8 장 연습문제 풀이

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8 정적분의 응용

8.3 평면곡선의 호의 길이

(1) $y = f(x)$ $(a \le x \le b)$ 의 호의 길이 :

$$s = \int_a^b \sqrt{1 + f'(x)^2}dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx.$$

(2) $x = g(y)$ $(c \le y \le d)$ 의 호의 길이 :

$$\int_c^d \sqrt{1 + g'(y)^2}dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy.$$

(3) 매개변수 방정식 $x = f(t)$, $y = g(t)$ $(t_1 \le t \le t_2)$, $(t_1 \le t \le t_2)$ 의 호의 길이 :

$$s = \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

(4) 극방정식 $r = f(\theta)$, $(\alpha \le \theta \le \beta)$ 의 호의 길이:

$$s = \int_\alpha^\beta \sqrt{r^2 + (r')^2}d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta.$$

연습문제 풀이

1. $3y = 2x^{\frac{3}{2}}, \quad 0 \le x \le 3.$

$$\begin{aligned} s &= \int_0^3 \sqrt{1+(y')^2}dx = \int_0^3 \sqrt{1+x}dx \\ &= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 \\ &= \frac{2}{3}(4^{\frac{2}{3}}-1) \\ &= \frac{2}{3}(8-1) = \frac{14}{3}. \end{aligned}$$

5. $4y = x^2 - 2\ln x,\ 1 \le x \le 3.$

$$\begin{aligned} s &= \int_1^3 \sqrt{1+(y')^2}dx = \int_1^3 \sqrt{1+(\frac{1}{2}(x-\frac{1}{x}))^2}dx \\ &= \int_1^3 \sqrt{1+\frac{1}{4}(x-\frac{1}{x})^2}dx = \int_1^3 \sqrt{1+\frac{1}{4}x^2+\frac{1}{4x^2}-\frac{1}{2}}dx \\ &= \int_1^3 \sqrt{\frac{1}{4}(x+\frac{1}{x})^2}dx \\ &= \frac{1}{2}\int_1^3 (x+\frac{1}{x})dx \\ &= \frac{1}{2}\left[\frac{1}{2}x^2+\ln x\right]_1^3 = \frac{1}{2}(\frac{7}{2}-\ln 3) \end{aligned}$$

11. $x = \frac{1}{8}y^4 + \frac{1}{4y^2},\ 1 \le y \le 2.$

$$\begin{aligned} s &= \int_1^2 \sqrt{1+(x')^2}dy = \int_1^2 \sqrt{1+(\frac{1}{2}y^3-\frac{1}{2y^3})^2}dy \\ &= \int_1^2 \sqrt{1+\frac{1}{4}y^6+\frac{1}{4y^6}-\frac{1}{2}}dy = \int_1^2 \sqrt{\frac{1}{4}(y^3+\frac{1}{y^3})^2}dy \\ &= \frac{1}{2}\int_1^2 (y^3+\frac{1}{y^3}dy \\ &= \frac{1}{2}\left[\frac{1}{4}y^4-\frac{1}{2y^2}\right]_1^2 \\ &= \frac{33}{16} \end{aligned}$$

25. $y = \sqrt{1-x^2} + \ln(1-\sqrt{1-x^2}) - \ln x, \;\; \frac{1}{2} \le x \le 1.$

$$\begin{aligned}
y' &= \frac{-x}{\sqrt{1-x^2}} + \frac{\frac{x}{\sqrt{1-x^2}}}{1-\sqrt{1-x^2}} - \frac{1}{x} \\
&= \frac{-x}{\sqrt{1-x^2}} + \frac{x}{(1-\sqrt{1-x^2})(\sqrt{1-x^2})} - \frac{1}{x} \\
&= \frac{x - x(1-\sqrt{1-x^2})}{(1-\sqrt{1-x^2})(\sqrt{1-x^2})} - \frac{1}{x} \\
&= \frac{x}{1-\sqrt{1-x^2}} - \frac{1}{x} \\
&= \frac{1+\sqrt{1-x^2}-1}{x} \\
&= \frac{\sqrt{1-x^2}}{x}
\end{aligned}$$

$$\begin{aligned}
s &= \int_{\frac{1}{2}}^{1} \sqrt{1+(y')^2}dx = \int_{\frac{1}{2}}^{1} \sqrt{1+\frac{1-x^2}{x^2}}dx \\
&= \int_{\frac{1}{2}}^{1} \sqrt{\frac{1}{x^2}}dx \\
&= \int_{\frac{1}{2}}^{1} \frac{1}{x}dx \\
&= \big[\ln x\big]_{\frac{1}{2}}^{1} = \ln 2
\end{aligned}$$

