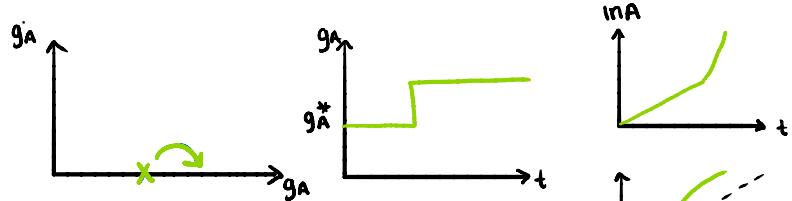


\*does not affect  $g_A'$  equation, but  $g_A$



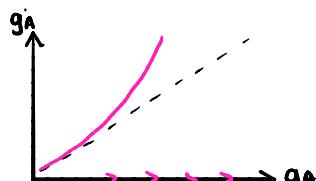
## Case 2: $\theta = 1, n = 0$

We have  $g_A(t) = B[\alpha_L L(t)]^\gamma$   
 $g_A'(t) = \gamma n g_A(t)$   
(linear growth model or AK model)

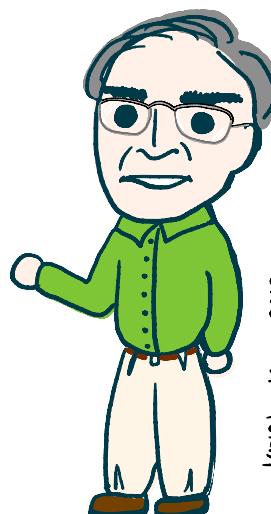


## Case 3: $\theta > 1$

Production of new knowledge rises more than proportionally w/existing stock  
\*or if  $\theta = 1 - n > 0, g_A$  grows w/o bound



Key determinant of LR growth: returns to scale to produced factors  
 $\rightarrow$  increasing, decreasing, constant



# (COUSIN) ROMER MODEL

## ENDOGENOUS GROWTH MODEL

### Assumptions

- How ideas + labor combine to produce output at a point in time
  - imperfect substitutes = downward sloping demand for ideas
  - At time  $t$ , inputs available are  $[0, A(t)]$
  - Output at  $t$ :  $Y = \left[ \sum_{i=0}^A L(i)^{\phi} d_i \right]^{\frac{1}{1-\phi}}$ ,  $0 < \phi < 1$  \* removed  $t$  from argument for now
  - MP of  $L(i)$ :  $\frac{\partial Y}{\partial L(i)} = \frac{1}{1-\phi} \left[ \sum_{j=0}^A L(j)^{\phi} d_j \right]^{\frac{1}{1-\phi}} \phi L(i)^{\phi-1}$  diminishing marginal product of  $L(i)$
  - Suppose there are equal amounts in all inputs s.t.  $L(i) = \frac{L_y}{A}$
  - $Y = \left[ A \left( \frac{L_y}{A} \right)^{\phi} \right]^{\frac{1}{1-\phi}} = A^{\frac{1-\phi}{\phi}} L_y$  \* CRS to  $L_y$  given  $A$ ; output is incr. in  $A$  given  $L_y$
- Implications for demand for the use of an idea
  - think of competitive output producer who can buy  $L(i)$  at price  $p(i)$  constant elasticity of demand
  - normalize  $p$  of final output to 1
  - profit max: produce until  $P \cdot MPC(i) = p(i) \Rightarrow \Gamma L(i)^{\phi-1} = p(i) \Rightarrow L(i) = \left[ \frac{p(i)}{\Gamma} \right]^{\frac{1}{\phi-1}} = B \cdot p(i)^{-\frac{1}{\phi-1}}$
- Market Structure
  - Given stock of ideas, how output is produced
    - exclusive right to use idea held by monopolists who charge  $\uparrow MC$  (fixed  $p(i)$  per unit)
    - output producers are competitive (zero  $\Pi$  in eq)
    - competitive labor market (going wage denoted  $w$  + workers produce their input)
    - final goods are competitive
    - Thm:  $p(i)^* = \frac{1}{\mu-1} MC(i)$  where  $\mu = \text{abs. value of elasticity of demand}$
    - $= \frac{1}{1-\phi} \left( \frac{1}{1-\phi} - 1 \right)^{-1} w(t) = \frac{1}{\phi} w(t)$
  - The production of new ideas
    - free entry w/ production func:  $A(t) = BL_A(t)A(t)$ ,  $B > 0$  \*  $\theta=1, S=1$  case
    - MC of producing an idea:  $\frac{w(t)}{BA(t)}$  (assume  $A(t)$  is used for free)
    - PV of profits from creating an idea at  $t$ 

$$\begin{cases} \frac{w(t)}{BA(t)} & \text{if } LA(t) > 0 \\ \leq \frac{w(t)}{BA(t)} & \text{if } LA(t) = 0 \end{cases}$$
- Rest of the Model
  - Population fixed at  $L=0$
  - Labor market eq:  $LA(t) + L_y(t) = L$   $\Rightarrow Y(t) = L \cdot C(t)$
  - Representative individuals have obj. func:  $\int_{t=0}^{\infty} e^{-rt} \ln C(t) dt$ ,  $r > 0$
  - Each person owns an equal share of patents on initial  $A(t)$  (given)

