CONSTRUCTING THE RATIONAL NUMBERS

A set is a collection of objects
→ write like $S = \{1, ☺, \square, \{2, ☺\}\}$ where there are 4 objects in this set
or $S = \{x : P(x) \text{ is true}\}$ where $P(x)$ is some statement of x

Shorthand Review:
→ $x \in S$ means "x is in S"
→ $x \notin S$ ——— not ———
→ $\emptyset$ is an empty set
→ $A \subset B$ means "A is a subset of B" which means if x is in A, x is in B (or write as $x \in A \to x \in B$)
→ If $A \subset B$ + $B \not\subset A$, then we call A a proper subset of B (in other words, strictly smaller)
→ If $A \subset B$ + $B \subset A$, then write $A = B$

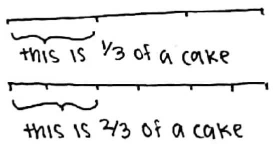
More sets:
1. Union: $A \cup B = \{x : x \in A \text{ OR } x \in B\}$
2. Intersection: $A \cap B = \{x : x \in A \text{ AND } x \in B\}$
3. Complement: $A^c = \{x : x \notin A\}$
4. Minus: $A \setminus B = \{x : x \in A \text{ AND } x \notin B\}$
5. Product: $A \times B = \{(a,b) : a \in A \text{ AND } b \in B\}$
where (a,b) are ordered pairs
↳ order matters!

A (binary) relation R is a subset of $A \times B$. If $(a,b) \in R$, write as aRb.
→ Example: A "is an ancestor of" is a relation on $P \times P$ where P = people
L "likes" ——— $P \times P$
S "is a sibling of" ——— $P \times P$
$<$ "less than" ——— $\mathbb{Z} \times \mathbb{Z}$ (integers)

An equivalence relationship on set S is a relation on $S \times S$ (often written as $\sim, \approx, \cong$, etc.) s.t.:
1. Reflexive: aRa
2. Symmetric: $aRb \to bRa$
3. Transitive: aRb + $bRc \Rightarrow aRc$

Aside: A function F from A to B is a relation s.t. if aFb + aFb', then $b = b'$; write $F(a) = b$.
→ rule that assigns to each $a \in A$ a unique $b \in B$

Construction of Rational Numbers $\mathbb{Q}$ (assume $\mathbb{Z}$, the integers, their arithmetic, + order)
→ What is $\mathbb{Q}$? Perhaps it's the set $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$ but not quite good enough b/c we don't know what this means
→ Motivation



write $(1,3) \sim (2,6)$ as equivalent ordered pairs

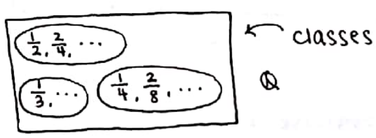
Idea: these belong to some equivalence class, we'll call "$\frac{1}{3}$"

Let $\mathbb{Q}$ = set of all such equivalence classes of ordered pairs in $\mathbb{Z} \times \mathbb{Z} - \{0\}$.
→ Want pairs to extend $\mathbb{Z}$, so that $\frac{n}{1}$ in $\mathbb{Q}$ corresponds to $n \in \mathbb{Z}$
→ See that $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$ where $\frac{m}{n}$ is an equivalence class of (m,n) with relation $(p,q) \sim (m,n)$ if $pn = qm$ + $q, n \neq 0$

Check $\sim$ is an equivalent relationship:
1. Check $(p,q) \sim (p,q)$ → $pq = qp$ + $p, q \neq 0$
2. Check $(p,q) \sim (m,n) \Rightarrow (m,n) \sim (p,q)$
3. Check $(p,q) \sim (m,n)$ + $(m,n) \sim (a,b)$ then $(p,q) \sim (a,b)$
↳ Try the cancellation law in $\mathbb{Z}$: If $ab = ac$ + $a \neq 0$ then $b = c$

PROPERTIES OF Q

Arithmetic Properties:
1. Addition
→ bad def: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$
$\left.\begin{array}{l} \frac{1}{2} + \frac{1}{3} = \frac{2}{5} \\ \frac{2}{4} + \frac{1}{3} = \frac{3}{7} \end{array}\right\}$ not equivalent + not well-defined!



→ well-defined: notion that does not depend on the representatives chosen (want!!); addition of classes should be well-defined
↳ ex: $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$ (well-defined + does not even depend on classes, but meaningless / boring)
→ good def: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$
↳ to show well-defined, must show: If $(a,b) \sim (a',b')$ + $(c,d) \sim (c',d')$ then $(ad + bc, db) \sim (a'd' + b'c', b'd')$
2. Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

To what extent does $\mathbb{Q}$ extend $\mathbb{Z}$?

→ Check that $\{\frac{n}{1} : n \in \mathbb{Z}\}$ behaves like $\mathbb{Z}$, correspondence is $\frac{n}{1} \longleftrightarrow n$

→ $\mathbb{Z}$ has an order. Does $\mathbb{Q}$?

An order on set S is a relation $<$ satisfying

1. Trichotomy: if $x, y \in S$ exactly one of these is true: $x<y$, $x=y$, $y<x$;
2. Transitivity: if $x, y, z \in S$ + if $x<y$, $y<z$ then $x<z$.

* call S an ordered set if it has an order

What is the ordering on $\mathbb{Z}$?

→ Ex: in $\mathbb{Z}$, say $m<n$ if $n-m$ is positive, i.e. in the set $\{1, 2, 3, 4 \dots\}$

→ Ex: in $\mathbb{Z} \times \mathbb{Z}$ (ordered pairs of integers), say $(a,b) < (c,d)$ if $[a<c]$ or $[a=c$ + $b<d]$ → dictionary order

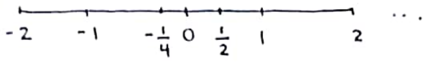
What is the ordering on $\mathbb{Q}$?

→ Ex: in $\mathbb{Q}$, say $\frac{m}{n}$ is positive if $mn>0$ (check if well-defined)

Then say $\frac{m}{n} < \frac{m'}{n'}$ if $\frac{m'}{n'} - \frac{m}{n}$ is positive

* write "$y>x$" for $x<y$
write "$x \le y$" to mean $x<y$ or $x=y$

New Picture of $\mathbb{Q}$:



→ Good enough to solve $5x = 3$

→ Not good enough to solve $x^2 = 2$.

Theorem: $x^2 = 2$ has no solution in $\mathbb{Q}$

→ Proof (by contradiction)

Assume $x^2=2$ has a solution in $\mathbb{Q}$, i.e. say $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, + assume p, q are in "lowest terms", i.e. have no common factors

So $(\frac{p}{q})^2 = 2$, hence $p^2 = 2q^2$

Then p^2 is even (divisible by 2)

Then p is even (b/c if p were odd, p^2 would be odd; if p^2 had a factor of 2, p would have factor 2)

So $p = 2m$, hence $p^2 = 4m^2$ + $4m^2 = 2q^2$, for some m in $\mathbb{Z}$, then $2m^2 = q^2$

Then q^2 is even, hence q is even

This contradicts p, q are in lowest terms

So $x^2 = 2$ must have no solution in $\mathbb{Q}$. qed

$\mathbb{Q}$ is a field, a set F with two operations $(+, \times)$ satisfying axioms

A1. F is closed (if add two things, still on set) under $+$

A2. $+$ is commutitive $(a+b = b+a)$

A3. $+$ is associative $(a+[b+c] = [a+b]+c)$

A4. F has an additive identity, call it 0 $(a+0=a)$

A5. Every element has an additive inverse (something you can add to element A to get identity)

M1. F is closed under $\times$

M2. $\times$ is commutative

M3. $\times$ is associative

M4. F has an additive identity, call it 1 (multiplicative)

M5. Every element except 0 has an additive inverse (multiplicative)

D. $\times$ distributes over $+$

→ in $\mathbb{Q}$, 0 element is the class $\frac{0}{1}$, 1 element is the class $\frac{1}{1}$) check these axioms hold through ordered pairs

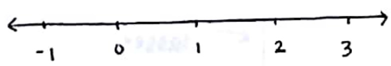
→ $\mathbb{Z}$ is not a field b/c it does not satisfy (M5)

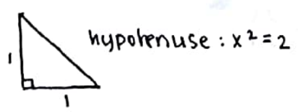
→ $\mathbb{Q}$ is an ordered field, a field w/an order so that the order is preserved by the field operations

1. $y<z \Rightarrow x+y < x+z$
2. $y<z$, $x>0 \Rightarrow xy < xz$

CONSTRUCTION OF THE REALS

Objective: Construct the Real Numbers via Dedekind "Cuts" (1872)





$x^2 = 2$ has no solution on $\mathbb{Q}$

Consider $A = \{x \in \mathbb{Q} : x^2 < 2\}$

↳ but no point right at the very end

Def. Say $E \subset S$ (ordered). If there exists $(\exists)$ $\beta \subset S$ s.t. for all $(\forall)$ $x \in E$ we have $x \le \beta$, then call β an upper bound for E, say E is bounded above

→ def. lower bound replace $\le$ with $\ge$

→ Ex: 2 is an upper bound (ub) for A

$\frac{3}{2}$ is an ub for A — why?

↳ If not, $\exists\, x \in A$ s.t. $x > \frac{3}{2}$, then $x^2 > (\frac{3}{2})^2 > 2$

$\Rightarrow$ (implies) $x \notin A$ (proof by contradiction)

If $\exists\, \alpha \in S$ s.t.

1. α is an u.b. of E and
2. $\gamma < A \Rightarrow \gamma$ is not an u.b. for E

then α is called the least upper bound (l.u.b.) of E or supremum of E (write $\alpha = \sup E$)

→ Ex: $S = \mathbb{Q}$

↳ $E = \{\frac{1}{2}, 1, 2\}$, $\sup E = 2$ (finite sets have l.u.b)

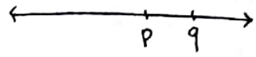
↳ $E = \mathbb{Q}-$, the negative rationals, $\sup E = 0$ (infinite set w/ l.u.b.)

↳ $E = \mathbb{Q}$, $\sup E$ does not exist (unbounded above) – "$\sup E = +\infty$"

↳ $E = A$ (before), $\sup A$ does not exist (even though it is bounded)

Proof? (naive: $\frac{p+\sqrt{2}}{2}$ ← but no def for $\sqrt{2}$ yet!)

Book: $q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2}$



we'll construct $\mathbb{R}$ + prove:

Theorem: $\mathbb{R}$ is an ordered field, w/ l.u.b property, + $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield

A set S has the least upper bound property (satisfies the completeness axiom) of every non-empty set of S that has an upper bound, also has a l.u.b (sup) in S

Dedekind: a cut α is a subset of $\mathbb{Q}$ s.t.

1. Nontrivial: $\alpha \notin \varnothing, \mathbb{Q}$ (not empty + not all rationals)
2. Closed downwards: If $p \in \alpha, q \in \mathbb{Q}$ + $q < p$, then $q \in \alpha$
3. No largest number: If $p \in \alpha$, then $p < r$ for some $r \in \alpha$

Ex: Set A (before) is not a cut (fails #2)

Ex: $\alpha = \mathbb{Q}-$ is a cut

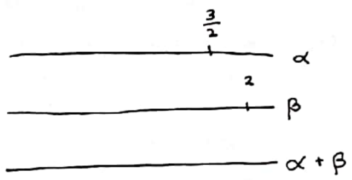
Ex: $p = \{r \in \mathbb{Q} : r \leq 2\}$ is not a cut (fails #3)

def. Let $\mathbb{R} = \{\alpha : \alpha \text{ is a cut}\}$ (some set we must show it has structure)

→ def. order: say $\alpha < \beta$ means $\alpha \subsetneq \beta$ (properly contained + not equal to)

↳ check order: trichotomy + transitivity

def Addition: $\alpha + \beta := \{r + s : r \in \alpha \text{ AND } s \in \beta\}$



→ check "cut": nontrivial b/c not empty + has upper bound

closed downward: if $p \in \alpha + \beta$, say $q < p = r + s$. If $q \in \alpha + \beta$, note $q - r < s$ + $q - s < r$. If r is in α, $q - s$ is in α. Then $(q-s) + s = q$ where $q - s \in \alpha$ + $s \in \beta$, as desired

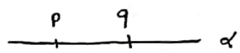
no largest #: check

→ show Axioms A1-5

↳ commutative: same $\alpha + \beta$ for $\beta + \alpha$ because addition itself is commutative for rationals ($r + s$)

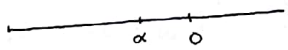
↳ associative (same principle as above). cut (already proven)

↳ additive identity: $0^* = \mathbb{Q}-$ (check $\alpha + 0^* = \alpha$. So verify $\alpha + 0^* \subset \alpha$ + $\alpha \subset \alpha + 0^*$)



$p = q + (q - p)$ in 0^*

↳ additive inverse for α: $\beta = \{p : \exists r < 0 \text{ s.t. } -p - r \notin \alpha\}$ → show $\alpha + \beta = 0^*$



def. Multiplication:

If $\alpha, \beta \in \mathbb{R}_+$ (positive or $\alpha > 0^*$): $\alpha\beta := \{p : p < rs \text{ for some } r \in \alpha, s \in \beta, rs > 0\}$

→ multiplicative identity: let $1^* = \{q < 1, q \in \mathbb{Q}\}$

* see book for negative #'s

Given a set of cuts A, let $\gamma = \bigcup\{\alpha : \alpha \in A\}$ → claim: γ is a cut + a sup A

union ↗

THE LEAST UPPER BOUND PROPERTY

$\mathbb{Q}$ has gaps; not every bounded set had a supremum (no l.u.b. property)

→ use cuts to describe; defined order (by inclusion) + arithmetic (+, x)

↳ check $\mathbb{R}$ is an ordered field

Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield

→ Associate to $q \in \mathbb{Q}$ the cut $q^* = \{r \in \mathbb{Q} : r < q\}$ which shows how $q/\mathbb{Q}$ is embedded in $\mathbb{R}$

↳ check $f: \mathbb{Q} \to \mathbb{R}$, $q \to q^*$ } preserves order +, x, <

e.g. If you add 2 rationals + look @ associated cut, same as looking @ the cut associated to one + to the other

or if one is less than the other, other is included in the one

+ is 1:1

↳ Then $\mathbb{Q}' = \{q^* : q \in \mathbb{Q}\}$ is a subfield in $\mathbb{R}$

Note: length "$\sqrt{2}$" sits in $\mathbb{R}$ as $\{q : q^2 < 2 \text{ or } q < 0\} = \gamma$. → check, using def of x, that $\gamma^2 = 2^*$ (cut associated w/2)

$\mathbb{R}$ has a l.u.b. property (whereas $\mathbb{Q}$ does not):

If A is a collection of cuts with u.b. β.

