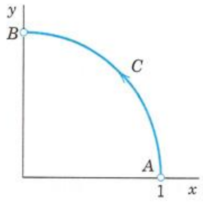
Chapter 10. Vector Integral Calculus

Line Integrals

- Definite integral
  - $\int_a^b f(x)dx$
- Integral along a curve C
  - Integration part, C: $\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$, $a \le y \le b$
    - † Smooth curve, closed path
- Line integral of $\boldsymbol{F}(\boldsymbol{r})$ over a curve C (integration path)
  - $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{r}'(t)dt \quad \leftarrow \boldsymbol{r}'(t) = \frac{d\boldsymbol{r}(t)}{dt}$ and $d\boldsymbol{r} = (dx, dy, dz)$
    - † $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')dt$

Line Integrals



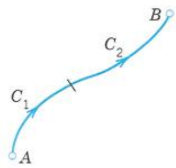
- Example 1. $\boldsymbol{F}(\boldsymbol{r}) = (-y, -xy)$ with C … circular arc
  - $\boldsymbol{r}(t) = (\cos t, \sin t)$, $t \in \left[0, \frac{\pi}{2}\right]$ $\Rightarrow$ $x(t) = \cos t$ and $y(t) = \sin t$
  - $\boldsymbol{F}(\boldsymbol{r}) = (\sin t, -\cos t \cdot \sin t)$
  - $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{t=0}^{\pi/2} (\sin t, -\cos t \cdot \sin t) \cdot (-\sin t, \cos t)\, dt = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t) dt$

- Example 2. $\boldsymbol{F}(\boldsymbol{r}) = (z, x, y)$ with C … circular helix
  - $\boldsymbol{r}(t) = (\cos t, \sin t, 3t)$, $t \in [0, 2\pi]$ $\Rightarrow$ $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$
  - $\boldsymbol{F}(\boldsymbol{r}) = (3t, \cos t, \sin t)$
  - $\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = (3t, \cos t, \sin t) \cdot (-\sin t, \cos t, 3) = -3t \sin t + \cos^2 t + 3 \sin t$
  - $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{t=0}^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t) dt$

Line Integrals

- Properties

  1. $\int_C k\boldsymbol{F} \cdot d\boldsymbol{r} = k \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$
  2. $\int_C (\boldsymbol{F} + \boldsymbol{G}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} + \int_C \boldsymbol{G} \cdot d\boldsymbol{r}$
  3. $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\{C_1, C_2\}$ is a partition of C.



- Example $\boldsymbol{F} = (0, xy, 0)$
  - C_1: $\boldsymbol{r}_1(t) = (t, t, 0)$, $t \in [0, 1]$, $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{t=0}^{1} (0, t^2, 0) \cdot (1, 1, 0) dt = \frac{1}{3}$
  - C_2: $\boldsymbol{r}_2(t) = (t, t^2, 0)$, $t \in [0, 1]$, $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{t=0}^{1} (0, t^3, 0) \cdot (1, 2t, 0) dt = \frac{2}{5}$
  - Different path could result in different line integral.

More work in longer path.

Line Integrals: Motivation

- Work done by a force
  - Work W done by a constant force $\boldsymbol{F}$ in the direction along a line segment $\boldsymbol{d}$
    - $W = \boldsymbol{F} \cdot \boldsymbol{d}$
  - Work W done by a time-varying force $\boldsymbol{F}$ in the direction along a curve $C\colon \boldsymbol{r}(t)$
    - $W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$
- Example 4. $\boldsymbol{F} = m\boldsymbol{a}$
  - A time-varying force $\boldsymbol{F}$ in the direction along a curve $C\colon \boldsymbol{r}(t)$
    - Velocity, $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$
    - $W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b m\boldsymbol{v}' \cdot \boldsymbol{v}(t)dt = \int_a^b \frac{m}{2}(\boldsymbol{v} \cdot \boldsymbol{v})'\, dt = \frac{m}{2}|\boldsymbol{v}|^2|_{t=a}^b$

Line Integral: Path Independence

- Theorem 1. Path independence (1)

  A line integral with continuous F_1, F_2, F_3 in a domain D is path independent in D, if and only if $\boldsymbol{F} = (F_1, F_2, F_3) = grad(f)$ for some function $f \in D$.
  - If $\boldsymbol{F} = (F_1, F_2, F_3) = grad(f)$, $F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$, $F_3 = \frac{\partial f}{\partial z}$ and so
    - $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt = \int_a^b \frac{df}{dt} dt = f(B) - f(A)$
    - Line integral is path independent, if and only if $\boldsymbol{F}$ is a gradient of a potential in D.

