1. Vectors

1.2 The dot product

$u = (u_1, \ldots, u_n)$,

$v = (v_1, \ldots, v_n)$

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Thm. a) $v \cdot u = u \cdot v$

b. $u \cdot (v + w) = u \cdot v + u \cdot w$
(distritutive law)

c. $(cu) \cdot v = c(u \cdot v)$

d. $u \cdot u \geq 0$ and $(u \cdot u) = 0$ iff
$u = 0$

Def. $\|v\| = \sqrt{v \cdot v}$

Rem. $e_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ k-th position

Thm. $|u, v| \le \|u\| \, \|v\|$

Pf $(u+\alpha v, u+\alpha v)$

$= \|u\|^2 + 2(u, v)\alpha + \alpha^2 \|v\|^2 \ge 0$

$\Rightarrow D = (u, v)^2 - \|u\|^2 \|v\|^2 \le 0$

$\Rightarrow \|u\| \, \|v\| \ge |(u, v)|$

Thm. $\|u+v\| \le \|u\| + \|v\|$

Def. The distance $d(u, v)$ between vectors u and v in $\mathbb{R}^n$ is defined by

$d(u, v) = \|u - v\|$

ex. $u = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

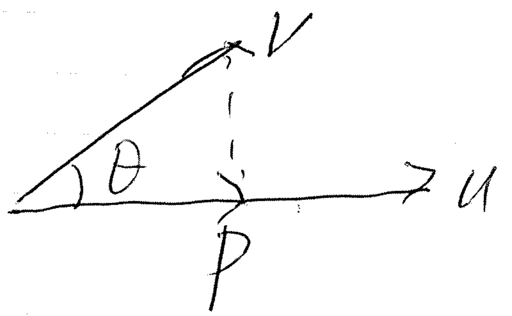
$d(u, v) = \sqrt{\sqrt{2}^2 + 1^2 + 1^2} = \sqrt{4} = 2$

Def. $\cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$

Def. u and v are said to be orthogonal if $(u, v) = 0$.

Thm. $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ iff u and v are orthogonal

- Projections



p : projection of v onto u

$$P = \|v\| \cos\theta \frac{\vec{u}}{\|u\|}$$

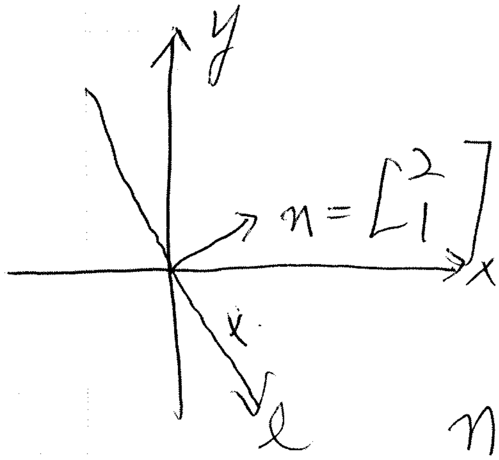
$$= \|v\| \frac{(u, v)}{\|u\| \|v\|} \frac{\vec{u}}{\|u\|}$$

$$= \frac{(u, v)\vec{u}}{\|u\|^2} = \frac{(u, v)}{(u, u)} \vec{u}$$

ex) (a) $v=\begin{bmatrix}-1\\3\end{bmatrix}$, $u=\begin{bmatrix}2\\1\end{bmatrix}$

$$\mathrm{proj}_u(v)=\frac{(u\cdot v)}{(u\cdot u)}u=\frac{1}{5}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2/5\\1/5\end{bmatrix}$$

1.3 Lines and Planes



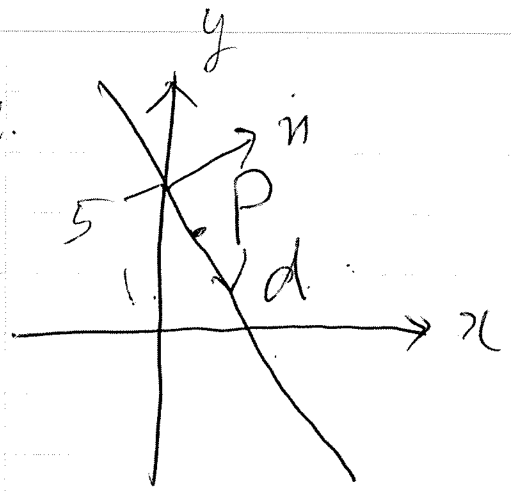
$n\cdot x=0$

n: normal vector to the line

or $x=t\vec{d}$, $\vec{d}=\begin{bmatrix}\cancel{0}\,1\\-2\end{bmatrix}$

d: direction vector.

ex.



