6 장 연습문제 풀이

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6.4 삼각치환법과 쌍곡치환법

피적분 함수가

$$\sqrt{a^2-u^2},\quad \sqrt{a^2+u^2},\quad \sqrt{u^2-a^2},\quad a^2+u^2,\quad a^2-u^2$$

를 포함하는 부정적분.

1. $\sqrt{a^2-u^2}$ 를 포함하는 경우.

   $u = a\sin\theta$ 로치환

   $du = a\cos\theta d\theta$

   $a^2 - a^2\sin^2\theta = a^2\cos^2\theta$

   기본예제.

   $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

2. $\sqrt{a^2+u^2}$ 를 포함하는 경우.

   $u = a\sinh x$ 로치환

   $du = a\cosh x dx$

   $a^2 + a^2\sinh^2 x = a^2\cosh^2 x$

   기본예제.

   $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} + C$$

3. $\sqrt{u^2-a^2}$ 를 포함하는 경우.

   $u = a\cosh x$ 로치환

   $du = a\sinh x dx$

   $a^2\cosh^2 x - a^2 = \sinh^2 x$

   기본예제.

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a} + C$$

4. a^2+u^2 를 포함하는 경우.

   $u = a\tan\theta$ 로 치환

   $du = a\sec^2\theta d\theta$

   $a^2 + a^2\tan^2\theta = a^2\sec^2\theta$

   기본예제.

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

5. a^2-u^2 를 포함하는 경우.

   $u = a\tanh x$ 로 치환

   $du = a \operatorname{sech}^2 x dx$

   $a^2 - a^2\tanh^2 x = a^2 \operatorname{sech}^2 x$

   기본예제.

$$\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C$$

6. $u\sqrt{u^2-a^2}$, $u\sqrt{a^2-u^2}$ 또는 $u\sqrt{u^2+a^2}$ 를 포함하는 경우.

   각각 $u = a\sec\theta$, $u = a \operatorname{sech} x$, $u = a \operatorname{csch} x$ 로 치환한다.

   기본예제.

$$\int \frac{du}{u\sqrt{u^2-a^2}}, \quad \int \frac{du}{u\sqrt{a^2-u^2}}, \quad \int \frac{du}{u\sqrt{u^2+a^2}}$$

연습문제 풀이

9.

$$\int \frac{1}{\sqrt{21+12x-9x^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{(\frac{5}{3})^2-(x-\frac{2}{3})^2}}dx$$

$$\left\{\begin{array}{c} x-\frac{2}{3}=\frac{5}{3}\sin\theta \text{ 치환} \\ dx=\frac{5}{3}\cos\theta d\theta \\ (\frac{5}{3})^2-(x-\frac{2}{3})^2=(\frac{5}{3})^2\cos^2\theta \end{array}\right\} \quad = \frac{1}{3}\int \frac{\frac{5}{3}\cos\theta}{\frac{5}{3}\cos\theta}d\theta$$

$$= \frac{1}{3}\theta + C$$

$$= \frac{1}{3}\sin^{-1}(\frac{3}{5}(x-\frac{2}{3})) + C$$

$$= \frac{1}{3}\sin^{-1}(\frac{3}{5}x-\frac{2}{5}) + C$$

10.

$$\int \frac{\sinh x}{\sqrt{4-\cosh^2 x}}dx \underset{\cosh x = t}{=} \int \frac{dt}{\sqrt{4-t^2}}$$

$$\underset{t=2\sin\theta}{=} \sin^{-1}(\frac{t}{2}) + C$$

$$= \sin^{-1}(\frac{\cosh x}{2}) + C$$

11.

$$\int \frac{1}{x^2+4x+5}dx = \int \frac{1}{1+(x+2)^2}dx$$

$$= \tan^{-1}(x+2) + C$$

12.

$$\int \frac{\cos x}{1+\sin^2 x}dx \underset{\sin x = t}{=} \int \frac{dt}{1+t^2}dt$$

$$= \tan^{-1} t + C$$

$$= \tan^{-1}(\sin x) + C$$

13.

