Study energy transfer in 2D fluid turbulence

Smit Zaveri[1], Mahendra K. Verma[2] and Abhishek Kumar[3], Santosh Akoijam[4]

[1]Department of physics, IIT Kanpur, Kanpur, 208016, India
[2]Department of physics, IIT Kanpur, Kanpur, 208016, India
[3]Department of physics, IIT Kanpur, Kanpur, 208016, India
[4]Department of physics, IIT Kanpur, Kanpur, 208016, India

An inviscid 2-D fluid flow conserves both kinetic energy and the enstrophy. 2-D fluid turbulence consists of dual cascade, constant inverse energy cascade for small wavenumbers and constant enstrophy cascade for larger wavenumbers. Here we will analyze the energy spectrum for 2-D turbulence for various grid size and try to verify the behavior of energy spectrum for different wavenumbers as mentioned.

I. INTRODUCTION:

Two dimensional turbulence has very wide aspects. We can study the large scale motion of in atmosphere and oceans by approximating it to two dimensional turbulent fluid flow. 2-D analysis of turbulence can numerically achieve much higher spatial and temporal resolution as compared to 3-D which provides simplified framework to the solution of Navier- Stokes equation. As a result, it is crucial to understand the energy and flux transfer in 2-D turbulent flow.
This review is brief analysis of energy transfer in 2-D turbulent flow for inviscid flow and forced- dissipative flow for various grid size and initial conditions. We will first introduce the Navier-Stokes equation and its fourier transform. Then we will discuss the Kolmogorov theory and its predictions briefly and we will move to methods and approaches for verification of the theory proposed. We will discuss the simulations obtained and observations made. Further we will discuss the error and discrepancies observed during simulations and analysis.

II. EQUATION OF MOTION AND KOLMOGOROV'S THEORY

Fluid flow as described by incompressible Navier-Stokes(NS) equation is

$$\frac{\partial u}{\partial t} + u.\nabla u = -\frac{1}{\rho}\nabla \mathrm{p} + \frac{1}{\rho}F + \vartheta\nabla^2 u \tag{1}$$

$$\nabla.u = 0 \tag{2}$$

The non-dimensionalised form of the equation is

$$\frac{\partial u}{\partial t} + u.\nabla u = -\nabla \mathrm{p} + F + \vartheta\nabla^2 u$$

$$\nabla.u = 0$$

Assuming divergenceless force, by taking divergence of NS equation, we get $-\nabla^2 p = \nabla.[(u.\nabla)u]$.

The velocity field **u** is decomposed in Fourier space as $u(r,t) = \sum_n u(k,t)\exp(ik.r)$ and the NS equation in Fourier space is

$$\left(\frac{d}{dt} + \vartheta k^2\right)u_i(k,t) = -ik_i p(k,t) - N_i(k) + f_i(k) \tag{4}$$

$$k.u(k,t) = 0$$

where

$$N(k) = i\sum_p [k.u(k-p)]u(p)$$

is the nonlinear term, and the pressure term is

$$\mathrm{p(k,t)} = \mathrm{i}\frac{k.N(k)}{k^2}$$

Energy transfer from Fourier mode p to k' with q as mediator is given by

$$S^{uu}(k'|p|q) = -\Im([k'.u(q)][u(p).u(k')]) \tag{5}$$

Here $\Im$ is for the imaginary part.

The equation for shell-to-shell energy transfer from shell m to shell n is given as

$$T_{nm}^{uu} = \sum_{\boldsymbol{p}\in \boldsymbol{m}}\sum_{\boldsymbol{k}\in \boldsymbol{n}} \boldsymbol{S^{uu}}(\boldsymbol{k}|\boldsymbol{p}|\boldsymbol{q}) \tag{6}$$

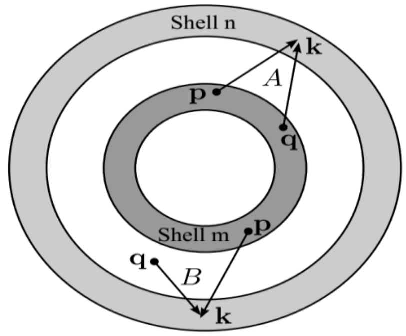


Fig.1 Shell-to-shell energy transfers from the wavenumber-shell *m* to wavenumber-shell *n*.

Kolmogorov's Theory of Turbulence:
For a homogeneous, isotropic, incompressible and steady fluid turbulence with $v \to 0$, Kolmogorov gave the relation

$$\langle(\Delta u)_{\parallel}^{3}\rangle = -\frac{4}{5}\varepsilon l \tag{7}$$

where $(\Delta u)_{\parallel}$ is the component of **u**(x+**l**)-**u**(x) along **l**, ε is the dissipation rate, and l lies between forcing scale (L) and dissipative scales l_d i.e. $l_d \ll l \ll L$. $\langle\rangle$ resembles ensemble averaging. This range of intermediate scale is called inertial range. The above relationship is universal and is independent of the forcing and dissipative mechanism, the fluid properties like viscosity and the initial conditions.
If we assume Δu to follow a fractal structure and ε to be independent of l then

$$\langle(\Delta u)\rangle \propto \varepsilon^{\frac{2}{3}} l^{\frac{2}{3}} \tag{8}$$

And the Fourier transform of this yield

$$E(k) = K_{K_0} \varepsilon^{\frac{2}{3}} l^{-\frac{5}{3}} \tag{9}$$

where $K_{K_0}$is a universal constant commonly known as Kolmogorov's constant.
In two dimensional turbulences we observe that the enstrophy conservation has significant effect on the flow.