$2x+y=5$

$$\vec{n}\cdot(x-p)=0 \Rightarrow n\cdot x = n\cdot p$$

Def. The normal form of the equation of a line l in $\mathbb{R}^2$ is

$n\cdot(x-p)=0$ or $n\cdot x = n\cdot p$

, p is a specific point on l

The general form of the equation of l is

$ax+by=c$, where $n=\begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for l.

Def. The vector form of the equation of a line ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$x = p + td$, where $p \in \ell$, $d \neq 0$ is a direction vector for ℓ.

ex. Find vector and parametric equations of the line in $\mathbb{R}^3$ through the point $P = (1, 2, -1)$ parallel to the vector $d = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

Sol.

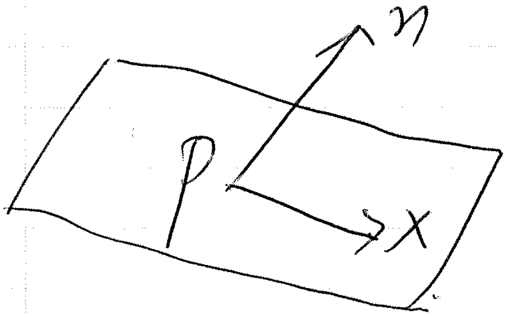
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

$x = 1 + 5t$

$y = 2 - t$

$z = -1 + 3t$

- Planes in $\mathbb{R}^3$



$(x-p) \cdot n = 0 \Rightarrow n \cdot x = n \cdot p.$

Def. The normal form of the equation of a plane P in $\mathbb{R}^3$ is

$n \cdot (x-p) = 0$

The general form is

$ax + by + cz = d, \quad n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

ex. Find the normal and general forms of the equation of the plane that contains $P=(6,0,1)$ and has normal vector $n=\begin{bmatrix}1\\2\\3\end{bmatrix}$

sol. $n \cdot x = n \cdot p$

$\Rightarrow x+2y+3z=9$

Def: The vector form of the equation of a plane P in $\mathbb{R}^3$ is

$x = p + su + tv$, where $p \in P$, u, v are direction vectors for P.

ex. Find the distance from the point $B=(1,0,2)$ to the line ℓ though the point $A=(3,1,1)$ with direction vector $d=\begin{bmatrix}-1\\1\\0\end{bmatrix}$

sol

$v = \cancel{B}\overrightarrow{AB} = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$

The length $= \| v - proj_d(v) \|$

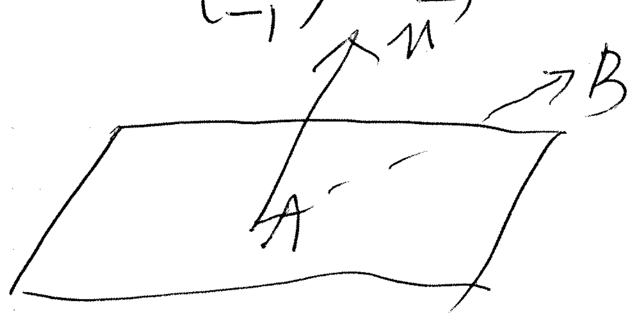
$proj_d(v) = \left(\frac{d \cdot v}{d \cdot d}\right) d = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix}$

$Ans = \left\| \begin{matrix}-3/2\\-3/2\\1\end{matrix} \right\| = \frac{1}{2}\sqrt{22}$

$\| \overline{AB} \| = \| v - p \|$

ex. Find the distance from $B=(1,0,2)$ to the plane $x+y-z=1$.

sol. $n=\begin{pmatrix}1\\1\\-1\end{pmatrix}$



$A=(1,0,0)$, $B=(1,0,2)$

$v=B-A=(0,0,2)$

$\text{length}=|\text{proj}_n v|=\left|\frac{(n.v)}{(n,n)}n\right|$

$=\left|\frac{-2}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}\right|=\frac{2}{3}\sqrt{3}$

◦ The Cross Product

Def. The cross product of

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is $u \times v$ defined by

$$\begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

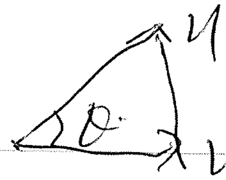
$$= i \times \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Rem. a) $(u \times v) . u = 0$