29. $r = a(1+\sin\theta).$

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{r^2+(r')^2}d\theta = \int_0^{2\pi} \sqrt{a^2(1+\sin\theta)^2 + a^2\cos^2\theta}d\theta \\
&= a\int_0^{2\pi} \sqrt{2+2\sin\theta}d\theta = \sqrt{2}a\int_0^{2\pi} \sqrt{1-\cos(\frac{\pi}{2}+\theta)}d\theta \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{2\sin^2(\frac{\pi}{4}+\frac{\theta}{2})}d\theta = 2a\int_0^{2\pi} \sin(\frac{\pi}{4}+\frac{\theta}{2})d\theta \\
&= -2a\left[2\cos(\frac{\pi}{4}+\frac{\theta}{2})\right]_0^{2\pi} \\
&= 8\sqrt{2}a
\end{aligned}$$

43. $r = e^{2\theta}, \;\; 0 \le \theta \le 2\pi.$

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{r^2 + (r')^2} d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta \\
&= \sqrt{5} \int_0^{2\pi} e^{2\theta} d\theta \\
&= \left[\frac{1}{2} e^{2\theta} \right]_0^{2\pi} \\
&= \frac{\sqrt{5}}{2} (e^{4\pi} - 1).
\end{aligned}$$

49. $r = \frac{1}{1+\theta}$, $\theta \ge 0$.

$$\begin{aligned}
s &= \int_0^{\infty} \sqrt{r^2 + (r')^2} d\theta \\
&= \int_0^{\infty} \sqrt{(\frac{1}{1+\theta})^2 + (\frac{1}{1+\theta})^4} d\theta \\
&= \int_0^{\infty} \frac{1}{1+\theta} \sqrt{1 + (\frac{1}{1+\theta})^2} d\theta > \int_0^{\infty} \frac{1}{1+\theta} d\theta = \infty
\end{aligned}$$

8.4 회전곡면의 겉넓이

기본공식: 회전곡면의 겉넓이$= \int 2\pi\rho ds$, $\rho =$ 회전축으로 부터 곡선 까지의 거리 (회전체의 반지름), $s =$ 호의길이.

(1) $y = f(x)$ $(a \le x \le b)$ 를 x 축에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}dx = \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

(2) $x = g(y)$ $(c \le y \le d)$ 를 y 축에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_a^b g(y)\sqrt{1+g'(y)^2}dy = \int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy.$$

(3) 매개변수 방정식 $x = f(t)$, $y = g(t)$ $(t_1 \le t \le t_2)$, $(t_1 \le t \le t_2)$ 를 x 축에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2+g'(t)^2}dt = 2\pi \int_{t_1}^{t_2} y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt.$$

(4) 매개변수 방정식 $x = f(t)$, $y = g(t)$ $(t_1 \le t \le t_2)$, $(t_1 \le t \le t_2)$ 를 y 축에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2+g'(t)^2}dt = 2\pi \int_{t_1}^{t_2} x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt.$$

(5) 극방정식 $r = f(\theta)$, $(\alpha \le \theta \le \beta)$ 를 x 축 (기선)에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_\alpha^\beta y\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta = 2\pi \int_\alpha^\beta r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta.$$

(6) 극방정식 $r = f(\theta)$, $(\alpha \le \theta \le \beta)$ 를 y 축 $(\theta = \frac{\pi}{2})$에 관하여 회전했을 때 생긴 회전곡면의 겉넓이:

$$S = 2\pi \int_\alpha^\beta x\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta = 2\pi \int_\alpha^\beta r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta.$$