Line Integral: Path Independence

- Example 1. $\int_C (2xdx + 2ydy + 4zdz)$: Path from $A: (0,0,0)$ to $B: (2,2,2)$
  - Check, $\boldsymbol{F} = (2x, 2y, 4z) = grad(f) \Rightarrow \frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = 4z \Rightarrow f = x^2 + y^2 + 2z^2 \Rightarrow$ path independent
  - $\int_C (2xdx + 2ydy + 4zdz) = f(B) - f(A) = 16$
- Example 2. $\int_C (3x^2dx + 2yzdy + y^2dz)$: Path from $A: (0,1,2)$ to $B: (1,-1,7)$
  - $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f = x^3 + g(y,z)$,
  - $\frac{\partial f}{\partial y} = 2yz = g_y \Rightarrow g(y,z) = y^2z + h(z)$,
  - $\frac{\partial f}{\partial z} = y^2 = y^2 + h_z \Rightarrow h(z) = c$ (const.): $f = x^3 + y^2z + c$
  - $\int_C (3x^2dx + 2yzdy + y^2dz) = f(B) - f(A) = 6$

Line Integral: Path Independence

- Theorem 2. Path independence (2)

  A line integral is path independent in a domain D, if and only if $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$, for all closed path C.
  - Work along a closed path C is zero: Conservative vector field
- Theorem 3. Path independence (3)

  A line integral is path independent in a domain D, if and only if the differential form $\boldsymbol{F} \cdot d\boldsymbol{r} = F_1dx + F_2dy + F_3dz$ has continuous coefficient functions F_1, F_2, F_3 and is exact in D.
  - Differential form, $\boldsymbol{F} \cdot d\boldsymbol{r}$ is exact, if $\exists f \in D$, such that

    $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = grad(f) \cdot d\boldsymbol{r} \ (\leftrightarrow \boldsymbol{F} \cdot d\boldsymbol{r} = df)$$
  - Differential form, $\boldsymbol{F} \cdot d\boldsymbol{r} = F_1dx + F_2dy + F_3dz$ is exact, if and only if $\exists f \in D$, such that $\boldsymbol{F} = grad(f)$.

Exact & Path Independence

- Theorem 3. Criterion for exactness and path independence

  Let F_1, F_2, F_3 in the line integral, $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ be continuous and have continuous first partial derivatives in a domain D.

  ① If the differential form $\boldsymbol{F} \cdot d\boldsymbol{r} = F_1 dx + F_2 dy + F_3 dz$ is exact in D (and thus path independent), then $curl(\boldsymbol{F}) = \boldsymbol{0}$;

  $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

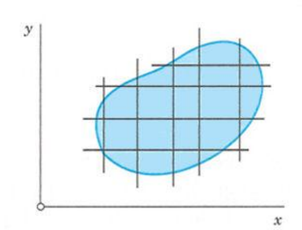
  ② If $curl(\boldsymbol{F}) = \boldsymbol{0}$ in a simply connected domain D, then the differential form is exact in D.

  † For the line integral in 2-D space, $\boldsymbol{F} = (F_1, F_2)$. ⇒ Exact, only if $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

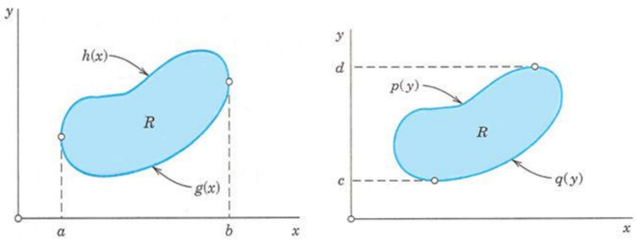
Exact & Path Independence

- Example 3. $F = (2xyz^2, x^2z^2 + z\cos yz, 2x^2yz + y\cos yz)$
  - ∨ $(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$, $(F_1)_z = (F_3)_x$, $(F_2)_x = (F_1)_y$ ⇒ exact
  - ∨ $F_2 = \frac{\partial f}{\partial y} \Rightarrow f = \int F_2 dy = x^2yz^2 + \sin yz + g(x,y)$
  - ∨ $f_x = 2xyz^2 + g_x = F_1 \Rightarrow g_x = 0$ and $g = h(z)$
  - ∨ $f_z = 2x^2yz + y\cos yz + h_z = F_3 \Rightarrow h_z = 0$ and $h(z) = c$
  - ∨ $f = x^2yz^2 + \sin yz$
  - ∨ Line integral from A: $(0,0,1)$ to $B: \left(1, \frac{\pi}{4}, 2\right)$, $I = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(B) - f(A) = \pi + 1$