$(u \times v) . v = 0$

b) $\|u \times v\| = \|u\| \, \|v\| \sin\theta$

$A = \frac{1}{2} \|u \times v\|$

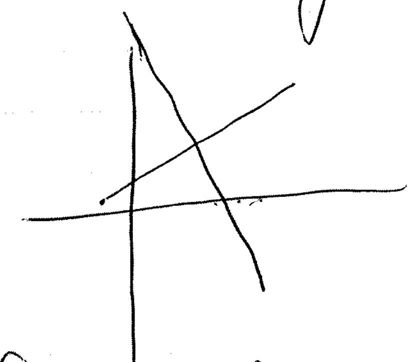


1.4 Code vectors and modular arithmetic

2. Systems of Linear equations

$2x + y = 8$

$x - 3y = -3$



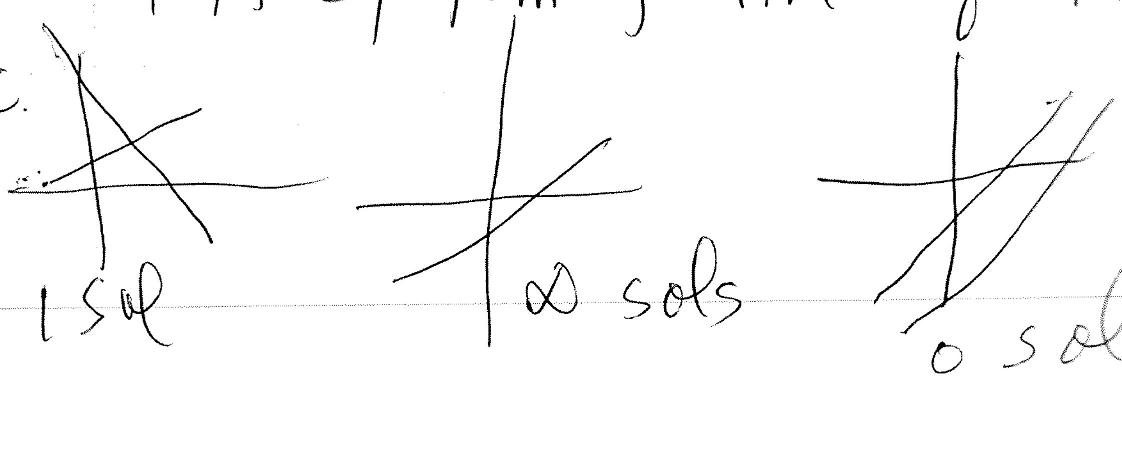
Def. $a_{11}x_1 + \cdots + a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$

: A system of linear equations

ex.



- A system of linear equations is called consistent if it has at least one solution. Otherwise it is inconsistent.
- A system of linear equations with real coefficients has either a) unique solution
  b) ∞ solutions
  c) no solution (inconsistent)
- Two linear systems are called equivalent if they have the same solution sets.

ex. $x - y = 1$ $\quad$ $x - y = 1$

$x + y = 3$ $\quad$ $y = 1$

ex. Solve

$$x - y - z = 2$$
$$y + 3z = 5$$
$$5z = 10$$

$z = 2$, $y = 5 - 3z = -1$,

$x = 2 + (-1) + 2 = 3$

$\Rightarrow$ back substitution.

ex. Solve

$$x - y - z = 2$$
$$3x - 3y + 2z = 16$$
$$2x - y + z = 9.$$

$$\begin{matrix} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{matrix}$$

$$\Rightarrow \begin{matrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{matrix}$$

$$\Rightarrow \begin{matrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{matrix}$$

$$\Rightarrow \begin{matrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{matrix}$$

$\Rightarrow$ Easier to solve.

2.2 Direct methods for solving linear systems

- Augmented matrix

$$2x + y - z = 3$$
$$x + 5z = 1$$
$$-x + 3y - 2z = 0$$

coefficient matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & \cancel{0}5 \\ -1 & 3 & -2 \end{bmatrix}$

Augmented matrix $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{bmatrix}$

Def. A matrix is in row echelon form if

1. Any rows consisting entirely of zeros are at the bottom

2. In each nonzero row, the first nonzero entries is in a column to the left of any ledding entries below it.

ex.

$$\begin{matrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$$

ex.

$$2x_1 + 4x_2 = 1$$

$$-x_2 = 2$$

$$\Rightarrow x_2 = -2, \quad x_1 = \frac{1}{2}(1 - 4x_2) = 9/2$$

Def. Elementary row operations

1. Interchange two rows
2. Multiply a row by a nonzero constant.
2. Add a multiple of a row to another row

ex.
$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

$$\begin{array}{ccccc} -1 & 1 & 3 & 6 & 5 \\ \hline 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array}$$

$$\begin{matrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{matrix}$$

$$\rightarrow \begin{matrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{matrix}$$

$$\rightarrow \begin{matrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{matrix}$$

$$\to \begin{array}{ccccc} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 0 & 0 & 24 \end{array}$$

Rem. i) The row echelon form of a matrix is not unique.

