Chapter 31 Electromagnetic Oscillations and Alternating Current

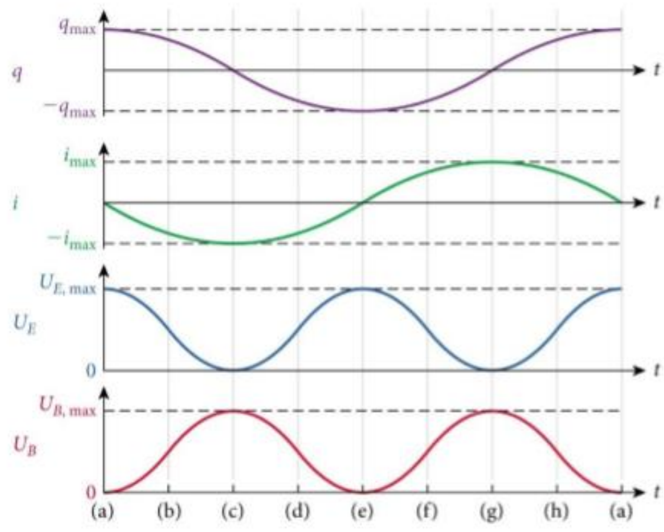
LC 회로

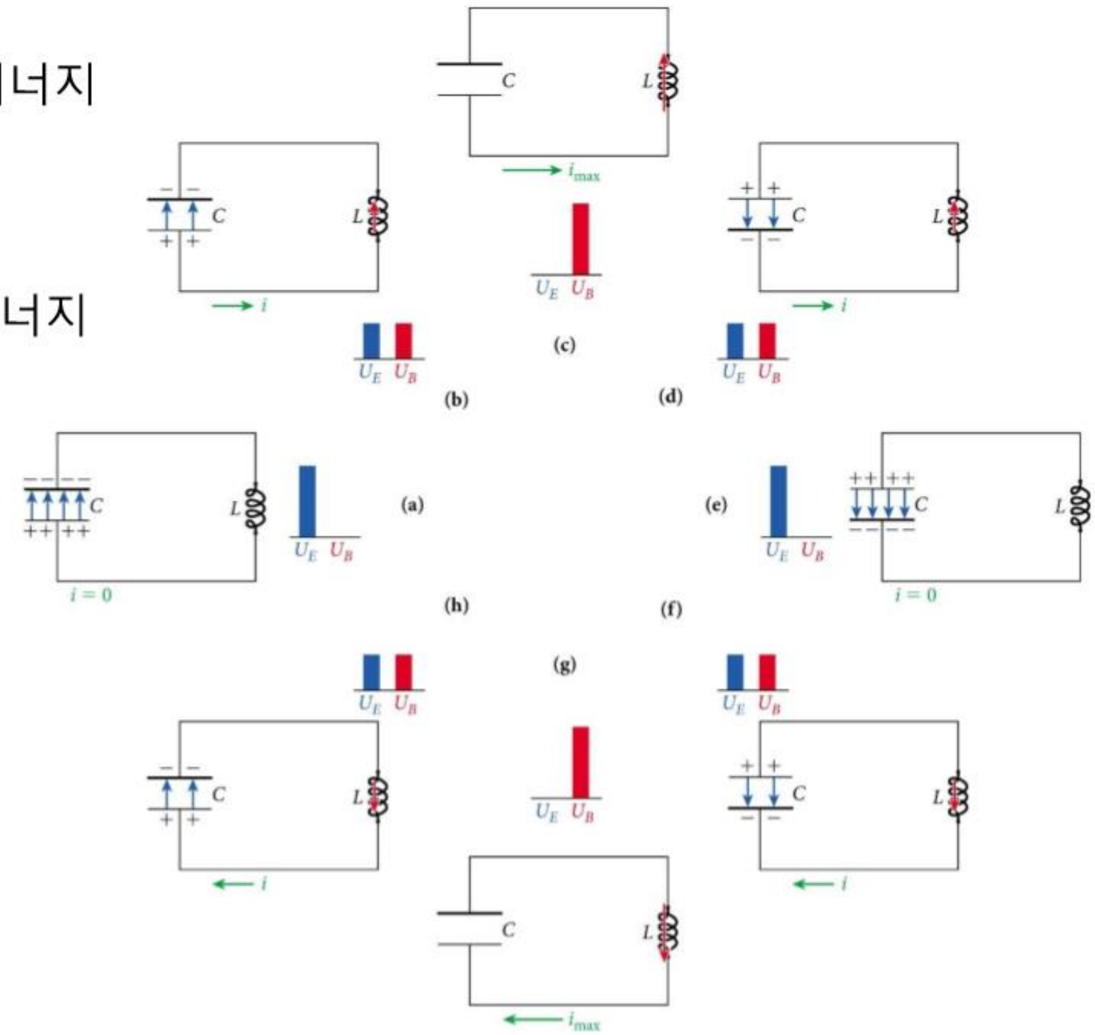
- 축전기의 전기장에 저장된 에너지

$$U_E = \frac{1}{2}\frac{q^2}{C}$$

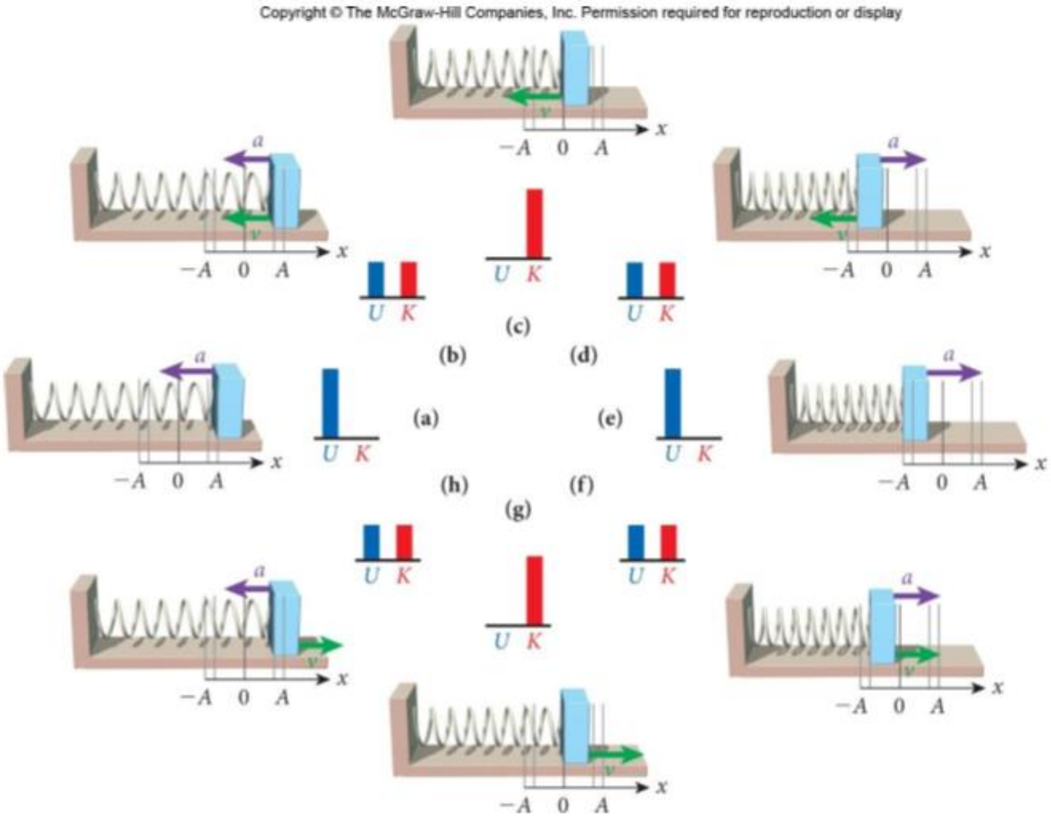
- 유도기의 자기장에 저장된 에너지

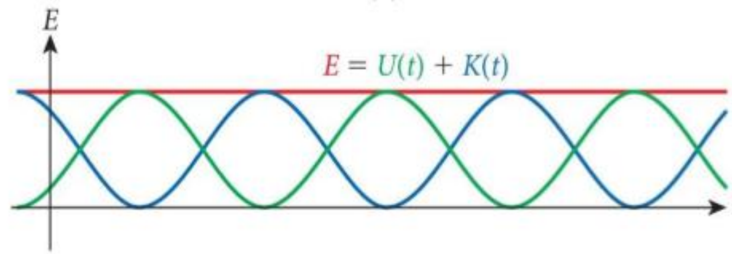
$$U_B = \frac{1}{2}L\,i^2$$





조화단진동와의 유사성





$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\omega_0 A\sin(\omega_0 t + \theta_0)\right)^2$$
$$= \frac{1}{2}kA^2\sin^2(\omega_0 t + \theta_0)$$
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \theta_0)$$

$$K + U = \frac{1}{2}kA^2 = E$$

LC 진동의 분석

$$U = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

$i = \dfrac{dq}{dt}$ 이므로, 위 식은

$$\frac{dq}{dt}\left(\frac{q}{C} + L\frac{d^2q}{dt^2}\right) = 0$$

$$\therefore \boxed{\frac{d^2q}{dt^2} + \frac{q}{LC} = 0}$$

