Chapter 1. 1st order ODE

1. Basic concepts: Modeling
2. Separable ODE
3. Integrating factors
4. Linear ODE
5. Orthogonal trajectories

Introduction: Modeling

- Solving scientific problems

❖ The typical steps in solving scientific problems:

Step 1. Modeling: set up a mathematical expression for a physical phenomenon

Step 2. Solving: solve the mathematical equations

Step 3. Physical interpretation of differential equations and their applications

- Types of equations
  - Algebraic equation: $ax^2 + bx + c = 0$
  - Differential equation: $\frac{dy}{dx} + ay = f(x)$
  - Difference equation: $f_{n+2} = f_{n+1} + f_n$

Introduction: Modeling

- Falling stone, $y'' = g$

  where y is the displacement and $g = 9.8\ [{}^{m}/_{s^2}]$ is constant (중력 가속도).

- Parachute, $mv' = mg - bv^2$

  where v is the velocity.

- Vibrating mass on a spring, $my'' + ky = 0$

  where y is the displacement, m is mass, and $k > 0$ is the spring constant.

- RLC series circuit, $Li'' + Ri' + \frac{1}{C}i = v$

  where i is the current and v is the source.

- Pendulum, $L\theta'' + g \cdot \sin\theta = 0$

  - where θ is the angle.

Types of Differential Eqns.

- **Differential Equation** : An equation containing derivatives of an unknown function

  - Ordinary Differential Equation (ODE, 상미분 방정식): An equation that contains one or several derivatives (도함수) of an unknown function of one independent variable

    † $\frac{dy}{dx} = \cos x$, $\frac{d^2y}{dx^2} + 9y = e^{-2x}$, $y'y^{(3)} - \frac{3}{2}(y')^2 = 0$

  - Partial Differential Equation (PDE, 편미분 방정식): An equation involving partial derivatives of an unknown function of two or more variables

    † $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (2D Laplace equation), $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (1D wave equation)

Basic Notations

- Order, 계 : The highest derivative of the unknown function

  ① $\frac{dy}{dx} = \cos x$ … 1st-order (1계)

  ② $\frac{d^2y}{dx^2} + 9y = e^{-2x}$ … 2nd-order

  ③ $y'y^{(3)} - \frac{3}{2}(y')^2 = 0$ … 3rd-order

- First-order ODE (1계 상미분 방정식): Equation contain only the 1st derivative y' and may contain y and any given functions of x.

  ① Explicit form: $\frac{dy}{dx} = f(x, y)$

  ② Implicit (음함수) form: $F(x, y, y') = 0$

Basic Notations

- **Solution** : Functions that make the equation hold true
  - General solution (일반해): a solution containing an arbitrary constant
  - Particular solution (특수해): a solution that we choose a specific constant
  - Singular Solution(Problem 16, 특이해): an additional solution that cannot be obtained from the general solution

- Example: $(y')^2 - xy' + y = 0$
  - General solution: $y = cx - c^2$, where c is a constant.
  - Particular solution: $y = 2x - 4$
  - Singular solution: $y = \frac{x^2}{4}$

Initial value problem

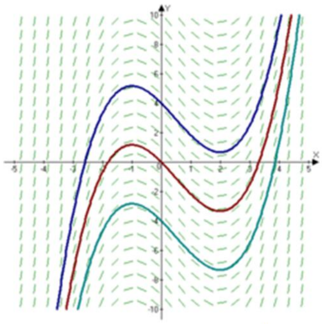
- An ordinary differential equation together with specified value of the unknown function at a given point in the domain of the solution.
  - $y' = f(x, y)$, $y(x_0) = y_0$
- Example: $\frac{dy}{dx} = 3y$, $y(0) = 5.7$
  1. Find the general solution: $y(x) = ce^{3x}$, where c is a constant.
  2. Apply the initial condition: $y(0) = c = 5.7$, so that the particular solution is $y(x) = 5.7e^{3x}$

Initial value problem

- Example Given an amount of a radioactive substance, say 0.5 g(gram), find the amount present at any later time
  - Physical information: Experiments show that at each instant a radioactive substance decomposes at a rate proportional to the amount present.
  - Step 1: Setting up a mathematical model(a differential equation) of the physical process.
  - $\frac{dy}{dt} = ky$, k constant, with $y(0) = 0.5$
  - Step 2: Mathematical solution
    - General solution: $y(t) = ce^{kt}$
    - Particular solution: $y(0) = c = 0.5$, so that $y(t) = 0.5e^{kt}$, $t \geq 0$
    - Check: $\frac{dy}{dt} = 0.5ke^{kt} = ky(t)$
  - Step 3: Interpretation: $\lim_{t \to \infty} y(t) = 0$

Direction Field

- Geometric meaning of $y' = f(x,y)$
- Direction field: The graph includes pairs of grid points and line segments that the line segment at grid point coincides with the tangent line to the solution.
  - We can understand the solution without actually solving the ODE.
  - The method shows the whole family of solutions and their typical properties.