Def. Matrices A and B are row equivalent if there is a series of elementary row operations that converts A into B.

Thm Matrices A and B are row equivalent if and only if they can be reduced to same row echelon form

- Gaussian elimination.

1. Write the augmented matrix of the linear equations
2. Reduce to row echelon form
3. Use back substitution for the solutions.

ex. $2x_2 + 3x_3 = 8$

$2x_1 + 3x_2 + x_3 = 5$

$x_1 - x_2 - 2x_3 = -5$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

$$\rightarrow \begin{array}{cccc} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}$$

$$\rightarrow \begin{array}{cccc} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}$$

$$\rightarrow \begin{array}{cccc} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}$$

$$\rightarrow \begin{matrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{matrix}$$

$$\rightarrow \begin{aligned} x_1 - x_2 - 2x_3 &= -5 \\ x_2 + x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

$\rightarrow x_3 = 2, \quad x_2 = 3 - x_3 = 1$

$x_1 = -5 + x_2 + 2x_3 = 0$

ex.

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

ibis

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\rightarrow \begin{array}{ccccc} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}$$

$$\rightarrow \begin{array}{ccccc} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$w - x - y + 2z = 1$$

$$y - z = 1$$

w, y; leading variables

x, z; free variables

$$y = z + 1$$

$$w = 1 + x + y - 2z$$

$$= 2 + x - z$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + x - z \\ x \\ z + 1 \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Def. The rank of a matrix is # of nonzero rows in its row echelon form

Thm. number of free variables $= n - \text{rank}(A)$

∘ Gauss-Jordan elimination

Def. A matrix is in reduced row echelon form if

1. It is in row echelon form
2. The leading entry in each nonzero row is 1
3. Each column containing a leading 1 has zeros in all other rows.

ex

$$\begin{matrix} 1 & 2 & 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Rem. The reduced row echelon form of a matrix is unique.

∘ Gauss-Jordan elimination.

1. Form the augmented matrix
2. Use elementary row operations to reduce the augmented matrix to reduced row echelon form

3. Solve the resulting system

ex. $\begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{matrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & & 0 \end{matrix}$$

$w = 2 + x - z$

$y = 1 + z$

ex. Find the line of intersection of the planes

$x + 2y - z = 3, \; 2x + 3y + z = 1$

ex. $x+2y-z=3$
$2x+3y+z=1$

$$\begin{matrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{matrix} \to \begin{matrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & -5 \end{matrix}$$

$$\to \begin{matrix} 1 & 0 & 5 & -7 \\ 0 & 1 & -3 & 5 \end{matrix}$$

$\to x+5z=-7$
$y-3z=5$

$x=-7-5t,\ y=5+3t,\ z=t$

$$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

ex. Let $p=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $q=\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $u=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$v=\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$. Determine whether the lines $x=p+tu$ and $x=q+tv$ intersect.

$\rightarrow su - tv = q - p$

$$\begin{aligned} s - 3t &= -1 \\ s + t &= 2 \\ s + t &= 2 \end{aligned} \Rightarrow s = \frac{5}{4},\ t = \frac{3}{4}$$

$$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 5/4 \\ 1/4 \end{bmatrix}$$

- Homogeneous system

$\rightarrow$ the constant terms are 0.

~~Def. A system of linear~~ equations

Thm. If $[A|0]$ is homogeneous,

$A: m \times n$, $m < n$,

it has ∞ sols.

2.3 Spanning Sets and Linear Independence ex. (a) is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$?

(b) Is $\begin{bmatrix}2\\3\\4\end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$?

(a) $x\begin{bmatrix}1\\0\\3\end{bmatrix}+y\begin{bmatrix}-1\\1\\-3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$$\begin{bmatrix}1 & -1 & 1\\0 & 1 & 2\\3 & -3 & 3\end{bmatrix}\to\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

$y=2,\ x=3.$

(b) $$\left[\begin{array}{cc|c}1 & -1 & 2\\0 & 1 & 3\\3 & -3 & 4\end{array}\right]\to\begin{bmatrix}1 & 0 & 5\\0 & 1 & 3\\0 & 0 & -2\end{bmatrix}$$

$\Rightarrow$ inconsistent

$\to$ Not a linear combination.

