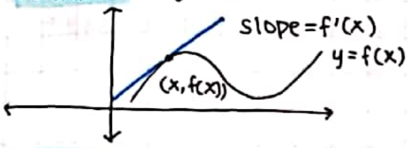
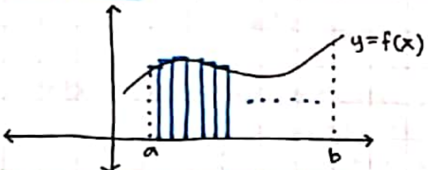
LECTURE 1.1.1 | Review + Introduction

Derivative of $y=f(x)$ function $\rightarrow \frac{dy}{dx}(x)=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ if this limit exists



Uses of Derivatives:
- derivative is the rate at which y is changing as x increases
- min or max of $f(x)$ occurs where $f'(x)=0$ (or on boundary of domain where x is defined or where $f'(x)$ is not defined)

Integration of $y=f(x) \rightarrow \int_a^b f(x)dx$ = "area under the graph" (if $f>0$)



↳ approx. with rectangles divided by n subintervals as $n\to\infty$

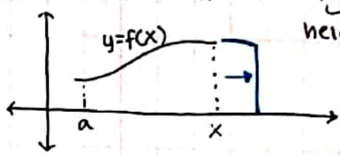
n subintervals $\rightarrow a=x_0<x_1<x_2<\dots<x_n=b$

width of subinterval $\rightarrow x_i - x_{i-1} = \frac{b-a}{n}$ (equal length) $=\Delta x$

height of subinterval $\rightarrow x_i^* =$ sample point w/ $x_{i-1}\le x_i^* \le x_i$

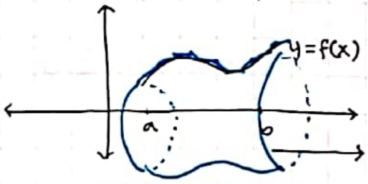
↳ $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height}}\underbrace{\Delta x}_{\text{width}}$ ← well defined if x is continuous

Fundamental Theorem of Calculus
1. $\int_a^b f'(x)dx = f(b)-f(a)$
2. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$



*the rate at which the area incr. as x moves right is the height

Other Useful Formula

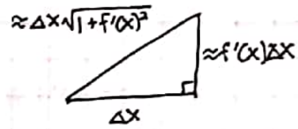


adding 3rd dimension z

Length of the graph $y=f(x)$ from $(a,f(a))$ to $(b,f(b))$

$L=\int_a^b\sqrt{1+f'(x)^2}\,dx$

↳ approx graph w/ n line segments as $n\to\infty$



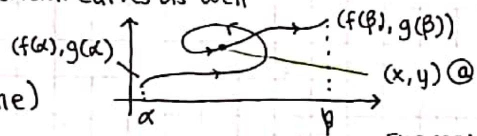
→ **Area of surface of revolution around the x axis**: $A=\int_a^b \underbrace{2\pi f(x)}_{\text{circumference}=2\pi r}\sqrt{1+f'(x)^2}\,dx$

↳ approx by many rings which is calculated by length of line segment times circumference of the circle

LECTURE 1.1.2 | Intro to Parametrized Curves

Parametric curves include general curves as well

$x=f(t)$
$y=g(t)$
$\alpha\le t\le\beta$ (parameter time)



Example 1

$x=\cos t$
$y=\sin t$
$0\le t\le 2\pi$

} unit circle going counterclockwise (at unit speed)

Example 2

$x=\cos(3t)$
$y=\sin(3t)$
$0\le t\le 2\pi$

} unit circle going cc three times (3x as fast)

Example 3

$x=\cos(-t)=\cos(t)$
$y=\sin(-t)=-\sin(t)$
$0\le t\le 2\pi$

} unit circle going clockwise

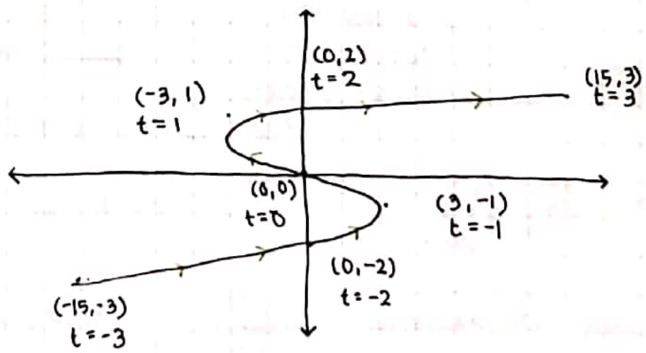
Parametrization is pair of functions $(f(t), g(t))$ that tells you where on the curve you are at a given time (timetable)

How to sketch a Parametrized curve from equations

Method 1: Plot points + connect the dots

t	x	y
-3	-15	-3
-2	0	-2
-1	3	-1
0	0	0
1	-3	1
2	0	2
3	15	3

Sketch $x=t^3-4t$
$y=t$
$-3\le t\le 3$



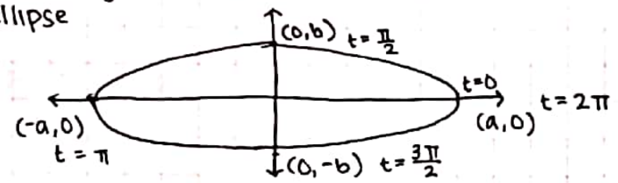
LECTURE 1.1.3 | More Techniques for Sketching Parametrized Curves

Method 2: Eliminating the parameter t

→ Example: sketch $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ where $a, b > 0$ (constants)

Trig Identity: $\sin^2 t + \cos^2 t = 1 \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ * every point on a curve must satisfy this equation!

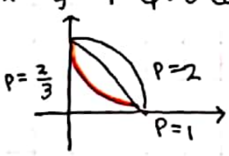
Ellipse



→ Example 2: Sketch the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$

Trig Identity: $\sin^2 t + \cos^2 t = 1 \rightarrow x^{2/3} + y^{2/3} = 1$

$x^p + y^p = 1$ ($p > 0$ constant, $x, y \ge 0$)



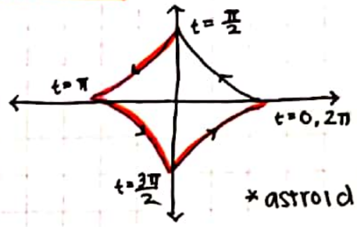
$\frac{d^2y}{dx^2} > 0$

convex

$\frac{d^2y}{dx^2} < 0$

concave

Method 3: using symmetry



* astroid

LECTURE 1.1.4 | Slope of a Parametrized Curve

We have $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ → **Heuristic**: slope $= \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$ if $f'(t) \neq 0$

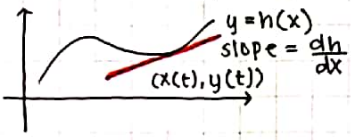
→ If $f'(t) = 0$ + $g'(t) \neq 0$, slope $= \infty$ (tangent is vertical)

→ **Justification**: If $f'(t) \neq 0$, then locally the curve is a graph $y = h(x)$

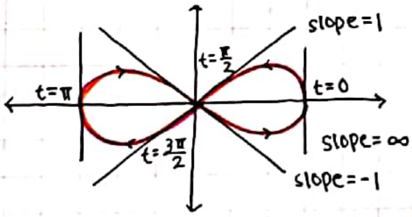
$y = h(x)$

$g(t) = h(f(t))$ ← chain rule

$g'(t) = \underbrace{\frac{dh}{dx}}_{\text{slope}} f'(t) \rightarrow$ **slope $= \frac{g'(t)}{f'(t)}$**



Example: Sketch $x = \cos t$, $y = \sin t \cos t$, $0 \le t \le 2\pi$



slope $= \frac{dy/dt}{dx/dt} = \frac{\cos^2 t - \sin^2 t}{-\sin t}$

$t = \frac{\pi}{2} \rightarrow$ slope $= \frac{0-1}{-1} = 1$

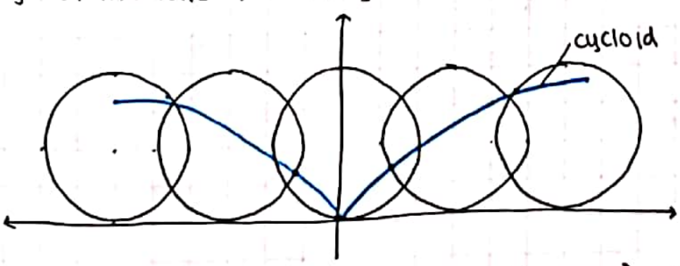
$t = \frac{3\pi}{2} \rightarrow$ slope $= \frac{0-1}{-(-1)} = -1$

$t = 0 \rightarrow (x,y) = (1,0) \rightarrow$ slope $= \infty$

$t = \pi \rightarrow (x,y) = (-1,0) \rightarrow$ slope $= \infty$

LECTURE 1.1.5 | Cycloid Example

A wheel of radius R rolls on top of the x axis. The cycloid consists of the positions of a point on the edge of the wheel, starting at its origin.



Center of the wheel:

$x = Rt$, $y = R$, t = rotation of wheel in radians

Displacement from center

Left $R\sin t$

Down $R\cos t$

$(-R\sin t, -R\cos t)$

$x = Rt - R\sin t$, $y = R - R\cos t \rightarrow x = R(t - \sin t)$, $y = R(1 - \cos t)$

Calculating slope:

$\left.\begin{array}{l} x' = R - R\cos t \\ y' = R\sin t \end{array}\right\}$ slope $= \frac{R\sin t}{R - R\cos t} = \frac{\sin t}{1 - \cos t}$ → at $(0,0)$ $\lim_{t \to 0} \frac{\sin t}{1 - \cos t} \rightarrow \lim_{t \to 0} \frac{\cos t}{\sin t} = \infty$

L'Hospital Rule ↗

Check for concavity: $\frac{d^2y}{dx^2} < 0$

$\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx} = \frac{d(dy/dx)/dt}{dx/dt}$ by chain Rule $= \frac{\frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right)}{R(1 - \cos t)}$ continue next page...

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(\frac{dy}{dx})/dt}{\frac{dx}{dt}} \text{ by chain Rule} \Rightarrow \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{R(1-\cos t)} = \frac{\frac{(1-\cos t)\cos t - \sin t(\sin t)}{(1-\cos t)^2}}{R(1-\cos t)}$$

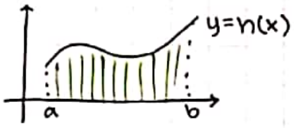
$$= \frac{\cos t - \cos^2 t - \sin^2 t}{R(1-\cos t)^3} = \frac{\cos t - 1}{R(1-\cos t)^3}$$

$$= \frac{-1}{R(1-\cos t)^2} < 0$$

i.e. curve is the graph of a concave function (between points where it touches the x axis)

LECTURE 1.1.6 | Area Under a Parametrized Curve w/o Vertical Tangents

We have $x = f(t)$
$y = g(t)$
$\alpha \le t \le \beta$

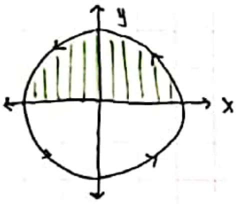


Area $= \int_a^b h(x)dx$ ← substitute
Area $= \pm\int_\alpha^\beta g(t)f'(t)dt$ $\left[\begin{array}{l} dx = f'(t)dt \\ h(x) = g(t)\end{array}\right.$

(+ if $a = x(\alpha)$ + $b = x(\beta)$ curve is going right

(− if $a = x(\beta)$ + $b = x(\alpha)$ curve is going left

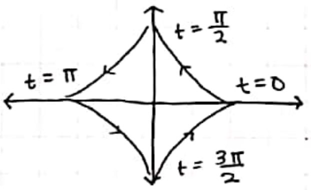
Example: Calculate the area enclosed by the unit circle : $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$



$$\text{Area} = -2\int_0^\pi y(t)x'(t)\,dt$$
$$= -2\int_0^\pi \sin t(-\sin t)dt = 2\int_0^\pi \sin^2 t\,dt$$
$$= 2\int_0^\pi \frac{(1-\cos 2t)}{2}dt$$
$$= \int_0^\pi (1-\cos 2t)dt$$
$$= \left(t - \frac{\sin 2t}{2}\right)\Big|_{t=0}^{t=\pi} = \pi$$

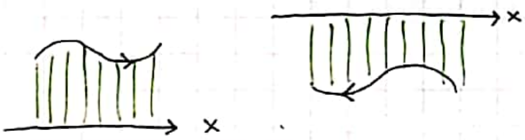
$\cos^2 t + \sin^2 t = 1$
$\cos^2 t - \sin^2 t = \cos 2t$
$\sin^2 t = \frac{1-\cos 2t}{2}$

Example 2: Calculate the area enclosed by astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$

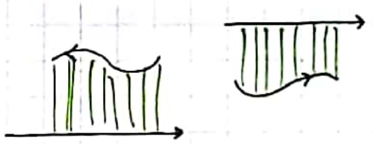


$$\text{Area} = -2\int_0^\pi \sin^3 t(-3\cos^2 t\sin t)dt$$
$$= 6\int_0^\pi \sin^4 t\cos^2 t\,dt$$
$$= 6\int_0^\pi \frac{\sin^2(2t)}{4}\left(\frac{1-\cos 2t}{2}\right)dt$$
$\vdots$ (see left.)
$$= \frac{3\pi}{8}$$

$\sin 2t = 2\sin t\cos t$



$= +\int_\alpha^\beta y x'(t)dt$



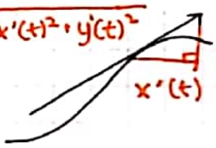
$= -\int_\alpha^\beta y x'(t)dt$

* If finding area of closed curve, it is positive if cc + negative if clockwise

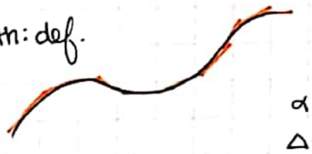
LECTURE 1.1.7 | Length of a Parametrized Curve

$x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ → $L = \int_\alpha^\beta \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{\text{length of velocity vector = speed}}dt$

Logic: $\sqrt{x'(t)^2 + y'(t)^2}$

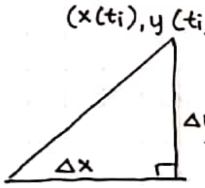


Length: def.



$\alpha = t_0 < t_1 < \cdots < t_N = \beta$ ← N subintervals
$\Delta t = t_i - t_{i-1} = \frac{\beta - \alpha}{N}$

→ Define $L = \lim_{N\to\infty} \sum_{i=1}^N$ length (line segment from $(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$)
$= \lim_{N\to\infty} \sum_{i=1}^N \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$



If Δt is small, we know $\Delta x_i \approx x'(t)\Delta t$ ⇐ $x'(t) = \lim_{\Delta t\to 0} \frac{\Delta x_i}{\Delta t}$
$\Delta y_i \approx y'(t)\Delta t$

$$L = \lim_{N\to\infty} \sum_{i=1}^N \Delta t\sqrt{x'(t)^2 + y'(t)^2} = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}$$

Example: Find the length of the unit circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$
$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{1}\,dt = 2\pi$

* should not be negative!

Fact: Length of curve does not depend on parametrization (as long as you do not cover the same part of the curve more than once)

Example 2: $x = \cos 3t$, $y = \sin 3t$, $0 \le t \le \frac{2\pi}{3}$
$L = \int_0^{2\pi/3} \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2}\,dt = \int_0^{2\pi/3} \sqrt{9}\,dt = \int_0^{2\pi/3} 3\,dt = 3\left(\frac{2\pi}{3}\right) = 2\pi$

* **length** is the integral of speed w/respect to time

Example 3: Find the length of astroid $x=\cos^3 t$, $y \sin^3 t$, $0 \le t \le 2\pi$

$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y(t)^2}\, dt = \int_0^{2\pi} \sqrt{(3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt = \int_0^{2\pi} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{9\sin^2 t \cos^2 t(\cos^2 t + \sin^2 t)}\, dt = \int_0^{2\pi} \sqrt{9 \sin^2 t \cos^2 t}\, dt$

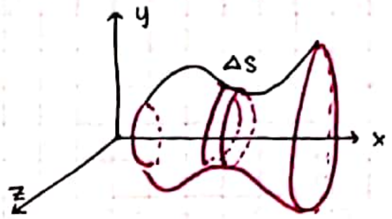
[TRAP!
$\int_0^{2\pi} 3\sin t \cos t\, dt = \frac{3}{2}\sin^2 t \Big|_0^{2\pi} = 0$]

* Remember an astroid has symmetry!

$= 4\int_0^{\pi/2} \sqrt{9\sin^2 t \cos^2 t}\, dt$

$= 4\int_0^{\pi/2} 3\sin t \cos t\, dt = 6\sin^2 t \Big|_0^{\pi/2} = 6$

LECTURE 1.1.8 | Area of Surface of Revolution of a Parametrized Curve



We have $x=f(t)$, $y=g(t)$, $\alpha \le t \le \beta$

$L = \int_\alpha^\beta \sqrt{x'(t)^2 + y(t)^2}\, dt$

ds = "element of arc length"

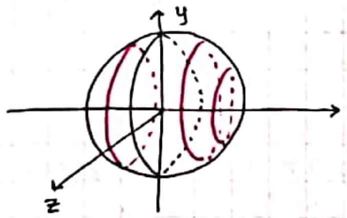
Surface of revolution around the x axis

Area = approx w/ thin ribbons (two consecutive points of length Δ rotated along x axis)

Area $= \lim_{\Delta s \to 0} \sum$ Area(ribbon) $\Rightarrow$ Area(ribbon) $\approx 2\pi y \Delta s$

$= \int_\alpha^\beta 2\pi y\, ds = \int_\alpha^\beta 2\pi y \sqrt{x'(t)^2 + y'(t)^2}\, dt$

Example: Calculate the area of unit sphere $x^2+y^2+z^2=1$



$x = \cos t$

$y = \sin t$

$0 \le t \le 2\pi$

[TRAP!
$A = \int_0^{2\pi} 2\pi y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$
* this will cause the circle to revolve twice!]

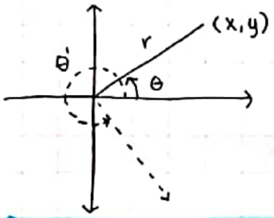
Area $= \int_0^\pi 2\pi \sin t \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$ds = dt \leftarrow \cos^2 x + \sin^2 x = 1$

$= \int_0^\pi 2\pi \sin t\, dt = -2\pi \cos t \Big|_0^\pi = -2\pi(-1-1) = 4\pi$

LECTURE 1.2

1.2.1. Introduction to Polar Coordinates



$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$

Polar coordinates: (r, θ)

* θ from positive x axis

$r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

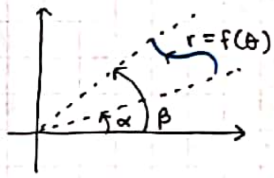
[⚠ 1) θ is defined only up to adding integer multiples of 2π
2) sometimes we allow $r<0$]

Curves in Polar Coordinates

We have $r = f(\theta)$, $\alpha \le \theta \le \beta$

→ like parametrized curve in x, y coordinates

$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$ $\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}$ $\alpha \le \theta \le \beta$

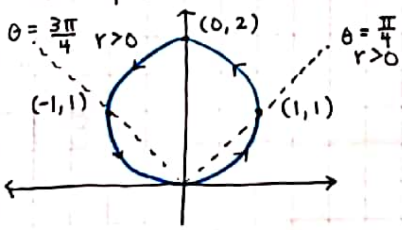


How to sketch $r=f(\theta)$, $\alpha \le \theta \le \beta$

Method 1: Plot points

→ Example: $r = 2\sin\theta$, $0 \le \theta \le \pi$

θ	r
0	0
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$\sqrt{2}$
π	0



Method 2: Convert to Cartesian Coordinates (x, y)

→ Example: same as left

$r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta$

$x^2+y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2+y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$

circle of radius 1 w/center (0, 1)

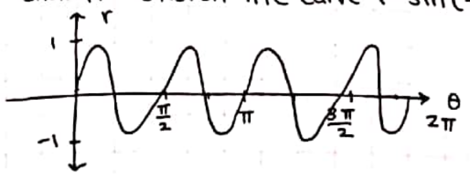
Example: $r = 2\sin\theta$, $\pi \le \theta \le 2\pi$

→ because $r<0$, we get flipped to top part of the plane (if $r>0$, would be in lower half)

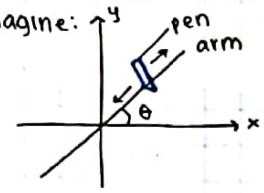
→ same curve! — no assumption on θ

↳ only question we ask if if we cover the curve more than once

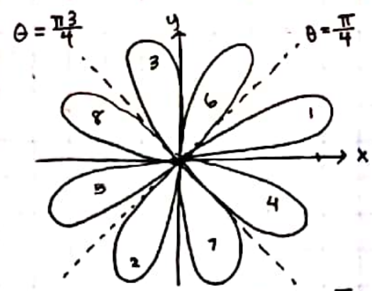
Example: Sketch the curve $r = \sin(4\theta)$ [Default: $(0 \le \theta \le 2\pi)$]



$\Rightarrow$ imagine:



$\Rightarrow$

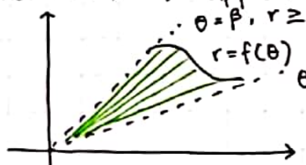


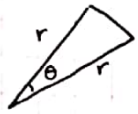
1.2.2. Slope + area in Polar Coordinates

We have $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

$\rightarrow$ slope $= \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d/d\theta\,(f(\theta)\sin\theta)}{d/d\theta\,(f(\theta)\cos\theta)}$

Area "under" the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$:
(assume $f \ge 0$) – approx w/ thin pie slices





$\text{Area(wedge)} = \frac{r^2 \Delta\theta}{2}$

$\text{Area} = \lim_{\Delta\theta \to 0} \Sigma \text{Area(pie slice)}$
$= \lim_{\Delta\theta \to 0} \Sigma \frac{r^2 \Delta\theta}{2} = \int_\alpha^\beta \frac{r^2}{2} d\theta = \int_\alpha^\beta \frac{f(\theta)^2}{2} d\theta$

Example: Calculate slope of $r = 1 - 2\cos\theta$ @ $\theta = \pi/2$

$x = (1 - 2\cos\theta)\cos\theta$
$y = (1 - 2\cos\theta)\sin\theta$

$\rightarrow$ slope $= \frac{\frac{d}{d\theta}[(1-2\cos\theta)\sin\theta]}{\frac{d}{d\theta}[(1-2\cos\theta)\cos\theta]}$

$= \frac{(2\sin\theta)\sin\theta + (1-2\cos\theta)\cos\theta}{2\sin\theta(\cos\theta) + (1-2\cos\theta)(-\sin\theta)}$

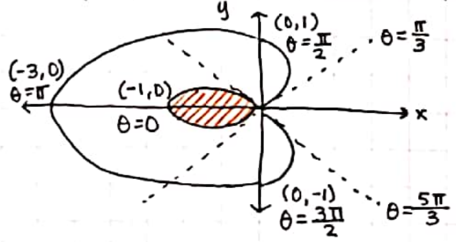
$= \frac{2\sin^2\theta + \cos\theta - 2\cos^2\theta}{4\sin\theta\cos\theta - \sin\theta}$

when $\theta = \frac{\pi}{2}$, slope $= -2$

If $\Delta\theta = 2\pi$, then area $= \pi r^2$ = area of whole figure / circle

1.2.3. An Example of Computing Area in Polar Coordinates

Example: Calculate the area enclosed by the inner loop of the curve $r = 1 - 2\cos\theta$



$\text{Area} = \int_{-\pi/3}^{\pi/3} \frac{(1-2\cos\theta)^2}{2} d\theta$

$= \int_{-\pi/3}^{\pi/3} \frac{1 - 4\cos\theta + 4\cos^2\theta}{2} d\theta$

$= \pi - \frac{3\sqrt{3}}{2}$

[⚠ $\int_{\pi/3}^{5\pi/3} \frac{r^2}{2} d\theta$ = Area of the outer circle]

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

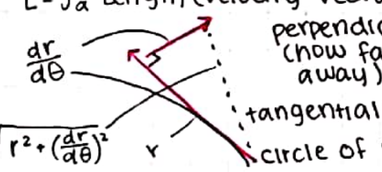
1.2.4. Length in Polar Coordinates

Length of a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$L = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$

$\Downarrow$

$L = \int_\alpha^\beta$ length (velocity vector) $d\theta$



$x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$

$L = \int_\alpha^\beta \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$(\frac{dx}{d\theta})^2 = (\frac{dr}{d\theta})^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

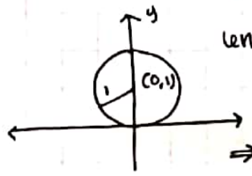
$(\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2\sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta$

$(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2(\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$
$= (\frac{dr}{d\theta})^2 + r^2$

Example: Calculate the length of curve $r = 2\sin\theta$

$L = \int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$
$= \int_0^{2\pi} \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$
$= \int_0^{2\pi} \sqrt{4}\, d\theta$
$= \int_0^{2\pi} 2\, d\theta = 4\pi$

BUT!

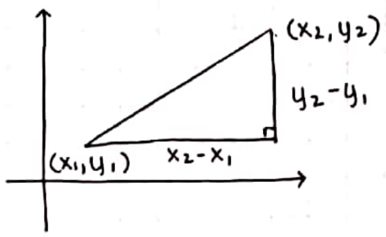


length is 2π!
$\rightarrow$ interval $[0, 2\pi]$ goes around the curve twice
$\Rightarrow \int_0^\pi 2\, d\theta = 2\pi$

LECTURE 1.3

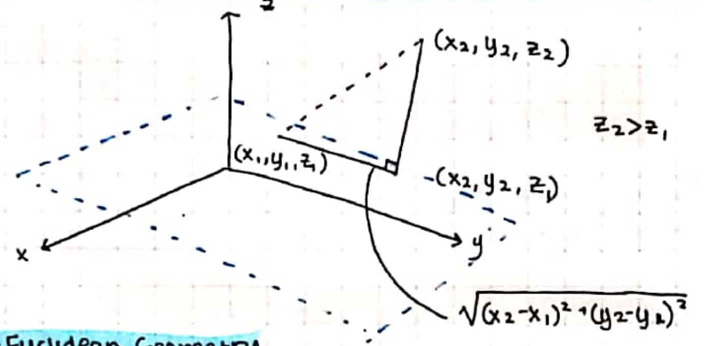
1.3.1. Distance in Two- & Three-Dimensional Euclidean Space

2-D space: $\text{dist}((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



3-D space: $\text{dist}((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3-D Space : Distance $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)}$

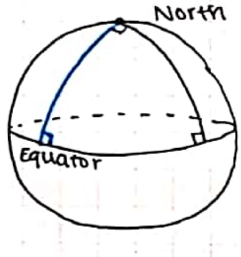


$z_2 > z_1$

⚠ This is not a rigorous proof
Assumptions:
→ vertical line where only z Δ's is perpendicular to a horizontal line where z is constant
→ Pythag Theorem holds in 3-D
Logically best to take distance formula as an axiom defining 3-D Euclidean geometry

× can add multiple dimensions & square the differences to add to sum

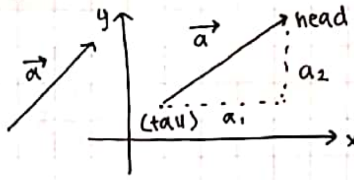
Non-Euclidean Geometry
→ 2D spherical geometry



* can draw a straight line from North Pole to equator w/90° degree line
Σ angles $= \frac{3\pi}{2} > \pi$
↳ cannot occur in Euclidean geometry
↳ in sphere Σangles $> \pi$ but in Euclid. $< \pi$

1.3.2 Introduction to Vectors

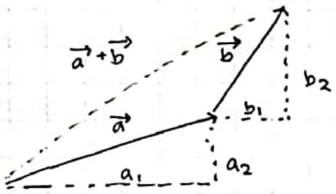
Vectors in 2D : $\vec{a} = \langle a_1, a_2 \rangle$
vector ↗ ↖ ↗ components real #'s

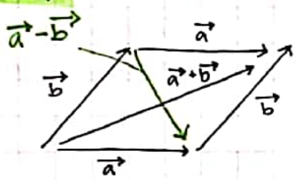


$a_1, a_2 > 0$
↳ if negative, move left or down

* only care about the difference between head & tail (can translate vector anywhere)

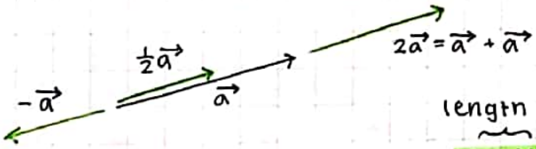
Addition of vectors
→ Define $\vec{a}+\vec{b} = \langle a_1+b_1, a_2+b_2 \rangle$



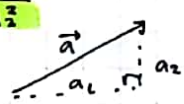


Subtraction of vectors :
$\vec{a}-\vec{b} = \langle a_1-b_1, a_2-b_2 \rangle$

Multiplication of vector $\vec{a} = \langle a_1, a_2 \rangle$ by scalar c (real number)
→ Define: $c\vec{a} \langle ca_1, ca_2 \rangle$
If $c > 0$ then $c\vec{a}$ points in the same direction as $\vec{a}$, length is multiplied by c.
If $c < 0$ ——— opposite ——— $|c|$.