$$\int \frac{1}{(x-2)\sqrt{4x^2-8x+40}}dx = \frac{1}{2}\int \frac{dx}{(x-1)\sqrt{9+(x-1)^2}}$$

$$\begin{Bmatrix} x-1 = 3\text{csch } t \text{ 치환} \\ dx = -3\text{csch } t\coth t dt \\ 9+(x-1)^2 = 9\coth^2 t \end{Bmatrix} \quad = \frac{1}{2}\int \frac{-3\text{csch } t\coth t}{9\text{csch } t\coth t}dt$$

$$= -\frac{1}{6}t + C$$

$$= -\frac{1}{6}\text{csch}^{-1}(\frac{x-1}{3}) + C$$

14.

$$\int \frac{1}{\sqrt{e^{2x}-1}}dx \underset{e^x=t}{=} \int \frac{1}{t\sqrt{t^2-1}}dt$$

$$\begin{Bmatrix} t = \sec\theta \text{ 치환} \\ dt = \sec\theta\tan\theta d\theta \\ t^2-1 = \tan^2\theta \end{Bmatrix} \quad = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta$$

$$= \theta + C$$

$$= \sec^{-1} e^x + C$$

15.

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx \underset{e^x=t}{=} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sin^{-1} t + C$$

$$= \sin^{-1} e^x + C$$

16.

$$\int \frac{1}{\sqrt{4x-x^2}}dx = \int \frac{1}{\sqrt{4-(x-2)^2}}dx$$

$$\begin{Bmatrix} x-2 = 2\sin\theta \text{ 치환} \\ dx = 2\cos\theta d\theta \\ 4-(x-2)^2 = 4\cos^2\theta \end{Bmatrix} \quad = \int \frac{2\cos\theta}{2\cos\theta}d\theta$$

$$= \theta + C$$

$$= \sin^{-1}(\frac{x-2}{2}) + C$$

17.

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}dx &= \frac{1}{3}\int \frac{1}{\sqrt{(\frac{2}{3})^2-x^2}}dx \\ &= \frac{1}{3}\sin^{-1}(\frac{3x}{2})+C\end{aligned}$$

18.

$$\begin{aligned}\int \frac{1}{\sqrt{9-4x^2}}dx &= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{3}{2})^2-x^2}}dx \\ &= \frac{1}{2}\sin^{-1}(\frac{2x}{3})+C\end{aligned}$$

23.

$$\int \sqrt{\sin^2 x+5}\cdot\cos x dx \underset{\sin x=u}{=} \int \sqrt{u^2+5}\ du$$

$$\left\{\begin{matrix} u=\sqrt{5}\sinh t \text{ 치환} \\ du=\sqrt{5}\cosh t dt \\ u^2+5=5\cosh^2 t\end{matrix}\right\} \quad = \int \sqrt{5}\cosh^2 t dt$$

$$\begin{aligned}&= 5\int \frac{1+\cosh 2t}{2}dt \\ &= 5\left(\frac{t}{2}+\frac{1}{4}\sinh 2t\right)+C \\ &= \frac{5}{2}(t+\sinh t\cosh t)+C \\ &= \frac{5}{2}(t+\sinh t\sqrt{1+\sinh^2 t})+C \\ &= \frac{5}{2}\left(\sinh^{-1}(\frac{u}{\sqrt{5}})+\frac{u}{\sqrt{5}}\sqrt{1+\frac{u^2}{5}}\right)+C \\ &= \frac{5}{2}\left(\sinh^{-1}(\frac{\sin x}{\sqrt{5}})+\frac{\sin x}{\sqrt{5}}\sqrt{1+\frac{\sin^2 x}{5}}\right)+C\end{aligned}$$

31. $\sqrt{y-1}=t$ 로 치환하면

$$\frac{1}{2\sqrt{y-1}}dy=dt, \quad y-1=t^2$$

이므로

$$\int \frac{1}{y\sqrt{y-1}}dy = \int \frac{2}{1+t^2}dt = 2\tan^{-1} t + C = 2\tan^{-1}\sqrt{y-1} + C$$

33.

$$\begin{aligned}\int x^2 \text{sech}\frac{1}{3}x^3 dx &\underset{\frac{1}{3}x^3=t}{=} \int \text{sech } t\ dt \\ &= \int \frac{2}{e^t+e^{-t}}dt \\ &= \int \frac{2e^t}{1+e^{2t}}dt \\ &\underset{e^t=u}{=} \int \frac{2}{1+u^2}du \\ &= 2\tan^{-1} u + C \\ &= 2\tan^{-1} e^t + C \\ &= 2\tan^{-1} e^{\frac{1}{3}x^3} + C\end{aligned}$$