The direction (slope) field of $\frac{dy}{dx} = x^2 - x - 2$, with the blue, red, and turquoise lines being $\frac{x^3}{3} - \frac{x^2}{2} - 2x + 4$, $\frac{x^3}{3} - \frac{x^2}{2} - 2x$, and $\frac{x^3}{3} - \frac{x^2}{2} - 2x - 4$, respectively

Separable ODE

- Separable ODE, 변수 분리형, 혹은 분리 상미방
  - A differential equation to be separable if the ODE has the following form:
  - $g(y)\frac{dy}{dx} = f(x)$
    - † $g(y)dy = f(x)dx \Rightarrow \int g(y)dy = \int f(x)dx + c$, c constant
- Example 1.3.1 $y' = 1 + y^2$
  - $\frac{dy}{1+y^2} = dx \Rightarrow \int \frac{dy}{1+y^2} = \int dx + c$
  - $\tan^{-1} y = x + c$ or $y = \tan(x + c)$

Separable ODE

- Example 1.3.2 $y' = (x+1)e^{-x}y^2$

  $\frac{dy}{y^2} = (x+1)e^{-x}dx \Rightarrow \int \frac{dy}{y^2} = \int (x+1)e^{-x}dx + c$

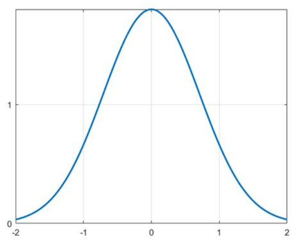
  $-\frac{1}{y} = -(x+2)e^{-x} + c$

  $\frac{d}{dx}(x^n e^{ax}) = nx^{n-1}e^{ax} + ax^n e^{ax}$
  $\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$

- Example 1.3.3 $y' = -2xy,\ y(0) = 1.8$

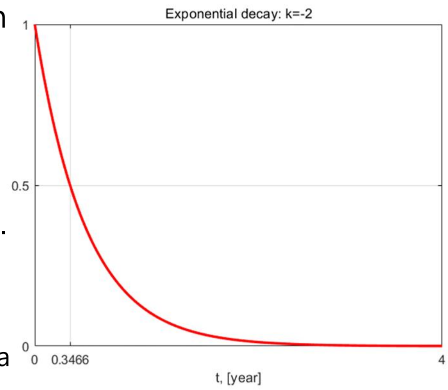
  $\frac{dy}{y} = -2xdx \Rightarrow \ln y = -x^2 + c \Rightarrow y = e^{-x^2+c} = ke^{-x^2}$

  $y(0) = k = 1.8$ and $y = 1.8e^{-x^2}$



Separable ODE

- Example 1.3.4 $y' = ky,\ y(0) = y_0$ (radiocarbon dating)
  - $\frac{dy}{y} = kdx \Rightarrow \ln y = kx + c \Rightarrow y = y_0 e^{kx}$
  - $y' = ky$: 도함수와 원래의 함수 모습이 같다.
    - 이러한 특성을 갖는 유일한 함수는 지수함수이다.
    - Guess $y(x) = e^{ax}$: $ae^{ax} = ke^{ax} \Rightarrow a = k$, so that $y(x) = e^{ax}$ is a solution.
    - If $y(x) = e^{ax}$ is a solution, $y(x) = be^{ax}$ is also a solution.



Separable ODE

- Example 1.3.5 (Mixing problem: pollutants in lake and drugs in organs)
  - A chemical tank contains 1000 [gal] of water in which initially 100 [lb] of salt is dissolved. Brine (소금물) runs in at a rate of 10 [gal/min], and each gallon contains 5 [lb] of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 [gal/min]. Find the amount of salt in the tank at any time t, say $y(t)$.
  - Balance law: Salt's time rate of change, $\frac{dy}{dt}$ = Salt inflow rate – Salt outflow rate
    - Salt inflow rate = 10 [gal/min] × 5 [lb/gal] = 50 [lb/min]
    - Salt outflow rate = 10 [gal/min] × $\frac{y}{1000}$ [lb/gal] = $\frac{y}{100}$ [lb/min]

$y' = 50 - 0.01y = -0.01(y - 5000)$, $y(0) = 100$

$\frac{dy}{y-5000} = -0.01dt \Rightarrow y(t) = 5000 - 4900e^{-0.01t}$, $t \geq 0$