→ check γ is a cut

1. nontrivial: union of cuts + bounded above
2. closed down: ———— closed down
3. no largest #: each $x \in \gamma$ is in some α

→ Check $\gamma = \sup A$

↳ contains everything ⇒ u.b.

γ contains all $\alpha \in A$ (order contains inclusion)

↳ γ is a l.u.b b/c if $\delta < \gamma$, $\exists\, x \in \gamma \setminus \delta$.

Then $x \in$ some $\alpha \in A$, not in δ, so δ is not u.b. for A

Theorem: $\mathbb{R}$ is an ordered field, contains $\mathbb{Q}$, has l.u.b. property

→ Fact: $\mathbb{R}$ is the only ordered field w/ l.u.b. property

Consequence: length "$\sqrt{2}$" $= \sup\{1, 1.4, 1.41, 1.414, 1.4147, \dots\} = 1.4142135\dots$

more generally, $a^{\frac{1}{n}} =_{(\text{def})} \sup\{r : r^n < a\}$ ⇒ roots exist

↳ Check $(a^{\frac{1}{n}})^n = a$ ↳ need $a > 0$ or set is empty + sup does not exist (if n is even)

Greatest lower bound (glb) of a set A or infimum (write inf A): $\inf A = -\sup(-A)$

→ in $\mathbb{R}$ (not $\mathbb{Q}$), inf A exists if set is bounded below [$\mathbb{R}$ satisfies glb property]

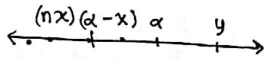
Consequences of L.U.B. Property

→ Archimedean property of $\mathbb{R}$: If $x, y \in \mathbb{R}$, $x > 0$, then $\exists$ positive integer s.t. $nx > y$.

↳ Equivalent: If $x > 0$ then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$.

↳ Proof (by contradiction): Consider $A = \{nx : n \in \mathbb{N}\}$.

If A were bounded by y (e.g. $nx < y$, $\forall n \in \mathbb{N}$), A has a l.u.b, call it α.



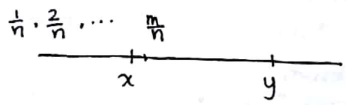
Then $\alpha - x$ is not an u.b. for A.

Hence $\alpha - x < mx$ for some $m \in \mathbb{N}$

So $\alpha < (m+1)x$, so α is not an u.b. for A (contradiction)

Theorem: Between $x, y \in \mathbb{R}$, $x < y$, $\exists\, q \in \mathbb{Q}$ s.t. $x < q < y$ ($\mathbb{Q}$ is dense in $\mathbb{R}$).

i.e. if you take any interval, you can always find a rational in that interval



→ Proof (by contradiction): choose n s.t. $\frac{1}{n} < y - x$ (Archimedean property)

Consider multiples of $\frac{1}{n}$, these are unbounded

Choose first multiple s.t. $\frac{m}{n} > x$.

Claim $\frac{m}{n} < y$. If not, then: $\frac{m-1}{n} < x$ + $\frac{m}{n} > y$

But these imply $\frac{1}{n} > y - x$

Properties of Sup:

1. γ is an u.b. for A ⇔ $\sup A \leq \gamma$
2. $\forall\, a \in A$, $a \leq \gamma$ ⇔ $\sup A \leq \gamma$
3. $\forall\, a \in A$, $a < \gamma$ ⇒ $\sup A \leq \gamma$
4. $\gamma < \sup A$ ⇒ $\exists\, a < A$ s.t. $\gamma < a \leq \sup A$
5. If $A \subset B$, then $\sup A \leq \sup B$. [$\forall$ of $A \in a$, $a \in B$ so $a \leq \sup B$; by #2, $\sup A \leq \sup B$]
6. To show $\sup A = \sup B$,

(one strategy): show $\forall a \in A$, $\exists\, b \in B$ s.t. $a \leq b$ ⇒ $\sup A \leq \sup B$.

similar method, show $\sup A \geq \sup B$

COMPLEX NUMBERS

Extended Reals: $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

→ Put order: $\forall\, x \in \mathbb{R}$, $-\infty < x < +\infty$

and arithmetic: $x + (+\infty) = +\infty$ if $x > 0$, $x \cdot (+\infty) = +\infty$ $\frac{x}{+\infty} = 0$

$x + (-\infty) = -\infty$ if $x < 0$, $x \cdot (-\infty) = -\infty$, etc... } not a field! $(+\infty) + (-\infty)$? $0 \cdot (+\infty)$?

→ why care?

↳ convenient, e.g. every subset in $\bar{\mathbb{R}}$ has a sup (possibly $+\infty$)

Euclidean Space: $\mathbb{R}^k := \{(x_1, x_2, \dots, x_k) : \text{every } x_i \in \mathbb{R}\}$

→ Define $(x_1, \dots, x_k) + (y_1, \dots, y_k) = (x_1 + y_1, \dots, x_k + y_k)$ as addition (also $\vec{x} + \vec{y}$)

→ scalar multiplication: $\alpha(x_1, \dots, x_k) = (\alpha x_1, \dots, \alpha x_k)$ where α is a scalar in $\mathbb{R}$

↳ Turns $\mathbb{R}^k$ into a vector space (comm, distributive, associative, etc.)

*multiplication not nice b/c no inverse → does not turn it into a field

→ $\mathbb{R}^k$ has an inner product or dot product: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$

↳ define a norm (notion of length): $|\vec{x}| := (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$

Complex Number Field: $\mathbb{R}^2$ can be given a field structure: $(a, b) + (c, d) = (a + c, b + d)$

$(a, b) \times (c, d) = (ac - bd, ad + bc)$

→ here the zero of the field (additive identity) + the one (multiplicative identity) is $(1, 0)$

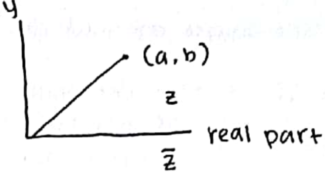
↳ is $(0, 0)$

→ Write $\mathbb{C}$, the set $\mathbb{R}^2$ with + + × as above

↳ $\mathbb{C}$ extends $\mathbb{R}$ · $\{(a, 0) : a \in \mathbb{R}\}$ ← behaves like $\mathbb{R}$, is a subfield to $\mathbb{R}$

↳ Note: $(0,1)\cdot(0,1) = (-1,0)$
 i so $i^2 = -1$, a real number

So write $a + bi$ for (a,b), i.e. $(a,0) + b(0,1)$
If $z = \underset{re(z)}{a} + \underset{im(z)}{b}i$, let $\bar{z} = a - bi$, the conjugate of z
 * conjugate of $\bar{z}$ is z

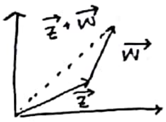


Check: $\overline{z + w} = \bar{z} + \bar{w}$
$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
$z + \bar{z} = 2\,\mathrm{Re}(z)$ + $z - \bar{z} = 2i\,\mathrm{Im}(z)$
$z \cdot \bar{z} = a^2 + b^2$, real ≥ 0 → define length / abs value $|z| = (z \cdot \bar{z})^{\frac{1}{2}}$

↳ suggests in $\mathbb{C}^k = \{(z_1, \dots, z_k), z_i \in \mathbb{C}\}$, the inner product $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{k} x_i \bar{y_i}$ (complex space)
↳ properties of length: $|z| \geq 0$, $|\bar{z}| = |z|$, $|zw| = |z||w|$, $\mathrm{Re}(z) \leq |z|$, + $|z + w| \leq |z| + |w|$

based on:
$(ac - bd)^2 + (ad + bc)^2$
$= (a^2 + b^2)(c^2 + d^2)$

form of the triangle inequality



→ proof $|z + w| \leq |z| + |w|$:
$|z + w|^2 = (z + w)\cdot(\bar{z} + \bar{w}) = (z \cdot \bar{z}) + (z \cdot \bar{w}) + (w\bar{z}) + (w \cdot \bar{w})$
$= |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2$
$\leq |z|^2 + 2|z||w| + |w|^2$
$= (|z| + |w|)^2$. This yields the desired inequality.

Cauchy-Schwartz inequality
If you have $a_1 - a_n$ + $b_1 - b_n$ as complex numbers, then $|\sum_{i=1}^{n} a_i \bar{b_i}|^2 = \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2$ ← inner product
→ in $\mathbb{R}^k$, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$; in $\mathbb{C}$, $|\langle \vec{v}, \vec{w} \rangle|^2 \leq \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle$

* if real #'s, can just use length

Proof: Let $\vec{a}, \vec{b} \in \mathbb{C}^n$
Note $0 \leq |\vec{a} - y\vec{b}|^2 = \langle \vec{a} - y\vec{b}, \vec{a} - y\vec{b} \rangle = \sum_{i=1}^{n} (a_i - y b_i)(\bar{a_i} - \bar{y}\bar{b_i})$
$= \langle \vec{a}, \vec{a} \rangle - \bar{y}\langle \vec{a}, \vec{b} \rangle - y\langle \vec{b}, \vec{a} \rangle + |y|^2 \langle \vec{b}, \vec{b} \rangle$

Let $y = \dfrac{\langle \vec{a}, \vec{b} \rangle}{\langle \vec{b}, \vec{b} \rangle}$. $= \langle \vec{a}, \vec{a} \rangle - \dfrac{|\langle \vec{a}, \vec{b} \rangle|^2}{\langle \vec{b}, \vec{b} \rangle}$, yields desired inequality after multiplying $\langle \vec{b}, \vec{b} \rangle$

PRINCIPLE OF INDUCTION

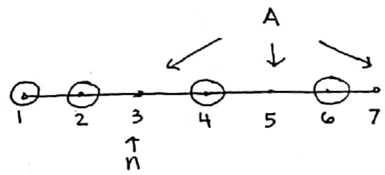
Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, rational numbers

Well-ordering Property of $\mathbb{N}$ (WOP)
→ $\mathbb{N}$ is well-ordered, meaning every nonempty subset of $\mathbb{N}$ has a least amt
→ can take WOP to be an axiom of $\mathbb{N}$ (don't have to prove)
} WOP ⇔ POI

Principle of Induction: Let S be a subset of $\mathbb{N}$ such that:
1. $1 = S$
2. If $k \in S$ then $k+1 \in S$

then $S = \mathbb{N}$

Proof (by contradiction): WOP ⇔ POI
Suppose S exists with the given properties in POI, but $S \not\subset \mathbb{N}$
Then $A = \mathbb{N} \setminus S$ has a least element by WOP. Call it n
Notice $n > 1$ since $1 \in S$ so $1 \notin A$, by property 1
Consider $n-1$, it is not in A so it is in S; by property 2, $(n-1) + 1 \in S \Rightarrow n \in S$; because $n \in A \Rightarrow$ contradiction

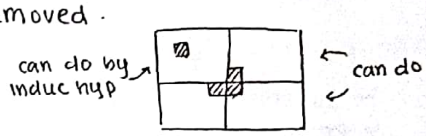


Proof by Induction Methods:
→ Let $P(n)$ be statements indexed by $n \in \mathbb{N}$
→ Idea: show $P(n)$ is true for all n
 ↳ We'll show $P(1)$ is true (base case) + $\underbrace{\text{if } P(k) \text{ is true then } P(k+1) \text{ is true}}_{\text{inductive hypothesis}}$ (inductive step)
 ↳ Then by POI, $P(n)$ is true for all n
→ Look at $S = \{n : P(n) \text{ is true}\}$

Principle of Strong Induction: use b'- if $P(1), P(2), \dots, P(k)$ is true, then $P(k+1)$ is true

Style of Proof by Induction:
→ at start, tell reader (Proof by ...) on ____ ← (variable inducted on); serves as index
→ tell reader when doing base case + inductive state
→ remind reader of conclusion at the end

Ex: Every $2^n \times 2^n$ chessboard w/one square removed can be filled by L-shaped tiles
Proof by induction on n:
For the base case, one ⇒ statement holds as desired
For the inductive step, I can assume any $2^k \times 2^k$ board w/square removed can be tiled w/L-shaped tiles
→ So consider, $2^{k+1} \times 2^{k+1}$ board w/square removed
Can divide board in 4 parts: 1 w/one square removed + 3 w/ full $2^k \times 2^k$ board
First can be tiled by inductive hypothesis.
Remaining 3 can be tiled by induct. hyp once one tile has been removed.
Let's remove one in an L-shape (see figure).
So $2^{k+1} \times 2^{k+1}$ board w/one square removed can be tiled
By POI, statement holds



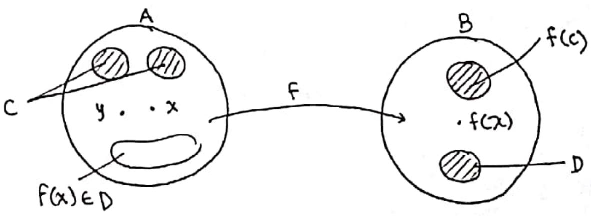
Theorem: Prove $S_n = 1+3+5+\cdots+(2n+1)$ is a perfect square
Scratch Proof: base case (n=1) holds because $S_1 = 1$ + $1^2 = 1$
for inductive step, we can assume S_n is a square for some k
→ we wish to show S_{n+1} is a square. $S_{n+1} = 1+3+\cdots+2n-1+2n+1 = S_n + (2n+1) = k^2+2n+1$
→ inductive hypothesis is too weak!
→ strengthen claim $S_n = n^2 \Rightarrow n^2+2n+1 = (n+1)^2$ which is a square as desired

Theorem: All natural numbers are even (n is even)
Proof by Induction on n:
Strong induction: assume all numbers $\leq n$ are even
Notice $n+1 = (n-1)+2$; by induct. hyp., n-1 is even so n+1 is the sum of two even numbers, as desired
→ but no base case!!! base case does not hold.

