Analyzing Characteristic equation of time delay systems

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1 Lambert W function

In mathematics, the Lambert-W function, also called the omega function or product logarithm, is a set of functions, namely the branches of the inverse relation of the function $f(z) = ze^z$ where e^z is the exponential function and z is any complex number. In other words

$$z = f^{-1}(ze^z) = W(ze^z)$$

By substituting $z_0 = ze^z$ into the equation above, we get the defining equation for the W function (and for the W relation in general):

$$z_0 = W(z_0) * e^{W(z_0)}$$

Now, Taking equation $ze^z = c$.
This imply $z = W_k(c)$. where W_k for k an integer are countably many branches of the W function. W_0 being the principal branch, see fig 1,. The branch point for the principal branch is at $z = \frac{-1}{e}$, with a branch cut that extends to $-\infty$ along the negative real axis. This branch cut separates the principal branch from the two branches W_{-1} and W_1 In all other branches, there is a branch point at $z = 0$ and a branch cut along the whole negative real axis.

Subsituting, $z = x + iy$, and seperating real and imaginary parts we get

$$e^x(x \cos y - y \sin y) = c \ \ \& \ \ ie^x(x \sin y + y \cos y) = 0$$

. After further simplification we get

$$x = -y \cot y \ \ \& \ \ x = \log - \frac{c \sin y}{y}$$

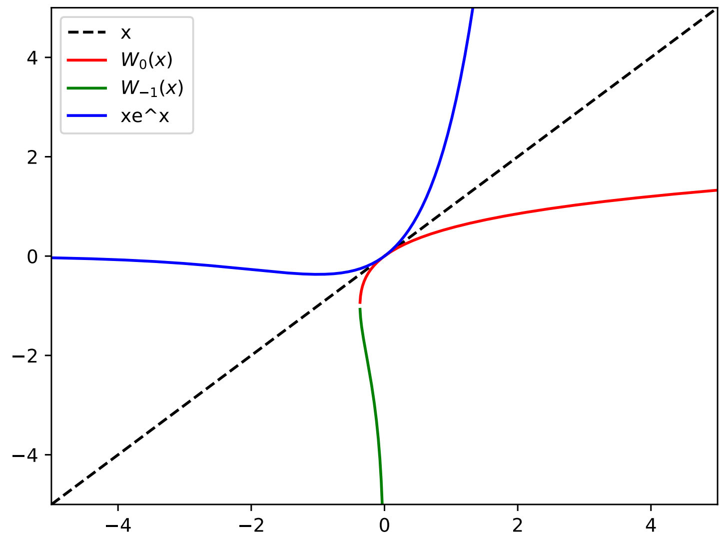

Figure 1

From fig 2., I calculated the point of intersection of this graph for $c = 1$. To find out y coordinate of point of intersection.

$$FindRoot[-y * \cot[y] - Log[-1 * \sin[y]/y], (y, \frac{3\pi}{2})]$$

this command gives root near $y = \frac{3\pi}{2}$ as it is can be seen in graph that we have root near $y = \frac{3\pi}{2}$ and the output is $(y \to 4.37519)$. putting this output in $-y\cot[y]$ we get $x \to -1.53391$, Thus, (-1.53391,4.37519) will satisfy $(x+iy) * e^{-(x+iy)} = 1$.
And from the symmetry of graph (-1.53391,-4.37519) is also a solution.
The values of lambert W function can be easily calculated in python.

$$-2.401 - 10.77I, -1.533 - 4.375I, 0.5671, -1.533 + 4.375I, -2.401 + 10.776I$$

These are values of $W_k(1)$ from $k = -2$ to $k = 2$.
Now, we can see the point of intersection exatly matches with values of Lambert W function. From here we can attempt to write general expression of W_k.

$$Im(W_k(c)) = y : -y\cot y = \log - \frac{c\sin y}{y}, \begin{cases} 2k\pi < y < (2k+1)\pi & for \quad k \geq 0 \\ (2k+1)\pi < y < (2k+2)\pi & for \quad k < 0 \end{cases}$$

$$Re(W_k(c)) = -y\cot y \ or \ \log - \frac{c\sin y}{y} : y \ is \ the \ Im(W_k(c))$$