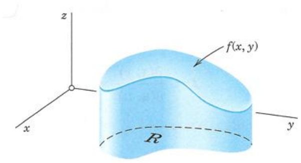
Double Integrals



- Volume of the region between the surface and plane.
  - ∨ $\iint_R f dxdy$
  - ∨ $\iint_R f dxdy = \int_{x=a}^{b} \int_{g(x)}^{h(x)} f(x,y)dydx$
  - ∨ $\iint_R f dxdy = \int_{y=c}^{d} \int_{p(y)}^{q(y)} f(x,y)dxdy$



Double Integrals

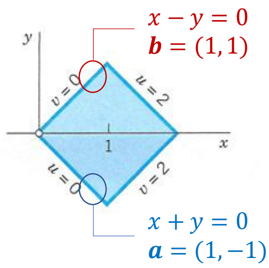


- Area
  - ∨ Area of a region R in xy-plane: $A = \iint_R dxdy$
- Volume
  - ∨ Volume beneath the surface $z = f(x,y) > 0$ and above R in xy-plane:
  - ∨ $V = \iint_R f(x,y)dxdy$
- When $f(x,y)$ is the density of a distribution of mass in xy-plane,
  - ∨ Total mass in R, $M = \iint_R f(x,y)dxdy$
  - ∨ Center of gravity, $\bar{x} = \frac{1}{M}\iint_R xf(x,y)dxdy$, $\bar{y} = \frac{1}{M}\iint_R yf(x,y)dxdy$
  - ∨ Moment of inertia, $I_x = \iint_R y^2 f(x,y)dxdy$, $I_y = \iint_R x^2 f(x,y)dxdy$

Double Integrals

- Jacobian: Change of variables in double integrals
  - ∨ From (x, y) to (u, v)
  - ∨ $\iint_R f(x,y)dxdy = \iint_{R^*} f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$
    - † Jacobian, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$
  - ∨ Polar coordinate, $x = r\cos\theta$, $y = r\sin\theta$; $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$
    - † $J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$
    - † $\iint_R f(x,y)dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta) r dr d\theta$

Double Integrals



- Example 1. $\iint_R (x^2+y^2)dxdy$, R … square
  - ∨ $x + y = u$, $x - y = v \Rightarrow R^* = \{(u,v): 0 \le u \le 2, 0 \le v \le 2\}$
  - ∨ $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v) \Rightarrow x^2+y^2 = \frac{1}{2}(u^2+v^2)$
  - ∨ $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$
  - ∨ $I = \iint_R (x^2+y^2)dxdy = \int_{v=0}^{2}\int_{u=2}^{0} \frac{1}{2}(u^2+v^2)\left(-\frac{1}{2}\right)dudv = \frac{8}{3}$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\boldsymbol{a} = (1,-1) \to T(\boldsymbol{a}) = (0,2)$
$\boldsymbol{b} = (1,1) \to T(\boldsymbol{b}) = (2,0)$
$\boldsymbol{c} = (2,0) \to T(\boldsymbol{c}) = (2,2)$

$$\boxed{I = \int_{x=0}^{1}\int_{y=-x}^{x}(x^2+y^2)dy\,dx + \int_{x=1}^{2}\int_{y=x-2}^{-x+2}(x^2+y^2)dy\,dx}$$

Double Integrals

- Example 2. Density, $f(x,y) = 1$, R … unit circle in 1st quadrant
  - $x = r\cos\theta,\ y = r\sin\theta;\ r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\frac{y}{x}$, and $J = r$
  - Mass, $M = \iint_R\ dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r dr\, d\theta = \frac{\pi}{4}$
  - Center of gravity, $\bar{x} = \frac{1}{M}\iint_R\ xdxdy = \frac{4}{\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r\cos\theta)\, rdr\, d\theta = \frac{4}{3\pi}$
  - Moment of inertia, $I_x = \iint_R\ y^2 dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r\sin\theta)^2 rdr\, d\theta = \frac{\pi}{16}$

Green's Theorem in the Plane

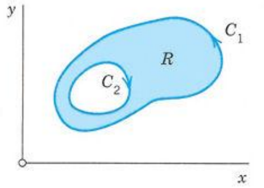
- Theorem 1. Green's theorem

  Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ everywhere in some domain containing R. Then

  $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C\ (F_1 dx + F_2 dy)$$

  Here we integrate along the entire boundary C of R in such a sense that R is on the left as we advance in the direction of integration.