Length of $\vec{a} = \langle a_1, a_2 \rangle$ (aka norm or magnitude)
$|\vec{a}| = \sqrt{a_1^2+a_2^2}$

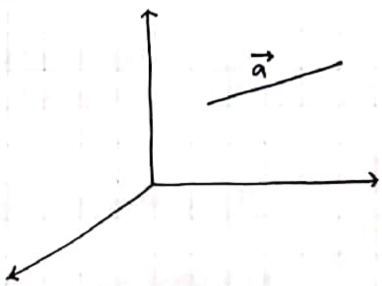


Fact : $\underbrace{|c\vec{a}|}_{\text{length}} = \underbrace{|c|}_{\text{abs value}} \underbrace{|\vec{a}|}_{\text{length of vector}}$

Proof : $|c\vec{a}| = |\langle ca_1, ca_2 \rangle| = \sqrt{(ca_1)^2+(ca_2)^2} = \sqrt{c^2(a_1^2+a_2^2)}$
$= |c|\sqrt{a_1^2+a_2^2} = |c||\vec{a}|$

⚠ ususally $|\vec{a}+\vec{b}| \neq |\vec{a}|+|\vec{b}|$ BUT Triangle inequality: $|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|$

Vectors in 3D : $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$



$\vec{a}+\vec{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$
$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$
$|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$
$|c\vec{a}| = |c||\vec{a}|$

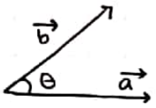
1.3.3 Dot Product

How to multiply two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$

→ **Dot product** inputs two n-component vectors & outputs a number:

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \ (+ \dots + a_nb_n)$$

Geometric Interpretation: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$



e.g. $\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$ $(\theta = 0, \cos\theta = 1)$ where $\vec{b} = \vec{a}$

e.g. If $\vec{a} \perp \vec{b}$ (so that $\cos\theta = 0$) then $\vec{a}\cdot\vec{b} = 0$

Define: "$\vec{a} \perp \vec{b}$" (perpendicular) means $\vec{a}\cdot\vec{b} = 0$

Dot Product Algebraic Properties:

1. **Communitive & distributive over addition**

   → comm.: $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$; distrib: $\vec{a}(\vec{b}+\vec{c}) = (\vec{a}\,\vec{b}) + (\vec{a}\cdot\vec{c})$

   Check: $\vec{a}\cdot(\vec{b}+\vec{c}) = \langle a_1, a_2, a_3\rangle(\langle b_1, b_2, b_3\rangle + \langle c_1, c_2, c_3\rangle)$

   $= \langle a_1, a_2, a_3\rangle(\langle b_1+c_1, b_2+c_2, b_3+c_3\rangle)$

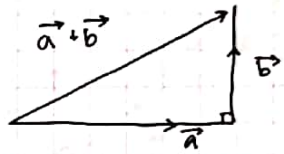
   $= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

   $= a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 = \vec{a}\cdot\vec{b} + \vec{a}\cdot c$

2. ⚠ **Not associative**: $\underbrace{\vec{a}}_{\text{vector}}\cdot\underbrace{(\vec{b}\cdot\vec{c})}_{\text{scalar}} \neq \underbrace{(\vec{a}\cdot\vec{b})}_{\text{scalar}}\underbrace{\vec{c}}_{\text{vector}}$

   *neither side is well-defined

Pythagorean Theorem: If $\vec{a} \perp \vec{b}$ then $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$



Proof: $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$= (\vec{a}\cdot\vec{a}) + \underbrace{(\vec{a}\cdot\vec{b})}_{=0 \text{ b/c } \vec{a}\perp\vec{a}=0} + \underbrace{(\vec{b}\cdot\vec{a})}_{=0 \text{ b/c } \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}} + (\vec{b}\cdot\vec{b})$

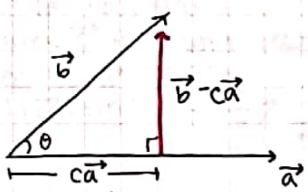
$= |\vec{a}|^2 + |\vec{b}|^2$

1.3.4. The Geometric Interpretation of Dot Product

Why $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ (assume $\vec{a}, \vec{b} \neq 0$)

Claim: There is a unique scalar c such that

$\vec{b} - c\vec{a} \perp \vec{a}$



Proof: Solve for c.

$(\vec{b} - c\vec{a})\cdot\vec{a} = 0$

$\vec{b}\cdot\vec{a} - (c\vec{a})\cdot\vec{a} = 0$

$\vec{b}\cdot\vec{a} - c(\vec{a}\cdot\vec{a}) = 0$

$c = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2} \Longrightarrow \cos\theta = \left(\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\dfrac{|\vec{a}|}{|\vec{b}|}$ ⟸

$\cos\theta = \dfrac{|c\vec{a}|}{|\vec{b}|}$ if $c \geq 0$

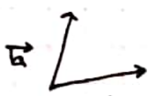
$\cos\theta = -\dfrac{|c\vec{a}|}{|\vec{b}|}$ if $c \leq 0$

$c\vec{a}$ = orthogonal projection of $\vec{b}$ onto $\vec{a}$

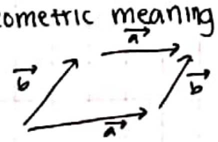
1.3.5. Determinants

Determinants in 2D & 3D

2D: $\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ (multiply across & subtract)



Geometric meaning



$\vec{a} = \langle a_1, a_2\rangle$

$\vec{b} = \langle b_1, b_2\rangle$

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ = ± Area (Parallelogram)

→ + when $\vec{a}$ points to the right of $\vec{b}$

→ − when $\vec{a}$ points to the left of $\vec{b}$

3D: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$

*diagonal tricks only work for 2×2 & 3×3

Geometric interpretation

$\vec{a} = \langle a_1, a_2, a_3\rangle$

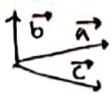
$\vec{b} = \langle b_1, b_2, b_3\rangle$

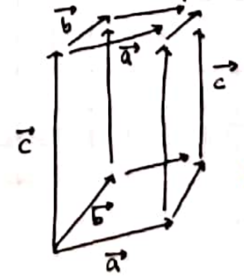
$\vec{c} = \langle c_1, c_2, c_3\rangle$

det = ± Volume (Parallelopiped)

+ when $\vec{a}, \vec{b}, \vec{c}$ satisfy "right hand rule"

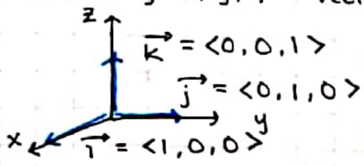
− otherwise





Parallelopiped

When drawing x, y, & z vectors, draw like this:



$\vec{i}, \vec{j}, \vec{k}$ satisfy the right hand rule

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

1.3.6. Cross Product

Cross products input two vectors + outputs a vector (only defined in three dimensions)

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \text{"}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\text{"}$$

* determinant matrix in "" b/c every entry should be a #

$= \vec{i}(a_2b_3 - a_3b_2) + \vec{j}(a_3b_1 - a_1b_3) + \vec{k}(a_1b_2 - a_2b_1)$

$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$

e.g. $\vec{i} \times \vec{j} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \langle 0,0,1 \rangle = \vec{k}$

$\vec{j} \times \vec{i} = \langle 0,1,0 \rangle \times \langle 1,0,0 \rangle = \langle 0,0,-1 \rangle = -\vec{k}$

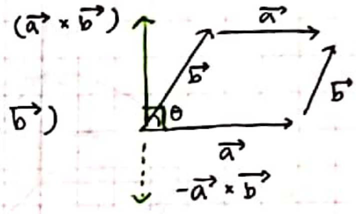
$\vec{i} = \langle 1,0,0 \rangle$
$\vec{j} = \langle 0,1,0 \rangle$
$\vec{k} = \langle 0,0,1 \rangle$

Cross product is not communitive! $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

↳ $\vec{a} \times \vec{a} = 0 \Rightarrow \vec{a} \times \vec{a} = -\vec{a} \times \vec{a}$

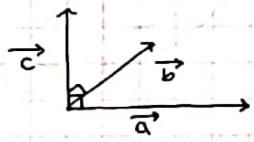
Geometric Interpretation of Cross Product

1. $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ & $\vec{b}$
2. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ = Area (Parallelogram generated by $\vec{a}, \vec{b}$)
3. $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy the right hand rule



proof:

observe: $\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ ← volume w/no height



e.g. $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{a} \times \vec{b} \perp \vec{a} \times \vec{b}$

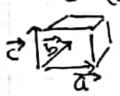
Let $\vec{c}$ be a unit vector perpendicular to $\vec{a}, \vec{b}$ such that $\vec{a}, \vec{b}, \vec{c}$ satisfy the right hand rule. (Assume $\vec{a}, \vec{b} \neq 0$, not parallel)

Know: $\vec{a} \times \vec{b} = \lambda\vec{c}$ for some scalar λ. (show $\lambda = |\vec{a}||\vec{b}|\sin\theta$)

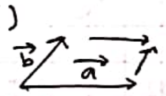
$\vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{c}(\lambda\vec{c}) = \lambda(\vec{c} \cdot \vec{c}) = \lambda$ (assume $\vec{c}$ is a unit vector $= 1$)

$\vec{c} \cdot (\vec{a} \times \vec{b})$ = volume (parallelopiped generated by $\vec{a}, \vec{b}, \vec{c}$)

= Area (parallelogram $\vec{a}, \vec{b}$) $= |\vec{a}||\vec{b}|\sin\theta$ →



satisfies the right hand rule = positive



LECTURE 1.4

1.4.1. Lines in 3D Space

→ Lines are like parametrized curves w/3 variables

→ Identify {points in 3D space} = {3 component vectors}

↳ geometrically point p w/vector that corresponds from the origin to p (an arrow)

Parametrization of line in space:

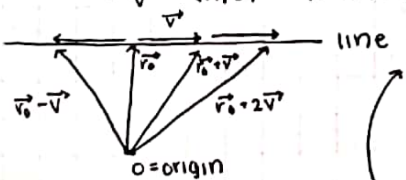
$\vec{r}(t) = \vec{r_0} + t\vec{v}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a point on the line depending on t (time parameter)

→ t = time parameter

→ $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is a base point on the line

→ $\vec{v} = \langle a,b,c \rangle$ is a tangent vector to the line



$\vec{r_0} + t\vec{v} = \vec{r}$

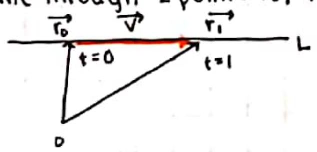
as t ranges through all real numbers, we will get all the points in the line

$\langle x(t), y(t), z(t) \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a,b,c \rangle$

vector version

$$\left[\begin{array}{l} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{array}\right.$$

Line through 2 points $\vec{r_0}, \vec{r_1}$

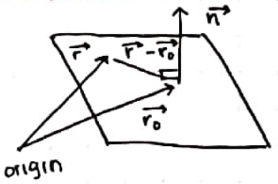


$\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0})$

$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$

Line segment from $\vec{r_0}$ to $\vec{r_1}$: same equation $0 \leq t \leq 1$

1.4.2. Planes



$\vec{r_0}$ = a point on the plane
$\vec{n}$ = a normal (perpendicular) vector to the plane
$\vec{r}$ is on the plane when $\vec{r}-\vec{r_0} \perp n$
$(\vec{r}-\vec{r_0})\cdot\vec{n}=0$

$\vec{r_0}=\langle x_0, y_0, z_0\rangle$
$\vec{n}=\langle a,b,c\rangle$
$\vec{r}=\langle x,y,z\rangle$

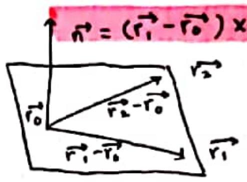
$\langle x-x_0, y-y_0, z-z_0\rangle\cdot\langle a,b,c\rangle=0$
$ax+by+cz=d$
where $d=ax_0+by_0+cz_0$

Plane through 3 noncolinear points: $\vec{r_0}, \vec{r_1}, \vec{r_2}$
$(\vec{r}-\vec{r_0})\cdot[(\vec{r_1}-\vec{r_0})\times(\vec{r_2}-\vec{r_0})]=0$
equivalent to ↓

$$\det\begin{pmatrix}\vec{r}-\vec{r_0}\\ \vec{r_1}-\vec{r_0}\\ \vec{r_2}-\vec{r_0}\end{pmatrix}=0$$

★ volume = 0 because all points lie on one plane

$\vec{n}=(\vec{r_1}-\vec{r_0})\times(\vec{r_2}-\vec{r_0})$

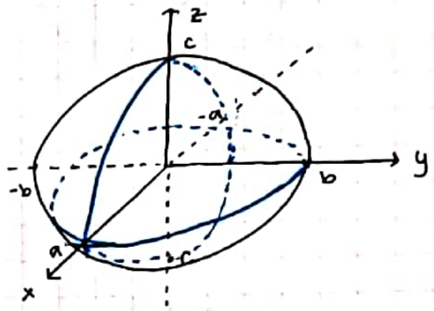


Remark: we can also describe this plane as a parametrized surface w/2 parameters (t_1, t_2)
$\vec{r}(t_1,t_2)=\vec{r_0}+t_1(\vec{r_1}-\vec{r_0})+t_2(\vec{r_2}-\vec{r_0})$

1.4.3. Quadratic Surfaces 1: Ellipsoid, Hyperboloid

$ax^2+by^2+cz^2+dxy+exz+fyz+\underbrace{gx+hy+iz+j}_{\text{eq of plane}}=0$

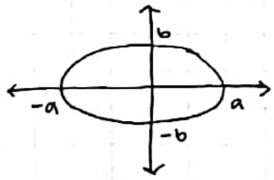
How to Sketch
→ set $z=0$ to find the intersection w/ the x, y plane
→ $y=0$ ——— x, z ———
→ $x=0$ ——— y, z ———



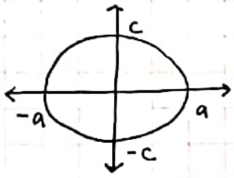
Ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where a, b, c are constants >0

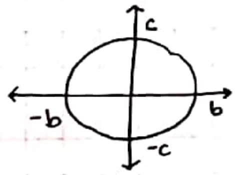
$z=0$



$y=0$



$x=0$

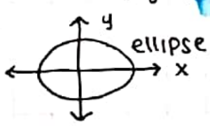


Hyperboloid

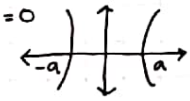
$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1$ (not all signs are the same: all + = ellipsoid & all − = empty set)
$a, b, c > 0$

→ case 1: Two + signs $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}$ hyperboloid of one sheet

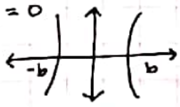
$z=0$

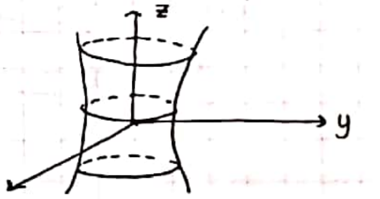


$y=0$



$x=0$





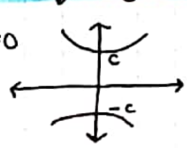
if z is a constant, we can see the different x, y planes (increases as z moves away from 0)

→ case 2: Two − signs $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

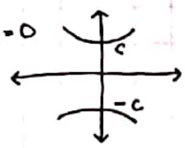
hyperboloid of two sheets

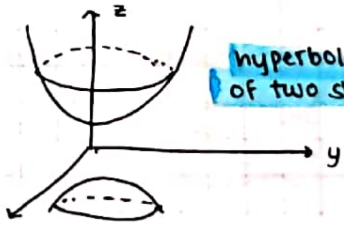
$z=0$: empty set

$y=0$



$x=0$





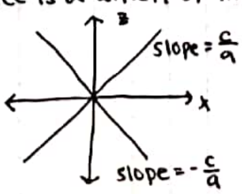
Horizontal slices: $z=d$
$|d|<c$: empty set
$|d|=c$: $x=0, y=0$
$|d|>c$: ellipse

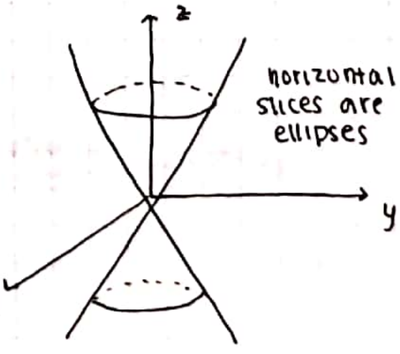
1.4.4. Quadric Surfaces 2: Cone, Elliptic Paraboloid

Cone: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ (all terms are quadratic) $a,b,c>0$

→ invariant under scaling: if (x,y,z) is a solution & λ is a scalar then $(\lambda x, \lambda y, \lambda z)$ is also a solution ⇒ surface is a union of lines through the origin

$z=0$: one point $(x=0, y=0)$
$y=0$: $\frac{z^2}{c^2}=\frac{x^2}{a^2}\Rightarrow\frac{z}{c}=\pm\frac{x}{a}$ two lines
$x=0$: two lines



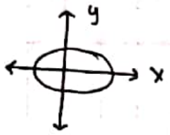


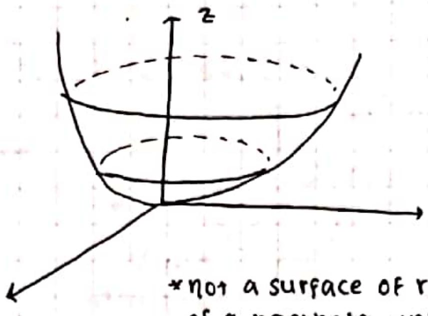
horizontal slices are ellipses

Elliptic Paraboloid: one linear & two quadratic terms

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $a, b, c > 0$

$z = 0$: point @ the origin
$z =$ constant > 0: ellipse →
$z =$ constant < 0: empty set
$y = 0$: parabola
$x = 0$: parabola





* c- = upside-down elliptic parabola below x,y plane

* not a surface of revolution of a parabola unless $a = b$

1.4.5. Quadric Surfaces 3: Hyperbolic Paraboloid

Hyperbolic Paraboloid: one linear & 2 quadratic terms w/opp. signs

$\frac{z}{c} = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$ where $a, b, c > 0$

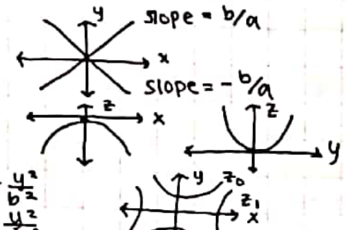
→ $z = 0$: $\frac{y^2}{b^2} = \frac{x^2}{a^2}$
$\frac{y}{b} = \pm\frac{x}{a}$

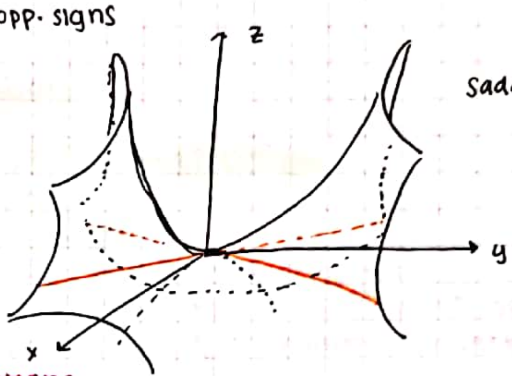
→ $y = 0$: $\frac{z}{c} = \frac{-x^2}{a^2}$

→ $x = 0$: $\frac{z}{c} = \frac{y^2}{b^2}$

→ $z = z_0 > 0$: $\frac{z_0}{c} = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

→ $z = z_1 < 0$: $\frac{z_1}{c} = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$





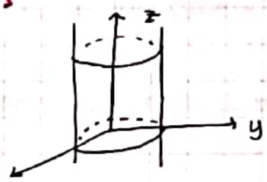
Saddle

1.4.6. Quadric Surfaces 4: Quadric Cylinders & Translations

Quadric Cylinder: one variable missing

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

* intersection is an ellipsoid



→ draw lines in direction of missing variable

General Quadric Surfaces

$$ax^2 + by^2 + cz^2 + \underbrace{dxy + exz + fyz}_{\text{rotation}} + \underbrace{gx + hy + iz + j}_{\text{get rid of (some of) these terms by completing the square}} = 0$$

⇒ translate surface effect

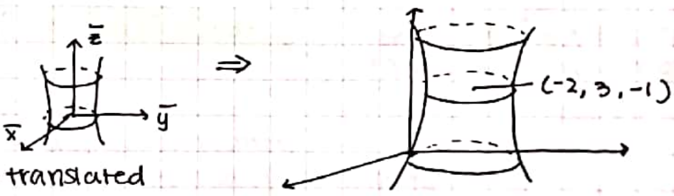
e.g. $x^2 + y^2 - 4z^2 + 4x - 6y - 8z = 13$

$(x+2)^2 + (y-3)^2 - 4(z+1)^2 = 22$

$\bar{x} = x + 2$ $\bar{y} = y - 3$ $\bar{z} = z + 1$

$\bar{x}^2 + \bar{y}^2 - 4\bar{z}^2 = 22$

↳ hyperboloid of one sheet



If you have extra linear terms = translated

LECTURE 1.5

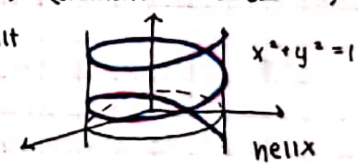
1.5.1. Introduction to Space Curves

Parametrized curves in space & vector-valued functions

$x = f(t)$
$y = g(t)$ $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$
$z = h(t)$ $= \langle x(t), y(t), z(t) \rangle$
$\alpha \le t \le \beta$

Ex: sketch $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ (default $-\infty \le t \le \infty$)

we know $x = \cos t$, $y = \sin t$ } around unit circle
$z = t$ (move up at unit speed)



Vector-valued function: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity vector at time t is $\vec{r}\,'(t) = \lim_{\varepsilon \to 0} \frac{\vec{r}(t+\varepsilon) - \vec{r}(t)}{\varepsilon}$ if this limit exists

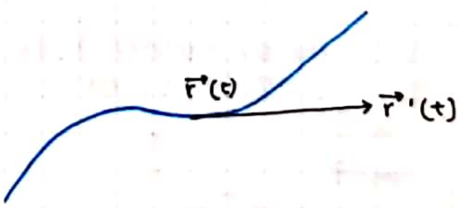
$= \langle x'(t), y'(t), z'(t) \rangle$

Geometric Interpretation

- $\vec{r}\,'(t)$ points in a direction tangent to a curve
- $|r'(t)|$ (length of velocity vector) = speed at time t

Tangent line to the curve at time t goes through $\vec{r}(t)$ & has tangent direction $r'(t)$.

$\vec{L}(\vec{s}) = \vec{r}(t) + s\vec{r}\,'(t)$



Ex: Find tangent line to $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at $\langle 1,0,0 \rangle$

At $t=0$: $\vec{r}(t) = \langle 1,0,0 \rangle \Rightarrow \vec{r}\,'(0) = \langle -\sin t, \cos t, 1 \rangle |_{t=0} = \langle 0,1,1 \rangle$

Tangent line is $\vec{L}(s) = \langle 1,0,0 \rangle + s\langle 0,1,1 \rangle$

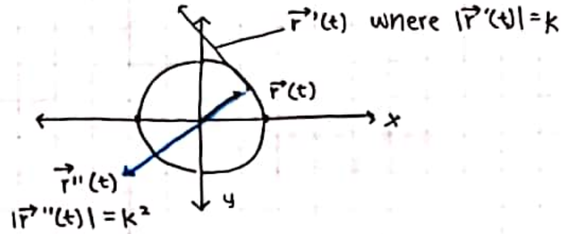
Acceleration at time t is $\vec{r}\,''(t)$

Ex: $\vec{r}(t) = \langle \cos(kt), \sin(kt), 0 \rangle$ ← unit circle @ k speed

Velocity: $\vec{r}\,'(t) = \langle -k\sin(kt), k\cos(kt), 0 \rangle$

Acceleration: $\vec{r}\,''(t) = \langle -k^2\cos(kt), -k^2\cos(kt), 0 \rangle$

e.g. If you drive a car in a circle @ 30 mph, car accelerate toward the center & push in other direction, If you drive 2x fast, then you will feel 4x the acceleration



1.5.2. Calculus of Vector-Valued Functions

Length of curve $\vec{r}(t)$, $\alpha \le t \le \beta$ is **Length $= \int_\alpha^\beta |\vec{r}\,'(t)|\,dt$**

Three versions of Product Rule

1. Multiplication of vector by a scalar: $\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$
2. Dot Product: $\frac{d}{dt}(\vec{r}(t)\cdot\vec{s}(t)) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$
3. Cross Product: $\frac{d}{dt}(\vec{r}(t)\times\vec{s}(t)) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$

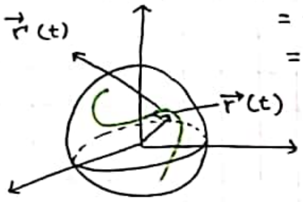
* To prove these, expand in components & use regular product rule for each component

Ex: Show that if the curve $\vec{r}(t)$ is on the unit sphere $x^2+y^2+z^2=1$ then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$ for all t.

Solution: $\vec{r}(t)\cdot\vec{r}\,'(t) = \frac{1}{2}\frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t))$ ← product rule & communitive property

$= \frac{1}{2}\frac{d}{dt}|\vec{r}(t)|^2$ } $\vec{r}(t)$ is alsways on unit sphere so length = 1

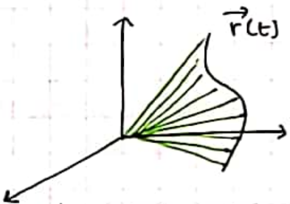
$= \frac{1}{2}\frac{d}{dt}(1) = 0$



$\vec{r}\,'(t) \perp \vec{r}(t)$: tangent plane to sphere is $\perp$ to point $\vec{r}(t)$.

Integration of Vector-Valued Functions: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\int_a^b \vec{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$



Note: This is not the area under the curve — it is a vector!

Remark: Area (ribbon) $= \frac{1}{2}\int_a^b |\vec{r}(t)\times\vec{r}\,'(t)|\,dt$ ← If on x,y plane & θ is incr. = generalize formula for areas of regions in polar coord.

Fundamental Theorem of Calc:

$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$

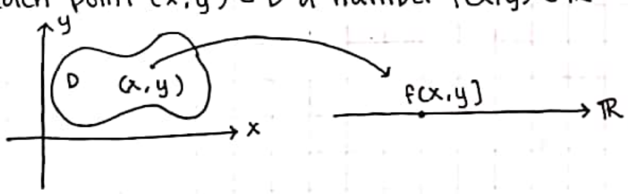
- can integrate velocity to obtain change in position
- can also integrate acceleration to obtain change in velocity

LECTURE 2.1

2.1.1 Functions of Two Variables

Functions of more than one variable

Let D be a region in $\mathbb{R}^2$. A function from D to $\mathbb{R}$, or a "function on D" is a rule f which assigns to each point $(x,y) \in D$ a number $f(x,y) \in \mathbb{R}$



D = **Domain** of f; this is a subset of $\mathbb{R}^2$

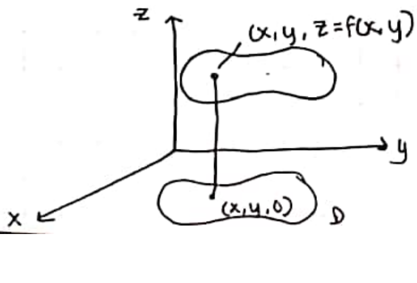
Range: $R = \{f(x,y) \mid (x,y) \in D\}$, i.e. the set of all values of f; this is a subset of $\mathbb{R}$

Two basic ways to visualize a function of two variables

1) **Draw the graph.**

Graph $= S = \{(x,y,z) \mid (x,y) \in D, z = f(x,y)\}$

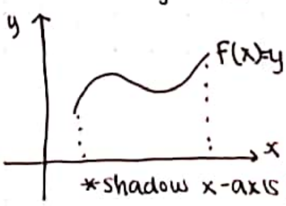
This is a surface in $\mathbb{R}^3$



* x,y plane like a shadow

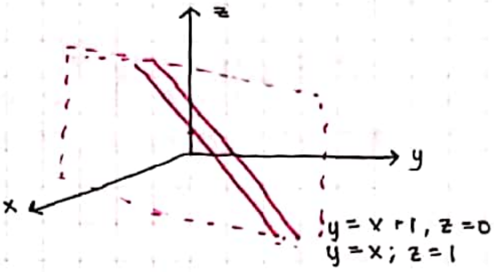
[Single-variable version: $y = f(x)$

Graph $= \{(x,y) \mid f(x) \text{ is defined},\ y = f(x)\}$



*shadow x-axis (domain in which f is defined)]

2.1.2. Examples of Graphs

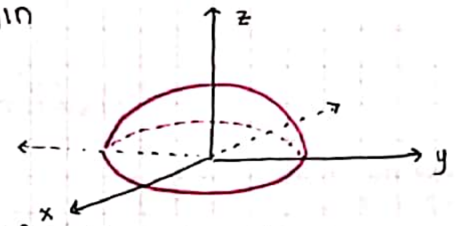


Ex 1: $f(x,y) = x - y + 1$ Domain $D = \mathbb{R}^2$

The graph is the surface $z = x - y + 1 \Rightarrow$ Plane
(see right for graph)

In general, the graph of a linear function
$f(x,y) = ax + by + c$ is a plane $z = ax + by + c$

Ex 2: $f(x,y) = \sqrt{9-x^2-y^2}$ $D = \{(x,y) \mid x^2+y^2 \le 9\}$
↳ disk of radius 3 centered @ origin



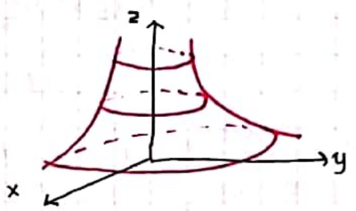
Graph is $z = \sqrt{9-x^2-y^2} \Rightarrow z^2 = 9 - x^2 - y^2$
$x^2+y^2+z^2 = 9$

a sphere of radius 3 centered @ origin

Because $z \ge 0$, graph is upper half of sphere
$x^2+y^2+z^2 = 9,\ z \ge 0$

* The whole sphere is not a graph of a function b/c it fails the vertical line test

Ex 3: $f(x,y) = \frac{1}{x^2+y^2}$ Domain = all of $\mathbb{R}^2$ except (0,0)
$= \mathbb{R}^2 \setminus \{0,0\}$



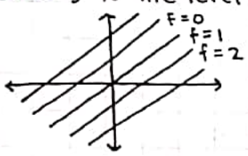
On a circle $x^2+y^2 = r^2$ where $f = \frac{1}{r^2}$, constant
→ on large circles, f is small
→ on small circles f is large
→ circle of radius r in x, y plane has height $\frac{1}{r^2}$

2.1.3. Level Curves

Method 2: Level curves (ex: weather map which assigns temp. to curves or contour map which assigns elevations to lines)

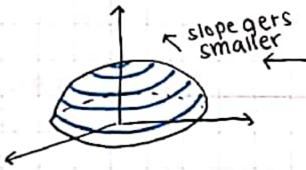
Let f be a function of two variables & D = domain of f
Given k, the set $\{(x,y) \in D \mid f(x,y) = k\}$ is the level curve or level set

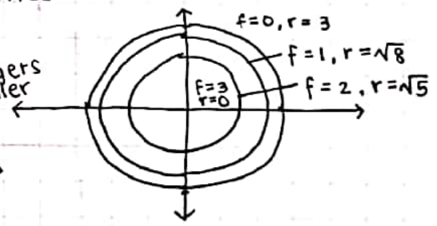
Ex: $f(x,y) = x - y + 1$
$f = 0$ when $y = x + 1$
$f = 1$ when $x = y$
$f = 2$ when $y = x - 1$



* plane that is not horizontal or vertical = level sets of corresponding function are equally spaced parallel lines

Ex2: $f(x,y) = \sqrt{9-x^2-y^2}$
$f = 0$ when $x^2+y^2 = 9 \Rightarrow r = 3$
$f = 1$ — $x^2+y^2 = 8 \Rightarrow r = 2\sqrt{2} \approx 2.8$
$f = 2$ — $x^2+y^2 = 5 \Rightarrow r = \sqrt{5} \approx 2.2$
$f = 3$ — $x^2+y^2 = 0 \Rightarrow$ origin





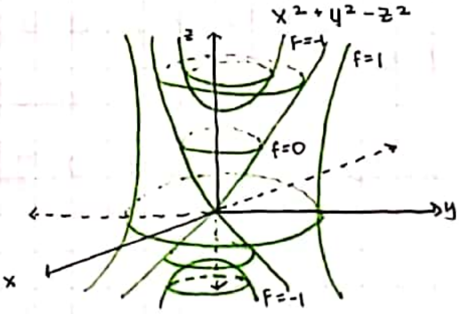
2.1.4. Functions of Three Variables

$D \ni (x,y,z) \xrightarrow{f} f(x,y,z) \in \mathbb{R}$

* D is a subset of $\mathbb{R}^3$

Graph of f is $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = f(x,y,z)\}$
a 3-D hypersurface in 4-D space
*Too hard to draw, but can instead draw level sets or level surfaces
$\{(x,y,z) \in D \mid f(x,y,z) = k\}$

Ex: Sketch level surface of $f(x,y,z) = x^2+y^2-z^2$



$f=0: x^2+y^2 = z^2$ (cone)
$f=1: x^2+y^2 = z^2+1$ (hyperboloid of 1 sheet)
$f=-1: x^2+y^2 = z^2-1$ (hyperboloid of 2 sheets)

2.1.5. Definition of a Limit

Let f be a function on domain D in $\mathbb{R}^2$.