Theorem: All horses are the same color
Proof by induction on # horses in given set
For base case, w/set of one horse, statement holds
For induc. step, let $S = \{h_1, \dots, h_{n+1}\}$
→ Then $S' = \{h_1, \dots, h_n\}$ + $S'' = \{h_2 + \cdots, h_{n+1}\}$ have n horses + by induc. hyp, each have the same color (within a set); both have h_2 so all have the same color, as desired
↳ but assumed ≥3 horses (h_2 may not exist)
↳ inductive step fails when n=2

COUNTABLE + UNCOUNTABLE SETS

Objective: How to count?



Recall: $f: A \longrightarrow B$ this maps $x \to f(x)$
domain codomain
If $C \subset A$, $D \subset B$, define $f(C) = \{f(x) : x \in C\}$, the image of C
$f^{-1}(D) = \{x : f(x) \in D\}$, the inverse image of D $f(x) \in D$
→ not necessarily a function (can be multiple points on the domain)
When $f(A) = B$, say f is onto (a surjection) $\twoheadrightarrow$
when $f(x) = f(y)$ implies $x = y$, say f is 1:1 (an injection) $\hookrightarrow$
When f is 1:1 + onto, call f a bijection, + say A + B are in "1:1 correspondence" (write $A \sim B$)
Elementary Counting: use $A = J_n = \{1, 2, \dots, n\}$
→ Ex: {1, ☺, ▱, △}, say $|A| = 4$
↑ ↑ ↑ ↑ $= A \leftarrow J_4$
1 2 3 4
→ Definition: Call A finite if $A \sim J$, else A is infinite
Call A countable if $A \sim \mathbb{N}$

→ Ex: $\mathbb{N}$ is countable, use $f: \mathbb{N} \to \mathbb{N}$ where $f(n) = n$
A unique $x_1, x_2, x_3, \dots$ of distinct terms is countable: $f(n) = x_n$ provides $f(\mathbb{N} \to x_i$ as a set)
A set that can be "listed" in a sequence is countable!
→ Ex: $\{2, 3, 4, 5 \dots\}$ is countable; use $f(n) = n+1$
$\{1, 2, 3, \dots, k-1, k+1, k+2, \dots\}$ is countable; use $f(n) = n$ if $n < k$, $f(n) = n+1$ if $n \geq k$

Theorem: $\mathbb{N}$ is infinite
Proof (sketch) by induction on n + show $\mathbb{N} \not\sim J_n$
Base case; if $\mathbb{N} \hookrightarrow\!\!\!\twoheadrightarrow \{1\}$ then consider $\mathbb{N} \setminus g(1)$ so g is not onto
Inductive step: we'll show if $J_n \hookrightarrow\!\!\!\twoheadrightarrow \mathbb{N}$ then $J_{n+1} \hookrightarrow\!\!\!\twoheadrightarrow \mathbb{N}$
$J_{n+1} \hookrightarrow\!\!\!\twoheadrightarrow \mathbb{N}$ then $J_n \hookrightarrow\!\!\!\twoheadrightarrow \mathbb{N}$ ← correct

If there were $\mathbb{N} \xleftarrow{h} J_{n+1} = \{1, \dots, n, n+1\}$ (bijection)
then $\exists$ bijection $\mathbb{N} \setminus \{h(n+1)\} \xleftarrow{\hat{h}} J_n = \{1, \dots, n\}$
$\exists$ bijection f by previous example between ↑ + $\mathbb{N}$ so there is a bijection bt } contrapositive

Can show set is ∞ if it can be put into a subset in itself

Ex: $\mathbb{N} \sim 2\mathbb{N}$ (even integers) $= \{2, 4, 6, 8, \dots\}$ → use $f(n) = 2n$
→ $\mathbb{N}$, $2\mathbb{N}$ has some cardinality ("size")

Ex: $\mathbb{Z} - \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is countable
$\quad$ 7 5 3 1 2 4 6

Theorem: Every infinite subset E of a countable set A is countable.
→ Proof idea: Have $A = \{x, x_1, x_2, x_3, \dots\}$
Let $n_1 = \inf\{i : x_i \in E\}$ [exists by well-ordered principle for $\mathbb{N}$]
$n_2 = \inf\{i : x_i \in E, i > n_1\}$
$n_k = \inf\{$ ______ $i > n_{k-1}\}$
Then $E = \{x_{n_1}, x_{n_2}, x_{n_3} \dots\}$

Theorem: $\mathbb{Q}_+$ is countable

$$\begin{array}{cccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} \end{array}$$

Some rationals repeat but doesn't matter b/c can omit

Theorem: A is countable $\Rightarrow$ $A \times A$ is countable
↳ ordered quadruples can be countable too
→ show by induction

What about $\mathbb{R}$?

$1 \xleftrightarrow{f} 0.1234567890123\dots$
$2 \longleftrightarrow 0.30+1592653584\dots$ } "proposed" listing
$3 \longleftrightarrow 0.14142135 11111 \dots$

Theorem: $\mathbb{R}$ is not countable
Proof by contradiction:
Suppose $\exists$ bijection, e.g. ↗
Will show f is not onto.
x^* differs from $f(n)$
in the n^{th} decimal place!
Set is bigger than the rational numbers

say we get n^{th} # of n^{th} term (1 if a 7, 7 if a 1) → $0.77717 = x^*$

claim: x^* is not $f(n)$ for any n

Have sets of different cardinalities $[J_0, J_1, J_2 \dots J_n]$, $\mathbb{N}/\mathbb{Q}$, $\mathbb{R}$

Theorem: For any set A, $A \not\sim 2^A$, the power set of A (set of all subsets of A)
→ Proof: diagonization argument

CANTOR DIAGONALIZATION + METRIC SPACES

Theorem: The countable union of countable sets is countable (same w/ finite)
Proof: Say each A_1, A_2, A_3 are countable sets
Then $A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \dots\}$
$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, a_{25} \dots\}$
$A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \dots\}$

So $\bigcup_{i=1}^{\infty} A_i$ is countable

Notation: Use $\bigcup_{\alpha \in J} A_\alpha$ (where J is an index set) for possibly uncountable collection

Ex: Set of computer programs is countable
Recall $\mathbb{R}$ is not contable (say $\mathbb{R}$ is uncountable, infinite + not countable)
means some real #'s are not count/computable (can't specify to arbitrary precision)

Given set A, the power set 2^A is the set of all subsets of A

Ex: $A = \{☺, \square, \triangle\}$ then $D = \{☺, \triangle\}$, $E = \{\triangle\}$, $\emptyset$ are elements of 2^A, which has 2^3 elements

$D \leftrightarrow 1 \; 0 \; 1$
$E \leftrightarrow 0 \; 0 \; 1$ } specified by 1's + 0's
$\emptyset \leftrightarrow 0 \; 0 \; 0$

Cantor's Theorem (diagonal argument): For any set A, $A \not\sim 2^A$
See proof on next page

Proof (by contradiction): Suppose $\exists$ bijection $f: A \hookrightarrow 2^A$

Then every $a \leftarrow f(x)$, a subset of A (Ex: ◡̈ $\leftarrow$ {◡̈, △})

Idea: $a \in A$ $\quad$ $f(a) = 2^A$

☺ $\xrightarrow{f}$ {△ □ ☺ ◡̈}

◡̈ $\xrightarrow{f}$ {△ ☺ ★ ☆} → no ◡̈

★ $\xrightarrow{f}$ {★ □ ☆}

Want: subset B that is not in the image of F

Let $B = \{a : a \notin f(a)\}$

So if $B = f(x)$ for some $x \in A$, consider x.

Is $x \in B$?

No bc then $x \notin f(x) = B$, contradiction.

Then $x \notin B$, but then $x \notin f(x) = B$ so $x \in B$, contradiction

So B is no $f(x)$ for any $x \in A$ (qed)

Ex: $2^A \sim$ all $\{f : A \to \{0,1\}\}$ by membership relation

Thm: $2^{\mathbb{R}} \sim$ all functions from $\mathbb{R} \to \{0,1\}$

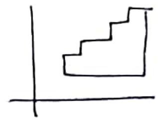
→ Consequence: There are infinitely many cardinalities $(0, 1, 2, 3, 4, \ldots \aleph_0, \aleph_1, \aleph_2, \ldots \aleph_\alpha)$ ordinal numbers

↳ cardinality of $\mathbb{R} = c$ (continuum) $\quad$ cardinality $\mathbb{Z}$ $\quad$ cardinality of $\mathbb{R}$

Continuum Hypothesis: $\aleph_1 = c$ is independent of axioms of set theory, provably undecidable.

Metric Spaces:

1. How to measure distance?
   - → ex: in $\mathbb{R}^n$, genome seq.?
   - → Def: A set A is a metric space if $\exists$ metric $d: X \times X \to \mathbb{R}$ such that $\forall p, q \in X$
     1. Distance $d(p,q) \geq 0$ & $= 0$ iff $p = q$ Non-negativity
     2. $d(p,q) = d(q,p)$ Symmetry
     3. $d(p,q) \leq d(p,r) + d(r,q)$ for all r in X triangle inequality

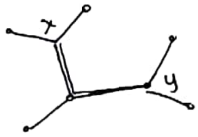


→ Ex: $\mathbb{R}$ w/ $d(x,y) = |x - y|$.

Ex: $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_m + y_m)^2}$ (Euclidean metric on $\mathbb{R}^n$)

Ex: $\mathbb{R}^2$ w/ $d_{sr}(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ (staircase metric)

Ex:



tree

$d_T(x,y)$ = shortest path taken (satisfies Δ inequality)

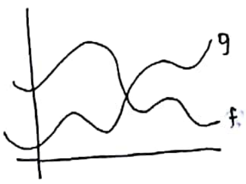
Ex: X = {genome sequences}

$\vec{x}$ = G A T T A C A

$\vec{y}$ = A G A T C A T

$\}$ $d(\vec{x}, \vec{y})$ = number of differences

Ex: space of functions



$d(f,g) = \int_a^b |f - g| \, dx$ on continuous funcrion on $[a,b]$

$d(f,g) = \sup |f(x) - g(x)|$ for all $\mathbb{R}$ on space of continuous, bounded, func.

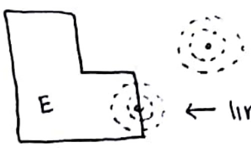
Sup norm · dsup

Open ball: $N_r(x) = \{y : d(x,y) < r\}$

neighborhood of radius r

Closed ball: $\overline{N_r(x)} = \{y : d(x,y) \leq r\}$

Define:

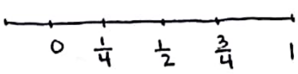


limit point

← limit ball touches E no matter how small

Say $p \in X$ is a limit point of E if every neighborhood of p contains point, $q \neq p$ s.t. $q \in E$

Ex: 0 is a limit point of the set

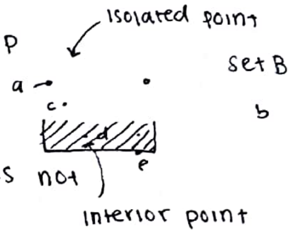


LIMIT POINTS

Recall: (X, d) metric space
set ↗ ↖ metric (symm, non-neg, Δ ineq)
→ Ex: $(\mathbb{R}^n$, Euclidean), staircase, $(\mathbb{R}^n/X$, discrete metric$) \to f(p,q) = \begin{cases} 1 & p=q \\ 0 & \text{otherwise} \end{cases}$

Recall: "open ball" or "neighborhood" – tells us which points are closed
→ ex: in (X, discrete metric), open balls are single points (if $r \leq 1$) or all of X (if $r > 1$)

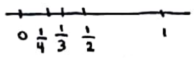
Running Example:



When does a set E "approach" a point p?