  - If $\boldsymbol{F} = (F_1, F_2, 0)$, $\iint_R\ curl(\boldsymbol{F}) \cdot \boldsymbol{k} dxdy = \oint_C\ \boldsymbol{F} \cdot d\boldsymbol{r}$
    - † Transform between double integral and line integral



Green's Theorem in the Plane

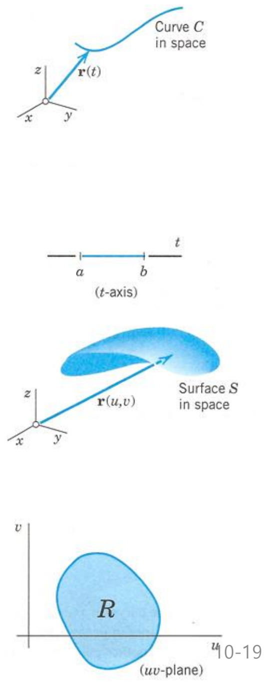
- Example 1. $F_1 = y^2 - 7y,\ F_2 = 2xy + 2x,\ C: x^2 + y^2 = 1$
  - $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R (2y + 2 - (2y - 7))dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 9r dr\, d\theta = 9\pi$
    - C: $\boldsymbol{r}(t) = (\cos t, \sin t)$, $\boldsymbol{dr} = (-\sin t, \cos t)$, $0 \le t \le 2\pi$
    - $\boldsymbol{F} = (\sin^2 t - 7\sin t, 2\cos t \cdot \sin t + 2\cos t)$
  - $\oint_C \boldsymbol{F} \cdot \boldsymbol{dr} = \int_{t=0}^{2\pi} (-\sin^3 t - 7\sin^2 t + 2\cos^2 t \cdot \sin t + 2\cos^2 t)dt = 0 + 7\pi - 0 + 2\pi$
- Example 2. Area of a plane region over the boundary C
  - Choose $F_1 = 0,\ F_2 = x$:
    - $A = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R dxdy = \oint_C (F_1 dx + F_2 dy) = \oint_C xdy$
  - Choose $F_1 = -y,\ F_2 = 0$:
    - $A = \iint_R dxdy = -\oint_C ydy$

Green's Theorem in the Plane

- Example 2. Area of a plane region over the boundary C
  - $A = \frac{1}{2}\oint_C (xdy - ydx)$ ← Green's theorem with $F_1 = -y,\ F_2 = x$
  - When R is a region covered by an ellipse, $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
    - $C: \boldsymbol{r}(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi)$ $A = \frac{1}{2}\int_{t=0}^{2\pi} (a\cos t \cdot b\cos t + b\sin t \cdot a\cos t)dt = \pi ab$
- Example 3. Area of a plane region (polar coordinates)
  - $x = r\cos\theta,\ y = r\sin\theta$: $dx = \cos\theta\, dr - r\sin\theta\, d\theta$, $dy = \sin\theta\, dr + r\cos\theta\, d\theta$,
    - $xdy - ydx = r\cos\theta\,(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta\,(\cos\theta\, dr - r\sin\theta\, d\theta) = r^2 d\theta$
  - $A = \frac{1}{2}\oint_C r^2 d\theta$
  - Cardioid, $r = a(1 - \cos\theta)$, $\theta \in [0, 2\pi)$
    - $A = \frac{a^2}{2}\int_{\theta=0}^{2\pi} (1 - \cos\theta)d\theta = \frac{3\pi}{2}a^2$

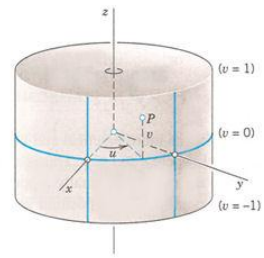
Surfaces

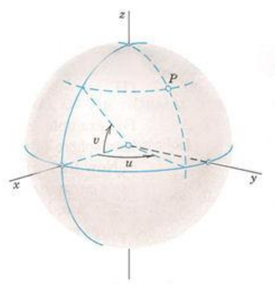
- Parametric representation of surfaces
  - ∨ $z = f(x, y)$ or $g(x, y, z) = 0$
    - † e.g., hemisphere 반구, $z = \sqrt{a^2 - x^2 - y^2}$ $(z \geq 0)$
  - ∨ $\boldsymbol{r}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ ← two parameters u and v
    - † Extension of a curve, $\boldsymbol{r}(t) = (x(t), y(t), z(t))$