Definition "$\lim_{(x,y)\to(a,b)} f(x,y) = L$" means:
For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $(x,y) \in D$ & $(x,y) \ne (a,b)$ & $\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

ε = "error tolerance"
δ = "how close (x,y) needs to be to (a,b) to guarantee that f(x,y) is within the error tolerance of L"

Ex: Prove that $\lim_{(x,y)\to(0,0)} 2x = 0$

Given $\varepsilon > 0$, we need to find $\delta > 0$ such that if $(x,y) \neq 0$ & $\sqrt{x^2+y^2} < \delta$ then $|2x| < \varepsilon$.

Choose $\delta = \frac{\varepsilon}{2}$. This works because if $\sqrt{x^2+y^2} < \frac{\varepsilon}{2}$ then $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} < \frac{\varepsilon}{2}$ so $|2x| < \varepsilon$

* since we're multiplying by 2, in order to guarantee we are within the error tolerance, we have to be twice as close
* can also choose $< \frac{\varepsilon}{2}$

2.1.6. Properties of Limits & Continuous Functions

Properties of Limits

1) $\lim_{(x,y)\to(a,b)} x = a$ 2) $\lim_{(x,y)\to(a,b)} y = b$ 3) $\lim_{(x,y)\to(a,b)} c = c$

4) $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$

5) $\lim (fg) = \lim(f) \lim(g)$ & $\lim\left(\frac{f}{g}\right) = \frac{\lim f}{\lim g}$ if $\lim g \neq 0$

} if both limits on the right of the eq. are defined

6) If $h: \mathbb{R} \to \mathbb{R}$ is continuous & $\lim_{(x,y)\to(a,b)} f(x,y) = L$ then $\lim_{(x,y)\to(a,b)} h(f(x,y)) = h(L)$

Ex: $\lim_{(x,y)\to(a,b)} e^{x+y} = e^{a+b}$ where $h(t) = e^t$ & $f(x,y) = x+y$

A function f of two variables is **continuous** at (a,b) if $f(a,b)$ is defined & $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

→ continuous functions: constants, coordinate functions, sum/products/quotients of continuous functions, & compositions w/continuous functions $h: \mathbb{R} \to \mathbb{R}$

ex: $f(x,y) = \begin{cases} 1 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$] not continuous @ $(0,0)$ b/c $\lim_{(x,y)\to(a,b)} f(x,y) = 1 \neq 0 = f(0,0)$ (we evaluate limits close to a & b but not at a & b)

2.1.7. Proving that Limits Do Not Exist

Property: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ & $(x(t), y(t))$ is a parametrized curve with $\lim_{t\to t_0} x(t) = a$, $\lim_{t\to t_0} y(t) = b$ & $(x(t), y(t)) \neq (a,b)$ for $t \neq t_0$, then $\lim_{t\to t_0} f(x(t), y(t)) = L$.

Consequence: If f has an undefined limit along some curve approaching (a,b) or if f has two different limits along two different curves approaching (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y)$ is not defined.

Ex: Prove $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist where $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

Approach $(0,0)$ along x-axis (fix y as zero):

$\lim_{x\to0} f(x,0) = \lim_{x\to0} \frac{x^2}{x^2} = \lim_{x\to0} 1 = 1$

Approach $(0,0)$ along the y-axis (fix x as zero):

$\lim_{y\to0} f(0,y) = \lim_{y\to0} \frac{-y^2}{y^2} = \lim_{y\to0} -1 = -1$

Since $1 \neq -1$, $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ is not defined.

Ex: Is $\lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$ defined?

Approach along x axis: $\lim_{x\to0} f(x,0) = \lim_{x\to0} 0 = 0$ | NOT defined (not sufficient to just consider the axis)

Approach along y axis: $\lim_{y\to0} f(0,y) = \lim_{y\to0} 0 = 0$

Approach along line $x=t$, $y=t$ $(x=y)$: $\lim_{t\to0} f(t,t) = \lim_{t\to0} \frac{t^2}{2t^4} = \lim_{t\to0} \frac{1}{2t^2} = \infty$

Ex: Is $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ defined?

Approach along axes: limit is 0

Approach along line $y = mx$ where $m \neq 0$ $(x=t, y=mt)$

$\lim_{t\to0} f(t,mt) = \lim_{t\to0} \frac{m^2t^3}{t^2+m^4t^4} = \lim_{t\to0} \frac{m^2t}{1+m^4t^2} = 0$

Approach along parabola $x = y^2$ where $x = t^2$ & $y = t$.

$\lim_{t\to0} f(t^2,t) = \lim_{t\to0} \frac{t^4}{2t^4} = \frac{1}{2}$ ⟹ NOT DEFINED

Takeaway: To prove a limit exists, use the principles & theorems.

______ does not exist, you can find clever ways to show the discrepancy

LECTURE 2.2

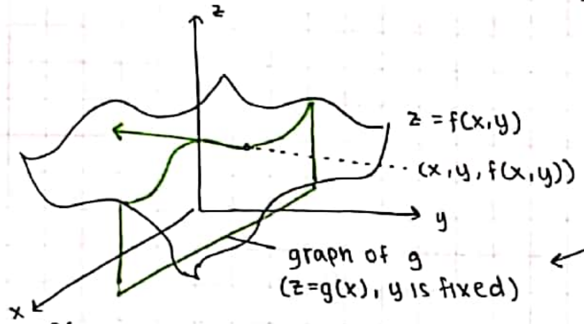
2.2.1. Introduction to Partial Derivatives

Partial Derivatives Definition: (Let f be a function of x & y)

The partial derivative of f with respect to x at (x,y) is $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ if this limit exists

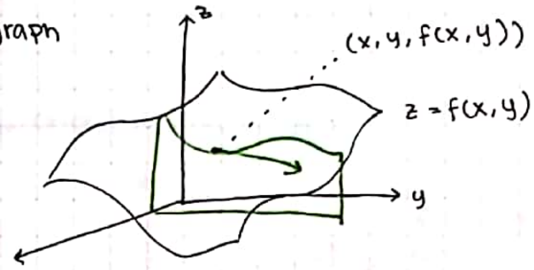
∂ = "curly a"

= "rate at which f changes as we vary x, holding y fixed"



Fix y, define $g(x) = f(x,y)$ then $\frac{\partial f}{\partial x}(x,y) = \frac{dg}{dx}(x)$ (looking at slope in x direction)

slope of this graph



To compute $\frac{\partial f}{\partial x}$ regard y as a constant & differentiate w/respect to x (vice versa)

Ex: $f(x,y) = x^3 + y^4 + x^2y$

$\frac{\partial f}{\partial x} = 3x^2 + 2xy \qquad \frac{\partial f}{\partial y} = 4y^3 + x^2$

Define: $\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ if this limit exists

= "rate at which f changes as we vary y, holding x fixed"

2.2.2. Higher Partial Derivatives

Let f be a function of $x, y,$ & z

Define: $\frac{\partial f}{\partial x}(x,y,z) = f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}$

= "rate at which f changes as we vary x, holding y & z fixed"

To compute, regard y & z as constants & differentiate w/respect to x.

Likewise: $\frac{\partial f}{\partial y}(x,y,z) = f_y(x,y,z) = \lim_{h\to 0} \frac{f(x,y+h,z)-f(x,y,z)}{h}$

$\frac{\partial f}{\partial z}(x,y,z) = f_z(x,y,z) = \lim_{h\to 0} \frac{f(x,y,z+h)-f(x,y,z)}{h}$ } if the limit exists

Ex: $f(x,y,z) = x^3 + xyz + yz^2 \Rightarrow f_x = 3x^2 + yz$; $f_y = xz + z^2$; $f_z = xy + 2yz$

Second Partial Derivatives:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = f_{yy}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx} \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

} if we have 3 variables, there are nine possibilities

Ex: we have $f(x,y) = x^2\cos(x+y)$

$f_x = 2x\cos(x+y) - x^2\sin(x+y) \qquad f_y = -x^2\sin(x+y)$

$f_{xx} = 2\cos(x+y) - 2\sin(x+y) - 2x\sin(x+y) - x^2\cos(x+y)$

$f_{xy} = -2x\sin(x+y) - x^2\cos(x+y)$

$f_{yx} = -2x\sin(x+y) - x^2\cos(x+y)$

} here $f_{xy} = f_{yx}$

$f_{yy} = -x^2\cos(x+y)$

Clairaut's Theorem: If f_{xy} & f_{yx} are defined in a neighborhood of (x,y) then $f_{xy}(x,y) = f_{yx}(x,y)$.

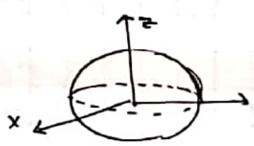
* for func. of 3 variables: $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$.

* can also take higher partial derivatives $f_{xyyz} = (((f_x)_y)_y)_z = f_{yxyz}$ = etc.

2.2.3. Implicit Partial Derivatives

Explicit Formula: $z = f(x,y)$ $\longleftrightarrow$ **Implicit Equation**: $F(x,y,z) = 0$

ex: $z = \sqrt{9-x^2-y^2}$ or $-\sqrt{9-x^2-y^2}$ ex: $x^2+y^2+z^2=9$



How do we calculate $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$?

Think of everything in this equation as a function of x & y.

$\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{\partial}{\partial x}(9) \Rightarrow 2x + 0 + 2z\frac{\partial z}{\partial x} = 0 \rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$

when z is small, steep slope

$\frac{\partial}{\partial y}(x^2+y^2+z^2) = \frac{\partial}{\partial y}(9) \Rightarrow 2y + 2z\frac{\partial z}{\partial y} = 0 \rightarrow \frac{\partial z}{\partial y} = \frac{-y}{z}$

Other Method: $z = \sqrt{9-x^2-y^2}$ (ignore negative)

$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{9-x^2-y^2}}(-2x) = \frac{-x}{\sqrt{9-x^2-y^2}} = -\frac{x}{z}$

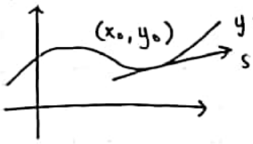
Ex 2: next page

Ex: Suppose z is defined implicitly by $zx^2+y^2+xyz^2=0$. Calculate $\frac{\partial z}{\partial x}$.

$\frac{\partial}{\partial x}(zx^2+y^2+xyz^2)=0 \Rightarrow \frac{\partial z}{\partial x}(x^2)+2zx+0+yz^2+2xyz\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=\frac{-2zx-yz^2}{x^2+2xyz}$

2.2.4. Tangent Planes

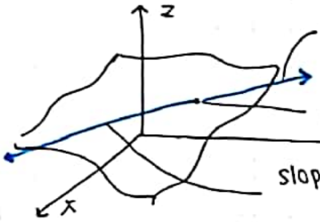
How to find the tangent line to $y=f(x)$ at point $(x_0, y_0=f(x_0))$



$y-y_0=f'(x_0)(x-x_0)$

"y displacement from y_0 is = x displacement from x_0 times slope"

How to find the tangent plane to $z=f(x,y)$ at $(x_0, y_0, z_0=f(x_0,y_0))$



$$z-z_0=\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

Ex: Find the tangent plane to $x^2+y^2+z^2=1$

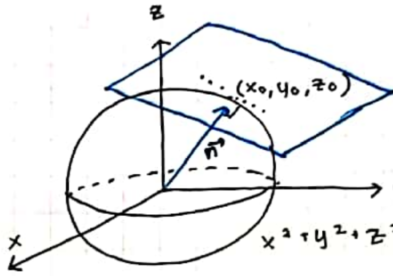
$z-z_0=\frac{\partial z}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial z}{\partial y}(x_0,y_0)(y-y_0)$

$z-z_0=-\frac{x_0}{z_0}(x-x_0)+-\frac{y_0}{z_0}(y-y_0)$

$z_0(z-z_0)+x_0(x-x_0)+y_0(y-y_0)=0$

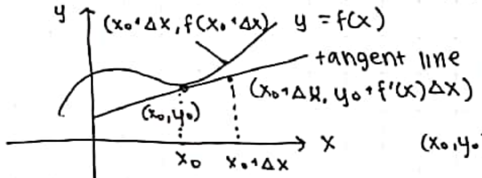
$x_0x+y_0y+z_0z=x_0^2+y_0^2+z_0^2$

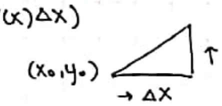
$x_0x+y_0y+z_0z=1 \Rightarrow$ normal vector is $\langle x_0, y_0, z_0\rangle$



$\vec{n}$ is $\perp$ to tangent plane

2.2.5. Linear Approximation of Functions of One Variable





Tangent line to graph is $y-y_0=f'(x_0)(x-x_0)$ ↙ Δy
contains point $(x_0+\Delta x, y_0+f'(x_0)\Delta x)$

Linear approximation:

$f(x_0+\Delta x)\approx y_0+f'(x_0)\Delta x$

Ex: Approximate $\sqrt{1.1}$ $\Rightarrow$ We have $f(x)=\sqrt{x}$ where $x_0=1$ & $y_0=f(x_0)=1$, $\Delta x=0.1$

$f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow f'(1)=\frac{1}{2}$ Thus $f(1.1)\approx f(1)+f'(1)(0.1)=1+\frac{1}{2}(0.1)$

Calculator: $\sqrt{1.1}=1.0488\ldots \longrightarrow f(1.1)=\sqrt{1.1}\approx 1.05$

2.2.6. Differentiability

Let f be a function of two variables

Definition: f is differentiable at (x_0,y_0) if (1) $\underbrace{f_x(x_0,y_0) \text{ \& } f_y(x_0,y_0)}_{\text{partial derivatives}}$ are defined, & there exists functions $\varepsilon_1, \varepsilon_2$ such that

$$f(x_0+\Delta x, y_0+\Delta y)=f(x_0,y_0)+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+\varepsilon_1(\Delta x,\Delta y)\Delta x+\varepsilon_2(\Delta x,\Delta y)\Delta y$$

where $\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_1(\Delta x,\Delta y)=\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_2(\Delta x,\Delta y)=0$

↑ if $\varepsilon=0$, this eq is the graph of f is a plane = to tangent plane

Observe: If $\varepsilon_1=\varepsilon_2=0$ this equation says that the graph $z=f(x,y)$ is the same as the tangent plane to the graph at $(x_0,y_0,z_0=f(x_0,y_0))$.

→ write $\Delta x=x-x_0$, $\Delta y=y-y_0$, $\Delta z=z-z_0$ → Eq. for a tangent plane is $\Delta z=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y$)

*The deviation from a plane by a graph is given by $\varepsilon_1\Delta x$ & $\varepsilon_2\Delta y$ where $\varepsilon_1, \varepsilon_2\to 0$ as $\Delta x,\Delta y\to 0$

Means: Graph f is well approx by tangent plane

Slogan: "f is differentiable at (x_0,y_0)" means "graph of f near $(x_0,y_0,z_0=f(x_0,y_0))$ is well-approx by the tangent plane to the graph at (x_0,y_0,z_0)"

⚠WARNING: $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ defined is not sufficient to check if partial derivatives are defined at the point b/c def. has additional condition ε_1 & ε_2

Ex: $f(x,y)=x^{\frac{1}{3}}y^{\frac{1}{3}}$ $(x_0,y_0)=(0,0)$

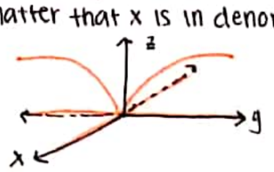
$f_x=\frac{1}{3}x^{-2/3}y^{1/3}$ ← (by def, y is fixed @ 0 so it doesn't matter that x is in denominator)

$f_x(0,0)=f_y(0,0)=0$ (because $f=0$ on the axes)

But f is not differentiable at $(0,0)$.

Consider the curve $x=t$, $y=t$

$f(t,t)=t^{1/3}t^{1/3}=t^{2/3}\Rightarrow \frac{d}{dt}f(t,t)=\frac{d}{dt}t^{2/3}=\frac{2}{3}t^{-1/3}\Rightarrow$ not defined at $t=0$



2.2.7. Linear Approximation of Functions of Two Variables

Fact: If f_x & f_y are defined & continuous in a neighborhood of (x_0, y_0) then f is differentiable at (x_0, y_0)

Linear Approximation: If f is differentiable at (x_0, y_0) then

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

Ex: Approximate $\sqrt{(3.012)^2 + (3.997)^2}$

$f(x,y) = \sqrt{x^2+y^2}$, $(x_0, y_0) = (3,4)$

$f(x_0, y_0) = 5$

$\Delta x = 0.012$

$\Delta y = -0.003$

$f(3.012, 3.997) \approx f(3,4) + f_x(3,4)(0.012) + f_y(3,4)(-0.003)$

$f_x = \frac{x}{f}$, $f_y = \frac{y}{f}$ $\quad f_x(3,4) = 3/5$, $f_y(3,4) = 4/5$

$\sqrt{(3.012)^2 + (3.997)^2} \approx 5 + \frac{3}{5}(0.012) + \frac{4}{5}(-0.003) = 5.0048$

Calculator: 5.004812784

LECTURE 2.3

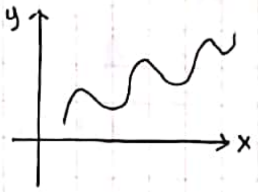
2.3.1. Review of the Chain Rule in Single Variable Calculus

Single variable: y is a function of x

x is a function of t

Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$\frac{dy}{dt}(t) = \frac{dy}{dx}x(t)\,\frac{dx}{dt}(t)$



y = altitude

x = position

t = time

rate of climb = slope · horizontal velocity

2.3.2. Multivariable Chain Rule #1

Suppose f is a differentiable function of x & y. Suppose x & y are differentiable functions of t.

Then $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

* Intuitively: think of f as height function & $t \to (x(t), y(t))$ as a parametrized curve

⇒ rate of Δ of altitude w/respect to time

More precisely, write $z(t) = f(x(t), y(t))$. Then

$$\frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}(t)$$

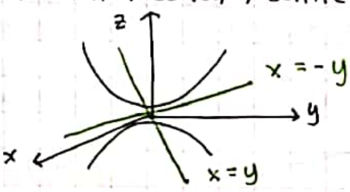
$\frac{dx}{dt}$ = velocity vector (x component); $\frac{\partial f}{\partial x}$ = slope in x direction; same for y

Ex 1: $f(x,y) = y^2 - x^2$, $x(t) = t$, $y(t) = t$

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

$= (-2x) + (2y)(1)$

$= -2t + 2t = 0$



contains line $x=y$ & $x=-y$

Ex 2: $f(x,y) = y^2 - x^2$, $x(t) = t$, $y(t) = t^2$

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

$= (-2x) + (2y)(2t)$

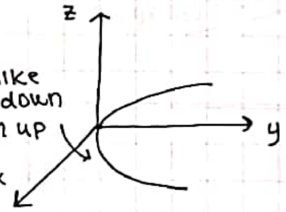
$= -2t + 4t^3$

$= 2t(2t^2 - 1)$

positive if $t > \frac{1}{\sqrt{2}}$

negative if $0 < t < \frac{1}{\sqrt{2}}$

*looks like we go down & then up



$z = f(x(t), y(t))$

Expect:

$\frac{dz}{dt} < 0$ for $t > 0$ small

$\frac{dz}{dt} > 0$ for $t > 0$ large

Note: $z = t^4 - t^2$

2.3.3. More About Multivariable Chain Rule #1

Proof $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$\Delta z = z(t+\Delta t) - z(t)$

$\frac{dz}{dt} = \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\frac{(f_x\Delta x + f_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y)}{\Delta t}$

definition of differentiability

ε_1 & $\varepsilon_2 \to 0$

$= \lim_{\Delta t \to 0} f_x\frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} f_y\frac{\Delta y}{\Delta t}$

$= f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$

Ex: $f(x,y) = x^2 + y^2$, $x = e^t$, $y = \ln t$

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= 2xe^t + 2y\left(\frac{1}{t}\right)$

$= 2e^{2t} + \frac{2\ln t}{t}$

$z = f(x(t), y(t))$

$= x(t)^2 + y(t)^2$

$= e^{2t} + (\ln t)^2$

$\frac{dz}{dt} = 2e^{2t} + \frac{2\ln t}{t}$

2.3.4. Multivariable Chain Rule #2

$z=f(x,y)$
$x=g(s,t)$
$y=h(s,t)$

think of z as a function of s & t $\Rightarrow$ $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

*assume differentiable functions

__Ex:__ $z=e^x\sin y$ $\quad x=st^2$ $\quad y=s^2t$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial s}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$=e^x(\sin y)t^2+e^x(\cos y)2st$$
$$=e^{st^2}\sin(s^2t)t^2+e^{st^2}\cos(s^2t)2st$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$=e^x\sin y\,(2st)+e^x(\cos y)s^2$$
$$=e^{st^2}\sin(s^2t)\,2st+e^{st^2}\cos(s^2t)s^2$$

2.3.5. General Chain Rule

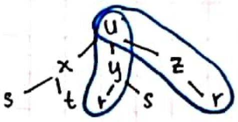
Suppose u is a differentiable function of $x_1,\dots,x_n$
Suppose each x_i is a differentiable function of $t_1,\dots,t_m$.
Then for each i from 1 to m,

$$\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\dots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$\frac{\partial u}{\partial t_i}=\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

previous case: $n=2$, $m=1$ or 2

__Ex:__ $u=x^3+xyz+z^3$
$x=s^2+t^2$
$y=rs$
$z=r^2$
$(n=3, m=3)$



$$\frac{\partial u}{\partial r}=\cancel{\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$
$$=(xz)s+(xy+3z^2)(2r)$$

2.3.6. Implicit Partial Differentiation Revisited

Suppose z is a function of x,y defined implicitly by $F(x,y,z)=0$ where F is a differentiable func. of three variables. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

We have $F(x,y,z(x,y))=0$ as functions of x,y

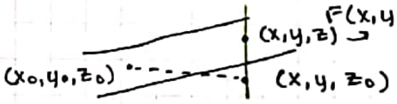
Apply $\frac{\partial}{\partial x}$: $\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1}+\cancel{\frac{\partial F}{\partial y}}\underbrace{\frac{\partial y}{\partial x}}_{0}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$ ← what we want to solve for

If $\frac{\partial F}{\partial z}\neq 0$ then $\frac{\partial z}{\partial x}=\frac{-\partial F/\partial x}{\partial F/\partial z}$ and $\frac{\partial z}{\partial y}=\frac{-\partial F/\partial y}{\partial F/\partial z}$.

How do we know that z is a differentiable function of x,y?

Implicit Function Theorem: Suppose F is a differentiable function of x,y,z. Suppose $F(x_0,y_0,z_0)=0$ & $\frac{\partial F}{\partial z}(x_0,y_0,z_0)\neq 0$. Then:

- Can solve $F(x,y,z)=0$ for z as a function of x,y when (x,y) is close to (x_0,y_0) with z close to z_0
- z is a differentiable function of x,y.



← get z up to get back on the plane (where $F=0$) b/c $\frac{\partial F}{\partial z}\neq 0$ we can

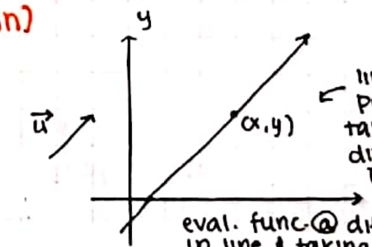
LECTURE 2.4

2.4.1. Definition of Directional Derivatives & Gradient (old version)

Let f be a function of x,y.
Let $\vec{u}=\langle a,b\rangle$ be a unit vector so $|\vec{u}|=\sqrt{a^2+b^2}=1$
Define the directional derivative of f in the direction $\vec{u}$ at (x,y) by

$$D_{\vec{u}}f(x,y)=\frac{d}{dt}\Big|_{t=0}\underbrace{f(x+at,\,y+bt)}_{t(\vec{u})+x,y}$$

↳ taking derivative of function as we move along the line



← line thru point whose tangent direction is $\vec{u}$

eval. func @ diff points in line & taking derivative

By the chain rule, if f is differentiable ($\frac{d}{dt}$ is defined) then

$$D_{\vec{u}}f(x,y)=\frac{\partial f}{\partial x}(x,y)\underbrace{\frac{d}{dt}\Big|_{t=0}(x+at)}_{a}+\frac{\partial f}{\partial y}(x,y)\underbrace{\frac{d}{dt}(y+bt)\Big|_{t=0}}_{b}$$

$$D_{\vec{u}}f=a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}$$ e.g. $D_{\langle 1,0\rangle}=\frac{\partial f}{\partial x}$ $\quad D_{\langle 0,1\rangle}=\frac{\partial f}{\partial y}$

↳ linear combo of partial derivatives w/ respect to x & y

*general directional derivatives allow you to measure the rate at which f Δs as you move in any direction

If f is a function of x,y,z & $\vec{u}=\langle a,b,c\rangle$ is a unit vector then define

$$D_{\vec{u}}f(x,y,z)=\frac{d}{dt}\Big|_{t=0}(f(x+at,\,y+bt,\,z+ct))$$

(can get partial derivatives w/ respect to x,y,z by taking directional derivative when $\vec{u}$ is one of the coordinate vectors)

If f is differentiable then $D_{\vec{u}}f=a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}+c\frac{\partial f}{\partial z}$

__Gradient:__ If f is a function of two variables, define the gradient of f at (x,y) to be

$$\operatorname{grad}f(x,y)=\nabla f(x,y)=\left\langle\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)\right\rangle$$

If f is a function of three variables, then $\nabla f(x,y,z)=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle(x,y,z)$

If $\vec{u}$ is a unit vector & f is differentiable then $\boxed{D_{\vec{u}}f = \vec{u}\cdot\nabla f}$

Ex: If f is a function of three variables: $\vec{u} = \langle a,b,c\rangle$

$$\left.\begin{aligned} D_{\vec{u}}f &= f_x a + f_y b + f_z c \\ \nabla f &= \langle f_x, f_y, f_z\rangle \end{aligned}\right\} \boxed{D_{\vec{u}}f = \vec{u}\cdot\nabla f}$$

Gradient is a vector which is a function of x, y, & z ; so it depends on what point you're on

Ex: $f(x,y) = xy^3 + \sin(x+y)$

$\nabla f = \langle y^3 + \sin(x+y),\ 3xy^2 + \sin(x+y)\rangle$

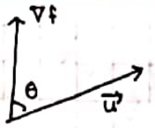
2.4.2. Properties of the Gradient

Let f be a differentiable function of 2 or 3 variables.

1) If $\nabla f \neq 0$ then ∇f points in the direction in which the directional derivative of f is longest

Proof: Let $\vec{u}$ be a unit vector, then

$$\begin{aligned} \boxed{D_{\vec{u}}f} &= \vec{u}\cdot\nabla f \\ &= |\vec{u}||\nabla f|\cos\theta \quad \text{) } \vec{u}=1 \\ &= |\nabla f|\cos\theta \quad \text{) } -1\le\cos\theta\le 1 \\ &\boxed{\le |\nabla f|} \end{aligned}$$



Equality holds when $\cos\theta = 1$, i.e. $\theta = 0$, i.e when $\vec{u}$ & ∇f point in the same direction

gradient in direction of most rapid increase & length is rate of increase

2) vector version of the chain rule: If $\vec{r}(t)$ is a vector-valued differentiable function of t then

$$\boxed{\frac{d}{dt}f(\vec{r}(t)) = \nabla f\cdot\vec{r}\,'(t)}$$

(how fast is f increasing as we move along parametrized curve $\vec{r}(t)$ = dot product of gradient of f & velocity vector of the curve)

Proof: write $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. Then by the chain rule,

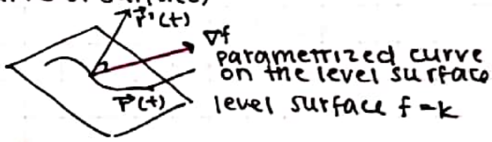
$$\begin{aligned} \frac{d}{dt}f(\vec{r}(t)) &= \frac{d}{dt}f(x(t), y(t), z(t)) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle\cdot\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = \nabla f\cdot\vec{r}\,'(t) \end{aligned}$$

3) If $\nabla f \neq 0$ then ∇f is perpendicular to the level sets $\{f = k\}$ (curve or surface)

i.e. If $\vec{r}(t)$ is a curve on the level set $\{f=k\}$ then $\nabla f \perp \vec{r}\,'(t)$

Proof: $\frac{d}{dt}(f(\vec{r}(t))) = \frac{d}{dk}(k) = 0$

$= \nabla f\cdot\vec{r}\,'(t)$



2.4.3. Tangent Planes Revisited

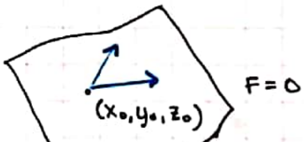
Tangent plane to a surface $F(x,y,z)=0$ at point (x_0,y_0,z_0)

Assume: $\nabla F(x_0,y_0,z_0) \neq 0$

know: $\nabla F(x_0,y_0,z_0) \perp$ tangent plane

$\vec{n} = \langle F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0), F_z(x_0,y_0,z_0)\rangle$

$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0 \qquad \vec{r} = \langle x,y,z\rangle \qquad \vec{r}_0 = \langle x_0,y_0,z_0\rangle$



$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

Ex: Find tangent plane to $x^2+y^2+z^2=1$ @ (x_0,y_0,z_0)

Tangent plane: $x_0(x-x_0) + y_0(y-y_0) + z_0(z-z_0) = 0$

$xx_0 + yy_0 + zz_0 = 1$ ← same eq as

$F = x^2+y^2+z^2-1$

$F_x = 2x \quad F_y = 2y \quad F_z = 2z$

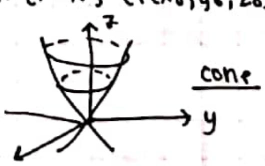
Implicit Function Theorem: If $\nabla F(x_0,y_0,z_0) \neq 0$ then the level set $K\ \{F=K\}$ $(F(x_0,y_0,z_0)=K)$ is a smooth surface near (x_0,y_0,z_0).