Def: A point $p \in X$ is a limit point of E if every neighborhood of p contains a pt $q \in E$, $q \neq p$

→ Ex: in $\mathbb{R}$, $G = \{\frac{1}{n}, n \in \mathbb{N}\}$
Claim: 0 is a limit point



↳ neighborhoods of p: $(p-\varepsilon, p+\varepsilon)$

→ Ex: in $\mathbb{R}^2$, consider B
c, d, e are limit points
a, b, are not

→ so a point p is not a limit point of E if $\exists$ neighborhood N of p such that N does not contain any other points of p

Logic Practice
A horse is superior if every leg is strong
________ not superior if $\exists$ leg that is not strong

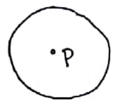
A horse is lame if it has a broken leg.
________ not lame if every leg is not broken

Def: p is an isolated point of E if $p \in E$ + p is not a limit point of E.
→ Ex: all points of G are isolated

Def: p is an interior point of E if $\exists N$ of p such that N is completely contained in E $(N \subset E)$
→ Ex: G has no interior points

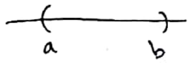
In $\mathbb{R}$, consider $\varnothing$, $\mathbb{R}$, $\mathbb{Q}$, + ($\mathbb{R}$, discrete) – what are the limit/interior/isolated points?
→ $\varnothing$: no limit point in $\mathbb{R}$ or ($\mathbb{R}$, discrete)
→ $\mathbb{R}$: in $\mathbb{R}$, all points are limit points; in ($\mathbb{R}$, discrete), there are neighborhoods that do not contain p = no limit
in $\mathbb{R}$, every point is interior
→ $\mathbb{Q}$: in $\mathbb{R}$, every point is a limit point (no isolated point); in ($\mathbb{R}$, discrete), no limit points

Theorem: If p is a limit point of E, then every neighborhood of p contains infinitely many points of E
→ Proof (by contradiction):
$\exists$ a neighborhood N of p w/only finitely many points of E $(E_1 \dots E_n)$
Let $r = \min_{i=1}^{N} \{d(p, e_i)\}$ ← exists b/c set is finite



But neighborhood around p has no points, contradiction

Def: A set E in metric space X is open if every point of E is an interior point
→ Ex: none of B is open
→ in $\mathbb{R}$



open interval $(a,b) = \{x : a < x < b\}$
→ in $\mathbb{R}$, $\varnothing$ is open (vacuosly true) + $\mathbb{R}$ is open

Def: A set E is closed if E contains all its limit points } not the negation of
→ in $\mathbb{R}$, $\{p\}$ is closed
interval $[a,b] = \{x : a \leq x \leq b\}$ is closed
$\varnothing$ is closed (vacuously true)
$\mathbb{R}$ is closed

→ $(a,b]$, $[a,b)$ are "half open" intervals – not open or closed

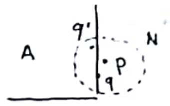
Ex: Mouth of set B is not closed – can we close it?
→ Def: let E' = set of all limit points of E.
The closure of E is $\bar{E} = E \cup E'$ (closure)
↳ is $\bar{E}$ closed? Yes!

THE RELATIONSHIP BETWEEN OPEN + CLOSED SETS

Recall: A set E in metric space X is open if every point is an interior point
A set K ———————— closed if K contains all its limit points

The closure of set A is $\bar{A} = A \cup A'$ (limit points of A)

Theorem: $\bar{A}$ is a closed set



Proof: Consider p a limit point of $\bar{A}$. Want to show $p \in \bar{A}$.
Consider a neighborhood N of p. Assume $p \notin A$, we'll show N contains a point of A (q')
Since p is a limit point of $\bar{A}$, N contains a point q of $\bar{A}$.
If $q \in A$, we will have found the desired point, $q = q'$
If $q \notin A$, then q is a limit point of A
Consider nbhd N' of q such that $N' \subset N$ (can do b/c nbhds are open)
So $q' \in N$, the desired point (qed)

Lemma: Neighborhoods are open

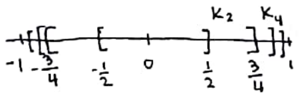
Note: $a = d(p,q) < r$
Let $r' = r - a$
Claim: $N_{r'}(q) \subset N_r(p)$ ← check: if $d(x,q) < r'$ then $d(x,p) \leq d(x,q) + d(q,p) < r' + a$
triangle inequality

Theorem: E is closed $\xleftrightarrow{\text{iff}}$ $E = \bar{E}$ (adding limit points doesn't Δ the set)
Proof: If E is closed $\rightarrow E' \subset E$ so $E \cup E' \subset E$ so $\bar{E} \subset E$. since $E \subset \bar{E}$. $E = \bar{E}$
———————— $\Leftarrow E = \bar{E} \Rightarrow E$ contains all of its limit points

Theorem: If $E \subset$ closed set F, then $\bar{E} \subset F$.
→ $\bar{E}$ is the smallest closed set containing E
Proof: If p is a l.p. of $E \Rightarrow p$ is a l.p. of F but F contains all its l.p. $\Rightarrow$ all l.p. of E are in F + $E \subset F$

Theorem: E is open $\Leftrightarrow E^c$ is closed (E^c is the compliment of E; $E^c = X \setminus E = \{p \in X : p \notin E\}$ ← metric space)
Proof: E is open $\Leftrightarrow$ any pt $x \in E$ is an interior point
$\Leftrightarrow \forall x \in E, \exists$ nbhd N of x s.t. N is disjoint from E^c
$\Leftrightarrow \forall x \in E$, x is not a limit point of E^c
$\Leftrightarrow E^c$ contains all its limit pts (qed)

Unions + Intersections: $K_i = [-1+\frac{1}{i}, 1-\frac{1}{i}]$
$\bigcup_{i=1}^{\infty} K_i = (-1,1)$ ← not closed



Lemma: $\{E_\alpha\}$ collection of sets
$(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha E_\alpha^c$ (compliment of union = intersection of compliments)
Proof: $x \in$ left hand side (LHS) $\Leftrightarrow x \notin$ any E_α
$\Leftrightarrow x \in E_\alpha^c$, $\forall \alpha \in A$
$\Leftrightarrow x \in \bigcap_\alpha E_\alpha^c$ (qed)

Theorem: Arbitrary union of open sets are open (a)
Arbitrary intersection of closed sets are closed (b)
Finite intersection of open sets are open (c)
Finite union of closed sets are closed (d)

$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ closed
← open

Proof: (a) $x \in \bigcup_\alpha U_\alpha$ + U_α is open $\Rightarrow x \in$ some U_α (open)
x has a nbhd N_x s.t. $x \in N_x \subset U_\alpha \subset \bigcup_\alpha U_\alpha$, as desired (found neighborhood)
(b) Say B_α (sets) are closed then U_α (compliments of B_α) are open
Use lemma noting $U_\alpha^c = B_\alpha$; LHS of lemma is compliment of arbitrary union, RHS arbitrary $\cap$ of closed sets
(c) $\exists N_{r_i}(x)$ for each U_i. Let $r = \min(r_1, \dots, r_n)$.
Then $N_r(x)$ shows x is interior to $\bigcap_{i=1}^{n} U_i$

E is dense in metric space X if every point of X is a l.p. of E or in E
$\Leftrightarrow \bar{E} = X$
$\Leftrightarrow$ every open set of X contains a point of E

→ Ex: $\mathbb{Q}$ is dense in $\mathbb{R}$

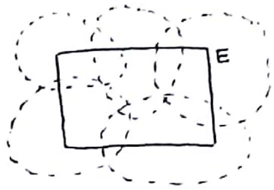
COMPACT SETS

* X will refer to a metric space

Finite sets are nice because:

1. They are small sets (bounded)
2. ——— closed sets
3. In $\mathbb{R}$, they contain their sup + inf (max/min)
4. When doing things, the process ends

} compact sets are the next best thing to being finite

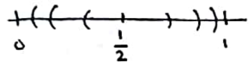


Given a cover, do we still need all the sets to still cover?

An (open) cover of E in X is a collection of open sets $\{G_\alpha\}$ whose union "covers" (contains) E.

→ A subcover of $\{G_\alpha\}$ is a subcollection $\{G_{\alpha_\gamma}\}$ that still covers E.

→ Ex: In $\mathbb{R}$, $[\frac{1}{2}, 1)$ has cover $\{V_n\}_{n=1}^{\infty}$ when $V_n = (\frac{1}{n}, 1-\frac{1}{n})$



$\{(0,2)\}$ ← one set cover

$\{W_x\}_{x\in[\frac{1}{2},1)}$ where $W_x = N_{\frac{1}{2}}(x)$

$\{V_n\}_{n=3}^{\infty}$ has a subcover $\{V_n\}_{n=12}^{\infty}$

$\{W_x\}$ ——— $\{W_{5/10}, W_{6/10}, W_{7/10}, W_{8/10}, W_{9/10}\}$ ← finite subcover

Ex: $[0,1]$ in $\mathbb{R}$ has cover by $\{V_n\} \cup \{W_0, W_1\}$. A finite subcover is $\{W_0, W_1, V_{11}\}$

Say a set K is compact (in X) if every open cover of K contains a finite subcover.

→ so not compact means $\exists$ some open cover w/no finite subcover (ex: $[\frac{1}{2},1)$, see $\{V_n\}$)

→ Warning: not saying there's a finite cover

Ex: $\mathbb{Z}$ in $\mathbb{R}$ is not compact

$[0,1]$ may be compact but need to check every open cover or prove theorem

Theorem: Finite sets are compact

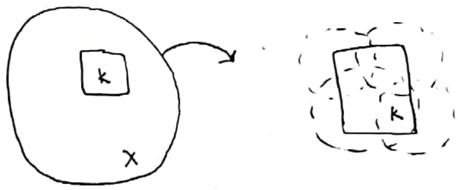
Proof: Consider an open cover $\{G_n\}$ covering $x_1, \dots x_N$.

$\forall x_i$, choose one G_{α_i} that contains x_i

Then $\{G_{\alpha_i}\}_{i=1}^{N}$ covers the set (qed)

Theorem: Compact sets are bounded

(A set K is bounded if $K \subset N_r(x)$ for some $x \in X$)



Proof: Let K be compact.

Notice we'll let $B(x) = N_1(x)$, ball of radius 1

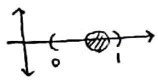
$\{B(x)\}_{x\in K}$ is an open cover of K.

By compactness of K, $\exists$ finite subcover $\{B(x_i)\}_{i=1}^{N}$

Let $R = \max_{1\leq x_i, j \leq N} \{d(x_i, x_p)\}$; this max exists because set $x_1, \dots x_N$ is infinite.

Then $N_{R+2}(x_1)$ contains all K (qed)

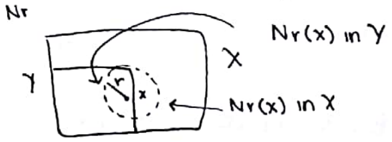
What does it mean to be open in a particular metric space?

Ex: 

open in $\mathbb{R}$ but not open in $\mathbb{R}^2$, $\{(a,0) : a \in (0,1)\}$ → openness depends on what set you're in!

"Relative" open sets

→ If $Y \subset X$ metric, then Y "inherets" metric from X



A set U is open in Y ("relative" to Y) if every pt of U is an interior point of U.

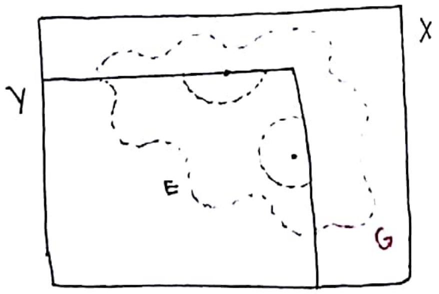
↳ using neighborhoods in Y

Theorem: $E \subset Y \subset X$. E is open in $Y \iff E = Y \cap G$ for some G open in X

Proof Idea ($\Leftarrow$): Use fact if $N_r(x) \subset G$ then $N_F(x) \cap Y$ is nbhd of x in Y. + in G.

($\Rightarrow$): Every pt x has $N_r(x) \subset Y \cap E$.

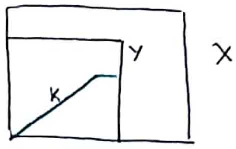
Then $\bigcup_{x\in E} N_r(x)$ in X is open, call G (qed)



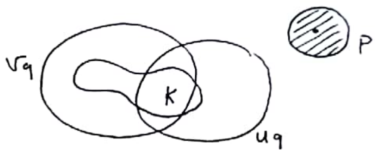
RELATIONSHIP OF COMPACT SETS TO CLOSED SETS

Recall: A set K in the metric space X is **compact** if every open cover of K (in X) has a finite subcover
Thm: $E \subset Y \subset X$
E is open in $Y \Leftrightarrow E = Y \cap G$ for some G open in X

Theorem: $K \subset Y \subset X$.
K compact in $Y \longleftrightarrow K$ compact in X



Proof: ($\Rightarrow$) (Assume K is compact in Y). Consider open cover $\{U_\alpha\}$ of K in X.
Let $V_\alpha = U_\alpha \cap Y$. Then $\{V_\alpha\}$ cover K in Y
By compactness of K in Y, $\exists$ finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$
Then $\{U_{\alpha_1}, \dots U_{\alpha_n}\}$ cover K in X (finite subcover of original cover), as desired
($\Leftarrow$) Consider open cover $\{V_\alpha\}$ of K in Y
By earlier theorem, $\exists$ open U_α s.t. $U_\alpha \cap Y = V_\alpha$
$\{U_\alpha\}$ cover K in X, so $\exists$ finite subcover $\{U_{\alpha_1} \dots U_{\alpha_n}\}$.
Then $\{V_{\alpha_1}, \dots V_{\alpha_N}\}$ are finite subcover of $\{V_\alpha\}$ for K in Y

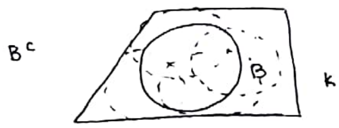


Compactness is an intrinsic property of the set.