$$\Leftrightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

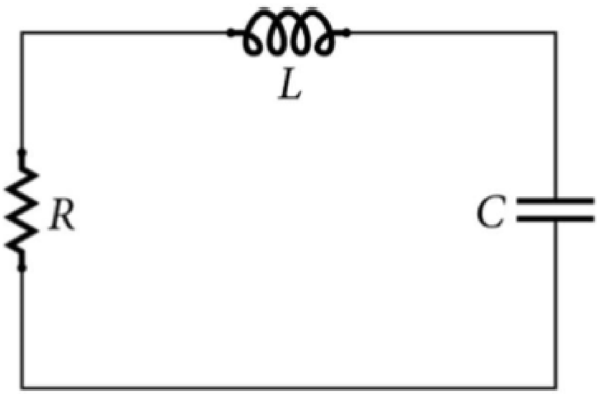
해) $\underline{q = q_{최대}\cos(\omega_0 t - \phi)}$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}$$

$$\boxed{\begin{aligned} U_E &= \frac{q^2_{최대}}{2C}\cos^2(\omega_0 t - \phi) \\ U_B &= \frac{Li^2_{최대}}{2}\sin^2(\omega_0 t - \phi) \end{aligned}}$$

$$\boxed{U = U_E + U_B = \frac{q^2_{최대}}{2C} = \frac{Li^2_{최대}}{2}}$$

RLC 회로의 감쇠진동



- 키르히호프 규칙 적용 : $L\frac{di}{dt} + Ri + \frac{q}{C} = 0$
- RLC 회로 미분 방정식

$$\boldsymbol{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0}$$

- 감쇠조화운동의 운동방정식과 비교

$$\boldsymbol{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0}$$

- RLC 회로 미분 방정식 풀기 : $q(t) = Ae^{\alpha t}$

$$A\left(L\alpha^2 + R\alpha + \frac{1}{C}\right) = 0 \quad \Rightarrow \quad \alpha = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Chap. 31