Ex: $F = z^2-x^2-y^2 \quad K=0 \quad F=K \Rightarrow$ cone

$\nabla F = \langle -2x, -2y, 2z\rangle$

$\nabla F(x_0,y_0,z_0) \neq 0$ unless $(x_0,y_0,z_0) = (0,0,0)$

singularity at (0,0,0)



where smooth, tangent plane $\perp \nabla F$

LECTURE 2.5

2.5.1. Review of Optimization of One Variable

Maxima & Minima of single variable $y=f(x)$

a is a (global) maximum if $f(a) \ge f(x)$ for all x

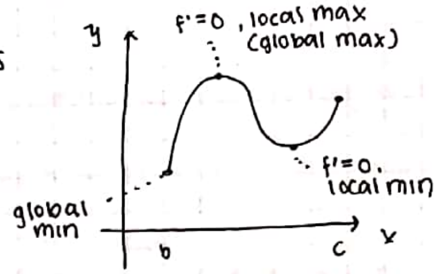
a is a (global) minimum if $f(a) \le f(x)$ for all x

a is a local maximum if $f(a) \ge f(x)$ for all x in the neighborhood of a

local minimum if $f(a) \le f(x)$

} If a is a local min or ma, if a is not on the boundary of domain of f, & if $f'(a)$ is defined, then $f'(a)=0$

If a is a local min or local max, then at least one of the following holds
1) $f'(a)=0$
2) $f'(a)$ undefined
3) a is on the boundary of the domain of f

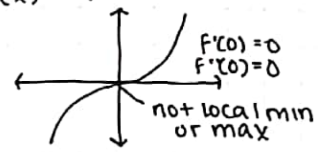


Extreme Value Theorem: If f is a continuous function defined on a closed interval then f has a global max & a global min
e.g. $f(x)=x$ defined on $\mathbb{R}$ has no global max or min

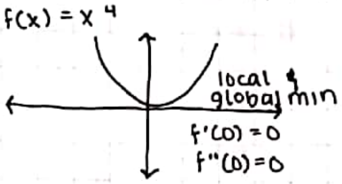
Second Derivative Test: Suppose $f'(a)=0$
→ If $f''(a)>0$ then a is a local min
→ If $f''(a)<0$ ——— local max

→ If $f''(a)=0$ or $f''(a)$ undefined, it could be local min, local max, or neither

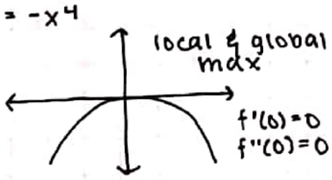
Ex: $f(x)=x^3$



$f(x)=x^4$



$f(x)=-x^4$

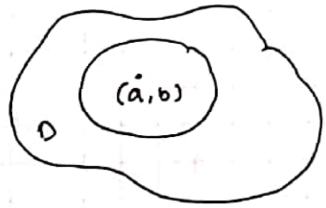


2.5.2. Introduction to Optimization in Two Variables

Let f be a function on a domain $D\subset\mathbb{R}^2$; let $(a,b)\in D$

(a,b) is a (global) maximum of f if $f(a,b)\geq f(x,y)$ for all $(x,y)\in D$.

(a,b) is a local maximum of f if $f(a,b)\geq f(x,y)$ for all (x,y) in some disk centered at (a,b)



local & global min is defined the same but with $f(a,b)\leq f(x,y)$

Local extremum: local min or local max

(a,b) is a critical point of f if $f_x(a,b)=f_y(a,b)=0$

Theorem: If (a,b) is a local extremum, if (a,b) is not on the boundary of D, & if $f_x(a,b)$ & $f_y(a,b)$ are defined, then $f_x(a,b)=f_y(a,b)=0$

Proof: Define $g(x)=f(x,b)$ [fixing $y=b$ & varying only x].
Then g has a local extremum @ a. By the single variable version, $g'(a)=0$ but $g'(a)=f_x(a,b)$
So $f_x(a,b)=0$ & similarly $f_y(a,b)=0$

If (a,b) is a local extremum then at least one of the following holds:
1) (a,b) is a critical point of f
2) $f_x(a,b)$ & $f_y(a,b)$ are not both defined
3) (a,b) is on the boundary of D

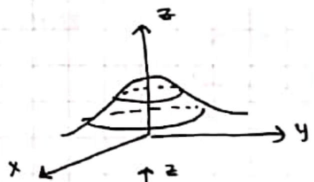
2.5.3. Examples of Critical Points

Ex: Find the critical points of $f(x,y)=e^{-x^2-y^2}$ $(D=\mathbb{R}^2)$

$f_x=-2xe^{-x^2-y^2}\ \Rightarrow\ f_x=0\Leftrightarrow x=0$

$f_y=-2ye^{-x^2-y^2}\ \Rightarrow\ f_y=0\Leftrightarrow y=0$

one critical point: $(0,0)$ Global max



Ex: Find critical points of $f(x,y)=y^2-x^2$

$f_x=-2x\ \Rightarrow\ f_x=0\Leftrightarrow x=0$

$f_y=2y\ \Rightarrow\ f_y=0\Leftrightarrow y=0$

one critical point: $(0,0)$ saddle Point

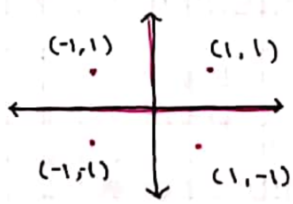
Ex: Find critical points of $f(x,y)=x^2y^2e^{-x^2-y^2}$

$f_x=2xy^2e^{-x^2-y^2}-2x^3y^2e^{-x^2-y^2}$
$=2xy^2(1-x^2)e^{-x^2-y^2}$

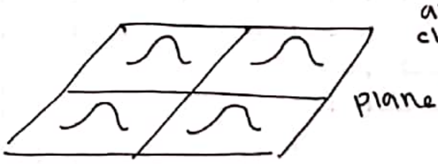
$f_y=2x^2y(1-y^2)e^{-x^2-y^2}$

Critical points: $x=0$, $y=$ anything] global
$y=0$, $x=$ anything] minima

$\underbrace{x=\pm1\ \&\ \pm1=y}_{\text{4 points global maxima}}$



axis: $f(x,y)=0$
close to axis: very small



2.5.4. The Second Derivative Test

Suppose (a,b) is a critical point of f. Assume f_{xx}, f_{xy}, f_{yy} are defined & continuous in a neighborhood of (a,b).

Define $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$ ... must both be positive or neg for $D>0$

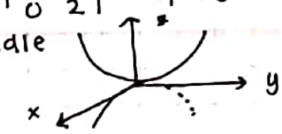
- $D>0$, $f_{xx}(a,b)>0 \Rightarrow$ local min .... both eigenvalues positive
- $D>0$, $f_{xx}(a,b)<0 \Rightarrow$ local max .... negative
- $D<0 \Rightarrow$ saddle point (neither local min nor max) ..... one positive & one negative eigenvalues
- Test is inconclusive in all other cases .... zero eigenvalue

Three Basic Examples

1) $f(x,y) = x^2+y^2$
$(a,b) = (0,0)$
$f_x = 2x \quad f_y = 2y$
$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0$
$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$
local min

2) $f(x,y) = -x^2-y^2$
$(a,b) = (0,0)$
$D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$
local max

3) $f(x,y) = y^2-x^2$
$(a,b) = 0$
$f_x = -2x \quad f_y = 2y$
$f_{xx} = -2 \quad f_{yy} = 2 \quad f_{xy} = 0$
$D = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0$
saddle



2.5.5. An Example of the Second Derivative Test

Ex: Find & classify the critical points of $f(x,y) = x^5 + y^4 - 5x - 32y - 3$ $(D = \mathbb{R}^2)$.
Is there a global max or global min?

$f_x = 5x^4 - 5 \quad f_{xx} = 20x^3 \quad f_{xy} = 0$
$f_y = 4y^3 - 32 \quad f_{yy} = 12y^2$
$f_x = 0 \Rightarrow x^4 - 1 \Rightarrow x = \pm 1$
$f_y = 0 \Rightarrow y^3 = 8 \Rightarrow x = 2$
critical points: $(1,2), (-1,2)$

Analyze $(1,2)$
$D = \begin{vmatrix} 20 & 0 \\ 0 & 48 \end{vmatrix} > 0$
local min

Analyze $(-1,2)$
$D = \begin{vmatrix} -20 & 0 \\ 0 & 48 \end{vmatrix} < 0$
saddle

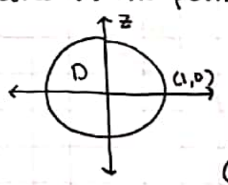
→ No local max ⇒ No global max
→ No global min because $f(x,0)$ can be arbitrarily negative if x is sufficiently negative

2.5.6. The Extreme Value Theorem

Extreme Value Theorem: Let f be a continuous function on $D \subset \mathbb{R}^2$. Assume D is closed & bounded. Then f has a global min & global max on D

Bounded means there exists $r>0$ such that D is contained in the disk of radius r centered at the origin (if you draw a sufficiently big circle, it will surround the entire domain)
→ there is an upper bound on the abs. values of x & y coordinates in all points in D

Closed means roughly that D contains all of its boundary points
e.g. $D = \{(x,y) \mid x^2+y^2 \le 1\}$ "closed unit disk" is closed
e.g. $D = \{(x,y) \mid x^2+y^2 < 1\}$ "open unit disk" is not closed
e.g. $f(x,y) = x$ has no global max or global min on the open unit disk

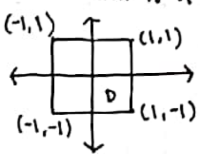


(0.999, 0)
(0.99999, 0)

* in general, closed sets are continuous & have $\le, \ge$
Strict inequalities like $>$ or $<$ are unlikely to have a closed set

2.5.7. Optimization Example #1

Example: Find the global maximum of $f = x^2 + y^2 + x^2y + 4$ on the square D w/ vertices $(\pm 1, \pm 1)$
Max is a critical point or a point on the boundary of D.



Critical Points: $f_x = 2x + 2xy = 2x(1+y)$
$f_y = 2y + x^2$
$f_x = 0 \Rightarrow x = 0$ or $y = -1$
$\swarrow f_y = 0 \quad \searrow f_y = 0$
$y = 0$ or $x = \pm\sqrt{2}$

3 critical points:
$(0,0)$, $\cancel{(\sqrt{2},-1)}$, $\cancel{(-\sqrt{2},-1)}$
not in domain
Candidate: $f(0,0) = 4$

Maximize f on boundary of D

Upper Edge: $y = 1$ & $-1 \le x \le 1$
$f(x,1) = 2x^2 + 5$
$f(-1,1) = f(1,1) = 7$

Left Edge: $x = -1$, $-1 \le y \le 1$
$f(-1,y) = y^2 + y + 5$
$f(-1,1) = 7$, $f(-1,-1) = 5$

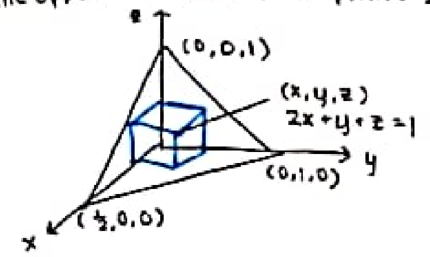
Right Edge: $x = 1$, $-1 \le y \le 1$
$f(1,y) = y^2 + y + 5$
$f(1,1) = 7$, $f(1,-1) = 5$

Bottom Edge: $y = -1$, $-1 \le x \le 1$
$f(x,-1) = 5$
max here is 5, attained everywhere

Global maxima of f on D are $(-1,1)$ & $(1,1)$ where $f = 7$

2.5.8. Optimization Example #2

Find the max possible volume of a rectangular box w/edges parallel to the axes, one corner at $(0,0,0)$ & the opposite corner on the plane $2x+y+z=1$ where $x,y,z \geq 0$



Volume $= xyz = xy(1-2x-y)$

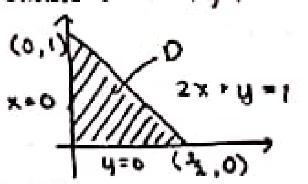
$f(x,y) = xy(1-2x-y)$

Maximize f

→ $f=0$ on every point on boundary D

→ Maximum must be @ a critical point in interior of D

Shadow on x,y plane



$f(x,y) = xy(1-2x-y) = xy - 2x^2y - xy^2$

$f_x = y - 4xy - y^2 = y(1-4x-y) \;\to\; f_x = 0 \Rightarrow \cancel{y=0}$ or $1-4x-y=0$

$f_y = x - 2x^2 - 2xy = x(1-2x-2y) \;\to\; f_y = 0 \Rightarrow \cancel{x=0}$ or $1-2x-2y=0$

$\Big\} \; x = \frac{1}{6}$ & $y = \frac{1}{3}$, $z = 1-2x-y = \frac{1}{3}$

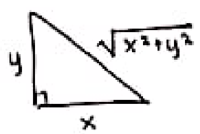
Maximum volume $= xyz = (\frac{1}{6})(\frac{1}{3})(\frac{1}{3}) = \frac{1}{54}$ (attained where $x=\frac{1}{6}$, $y=z=\frac{1}{3}$)

LECTURE 2.6

2.6.1. Introduction to Optimization with Constraints

Minimize or maximize $f(x,y)$ subject to constraint $g(x,y)=k$

Ex: Find the isoceles right triangle w/hypothenuse 10 w/largest possible area



Maximize $f(x,y) = \frac{xy}{2}$

Constraint: $g(x,y) = \sqrt{x^2+y^2} = 10$

Simplest approach: use the constraint to eliminate a variable

$y = \sqrt{100-x^2}$ $\quad g(x) = f(x,y) = \frac{x\sqrt{100-x^2}}{2}$ $\quad 0 \leq x \leq 10$

Maximize $g(x) = \frac{x\sqrt{100-x^2}}{2}$ for $0 \leq x \leq 10$

$g(0) = g(10) = 0$

$g'(x) = \frac{\sqrt{100-x^2}}{2} + \frac{x}{2}(2\sqrt{100-x^2})^{-1}(-2x)$

$= \sqrt{100-x^2}/2 - x^2/2\sqrt{100-x^2}$

$= 100-x^2-x^2/2\sqrt{100-x^2}$

$= 50-x^2/\sqrt{100-x^2}$

$g'(x) \Leftrightarrow 100-2x^2 = 50-x^2=0$

$\Leftrightarrow x^2=50$

$\Leftrightarrow x=\sqrt{50}$ $\quad$ * $-\sqrt{50}$ is out of domain

Max is $f(\sqrt{50},\sqrt{50}) = 25$

2.6.2. Lagrange Multipliers

To minimize or maximize $f(x,y)$ w/constraint $g(x,y)=k$ solve the equations $\nabla f = \lambda \nabla g$ & $g=k$ (λ = scalar, called **lagrange multiplier**) [Also need to check where $\nabla g = 0$]

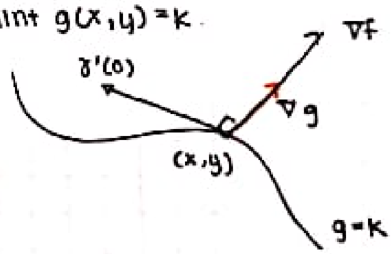
Claim: Suppose (x,y) minimizes or maximizes f subject to constraint $g(x,y)=k$.

Suppose $\nabla g(x,y) \neq 0$. Then $\nabla f(x,y) = \lambda \nabla g(x,y)$.

Let $\gamma(t)$ be a parametrized curve on $g=k$ w/ $\gamma(0)=(x,y)$

Since (x,y) is optimal, $\underbrace{\frac{d}{dt}\Big|_{t=0} f(\gamma(t))}_{\nabla f(x,y)\cdot\gamma'(0)} = 0$

Also $g(\gamma(t))=k$ for all t. So $\underbrace{\frac{d}{dt}\Big|_t g(\gamma(t))}_{\nabla g(x,y)\cdot\gamma'(t)} = 0$



∇f has to be $\perp$ to all tangent vectors to the level set which are also $\perp$ to ∇g

Ex: Maximize $f(x,y) = \frac{xy}{2}$ w/the constraint $g(x,y) = x^2+y^2 = 100$ (Domain: $x,y \geq 0$)

$\nabla f = \lambda \nabla g$, $\quad \nabla f = \langle \frac{y}{2}, \frac{x}{2} \rangle$, $\quad \nabla g = \langle 2x, 2y \rangle$

$\langle \frac{y}{2}, \frac{x}{2} \rangle = \lambda \langle 2x, 2y \rangle \Rightarrow \frac{y}{2} = \lambda 2x \to y = 4\lambda x$

$\frac{x}{2} = \lambda 2y \to x = 4\lambda y$

$\Rightarrow y = 16\lambda^2 y \Rightarrow y(1-16\lambda^2) = 0$

↳ $y=0$ (not max) or $\lambda = \frac{1}{4}, \cancel{-\frac{1}{4}}$ (out of domain)

$x=y \Rightarrow 2x^2 = 100 \Rightarrow x=y=\sqrt{50}$

2.6.3. More Lagrange Multipliers

To minimize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, solve $\nabla f = \lambda \nabla g$, $g=k$ [also check where $\nabla g = 0$]

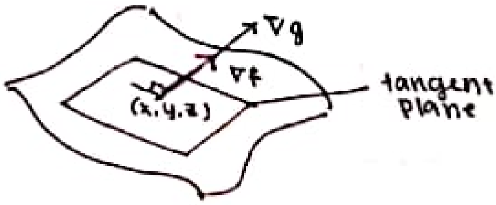
i.e. if (x,y,z) minimizes or maximizes $f(x,y,z)$ w/the constraint $g(x,y,z)=k$ & if $\nabla g(x,y,z) \neq 0$

then $\nabla f = \lambda \nabla g$ for some scalar λ.

Let $\vec{r}(t)$ be a paramerized curve on $g=k$ w/ $\vec{r}(0) = \langle x,y,z \rangle$

$0 = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(x,y,z)\cdot\vec{r}'(t)$ $\quad$ because (x,y,z) is min/max

$0 = \frac{d}{dt}\underbrace{g(\vec{r}(t))}_{k} = \nabla g(x,y,z)\cdot\vec{r}'(t)$



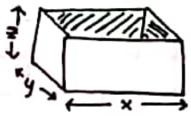
Ex: Maximize $f=xyz$ w/ the constraint $g=2x+y+z=1$ (Require $x,y,z>0$). Solve $\nabla f=\lambda\nabla g$, $g=k$

$\left.\begin{array}{l}\nabla f=\langle yz, xz, xy\rangle \\ \nabla g=\langle 2,1,1\rangle\end{array}\right\}$ $\nabla f=\lambda\nabla g \left\{\begin{array}{l} yz=2\lambda \\ xz=\lambda \\ xy=\lambda\end{array}\right.$ $\longrightarrow y=2x$, $\longrightarrow z=y$

$g(r)=1 \Rightarrow 2x+y+z=1 \rightarrow 2x+2x+2x=1 \rightarrow x=\frac{1}{6}, y=\frac{1}{3}, z=\frac{1}{3}$

2.6.4. A more Complicated Example of Lagrange

Problem: Build a topless rectangular box w/volume 32000 & the smallest possible area



Volume $= xyz = g(x,y,z)$

Area $= xy+2xz+2yz = f(x,y,z)$

$\nabla f=\lambda\nabla g$. where $g=32000$, require $x,y,z>0$

$\nabla f = xy+2xz+2yz$ & $g=xyz$

$\nabla f=\langle y+2z, x+2z, 2x+2y\rangle$ & $\nabla g=\langle yz, xz, xy\rangle$

$\nabla f=\lambda\nabla g \Rightarrow y+2z=\lambda yz$ (1)

$x+2z=\lambda xz$ (2)

$2x+2y=\lambda xy$ (3)

(1)-(2): $y-x=\lambda yz-\lambda xz$

$(y-x)(1-\lambda z)=0$ $\longrightarrow$ $x=y$ or $\lambda z=1$

If $\lambda z=1$ then (2) $\Rightarrow z=0$ impossible

(3) $4x=\lambda x^2 \Rightarrow 4=\lambda x$

(2) $x+2z=4z \Rightarrow x=2z$

$g=xyz=32000$

$\hookrightarrow z^3=8000$

$z=20$, $x=y=40$, & $\lambda=\frac{1}{10}$

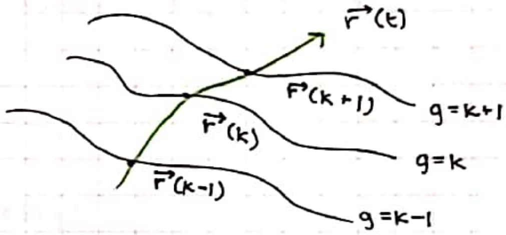
2.6.5. The Meaning of Lagrange Multipliers

Follow up question above: Suppose we want the volume of the box to be 32001 instead of 32000. How much more area do we need? $\longrightarrow$ Answer: Approximately $\frac{1}{10}$ (λ)

Claim: Let $M(k)$ be the minimum (maximum) value of f subject to the constraint $g=k$.

(suppose that $\nabla g\neq 0$ here & the minimum is unique)

Then $\frac{dM(k)}{dk}=\lambda$



Proof: Let $\vec{r}(t)$ be a parametrized curve such that $\vec{r}(k)$ is the point on $g=k$ where f is minimized

$\frac{d}{dk}M(k)=\frac{d}{dk}f(\vec{r}(k)) = \nabla f\cdot\vec{r}\,'(k)$

$=\lambda\nabla g\cdot\vec{r}\,'(k)=\lambda\frac{d}{dk}g(\vec{r}(k))=\lambda\frac{d}{dk}k=\lambda$

LECTURE 3.1

3.1.1. Definition of Double Integrals Over Rectangles

Single Variable Integration: $f:[a,b]\rightarrow\mathbb{R}$

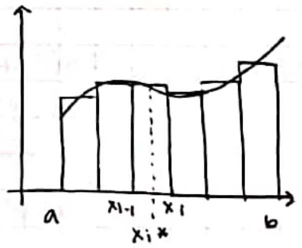
$\int_a^b f(x)dx$ = "area under the curve"

$a=x_0<x_1<\cdots<x_n=b$

$x_i-x_{i-1}=\frac{b-a}{n}=\Delta x$

Choose a "sample point" x_i^* between x_{i-1} & x_i

Define $\int_a^b f(x)dx=\lim\limits_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x$

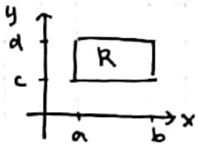


Theorem: This is well defined if f is continuous (limit converges & does not depend on sample points)

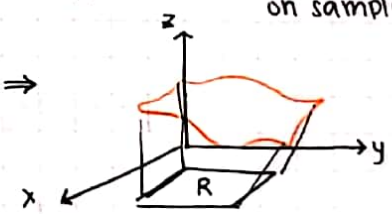
Two variable integration:

$R=[a,b]\times[c,d]=\left\{(x,y)\,\middle|\,\begin{array}{l}a\le x\le b\\ c\le y\le d\end{array}\right\}$

Cartesian product



$\Rightarrow$



$f:R\rightarrow\mathbb{R}$ (positive so above x,y)

Idea: define $\iint_R f\,dA$ = "volume under the graph" (when $f>0$)

$\hookrightarrow dA$ = "element of area"

Divide R into n^2 subrectangle

$a=x_0<x_1<\cdots<x_n=b$

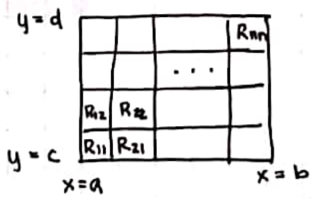
$b=y_0<y_1<\cdots<y_n=d$

$x_i-x_{i-1}=\frac{b-a}{n}=\Delta x$

$y_i-y_{i-1}=\frac{d-c}{n}=\Delta y$

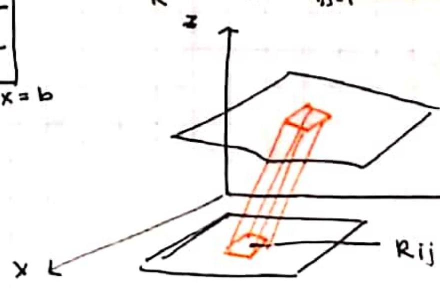
$R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$

Area$(R_{ij})=\Delta x\Delta y$



Choose a sample point $(x_{ij}^*, y_{ij}^*)\in R_{ij}$

Def $\iint_R f\,dA=\lim\limits_{n\to\infty}\sum_{i,j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y$



Theorem: If f is continuous then $\iint_R f\,dA$ is well defined

$$\iint_R f\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

Basic Properties

$\iint_R 1\,dA = \text{Area}(R)$

$\iint_R cf\,dA = c\iint_R f\,dA$

$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$

If $f \geq g$ (i.e. $f(x,y) \geq g(x,y)$ for every (x,y) in R) then $\iint_R f\,dA \geq \iint_R g\,dA$

3.1.2. How to Compute Double Integrals Over Rectangles

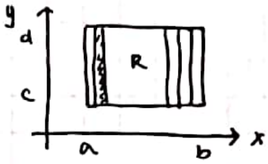
How to compute $\iint_R f\,dA$ where $R = [a,b]\times[c,d]$

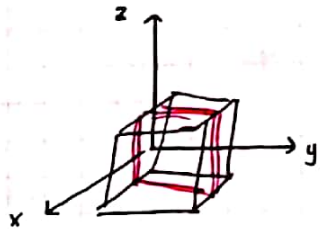
→ slice R into vertical strips

$a = x_0 < x_1 < \dots < x_n = b \qquad x_i - x_{i-1} = \frac{b-a}{n}$

→ choose a sample point x_i^* between x_{i-1} & x_i

→ i^{th} vertical strip is $[x_{i-1}, x_i]\times[c,d]$





$$\iint_R f\,dA = \sum_{i=1}^{n} \iint_{[x_{i-1},x_i]\times[c,d]} f\,dA \approx \sum_{i=1}^{n} \underbrace{\int_c^d f(x_i^*, y)\,dy}_{\substack{\text{area under } z=f(x_i^*,y) \\ \text{where } x_i^* \text{ is fixed}}} \underbrace{\Delta x}_{\text{thickness of slice}}$$

$$\iint_R f\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\int_c^d f(x_i^*, y)\,dy\right)\Delta x \longrightarrow \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

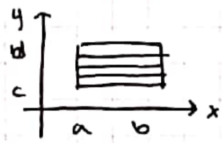
Fubini's Theorem: $\iint_R f\,dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{\substack{x \text{ is a constant} \\ \text{(integral is a func. of } x)}} dx$

Ex: $R = [0,1]\times[0,1]$. Calculate $\iint_R xe^{xy}\,dA$

$= \int_0^1 \left(\int_0^1 xe^{xy}\,dy\right)dx = \int_0^1 \left(e^{xy}\Big|_{y=0}^{y=1}\right)dx = \int_0^1 (e^x - 1)\,dx = (e^x - x)\Big|_{x=0}^{x=1} = e - 1 - 1 = e - 2$

can also integrate in other order!

$$\iint_R f\,dA = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{\substack{\text{regard } y \text{ as constant} \\ \text{integrate over } x}} dy$$

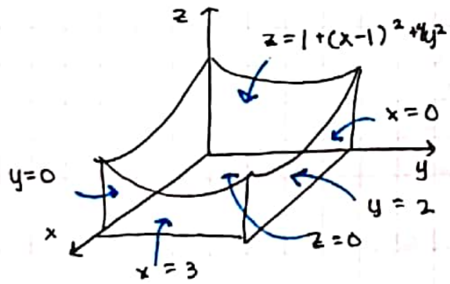


Same Ex: $\iint_R xe^{xy}\,dA = \int_0^1 \left(\int_0^1 xe^{xy}\,dx\right)dy$

$= \int_0^1 \left(\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right)\Big|_{x=0}^{x=1} dy$ ← messy!

3.1.3. Example of Double Integration Over a Rectangle

Ex: Find the volume of the solid region bounded by the elliptic paraboloid $z = 1 + (x-1)^2 + 4y^2$, the planes $x = 3$, $y = 2$, & the coordinate planes.