Theorem: Compact sets are closed
Proof: Let K be compact. Consider $p \notin K$. We'll show p has a nbhd not intersecting K (p is not a limit pt of K) so p is interior to K
$\forall q \in K$, let $V_q = N_{R/2}(q)$, $U_q = N_{R/2}(p)$ where $R = d(p,q)$
Notice $\{V_q\}$ open cover of K. By compactness of K, $\exists$ finite subcover $\{V_{q_1} \dots V_{q_n}\}$.
Then $W = U_{q_1} \cap U_{q_2} \cdots U_{q_n}$ is open (ball of radius min $d(p,q_i)$)
Claim: $W \cap V_{q_i}$ is $\emptyset$ for each i because $W \subset U_{q_i}$ + $U_{q_i} \cap V_{q_i} = \emptyset$. (qed)

Ex: $(0,1)$ is not compact
$\mathbb{R}$ (in $\mathbb{R}$) is not compact b/c not bounded, though it is closed (so closed sets not always compact)

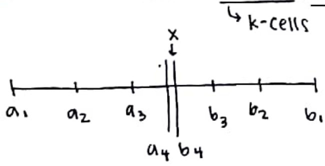
Theorem: A closed subset B of a compact set K is compact.



Proof: Let $\{U_\alpha\}$ be an open cover of B.
Notice, B^c is open (b/c B is closed).
So $\{U_\alpha\} \cup \{B^c\}$ is an open cover of K.
By compactness, $\exists$ finite subcover $\{U_{\alpha_1}, \dots U_{\alpha_n}\}, \{B^c\}$ of $\{U_\alpha\} = \{B^c\}$
Notice $B^c \cap B = \emptyset$ so $\{U_{\alpha_1}, \dots U_{\alpha_n}\}$ covers B + is finite subcover of original cover (qed)

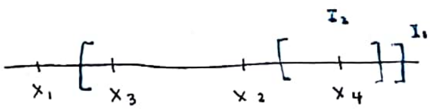
Corrolary: F closed, K cpt in metric space X, then $F \cap K$ is compact

Theorem: Nested closed intervals in $\mathbb{R}$ are not empty ($I_n = [a_n, b_n]$)
↳ k-cells ↳ in $\mathbb{R}^k$
nested: if $m > n$, $a_n \le a_m \le b_m \le b_n$



Proof: Let $x = \sup\{a_i\}$, exists b/c they're bounded by b_1
Clearly $x \ge a_i$ for all i b/c it's the sup
And $x \le b_n$ for all n, b/c b_n is an u.p. for all a_n by nested def. (qed)

Aside: Prove $\mathbb{R}$ is uncountable
Suppose $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$ is countable
Choose interval I_1 that misses x_1
$I_2 \subset I_1$, that misses x_1, x_2
$I_3 \subset I_2$ ______ x_3

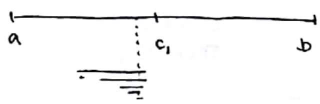


Nested sequence $\Rightarrow \exists x \in I_n$, x is not on list

COMPACTNESS + THE HEINE-BOREL THEOREM

Last lecture:
1. Compact sets are bounded
2. ______ closed
3. Closed subsets of compact sets are compact
4. For nested short intervals, in $\mathbb{R}$, intersection is non-empty (k-cells in $\mathbb{R}^k$)

Theorem: $[a,b]$ is compact (in $\mathbb{R}$)
(k-cells) (in $\mathbb{R}^k$)



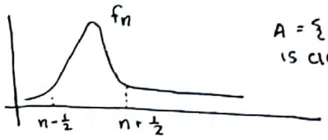
Proof: Suppose not (Proof by contradiction). Then $\exists$ open cover $\{G_\alpha\}$ that has no finite subcover.
Then $\{G_\alpha\}$ covers $[a, c_1]$ + $[c_1, b]$, at least one has no finite subcover, call it I_1.
Without loss of generality (WLOG), say $I_1 = [a, c_1]$ has no finite subcover (FS)
Then subdivide (half) using c_2, note $[a, c_2]$ or $[c_1, c_2]$ or both has no FS
Continue, obtain sequence $I_1 \supset I_2 \supset I_3 \cdots$ nested closed intervals, each halved @ each step*
+ w/no FS of $\{G_\alpha\}$ ☺

By nested interval theorem, $\exists\ x \in I_n$ for all n.
But x same G_α of cover. So $\exists\ r>0$ s.t. $N_1(r) < G_\alpha$
By * some I_n is completely contained $N_r(x) \Rightarrow$ single G_α covers I_n, contradicts ☺

Heine - Borel Theorem: In $\mathbb{R}$ (or $\mathbb{R}^n$), K is compact $\Leftrightarrow$ K is closed + bounded
Proof: ($\Rightarrow$) already
($\Leftarrow$) not true in arbitrary metric spaces
K bound $\Rightarrow K \subset [-r,r]$ for some $r>0$
But K is closed, $[-r,r]$ is compact, so K is compact (qed)

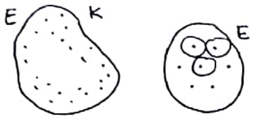
* in $\mathbb{R}^n$, show k cells are compact

Ex: discrete metric on infinite set $A \rightarrow A$ is closed, bounded, but not compact
Ex: $C^b(\mathbb{R})$ = set of continuous func. from $\mathbb{R} \rightarrow \mathbb{R}$
$d(f,g) = \sup_{x\in\mathbb{R}} |f(x)-g(x)|$



$A = \{f_n\}_{n\in\mathbb{Z}}$ is closed

Theorem: K is compact $\Leftrightarrow$ every infinite subset of K has a limit point in K
Proof (by contradiction): ($\Rightarrow$) If not point of K is a l.p. of E then each $q \in K$ has nbhd V_q contains exactly one point q of K
$\{V_q\}$ cover E, with no finite subcover



($\Leftarrow$) [proof for $\mathbb{R}^n$ but true for all metric spaces]
We'll show K is closed + bounded
Suppose K is not bounded. Choose x_n s.t. $|x_n| > n$. These have no limit points
Suppose K is not closed. $\exists\ p \notin K$ that is a l.p. of E.
Choose x_n s.t. $d(x_n, z) < \frac{1}{n}$. $\{x_n\}$ has l.p. at p + no other

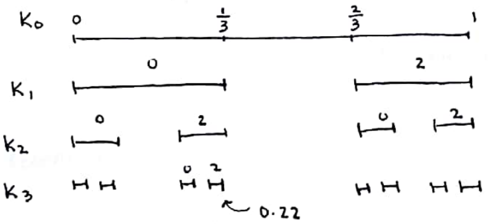
Bolzano-Weierstrass Theorem: Every bounded infinite subset of $\mathbb{R}^n$ has a limit point
Proof: If subset E is bounded, then $E \subset$ compact k-cell, so has limit point on k-cell (qed)

Finite Intersection Property: $\{K_\alpha\}$ compact subset of metric space X. If any finite subcollection has non-empty intersection then $\bigcap_\infty K_\alpha \neq \emptyset$. $\rightarrow$ gen. nested interval theorem
Proof: Let $U_\alpha = K_\alpha^c$, open.
Fix one K in $\{K_\alpha\}$. If $\bigcap_\infty K_\alpha = \emptyset$ then $\{U_\alpha\}$ cover K, compact.
$\Rightarrow \exists$ finite $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covering K.
$K \cap K_{\alpha_m} \cap \dots \cap K_{\alpha_n} = \emptyset$, by contradiction

CONNECTED SETS, CANTOR SETS

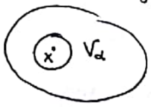
Finite Intersection Property Corollary: $\{K_n\}$ a sequence of compact sets, nested, then $\bigcap_{n=1}^\infty K_n$ is non-empty
Theorem: X (space) is compact $\Leftrightarrow$ any collection of closed sets $\{D_\alpha\}$ satisfies the finite intersection property (FIP)
$\rightarrow$ If every finite subcollection has non-$\emptyset$ intersection, then $\bigcap D_\alpha \neq \emptyset$
Proof: ($\Rightarrow$) cover $\{D_\alpha\}$, these are closed subset of compact X, so they're compact (apply prev. theorem).
($\Leftarrow$) exercise

Cantor Set: $C = \bigcap_{n=1}^\infty K_n$ (note K are closed, compact, nested)



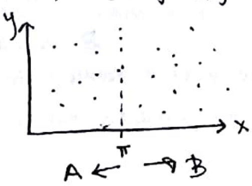
$\rightarrow$ C is closed (b/c $\cap$ of closed sets)
$\rightarrow$ C is perfect: closed + every pt is a l.p., follows from
$\rightarrow$ C consists of real #'s, whose ternary expansion contains only 0 + 2's.
↳ ternary: $\sum_{k=-\infty}^{\infty} a_k 3^k$ write $\dots a_{-2}, a_1 a_0 \cdot a_1 a_2 \dots$
$\rightarrow$ Shows C is uncountable (use diagonalization argument)
C has non (end pts of K_n) pts: 0.020202...
$\rightarrow$ Are totally disconnected
$\rightarrow$ C has measure 0, $\forall\ \varepsilon > 0$, C can be covered by intervals by total length $< \varepsilon$

A base (basis) for topology is $\{V_\alpha\}$ a collection s.t. $\forall\ x \in$ open U, $\exists\ V_\alpha$ s.t. $x \in V_\alpha \subset U$. So every open set is the union of base elements

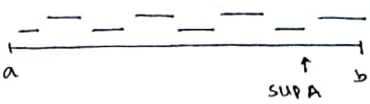


Def. say A, B in X are separated if both $A \cap \bar{B}$ + $\bar{A} \cap B$ are empty
say E is connected if E is not union of 2 separated sets
separation
↳ E is connect if E is not union of 2 open sets in E
2 closed sets in E
nonempty, nontrivial, relative

Ex: in $\mathbb{R}^2$, $E = \{(x,y) : x,y \in \mathbb{Q}\}$



Theorem: $[a,b]$ is connected.



Proof (by contradict.): If not then $\exists$ separation $A + B$ w/ $a \in A$.
Let s be $\sup A$. Then $s \in \bar{A}$, so $s \notin B$ then $s \subset A$, so $s \notin \bar{B}$.
Then $\exists (s-\varepsilon, s+\varepsilon)$, containing no sets of B, hence all in A, but this contradicts s as $\sup A$.

CONVERGENCE OF SEQUENCES

Finite Intersection Condition: a collection of sets has FIC if any finite subcollection has non-empty intersections.

Theorem: K compact $\Leftrightarrow$ Any collection of closed sets $\{K_\alpha\}$ that has the FIC has non-empty intersection

A sequence $\{p_n\}$ in X is a function ($f: \mathbb{N} \to X$) maps $n \to p_n$, a point in X

→ $\{p_n\}$ converges if $\exists p \in X$ s.t. $\forall \varepsilon > 0\ \exists N$ s.t. $n > N$ implies $d(p_n, p) < \varepsilon$ ☺

Write $p_n \Rightarrow p$ or $\lim_{n\to\infty} p_n = p$

Say "p_n converges to p" or "p is the limit of sequence p_n"

Ex: $p_n = \frac{n+1}{n}$ in $\mathbb{R}$ Scratch: given ε, find N that satisfies ☺

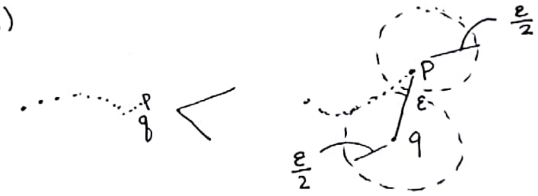
Claim: $p_n \to 1$

Proof: We have to bound $|\frac{n+1}{n} - 1| < \varepsilon \Rightarrow |\frac{1}{n}| < \varepsilon \to$ when?

Given $\varepsilon > 0$, choose $N = |\frac{1}{\varepsilon}| + 1$

For if $n \geq N$ then $n > \frac{1}{\varepsilon}$, hence $\frac{1}{n} \leq \varepsilon$.

So $|p_n - p| = |\frac{n+1}{n} - 1| = |\frac{1}{n}| < \varepsilon$. (qed)



TRUE: $p_n \to p$ + $p_n \to p'$ $\Rightarrow p = p'$

→ Assume $p_n \to p$, $p_n \to q$

Let $\varepsilon = d(p,q)$. Assume $p \neq q$ so $\varepsilon > 0$.

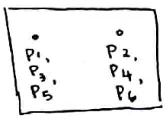
Then $\exists N_p$ s.t. $n \geq N_p$ implies $d(p_n, p) < \frac{\varepsilon}{2}$.

Also $\exists N_q$ s.t. $n \geq N_q$ implies $d(p_n, q) < \frac{\varepsilon}{2}$.