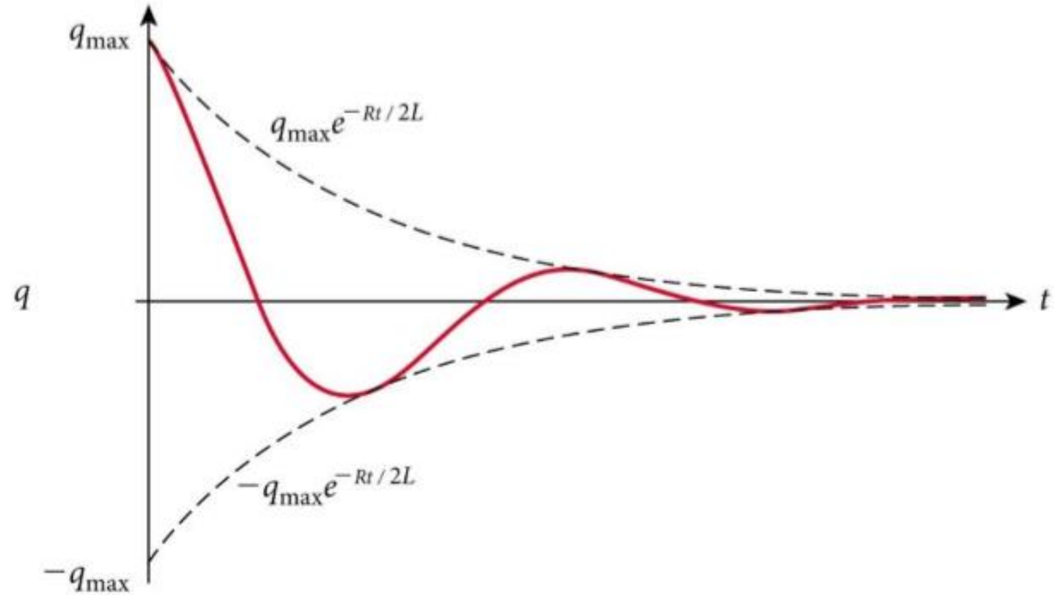
- R이 충분히 작은 경우 ($R < 2\sqrt{L/C}$): $\omega = \sqrt{\left(\frac{1}{LC} - \frac{R}{2L}\right)^2}$

$$q(t) = e^{-Rt/2L}\left(Ae^{i\omega t} + Be^{-i\omega t}\right) = q_{\max}e^{-Rt/2L}\cos\omega t$$

초기조건

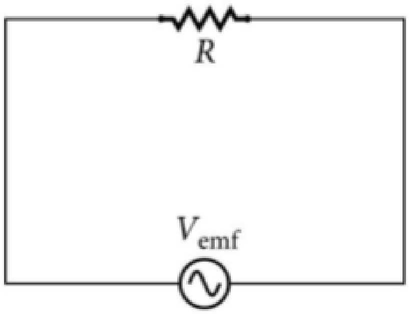
$$q(0) = q_{\max}$$

$$\dot{q}(0) = i(0) = 0$$



구동 AC 회로

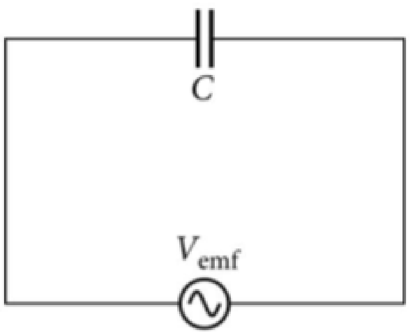
$$\boxed{V_{\mathrm{emf}} = V_0 \sin \omega t}$$ AC 기전력에 대한 반응



$$iR = V_{\mathrm{emf}} \qquad i = \frac{V_0}{R} \sin \omega t$$

저항

R

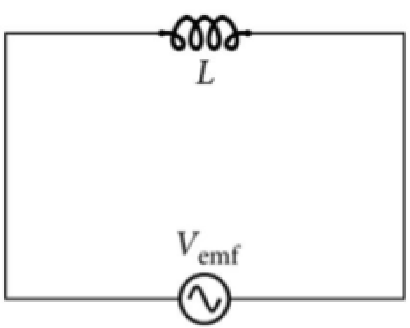


$$\frac{q}{C} = V_{\mathrm{emf}} \qquad i = \omega C V_0 \cos \omega t$$

용량형 반응저항

$X_C = (\omega C)^{-1}$

진동수에 따라 달라진다.



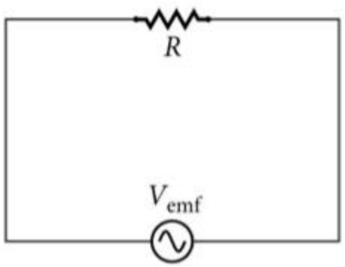
$$L\frac{di}{dt} = V_{\mathrm{emf}} \qquad i = -\frac{V_0}{\omega L} \cos \omega t$$