Volume $= \iint_R (1 + (x-1)^2 + 4y^2)\,dA$ where $R = [0,3]\times[0,2]$

$= \int_0^3 \int_0^2 (1 + (x-1)^2 + 4y^2)\,dy\,dx$

$= \int_0^3 \left(y + (x-1)^2 y + \frac{4}{3}y^3\right)\Big|_{y=0}^{y=2} dx$

$= \int_0^3 \left(2 + 2(x-1)^2 + 32/3\right)dx$

$= \left(2x + \frac{2}{3}(x-1)^3 + 32/3\,x\right)\Big|_{x=0}^{x=3}$

$= 6 + 6 + 32 = 44$

Check w/ $\int_0^2 \int_0^3 (1 + (x-1)^2 + 4y^2)\,dx\,dy = \int_0^2 \left(x + \frac{1}{3}(x-1)^3 + 4y^2 x\right)\Big|_{x=0}^{x=3} dy$

$= \int_0^2 (3 + 3 + 12y^2)\,dy = 6y + 4y^3 \Big|_{y=0}^{y=2} = 44$

3.1.4. Double Integrals Over More General Regions

Let D be a closed & bounded region in $\mathbb{R}^2$; let $f: D \to \mathbb{R}$

$\iint_D f\,dA$ = "volume under the graph over D"

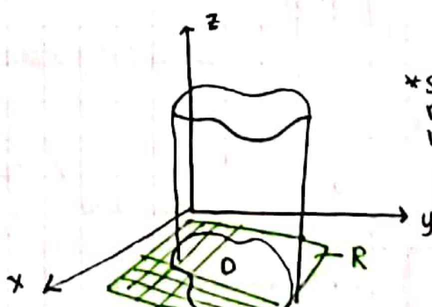
Let R be a rectangle containing D

Define $F: R \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

Define $\iint_D f\,dA = \iint_R F\,dA$

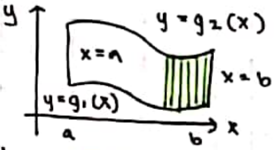
Ex: $\iint_D 1\,dA = \text{area}(D)$



*sum up rectangles in domain D

"Type I Region"

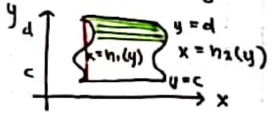
$D=\left\{(x,y)\ \middle|\ \begin{array}{l} a\le x\le b \\ g_1(x)\le y\le g_2(x)\end{array}\right\}$



$\iint_D f\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f\,dy\,dx$

"Type II Region"

$D=\left\{(x,y)\ \middle|\ \begin{array}{l} h_1(y)\le x\le h_2(y) \\ c\le y\le d\end{array}\right\}$



$\iint f\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f\,dx\,dy$

*red = integrate y first

If a region is both Type I & II, then you can integrate in either order

3.1.5. Examples of Double Integrals Over More General Regions

Calculate $\iint_D x\cos y\,dA$

Type I: $\iint_D x\cos y\,dA=\int_0^1\int_0^{x^2} x\cos y\,dy\,dx$

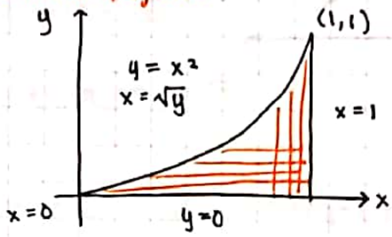
$=\int_0^1 x\sin y\Big|_{y=0}^{y=x^2}dx=\int_0^1 x\sin(x^2)\,dx$

$=-\frac{1}{2}\cos(x^2)\Big|_{x=0}^{x=1}=-\frac{1}{2}\cos(1)+\frac{1}{2}$

Type II: $\int_0^1\int_{\sqrt{y}}^1 x\cos y\,dx\,dy=\int\frac{1}{2}x^2\cos y\Big|_{x=\sqrt{y}}^{x=1}dy$

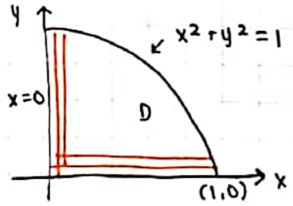
$=\int_0^1(\frac{1}{2}\cos y-\frac{1}{2}y\cos y)\,dy$

$=\ldots$ integration by parts



D is both Type I & Type II

Calculate $\iint_D xy\,dA$



*D is both Type I & II

Type I: $\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx=\int_0^1\frac{xy^2}{2}\Big|_{y=0}^{y=\sqrt{1-x^2}}dx$

$=\int_0^1\frac{1}{2}(x(1-x^2))\,dx$

$=\int_0^1(\frac{x}{2}-\frac{x^3}{2})\,dx=\frac{x^2}{4}-\frac{x^4}{8}\Big|_{x=0}^{x=1}$

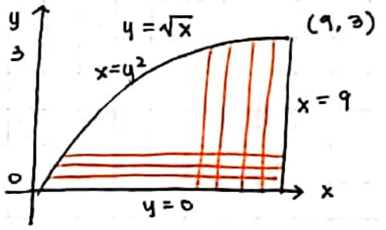
$=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}$

Evaluate $\int_0^3\int_{y^2}^9 y\cos(x^2)\,dx\,dy$: change the order of integration

$\int_0^9\int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx=\int_0^9\frac{1}{2}y^2\cos(x^2)\Big|_{y=0}^{y=\sqrt{x}}dx$

$=\int_0^9\frac{x}{2}\cos(x^2)\,dx=\frac{\sin(x^2)}{4}\Big|_{x=0}^{x=9}$

$=\frac{\sin(81)}{4}$



LECTURE 3.2

3.2.1. Double Integrals Over Polar Rectangles

"Polar Rectangle": $R=\{(r,\theta)\,|\,\alpha\le\theta\le\beta,\ a\le r\le b\}$

How to evaluate $\iint_R f\,dA$?

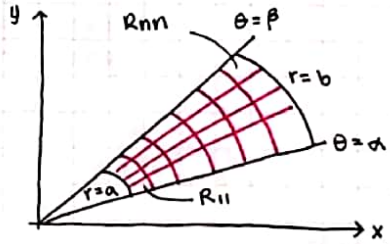
Partition R into smaller polar rectangles

$\alpha=\theta_0<\theta_1<\cdots<\theta_n=\beta \qquad \Delta\theta=\theta_i-\theta_{i-1}=\frac{\beta-\alpha}{n}$

$a=r_0<r_1<\cdots<r_n=b \qquad \Delta r=r_i-r_{i-1}=\frac{b-a}{n}$

$R_{ij}=\{(r,\theta)\,|\,\theta_{i-1}\le\theta\le\theta_i,\ r_{j-1}\le r\le r_j\}$

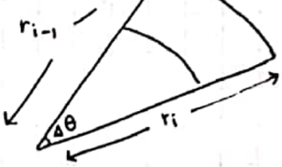
$\iint_R f\,dA=\lim_{n\to\infty}\sum_{i,j=1}^{n}\text{Area}(R_{ij})f(r_{ij}^*,\theta_{ij}^*)$



$(r_{ij}^*,\theta_{ij}^*)\in R_{ij}$ sample point

Area(R_{ij}) = Area(big pizza) − Area small pizza $=\frac{1}{2}r_i^2\Delta\theta-\frac{1}{2}r_{i-1}^2\Delta\theta$

$=\frac{1}{2}(r_i+r_{i-1})\Delta r\Delta\theta$



$\iint_R f\,dA=\lim_{n\to\infty}\sum_{i,j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\frac{r_i+r_{i-1}}{2}\Delta r\Delta\theta$

$=\int_\alpha^\beta\int_a^b f r\,dr\,d\theta$

$=\int_\alpha^\beta\int_a^b f(r,\theta)\,r\,dr\,d\theta$

"magnification factor"

"Area (infintesimal polar rectangle)"

$dr\,d\theta$

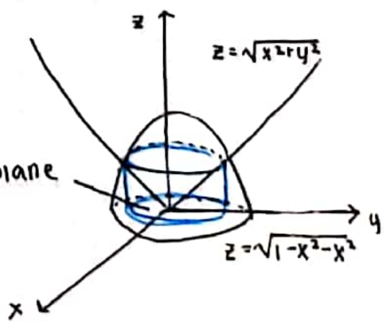
Area(small polar rectangle) $\approx r\Delta r\Delta\theta$

3.2.2. Integration in Polar Coordinates Example #1

Ex: Find the volume of the region between the surfaces $\underbrace{z=\sqrt{x^2+y^2}}_{\text{upper half of the cone}}$ & $\underbrace{z=\sqrt{1-x^2-y^2}}_{\text{upper half of the unit sphere}}$

Volume $=\iint_D$[(height of upper boundary) − (height of lower boundary)]dA

D = shadow of the region in the xy plane



Boundary of D is where

$\sqrt{x^2+y^2}=\sqrt{1-x^2-y^2}$

$r=\sqrt{1-r^2}$

$r^2=1-r^2$

$r=\frac{1}{\sqrt{2}}$

D = disk of radius $\frac{1}{\sqrt{2}}$ centered @ the origin

$\text{Volume} = \iint_{\text{disk of radius } 1/\sqrt{2}} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$

$= \int_0^{2\pi}\int_0^{1/\sqrt{2}} (\sqrt{1-r^2} - r)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{r^3}{3}\right]_{r=0}^{r=1/\sqrt{2}} d\theta$

$= -\frac{1}{3}\int_0^{2\pi} \left[(1-r^2)^{3/2} + r^3\right]_{r=0}^{r=1/\sqrt{2}} d\theta$

$= -\frac{1}{3}\int_0^{2\pi} \left[\left(\frac{1}{2}\right)^{3/2} + \left(\frac{1}{\sqrt{2}}\right)^3 - 1\right] d\theta$

$\frac{d}{dr}(1-r^2)^{3/2} = \frac{3}{2}(1-r^2)^{1/2}(-2r)$

$= -\frac{1}{3}\int_0^{2\pi} \left(\frac{1}{\sqrt{2}} - 1\right) d\theta$

$= -\frac{2}{3}\pi\left(\frac{1}{\sqrt{2}} - 1\right)$

$= \frac{2}{3}\pi\left(1 - \frac{1}{\sqrt{2}}\right)$

3.2.3. Integration in Polar Coordinates Example #2

Evaluate $A = \int_{-\infty}^{\infty} e^{-x^2} dx$

Clever trick: $A^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$

$= \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2} dA$

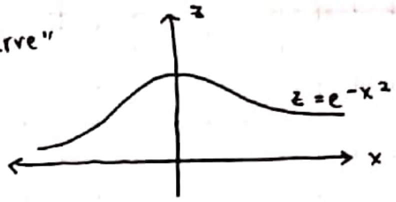
$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dA \longrightarrow$ improper integral converges because $e^{-x^2}e^{-y^2}$ decays rapidly away from the origin

$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=\infty}\right) d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$

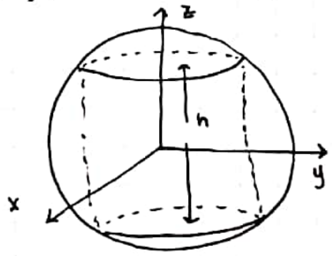
$A^2 = \pi \Rightarrow A = \sqrt{\pi}$

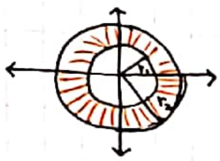
"bell curve"

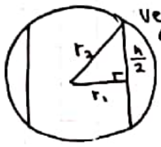


3.2.4. Integration in Polar Coordinates Example #3

Find the volume of the region obtained from a ball by cutting out a cylinder of height h (& parts above & below the cylinder). Let r_1 = radius of cylinder & r_2 = radius of ball







vertical cross section

$r_1^2 + \left(\frac{h}{2}\right)^2 = r_2^2$

$r_2^2 - r_1^2 = \left(\frac{h}{2}\right)^2$

$\text{Volume} = \iint_{\text{shadow}} (\text{upper height} - \text{lower height})\, dA$

$= \iint_{r_1 \le r \le r_2} \left(\sqrt{r_2^2-x^2-y^2} - (-\sqrt{r_2^2-x^2-y^2})\right) dA$

$= \iint_{r_1 \le r \le r_2} 2\sqrt{r_2^2-x^2-y^2}\, dA$

$= 2\iint_{r_1 \le r \le r_2} \sqrt{r_2^2-r^2}\, r\, dA$

$= 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2-r^2}\, r\, dr\, d\theta$

$= 2\int_0^{2\pi} -\frac{1}{3}(r_2^2-r^2)^{3/2}\Big|_{r=r_1}^{r=r_2} d\theta$

$= 2\int_0^{2\pi} -\frac{1}{3}\left(-(r_2^2-r_1^2)^{3/2}\right) d\theta$

$= \frac{4\pi}{3}(r_2^2-r_1^2)^{3/2}$

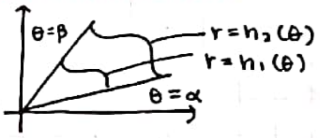
$= \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{3/2} = \frac{4\pi}{3}\left(\frac{h}{2}\right)^3 = \frac{\pi h^3}{6}$

3.2.5. Integration Over More General Regions in Polar Coordinates

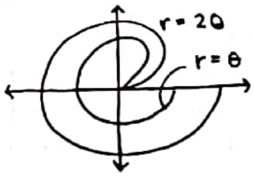
Analogue of a "Type I Region"

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, \; h_1(\theta) \le r \le h_2(\theta)\}$

$\iint_D f\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$



Ex: Calculate $\iint_D (x^2+y^2)\, dA$



$\alpha = 0, \beta = 2\pi$; $h_1(\theta) = \theta$, $h_2(\theta) = 2\theta$

$\iint_D (x^2+y^2)\, dA = \int_0^{2\pi}\int_\theta^{2\theta} r^2 \cdot r\, dr\, d\theta$

$= \int_0^{2\pi} \frac{r^4}{4}\Big|_{r=\theta}^{r=2\theta} d\theta$

$= \int_0^{2\pi} \left(\frac{16\theta^4}{4} - \frac{\theta^4}{4}\right) d\theta$

$= \int_0^{2\pi} \frac{15}{4}\theta^4 d\theta = \frac{3}{4}\theta^5\Big|_{\theta=0}^{\theta=2\pi}$

$= \frac{3}{4}(2\pi)^5 = 24\pi^5$

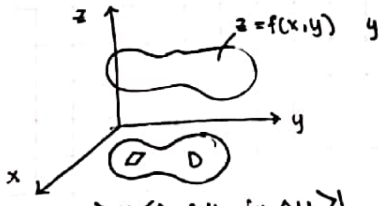
3.2.6. Surface Area

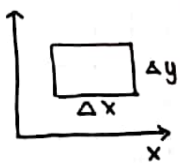
D domain in $\mathbb{R}^2$

$f: D \to \mathbb{R}$ differentiable

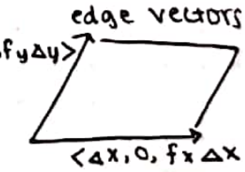
Find the area of the graph f

Divide D into small rectangles





edge vectors



Part of the graph over this rectangle is approximately a parallelogram.

Area of Parallelogram $= |\langle \Delta x, 0, f_x \Delta x\rangle \times \langle 0, \Delta y, f_y \Delta y\rangle|$

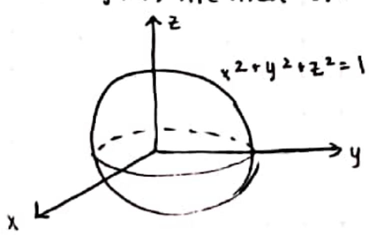
$= |\langle -f_x \Delta x \Delta y, -\Delta x f_y \Delta y, \Delta x \Delta y\rangle|$

$= \Delta x \Delta y |\langle -f_x, -f_y, 1\rangle| = \Delta x \Delta y \sqrt{1+f_x^2+f_y^2}$

Area of the Graph $\approx \sum_{\text{rectangles}} \Delta x \Delta y \sqrt{1+f_x^2+f_y^2}$ take limit as rectangles $\to 0$ then Δx & $\Delta y = dx, dy$

Area (Graph) $= \iint_D \sqrt{1+f_x^2+f_y^2}\, dA$

Ex: Find (again) the area of a sphere



Northern Hemisphere is $f(x,y) = \sqrt{1-x^2-y^2}$ over D = unit disk

Area (sphere) $= 2\iint_D \sqrt{1+f_x^2+f_y^2}\, dA$

$f_x = \frac{-x}{\sqrt{1-x^2-y^2}}$ $\quad f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$

$1 + f_x^2 + f_y^2 = 1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} = \frac{(1-x^2-y^2)+x^2+y^2}{1-x^2-y^2} = \frac{1}{1-x^2-y^2}$

Area (sphere) $= 2\iint_D \sqrt{\frac{1}{1-x^2-y^2}}\, dA = 2\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{1-r^2}}\, r\, dr\, d\theta$

$\frac{d}{dr}(1-r^2)^{1/2} = \frac{1}{2}(1-r^2)^{-1/2}(-2r) \rightarrow$ $= 2\int_0^{2\pi} -\sqrt{1-r^2}\Big|_{r=0}^{r=1} d\theta$

$= 2\int_0^{2\pi} 1\, d\theta = 4\pi$

LECTURE 3.3

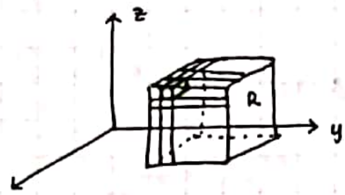
3.3.1. Definition of Triple Integrals

Rectangular box : $R = [a,b] \times [c,d] \times [r,s]$ as $f: R \to \mathbb{R}$

$a = x_0 < x_1 < \cdots < x_n = b$ $\quad x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$

$c = y_0 < y_1 < \cdots < y_n = d$ $\quad y_i - y_{i-1} = \Delta y = \frac{d-c}{n}$

$r = z_0 < z_1 < \cdots < z_n = s$ $\quad z_i - z_{i-1} = \Delta z = \frac{s-r}{n}$



Fact: $\iiint_R f\,dV$ well defined if f continuous

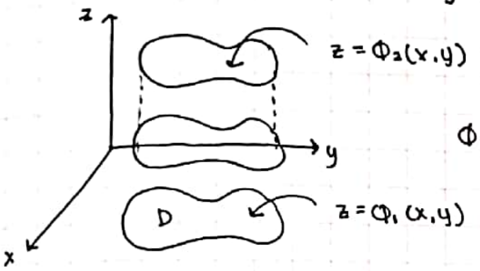
Define: $\underbrace{\iiint_R f\,dV}_{\text{volume}} = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \underbrace{\Delta x \Delta y \Delta z}_{\text{volume of sub-book}}$

where $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ sample point

Fubini's Theorem: $\iiint_R f\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$ ... 5 other orders

Analogue of Type I Region

$R = \left\{(x,y,z) \,\middle|\, \begin{array}{l} (x,y) \in D \\ \phi_1(x,y) \le z \le \phi_2(x,y) \end{array}\right\}$

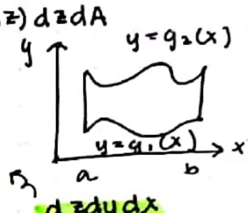


$\phi_1, \phi_2 : D \to \mathbb{R}$

Fact : $\iiint_R f\,dV = \iint_D \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dA$

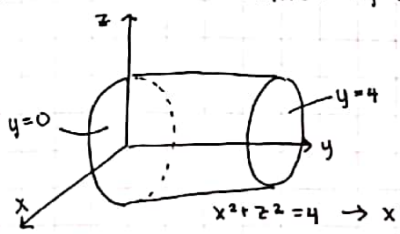
If D is a Type I Region

$D = \left\{(x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array}\right\}$

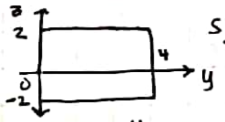


$\iiint_R f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

Ex: Let E be the solid bounded by the surfaces $x^2+z^2=4$, $y=0$, $y=4$. Write $\iiint_E f\,dV$ as an iterated integral.

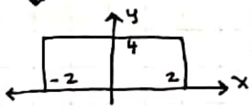


Order 1: $\int_{-2}^{2} \int_0^4 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f\,dz\,dy\,dx$



Shadow of x,y plane

Order 2: $\int_{-2}^{2} \int_0^4 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f\,dx\,dy\,dz$



Shadow of x,y plane

$x^2+z^2=4 \to x = \pm\sqrt{z^2+4}$ or $z = \pm\sqrt{4-x^2}$

3.3.2. An Example of a Triple Integral

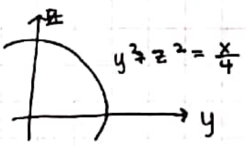
Compute $\iiint_E x\,dV$ where E is the region bounded by the paraboloid $x = 4y^2+4z^2$ & plane $x=4$

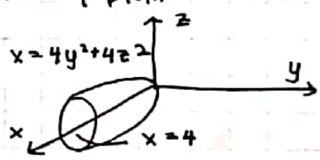
1. $\int_0^4 \iint_{(y^2+z^2 \le x/4)} x\,dA\,dx$

$= \int_0^4 x \text{ Area } (y^2+z^2 \le \frac{x}{4})\,dx$

disk of radius $\sqrt{\frac{x}{4}}$ $\sqrt{x/4}$

$= \int_0^4 x\,\pi(\frac{x}{4})\,dx = \frac{\pi x^3}{12}\Big|_0^4 = \frac{64\pi}{12} = \frac{16\pi}{3}$





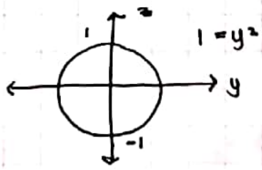
2. order of $dx\,dy\,dz$

$\int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{4y^2+4z^2}^{4} x\,dx\,dy\,dz$

$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{x^2}{2} \Big|_{x=4y^2+4z^2}^{x=4} dy\,dz$

$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} (8 - 8(y^2+z^2)^2)\,dy\,dz$

$= \cdots = \frac{16\pi}{3}$



$1 = y^2+z^2 \Rightarrow y = \pm\sqrt{1-z^2}$

3.3.3. Change the Order of Integration in Triple Integrals

Change the order of integration : $\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f\,dz\,dy\,dx = \int_?^? \int_?^? \int_?^? f\,dx\,dy\,dz$

Integration Region

$0 \le x \le \cancel{1}$

$0\ \sqrt{x} \le y \le \cancel{1}$ } redundant

$0 \le z \le 1-y$

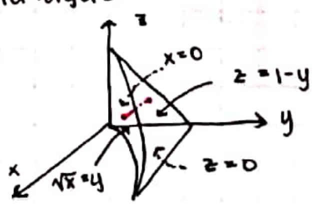
Boundary Surfaces:

$0 = x$

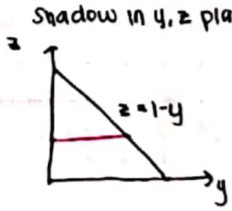
$\sqrt{x} = y$

$0 = z$ & $z = 1-y$

$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f\,dx\,dy\,dz$



Shadow in y,z plane



To change, see outer variable range; fix outer variable & see middle variable range; fix outer & middle for inner

3.3.4. Center of Mass

E = Solid region

$\rho : E \to \mathbb{R}^{>0}$

mass density $= \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$

small box : volume ΔV

mass Δm

mass of box $\approx \rho \Delta V$

Total Mass: $M = \iiint_E \rho\,dV$

Center of Mass: $\bar{x} = \frac{1}{M} \iiint_E x\rho\,dV$ "weighted average of x"

$\bar{y} = \frac{1}{M} \iiint_E y\rho\,dV$

$\bar{z} = \frac{1}{M} \iiint_E z\rho\,dV$

Ex: $E = [0,a] \times [0,a] \times [0,a]$ (cube side length a)

$\rho = x^2 + y^2 + z^2$

Calculate the center of mass: by symmetry, $(\bar{x}, \bar{y}, \bar{z}) = (c, c, c)$

$c < \frac{a}{2}$? $c = \frac{a}{2}$? $c > \frac{a}{2}$?

Does c depend on a?

Total Mass

$$M = \int_0^a\int_0^a\int_0^a x^2+y^2+z^2 \, dz\,dy\,dx$$
$$= \int_0^a\int_0^a x^2z + y^2z + \frac{z^3}{3}\Big|_{z=0}^{z=a} dy\,dx$$
$$= \int_0^a\int_0^a (ax^2 + ay^2 + \frac{a^3}{3})\,dy\,dx$$
$$= \int_0^a (ax^2y + \frac{ay^3}{3} + \frac{a^3y}{3})\Big|_0^a dx$$
$$= \int_0^a a^2x^2 + 2a^4(\tfrac{1}{3})\,dx$$
$$= \frac{a^2x^3}{3} + \frac{2a^4x}{3}\Big|_0^a = a^5$$

$$\bar{x} = \frac{1}{M}\int_0^a\int_0^a\int_0^a x(x^2+y^2+z^2)\,dz\,dy\,dx$$
$$= \frac{1}{a^5}\int_0^a\int_0^a x^3z + xy^2z + \frac{xz^3}{3}\Big|_{z=0}^{z=a} dy\,dx$$
$$= \frac{1}{a^5}\int_0^a\int_0^a ax^3 + axy^2 + \frac{xa^3}{3}\,dy\,dx$$
$$= \frac{1}{a^5}\int_0^a (ax^3y + \frac{axy^3}{3} + \frac{a^3}{3}xy)\Big|_{y=0}^{y=a} dx$$
$$= \frac{1}{a^5}\int_0^a (a^2x^3 + 2\frac{a^4}{3}x)\,dx$$
$$= \frac{1}{a^5}\left(\frac{a^2x^4}{4} + \frac{a^4}{3}x^2\right)\Big|_{x=0}^{x=a}$$
$$= \frac{1}{a^5}\left(\frac{a^6}{4} + \frac{a^6}{3}\right)$$
$$= \left(\frac{1}{4} + \frac{1}{3}\right)a = \frac{7}{12}a$$

center of mass $(\bar{x}, \bar{y}, \bar{z}) = (\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a)$

LECTURE 3.4

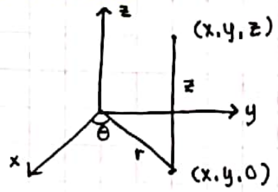
3.4.1. Triple Integrals in Cylindrical Coordinates

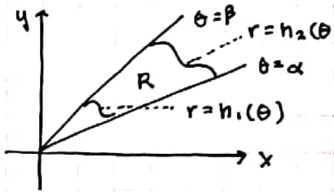
$(x,y,z) \longleftrightarrow (r,\theta,z)$

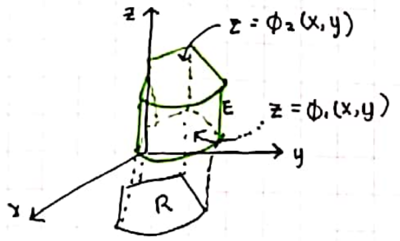
$x = r\cos\theta$ $\quad r^2 = x^2+y^2$

$y = r\sin\theta$ $\quad \tan\theta = y/x$

$z = z$







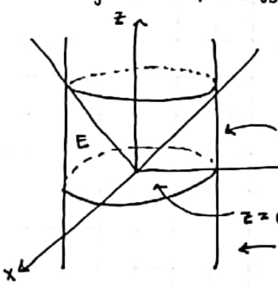
$R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

$E = \{(x,y,z) \mid (x,y) \in R,\ \Phi_1(x,y) \le z \le \Phi_2(x,y)\}$

$$\iiint_E f\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\Phi_1(\theta)}^{\Phi_2(\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$

3.4.2. Cylindrical Coordinates Example #1

Let E be the solid region inside the cylinder $x^2+y^2=1$, above the plane $z=0$, & below the surface $z^2 = 4x^2+4y^2$. Compute $\iiint_E x^2\,dV$.



$z^2 = 4x^2+4y^2$ (upper half)

$z^2 = 4r^2 \Rightarrow z = 2r$

r = 1

z = 0

$x^2+y^2=1$

Cartesian: $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{4x^2+4y^2}} x^2\,dz\,dy\,dx$

Cylindrical: $\int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,d\theta$

$$= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{2}{5}\cos^2\theta\,d\theta$$
$$= \frac{2}{5}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi}$$
$$= \frac{2\pi}{5}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

3.4.3. Cylindrical Coordinates Example #2

Find the volume of the region which is above the paraboloid $z = x^2+y^2$ & below the paraboloid $z = 36-3x^2-3y^2$.

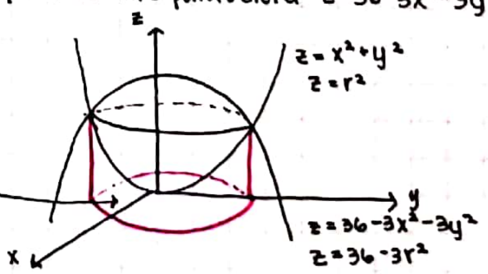
E = the region $\qquad$ Volume (E) = $\iiint_E 1\,dV$

Boundary of shadow = shadow of curve where surfaces meet

$z = r^2 = 36 - 3r^2 \Rightarrow r^2 = 9 \rightarrow r = 3$

$$\text{Volume }(E) = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} 1\cdot r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 r(36-4r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}(18r^2 - r^4)\Big|_{r=0}^{r=3} d\theta$$
$$= \int_0^{2\pi} 81\,d\theta = 162\pi$$

shadow in xy plane, disk of radius 3



3.4.4. Triple Integrals in Spherical Coordinates

$(x,y,z) \longleftrightarrow (\rho,\theta,\Phi)$

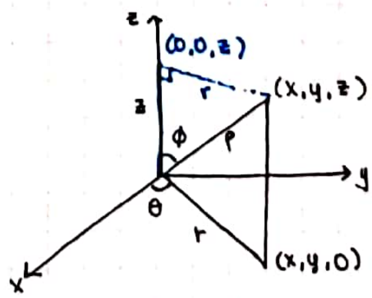
$z = \rho\cos\Phi$ $\qquad r = \rho\sin\Phi$

$x = \rho\sin\Phi\cos\theta$ $\qquad \rho = \sqrt{x^2+y^2+z^2}$

$y = \rho\sin\Phi\sin\theta$

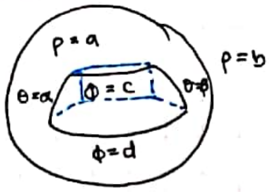
$0 \le \Phi \le \pi$ $\qquad \Phi = 0$: positive z axis

$\rho \ge 0$ $\qquad \Phi = \pi$: negative z axis



If $\rho > 0$ is fixed $\rightarrow$ sphere $\qquad \theta$ = longitude, $\Phi \approx$ latitude

Analogue of a box in spherical coordinates



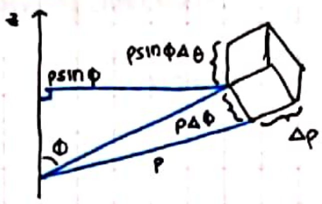
$$E = \left\{ (\rho, \theta, \phi) \;\middle|\; \begin{matrix} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ c \le \phi \le d \end{matrix} \right\}$$

$$\iiint_E f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f\,\overbrace{\rho^2 \sin\phi}^{\text{magnification factor}}\,d\rho\,d\theta\,d\phi$$

* can integrate using the other 5 orders

Magnification Factor

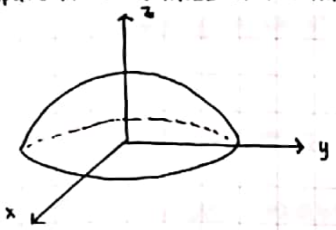
$$\lim_{(\Delta\rho, \Delta\theta, \Delta\phi)\to(0,0,0)} \frac{\text{Volume(spherical box, sides } \Delta\rho, \Delta\theta, \Delta\phi)}{\Delta\rho\,\Delta\theta\,\Delta\phi}$$



Volume of box $\approx (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi\,\Delta\theta) = \overbrace{\rho^2\sin\phi}^{\text{magnification factor}}\,\Delta\rho\,\Delta\phi\,\Delta\theta$

3.4.5. Spherical Coordinates Example #1

Compute the total mass of the half ball $x^2+y^2+z^2 \le 1$, $z>0$ with mass density $\mu = (x^2+y^2+z^2)^{3/2}$



Spherical coordinates: $\rho \le 1$ (unit ball)

$0 \le \phi \le \pi/2$ (upper half space)

$0 \le \theta \le 2\pi$

$M = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^3 \cdot \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

mass density ↗ ↖ magnification factor

$= \int_0^{\pi/2}\int_0^{2\pi} \frac{1}{6}\sin\phi\,d\theta\,d\phi = \int_0^{\pi/2} \frac{\pi}{2}\sin\phi\,d\theta = -\frac{\pi}{3}\cos\phi \Big|_{\phi=0}^{\phi=\pi/2} = \frac{\pi}{3}$

3.4.6. Spherical Coordinates Example #2

Find the volume of the region above the surface $\phi = \frac{\pi}{3}$ & below surface $\rho = 4\cos\phi$

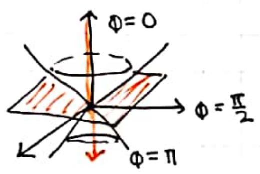
$z = \rho\cos\phi \Rightarrow z = \frac{1}{2}\rho$

$z^2 = \frac{1}{4}\rho^2$

$z^2 = \frac{1}{4}(x^2+y^2+z^2)$

$3z^2 = x^2+y^2$

(upper half of cone)



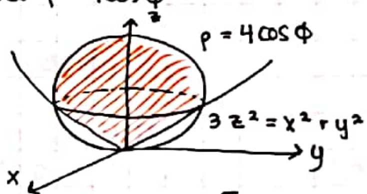
$\rho = 4\cos\phi$

$\rho^2 = 4\rho\cos\phi$

$x^2+y^2+z^2 = 4z$

$x^2+y^2+(z-2)^2 = 4$

(sphere of radius 2 centered @ (0,0,2))



Above $\phi = \frac{\pi}{3}$

Below $\rho = 4\cos\phi$

Volume $= \iiint_{\text{region}} 1\,dV$

$= \int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\phi} 1 \cdot \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$= \int_0^{\pi/3}\int_0^{2\pi} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=4\cos\phi} d\theta\,d\phi = \int_0^{\pi/3}\int_0^{2\pi} \frac{64}{3}\cos^3\phi\sin\phi\,d\theta\,d\phi$

$= \int_0^{\pi/3} \frac{128}{3}\cos^3\phi\sin\phi\,d\phi = -\frac{32\pi}{3}\cos^4\phi \Big|_{\phi=0}^{\phi=\pi/3}$

$= -\frac{32\pi}{3}\left[\frac{1}{16}-1\right] = -\frac{32\pi}{3}\left(-\frac{15}{16}\right) = 10\pi$

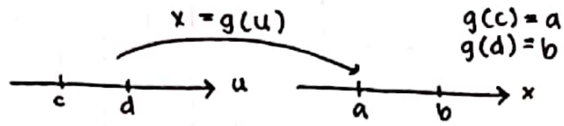
LECTURE 3.5

3.5.1. Change of Variables in Single Variable Calculus

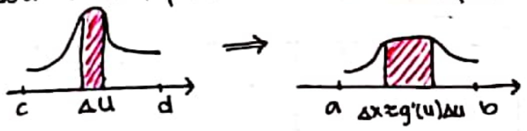
"u substitution": $\int_a^b f(x)\,dx = \int_c^d f(g(u))\underbrace{g'(u)\,du}_{\text{magnification factor}}$

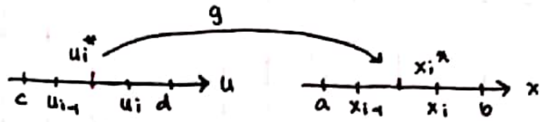
$g'(u)\,du = \frac{dx}{du}\,du$

(Also works if $c>d$ if you interpret $\int_c^d = -\int_d^c$)



Assume each point in $[a,b]$ corresponds to exactly one point in $[c,d]$.