Surfaces: Parametric Representation

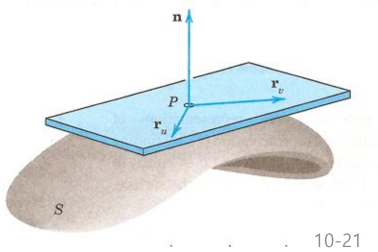
- Example 1. Circular cylinder
  - ∨ $x^2 + y^2 = a^2$, $-1 \leq z \leq 1$
  - ∨ $r(u, v) = (a\cos u\,, a\sin u\,, v)$, $u \in [0, 2\pi)$ and $v \in [-1, 1]$
    - † Fixed $u = u_0$, $r(u_0, v)$ … vertical line, while fixed $v = v_0$, $r(u, v_0)$ … circle
- Example 2. Sphere
  - ∨ $x^2 + y^2 + z^2 = a^2$, $-1 \leq z \leq 1$
  - ∨ $r(u, v) = (a\cos v \cos u\,, a\cos v \sin u\,, a\sin v)$, $u \in [0, 2\pi)$ and $v \in [-\pi, \pi)$
    - † Spherical coordinate system
  - ∨ $r(u, v) = (a\cos u \cos v\,, a\sin u \cos v\,, a\sin v)$, $u \in [0, 2\pi)$, $v \in [0, \pi)$





Surfaces: Tangent Plane

- Tangent plane:
  - A plane which is formed by tangent vectors of all the curves on a surface S through a point P in S.
    - Normal vector to the tangent plane
    - Given $S: \boldsymbol{r}(u, v)$, choose a curve on S by taking $u = u(t)$ and $v = v(t)$: $C: \tilde{\boldsymbol{r}}(t) = \boldsymbol{r}\big(u(t), v(t)\big)$.
    - Tangent vector of C on S, $\tilde{\boldsymbol{r}}'(t) = \frac{\partial \boldsymbol{r}}{\partial u} \cdot \frac{du}{dt} + \frac{\partial \boldsymbol{r}}{\partial v} \cdot \frac{dv}{dt} = \boldsymbol{r}_u u' + \boldsymbol{r}_v v'$
      - Linear combination of $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$.
      - $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ span the tangent plane of S at P.
    - Normal vector, $\boldsymbol{N}$ of S at P, $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$
      - Unit normal vector, $\boldsymbol{n} = \frac{\boldsymbol{N}}{|\boldsymbol{N}|}$



Surfaces: Tangent Plane

- When $S: g(x, y, z) = 0$
  - $\boldsymbol{n} = \frac{grad(g)}{|grad(g)|}$ … gradient is a surface normal vector for a surface, $S: g(x, y, z) = 0$

- Example 4. Sphere
  - $g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$
  - $grad(g) = (2x, 2y, 2z)$, $|grad(g)| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a^2$
  - $\boldsymbol{n} = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

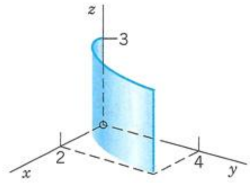
Surfaces: Tangent Plane

- Example 4. Sphere
  - $g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$
  - $\boldsymbol{r}(u,v) = (a\cos v \cdot \cos u\,, a\cos v \cdot \sin u\,, a\sin v)$
  - $\boldsymbol{r}_u = (-a\cos v \cdot \sin u\,, a\cos v \cdot \cos u\,, 0)$, $\boldsymbol{r}_v = (-a\sin v \cdot \cos u\,, -a\sin v \cdot \sin u\,, a\cos v)$
  - Normal vector, $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\cos v \cdot \sin u & a\cos v \cdot \cos u & 0 \\ -a\sin v \cdot \cos u & -a\sin v \cdot \sin u & a\cos v \end{vmatrix}$
    - † $\boldsymbol{N} = (a^2\cos^2 v \cdot \cos u\,, a^2\cos^2 v \cdot \sin u\,, a^2\cos v \cdot \sin v \cdot \sin^2 u + a^2\cos v \cdot \sin v \cdot \cos^2 u) = (a^2\cos^2 v \cdot \cos u\,, a^2\cos^2 v \cdot \sin u\,, a^2\cos v \cdot \sin v)$
    - † $|\boldsymbol{N}|^2 = a^4\cos^2 v\,(\cos^2 v \cdot \cos^2 u + \cos^2 v \cdot \sin^2 u + \sin^2 v) = a^4\cos^2 v$
  - Unit normal vector, $\boldsymbol{n} = (\cos v \cdot \cos u\,, \cos v \cdot \sin u\,, \sin v) = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