유도형 반응저항

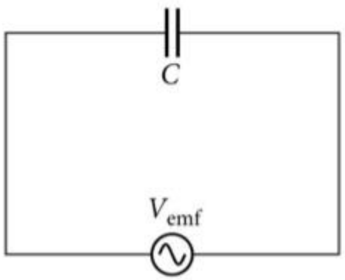
$X_L = \omega L$

Chap. 31

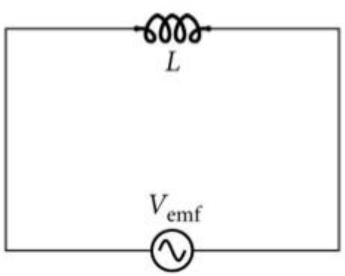
$$V_{\text{emf}} = V_0 \sin \omega t$$



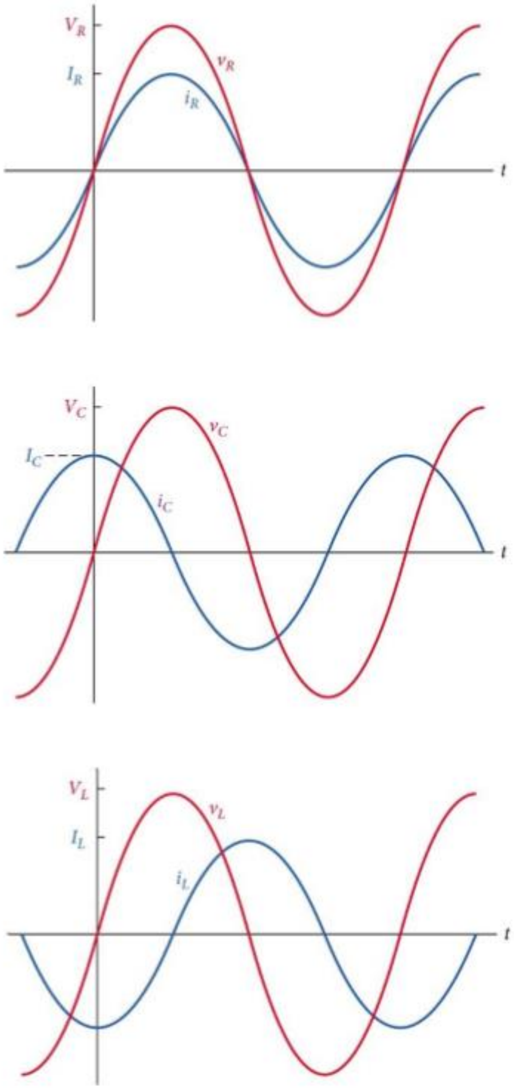
$$i = \frac{V_0}{R} \sin \omega t$$



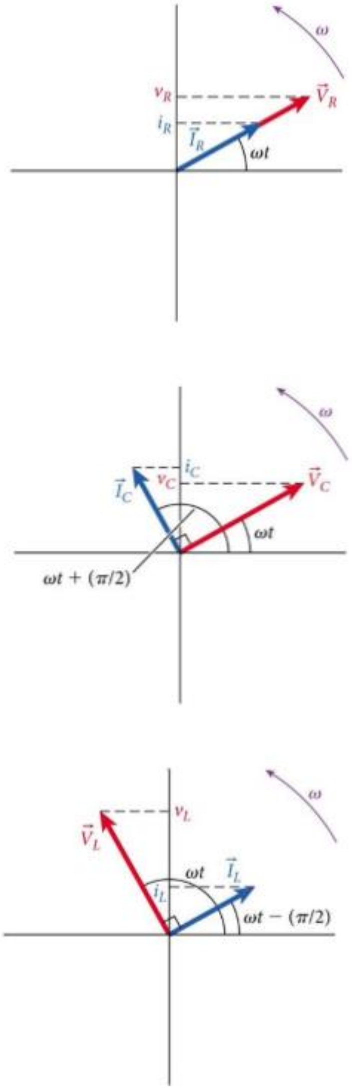
$$i = \omega C\, V_0 \cos \omega t$$



$$i = -\frac{V_0}{\omega L} \cos \omega t$$

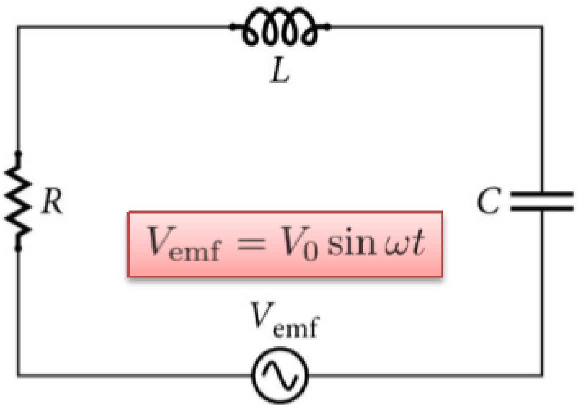


(a)



(b)

직렬 RLC 회로



- 키르히호프 규칙 적용 : $L\dfrac{di}{dt} + Ri + \dfrac{q}{C} = V_0 \sin \omega t$

- 직렬 RLC 회로의 미분 방정식

$$\boldsymbol{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \underbrace{V_0 \sin \omega t}_{\text{구동기전력}}}$$

- 미분 방정식의 해

$$q(t) = q_{\mathrm{h}}(t) + q_{\mathrm{p}}(t)$$

$L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = 0$ 의 해

- 초기조건을 맞추기 위해서 필요하다.
- 충분한 시간이 지나면 감쇠하여 사라진다.

왜 sine (또는 cosine) 함수 형태의 진동을 고려하는가?

Fourier 정리 : 주기함수는 진동수가 다른 sine, cosine 함수의 선형결합으로 쓸 수 있다.

Sine 함수에 대한 해를 구하면 모든 주기함수에 대한 해를 알 수 있다.

Chap. 31

- **특별해 찾기** : $q_p(t) = -A\cos(\omega t - \phi)$

$$X_L = \omega L\,,\quad X_C = (\omega C)^{-1}$$
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$\cos\phi_X = \frac{R}{Z},\ \sin\phi_X = \frac{X_L - X_C}{Z}$$

$$V_0 \sin\omega t = A\left[\left(\omega^2 L - \frac{1}{C}\right)\cos(\omega t - \phi) + \omega R\sin(\omega t + \phi)\right]$$
$$= A\omega\sqrt{R^2 + (X_L - X_R)^2}\,[\sin\phi_X\cos(\omega t + \phi) + \cos\phi_X\sin(\omega t + \phi)]$$
$$= A\omega\sqrt{R^2 + (X_L - X_R)^2}\,\sin(\omega t - \phi + \phi_X)$$

$$\Rightarrow\quad A = \frac{V_0}{\omega\sqrt{R^2 + (X_L - X_C)^2}}$$
$$\phi = \phi_X = \arctan\left(\frac{X_L - X_C}{R}\right)$$

$$\Rightarrow\quad q(t) = -\frac{V_0}{\omega\sqrt{R^2 + (X_L - X_C)^2}}\cos(\omega t - \phi)$$
$$i(t) = \frac{dq(t)}{dt} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}\sin(\omega t - \phi)$$

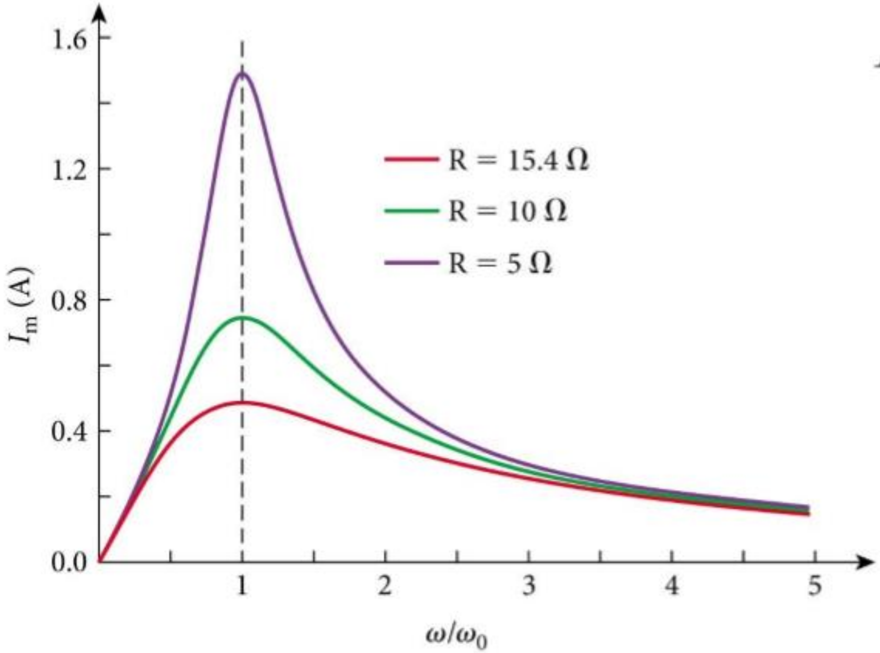
구동기전력 $V(t) = V_0 \sin\omega t$

회로에 흐르는 전류 $i(t) = \dfrac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}\sin(\omega t - \phi)$