(1) $y(t)$ increases with time
(2) $\lim_{t\to\infty} y(t) = 5000$

Separable ODE

- Certain first order equations that are not separable can be made separable by a simple change of variables.
  - A homogeneous ODE, $y' = f\left(\frac{y}{x}\right)$, can be reduced to separable form by the substitution of $y = ux$.
  - $\frac{dy}{dx} = \frac{du}{dx}x + u \Rightarrow u'x + u = f(u)$ or $\frac{du}{f(u)-u} = \frac{dx}{x}$ (assuming $f(u) - u \neq 0$)
- Example 1.3.8 $2xyy' = y^2 - x^2$
  - Not separable: Given $y' = \frac{y}{2x} - \frac{x}{2y}$, put $y = ux$ or $u = \frac{y}{x}$ to get

$u'x + u = \frac{u}{2} - \frac{1}{2u}$ or $\frac{2udu}{1+u^2} = -\frac{dx}{x}$

$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$

Integrating Factors

- If a function $u(x, y)$ has continuous partial derivatives, its differential is
  - $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$
    - † If $u(x, y) = c$, (c const.), then $du = 0$.
    - † If $u(x, y) = x + x^2y^3 = c$, $du = (1 + 2xy^3)dx + 3x^2y^2dy = 0$ or $y' = -\frac{1+2xy^3}{3x^2y^2}$.
- The ODE, $M(x, y)dx + N(x, y)dy = 0$, is an exact (완전) DE, if the differential form $M(x, y)dx + N(x, y)dy$ is exact, (i.e., $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$).
  - If exact, then the solution is $u(x, y) = c$ (implicit form).
  - $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$
  - Condition for exactness: $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \equiv \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$

Integrating Factors

- Exact ODE, $M(x, y)dx + N(x, y)dy = 0$, such that $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$:
  - $u(x, y) = \int M dx + k(y)$ or $u(x, y) = \int N dy + \ell(y)$
- Example 1.4.1 $\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))dy = 0$
  - Exact check: $\frac{\partial M}{\partial y} = -\sin(x + y)$ and $\frac{\partial N}{\partial x} = -\sin(x + y)$

    $u(x, y) = \int M dx + k(y) = \int \cos(x + y)\, dx + k(y) = \sin(x + y) + k(y)$

    $\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N(x, y) = 3y^2 + 2y + \cos(x + y) \Rightarrow \frac{dk}{dy} = 3y^2 + 2y$

    $u(x, y) = \sin(x + y) + y^3 + y^2 = c$ ← Check: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

Integrating Factors

- Reduction to exact form, Integrating factors (적분인자)
  - Some equations can be made exact by multiplication by some function, which is usually called the integrating factor.
  - Given a non-exact ODE, $P(x,y)dx + Q(x,y)dy = 0$, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, we want to find a function F, such that $FPdx + FQdy = 0$ is exact. Then the function F is an integrating factor.
- Example1.4.4. $-ydx + xdy = 0$, not exact
  - Choose $F = \frac{1}{x^2}$. Multiply F on both sides to get $-\frac{y}{x^2}dx + \frac{1}{x}dy = \frac{d}{dx}\left(\frac{y}{x}\right) = 0$ (exact).
  - $\frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) = -\frac{1}{x^2}$ and $\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

Integrating Factors: How to get?

- Given $FPdx + FQdy = 0$
  - Exact condition: $\frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x} \Rightarrow F_yP + FP_y = F_x + FQ_x$
    - † The integrating factor usually is a function of one variable.
    - † When $F = F(x)$, $F_y = 0$ and $P + FP_y = F_x + FQ_x$.
    - † Divide both sides by FQ to get $\frac{1}{F}\cdot\frac{dF}{dx} = R$, where $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$
- Integrating factor
  - If R is a function of x only, $\exists\ F(x) = \exp\left(\int R(x)dx\right)$.
  - If R is a function of y only, $\exists\ F(y) = \exp\left(\int R(y)dy\right)$, where $R = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$.

Integrating Factors

- Example. $y' = ay$ or $aydx - dy = 0$
  - ∨ $P = ay$ and $Q = -1$. $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ (not exact)
  - ∨ $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -a$ is constant and can be treated as a function of x only.
  - ∨ Integrating factor, $F = \exp\left(\int R(x)dx\right) = \exp\left(\int -adx\right) = e^{-ax}$
  - ∨ $FPdx + FQdy = aye^{-ax}dx - e^{-ax}dy = \frac{d}{dx}(ye^{-ax}) = 0$
    - † Given $M = aye^{-ax}$ and $N = -e^{-ax}$,
    - † $u = \int Mdx + k(y) = \int aye^{-ax}dx + k(y) = -ye^{-ax} + k(y)$
    - † $\frac{\partial u}{\partial y} = -e^{-ax} + \frac{dk}{dy} = N \Rightarrow \frac{dk}{dy} = 0$ and $k(y) = c$ (const)
    - † Solution: $u(x, y) = 0$, $ye^{-ax} = c$ or $y = ce^{ax}$

Integrating Factors

- Example1.4.5. $(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$, $y(0) = -1$
  - ∨ $P = e^{x+y} + ye^y$ and $Q = xe^y - 1$. $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ (not exact)
  - ∨ $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is not a function of x only.
  - ∨ $R = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y}+ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1$ is a function of y only.
  - ∨ Integrating factor: $F = \exp\left(\int R(y)dy\right) = \exp\left(\int(-1)dy\right) = e^{-y}$.
  - ∨ Multiply F on both sides to get $(e^x + y)dx + (x - e^{-y})dy = 0$ (exact).