Surface Integrals

- Surface
  - $S: \boldsymbol{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$, where $(u,v) \in R$
  - Unit normal vector, $\boldsymbol{n} = \frac{\boldsymbol{N}}{|\boldsymbol{N}|}$, where $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$
- Surface integral: $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}dA = \iint_R \boldsymbol{F}(u,v) \cdot \boldsymbol{N}(u,v)dudv$
  - $dA = |\boldsymbol{N}|dudv$ and $\boldsymbol{N}(u,v) = |\boldsymbol{N}|\boldsymbol{n}(u,v)$
  - When $\boldsymbol{F} = (F_1, F_2, F_3)$, $\boldsymbol{N} = (N_1, N_2, N_3)$, and $\boldsymbol{n} = (\cos\alpha\,, \cos\beta\,, \cos\gamma)$
  - $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}dA = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA = \iint_R (F_1N_1 + F_2N_2 + F_3N_3)dudv = \iint_S (F_1dydz + F_2dzdx + F_3dxdy)$
    - † $\boldsymbol{n} \cdot \boldsymbol{i} = \cos\alpha$, $\boldsymbol{n} \cdot \boldsymbol{j} = \cos\beta$, $\boldsymbol{n} \cdot \boldsymbol{k} = \cos\gamma \Rightarrow \boldsymbol{n} = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k}$
    - † $\cos\alpha\, dA = dydz$, $\cos\beta\, dA = dzdx$, $\cos\gamma\, dA = dxdy$

Surface Integrals



- Example 1. Flux through a surface
  - $\boldsymbol{F} = \boldsymbol{v} = (3z^2, 6, 6xz)$, $S: y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
  - $\boldsymbol{r}(u, v) = (u, u^2, v)$, $0 \le u \le 2$, $0 \le v \le 3$
    - † $\boldsymbol{r}_u = (1, 2u, 0)$, $\boldsymbol{r}_v = (0, 0, 1) \Rightarrow \boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = (2u, -1, 0)$
  - $\boldsymbol{F}(u, v) \cdot \boldsymbol{N} = (3v^2, 6, 6uv) \cdot (2u, -1, 0) = 6uv^2 - 6$
  - $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA = \iint_R \boldsymbol{F} \cdot \boldsymbol{N} dudv = \int_{v=0}^{3} \int_{u=0}^{2} (6uv^2 - 6) du\, dv = 72$

  - $\iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA = \int_{z=0}^{3} \int_{y=0}^{4} 3z^2 dy\, dz - \int_{x=0}^{2} \int_{z=0}^{3} 6 dz\, dx = 72$
    - † $\boldsymbol{N} = (2x, -1, 0) \Rightarrow \cos\alpha > 0$, $\cos\beta < 0$, and $\cos\gamma = 0$
    - † $\cos\alpha\, dA = dydz$, $\cos\beta\, dA = dzdx$, $\cos\gamma\, dA = dxdy$

Surface Integrals

- Example 2. Surface integral
  - $\boldsymbol{F} = (x^2, 0, 3y^2)$, $S: x + y + z = 1$, $0 \le x, y, z \le 1$ (1st octant)
  - Choose $u = x$ and $v = y$, S: $\boldsymbol{r}(u, v) = (u, v, 1 - u - v)$,
    - † $R = \{(u, v);\ 0 \le u \le 1, 0 \le v \le 1, u + v = 1\}$ … triangle in xy −plane (projection of surface S onto xy-plane)
    - † $\boldsymbol{r}_u = (1, 0, -1)$, $\boldsymbol{r}_v = (0, 1, -1) \Rightarrow \boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = (1, 1, 1)$
  - $\boldsymbol{F}(u, v) \cdot \boldsymbol{N} = u^2 + 3v^2$
  - $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA = \iint_R \boldsymbol{F} \cdot \boldsymbol{N} dudv = \int_{v=0}^{1} \int_{u=0}^{1-v} (u^2 + 3v^2) du\, dv = \frac{1}{3}$

  - $\iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA = \int_{z=0}^{1} \int_{y=0}^{1-z} (1 - y - z)^2 dy\, dz + \int_{y=0}^{1} \int_{x=0}^{1-y} 3y^2 dx\, dy$
    - † $\boldsymbol{N} = (1, 1, 1) \Rightarrow \cos\alpha > 0$, $\cos\beta > 0$, and $\cos\gamma > 0$

Surface Integrals: Second Type

- $\iint_S G(\boldsymbol{r})dA = \iint_R G(\boldsymbol{r}(u,v))|\boldsymbol{N}(u,v)|dudv$
  - $dA = |\boldsymbol{N}|dudv = |\boldsymbol{r}_u \times \boldsymbol{r}_v|dudv$
  - Surface area, $A(S) = \iint_S dA = \iint_R |\boldsymbol{r}_u \times \boldsymbol{r}_v|dudv$