위상차 $\phi = \arctan\left(\dfrac{X_L - X_C}{R}\right)$

$$R\,,\quad \underbrace{X_L = \omega L\,,\quad X_C = (\omega C)^{-1}}_{\text{반응저항 (reactance)}} \quad\Rightarrow\quad \underbrace{Z = \sqrt{R^2 + (X_L - X_C)^2}}_{\text{온저항 (impedence)}}$$

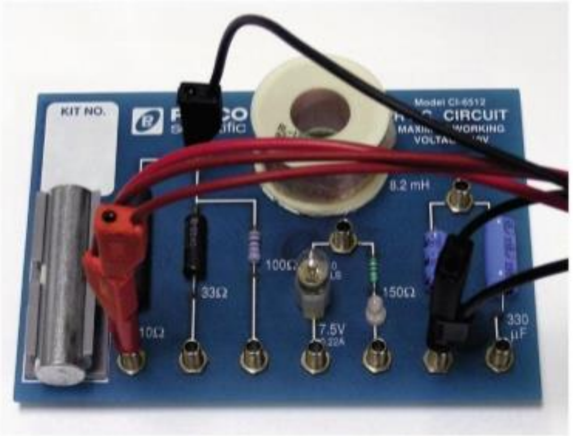
- **공명**

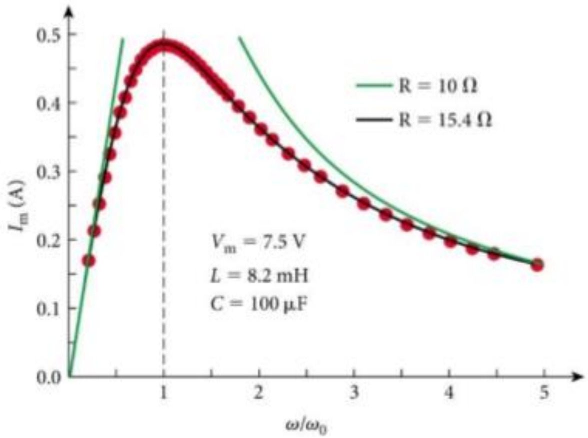


$$I_{\max} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

0일때 I_{max}가 최대가 된다.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$: **공명각진동수**





실제 RLC 회로

- 최대값은 공명진동수에서 생긴다.
- 실제 회로에서는 유도기가 저항을 가지기 때문에 공명진동수 근처에서 차이가 많이 생긴다.

AC 회로의 에너지와 전력

- RLC 회로에 공급된 에너지 ⇨ **저항기**, **축전기**, **유도기**

저항기	축전기	유도기
열로 소모	전기장에 저장	자기장에 저장

- 교류의 경우 저항기가 소비하는 평균전력 = 회로가 소비하는 평균전력

$$P = i^2 R = I_{\max}^2 R \sin^2(\omega t - \phi)$$

$$\langle P \rangle = \langle i^2 \rangle R = I_{\mathrm{rms}}^2 R = I_{\max}^2 R \langle \sin^2(\omega t - \phi) \rangle = \frac{1}{2} I_{\max}^2 R$$

- 제곱평균제곱근 (Root-mean-square) :

$$V_{\mathrm{rms}} = \sqrt{\langle V^2 \rangle} = \sqrt{V_{\max}^2 \langle \sin^2(\omega t) \rangle} = \frac{1}{\sqrt{2}} V_{\max}$$

$$I_{\mathrm{rms}} = \sqrt{\langle i^2 \rangle} = \sqrt{I_{\max}^2 \langle \sin^2(\omega t - \phi) \rangle} = \frac{1}{\sqrt{2}} I_{\max}$$

교류의 전압과 전류는 통상 V_{rms}와 I_{rms}로 표기한다.

Chap. 31

- RLC 회로가 소비하는 평균전력

$$V(t) = V_0 \sin \omega t$$
$$i(t) = \frac{V_0}{Z} \sin(\omega t - \phi)$$
$$\langle P \rangle = \langle V i \rangle = \frac{V_0^2}{Z} \langle \sin(\omega t) \sin(\omega t - \phi) \rangle = \frac{V_0^2}{Z} \frac{1}{2} \cos\phi = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\phi$$

- 저항이 소비하는 평균전력 = RLC 회로가 소비하는 평균전력

$$\langle P \rangle = \frac{1}{2} I_{\mathrm{max}}^2 R = \frac{1}{2} \left(\frac{V_0}{Z} \right)^2 R = \frac{1}{2} V_0 \frac{V_0}{Z} \frac{R}{Z} = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\phi$$

- 진동수의 함수로 본 RLC 회로가 소비하는 평균전력

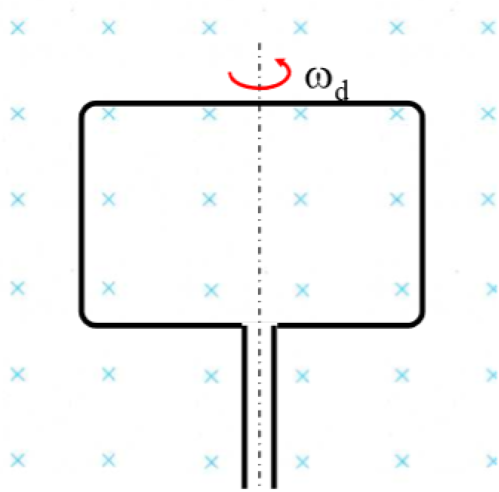
$$\langle P \rangle = \frac{V_0^2 R}{2Z^2} = \frac{V_{\mathrm{rms}}^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \frac{\left(V_{\mathrm{rms}}^2 / R\right) R^2 \omega^2}{R^2 \omega^2 + L^2 \left(\omega^2 - \omega_0^2\right)^2}$$

