Heat Equation with Spectral Methods

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1 Problem Formulation

$$\begin{cases} u_t = ku_{xx} & x \in [0, L] \\ u(0,t) = sin(t) & u(L,t) = 1.0 \\ u(x,0) = 0.5 \end{cases} \tag{1}$$

2 Discretizing the Heat Equation via Finite Differences

$$u_t = ku_{xx}$$

Partial derivatives of u(x,t) can be approximated using the following finite difference formulas:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} \\ \frac{\partial^2}{\partial x^2}u(x,t) \approx \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} \end{cases} \tag{2}$$

From which the iterative equation to approximate the next time step is derived:

$$\Rightarrow \frac{\partial}{\partial t}u(x,t) - k \cdot \frac{\partial}{\partial x^2}u(x,t) = 0$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} - k \cdot \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} = 0$$

$$\Rightarrow u_i^{n+1} = \frac{k \cdot \Delta t}{\Delta x^2} \cdot (u_{i-1}^n - 2u_i^n + u_{i+1}^n) \tag{3}$$

Equation [3] represents how to solve for the subsequent time step $t = n + 1$ of the heat equation at point $x = i$ given the value at $x = i, i \pm 1$ for $t = n$.

The residual form for this approximation is:

$$R_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} - k \cdot \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} \tag{4}$$

3 Discretizing the Heat Equation via Spectral Method

$$\mathbf{u}(x) = \begin{bmatrix} \mathbf{u}(x,0) \\ \mathbf{u}(x,\frac{T}{N}) \\ \mathbf{u}(x,2\frac{T}{N}) \\ \vdots \\ \mathbf{u}(x,(N-1)\frac{T}{N}) \end{bmatrix} \tag{5}$$

$$\mathbf{D}_{i,j}(T,N) = \begin{cases} \frac{2\pi}{T}(\frac{1}{2}(-1)^{i-j}\frac{1}{sin(\frac{\pi(i-j)}{N})}) & \text{i} \neq \text{j} \\ 0 & \text{i} = \text{j} \end{cases} \tag{6}$$

$$\frac{\partial}{\partial t}\mathbf{u}(x) \approx \mathbf{D}(T,N)\mathbf{u}(x) \tag{7}$$

$$\mathbf{D}(T,N) = \begin{bmatrix} 0 & \frac{-\pi}{T\cdot sin(-\pi/N)} & \frac{\pi}{T\cdot sin(-2\pi/N)} & \cdots & \frac{\pi}{T\cdot sin(-(N-1)\pi/N)} \\ \frac{-\pi}{T\cdot sin(\pi/N)} & 0 & \frac{-\pi}{T\cdot sin(-\pi/N)} & \cdots & \frac{-\pi}{T\cdot sin(-(N-2)\pi/N)} \\ \frac{\pi}{T\cdot sin(2\pi/N)} & \frac{-\pi}{T\cdot sin(\pi/N)} & 0 & \cdots & \frac{\pi}{T\cdot sin(-(N-3)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\pi}{T\cdot sin((N-1)\pi/N)} & \frac{-\pi}{T\cdot sin((N-2)\pi/N)} & \frac{\pi}{T\cdot sin((N-3)\pi/N)} & \cdots & 0 \end{bmatrix} \tag{8}$$

The Spectral Differentiation Matrix (Equation [8]) is the above for input scalars T, N corresponding to the total time of the simulation and the number of discretized timesteps, respectively.

3.1 Question:

Here is my progress for part 1.2:

given:

$$\frac{\partial}{\partial t}\mathbf{u}(x) \approx \mathbf{D}(T,N)\mathbf{u}(x)$$

let:

$$\frac{\partial}{\partial t}\mathbf{u}(x) \approx \begin{bmatrix} \mathbf{u}(x,\frac{T}{N}) \\ \mathbf{u}(x,2\frac{T}{N}) \\ \mathbf{u}(x,3\frac{T}{N}) \\ \vdots \\ \mathbf{u}(x,(N-1)\frac{T}{N}) \end{bmatrix} - \begin{bmatrix} \mathbf{u}(x,0) \\ \mathbf{u}(x,\frac{T}{N}) \\ \mathbf{u}(x,2\frac{T}{N}) \\ \vdots \\ \mathbf{u}(x,(N-2)\frac{T}{N}) \end{bmatrix}$$

$$\Rightarrow \mathbf{D}(T,N)\mathbf{u}(x) \approx \begin{bmatrix} \mathbf{u}(x,\frac{T}{N}) \\ \mathbf{u}(x,2\frac{T}{N}) \\ \mathbf{u}(x,3\frac{T}{N}) \\ \vdots \\ \mathbf{u}(x,(N-1)\frac{T}{N}) \end{bmatrix} - \begin{bmatrix} \mathbf{u}(x,0) \\ \mathbf{u}(x,\frac{T}{N}) \\ \mathbf{u}(x,2\frac{T}{N}) \\ \vdots \\ \mathbf{u}(x,(N-2)\frac{T}{N}) \end{bmatrix}$$

$$\Rightarrow \mathbf{D}(T,N)\mathbf{u}(x) \approx \mathbf{S}\mathbf{u}(x) - \mathbf{I}\mathbf{u}(x)$$

Where **S** denotes the circulant matrix.

** *This also ignores the fact that it is no longer full rank.* **

$$\Rightarrow \mathbf{D}(T,N)\mathbf{u}(x) \approx (\mathbf{S}-\mathbf{I})\mathbf{u}(x)$$

$$\Rightarrow (\mathbf{D}(T,N) - (\mathbf{S}-\mathbf{I}))\mathbf{u}(x) \approx 0$$

Therefore, the residual form of this spectral method approximation to the heat equation would be:

$$\mathbf{R} = (\mathbf{D}(T,N) - (\mathbf{S}-\mathbf{I}))\mathbf{u}(x)$$

Is this on the right track? I am not sure how to solve this equation while incorporating the initial conditions and boundary conditions. What I tried doing was a minimization of **R** by modifying **u**(x) with initial conditions and boundary conditions imposed on the problem as constraints on **u**(x). I also read through literature on the differentiation matrices of spectral methods, but this is all new to me so I am not sure where to go from here.