Let $N = \max\{N_p, N_q\}$. Then $n \geq N$ implies $\varepsilon = d(p,q) \leq d(p,p_n) + d(p_n,q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, hence $\varepsilon < \varepsilon$ (contradiction)

(or can show $d(p,q) = 0$)

FALSE: $\{p_n\}$ bounded (range of p_n is bounded) $\Rightarrow p_n$ converges

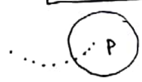


TRUE: p_n converges $\Rightarrow \{p_n\}$ bounded

→ Use $\varepsilon = 1$, then $\exists N$ s.t. $n > N$ $d(p_n, p) < 1$

Let r be $\max\{1, d(p, p_1), d(p, p_n)\}$

So all $\{p_n\}$ are in $B_R(p)$



FALSE: $p_n \to p \Rightarrow p$ is a limit point of range of $\{p_n\}$

TRUE: p is a limit point of $E \subset X$ $\Rightarrow \exists$ seq $\{p_n\}$ in E s.t. $p_n \to p$

→ choose a seq p_n $B_{R\frac{1}{n}}(p)$ then $p_n \to p$

TRUE: $p_n \to p \Leftrightarrow$ Every nbhd of p contains all but finitely many p_n

→ True b/c ☺

SUBSEQUENCES, CAUCHY SEQUENCES

Consider seq: $\{s_n\}\{t_n\} \in \mathbb{C}$ & $s_n \to s$, $t_n \to t$

Theorem: $\lim_{n\to\infty} (s_n + t_n) = s + t$

* to show convergence, you must find an N

Idea: bound $|(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t|$

Proof: Given $\varepsilon > 0$, $\exists N_1 + N_2$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$

Let $N = \max(N_1, N_2)$. Then for $n > N$, we have $|(s_n + t_n) - (s+t)| \leq |(s_n - s) + (t_n - t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ as desired. (qed)

Theorem: $\lim_{n\to\infty} cs_n = cs$ + $\lim_{n\to\infty} (c + s_n) = c + s$

Idea: $|cs_n - cs| = |c||s_n - s|$

Proof: Given $\varepsilon > 0$, $\exists N$ s.t. $n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{c}$

Then for this N, $n > N \Rightarrow |cs_n - cs| = |c||s_n - s| < \varepsilon$. So $cs_n \to cs$ (qed)

Theorem: $\lim_{n\to\infty} s_n t_n = st$

Idea: $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$

Proof: Given $\varepsilon > 0$, let $K = \max\{s, t, 1, \varepsilon\}$

$\exists N_1, N_2$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{3K}$ + $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{3K}$

Let $N = \max\{N_1, N_2\}$.

Then $|s_n t_n - st| \leq \frac{\varepsilon^2}{9K^2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$

$< \frac{\varepsilon}{9K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$, $K \geq 1$

$\{p_n\}$ a seq. Let $n_1 < n_2 < n_3 < \cdots <$ in $\mathbb{N}$, an increasing sequence
Then $\{p_{n_i}\}$ is a subsequence
Ex: $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\}$

If $p_n \to p$, must every subseq. converge to p? YES – every nbhd of p contains all but finitely many points of p

Ex: $\{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \ldots\}$ does not converge but subseq does! subsequential limits

Must every seq contain a converging subsequence? No – $\{1, 2, 3, 4 \ldots\}$
If seq. bounded, must it have a converging subseq? No – $\mathbb{Q}$, $\{3, 3.1, 3.14, 3.141 \ldots\}$ ★

A metric space is sequentially compact if every seq has a converging subseq.
Theorem: If X is compact, then X is sequentially compact (in compact metric space, every seq has subseq. converging to pt x)
→ seq compact ⟺ compact — CHALLENGE
→ Corollary: every bounded seq in $\mathbb{R}^k$ contains a converging subseq (Bolzano-Weirstrass Theorem)
Proof: Let $R = \text{range}\{p_n\}$
If R finite, then some p in $\{p_n\}$ is achieved infinitely many times (use this subseq)
If R infinite, then by prev theorem since X compact, R has l.p. called p.
→ Use this to construct a subseq.
(Alt $\exists$ seq in R converging to p

How to tell $\{p_n\}$ converges if limit is not known?
Idea: If they do converge, p_n must be getting closer to each other

$\{p_n\}$ is a Cauchy sequence means $\forall \varepsilon > 0$, $\exists N$ s.t. $m, n \geq N$ implies $d(p_m, p_n) < \varepsilon$

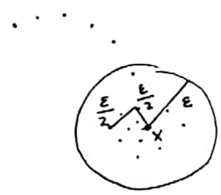
Idea: bound $d(p_n, p_m) = d(p_n, p) + d(p, p_m)$

Theorem: If $\{p_n\}$ converges then $\{p_n\}$ is Cauchy.
Proof: Given $\varepsilon > 0$, $\exists N$ s.t. $n > N \Rightarrow d(p, p_n) < \varepsilon/2$
So for this N, $n, m > N \Rightarrow d(p_n, p_m) < d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, as desired (qed)
Converse is not true: seq in $\mathbb{Q}$ is Cauchy but not convergent ★

A metric space X is complete if every Cauchy seq. converges to point of X
→ $\mathbb{Q}$ is not complete

COMPLETE SPACES

Theorem: compact metric spaces are complete
Proof: Let $\{x_n\}$ be Cauchy seq in X.
Since X is compact, it is sequentially compact
So $\exists$ subseq $\{x_{n_k}\}$ converging to a point x in X
Fix $\varepsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1$ s.t. $i, j > N \Rightarrow d(x_i, x_j) < \frac{\varepsilon}{2}$
$\{x_n\} \to x$ implies $\exists N_2$ s.t. $n_k > N_2 \Rightarrow d(x_{n_k}, x) < \frac{\varepsilon}{2}$
Let $N = \max(N_1, N_2)$
If $n > N$ then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$ for any $x_k > N$ (fix one)
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
So given $\varepsilon > 0$, found N that shows $x_n \to x$ (qed).



Corollary: $[0,1]$ is complete, k-cells in $\mathbb{R}^n$ are complete, closed subsets of compact, $\mathbb{R}^n$
Proof Idea: If $\{x_n\}$ Cauchy, it is bounded
Let $\varepsilon = 17$, $\exists N$ s.t. $d(x_n, x_m) < 17$ for $\forall n, m > N$
Let $R = \max\{d(x_N, x_1), d(x_N, x_{N-1}), 17\}$

cauchy → bounded → ball

Seq. bounded by $B_R(x_N)$
$B_R(x_N)$ is in some k-cell in $\mathbb{R}^n$, so $\{x_n\}$ converges b/c k-cell complete. So $\mathbb{R}^n$ is complete (qed)

Ex: Does $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ converge?
Consider $|x_n - x_m| = |\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}| \geq \frac{n-m}{n} = 1 - \frac{m}{n}$ where $n > m$
Let $n = 2n$ $|x_{2n} - x_n| > \frac{1}{2} \Rightarrow$ seq not Cauchy $\Rightarrow$ does not converge
Ex: $x_1 = 1$, $x_2 = 2$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$
easy to show it is cauchy $\Rightarrow$ converges
seq of $\mathbb{R}$
Don't need to find limit!

If X is not complete, can it be embedded in one that is?

Theorem: Every metric space (X,d) has a completion (X^*,d)

Idea: Given X, let X^* = set of all Cauchy seq in X under an equiv. relation $(\sim)$ where $\{p_n\} \sim \{q_n\}$ if $\lim_{n\to\infty} d(p_n,q_n) = 0$.

Let $\Delta(P,Q)$ (for $P,Q \in X^*$) $\Delta(P,Q) = \lim_{n\to\infty} d(p_n,q_n)$ where $\{p_n\}\{q_n\}$ representatives of $P + Q$

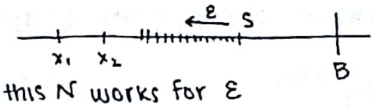
Then X^* is complete w/ X isometrically embedded in X^*.

[Another way to construct $\mathbb{R}$ from $\mathbb{Q}$]

monotonically increasing seq: $s_n \le s_{n+1}$

monotonically decreasing seq: $s_n \ge s_{n+1}$

Theorem: Bounded monotonic seq. converge [to their sup or inf]



Proof: Given $\{s_n\}$, let $s = \sup(\text{range } \{s_n\})$

$\forall \varepsilon > 0$, $\exists N$ s.t. $s - \varepsilon < s_N \le s$ but $n \ge N$, $s - \varepsilon \le s_N \le s$ so this N works for ε

Note: some seq diverge $s_n \to +\infty$ if $M \in \mathbb{R}$ $\exists N$ s.t. $n > N \Rightarrow s_n > M$

→ similarly $s_n \to -\infty$

Given $\{s_n\}$ let $E = \{\text{subseq. limit}\}$ ← allow $+\infty, -\infty$

→ Let $s^* = \sup E$ ← limsup, "upper limit"

$s_* = \inf E$ ← liminf, "lower limit"

Alternative: $\limsup s_n = \lim_{n\to\infty} (\sup_{k>n} s_n)$

$\liminf s_n = \lim_{n\to\infty} (\inf_{k>n} s_n)$

Ex: If $s_k \to s$ then $\liminf s_k = \limsup s_k = s$

Ex: $\{s_n\} = \{0.1, 2/3, 0.11, 4/3, 0.111, 5/4, 0.1111, 6/5, \ldots\}$, $s^* = 1$, $s_* = \frac{1}{9}$

SERIES

Some have learned:

$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1-\frac{1}{3}}$ (geometric series)

$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

$\left.\begin{array}{l} 1 + 2 + 4 + 8 + \cdots = \frac{1}{1-2} = -1 \\ 1 - 2 + 4 - 8 + \cdots = \frac{1}{1-(-2)} = \frac{1}{3} \\ 1 - 1 + 1 - 1 + \cdots = \frac{1}{1-(-1)} = \frac{1}{2} \end{array}\right\}$ Euler (What does this all mean??)

Given $\{a_n\}$, define $\sum_{n=p}^{q} a_n = a_p + \cdots + a_q$

Let $s_n = \sum_{n=1}^{n} a_k$, the n^{th} partial sum. Then $\{s_n\}$ is a sequence, sometimes written $\sum_{n=1}^{\infty} a_n$, called infinite series.

This may not converge, but if it does, say s, write $\sum_{n=1}^{\infty} a_n = s$

→ Notice: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ or $\lim_{n\to\infty} (\sum_{k=1}^{n} a_k)$.

A series converges when a seq of partial sums converges

Ex: $a_n = \frac{1}{n}$, does $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converge? (harmonic series)

Is $\{s_n\}$ Cauchy? $s_m - s_n = \cancel{a_n} + a_{n+1} + \cdots + a_m$ if $m > n$

$s_{2n} - s_n > \frac{1}{2}$ so not Cauchy + does not converge

The Cauchy Criterion (for series): $\sum a_n$ converges $\Leftrightarrow \forall \varepsilon > 0$ $\exists N$ s.t. $m,n > N \Rightarrow |\sum_{k=n}^{m} a_k| < \varepsilon$

Corollary: Let $m=n$, $\sum a_n$ converges $\Rightarrow \lim_{n\to m} a_n = 0$ (terms go to 0) ← term test for convergence

→ converse is false (ex: harmonic series)

Theorem (non-neg series): If $a_n \ge 0$, then $\sum a_n$ converges $\Leftrightarrow$ p sums are bounded

Proof: If $a_n \ge 0$, p sums are mono. incr. but bounded mono. seq. converge

Theorem (Comparison test)

(a) If $|a_n| \le c_n$ for if n large enough, + $\sum c_n$ converges, then $\sum a_n$ converges (say $M > N_2$

(b) If $a_n > d_n \ge 0$ ——————, if $\sum d_n$ diverges then $\sum a_n$ diverges

Proof: (a) since $\sum c_n$ converges, $\exists N$ s.t. $m,n \ge N_1 \Rightarrow |\sum_{k=n}^{m} a_k| < \varepsilon$

Let $N = \max\{N_1, N_2\}$, for $m,n > N_1$, $|\sum a_k| \le \sum |a_k| \le \sum c_k < \varepsilon$, as desired

(b) Use (a) if $\sum a_n$ converg, so does $\sum d_n$ to show (contrapositive) (qed)

What to compare to?

geometric series: If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$

If $|x| \ge 1$, then —— diverges

Proof: If $x \ne 1$

Let $s_n = 1 + x + \cdots + x^n$

Then $s_n = \frac{1 - x^{(x+1)}}{1-x}$

so $\lim_{n\to\infty} s_n = \frac{1}{1-x} \cdot \lim_{n\to\infty} (1 - x^{x+1}) = \frac{1}{1-x} \cdot 1 = \frac{1}{1-x}$

If x 1, terms $\to 0$

Q: $\Sigma \frac{1}{n^p}$ converge or diverge? For which p?