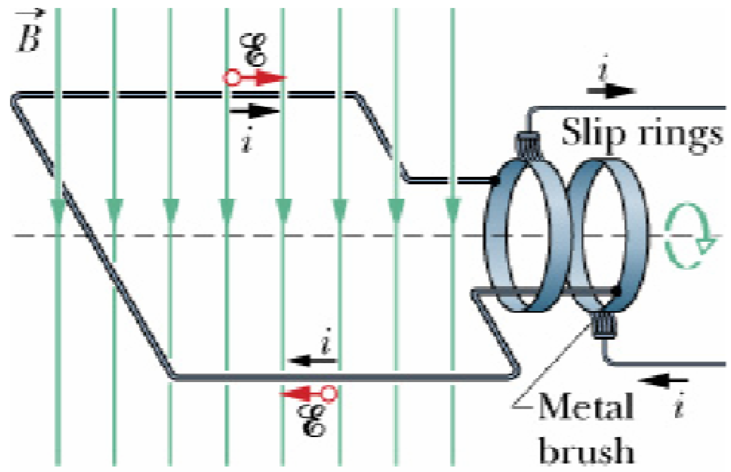
Quality Factor $\quad Q = \dfrac{\omega_0}{\Delta\omega} = \dfrac{\omega_0 L}{R} = \dfrac{1}{R} \sqrt{\dfrac{L}{C}}$

교류전류(Alternating Current)

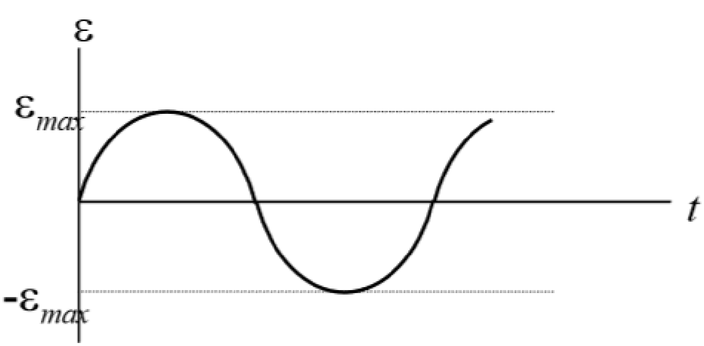
교류발전기

- ***The Alternating-Current (AC) Generator***





$$\Phi_B = BA\cos\theta = BA\cos\omega_d t \Rightarrow \mathcal{E} = -\frac{d\Phi_B}{dt} = BA\omega_d \sin\omega_d t$$



N-coils

$$\mathcal{E} = BAN\omega_d \sin\omega_d t$$

$$= \varepsilon_m \sin\omega_d t$$

$$(\varepsilon_m = BAN\omega_d) \Rightarrow i = I\sin(\omega_d t - \phi)$$

강제진동(Forced Oscillation)

LC, RLC 회로에서 외부 기전력이 없을 때 (자유진동):

$$\omega = \frac{1}{\sqrt{LC}} \longrightarrow \text{자연 각진동수}$$

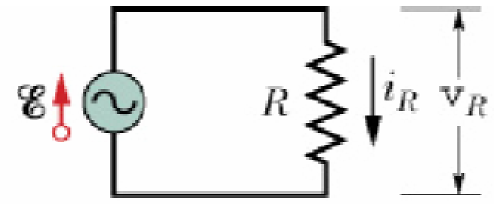
LC, RLC 회로에서 외부 기전력이 ω_d 진동수로 가해질 때 (강제진동):

$$\omega_{forced} = \omega_d \longrightarrow \text{강제 각진동수}$$

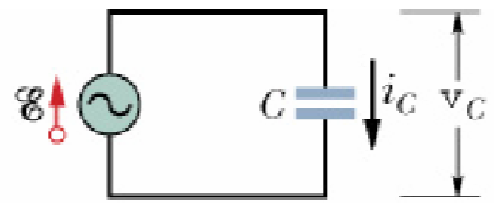
$\omega_d = \omega$ ⇨ **공명 (resonance)**

간단한 세가지 교류회로 (R, C, L)

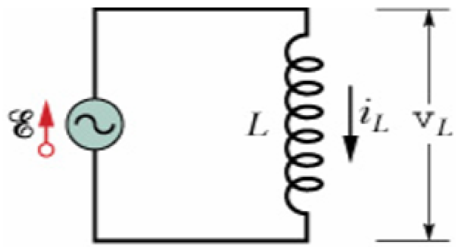
[R 회로]



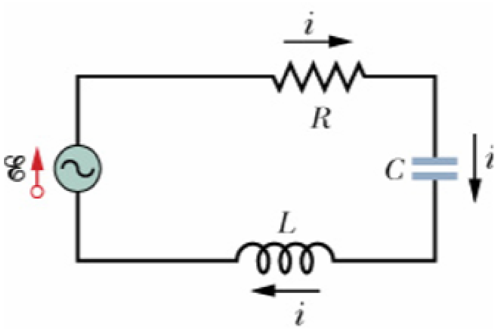
[C 회로]



[L 회로]



[RLC 직렬회로]

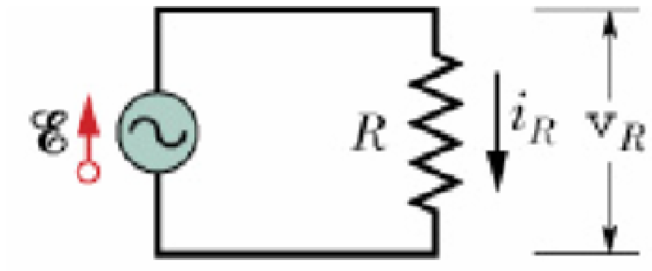


⟶ 저항, 축전기, 코일에 걸린 교류전압과 전류 사이의 관계: 특히 위상차

$$\varepsilon(t) = V_0 \sin(\omega_d t) \quad \Rightarrow \quad i(t) = I_0 \sin(\omega_d t + \phi)$$???