### Solving the Model

- Look for an equilibrium with  $L_y(t)$  constant
- Preliminary result about symm. among inputs
  - each monopolist faces some demand curve, elasticity of demand charges same  $p$  so  $L(i,t) = \frac{L_y(t)}{A(t)} = \frac{L-L_A}{A(t)}$  \* use equal amt of each input
- For a given  $L_A$ , find:
  - Growth rate of economy:  $\frac{A(t)}{A(t)} = BL_A$ 
    - Recall if all  $L(i,t) = \frac{L_y(t)}{A(t)}$ ,  $Y(t) = A(t)^{\frac{1-\phi}{\phi}} L_y(t) \Rightarrow g_Y = \frac{1-\phi}{\phi} g_A(t) = \frac{1-\phi}{\phi} BL_A$
  - Interest rate,  $r$ 
    - HT Euler eq:  $\frac{c(t)}{c(t)} = r(t) - p$
    - general eq:  $c(t) = Y(t) \Rightarrow \frac{c(t)}{c(t)} = \frac{Y(t)}{Y(t)} = r(t) = p + \frac{1-\phi}{\phi} BL_A$  \* constant
  - Flow of  $\Pi$  to a monopolist at given  $t$  from an input
 
$$(P-MC)Q = \left[ \frac{1}{\phi} w(t) - w(t) \right] \frac{L_y}{A(t)} \Rightarrow \frac{1-\phi}{\phi} w(t) \frac{L-L_A}{A(t)}$$
  - PV of  $\Pi$  from creating an idea at  $t$  =  $\int_{t=0}^{\infty} e^{-rt} \frac{1-\phi}{\phi} \frac{L-L_A}{A(t)} w(t+t) dt$ 
    - everything grows at constant rate ( $w$  grows at same rate as output)
- Find  $L_A$  s.t.  $MC$  of producing an idea = PV of profits
 
$$\int_{t=0}^{\infty} e^{-rt} \frac{1-\phi}{\phi} w(t) e^{g_w t} \frac{L-L_A}{A(t)} dt = \frac{1-\phi}{\phi} w(t) \frac{L-L_A}{A(t)} e^{-rt} e^{g_w t} e^{-g_A t}$$

$$\Rightarrow \frac{1-\phi}{\phi} \frac{w(t)}{A(t)} \frac{1}{P-BL_A} = \frac{w(t)}{BA(t)} \text{ if there is innovation}$$

$$L_A = \max \left\{ (1-\phi)L - \frac{(1-\phi)}{B} P, 0 \right\}$$

$$\frac{Y(t)}{Y(t)} = \max \left\{ \frac{(1-\phi)^2}{\phi} BL - (1-\phi)P, 0 \right\}$$

### Externalities for R&D

- Consumer-surplus effect
  - Business-stealing effect
  - R&D effects
- cancels out

Social Planner (max  $U$  of rep hh):  $L_A^{opt} = \max \left\{ L - \frac{1-\phi}{\phi} PB, 0 \right\}$

### Readings

#### Hall & Jones

Approach 1: Are there permanent  $\Delta's$  in growth rates? Are growth rates stationary?

Approach 2: Has growth been rising? If so, is it large?

Simple linear growth model:

$$Y(t) = BK(t), B > 0$$

$$\frac{Y(t)}{Y(t)} = \frac{K(t)}{K(t)} = \frac{S(t)Y(t) - \delta K(t)}{Y(t)/B}$$

$$= BS - \delta \frac{Bk}{Y} = Bs(t) + \delta$$

Conclusion: Abandon linear + incr. returns to scale to produced factors model: focus more on models like "Case 1"

"semi-endogenous"

#### Jones & Kim

Pareto distributions

Conditional on being above  $x_0$

$$1 - \text{CDF}(inx | inx > inx_0)$$

$$= \exp \{ -[inx - inx_0] \}$$

$$= \exp \{ \ln \left[ \frac{x}{x_0} \right]^\xi \} = \left( \frac{x}{x_0} \right)^\xi$$

1. High  $\xi$  = dies faster = more equal

2. If  $\xi$  low, mean is infinite

3. Pareto distrib. are fractal

4. But many things are just Pareto in nature

Toy model:

Entrepreneurs

- initial inc:  $y_0$
- inc. conditional on staying an entr. grows at  $\mu > 0$
- prob.  $\delta$  per unit of time, revert to be a regular worker
- # of entr. constant
- only stationary distrib. of entr. inc. is Pareto ( $y/y_0$ ) $^{-\xi}$
- log inc: exponential above ln  $y$ , decay parameter  $\xi/\mu$