Theorem (Cauchy): If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ (monoton. decreasing)
Then Σa_n converge $\Leftrightarrow$ $\Sigma 2^k a_k$ converge $\leftarrow a_1 + 2a_2 + 4a$

Proof Idea: Compare: Let $s_n = a_1 + a_2 + \cdots = a_n = a_1 + (a_2 + a_3) + (a_3 + \cdots a_7) + \cdots$
$t_k = a + 2a_1 + \cdots = 2a_k = a_1 + (a_2 + a_2) + (a_4 + \cdots + a_4) + \cdots$

If $n < 2^k$, then $s_n < t_k$. use comparison
$2s_n = 2a_1 + 2a_2 + 2(a_3 + a_4) + 2(a_5 + \cdots + a_8)$
$t_k = a_1 + 2a_2 + 4a_4 + 8a_8$

If $n > 2^k$, then $t_k < 2s_n$, use comparison

Application: $\Sigma \frac{1}{n^p}$ converges if $p > 1$ + diverges if $p \leq 1$
If $p \leq 0$, term $\not\to 0$ so series diverges
If $p > 0$, use $\Sigma 2^k \frac{1}{n^p} = \Sigma 2^{(1-p)k}$ = geometric $\to$ converges when $|2^{(1-p)}| < 1$
↳ true if $1 - p < 0$

Ex: $\Sigma_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$ converges (to e) $\to$ its partial sums are bounded by 3
$1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

convergence is rapid: $e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!}\underbrace{\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)}_{\text{geometric seq}} = \frac{1}{n!n}$

$\to$ n^{th} term is no farther from e than $\frac{1}{n!n}$

$\to$ see e is irrational
If $e = \frac{m}{n}$ then $\underbrace{(e - s_n)n!}_{\text{integer}} < \frac{1}{n}$ — but no integer b/t 0 + $\frac{1}{n}$

SERIES CONVERGENCE TESTS, ABSOLUTE CONVERGENCE

Root Test: Given Σa_n, let $\alpha = \limsup \sqrt[n]{|a_n|}$
Then $\alpha < 1 \Rightarrow$ series converg.
$\alpha > 1 \Rightarrow$ series diverg.
$\alpha = 1 \Rightarrow$ test inconclusive

more powerful

but easier to use

Ex: $1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \cdots$

Proof by Comparison (w/ geometric series)
If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$.
Then $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} < \beta$ so $|a_n| < \beta^n$ for $n = N$ (by def of limsup)
But $\Sigma \beta^n$ conv. so Σa_n conv also
If $\alpha > 1$, $\exists$ subseq $\sqrt[n]{|a_{n_k}|} \to \alpha > 1$ so $|a_{n_k}| > 1$ for infinitely many terms, so terms $\not\to 0$, so series diverg.
If $\alpha = 1$, note $\Sigma \frac{1}{n}$ div + $\Sigma \frac{1}{n^2}$ conv, both $\alpha = 1$ (qed)

Ratio Test: Σa_n conv. if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$
div if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for n large enough

Proof by Comparison: We have $\left|\frac{a_{n+1}}{a_n}\right| <$ some $\beta < 1$ for $x >$ some N
$|a_{n+1}| < \beta|a_n| < \beta^2|a_{n-1}| < \beta^3|a_{n-2}| < \cdots$
$|a_{N+k}| < \qquad < \beta^k a_N$
Tail $\Sigma_{k=0}^{\infty} a_{N+k} \leq a_N \Sigma \beta^k$ ↳ converges

For div, see tails $\not\to 0$

If c_n is complex, a **power series** is $\Sigma_{n=0}^{\infty} c_n z^n = c_1 + c_2 z + c_3 z^2 + \cdots$ in z, a complex variable

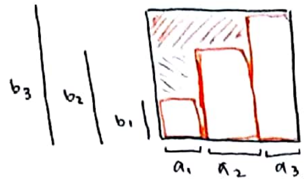
Theorem: If $\alpha = \limsup \sqrt[n]{c_n}$, let $R = \frac{1}{\alpha}$ then $\Sigma c_n z^k$ conv. if $|z| < R$
div. if $|z| > R$
radius of conv.

Proof: $\sqrt[n]{|a_n|} = |z| \cdot \sqrt[n]{|c_n|} < 1 \cdots$

Summation by Parts: Let $A_n = \Sigma_{k=0}^{n} a_k$ for $n = 0$. Let $A_1 = 0$.
Then $\Sigma_{n=p}^{q} a_n b_n = \Sigma_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$

$A \to v$ $\quad b \to u$
$a \to dv$

like integration by parts

Proof Idea: 

alg. proof: check both sides

Theorem: If A_n is bounded, b_n decreasing + $\to 0$, then $\Sigma a_n b_n$ converges.

Proof: Say $|A_n| \le M$. $\exists N$ s.t. $b_N \le \varepsilon/2M$ given $\varepsilon > 0$

For $q \ge p \ge N$

|bound summation by parts| $\le M \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_p + b_q \right|$

$\le 2Mb_p \le 2Mb_N \le \varepsilon$, as desired

[**Corollary:** $|c_1| \ge |c_2| \ge \cdots$, c_i has alt. signs + $\to 0$, Σc_n converges]

Proof: $a_n = (-1)^{n+1}$, $b_n = |c_n|$;

motivation: power series

Sums of Series: $\Sigma a_n + \Sigma b_n = \Sigma (a_n + b_n)$

Products: $(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 b_0) + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots$

Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, product series: Σc_n

Problem: Σc_n may not conv, even if $\Sigma a_n \Sigma b_n$ do!

But (**Theorem**): If $\underset{A}{\Sigma a_n}$, $\underset{B}{\Sigma b_n}$ converges absolutely then $\underset{AB}{c_n \text{ conv.}}$

Define: a series **converges absolutely** if $\Sigma |a_n|$ converges

→ Ex: $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$ conv. but not absolutely

Proof Idea: $\sum_{k=n}^{m} a_k \le |\sum_{k=n}^{m} a_k|$ ← small by Cauchy criterion for $\Sigma |a_k|$

Say $\Sigma a_n = A$. If I **rearrange** the terms, must it converge? No

→ If it does converge: it may not conv. to A

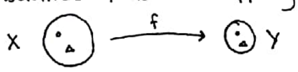
→ Ex: $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \ln 2$ (pos + neg terms get infinitely large!)

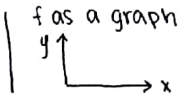
Riemann: If Σa_n conv (but not abs), then a rearrangement can have any limsup + liminf you like

If ——— absolutely, then all rearrangements have the same limit

FUNCTIONS: LIMITS + CONTINUITY

Functions: Let X, Y be metric spaces, $f: X \to Y$

→ Visualize: f as a mapping 

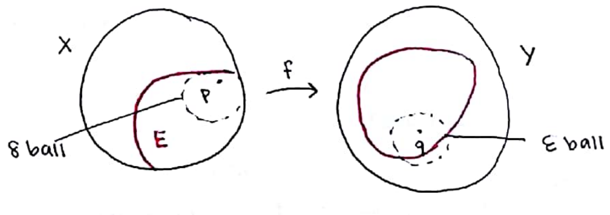
f as a graph 

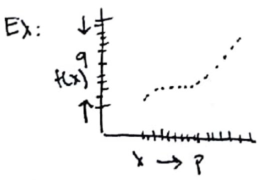
X, Y metric. $E \subset X$, p is a **limit point** of E. Let $f: E \to Y$. ↗ in E

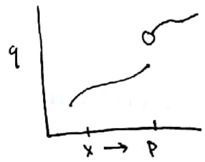
To say "$f(x) \to q$ as $x \to p$" or "$\lim_{x \to p} f(x) = q$" means $\exists q \in Y$ s.t.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in E$, $0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$

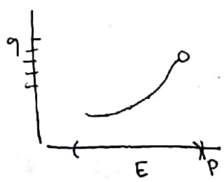
↖ excludes $x = p$



Ex: 



no limit as $x \to p$



p does not have to be on E

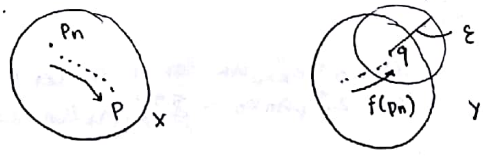
To show convergence given $\varepsilon > 0$, find a δ that works

Theorem: $\lim_{x \to p} f(x) = q$ iff **sequence characterization**: $\forall$ seq $\{p_n\}$, $p_n \ne p$, $p_n \to p$ (in E) we have $f(p_n) \to q$ (seq. convergence)

Proof ($\Rightarrow$) Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

So for a given $\{p_n\}$ as above,

$\exists N$ s.t. $d(p_n, p) < \delta$ so $n \ge N$ implies $d(f(p_n), q) < \varepsilon$



($\Leftarrow$) If $\lim_{x \to p} f(x) \not\to q$ then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$

$\exists x \in E$ s.t.

$0 < d(x,p) < \delta$ but $d(f(x), q) \ge \varepsilon$

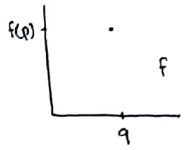
We propose bad seq: use $\delta_n = \frac{1}{n}$, choose x_n by

Then $x_n \to p$, but $d(f(x), q) \ge \varepsilon$ by

So $f(x_n) \not\to q$ (qed)

From theorems on sequences
→ limits are unique
→ limits of sums are sums of limits (can apply to products)

$Y = \mathbb{R}$

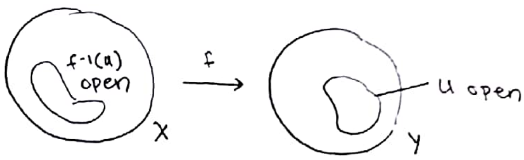


f is cont.

X, Y metric space. $p \in E \subset X$, $f: E \to Y$
Say f is **continuous** at p if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in E$, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$
→ x may be p + $p \in E$

Theorem: If p is a limit pt of E, then f continuous @ $p \Leftrightarrow \lim_{x \to p} f(x) - f(p)$
Also if x_n is a converg. seq, f continuous $\Leftrightarrow \lim_{n\to\infty} f(x_n) = f(\lim_{n \to \infty} x_n)$ ← limit + func. op. commute

Corollary: sums, prods of contin. func. are contin ($Y = \mathbb{R}$); quotient f/g as well ($g \neq 0$)
Corollary: $f, g: X \to \mathbb{R}^k$ s.t. $f = (f_1, f_2, f_3, \dots f_k)$
a) f contin $\Leftrightarrow$ each f_i contin
b) $f + g$, $f \cdot g$ (dot) contin
Proof Idea: a) use $|f_i(x) - f_i(p)| \leq \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^{n} |f_i(x) - f_i(p)|^2}$
b) use components from part (a)

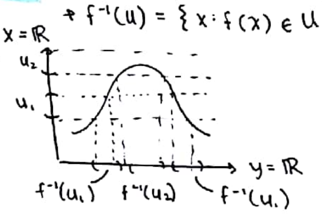


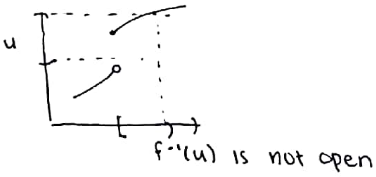
Theorem: $f: X \to Y$ is contin $\Leftrightarrow \forall$ open sets U in Y, $f^{-1}(U)$ is open in X

CONTINUOUS FUNCTIONS

One very useful characterization:
Theorem: $f: X \to Y$ is contin. $\Leftrightarrow \forall$ open set U in Y, $f^{-1}(U)$ is open in X
+ $f^{-1}(U) = \{x : f(x) \in U\}$

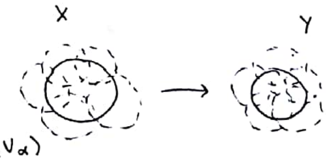
Ex:





Proof: Given U open in Y, consider $p \in f^{-1}(U)$. Will show p is interior point of $f^{-1}(U)$
Note $f(p) \in U$ open, so $\exists$ ball/nbhd $N_\varepsilon(f(p)) \subset U$
By continuity of f, $\exists$ δ-ball $N_\delta(p)$ that maps into $N_\varepsilon(f(p)) \subset U$.
This means $N_\delta(p) \subset f^{-1}(U)$. So p is an interior point of $f^{-1}(U)$.
($\Leftarrow$) Fix point $p \in X$, $\varepsilon > 0$
Let $B = \varepsilon$-ball about $f(p)$.
Then $p \in f^{-1}(B)$ which is open by the assumption
Since p is interior pt $f^{-1}(B)$, $\exists$ some ball $N_\delta(p) \subset f^{-1}(B)$
This δ has required property since $f(N_\delta(p)) \subset B$ so f is cont.