41. $x = 2\sin\theta$ 로 치환하면

$$\sin\theta = \frac{x}{2}, \quad dx = 2\cos\theta d\theta, \quad 4 - x^2 = 4 - \sin^2\theta = 4\cos^2\theta$$

이므로

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}dx &= \int \frac{2\cos\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}d\theta \\ &= \frac{1}{4}\int \csc^2\theta\ d\theta \\ &= -\frac{1}{4}\cot\theta + C \\ &= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C \\ &= -\frac{1}{4}\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} + C \\ &= -\frac{1}{4}\frac{\sqrt{1-\frac{x^2}{4}}}{\frac{x}{2}} + C = -\frac{\sqrt{4-x^2}}{4x} + C\end{aligned}$$

47. $x = a\text{csch } t$ 로 치환하면

$$\text{csch } t = \frac{x}{a},\ \ dx = -a\text{csch } t \coth t dt,\ \ x^2 + a^2 = a^2\text{csch}^2 t + a^2 = a^2\coth^2 t$$

이므로

$$\begin{aligned}
\int \frac{1}{x^3\sqrt{x^2+a^2}}dx &= \int \frac{-a\text{csch } t \coth t}{a^3\text{csch}^3 t \cdot a\coth t}dt \\
&= -\frac{1}{a^3}\int \frac{1}{\text{csch}^2 t}dt \\
&= -\frac{1}{a^3}\int \sinh^2 t \ dt \\
&= -\frac{1}{a^3}\int \frac{\cosh 2t - 1}{2}dt \\
&= -\frac{1}{2a^3}\left(\frac{1}{2}\sinh 2t - t\right) + C \\
&= -\frac{1}{2a^3}(\sinh t \cosh t - t) + C \\
&= -\frac{1}{2a^3}(\sinh t\sqrt{1+\sinh^t} - t) + C \\
&= -\frac{1}{2a^3}\left(\frac{a}{x}\sqrt{1+(\frac{a}{x})^2} - \text{csch}^{-1}\frac{x}{a}\right) + C \\
&= -\frac{1}{2a^3}\left(\frac{a\sqrt{x^2+a^2}}{x^2} - \text{csch}^{-1}\frac{x}{a}\right) + C
\end{aligned}$$

6.5 부분적분법

곱의 미분

$$(uv)' = u'v + uv'$$

으로부터

$$\int (uv)' = \int u'v + \int uv'$$

이므로

$$\int uv' = uv - \int u'v$$

연습문제 풀이

1. $u = \ln x$, $v' = x^2$ 이라하면

$$\begin{array}{cc} u = \ln x & v' = x^2 \\ u' = \frac{1}{x} & v = \frac{1}{3}x^3 \end{array}$$

이므로

$$\begin{aligned} \int x^2 \ln x \; dx &= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 \; dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \frac{1}{3}x^3 + C \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \end{aligned}$$

3. $\int x \cos x \; dx$

$u = x$, $v' = \cos x$ 라하면

$$\begin{array}{cc} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array}$$

$$\int x \cos x \; dx = x \sin x - \int \sin x \; dx = x \sin x + \cos x + C$$

8. $\int x^2\sqrt{x-1} \; dx$ 이문제는 좀 어려운데 (다른 풀이가 있을수도...) 우선

$$\begin{aligned} \int x\sqrt{x-1} \; dx &\underset{x-1=t}{=} \int (1+t)\sqrt{t} \; dt \\ &= \frac{2}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}} + C \\ &= \frac{2}{3}(x-1)^{\frac{3}{2}} + \frac{2}{5}(x-1)^{\frac{5}{2}} + C \end{aligned}$$

이므로 $u = x$, $v' = x\sqrt{x-1}$ 이라하면

$$\begin{array}{cc} u = x & v' = x\sqrt{x-1} \\ u' = 1 & v = \frac{2}{3}(x-1)^{\frac{3}{2}} + \frac{2}{5}(x-1)^{\frac{5}{2}} \end{array}$$