    $u = \int Mdx + k(y) = e^x + xy + k(y)$

    $\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N \Rightarrow \frac{dk}{dy} = -e^{-y}$ and $k(y) = e^{-y} + c$
  - ∨ Solution: $u(x, y) = e^x + xy + e^{-y} + c = 0$

Linear ODEs

- Linear ODE vs. non-linear ODE (선형 vs. 비선형 미분방정식)
  - ODEs which is linear in both the unknown function and its derivatives.
    - $y' + p(x)y = r(x)$: Linear differential equation, $r(x)$ … input, $y(x)$
    - $y' + p(x)y = r(x)y^2$: Non-linear differential equation
- Homogeneous vs. non-homogeneous ODE (제차 vs. 비제차 미분방정식)
  - Homogeneous: $y' + p(x)y = 0$ (1계 제차 상미분 방정식)
    - Solution: $\frac{dy}{y} = -p(x)$, $\ln y = -\int p(x)dx + c$, and $y(x) = a \cdot \exp\left(-\int p(x)dx\right)$
  - Non-homogeneous: $y' + p(x)y = r(x)$ or
    - Integrating factor,

Non-homogeneous Linear ODE

- $y' + p(x)y = r(x)$
  - $(py - r)dx + dy = 0$: $P = py - r$, $Q = 1$, and $\frac{\partial P}{\partial y} = p \neq 0 = \frac{\partial Q}{\partial x}$ (not exact)
  - Integrating factor: $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p$ and $F(x) = \exp\left(\int R(x)dx\right) = e^{\int pdx} = e^{h(x)}$
  - Exact ODE: $FPdx + FQdy = 0$
    - Choose an easier approach: $u = \int Ndy + \ell(x) \Rightarrow u = ye^h + \ell(x)$

$$\frac{\partial u}{\partial x} = pe^h y + \frac{d\ell}{dx} = FP = e^h(py - r) \leftarrow \boxed{\frac{d}{dx}\left(e^{\int pdx}\right) = pe^{\int pdx} = pe^h}$$

$$\frac{d\ell}{dx} = -re^h \text{ and } \ell(x) = -\int re^h dx + a$$

$$u = ye^h - \int re^h dx = c, \therefore y = e^{-h}\left(\int re^h dx + c\right), \ h(x) = \int pdx$$

Non-homogeneous Linear ODE

- Example. $y' - y = e^{2x}$
  - $p(x) = -1$ and $r(x) = e^{2x}$
  - $h(x) = \int p dx = -x$
  - $y(x) = e^{-h}\left(\int e^{h} r dx + c\right) = e^{x}\left(\int e^{-x}e^{2x}dx + c\right) = e^{2x} + ce^{x}$
- Example 1.5.1. (RL series circuit) $i' + \frac{R}{L}i = \frac{v}{L}$, v … DC voltage
  - $p(t) = \frac{R}{L}$ and $r(t) = \frac{v}{L}$
  - $h(t) = \frac{R}{L}t$
  - $i(t) = e^{-Rt/L}\left(\frac{v}{L}\int e^{\frac{Rt}{L}}dt + c\right) = \frac{v}{R} + ce^{-Rt/L}$

Existence and Uniqueness of Solutions

- Theorem 1. Existence theorem
  - Given an ODE, $y' = f(x,y)$, $y(x_0) = y_0$. Let $f(x,y)$ be

  1. continuous at all points (x,y) in the region, $R = \{(x,y): |x - x_0| < a, |y - y_0| < b\}$ and
  2. bounded in R: i.e., $\exists\, K > 0$, such that $|f(x,y)| \le K$, $\forall\, (x,y) \in R$.

  Then, the ODE has at least one solution.
- Theorem 2. Uniqueness theorem

  Let $f(x,y)$ and its partial derivative $\frac{\partial f}{\partial y}$ be continuous $\forall\, (x,y) \in R$ and bounded, say

  (a) $|f(x,y)| \le K$, and (b) $\left|\frac{\partial f}{\partial y}\right| \le M$, $\forall\, (x,y) \in R$.

  Then, the ODE has at most one solution.