A more formal definition:

$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$, $x_{i-1} \le x_i^* \le x_i$

$\int_c^d f(g(u))g'(u)\,du = \lim_{\max \Delta u_i \to 0} \sum_{i=1}^{n} \underbrace{f(g(u_i^*))}_{f(x_i^*)}\underbrace{g'(u_i^*)\Delta u_i}_{\approx \Delta x_i}$

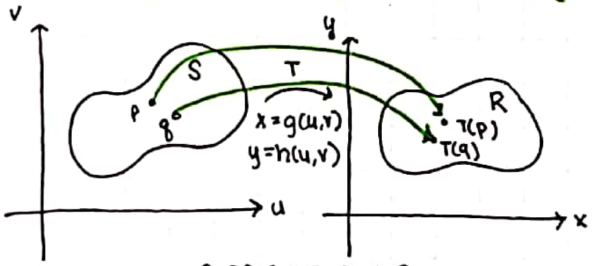
$c = u_0 < u_1 < \cdots < u_n = d$, $\Delta u_i = u_i - u_{i-1}$, $u_{i-1} \le u_i^* \le u_i$

$g(u_i) = x_i$

$g(u_i^*) = x_i^*$

$\Delta x_i \approx \Delta u_i \frac{dx}{du}(u_i^*)$

3.5.2. Injections, Surjections, & Bijections



Suppose $T: S \to R$

T is injective if $p \ne q$ are different points in S then $T(p) \ne T(q)$

T is subjective if every point in R is T of some point in S

T is bijective, or a bijection, or a one-to-one correspondence, if T is injective & subjective (one one point in S corresponds to exactly one point in S)

$f: R \to \mathbb{R}$ (R in the real number line)

Composition: $f \circ T : S \to \mathbb{R}$; $(f \circ T)(u,v) = f(g(u,v), h(u,v))$

Suppose T is a bijection: $\iint_R f\,dA = \iint_S (f\circ T)[??]\,dA$

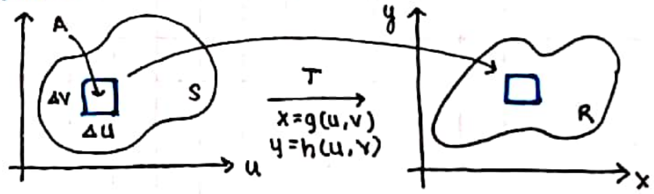
3.5.3. Change of Variables for Double Integrals

Suppose T is a differentiable bijection.

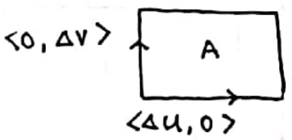
$f: R \to \mathbb{R}$ $\quad$ $f\circ T : S \to R$

$\iint_R f\,dA = \iint_S (f\circ T)[\text{magnification factor}]\,dA$

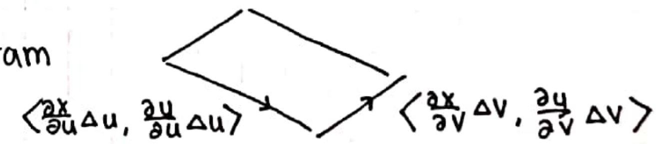
$\Delta u \Delta v = \text{Area}(A)$



Magnification factor $= \lim_{(\Delta u, \Delta v)\to(0,0)} \frac{\text{Area}(T(A))}{\text{Area}(A)}$



$\xrightarrow{T}$ Approx a parallelogram



Area (Parallelogram) = |det (edge vectors)|

$$\det \begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\Delta u\Delta v - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\Delta u\Delta v \Rightarrow \text{Area} = \underbrace{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|}_{\text{magnification factor}}\Delta u\Delta v$$

Jacobian: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem (Change of Variables in Double Integrals)

$$\iint_R f\,dA = \iint_S (f\circ T)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

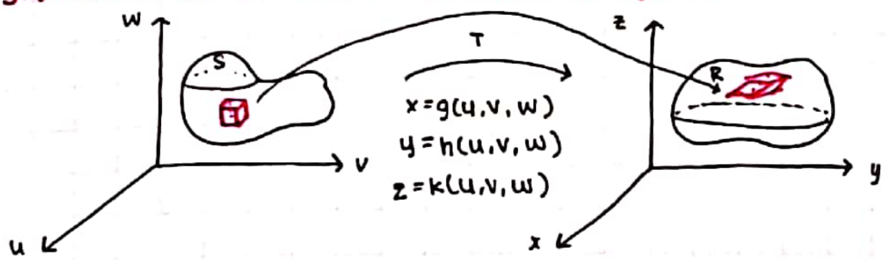
where T is a differentiable bijection from S (in the u,v plane) to R (in the x,y plane)

3.5.4. Integration in Polar Coordinates Revisited

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$
$= r$

$\iint_R f(x,y)\,dA = \iint_S f(r,\theta)\, r\, dA$

3.5.5. Change of Variables for Triple Integrals



$T: S \to R$ differentiable bijection

$$\iiint_R f\,dV = \iiint_S (f \circ T) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \lim_{(\Delta u, \Delta v, \Delta w) \to (0,0,0)} \frac{\text{Volume}(T(\text{Box}))}{\text{Volume}(\text{Box})}$$

Ex: Spherical Coordinates (magnification factor: $\rho^2 \sin\phi$)

$x = \rho\sin\phi\cos\theta$ $\quad \phi = w$
$y = \rho\sin\phi\sin\theta$ $\quad \theta = v$
$z = \rho\cos\phi$ $\quad u = \rho$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$= \rho^2\sin^3\phi\cos^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta$
$- \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta$
$= -\rho^2\sin^3\phi - \rho^2\sin\phi\cos^2\phi = -\rho^2\sin\phi$

Since $0 \le \phi \le \pi \to \sin\phi \ge 0$

Mag. factor $= |-\rho^2\sin\phi|$
$= \rho^2\sin\phi$

3.5.6. Change of Variables Example #1

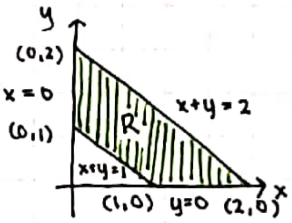
In general, to evaluate an integral by change of variables:

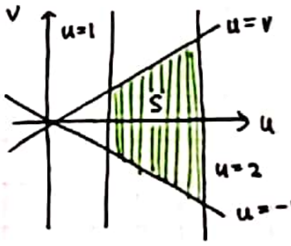
1. Choose a transformation which makes things nicer
2. Understand the change in geometry of the region
3. Compute the magnification factor
4. Evaluate the integral

Ex: Calculate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$

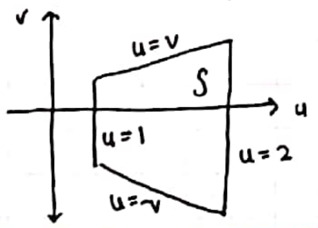
Change variables so that

$u = x + y$, $v = y - x$ $\xrightarrow[\text{in terms of } u, v]{\text{solve for } x, y}$ $\left.\begin{array}{l} x = \frac{u-v}{2} \\ y = \frac{u+v}{2} \end{array}\right\} T$





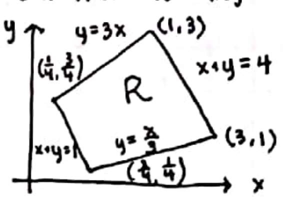
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$ magnification factor



$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint_S \cos\left(\frac{v}{u}\right)\left(\frac{1}{2}\right) dA$
$= \int_1^2 \int_{-u}^{u} \frac{1}{2}\cos\left(\frac{v}{u}\right) dv\,du$
$= \int_1^2 \frac{1}{2} u \sin\left(\frac{v}{u}\right)\Big|_{v=-u}^{v=u} du$
$= \int_1^2 \frac{1}{2}[u\sin(1) - u\sin(-1)]\, du$
$= \int_1^2 u\sin(1)\,du = \frac{\sin(1)}{2} u^2 \Big|_{u=1}^{u=2} = \frac{3}{2}\sin(1)$

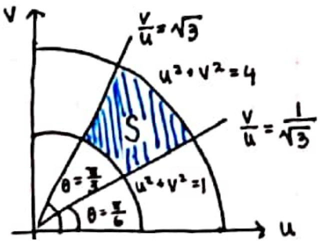
3.5.7. Change of Variables Example #2

Ex: Calculate $\iint_R \frac{1}{\sqrt{xy}}\, dA$ $\Rightarrow$ $\left(\int_{1/4}^{3/4} \int_{1-x}^{3x} + \int_{3/4}^{1} \int_{x/3}^{3x} + \int_1^3 \int_{x/3}^{4-x} \right) \frac{1}{\sqrt{xy}}\, dy\,dx =$ yucky!



Transformation: $x = u^2$, $y = v^2$

Mag. factor: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv$



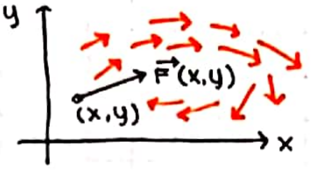
$\iint_R \frac{1}{\sqrt{xy}}\, dA = \iint \frac{1}{uv}(4uv)\, dA = 4\,\text{Area}(S)$
$= 4\left[\frac{\pi}{3} - \frac{\pi}{12}\right]$
$= 4\left[\frac{\pi}{4}\right]$
$= \pi$

Area(pie slice)
$= \frac{\pi}{6}\left(\frac{4}{2}\right) - \frac{\pi}{6}\left(\frac{1}{2}\right)$
θ $\quad \frac{r^2}{2}$

LECTURE 4.1

4.1.1. Introduction to Vector Fields

Definition: A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ associating to each point $(x,y) \in \mathbb{R}^2$ a vector $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$



Physical Examples: wind velocity
gravitational field
electric field

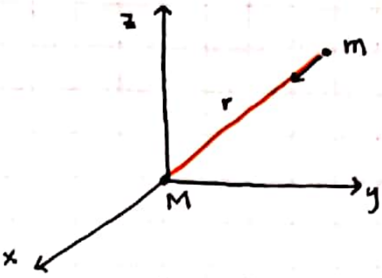
Ex: Gravitational Field

gravitational attraction $= \frac{GMm}{r^2} = \left(\frac{GM}{r^2}\right)m$
(points towards the origin)

gravitational field = vector pointing towards the origin of magnitude $\frac{GM}{r^2}$

$\vec{F}(\vec{r}) = \left(\frac{GM}{|\vec{r}|^2}\right)\left(-\frac{\vec{r}}{|\vec{r}|}\right) = \frac{-GM\vec{r}}{|\vec{r}|^3}$

→ unit vector pointing to origin $= -\frac{\vec{r}}{|\vec{r}|}$



Defined at all points except the origin (as it approaches 0, it goes to infinity)

4.1.2. Conservative Vector Fields

If f is a real-valued function on $\mathbb{R}^2$ or $\mathbb{R}^3$ then ∇f is a vector field.

On $\mathbb{R}^2$, $\nabla f = \langle f_x, f_y \rangle$

On $\mathbb{R}^3$, $\nabla f = \langle f_x, f_y, f_z \rangle$

Definition: A vector field $\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some function f (a potential)

e.g. $\vec{F} = \langle 3x^2\sin y, x^3\cos y \rangle$ is conservative so $\vec{F} = \nabla f$ where $f = x^3 \sin y$

e.g. $\vec{F} = \langle x^3 \sin y, y^2 \rangle$ is not conservative

Useful Fact: If f is a differentiable function on $\mathbb{R}^2$ then

$f(x,y) = f(0,y) + \int_0^x f_x(t,y)\,dt$

because for fixed y, define $g(t) = f(t,y)$ then $\frac{dg}{dt}(t) = f_x(t,y)$

Ex: Suppose $f_x = 3x^2 \sin y$, $f_y = x^3 \cos y$. Find f

$f(x,y) = g(y) + x^3 \sin y$

$f_y = g'(y) + x^3 \cos y$

$x^3 \cos y = f_y \rightarrow g'(y) = 0 \rightarrow g(y) = c \rightarrow f(x,y) = x^3 \sin y + c$

Ex: Try to find f w/ $f_x = x^3 \sin y$ & $f_y = y^2$

Integrate out x: $f = \frac{x^4}{4}\sin y + g(y)$

Differentiate w/r/t y: $f_y = \frac{x^4}{4}\cos y + g'(y) = y^2$

$\Rightarrow g'(y) = y^2 - \frac{x^4}{4}\cos y$

contradiction because $g'(y)$ depends only on y, not on x. $\vec{F}(x,y) = \langle x^3 \sin y, y^2 \rangle$ is not conservative.

Ex: The gravitational field $\vec{F} = -\frac{GM\vec{r}}{|\vec{r}|^3}$ is conservative.

$\vec{F} = \nabla f$ where $f = \frac{GM}{|\vec{r}|}$

$f = GM(x^2+y^2+z^2)^{-\frac{1}{2}}$

$f_x = GM\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = \frac{-GMx}{(x^2+y^2+z^2)^{3/2}} = \frac{-GMx}{|\vec{r}|^3}$

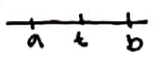
$f_y = \frac{-GMy}{|\vec{r}|^3}$, $f_z = \frac{-GMz}{|\vec{r}|^3}$

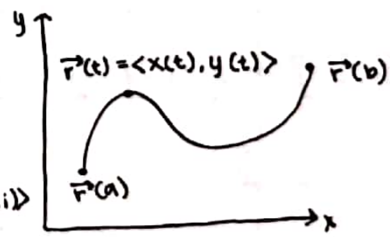
$\nabla f = -\frac{GM}{|\vec{r}|^3}\langle x, y, z \rangle = -\frac{GM\vec{r}}{|\vec{r}|^3}$

4.1.3. Line Integrals with respect to Arc Length

(Curve) Line Integrals in $\mathbb{R}^2$

Let C be a parametrized curve

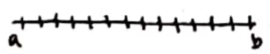




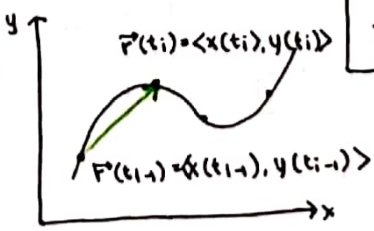
Three Different kinds of "Line Integrals" over C.

1. $\int_C f\,ds$ where $f: C \rightarrow \mathbb{R}$ (integration wrt arc length)
2. $\int_C f\,dx$, $\int_C f\,dy$
3. $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is a vector field

1. Integration wrt arc length

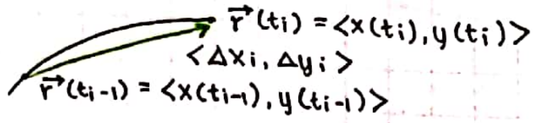


$t_i^* \in [t_{i-1}, t_i]$



$a = t_0 < t_1 < \cdots < t_n = b$, $\Delta t = t_i - t_{i-1} = \frac{b-a}{n}$

$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = |\vec{r}(t_i) - \vec{r}(t_{i-1})|$

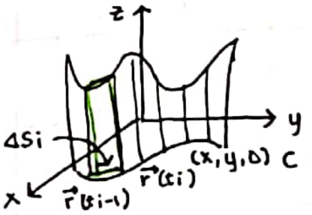


Define $\int_c f\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i^*))\Delta s_i$

$\int_c f\,ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt$

Ex: $\int_c 1\,ds = \text{length}(c)$

If $f>0$ on c, then $\int_c f\,ds =$ area under the graph of f over c



$\int_c f\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i^*))\Delta s_i$

[**Important Property**: $\int_c f\,ds$ does not depend on the parametrization as long as you don't "backtrack"]

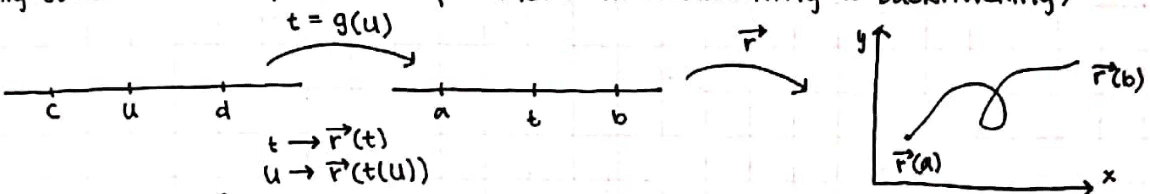
Suppose that the curve C is a wire w/mass density ρ

Mass (wire) is $M = \int_c \rho\,ds$

Center of mass of wire is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{M}\int x\rho\,ds$ & $\bar{y} = \frac{1}{M}\int y\rho\,ds$

4.1.4. Why Integration with respect to Arc Length is Well Defined

Why $\int_c f\,ds$ does not depend on the parametrization (assuming no backtracking)



Assume g is a differentiable bijection

Case 1: $g(c)=a$, $g(d)=b$, $g' \geq 0$

$\int_c^d f(\vec{r}(t(u)))|\frac{d}{du}\vec{r}(t(u))|\,du$

$\int_c^d f(\vec{r}(t(u)))|\underbrace{\frac{d}{dt}\vec{r}(t)}_{\text{vector}}\underbrace{\frac{dt}{du}}_{\text{non-negative scalar}}|\,du$

$= \int_c^d f(\vec{r}(t(u)))|\frac{d}{dt}\vec{r}(t)|\frac{dt}{du}\,du = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt$

Case 2: $g(c)=b$, $g(d)=a$, $g' \leq 0$

$\int_c^d f(\vec{r}(t(u)))|\frac{d}{dt}\vec{r}(t)\underbrace{\frac{dt}{du}}_{\text{nonpositive scalar}}|\,du$

$= -\int_c^d f(\vec{r}(t(u)))|\frac{d}{dt}\vec{r}(t)|\frac{dt}{du}\,du$

$= -\int_b^a f(\vec{r}(t))|\vec{r}\,'(t)|\,dt = \int_a^b \cdots$

4.1.5. Line Integrals with respect to x or y

Let C be a parametrized curve. $t \in [a,b] \mapsto \vec{r}(t) = \langle x(t), y(t)\rangle$

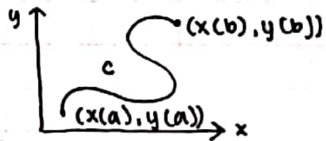
Let $f: C \to \mathbb{R}$

Define: $\int_c f\,dx = \int_a^b f(x(t),y(t))\underbrace{x'(t)}_{\text{x component of velocity vector}}\,dt$

$\int_c f\,dy = \int_a^b f(x(t),y(t))\underbrace{y'(t)}_{\text{y component of velocity vector}}\,dt$

Compare: $\int_c f\,ds = \int_a^b f(x(t),y(t))\underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{\text{length of velocity vector}}\,dt$

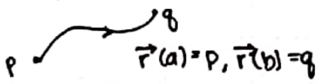
Ex:



$\left.\begin{array}{l}\int_c 1\,dx = x(b) - x(a)\\ \int_c 1\,dy = y(b) - y(a)\end{array}\right]$ displacement

$\int_c 1\,ds = \text{length}(c)$

Important Fact: $\int_c f\,dx$ & $\int_c f\,dy$ do not depend on the parametrization (even allowing backtracking), except that if you reverse direction, then $\int_c f\,dx$ & $\int_c f\,dy$ get multiplied by -1

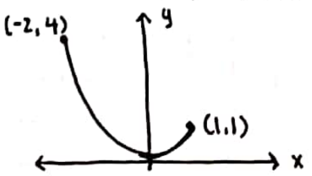


A curve with a chosen direction is called an oriented curve: $-c = c$ w/orient. reversed

$\int_{-c} f\,dx = -\int_c f\,dx \qquad \int_{-c} f\,dy = -\int_c f\,dy$

4.1.6. Example of Integration with respect to x

Compute $\int_c (x+y)\,dx$ where C is the portion of the parabola $y=x^2$ from $(-2,4)$ to $(1,1)$



Parametrization 1: $x=t$, $y=t^2$, $-2 \leq t \leq 1$

$\int_c (x+y)\,dx = \int_{-2}^{1}(x(t)+y(t))\,x'(t)\,dt = \int_{-2}^{1}(t+t^2)\,dt = \left(\frac{t^2}{2}+\frac{t^3}{3}\right)\Big|_{t=-2}^{t=1} = \frac{3}{2}$

Parametrization 2: $x=t^3$, $y=t^6$, $(-2)^{1/3} \leq t \leq 1$

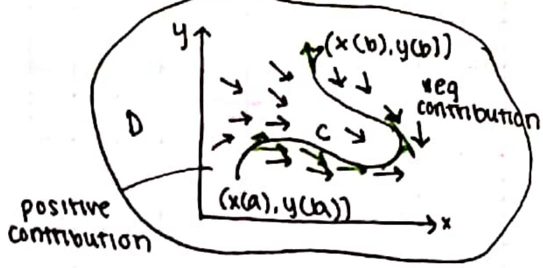
$\int_{-\sqrt[3]{2}}^{1}(t^3+t^6)(3t^2)\,dt = \int_{-\sqrt[3]{2}}^{1}(3t^5+3t^8)\,dt = \left(\frac{t^6}{2}+\frac{t^9}{3}\right)\Big|_{t=-\sqrt[3]{2}}^{t=1} = \frac{3}{2}$

Parametrization 3: $x=-t$, $y=t^2$, $-1 \leq t \leq 2$

$\int_{-1}^{2}(x(t)+y(t))x'(t)\,dt = \int_{-1}^{2}(-t+t^2)(-1)\,dt = \left(\frac{t^2}{2}-\frac{t^3}{3}\right)\Big|_{t=-1}^{t=2} = -\frac{3}{2}$

* this is negative because we are going backwards!

4.1.7. Line Integrals of Vector Fields



Let C be a parametrized curve in $\mathbb{R}^2$

$t \to \langle x(t), y(t) \rangle = \vec{r}(t)$ where $a \le t \le b$

Let $\vec{F}$ be a vector field defined in a domain D in $\mathbb{R}^2$ containing C

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

Define: $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$= \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t) \rangle dt = \int_a^b (P(\vec{r}(t))x'(t) + Q(\vec{r}(t))y'(t))dt = \int_a^b (Pdx + Qdy)$

Physical Interpretation: $\vec{F}$ = force field ; $\int_C \vec{F} \cdot d\vec{r}$ = work done as you go along the curve

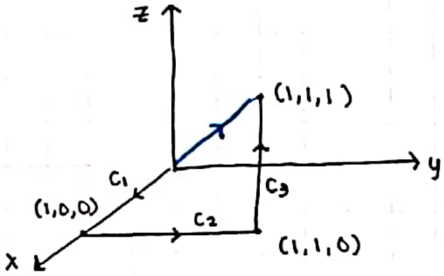
In $\mathbb{R}^3$: C (curve) where $t \to \langle x(t), y(t), z(t) \rangle = \vec{r}(t)$, $a \le t \le b$

$f: C \to \mathbb{R}$ $\qquad$ $F = \langle P, Q, R \rangle$ a vector field defined on C

1. $\int_C f ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_a^b f(\vec{r}(t)) |\vec{r}\,'(t)| dt$
2. $\int_C f dx = \int_a^b f(\vec{r}(t)) x'(t) dt$, $\int_C f dy = \int_a^b f(\vec{r}(t)) y'(t) dt$, $\int_C f dz = \int_a^b f(\vec{r}(t)) z'(t) dt$
3. $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt = \int_C (Pdx + Qdy + Rdz)$ where $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$

4.1.8. Examples of Line Integrals of Vector Fields

Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle z, xy, x+z \rangle$ & C consists of the line segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.



$\int_C \vec{F} \cdot d\vec{r} = (\int_{C_1} + \int_{C_2} + \int_{C_3}) \vec{F} \cdot d\vec{r}$

C_1: $x = t$, $y = 0$, $z = 0$ where $0 \le t \le 1$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 0, 0, t \rangle \cdot \langle 1, 0, 0 \rangle dt = \int_0^1 0\,dt = 0$

C_2: $x = 1$, $y = t$, $z = 0$ where $0 \le t \le 1$

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 0, t, 1 \rangle \cdot \langle 0, 1, 0 \rangle dt = \int_0^1 t\,dt = \frac{1}{2}$

C_3: $x = 1$, $y = 1$, $z = t$ $\quad 0 \le t \le 1$

$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 \langle t, 1, 1+t \rangle \cdot \langle 0, 0, 1 \rangle dt = \int_0^1 (1+t) dt = \frac{3}{2}$

Answer: $0 + \frac{1}{2} + \frac{3}{2} = 2$

Ex 2: Calculate $\int_{\tilde{C}} \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle z, xy, x+z \rangle$ & $\tilde{C}$ is the line seg. from $(0,0,0)$ to $(1,1,1)$.

Parametrize $\tilde{C}$: $x = t$, $y = t$, $z = t$ where $0 \le t \le 1$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t, t^2, 2t \rangle \cdot \langle 1, 1, 1 \rangle dt = (\frac{3}{2}t^2 + \frac{t^3}{3}) \Big|_{t=0}^{t=1} = \frac{11}{6} \ne 2$

Work done by $\vec{F}$ as you go from $(0,0,0)$ to $(1,1,1)$ depends on the curve from $(0,0,0)$ to $(1,1,1)$

LECTURE 4.2

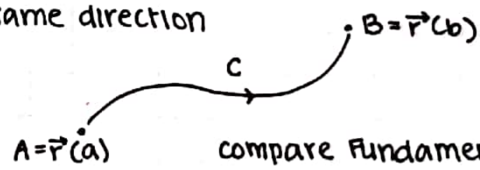
4.2.1. Statement of the Fundamental Theorem of Line Integrals

C parametrized curve, $\vec{r}(t)$, $a \le t \le b$; $\vec{F}$ a vector field defined on C

$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt$

Does not depend on parametrization if you go in the same direction

$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$



$\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some function f

Fundamental Theorem of Line Integrals

If $\vec{F} = \nabla f$ & if C is a parametrized curve from A to B, then

$\boxed{\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)}$

compare Fundamental Theorem of Calculus:

$\int_a^b \frac{dg}{dx} dx = g(b) - g(a)$

Corollary: If $\vec{F}$ is conservative then $\int_C \vec{F} \cdot d\vec{r}$ depends only on A, B, & not on C

Ex: Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle 2xyz + e^{x+z}, x^2z, x^2y + e^{x+z} \rangle$ & C is a curve from $(1,2,0)$ to $(1,0,5)$.