이고

$$\begin{aligned}
&\int x^2\sqrt{x-1}\ dx \\
&\quad = \frac{2}{3}x(x-1)^{\frac{3}{2}} + \frac{2}{5}x(x-1)^{\frac{5}{2}} - \int \left(\frac{2}{3}(x-1)^{\frac{3}{2}} + \frac{2}{5}(x-1)^{\frac{5}{2}}\right) dx \\
&\quad = \frac{2}{3}x(x-1)^{\frac{3}{2}} + \frac{2}{5}x(x-1)^{\frac{5}{2}} - \frac{2}{3}\cdot\frac{2}{5}(x-1)^{\frac{5}{2}} - \frac{2}{5}\cdot\frac{2}{7}(x-1)^{\frac{7}{2}} + C \\
&\quad = \frac{2}{3}x(x-1)^{\frac{3}{2}} + \frac{2}{5}x(x-1)^{\frac{5}{2}} - \frac{4}{15}(x-1)^{\frac{5}{2}} - \frac{4}{35}(x-1)^{\frac{7}{2}} + C
\end{aligned}$$

11. $\int \sin(\ln x)\ dx.$

$u = \sin(\ln x)$, $v' = 1$ 이라하면

$$\begin{matrix} u = \sin(\ln x) & v' = 1 \\ u' = \frac{\cos(\ln x)}{x} & v = x \end{matrix}$$

이므로

$$\begin{aligned}
\int \sin(\ln x)\ dx &= x\sin(\ln x) - \int \cos(\ln x)\ dx, \quad \begin{pmatrix} u = \cos(\ln x) & v' = 1 \\ u' = -\frac{\sin(\ln x)}{x} & v = x \end{pmatrix} \\
&= x\sin(\ln x) - \left\{ x\cos(\ln x) + \int \sin(\ln x)\ dx \right\} \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ dx
\end{aligned}$$

따라서

$$2\int \sin(\ln x)\ dx = x\sin(\ln x) - x\cos(\ln x) + C$$

$$\Longrightarrow \int \sin(\ln x)\ dx = \frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right) + C$$

27. $\int \tanh^{-1} x\ dx.$

$$\begin{array}{ll} u = \tanh^{-1} x & v' = 1 \\ u' = \frac{1}{1-x^2} & v = x \end{array}$$

$$\begin{aligned}
\int \tanh^{-1} x\ dx &= x \tanh^{-1} x - \int \frac{x}{1-x^2}\ dx \\
&= x \tanh^{-1} x + \frac{1}{2} \int \frac{-2x}{1-x^2}\ dx \\
&= x \tanh^{-1} x + \frac{1}{2} \ln |1-x^2| + C
\end{aligned}$$

30. $\int x\ \text{sech}^{-1} x\ dx$

$$\begin{array}{ll} u = \text{sech}^{-1} x & v' = x \\ u' = \frac{-1}{x\sqrt{1-x^2}} & v = \frac{1}{2}x^2 \end{array}$$

$$\begin{aligned}
\int x\ \text{sech}^{-1} x\ dx &= \frac{1}{2}x^2\ \text{sech}^{-1} x + \frac{1}{2} \int \frac{x}{\sqrt{1-x^2}}\ dx \\
&= \frac{1}{2}x^2\ \text{sech}^{-1} x - \frac{1}{2} \int \frac{-\frac{1}{2}(2x)}{\sqrt{1-x^2}}\ dx \\
&= \frac{1}{2}x^2\ \text{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C
\end{aligned}$$

31. $\int \sin^3 x \cos^3 x\ dx$

$$\begin{aligned}
\int \sin^3 x \cos^3 x\ dx &= \int (\frac{1}{2} \sin 2x)^3\ dx \\
&= \frac{1}{8} \int \sin^3 2x\ dx \\
&= \frac{1}{8} \int \sin 2x \cdot \sin^2 2x\ dx \\
&= \frac{1}{8} \int \sin 2x (1 - \cos^2 2x)\ dx \\
&= \frac{1}{8} \int \sin 2x\ dx - \frac{1}{8} \int \sin 2x \cdot \cos^2 2x\ dx \\
&= -\frac{1}{16} \cos 2x + \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{3} \cos^3 2x + C \\
&= -\frac{1}{16} \cos 2x + \frac{1}{48} \cos^3 2x + C
\end{aligned}$$