Theorem: $X \xrightarrow{f} Y \xrightarrow{g} Z$. f, g contin $\Rightarrow g$ composed w/ f is contin.
Proof: Given U open Z. $\Rightarrow g^{-1}(U)$ is open in Y (g is cont)
$\Rightarrow f^{-1}(g^{-1}(U))$ —— X (f is cont)
$= (g \circ f)^{-1}(U)$



cover $\{V_\alpha\}$

Theorem: $f: X \to Y$ contin $\Leftrightarrow \forall$ closed K in Y, $f^{-1}(K)$ is closed in X
Proof Idea: $f^{-1}(K) = [f^{-1}(K^c)]^c$

Theorem: $f: X \to Y$ contin, X compact $\Rightarrow f(X)$ is compact
Proof: Let $\{V_\alpha\}$ cover of $f(X)$. Let $U = f^{-1}(V_\alpha)$
By compactness of X, $\exists$ finite subcover $U_1, \dots U_n$. Then $V_1, \dots, V_n$ cover $f(X)$

Corollary: $f: X_{qk} \to \mathbb{R}^k$, then $f(X)$ is closed + bounded
Corollary: f: compact $X \to \mathbb{R}$, then f achieves its max + min
→ i.e. interval or disc

Theorem: If $f: X \to Y$ bijection + contin, X is compact $\Rightarrow f^{-1}$ is contin
Proof: Suppose U is open in $X \Rightarrow U^c$ is closed in X compact
$\Rightarrow U^c$ is compact $\Rightarrow f(U^c)$ is compact
$\Rightarrow f(U^c)$ is closed $\Rightarrow f(U)$ is open

UNIFORM CONTINUITY

To say function $f: X \to Y$ is continuous means
1. Close enough points map to close points
2. Func. preserve limit of seq.
3. Inverse images of open/closed sets are open/closed

Call $f: X \to Y$ **uniformly continuous** on X if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x + p$ in X, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$
→ same δ works for all p in X

Theorem: $f: X \to Y$ contin, X compact then f is uniformly contin.
Proof: (Goal - find a δ that works for all p)
Given $\varepsilon > 0$, each pt x has δ-ball s.t. $d(y,x) < \delta \Rightarrow d(f(y), f(x)) <$ ☺ $= \frac{\varepsilon}{2}$
These cover X
(Q: Can I find δ s.t. if $d(p,q) < \delta$ then p, q are the same cover set?)
→ For then $d(p,q) \leq d(p,x) + d(q,x) \leq$ ☺ + ☺ since $d(p,x) < \delta_x$ + $d(q,x) < \delta_x$

Lebesgue covering lemma: If $\{U_x\}$ is open cover of compact X then $\exists \delta > 0$ (lebesgue # of cover) s.t. $\forall x \in X$, $B_\delta(x)$ is contained in some U_x
Proof: Since X compact, $\exists$ finite subcover, $\{U_{\alpha_i}\}_{i=1}^n$
If K closed, define $d(x,K) = \inf_{y \in K} d(x,y)$
Claim: $d(x,K)$ is contin func. of x then $f(x) = \frac{1}{n} \sum_{i=1}^n d(x, U_{\alpha_i}^c)$ is contin. func. on compact set so it attains its own value δ

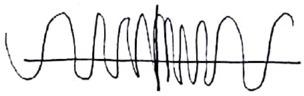
So if $f(x) \geq \delta$ then at least one of $d(x, U_{\alpha_i}^c) \geq \delta$, so for this i, $B(x) \subset U_x$ (qed)
(Notice $\delta > 0$ b/c $f(x) > 0$ @ each x since U_{α_i} cover

Theorem: $f: X \to Y$ contin, E connected $\subset X$, then $f(E)$ is conn
Proof: Suppose $f(E)$ is not conn, then $f(E) - A \cup B$ a separation (non-$\emptyset$, $\bar{A} \cap B + A \cap \bar{B} \neq \emptyset$)
Notice $K_A = f^{-1}(A)$, $K_B = f^{-1}(B)$ are closed (same f contin)

Let $E_1 = f^{-1}(A) \cap E$, $E_2 = f^{-1}(B) \cap E$ } disjoint, non-$\emptyset$ Claim: they separate E

Notice $E_1 \subset K_A$, $E_2 \subset K_B$ closed so $\bar{E_1} \subset K_A$, $\bar{E_2} \subset K_B$ + $K_A \cap E_2 = \emptyset$; $K_A = f^{-1}(\bar{A})$, $E_2 \subset f^{-1}(B)$ + $\bar{A} \cap B = \emptyset$ (similarly $K_B \cap E_1 = \emptyset$) E is separated (qed)

Intermediate Value Theorem: If $f: [a,b] \to \mathbb{R}$ contin, + $f(a) < c < f(b)$ then $\exists x \subset (a,b)$ s.t. $f(x) = c$
Proof: $[a,b]$ connected $\Rightarrow f([a,b])$ connect but if c not achieved then "c disconnects" $f([a,b])$ (qed)
Converse false: **topologist sine curve** $f(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \neq 0 \end{cases}$
not contin but satisfies IV Prop.

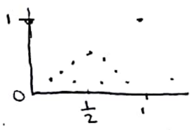


DISCONTINUOUS FUNCTIONS

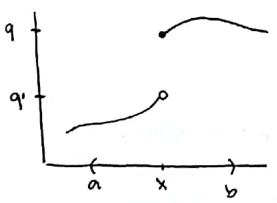
Discontinuous Functions

→ Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ f is not contin @ any p

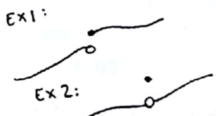
→ Ex: $f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ discontin. @ all rationals but contin at all irrationals



→ Ex: $f(a,b) \to \mathbb{R}$



Note: $\forall \{t_n\}$ in (x,b) w/ $t_n \to x$, we have $f(t_n) \to q$
Write $f(x^+) = q$ or $\lim_{t \to x^+} f(t) = q$.
Similarly $f(x^-) = q'$ or $\lim_{t \to x^-} f(t) = q'$.
$\lim_{t \to x} f(t)$ exists if $f(x^+) = f(x^-)$
↳ If not equal, say f has a discontinuity of the first kind (i.e. simple discontinuity) → $f(x^+) + f(x^-)$ exist



Discontinuity of the second kind: $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(\frac{1}{x}) & \text{if } x > 0 \end{cases}$
Dirichlet limits

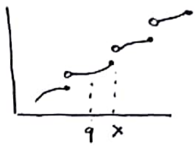
→ Ex: $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is contin @ 0

$f: (a,b) \to \mathbb{R}$
→ f is monotonically incr. if $x \leq y \Rightarrow f(x) \leq f(y)$
decr. ... $\Rightarrow f(x) \geq f(y)$

Theorem: f mono. incr. in $(a,b) \to f(x^+), f(x^-)$ exists $\forall x \in (a,b)$.

$\to$ In fact, $\sup_{t\in(a,x)} f(t) \le f(x) \le \inf_{t \in (x,b)} f(t)$] sup + inf exist b/c bounded

↑ call A, claim $A = f(x^-)$



Given $\varepsilon > 0$, consider $A-\varepsilon$. $\exists\, \delta$ s.t. $A-\varepsilon < f(x-\delta) \le A$ (same A as sup)
but then any $t \in (x-\delta, x)$ must satisfy $f(x-\delta) \le f(t) < A$.
So $f(t) \in (A-\varepsilon, A)$, as desired. Similar arg. on other side.

Corollary: Monotonic func. have no discontin. of 2nd kind

Theorem: f mono. in (a,b) set of pts where f is not contin. is countable
Proof: $\forall x$ (where f is discontin) $\in D$.
Pick $r(x) \in \mathbb{Q}$ s.t. $f(x^-) < f(x) < f(x^+)$
If $x, y \in D$, $r(x) \ne r(y)$ b/c f is mono. Get 1:1 corres. b/t D + subset of $\mathbb{Q}$ (qed)

THE DERIVATIVE AND MEAN VALUE THEOREM

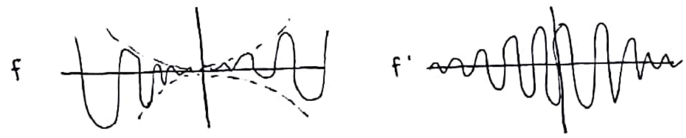
A function $f:[a,b] \to \mathbb{R}$ is differentiable at $x \in [a,b]$ if the limit exists: $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ $\quad t \in (a,b)$, $t \ne x$

If f contin = [a,b] is f differentiable on [a,b]? No [the derivative of f at x]

If f differentiable on [a,b], is f contin on [a,b]? No

$\to \lim_{t\to x} f(t) - f(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}(t-x) = f'(x) = 0$

$\to$ ex: $f(x) = \begin{cases} x^{4/3}\sin\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & x = 0 \end{cases}$



not always contin but f' always satisfies IVP + has a simple discontin

Call a function f a "$\mathscr{C}$-function" if f' exists + is contin.
"$\mathscr{C}^k$ function" if k^{th} derivative $f^{(k)}$ exists + is contin.
"$\mathscr{C}^0$" - continuous func.
"$\mathscr{C}^\infty$" - all derivatives exist (smooth)

If f' is a limit, then sum/prod/quot. must follow
$(f+g')' = f'+g'$ $\quad (fg)' = f'g + fg'$
sum(limits) = lim(sum)

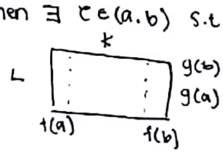
Theorem: There exists func. $\mathbb{R}\to\mathbb{R}$ that are contin. everywhere but differentiable nowhere.
$\to$ Ex: $f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$; $0<b<1$, a odd $\mathbb{Z}$, $ab > 1 + 3\pi/2$

Mean Value Theorem: If f is contin on [a,b], differentiable on (a,b) then $\exists$ point $c \subset (a,b)$ s.t.
$f(b) - f(a) = (b-a)f'(c)$
$\to$ slope $= f(b)-f(a)/(b-a)$
$\to$ connects value of f to value of f' (w/o using limits)

App: If $f'(x) > 0$ for all $x \in (a,b)$, then show $f(b) > f(a)$
Proof: $f(b) - f(a) = (b-a)\cdot f'(c) > 0$, as desired

Proof: If h on [a,b] has a local max at $c \in [a,b]$ + $h'(c)$ exists $\Rightarrow h'(c) = 0$ ★
Suppose $\frac{h(t)-h(c)}{t-c} < 0$ b/c c is a local max if $t > c$
Since left + right limit exist + are equal, limit must be 0

Generalized MVT: If $f(x), g(x)$ is contin on [a,b] + differentiable on (a,b) then $\exists\, c \in (a,b)$ s.t.
$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$ $\quad$ (Note: if $g(x) = x$, get MVT)
Idea: LHS: rate L sweeps area
RHS: —— K



Consider $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$
diff. in area swept by time x
clear: $h(a) = 0$
$h(b) = 0$
So ★ $\to \exists$ t s.t. $h'(t) = 0$ but $h'(x)$ = LHS − RHS (qed)

TAYLOR THEOREM, SEQUENCE OF FUNCTIONS

Taylor's theorem: Suppose know $f(a)$, want appropriate $f(b)$ "error" not precisely known

MVT says: $f(b) = f(a) + f'(c)(b-a)$

This suggests $f(b) = f(a) + f'(a)(b-a) +$ error In fact, $\frac{f''(c)}{2}(b-a)^2$, $c \in (a,b)$

More generally, if $P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)}$ ← poly. degree n-1

If $f^{(n-1)}$ contin on $[a,b]$, f^n contin on (a,b) then $P_{n-1}(x)$ approx $f(x)$ +
$f(x) = P_{n-1}(x) + \frac{f^n(c)}{n!}(x-a)^n$ where $c \in (a,b)$

→ when $n=1$, this is MVT

→ $P_n(x)$ is "best" poly approx. of order at n — same orders $\begin{cases} f, f', f'', f^{n-1} \\ P, P', P'', P^{n-1} \end{cases}$

Proof: Clearly for some number M, $f(b) = P_{n-1}(b) - M(b-n)^n$ ★

Let $g(x) = f(x) - P_{n-1}(x) - M(x-a)^n$

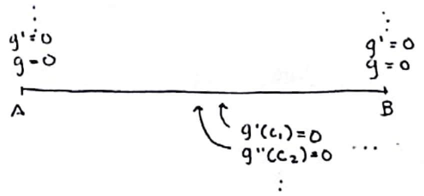
$g^{(n)}(x) = f^{(n)}(x) - Mn!$

Enough to show $g^{(n)}(c) = 0$ for some $c \in (a,b)$

Check $g(a)$ since $f(a) = P_{n-1}(a)$; $g'(a) = 0$, $g''(a) = 0$

Also $g(b) = 0$ by ★

This shows $M = f^{(n)}(c)/n!$



What does it mean for seq of func. to converge?

→ **Pointwise convergence**: fix x, does $\{f_n(x)\}$ converge? If so, **pointwise limit** $f(x) = \lim_{n\to\infty} f(x)$

What prop preserved for pointwise?

→ not continuity, derivatives, integrals, etc.

→ need stronger notion, let $\|f\| = \sup_{x\in E}|f(x)|$ ← usual convergence in metric space $\mathcal{C}_b(E)$ (contin, bounded func. in E) $d(f,g) = \|f-g\|$

Say $f_n \xrightarrow{u} f$ (**converges uniformly**) in E if $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \|f_x - f\| < \varepsilon$

Fact: $\mathcal{C}_b(E)$ is complete so we have Cauchy seq/criterion

Theorem: $f_n \to f$ on $E \iff \forall \varepsilon > 0$ $\exists N$ s.t. $\forall n, m > N$ + $\forall x \in E$, $|f_n(x) - f_m(x)| < \varepsilon$

Theorem: If $f_n \xrightarrow{u} f$, f_n contin, then f contin

Proof: use bound $|f(x) - f(y)| \leq \underbrace{|f(x) - f_n(x)|}_{☺} + \underbrace{|f_n(y) - f_n(y)|}_{☺} + \underbrace{|f_n(y) - f(y)|}_{☺}$

Fix x.

$\forall \varepsilon > 0$, choose f_n so $\|f_n - f\| < \frac{\varepsilon}{3}$. So ☺ + ☺ $< \frac{\varepsilon}{3}$

Then f_n contin. $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow$ ☺ $< \frac{\varepsilon}{3}$

So $\forall \varepsilon > 0$, we found $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{3}(3) = \varepsilon$ (qed)

Theorem: $\exists f: [0,1] \to [0,1]^2$ base that is space-filling.