Solution: $\vec{F} = \nabla f$ where $f = x^2yz + e^{x+z}$ (guess & check)

So by FTLI $\int_C \vec{F} \cdot d\vec{r} = f(1,0,5) - f(1,2,0)$

$= (0 + e^6) - (0 + e^{1+0})$

$= e^6 - e$

$= e(e^5 - 1)$

4.2.2. Proof of the Fundamental Theorem of Line Integrals

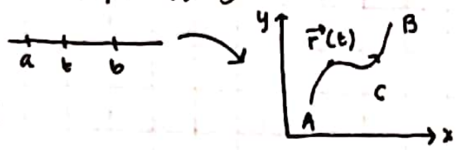
FTLI: If $\vec{F} = \nabla f$ & if C is a curve from A to B then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

<u>Proof</u>: Choose a parametrization of C as $\vec{r}(t)$ where $a \le t \le b$, $\vec{r}(a) = A$, $\vec{r}(b) = B$

Recall vector version of the chain rule:

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

(or $\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))\, y'(t)$)

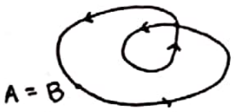


By the fundamental theorem of calculus,

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\text{vector field: } \nabla f(\vec{r}(t))} \cdot \vec{r}\,'(t)\,dt = \underbrace{\int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt}_{\text{vector version of chain rule}} = \underbrace{f(\vec{r}(b)) - f(\vec{r}(a))}_{\text{FTC}} = \boxed{f(B) - f(A)}$$

4.2.3. Conservative Vector Fields & Closed Curves

A curve C from A to B is closed if A=B



If $\vec{F}\,(=\nabla f)$ is conservative & C is closed then:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = (\text{FTLI}) = f(B) - f(A) = 0}$$

<u>Physical interpretation</u>: Work done around a closed curve is zero

e.g. $\vec{F}$ = gravitational field cannot look like the following picture:

For this picture $\int_C \vec{F}\cdot d\vec{r} > 0$

$\Rightarrow \vec{F}$ is not conservative

(cannot go downhill forever)

4.2.4. A Characterization of Conservative Vector Fields

<u>Theorem</u>: Let $\vec{F}$ be a vector field in a domain D. Then

$\vec{F}$ is conserative $\overset{\text{if \& only if}}{\Longleftrightarrow}$ $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C in D

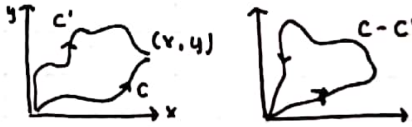
<u>Proof</u>: ($\Rightarrow$) Suppose $\vec{F}$ is conservative. We need to show $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve in D. We know this by FTLI.

($\Leftarrow$) Suppose $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C in D. We need to find a function f on D such that $\vec{F} = \nabla f$. Suppose $D = \mathbb{R}^2$ (general case similar)

Define $f(x,y) = \int_C \vec{F}\cdot d\vec{r}$ where C is a curve from (0,0) to (x,y).

This does not depend on C because if C' is another curve from (0,0) to (x,y) then

$\int_C \vec{F}\cdot d\vec{r} - \int_{C'} \vec{F}\cdot d\vec{r} = \int_{C-C'} \vec{F}\cdot d\vec{r} = 0$ because C−C' is closed



So $f(x,y) = \int_C \vec{F}\cdot d\vec{r}$ is well-defined

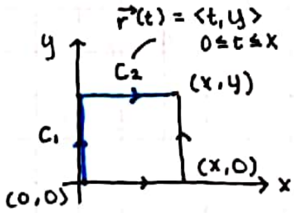
Still need to know $\nabla f = \vec{F}$

Show $\vec{F}(x,y) = \nabla f(x,y)$ suppose $x, y > 0$

Show $\frac{\partial f}{\partial x}(x,y) = P(x,y)$

$\frac{\partial f}{\partial y}(x,y) = Q(x,y)$

$\vec{F} = \langle P, Q \rangle$



$$f(x,y) = \underbrace{\int_{C_1} \vec{F}\cdot d\vec{r}}_{\text{constant if } y \text{ is fixed}} + \underbrace{\int_{C_2} \vec{F}\cdot d\vec{r}}_{\int_0^x P(t,y)\,dt}$$

Fix y. Think about varying x.

$\frac{\partial f}{\partial x} = \frac{d}{dx}\int_0^x P(t,y)\,dt$

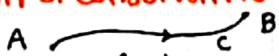
$= P(x,y)$) Fundamental Theorem of Calc

* To prove $\frac{\partial f}{\partial y} = Q$, fix x & vary y

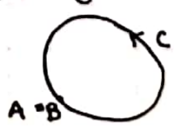
LECTURE 4.3

4.3.1. A More Practical Characterization of Conservative Vector Fields

<u>Recall</u>: FTLI $\int_C \nabla f\cdot d\vec{r} = f(B) - f(A)$



$\vec{F}$ is conservative (i.e. $\vec{F} = \nabla f$ for some f) $\Leftrightarrow \int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curve C



<u>Observation</u>: Let $\vec{F} = \langle P, Q \rangle$ be a differentiable vector field on a domain D in $\mathbb{R}^2$. If $\vec{F}$ is conservative then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftarrow f_{xy} = f_{yx}$

<u>Reason</u>: If $\vec{F} = \nabla f$ then $P = f_x$ & $Q = f_y$ (Clairaut's Theorem)

<u>Ex</u>: Is $\vec{F} = \langle x^2 \cos y,\ x + y^3 \rangle$ conservative?

If $\vec{F}$ is conservative then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

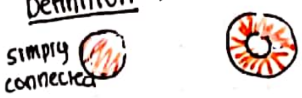
$\frac{\partial P}{\partial y} = -x^2 \sin y$

$\frac{\partial Q}{\partial x} = 1$

] $\vec{F}$ is not conservative

If $\vec{F} = \langle P, Q \rangle$ & $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, must $\vec{F}$ be conservative?

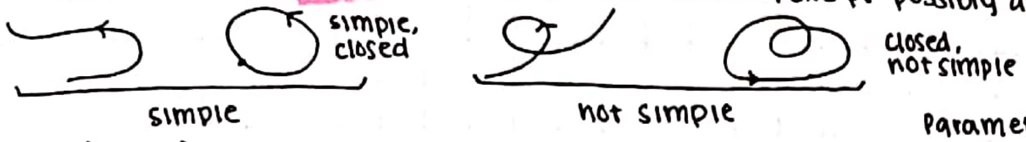
Definition: A domain D in $\mathbb{R}^2$ is simply connected if "it contains no holes"



Theorem: If D is simply connected & if $\vec{F} = \langle P, Q \rangle$ is a vector field on D such that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative

4.3.2. Statement of Green's Theorem

Definition: A curve C in $\mathbb{R}^2$ is simple if it does not self-intersect, except possibly at endpoints

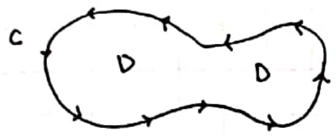


Parametrization: $\vec{r}(t)$ $a \le t \le b$

Simple means $\vec{r}(t_1) \neq \vec{r}(t_2)$ when $t_1 \neq t_2$ except possibly when $t_1 = a$, $t_2 = b$, or vice versa

Jordan Curve Theorem: A simple closed curve C in $\mathbb{R}^2$ is the boundary of a (unique) simply connected region D

Let C be a simple closed curve in $\mathbb{R}^2$ & let D be the region that it bounds. We say C is positively oriented if "D is to the left as you walk along C"

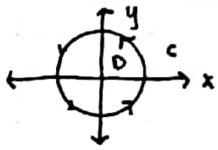


Notation: If C is a simple closed curve & P, Q are functions then
$$\oint_C (P\,dx + Q\,dy)$$
is the line integral $\int_C (P\,dx + Q\,dy)$ where C is positively oriented

Green's Theorem: Let C be a simple closed curve in $\mathbb{R}^2$. Let D be the region that it bounds. Let P, Q be differentiable functions on D. Then $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

4.3.3. Examples of Green's Theorem

Calculate $\oint_C (-x^2 y\,dx + xy^2\,dy)$ where C = unit circle



Direct Method: Parametrize C: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$$\oint_C (-x^2 y\,dx + xy^2\,dy) = \int_0^{2\pi} [-x(t)^2 y(t) x'(t) + x(t) y(t)^2 y'(t)]\,dt$$
$$= \int_0^{2\pi} [\cos(t)^2 \sin(t)^2 + \cos(t)^2 \sin(t)^2]\,dt$$
$$= 2\int_0^{2\pi} \cos(t)^2 \sin(t)^2\,dt$$
$$= \frac{1}{2}\int_0^{2\pi} \sin^2(2t)\,dt$$
$$= \frac{1}{4}\int_0^{2\pi} (1 - \cos(4t))\,dt = \frac{\pi}{2}$$

$\sin(2t) = 2\sin t \cos t$
$\sin^2(2t) = 4\sin^2 t \cos^2 t$
$\sin^2(2t) = \frac{1 - \cos(4t)}{2}$

Green's Theorem: $\oint_C (\underbrace{-x^2 y}_{P}\,dx + \underbrace{xy^2}_{Q}\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \iint_D (y^2 - (-x^2))\,dA = \iint_D x^2 + y^2\,dA = \int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{r^4}{4}\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

Ex 2: If C is a simple closed curve, & D is the region it bounds, then
$$\text{Area}(D) = \oint_C x\,dy = \oint_C (-y\,dx) = \oint_C \tfrac{1}{2}(x\,dy - y\,dx)$$

Proof: $\oint_C x\,dy \underset{\text{Green's Theorem}}{=} \iint_D 1\,dA = \text{Area}(D)$

$P = 0$
$Q = x$
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$

$\oint_C (-y\,dx) = \iint_D 1\,dA = \text{Area}(D)$

$P = -y$
$Q = 0$
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$

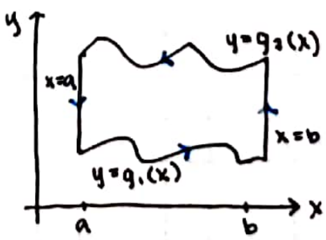
$x = \cos t$
$y = \sin t$
$0 \le t \le 2\pi$

e.g. C = unit circle ∴ $\text{Area}(C) = \oint_C \frac{1}{2}(x\,dy - y\,dx) = \int_0^{2\pi} \frac{1}{2}(x(t)y'(t) - y(t)x'(t))\,dt$
$$= \frac{1}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} 1\,dt = \frac{1}{2}(2\pi) = \pi$$

4.3.4. Proof of Green's Theorem

$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, C = simple closed curve, R = region bounded by C

Step 1: R = "Type I" region, Q = 0



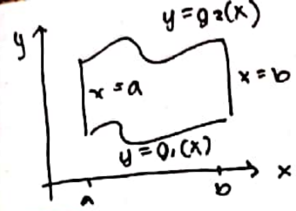
$\oint_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$

Parametrization of C_1: $x = t$, $y = g_1(x)$, $a \le t \le b$

$\int_{C_1} P\,dx = \int_a^b P(t, g_1(t))\,dt = \int_a^b P(x, g_1(x))\,dx$

$\int_{C_3} P\,dx = -\int_a^b P(x, g_2(x))\,dx$

$\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$



$$\oint_C P\,dx = \int_a^b [P(x,g_1(x)) - P(x,g_2(x))]\,dx$$
$$\iint_R -\frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} -\frac{\partial P}{\partial y}\,dy\,dx$$
$$= \int_a^b [-P(x,g_2(x)) + P(x,g(x))]\,dx$$

equivalent

Step 2: Special case where $P=0$ & R is Type II (similar to step 1)

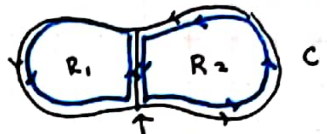
Step 3: R is both Type I & Type II; P, Q arbitrary

$$\oint_C (P\,dx + Q\,dy) = \underbrace{\oint_C P\,dx}_{-\iint_R \frac{\partial P}{\partial y}dA} + \underbrace{\oint_C Q\,dy}_{\iint_R \frac{\partial Q}{\partial x}dA} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

Step 4: General case. Divide R into subregions, each of which is both Type I & Type II

Green's Theorem for R_1 & Green's Theorem for R_2 $\Rightarrow$ Green's Theorem for R

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_{R_1}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA + \iint_{R_2}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$
$$= \oint_{C_1}(P\,dx + Q\,dy) + \oint_{C_2}(P\,dx + Q\,dy)$$
$$= \oint_C (P\,dx + Q\,dy)$$



because integrals along the dividing curve cancel out (opposite directions)

4.3.5. Proof of the Second Characterization of Conservative Vector Fields

Theorem: If $\vec{F} = \langle P, Q \rangle$ is a differentiable vector field defined on a simply connected domain D, then

$\vec{F}$ is conservative $\Longleftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Proof (sketch): ($\Rightarrow$ Know this Clairaut's Theorem)

($\Leftarrow$) Suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. We need to show that $\vec{F}$ is conservative

Enough to show $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C in D

Enough to do this when C is simple (& mild). If C is simple then C bound a region R. Since D is simply connected, $R \subset D$

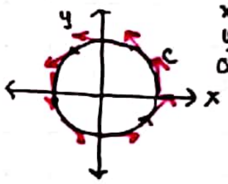
By Green's Theorem, $\underbrace{\oint \vec{F}\cdot d\vec{r}}_{P\,dx + Q\,dy} = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0 \text{ by assumption}} dA = 0$

4.3.6. An Interesting Nonconservative Vector Field

$\vec{F} = \langle \underbrace{-\frac{y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q} \rangle$

$\vec{F}$ is defined on $\mathbb{R}^2$ with $(0,0)$ removed.

Not simply connected!



$x = \cos t$
$y = \sin t$
$0 \le t \le 2\pi$

Claim: $P_y = Q_x$, but $\vec{F}$ is not conservative

Proof: $P_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

Not conservative b/c if C is the unit circle then:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{-y(t)}{x(t)^2+y(t)^2}\cdot x'(t)\,dt + \int_0^{2\pi} \frac{x(t)}{x(t)^2+y(t)^2}\,y'(t)\,dt$$
$$= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\,dt$$
$$= \int_0^{2\pi} 1\,dt = 2\pi$$

$\vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ Let C be a closed curve in $\mathbb{R}^2$ not going through $(0,0)$

$\int_C \vec{F}\cdot d\vec{r} = 2\pi \cdot$ (winding number of C around $(0,0)$)

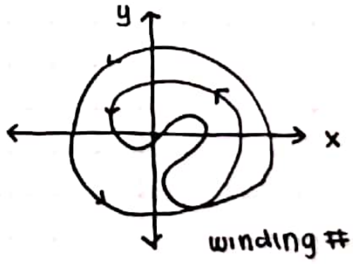
Parametrize C with domain $[a,b]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot \langle x', y' \rangle\,dt = \int_a^b \underbrace{\frac{-xy' + xy'}{x^2+y^2}}_{\text{"}\frac{d\theta}{dt}\text{"}}\,dt$$

$x = r\cos\theta$ $x' = r'\cos\theta - r\sin\theta\cdot\theta'$

$y = r\sin\theta$ $y' = r'\sin\theta + r\cos\theta\cdot\theta'$

$$\left.\begin{aligned} -yx' &= -rr'\cos\theta\sin\theta + y^2\theta' \\ xy' &= rr'\cos\theta\sin\theta + x^2\theta' \end{aligned}\right] \Rightarrow \theta' = \frac{xy' - yx'}{x^2+y^2}$$



LECTURE 4.4

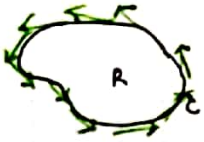
4.4.1. Review & Interpretation of Green's Theorem

Let $\vec{F} = \langle P, Q \rangle$ be a vector field in two dimensions

1. $\vec{F}$ is conservative $\Longleftrightarrow \int_C \vec{F}\cdot d\vec{r}$ for all closed curves C.
2. If $\vec{F}$ is conservative then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$
3. Green's Theorem: If C is a simple closed curve in $\mathbb{R}^2$ & R is the region it bounds, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$
4. If $\vec{F}$ is defined on a simply connected domain & $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then $\vec{F}$ is conservative

What is the interpretation of $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$?

$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{"circulation of } \vec{F} \text{ around } C\text{"}}$ $\quad \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{"local rotation" of } \vec{F}}$



e.g. $\vec{F} = \langle -y, x\rangle$ so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$

4.4.2. Definition of Curl

$\vec{F} = \langle P, Q\rangle$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ measures "location rotation" of $\vec{F}$

$\vec{F} = \langle P, Q, R\rangle$ vector field on some domain in $\mathbb{R}^3$

$$\text{curl } \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

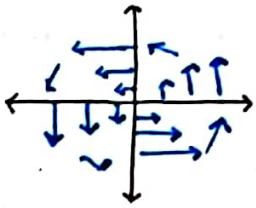
How to remember: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$

curl $\vec{F} = \nabla \times \vec{F}$ $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Interpretation: curl $\vec{F}$ measure "local rotation" of $\vec{F}$

direction of curl $\vec{F}$ = "local axis of rotation"

magnitude of curl $\vec{F}$ = "amount of local rotation"



Ex: $\vec{F} = \langle -y, x, 0\rangle$

curl $\vec{F} = \nabla \times \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \times \langle -y, x, 0\rangle = \langle 0, 0, 2\rangle$

points in the z direction with magnitude 2

4.4.3. Curl & Conservative Vector Fields

Recall: in 2D, if $\vec{F} = \langle P, Q\rangle$ is conservative then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Theorem: If $\vec{F} = \langle P, Q, R\rangle$ is conservative (& partial derivatives of P, Q, R defined, continuous) then $\nabla \times \vec{F} = 0$

Proof: suppose $\vec{F}$ is conservative. Then $\vec{F} = \nabla f$ for some function f. Then

$$\begin{aligned}\nabla \times \vec{F} &= \nabla \times \nabla f \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \\ &= \Big\langle \underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{0}, \underbrace{\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}}_{0}, \underbrace{\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}}_{0}\Big\rangle = 0\end{aligned}$$

by Clairaut's Theorem

Ex: Is $\vec{F} = \langle xz, \cos y, z\rangle$ conservative?

$\nabla \times \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \times \langle xz, \cos y, z\rangle = \langle 0, x, 0\rangle \neq 0$

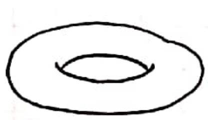
$\vec{F} \Rightarrow$ not conservative

Theorem: If curl $\vec{F} = 0$ & $\vec{F}$ is defined on a simply connected region, then $\vec{F}$ is conservative

$\mathbb{R}^3$: simply connected

$\mathbb{R}^3 \setminus \{0,0,0\}$

$\mathbb{R}^3$ with the origin removed: also simply connected



← not simply connected

4.4.4. Examples of Curl & Conservative Vector Fields

Ex1: $\vec{F} = \langle 2xyz, x^2z + z, x^2y + y\rangle$. Is $\vec{F}$ conservative?

$\nabla \times \vec{F} = \langle \frac{\partial}{\partial y}(x^2y + y) - \frac{\partial}{\partial z}(x^2z + z), \frac{\partial}{\partial z}(2xyz) - \frac{\partial}{\partial x}(x^2y + y), \frac{\partial}{\partial x}(x^2z + z) - \frac{\partial}{\partial y}(2xyz)\rangle$

$= \langle x^2 + 1 - (x^2+1), 2xy - 2xy, 2xz - 2xz\rangle = 0 \Rightarrow \vec{F}$ is conservative

Find f with $\nabla f = \vec{F}$.

$f_x = 2xyz \Rightarrow f = x^2yz + \overbrace{g(y,z)}^{f(0,y,z)}$

$x^2z + z = f_y = x^2z + g_y \Rightarrow g_y = z \Rightarrow g = \overbrace{yz + h(z)}^{g(0,z) = f(0,0,z)}$

$\Rightarrow f = x^2yz + yz + h(z)$

$x^2y + y = f_z = x^2y + y + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h = c$ ← $f(0,0,0)$

$f = x^2yz + yz + c$

Ex2: $\vec{F} = \langle y, z, x\rangle$. Is $\vec{F}$ conservative?

$\nabla \times \vec{F} = \langle \frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}, \frac{\partial y}{\partial z} - \frac{\partial x}{\partial x}, \frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\rangle = \langle -1, -1, -1\rangle \neq 0 \Rightarrow \vec{F}$ is not conservative

Try anyway to find f with $\nabla f = \vec{F}$

$f_x = y \rightarrow f = xy + g(y,z)$

$z = f_y = x + g_y \Rightarrow \underline{g_y = z - x}$

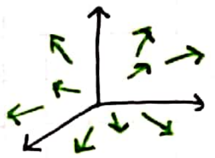
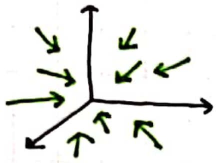
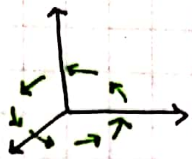
impossible because g does not depend on x so g_y does not depend on x either

4.4.5. Definition of Divergence

Divergence: $\vec{F} = \langle P, Q, R \rangle$

$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ (a function)

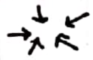
Ex $\operatorname{div}\langle x, y, z\rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ $\quad$ $\operatorname{div}\langle -x, -y, -z\rangle = -3$ $\quad$ $\operatorname{div}\langle -y, x, 0\rangle = 0$

Curl measures rotation. Divergence measures **expansion** & **compression**

$\operatorname{div} > 0$ $\quad$ $\operatorname{div} < 0$

e.g. $\vec{F}$ = velocity vector of air

- $\operatorname{div} < 0$ where pressure is increasing 
- $\operatorname{div} > 0$ ——— decreasing 
- $\operatorname{div} = 0$ ——— stays the same

4.4.6. Relations between Gradient, Curl, & Divergence

In 3D, know $\operatorname{curl}(\nabla f) = 0$

Theorem: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on a domain D in $\mathbb{R}^3$ then $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$

Proof: $\operatorname{curl}\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$\operatorname{div}(\operatorname{curl}\vec{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$
$$= R_{xy} - Q_{xz} + P_{yz} - R_{xy} + Q_{xz} - P_{yz} = 0$$

Fact: If $\vec{F}$ is a vector field on $\mathbb{R}^3$ & $\operatorname{div}\vec{F} = 0$ then $\vec{F} = \operatorname{curl}$(some other vector field)

Function
↓ gradient
Vector Field
↓ curl
Vector Field
↓ divergence
Function

(0 for gradient then curl; 0 for curl then divergence)

If f is a function on $\mathbb{R}^3$, define the **Laplacian** $\Delta f = \nabla^2 f$ by

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f$$
$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$\nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$$

e.g. If f is the electric potential then $\nabla^2 f = 0$ where there are no charges

4.4.7. Alternate Statement of Green's Theorem using Curl

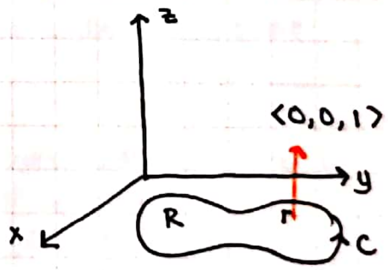
$\vec{F} = \langle P, Q, 0 \rangle$

$\vec{n} = \langle 0, 0, 1 \rangle$ = normal vector to the surface R

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \underbrace{(\operatorname{curl}\vec{F} \cdot \vec{n})}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} dA$$

$\operatorname{curl}\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle P, Q, 0 \rangle = \langle \cdots, \cdots, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$

Goal: Generalize to other surfaces in 3D



LECTURE 4.5

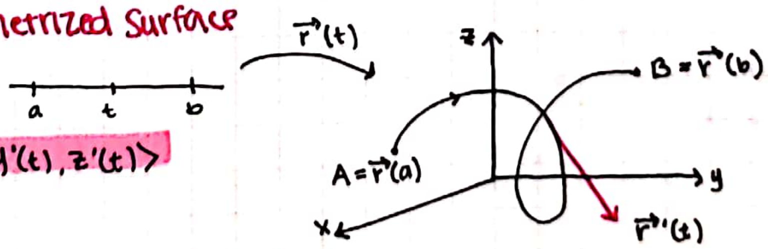
4.5.1. Definition of a Parametrized Surface

Parametrized curve:

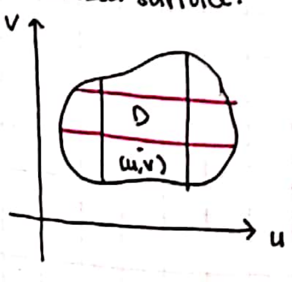
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

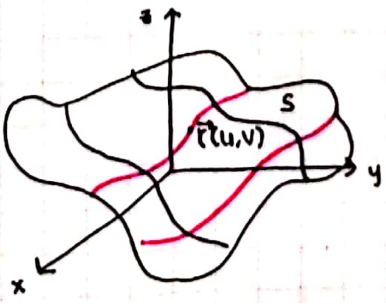
Velocity vector: $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

tangent to the curve



Parametrized surface:





4.5.2. Examples of Parametrized Surfaces

Ex 1: $x = r\sin u\cos v$ Eliminate Parameters:

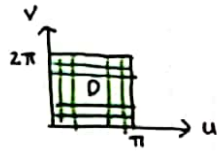
$y = r\sin u\sin v$ $\quad x^2+y^2 = r^2\sin^2 u(\cos^2 v+\sin^2 v)$

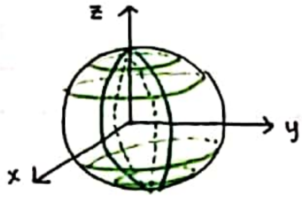
$z = r\cos u$ $\quad z^2 = r^2\cos^2 u$

$0 \le u \le \pi$ $\quad z^2+y^2+x^2 = r^2$

$0 \le v \le 2\pi$ $\quad$ Sphere of radius r centered @ origin

(r = constant $\quad r=\rho,\ u=\phi,\ v=\theta$





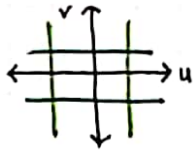
e.g. same equations except $0 \le u \le \frac{\pi}{2}$ instead of $0 \le u \le \pi$

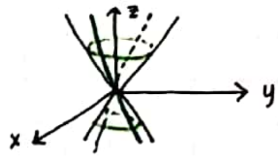
$\Rightarrow$ upper hemisphere of sphere ($z \ge 0$)

Ex 2: $x = u\cos v$ $\xrightarrow{\text{elimin. paramet.}}$ $x^2+y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2$

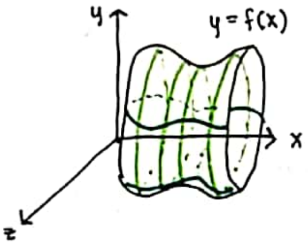
$y = u\sin v$ $\quad z^2 = u^2$

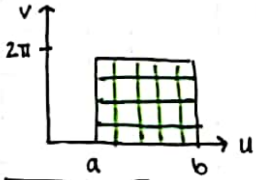
$z = u$ $\quad x^2+y^2 = z^2 \Rightarrow$ cone





Ex 3: Parametrize the surface of revolution of $y=f(x)$, $a \le x \le b$, around the x-axis.



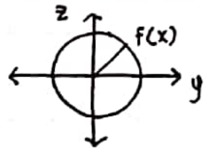


$\boxed{\begin{array}{l} a \le u \le b \\ 0 \le v \le 2\pi \end{array}}$

$\boxed{x=u}$

For fixed u, we have a circle in the plane $x=u$, centered at the origin with radius $f(x) = f(u)$

$\boxed{\begin{array}{l} y = f(u)\cos v \\ z = f(u)\sin v \end{array}}$



4.5.3. The Tangent Plane to a Parametrized Surface

How to find the tangent plane to a parametrized surface at a point on the surface.