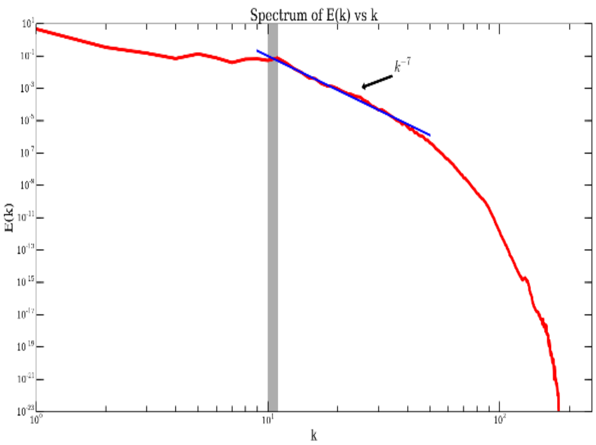
III. PHENOMENOLOGY FOR 2-D TURBULENCE

An inviscid 2-D fluid flow conserves both kinetic energy and the entropy. As a result, if the fluid is forced at a wavenumber $k = k_f$, the kinetic energy cascade show two type of characteristics. One is that the energy cascades backward from k_f to smaller wavenumbers $k < k_f$ and the second is that the enstrophy cascades to larger wavenumbers $k > k_f$. As the kinetic energy spectrum is consistent, for $k < k_f$, it follows Kolmogorov's spectrum. Therefore, 2-D fluid turbulence consists of dual cascade, constant inverse energy cascade for small wavenumbers and constant enstrophy cascade for larger wavenumbers. As a result, we can see that from small wavenumbers, inverse energy cascade leads to the formation of large scale structures.

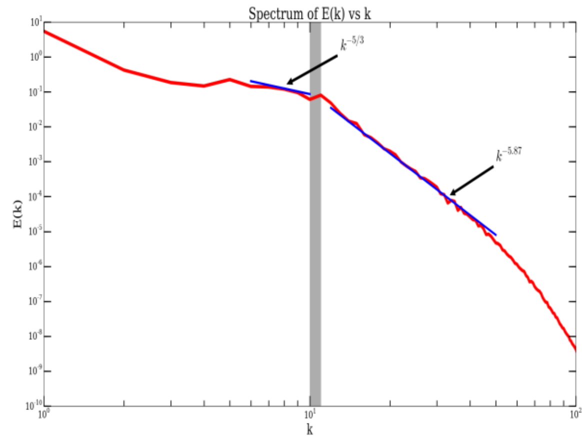
IV. METHOD AND ANALYSIS

For simulations of 2-D turbulent fluid flow we are using TARANG code is used which is based on pseudo- spectral method. In spectral simulation, instead of calculation of N(k) by involving convolution will require O(N^2) floating point operations. So, it is efficiently calculated using Fast Fourier Tranform(FFT). Boxes of size 2π is assumed which gives wavenumbers k = 2πn/L as integers. The aim is to compute field variables at a later time given u (k, t = 0). Grid size is estimated as $N_i \geq Re^{3/4}$. FFT takes O(N log N) floating point operations making it more efficient.
We will now analyze the obtained results for the energy spectrum after simulating the flow for different grid size. Initially we simulated data for 64*64 as trial and then for 128*128. After getting formalized with the simulations we simulated flow for 256*256 grid points and obtained the energy spectrum as show in FIG.2(a). Using the data of 256*256 we increased the

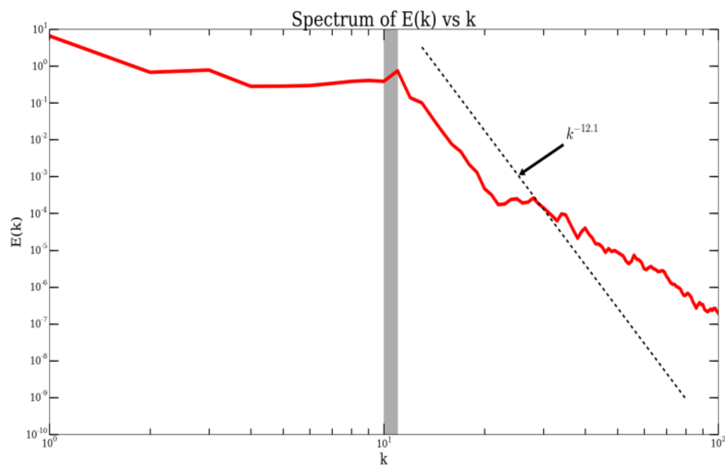
resolution of the grid to 512*512 FIG.2(b) and similarly using 512*512 we simulated flow for 1024*1024 FIG.2(c), and plotted the energy spectrum.



(a)



(b)



(c)

FIG.2 (a) spectrum plot for 256*256 grid points with $Re = 2 * 10^3$. (b) spectrum plot for 512*512 with $Re = 3.1 * 10^4$. (c) spectrum plot for 1024*1024 with $Re = 4.2 * 10^4$

256*256 data were inconsistent so we increased grid size and observed the power law as predicted and after further increase in grid size we were unable to observe power law. We need to change initial conditions and simulate again till we get the desired output and then with those data set increase the grid size to 2048*2048 and so on. This is done to improve simulation and further the efficiency of code we are using.

V. Acknowledgement

I would like to thank Prof. Mahendra K. Verma guiding us through the project and Mr. Abhishek Kumar for mentoring us throughout and teaching us where we were going wrong. Without their guidance we would not be able to simulate and complete the work. I would also thank Santosh Akoijam for being my project partner and helping me with the simulations.

VI. REFERENCES

1.Notes from Prof M.K.Verma.
2. Boffetta G. 2012. Two Dimensional Turbulence. Annu. Rev. Fluid Mech. 2012. 44:427–51.