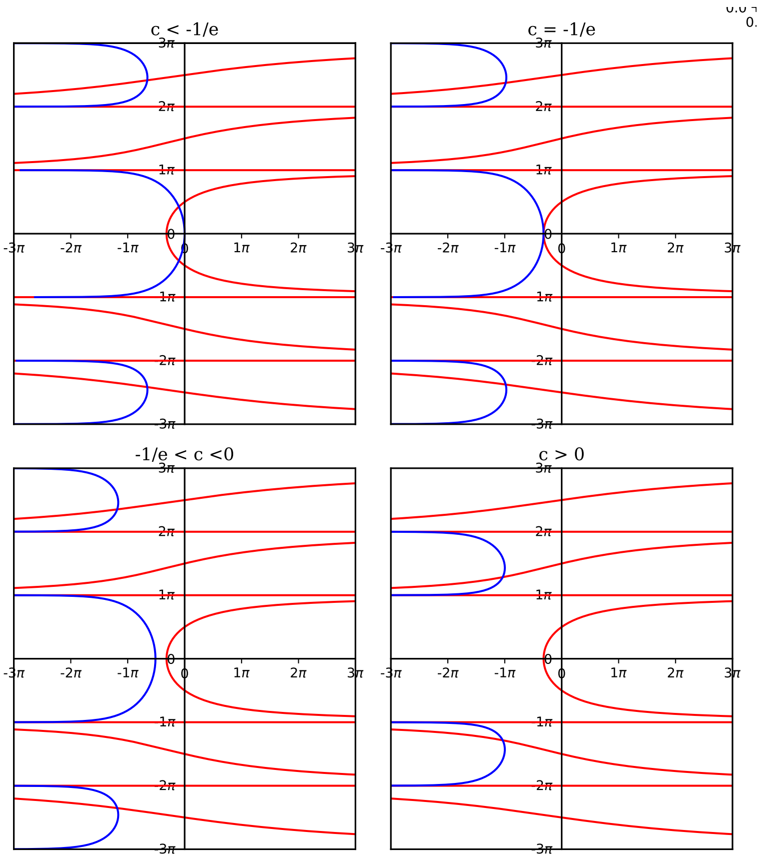

Figure 2

This expression is true for $c < \frac{-1}{e}$ where all roots are imaginary. For $\frac{-1}{e} < c < 0$ this expression is still true except for real roots W_{-1} and W_0.

For $c > 0$ only W_0 is real, thus expression is:

$$Im(W_k(c)) = y : -y \cot y = \log -\frac{c \sin y}{y}, \begin{cases} (2k-1)\pi < y < 2k\pi & for \quad k > 0 \\ (2k)\pi < y < (2k+1)\pi & for \quad k < 0 \end{cases}$$

$$Re(W_k(c)) = -y \cot y \ or \ \log -\frac{c \sin y}{y} : y \ is \ the \ Im(W_k(c))$$

From fig 2, we can infer that W_0 has maximum real part and real part of W_k decreases as k goes away from zero.

2 Transcendental equation for linear time delay system

The characteristic equation for linear time delay systems is given by Eq. looks like:

$$\lambda + \alpha e^{-\lambda\tau} = \beta \tag{1}$$

There are infinite solutions to above equation are given as below $\forall k \in \mathbb{Z}$,

$$\lambda_k = \beta + \frac{W_k(-\alpha\tau e^{-\beta\tau})}{\tau} \quad , \tag{2}$$

where W_k is the k^{th} branch of the Lambert W function with $k = 0$ as the principal brach. The stability of fixed point is governed by the eigenvalue corresponding to the principal branch(since it has maximum real part). Consequently we will focus on the solution λ_0 analytically. It is easy to check that the solution for $\alpha = 0$ is $\lambda_0 = \beta$. The eigenvalue corresponding to the principal branch is real if and only if W_0 is real. This implies,

$$-\alpha\tau e^{-\beta\tau} \geq -1/e; \quad \forall \quad \tau \geq 0 \quad . \tag{3}$$

This can be rewritten as,

$$\alpha \leq \frac{e^{\beta\tau}}{e\tau} \implies \begin{cases} \alpha \leq \frac{1}{e\tau}, & \text{if } \beta = 0 \\ \alpha \leq \frac{\beta}{e}\frac{e^{\beta\tau}}{\beta\tau}, & \text{if } \beta \neq 0 \end{cases} \tag{4}$$

Since α, β are known system parameters. From Eq. 4 it is clear that if $\alpha \leq \min[\frac{e^{\beta\tau}}{e\tau}]; \forall\tau > 0$ then the eigenvalue corresponding to the principal branch (λ_0) will always be real for all $\tau > 0$. One can note that the maximum value of α in this scenario can be given as,

$$\alpha_{max} = \begin{cases} \frac{\beta}{e}\max[\frac{e^{\beta\tau}}{\beta\tau}] = 0, & \text{for } \beta < 0 \\ \frac{\beta}{e}\min[\frac{e^{\beta\tau}}{\beta\tau}] = \beta, & \text{for } \beta > 0 \\ \min[\frac{1}{e\tau}] = 0, & \text{for } \beta = 0 \end{cases} \tag{5}$$

If the above doesnot hold true then, λ_0 will be complex for some $\tau > 0$.