Chap. 31

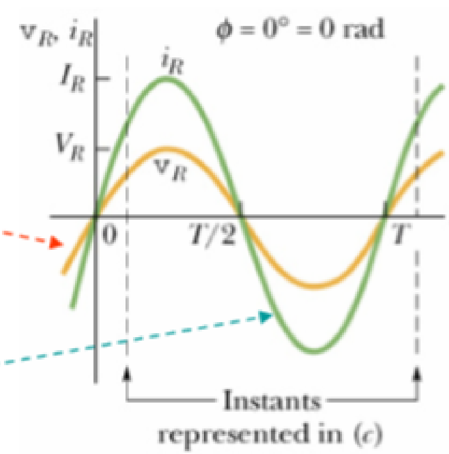
[저항형 회로]



- 전압: $v_R = V_R \sin\omega_d t$

- 전류: 오옴의 법칙 $i_R = \dfrac{v_R}{R}$

$$i_R = \frac{V_R}{R}\sin\omega_d t = I_R \sin\omega_d t$$

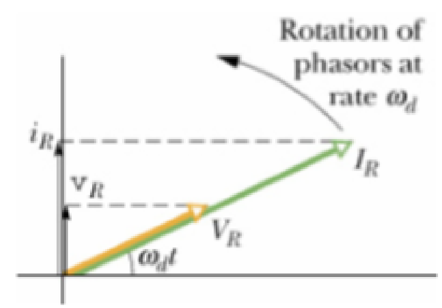


- 전압-전류 관계

① 위상: $\phi_R = 0^o$

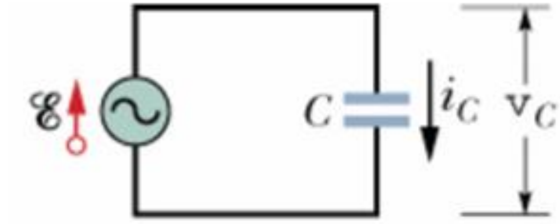
② 진폭: $V_R = I_R R$

위상자 (phasor)



진폭 (길이로)과 위상 (각도로)을 동시에 표현

[용량형 회로]

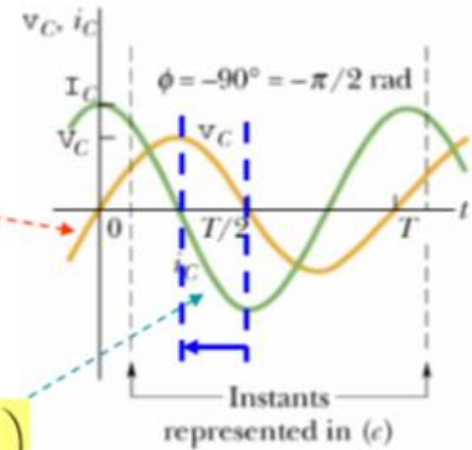


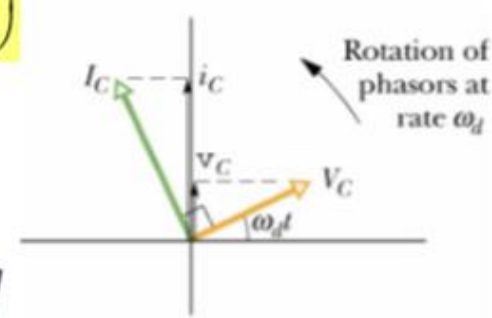
- 전압: $v_C = V_C \sin\omega_d t$
- 전류: 전기용량과 전류의 정의

$$q_C = Cv_C = CV_C \sin\omega_d t$$

$$\begin{aligned} i_C &= \frac{dq_C}{dt} \\ &= \omega_d C V_C \cos\omega_d t \\ &= \omega_d C V_C \sin(\omega_d t + 90^o) \\ &= I_C \sin(\omega_d t - \phi_C) \end{aligned}$$

$$t = \frac{\phi_c}{\omega_d} = -\left(\frac{90^\circ}{\omega_d}\right)$$



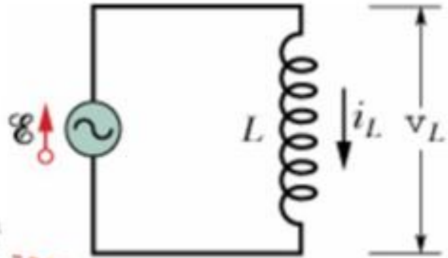


- 전압-전류 관계

① 위상: $\phi_C = -90^o$ $\longrightarrow$ **전류의 위상이 전압을 90° 앞서간다.**

② 진폭: $V_C = I_C X_C \longleftarrow X_C = \dfrac{1}{\omega_d C}$ (용량형 리액턴스)

[유도형 회로]



- 전압: $v_L = V_L \sin \omega_d t$
- 전류: 파라데이 유도법칙

$$v_L = L\frac{di_L}{dt}$$

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin\omega_d t$$

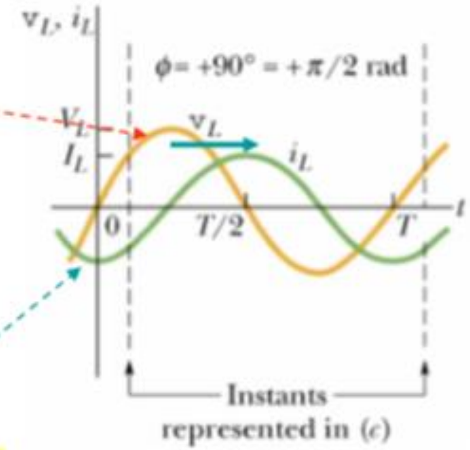
$$i_L = \int di_L = \frac{V_L}{L}\int \sin\omega_d t dt$$

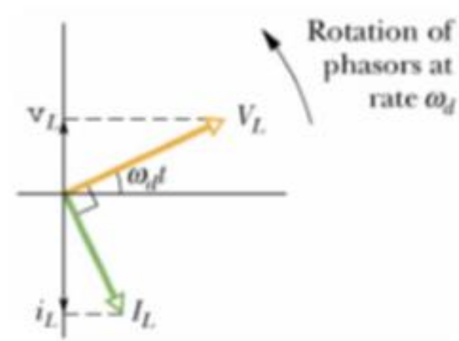
$$= -\left(\frac{V_L}{\omega_d L}\right)\cos\omega_d t$$

$$= \left(\frac{V_L}{\omega_d L}\right)\sin(\omega_d t - 90^o)$$

$$= I_L \sin(\omega_d t - \phi_L)$$

$$t = \frac{\phi_c}{\omega_d} = +\left(\frac{90^\circ}{\omega_d}\right)$$





- 전압-전류 관계

① 위상: $\phi_L = 90^o$ ⟶ 전류의 위상이 전압보다 90° 뒤처져 간다.