- Example 4. Surface area of a sphere
  - $\boldsymbol{r}(u,v) = (a\cos v \cdot \cos u, a\cos v \cdot \sin u, a\sin v)$
  - Normal vector, $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = a^2 \cos v\,(\cos v \cdot \cos u, \cos v \cdot \sin u, \sin v)$
    - † $|\boldsymbol{N}| = a^2|\cos v|$
  - $A(S) = a^2 \int_{v=-\pi}^{\pi}\int_{u=0}^{2\pi}|\cos v|du\,dv = 2\pi a^2 \int_{v=-\pi}^{\pi}\cos v\,dv = 4\pi a^2$

Triple Integral

- An integral of a function $f(x,y,z)$ over a closed bounded region T.

- Theorem 1. Divergence theorem

  Let T be a closed bounded region in the space whose boundary is a piecewise smooth orientable surface S. Let $\boldsymbol{F}(x,y,z)$ be a function that is continuous and has continuous partial derivatives in some domain containing T. Then

  $\iiint_T div(\boldsymbol{F})dV = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}dA$

  - If $\boldsymbol{F} = (F_1, F_2, F_3)$ and the outer unit normal vector $\boldsymbol{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ of S,

  $\iiint_T div(\boldsymbol{F})dV = \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \cdots$

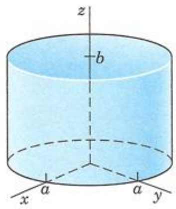
  $\ldots = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$

Divergence Theorem

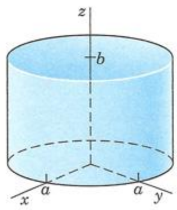
- It relates the flux of a vector field through a closed surface S to the divergence over the region T inside the surface S.
- The surface integral of a vector field over a closed surface S is equal to the volume integral of divergence over the region inside the surface.

• Example 1. $I = \iint_S (x^3 dydz + x^2 ydzdx + x^2 zdxdy)$

- $S: x^2 + y^2 = a^2$, $0 \le z \le b$ (surface of a cylinder and two circular disks)
- $div(\boldsymbol{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$
- $I = \iiint_T 5x^2 dxdydz = \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 5(r\cos\theta)^2 dr\, d\theta\, dz = \frac{5\pi}{4} a^4 b$



Divergence Theorem



• Example 1 (contd.). $I = \iint_S (x^3 dydz + x^2 ydzdx + x^2 zdxdy)$

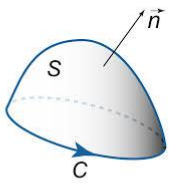
- $S_1: x^2 + y^2 = a^2$, $0 \le z \le b$ (surface of a cylinder)
  - † $x = a\cos u$, $y = a\sin u$, $z = v$, C_1: $\boldsymbol{r}_1(u, v) = (a\cos u, a\sin u, v)$
  - † $\boldsymbol{r}_{1u} = (-a\sin u, a\cos u, 0)$, $\boldsymbol{r}_{1v} = (0, 0, 1) \Rightarrow \boldsymbol{N} = \boldsymbol{r}_{1u} \times \boldsymbol{r}_{1v} = (a\cos u, a\sin u, 0)$
  - † $\boldsymbol{F} \cdot \boldsymbol{N} = (x^3, x^2 y, x^2 z) \cdot (x, y, 0) = x^4 + x^2 y^2 = a^4 \cos^2 u$
  - † $I_1 = \iint_R \boldsymbol{F} \cdot \boldsymbol{N} dudv = \int_{v=0}^{b} \int_{u=0}^{2\pi} \frac{1}{2} a^4 (1 + \cos 2u) du\, dv = \pi a^4 b$
- $S_2: x^2 + y^2 = a^2$, $z = b$ (top disk)
  - † $\boldsymbol{n}_2 = \boldsymbol{k} \Rightarrow \cos\alpha = \boldsymbol{i} \cdot \boldsymbol{n}_1 = 0 = \cos\beta$ and $dA = \cos\gamma\, dxdy = dxdy$
  - † $I_2 = \iint_S F_3\, dxdy = \iint_S x^2 zdxdy = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (r\cos\theta)^2 \cdot b \cdot rdr\, d\theta = \frac{\pi}{4} a^4 b$
- $S_3: x^2 + y^2 = a^2$, $z = 0$ (bottom disk), $\boldsymbol{n}_3 = -\boldsymbol{k}$ and $I_3 = 0$ (since $z = 0$)
- $I = I_1 + I_2 + I_3 = \pi a^4 b + \frac{\pi}{4} a^4 b + 0 = \frac{5\pi}{4} a^4 b$