Suppose, $\exists\ \tau = \tau_c$, for which $\lambda_k = i\omega$ for a particular $k \in \mathbb{Z}$. By putting this in Eq. 1 and separating real and imaginary part we get,

$$\begin{aligned} &\omega^2 = \alpha^2 - \beta^2 \quad , \\ &\tau_c = \frac{\pm 1}{\sqrt{\alpha^2 - \beta^2}}\cos^{-1}(\frac{\beta}{\alpha}) + \frac{2\pi m}{\sqrt{\alpha^2 - \beta^2}} \quad , \end{aligned} \tag{6}$$

where m is any integer. For τ_c to be real ω must be real, i.e., $|\alpha| > |\beta|$. The case m=0 corresponds to the principal solution and we denote it as τ_0. If we combine inequality 5 and Eq. 6 we can comment that for λ_0 to be purely imaginary we must have the following condition satisfied, $\forall\beta \in (-\infty, +\infty)$,

$$\alpha > \max\{\beta, 0\} \quad \text{and} \quad \alpha + \beta \in \mathbb{R}^+ \tag{7}$$

The nature of roots λ_0 and λ_{-1} is shown in figure 3. The proof for remaining cases is similar to above and we can infer the stability and existence of limit cycle from figure 3.

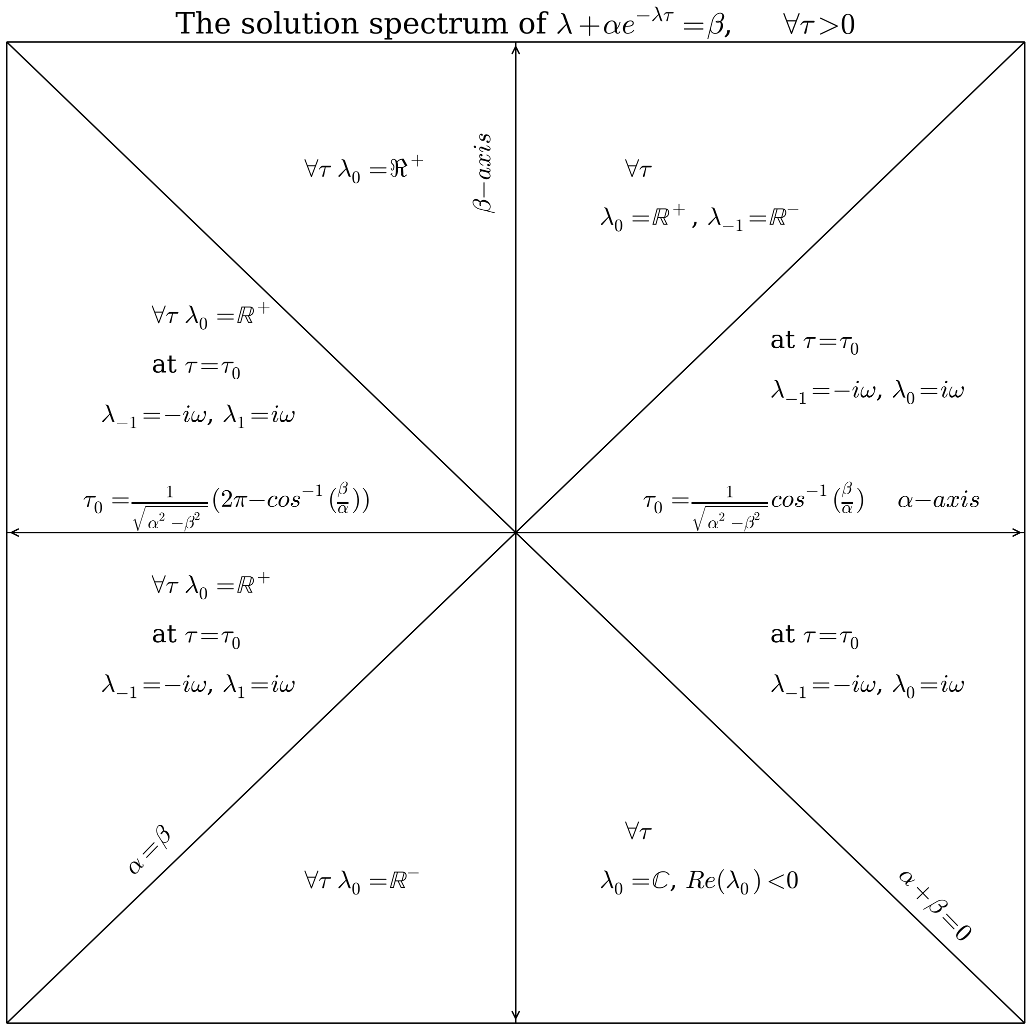

Figure 3