Thm: A system of linear eqns with augmented matrix $[A|b]$ is consistent iff b is a linear combination of the columns of A.

(a) $x - y = 1$
$x + y = 3$ $\Rightarrow x = 2, y = 1$

$$2\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

(c) $x - y = 1$
$x - y = 3$ has no sol

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} \neq \begin{bmatrix}1\\3\end{bmatrix}, \forall x, y.$$

Def. If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of linear combinations of $v_1, \ldots, v_k$ is called the span of S, and is denoted by span$\{v_1, \ldots, v_k\}$ or span$\{S\}$

ex. Show that span$\left(\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right) = \mathbb{R}^2$

sol. $\left[\begin{array}{cc|c}2 & 1 & a\\-1 & 3 & b\end{array}\right] \to \left[\begin{array}{cc|c}-1 & 3 & b\\2 & 1 & a\end{array}\right]$

$\to \begin{bmatrix}-1 & 3 & b\\0 & 7 & a+2b\end{bmatrix} \to y = \frac{a+2b}{7}, \ x = \frac{3a-b}{7}$

$\left(\frac{3a-b}{7}\right)\begin{bmatrix}2\\-1\end{bmatrix} + \frac{a+2b}{7}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

Rem. $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}5\\7\end{bmatrix}\right)$

If $x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$, then

$x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}1\\3\end{bmatrix} + 0\begin{bmatrix}5\\7\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

Any set of vectors containing a spanning set will also be a spanning set.

ex. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$= x e_1 + y e_2 + z e_3$

$\rightarrow \mathbb{R}^3 = \text{span}(e_1, e_2, e_3)$

ex. Find the span of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \rightarrow \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & z-3x \end{bmatrix}$$

$\rightarrow z - 3x = 0,$

- Linear Independence

Def. A set of vectors $v_1, \ldots, v_k$ is linearly dependent if there are scalars $c_1, \ldots, c_k$,

not all ~~zz~~ zero, such that
$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$.
Otherwise it is called linearly independent

Rem. a) In $\mathbb{R}^2$, u and v are linearly dependent if $u = cv$ for some c.

Thm. $v_1, \ldots, v_m$ are linearly ~~in~~dependent if at least ~~of~~ one of the vectors can be expressed as a linear combinati of the others.

ex. (a) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are lin indep.

(b) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{array}{cccc}1&0&1&0\\0&1&-1&0\\0&1&1&0\end{array} \rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&2&0\end{array}\right)$$

$\rightarrow c_1 = c_2 = c_3 = 0$

$\rightarrow$ lin indep.

(c) $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\;\; \| \qquad\qquad \| \qquad\qquad \|$

$\;\; x \qquad\qquad y \qquad\qquad z$

$\rightarrow x = -y - z$

$\rightarrow$ lin dep.

(d) $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\4\\2\end{bmatrix}$

$c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\left[\begin{bmatrix}1 & 1 & 1\\2 & 1 & 4\\0 & -1 & 2\end{bmatrix} \begin{matrix}0\\0\\0\end{matrix}\right]$

$\to \begin{bmatrix}1 & 1 & 1 & 0\\0 & -1 & 2 & 0\end{bmatrix}$
$0 \ -1 \ 2 \ 0$
$\to \begin{bmatrix}1 & 1 & 1 & 0\\0 & -1 & 2 & 0\\0 & 0 & 0 & 0\end{bmatrix}$

$\to c_2 = 2c_3, \ c_1 = -c_2 - c_3 = -3c_3$

$\to$ lin dep.

Thm. $v_1, \ldots, v_m$ are linerly dep. iff $[A|b]$ has a nontrivial solution, $A = [v_1|v_2|\ldots\ldots v_m]$.

Thm. Let $v_1, \ldots, v_m$ be row vectors in $\mathbb{R}^n$ and let A be the $m \times n$ matrix $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$. Then $v_1, v_2, \ldots, v_m$ are lin dep iff $\operatorname{rank}(A) < m$

ex. $[1, 2, 0]$, $[1, 1, -1]$, $[1, 4, 2]$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3'' = R_3' + 2R_2'} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$0 = R_3'' = R_3' + 2R_2' = (R_3 - R_1) + 2(R_2 - R_1)$$

$$= -3R_1 + 2R_2 + R_3$$

Thm. A set of m vectors in $\mathbb{R}^n$ is lin dep if $m > n$.