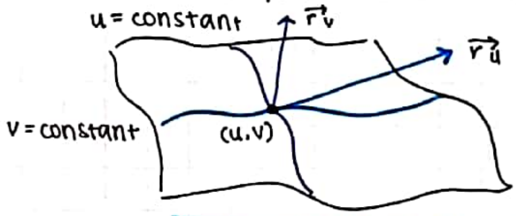
$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Fix (u,v). Find the tangent plane at $\vec{r}(u,v)$

Two tangent vectors:

$\vec{r}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$

$\vec{r}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$



$\vec{r}_0 = \vec{r}(u,v)$

The parametrized surface is smooth at (u,v) if $\vec{r}_u, \vec{r}_v$ are linearly independent (i.e. $\vec{r}_u \times \vec{r}_v \ne 0$; $\vec{r}_v$ is not a scalar multiple $\vec{r}_u$). Then $\vec{n} = \vec{r}_u \times \vec{r}_v$ is a normal vector to the tangent plane

Equation for Tangent Plane: $\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$

Ex: Find the tangent plane to the surface

$x = u^2$

$y = u - v^2 \quad u,v \ge 0$

$z = v^2 \quad$ at point $(1,0,1)$

1) $(x,y,z) = (1,0,1)$ corresponds to $(u,v) = (1,1)$

2) Calculate $\vec{r}_u, \vec{r}_v$ @ $(u,v) = (1,1)$

$\vec{r}_u = \langle 2u, 1, 0\rangle \xrightarrow{(u,v)=(1,1)} \langle 2,1,0\rangle$

$\vec{r}_v = \langle 0, -2v, 2v\rangle \longrightarrow \langle 0,-2,2\rangle$

We have $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle 2,1,0\rangle \times \langle 0,-2,2\rangle = \langle 2,-4,-4\rangle$

$\vec{r}_0 = \langle 1,0,1\rangle$ $\quad \langle 2,-4,-4\rangle \cdot \langle x-1, y, z-1\rangle = 0$

$\vec{r} = \langle x,y,z\rangle$ $\quad 2(x-1) - 4y - 4(z-1) = 0$

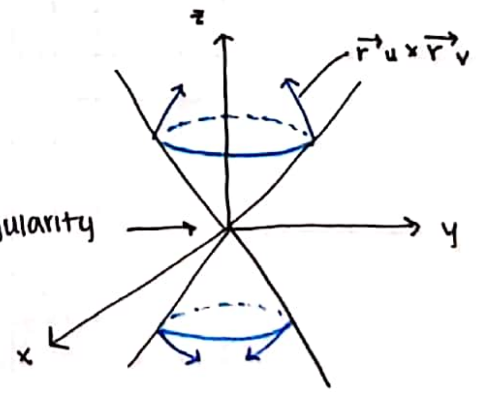
$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$ $\quad 2x - 4y - 4z = -2$

Ex 2: cone $x^2+y^2 = z^2$

$x = u\cos v$

$y = u\sin v$

$z = u$



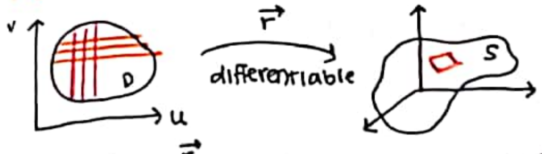
$\vec{r}_u = \langle \cos v, \sin v, 1\rangle$

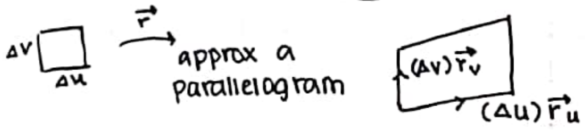
$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

↳ If $u \ne 0$ then $\vec{r}_u \times \vec{r}_v \ne 0$

↳ If $u = 0$ ($(x,y,z) = (0,0,0)$) then $\vec{r}_u \times \vec{r}_v = 0$

4.5.4. Area of Parametrized Surfaces





Area of Parallelogram $= |(\Delta u)\vec{r}_u \times (\Delta v)\vec{r}_v|$
$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

Area of surface $= \lim\limits_{(\Delta u, \Delta v)\to(0,0)} \sum\limits_{\text{rectangles}} \underbrace{\text{Area(Parallelog.)}}_{|\vec{r}_u \times \vec{r}_v| dA}$

$= \iint_D |\vec{r}_u \times \vec{r}_v| dA$

Ex: S = unit sphere

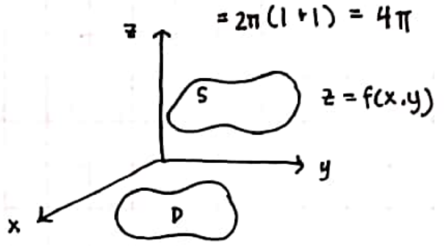
$x = \sin u \cos v$
$y = \sin u \sin v$
$z = \cos u$
$0 \le u \le \pi$
$0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$
$\vec{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$
$\vec{r}_u \times \vec{r}_v = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle$
$= \sin u \langle \sin u \cos v, \sin v \sin u, \cos u \rangle$
$= \sin u \langle x, y, z \rangle$

Area(S) $= \int_0^\pi \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, dv\, du$
$= \int_0^\pi \int_0^{2\pi} \sin u \, dv\, du$
$= 2\pi \int_0^\pi \sin u \, du$
$= 2\pi (-\cos u) \Big|_{u=0}^{u=\pi}$
$= 2\pi (1+1) = 4\pi$

Ex2: S = graph of $z = f(x,y)$ where $f: D \to \mathbb{R}$ (domain in $\mathbb{R}^2$)

Parametrization: $x = u$
$y = v$ $\quad (u,v) \in D$
$z = f(u,v)$



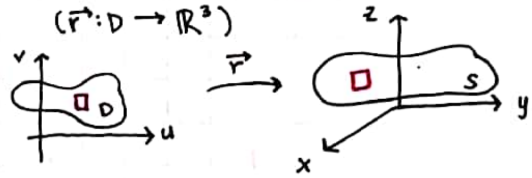
$\vec{r}_u = \langle 1, 0, f_x \rangle$
$\vec{r}_v = \langle 0, 1, f_y \rangle$
$\vec{r}_u \times \vec{r}_v = \langle -f_x, -f_y, 1 \rangle$

Area $= \iint_D |\vec{r}_u \times \vec{r}_v| dA = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$

4.5.5. Integration with respect to Surface Area

Integrals over Surfaces: S = parametrized surface in $\mathbb{R}^3$ $\quad (\vec{r}: D \to \mathbb{R}^3)$

1) $\iint_S f \, dS$ where $f: S \to \mathbb{R}$
"integration with respect to surface area"
2) $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}$ is a 3D vector field defined on S



Define: $\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\text{magnification factor}} dA$

e.g. $\iint_S 1 \, dS = \text{area}(S)$

If S is a sheet with mass density $\rho: S \to \mathbb{R}$, then $M = \iint_S \rho \, dS$ is the total mass of the sheet

Ex: Calculate $\iint_S yz \, dS$ where S is given by $x = uv$, $y = u+v$, $z = u-v$ $\quad \underbrace{u^2 + v^2 \le 1}_{D}$ (unit circle)

$\vec{r} = \langle uv, u+v, u-v \rangle$

$\vec{r}_u = \langle v, 1, 1 \rangle$
$\vec{r}_v = \langle u, 1, -1 \rangle$
$\vec{r}_u \times \vec{r}_v = \langle -2, u+v, v-u \rangle$
$|\vec{r}_u \times \vec{r}_v| = \sqrt{4 + (u+v)^2 + (v-u)^2} = \sqrt{4 + 2(u^2+v^2)}$

[domain D stays the same
converts one integral into the other]

$\iint_S yz \, dS = \iint_D \underbrace{(u+v)}_{y}\underbrace{(u-v)}_{z} \underbrace{\sqrt{4 + 2(u^2+v^2)}}_{\text{mag. factor}} dA$

Change of var: $u \to v$, $v \to u$

$\frac{\partial(v,u)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$= \iint_D u^2 \sqrt{4 + 2(u^2+v^2)}\, dA - \iint_D v^2 \sqrt{4 + (2)(u^2+v^2)}\, dA = 0$

4.5.6. Orientation of Surfaces

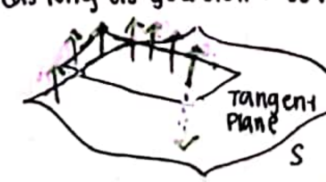
$\int_C f \, ds$ does not depend on parametrization of C (no backtracking)

$\int_C \vec{F} \cdot d\vec{r}$ depends on orientation of C

$\iint_S f \, dS$ does not depend on parametrization of S (as long as you don't covert part of S more than once)

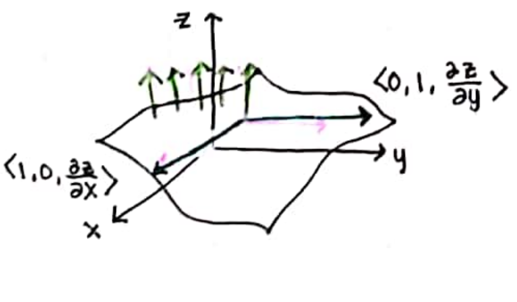
$\iint_S \vec{F} \cdot d\vec{r}$ depends on an orientation of S

An orientation of a smooth surface is a choice of a unit normal vector at each point, which varies continuously as you move along S



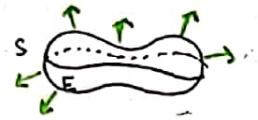
unit normal vector is perpendicular to tangent plane & has length one.

Ex 1: A graph $z = f(x,y)$ has a distinguished "upward orientation"



$\vec{n} = \dfrac{\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}}$ $\quad$] $\vec{n} \cdot \vec{r}_1 = 0 = \vec{n} \cdot \vec{r}_2$] unit vector

Ex2: If S is the boundary of a solid region E, then S has a distinguished "outward" orientation



If S is connected (one piece) then S has either 2 or 0 orientations.

e.g. Mobius strip has no orientation (not orientable) (one sided)

4.5.7. Integration of a Vector Field Over a Surface

S oriented surface $\to \vec{n}$ unit normal vector

$\vec{F}$ = a vector field defined on S

Define: $\iint_S \underbrace{\vec{F}}_{\text{vector field}}\cdot d\vec{S} = \iint_S \underbrace{(\vec{F}\cdot\vec{n})}_{\text{function}}\underbrace{dS}_{\text{element of surface area}}$

[$\iint_S \vec{F}\cdot d\vec{S}$ depends on the orientation

opposite direction: $\vec{n} \to -\vec{n}$

$\iint_S (\vec{F}\cdot -\vec{n})dS = -\iint_S (\vec{F}\cdot\vec{n})dS$

$\iint_{-S} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S}$]

Physical interpretation: $\iint_S \vec{F}\cdot d\vec{S}$ = flux of $\vec{F}$ through S

rate water is passing thru the surface

normal vector dot product by velocity of the water] integration = total flow of water thru the surface

How to compute $\iint_S \vec{F}\cdot d\vec{S}$?

If $\vec{r}$ is a smooth parametrization (domain D in the u,v plane) ($\vec{r}_u \times \vec{r}_v \neq 0$) then this determines an orientation: $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$. With this orientation,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)dS = \iint_D \vec{F}\cdot\left(\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\right)\cancel{|\vec{r}_u\times\vec{r}_v|}\,dA$$

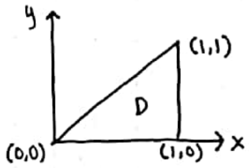
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

Three Steps to Compute: (1) Choose a smooth parametrization; (2) Check the orientation is correct; (3) Calculate the above integral

4.5.8. Example of Integration of a Vector Field Over a Surface

S = triangle with vertices $(0,0,0)$, $(1,0,1)$, $(1,1,2)$, oriented upward. Calculate $\iint_S \langle 3,4,5\rangle\cdot d\vec{S}$

Parametrization of S: S is the graph of $g(x,y) = x+y$ over D where D is the triangle.



$\vec{r}(x,y) = \langle x, y, x+y\rangle$ — positive ⇒ upward

$\vec{r}_x \times \vec{r}_y = \langle 1,0,1\rangle \times \langle 0,1,1\rangle = \langle -1,-1,1\rangle$

$\iint_S \langle 3,4,5\rangle\cdot d\vec{S} = \iint_D \langle 3,4,5\rangle\cdot\langle -1,-1,1\rangle dA$

$= \iint_D (-2)\,dA = -2\,\text{Area}(D) = -1$

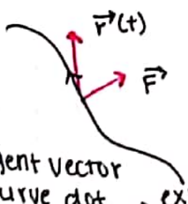
More generally, $\iint_S \langle P,Q,R\rangle\cdot d\vec{S}$ where S is the graph of $z = g(x,y)$ over D (w/upward orientation) can be computed as follows: Parametrization $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$ $x,y \in D$

$\vec{r}_x \times \vec{r}_y = \langle 1,0,g_x\rangle \times \langle 0,1,g_y\rangle = \langle -g_x, -g_y, 1\rangle$ positive ⇒ upward orientation

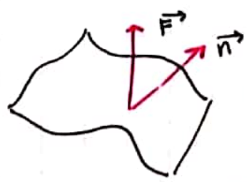
$\iint_S \langle P,Q,R\rangle\cdot d\vec{S} = \iint_D \langle P,Q,R\rangle\cdot\langle -g_x,-g_y,1\rangle dA = \iint_D (-Pg_x - Qg_y + R)\,dA$

4.5.9. Summary of Integrals Over Curves & Surfaces

Integrals over curves	Integration over Surfaces	Properties
$\int_C f\,ds$ integration w.r.t arc length	$\iint_S f\,dS$ integration wrt surface area	does not depend on the parametrization (bijective)
$\int_C f\,dx$, $\int_C f\,dy$	$\iint_S f\,dx\,dy$, $\iint_S f\,dx\,dz$, $\iint_S f\,dy\,dz$	depends on orientation, otherwise does not depend on (bijective) parametrization
$\int_C \vec{F}\cdot d\vec{r}$ measures circulation of $\vec{F}$ along C (use tangent vector to C)	$\iint_S \vec{F}\cdot d\vec{S}$ measures flux across S use normal vector to S	switching orientation multiplies integral by -1



tangent vector of curve dot product $\vec{F}$ ⇒ extent to which they point in same direction



normal vector dot product ⇒ extent to which $\vec{F}$ is pointing in normal vector

* If everywhere vector field was tangent to surface, then flux = 0

* If vector is everywhere ⊥ to curve then circulation = 0

LECTURE 4.6

4.6.1. Statement of Stokes' Theorem

Let S be an oriented surface with boundary curve C

C is positively oriented if it is oriented as shown

Stokes' Theorem: Let S & C be as above. Let $\vec{F}$ be a differentiable vector field defined on S. Then

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{"circulation" of } \vec{F} \text{ around } C} = \iint_S \underbrace{(\nabla\times\vec{F})}_{\text{rotation of } \vec{F}}\cdot \underbrace{d\vec{S}}_{\vec{n}\,dS}$$

If boundary of S has several curves, then Stokes' Theorem is true where on the left hand side you add up the integrals over all the boundary curves.

→ component of curl ⊥ to surface (rotation which is tangential to the surface) ⇒ axis of rotation ⊥ to surface ⇐ integrate over whole surface = net rotation around boundary

4.6.2. Basic Examples of Stokes' Theorem

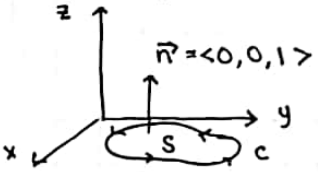
Ex 1: $\vec{F}=\langle P,Q,0\rangle$ where P,Q depend only on x,y. S = domain in x,y plane w/ upward orientation

Stokes' Theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS \leftarrow d\vec{S}$

$\nabla\times\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \times \langle P,Q,0\rangle = \langle \cdots, \cdots, Q_x - P_y\rangle$

$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \iint_S (Q_x-P_y)\,dA$

Green's Theorem : " $\int_C \vec{F}\cdot d\vec{r}$



Ex 2: Suppose $\vec{F}$ is conservative, i.e. $\vec{F}=\nabla f$

Stokes' Theorem : $\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{0 \text{ by FTLI}} = \iint_S \underbrace{(\nabla\times\nabla f)}_{0}\cdot d\vec{S} \quad \Rightarrow \quad 0=0$

Theorem: If $\vec{F}$ is defined on $\mathbb{R}^3$ & $\nabla\times\vec{F}=0$ then $\vec{F}$ is conservative

Proof: Enough to show that $\int_C \vec{F}\cdot d\vec{r}=0$ for all closed curve C.

Can restrict attention to the case where C is simple & the boundary of an oriented surface S. Then if we give C the positive orientation $\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(\nabla\times\vec{F})}_{0}\cdot d\vec{S}=0$

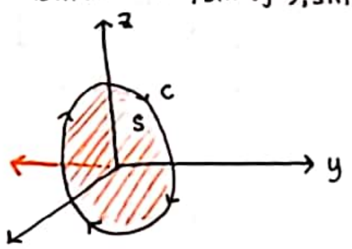
4.6.3. A Computational Example using Stokes' Theorem

Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $C = (\vec{r}(t)=\langle \cos t, 0, \sin t\rangle, 0\le t\le 2\pi)$; $\vec{F}=\langle \sin(x^3)+z^3, \sin(y^3), \sin(z^3)-x^3\rangle$

C = unit circle in x,z plane

C = boundary of S where S = unit disk in x,z plane

$\vec{n}=\langle 0,-1,0\rangle \Rightarrow C$ is positively oriented



$\int_C \vec{F}\cdot d\vec{s} = \iint_{x^2+z^2\le 1} (\nabla\times\vec{F})\cdot\langle 0,-1,0\rangle\,dA$

↳ $\nabla\times\vec{F} = \langle \cdots, \cdots, \frac{\partial}{\partial z}(\sin(x^3)+z^3) - \frac{\partial}{\partial x}(\sin(x^3)-x^3)\rangle$

$= \langle \cdots, \cdots, 3x^2+3z^2\rangle$

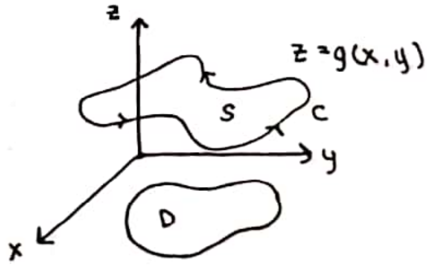
$\int_C \vec{F}\cdot d\vec{s} = \iint_{x^2+z^2\le 1} (-3x^2-3z^2)\,dA = \int_0^{2\pi}\int_0^1 (-3r^2)\,r\,dr\,d\theta = 2\pi(-\frac{3}{4}) = -\frac{3}{2}\pi$

4.6.4. Proof of Stokes' Theorem

Special Case: S is the graph of $g(x,y)$ over a domain D in the x,y plane (upward orientation)

$\vec{F}=\langle P,Q,R\rangle$

Show $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_D (\nabla\times\vec{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$

$= \iint_D [g_x(\underline{Q_z - R_y}) + g_y(\underline{R_x - P_z}) + (\underline{Q_x - P_y})]\,dA$

evaluated at $(x,y,g(x,y))$

$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \langle P,Q,R\rangle\cdot\langle x', y', \underbrace{g_x x' + g_y y'}_{z' \text{ where } z(t)=g(x(t),y(t))}\rangle\,dt$

$= \int_a^b (Px' + Qy' + Rg_x x' + Rg_y y')\,dt$

$= \int_{C_0} [(P+Rg_x)\,dx + (Q+Rg_y)\,dy]$ — Green's Theorem → $= \iint_D [(Q+Rg_y)_x - (P+Rg_x)_y]\,dA$

↳ $(Q+Rg_y)_x = \frac{\partial}{\partial x}(Q(x,y,g(x,y))) + R(x,y,g(x,y))g_y(x,y))$

$= Q_x + Q_z g_x + (R_x + \cancel{R_z g_x})g_y + \cancel{R g_{yx}}$

↳ $(P+Rg_x)_y = P_y + P_z g_y + (R_y + \cancel{R_z g_y})g_x + \cancel{R g_{xy}}$

$\Big\} = \iint_D [(Q_z - R_y)g_x + (R_x - P_z)g_y + (Q_x - P_y)]\,dA$

Step 2: Also works if S is a graph $y=g(x,z)$ or $x=g(y,z)$ [similar calculation]

Step 3: For a general surface S, divide it into simpler surfaces S_i, each of which is a graph as in Step 1 or Step 2 (C_i = boundary of S_i, positively oriented)

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \Sigma_i \iint_{S_i} (\nabla\times\vec{F})\cdot d\vec{S} = \Sigma_i \int_{C_i}\vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$$

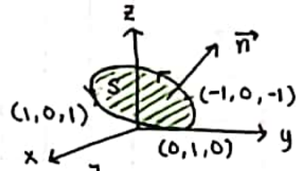
interior boundary terms cancel in the sum

4.6.5. Another Example of Stokes' Theorem

Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle \sin(\sin z), z^3, -y^3\rangle$ & C is the curve $x^2+y^2=1$, $z=x$, oriented counterclockwise when viewed from above

C is in the plane $z=x$

Let S = part of the plane $z=x$ bounded by C, oriented upward



Parametrization of S:

$x=u$

$y=v$ $\quad u^2+v^2\leq 1$

$z=u$

$\vec{r}_u\times\vec{r}_v = \langle 1,0,1\rangle\times\langle 0,1,0\rangle = \langle -1,0,1\rangle$

$\vec{F} = \langle \sin(\sin z), z^3, -y^3\rangle$

curl $\vec{F} = \langle -3y^2-3z^2, \cdots, 0\rangle$

$= \langle -3v^2-3u^2, \ldots, 0\rangle$

$$\iint_{u^2+v^2\leq 1} 3(u^2+v^2)\,dA = \int_0^{2\pi}\int_0^1 3r^2\cdot r\,dr\,d\theta = 2\pi\left(\tfrac{3}{4}\right) = \tfrac{3}{2}\pi$$

Alternative Parametrization of S:

$x = u\cos v \quad 0\leq u\leq 1$

$y = u\sin v \quad 0\leq v\leq 2\pi$

$z = u\cos v$

$\vec{r}_u = \langle \cos v, \sin v, \cos v\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, -u\sin v\rangle$

$\vec{r}_u\times\vec{r}_v = \langle -u, 0, u\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \iint_D \langle -3u^2,\cdots,0\rangle\cdot\langle u,0,u\rangle\,dA$

LECTURE 4.7

4.7.1. Statement of the Divergence Theorem

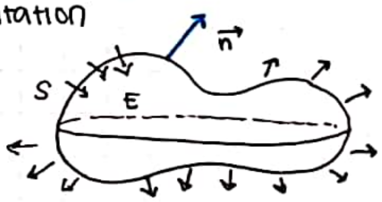
Let E be a bounded solid region in $\mathbb{R}^3$.

Let S be the boundary surface of E, with the outward orientation

Let $\vec{F}$ be a differentiable vector field defined on E.

Then $\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{flux of } \vec{F} \text{ out of } S} = \iiint_E (\operatorname{div}\vec{F})\,dV$

positive where $\vec{F}$ is expanding; negative where $\vec{F}$ is contracting



i.e. an elevator (1) (2)

(1) outward flux & integral of divergence < 0

e.g. $\vec{F} = \text{curl}\,\vec{G}$

Divergence Theorem $\Rightarrow$ $\underbrace{\iint_S (\text{curl}\,\vec{G})\cdot d\vec{S}}_{\text{0 by Stokes' Theorem b/c S has no boundary}} = \underbrace{\iiint_E \operatorname{div}(\text{curl}\,\vec{G})\,dV}_{0}$

$$\iint_S (\text{curl}\,\vec{G})\cdot d\vec{S} = \lim_{\varepsilon\to 0}\iint_{S \text{ w/ disk of radius } \varepsilon \text{ removed}} (\text{curl}\,\vec{G})\cdot d\vec{S} = \lim_{\varepsilon\to 0}\int_{\text{boundary of disk of radius } \varepsilon} \vec{G}\cdot d\vec{r} = 0$$

4.7.2. An Example of the Divergence Theorem

Suppose $\vec{F}$ is a vector field on $\mathbb{R}^3\setminus\{0,0,0\}$ (exclude origin) w/ div $\vec{F} = \sqrt{x^2+y^2+z^2}$.

Let S_1 be the sphere $x^2+y^2+z^2=1$, oriented outward

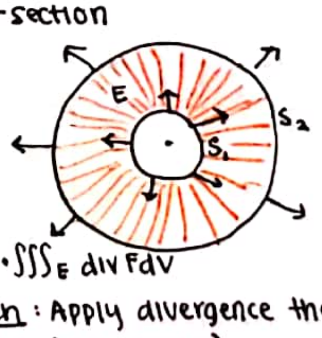
Let S_2 be the sphere $x^2+y^2+z^2=4$, oriented outward.

Suppose $\iint_{S_1}\vec{F}\cdot d\vec{S} = 2\pi$. Calculate $\iint_{S_2}\vec{F}\cdot d\vec{S}$

$E = \{(x,y,z) \mid 1\leq x^2+y^2+z^2\leq 4\}$

$$\underbrace{\iiint_E (\operatorname{div}\vec{F})\,dV}_{\text{Need to calculate this}} = \underbrace{\iint_{S_2}\vec{F}\cdot d\vec{S}}_{\substack{\text{oriented out (from origin),} \\ \text{want this}}} - \underbrace{\iint_{S_1}\vec{F}\cdot d\vec{S}}_{\substack{\text{oriented out (from origin)} \\ \text{given } 2\pi}}$$

cross-section



div $\vec{F} = \sqrt{x^2+y^2+z^2} = \rho$

$E = (1\leq x^2+y^2+z^2\leq 4)$

$(1\leq\rho\leq 2)$

Answer $= 2\pi + \iiint_E \operatorname{div}\vec{F}\,dV$

$$\iiint_E (\operatorname{div}\vec{F})\,dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{\rho^4}{4}\sin\phi\Big|_{\rho=1}^{\rho=2}\,d\theta\,d\phi = \frac{15}{4}\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi$$

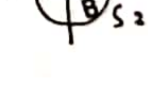
$$= \frac{15}{2}\pi\int_0^{\pi}\sin\phi\,d\phi = 15\pi$$

Answer: $2\pi + 15\pi = 17\pi$

Wrong Approach: Apply divergence theorem to ball $(x^2+y^2+z^2\leq 4) = B = (\rho\leq 2)$

$\iint_{S_2}\vec{F}\cdot d\vec{S} = \iiint_B \operatorname{div}\vec{F}\,dV$

$= 16\pi$ (must exclude origin)



4.7.3. Another Example of Divergence Theorem

Calculate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle z, y, x\rangle$ & S is the upper hemisphere of the unit sphere ($x^2+y^2+z^2=1$, $z \geq 0$), oriented upward

Direct Calculation : S = graph of $g(x,y) = \sqrt{1-x^2+y^2}$ over unit disk = D

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle z,y,x\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA$

$= \iint_D \langle \sqrt{1-x^2-y^2}, y, x\rangle \cdot \langle \frac{x}{\sqrt{\cdots}}, \frac{y}{\sqrt{\cdots}}, 1\rangle\, dA$

$= \iint_D \left(x + \frac{y^2}{\sqrt{1-x^2-y^2}} + x\right) dA = \int_0^{2\pi}\int_0^1 \frac{r^3 \sin^2\theta}{\sqrt{1-r^2}}\, dr\, d\theta$

$= \int_0^{2\pi} \left(-r^2\sqrt{1-r^2} - \frac{2}{3}(1-r^2)^{3/2}\right)\sin^2\theta \Big|_{r=0}^{r=1} d\theta$

$= \frac{2}{3}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{2}{3}\pi$

$\frac{d}{dr} r^2\sqrt{1-r^2} = \frac{-r^3}{\sqrt{1-r^2}} + 2r\sqrt{1-r^2}$

$\frac{d}{dr}(1-r^2)^{3/2} = -3r\sqrt{1-r^2}$

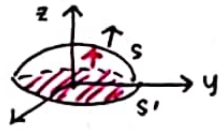
$\frac{d}{dr}\left(-r^2\sqrt{1-r^2} - \frac{2}{3}(1-r^2)^{3/2}\right) = \frac{r^3}{\sqrt{1-r^2}}$

Divergence Theorem : $\iint_S \langle z,y,x\rangle\cdot d\vec{S} - \iint_{S'} = \iiint_{\substack{\text{upper half}\\ \text{of unit ball} = E}} \operatorname{div}\langle z,y,x\rangle\, dV$

$\iint_{S'} \langle z,y,x\rangle \cdot d\vec{S} = \iint_{D^2} \langle 0,y,x\rangle\cdot\langle 0,0,1\rangle\, dA$

$= \iint_{D^2} x\, dA = 0$

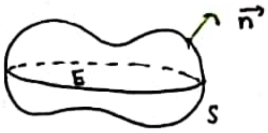
$\iiint_E 1\, dV = \operatorname{vol}(E) = \frac{1}{2}\operatorname{vol}(\text{unit ball}) = \frac{1}{2}\left(\frac{4\pi}{3}\right) = \frac{2}{3}\pi$



$\iiint_E = \underbrace{\iint_S}_{\text{upward}} + \underbrace{\iint_{S'}}_{\text{downward}}$

4.7.4. Proof of the Divergence Theorem

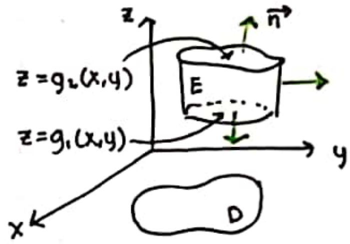
We have $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\, dV$



Special Case : $\vec{F} = \langle 0, 0, R\rangle$

E is a "Type I" region

$E = \{(x,y,z) \mid (x,y)\in D \text{ \& } g_1(x,y) \leq z \leq g_2(x,y)\}$



$\iint_S \vec{F}\cdot d\vec{S} = \Big(\underbrace{\iint_{top}}_{\text{(oriented up)}} - \underbrace{\iint_{bottom}}_{\text{(oriented up)}} + \underbrace{\iint_{side}}_{\text{(oriented out)}}\Big)\vec{F}\cdot d\vec{S}$

$\iint_{top} \vec{F}\cdot d\vec{S} = \iint_D \langle -\frac{\partial g_2}{\partial x}, -\frac{\partial g_2}{\partial y}, 1\rangle \cdot \langle 0,0,R\rangle\, dA$

$= \iint_D R(x,y,g_2(x,y))\, dA$

$\iint_{side} \vec{F}\cdot d\vec{S} = 0$ because $\langle 0,0,1\rangle$ is tangent to the side, so $\vec{n}$ has no z component

$\vec{F} = \langle 0,0,R\rangle$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D R(x,y,g_2(x,y))\, dA - \iint_D R(x,y,g_1(x,y))\, dA$

$\iiint_E (\operatorname{div}\vec{F})\, dV = \iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \underbrace{\int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)\, dz}_{R(x,y,g_2(x,y)) - R(x,y,g_1(x,y)) \text{ by FTC}}\, dA$

Other Special Case: [E is between the graphs of two functions y of x, z
$\vec{F} = \langle 0, Q, 0\rangle$

[E is __________________________ x of y, z
$\vec{F}$ is $\langle P, 0, 0\rangle$

General Case: For a general region E, divide it into "simple" regions E_i to which all of the previous special cases apply. Know divergence theorem for each E_i ; S_i = boundary of E_i, oriented out

$\iiint_E (\operatorname{div}\vec{F})\, dV = \Sigma_i \iiint_{E_i} (\operatorname{div}\vec{F})\, dV$

$= \Sigma_i \iint_{S_i} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S}$

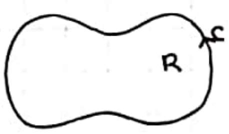
∵ interior boundary fluxes cancel out

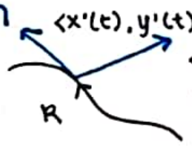
4.7.5. Two-Dimensional Version of the Divergence Theorem

$\vec{F} = \langle P, Q\rangle$ defined on all of R

$\operatorname{div}\vec{F} = P_x + Q_y$

$\vec{n}$ = outward unit normal vector to C





$\vec{n} = \frac{\langle y', -x'\rangle}{\sqrt{(x')^2 + (y')^2}}$

Theorem : $\iint_R (\operatorname{div}\vec{F})\, dA = \int (\vec{F}\cdot\vec{n})\, ds$

Proof: Parametrize C as $(x(t), y(t))$, $\alpha \leq t \leq \beta$.

$\int_C (\vec{F}\cdot\vec{n})\, ds = \int_\alpha^\beta \langle P, Q\rangle \cdot \frac{\langle y'(t), -x'(t)\rangle}{\sqrt{x'(t)^2 + y'(t)^2}} \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\, dt}_{ds} = \int_\alpha^\beta (Py' - Qx')\, dt$

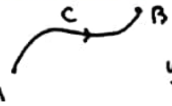
$= \int_C (P dy - Q dx) = \iint_R \underbrace{(P_x - (-Q_y))}_{\operatorname{div}\langle P,Q\rangle} dA$

$\int_C \vec{F}\cdot d\vec{r} = \int_C (P dx + Q dy) = \iint_R (Q_x - P_y)\, dA$

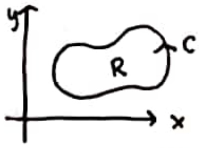
$= \int_C (\vec{F}\cdot\vec{n})\, ds = \int_C (P dy - Q dx) = \iint_R (P_x + Q_y)\, dA$

4.7.6. Summary of the Four Main Theorems of Vector Calculus

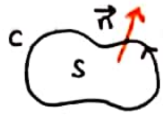
FTLI $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$



Green's Theorem $\int_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dA$



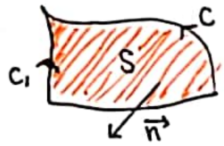
Stokes' Theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$



Divergence Theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV$

Integral over boundary = integral of derivatives over interior

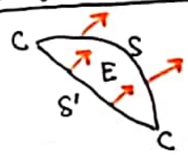
Stokes' Theorem: Find $\int_C \vec{F} \cdot d\vec{r}$ (closed curve) by choosing an oriented surface S w/ boundary C & integrating curl $\vec{F}$ over S



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} + \int_{C'} \vec{F} \cdot d\vec{r}$$

Divergence Theorem: $\iint_S \vec{F} \cdot d\vec{S}$ if S is a closed surface which is the boundary of E then

$\iint_S \vec{F} \cdot d\vec{S} = \pm \iiint_E (\text{div}\,\vec{F})\,dV$ (+ if S is oriented out of E)



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV + \iint_{S'} \vec{F} \cdot d\vec{S}$$