Divergence Theorem

- Example 2. $\boldsymbol{F} = (7x, 0, -z)$ and $S: x^2 + y^2 + z^2 = 4$ (sphere)
  - $I = \iiint_T \, div(\boldsymbol{F}) dV = \iint_S \, \boldsymbol{F} \cdot \boldsymbol{n} dA$
    - † $div(\boldsymbol{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 7 - 1 = 6$
  - $I = \iiint_T \, 6dV = 6 \cdot \frac{4}{3}\pi(2^2) = 64\pi$

  - $S: \boldsymbol{r}(u, v) = (a \cos v \cdot \cos u, a \cos v \cdot \sin u, a \sin v)$
    - † $\boldsymbol{r}_u = (-a \cos v \cdot \sin u, a \cos v \cdot \cos u, 0)$, $\boldsymbol{r}_v = (-a \sin v \cdot \cos u, -a \sin v \cdot \sin u, a \cos v)$
    - † $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = 4 \cos v \,(\cos v \cdot \cos u, \cos v \cdot \sin u, \sin v)$
    - † $\boldsymbol{F} \cdot \boldsymbol{N} = (14 \cos v \cdot \cos u, 0, -2 \sin v) \cdot \boldsymbol{N} = 56 \cos^3 v \cdot \cos^2 u - 8 \cos v \cdot \sin^2 v$
  - $I = \iint_S \, \boldsymbol{F} \cdot \boldsymbol{n} dA = \iint_R \, \boldsymbol{F} \cdot \boldsymbol{N} dudv = \int_{v=-\pi}^{\pi} \int_{u=0}^{2\pi} (56 \cos^3 v \cdot \cos^2 u - 8 \cos v \cdot \sin^2 v) du\, dv$

Stoke's Theorem



- Theorem 1. Stoke's theorem

  Let S be a piecewise smooth orientable surface and let the boundary of S be a piecewise smooth simple closed curve C. Let $\boldsymbol{F}(x, y, z)$ be a function that is continuous and has continuous partial derivatives in some domain containing S. Then

  $\iint_S \, curl(\boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_C \, \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds$

  Here $\boldsymbol{n}$ is a unit normal vector of S and $\boldsymbol{r}'(s) = \frac{d\boldsymbol{r}}{ds}$ is the unit tangent vector and s the arc length of C.

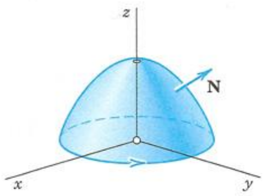
  - If $\boldsymbol{F} = (F_1, F_2, F_3)$, $\boldsymbol{N} = (N_1, N_2, N_3)$, $\boldsymbol{n} dA = \boldsymbol{N} dudv$, $\boldsymbol{r}' ds = (dx, dy, dz)$

$$\iint_R \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right) dudv = \oint_{\bar{C}} \, (F_1 dx + F_2 dy + F_3 dz)$$

  - † R is the region with boundary curve $\bar{C}$ in the uv-plane corresponding to S represented by $r(u, v)$.

Stoke’s Theorem

- Example 1. $\boldsymbol{F}=(y,z,x)$, $S: z=1-(x^2+y^2)$, $z\geq 0$ … paraboloid



  - $C: \boldsymbol{r}(s)=(\cos s, \sin s, 0)$
    - $\boldsymbol{r}'(s)=(-\sin s, \cos s, 0)$, $\boldsymbol{F}(\boldsymbol{r}(s))=(\sin s, 0, \cos s)$
  - $\oint_C \boldsymbol{F}\cdot\boldsymbol{r}'(s)ds=\int_{s=0}^{2\pi}(-\sin^2 s)ds=-\pi$
  - $curl(\boldsymbol{F})=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)=(-1,-1,-1)$
    - Given $S: g(x,y,z)=x^2+y^2+z-1=0$, $\boldsymbol{N}=grad(g)=(2x,2y,1)$
    - $curl(\boldsymbol{F})\cdot\boldsymbol{N}=-2x-2y-1$
  - $\iint_S curl(\boldsymbol{F})\cdot\boldsymbol{n}dA=\iint_R curl(\boldsymbol{F})\cdot\boldsymbol{N}dxdy=\iint_R(-2x-2y-1)dxdy$ …
    - … $\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(-2r(\cos\theta+\sin\theta)-1)r dr\, d\theta=-\pi$