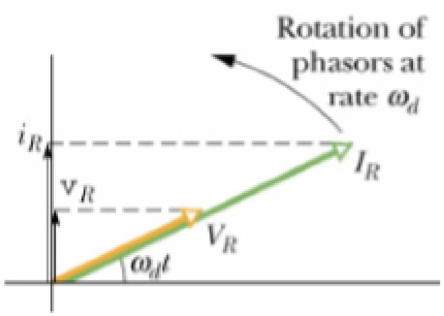
② 진폭: $V_L = I_L X_L$, $X_L \equiv \omega_d L$ (유도형 리액턴스)

(요약) 교류전압과 전류에 대한 R, C, L의 위상과 진폭 관계

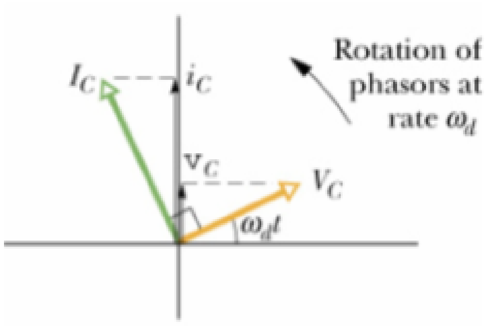
소자(기호)	저항 또는 리액턴스	전류의 위상
저항(R)	R	v_R과 같음
축전기(C)	$X_C = 1/\omega_d C$	v_C에 90^o 앞섬
유도코일(L)	$X_L = \omega_d L$	v_L에 90^o 뒤짐

위상자 (phasor)로 비교해 보면

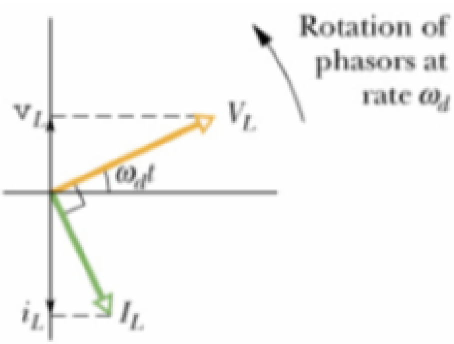
R



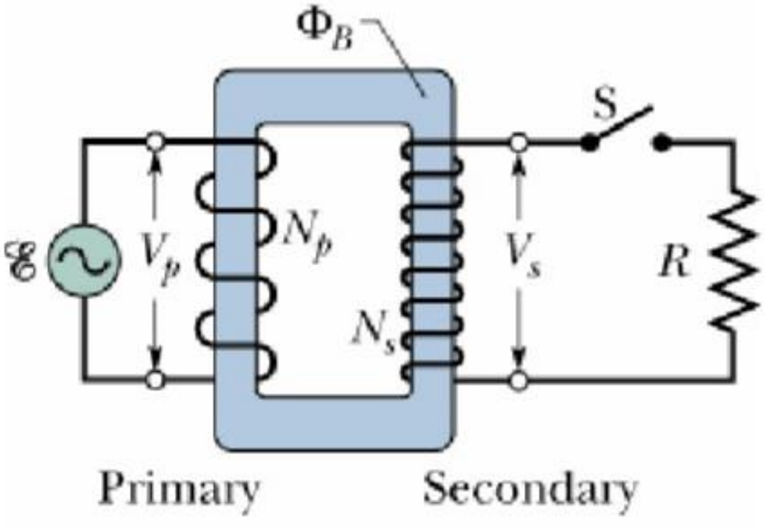
C



L



변압기 (Transformer)



왜 변압기가 필요한가?

$$P_{avg} = \varepsilon_{rms} I_{rms} \cos\phi = \varepsilon I \quad : (\cos\phi = 1 : \text{회로에 R만 있다고 가정})$$

$$I = \frac{P_{avg}}{\varepsilon}$$

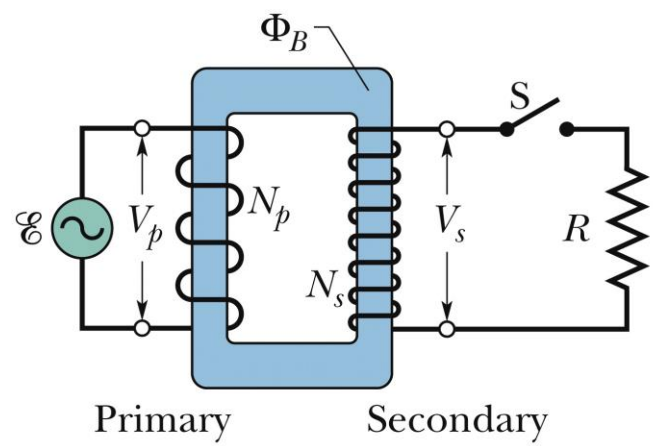
전송선에서의열에너지소모량 : $P_{avg} = I^2 R$

➡ **ε를 높여 I 를 낮출수록 전력전송에 유리하다.**

Chap. 31

A transformer (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p \frac{N_s}{N_p}$$



An ideal transformer (two coils wound on an iron core) in a basic trans- former circuit. An ac generator produces current in the coil at the left (the primary). The coil at the right (the secondary) is connected to the resistive load R when switch S is closed.

Energy Transfers. The rate at which the generator transfers energy to the primary is equal to I_pV_p. The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is I_sV_s. Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_pV_p = I_sV_s. \quad \Rightarrow \quad I_s = I_p \frac{N_p}{N_s}$$

The equivalent resistance of the secondary circuit, as seen by the genera

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R.$$