연습문제 풀이

1. $y=\sqrt{x}$, $0\le x\le 1$, x-축회전 (별지 그림참고)

$$\begin{aligned}
S &= 2\pi\int_0^1 y\sqrt{1+(y')^2}dx = 2\pi\int_0^1 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}dx \\
&= 2\pi\int_0^1 \sqrt{x}\sqrt{1+\frac{1}{4x}}dx = 2\pi\int_0^1 \sqrt{x+\frac{1}{4}}dx \\
&= 2\pi\left[\frac{2}{3}(x+\frac{1}{4})^{\frac{3}{2}}\right]_0^1 = \frac{4\pi}{3}\left((\frac{5}{4})^{\frac{3}{2}}-(\frac{1}{4})^{\frac{3}{2}}\right) \\
&= \frac{\pi}{6}(5\sqrt{5}-1)
\end{aligned}$$

5. $x=a(\theta-\sin\theta),\;\; y=a(1-\cos\theta),\quad 0\le\theta\le 2\pi.$ x-축회전 (별지 그림 참고)

$$\begin{aligned}
S &= 2\pi\int_0^{2\pi} y\sqrt{(\frac{dx}{d\theta})^2+(\frac{dy}{d\theta})^2}d\theta \\
&= 2\pi\int_0^{2\pi} a(1-\cos\theta)\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta d\theta} \\
&= 2\pi a^2\int_0^{2\pi}(1-\cos\theta)\sqrt{2(1-\cos\theta)}d\theta \\
&= 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}d\theta \\
&= 2\sqrt{2}\pi a^2\int_0^{2\pi}(2\sin^2\frac{\theta}{2})^{\frac{3}{2}}d\theta = 2\sqrt{2}\pi a^2\int_0^{2\pi}2\sqrt{2}\sin^3\frac{\theta}{2}d\theta \\
&= 8\pi a^2\int_0^{2\pi}\sin\frac{\theta}{2}\sin^2\frac{\theta}{2}d\theta = 8\pi a^2\int_0^{2\pi}\sin\frac{\theta}{2}(1-\cos^2\frac{\theta}{2})d\theta \\
&= 8\pi a^2\int_0^{2\pi}\sin\frac{\theta}{2}d\theta - 8\pi a^2\int_0^{2\pi}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta \\
&= 8\pi a^2\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} - 8\pi a^2\left[-2\cdot(\frac{1}{3}\cos^3\frac{\theta}{2})\right]_0^{2\pi} \\
&= -16\pi a^2(\cos\pi-\cos 0)+\frac{16\pi a^2}{3}(\cos^3\pi-\cos^3 0) = \frac{64\pi a^2}{3}
\end{aligned}$$

11. $2y = x^2 + 1,\ \ 0 \le x \le 1$. y-축회전 (별지 그림참고) 위식을 x 의 식으로 바꾸면

$$x = \sqrt{2y-1}, \quad \frac{1}{2} \le y \le 1.$$

$$\begin{aligned}
S &= 2\pi \int_{\frac{1}{2}}^{1} x\sqrt{1+(x')^2}dy \\
&= 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y-1}\sqrt{1+\left(\frac{2}{2\sqrt{2y-1}}\right)^2}dy \\
&= 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}dy \\
&= 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y-1+1}dy = 2\pi \int_{\frac{1}{2}}^{1} \sqrt{2y}dy \\
&= 2\sqrt{2}\pi \left[\frac{2}{3}y^{\frac{3}{2}}\right]_{\frac{1}{2}}^{1} \\
&= \frac{2\pi}{3}(2\sqrt{2}-1)
\end{aligned}$$

19. $r = a(1-\cos\theta)$, $\theta = \pi$ (즉, x-축) 회전. (별지 그림참고) (주의: 이 그림은 x-축 대칭이므로 그림 전체를 돌려서는 안되고 $0 \le \theta \le \pi$ 만 돌린다.

$$\begin{aligned}
S &= 2\pi \int_0^{\pi} y\sqrt{r^2+(r')^2}d\theta = 2\pi \int_0^{\pi} r\sin\theta\sqrt{r^2+(r')^2}d\theta \\
&= 2\pi \int_0^{\pi} a(1-\cos\theta)\sin\theta\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta d\theta} \\
&= 2\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta\sqrt{2(1-\cos\theta)}d\theta \\
&= 2\sqrt{2}\pi a^2 \int_0^{\pi} \sin\theta(1-\cos\theta)^{\frac{3}{2}}d\theta \quad (\ 1-\cos\theta \text{ 를 치환}) \\
&= 2\sqrt{2}\pi a^2 \left[\frac{2}{5}(1-\cos\theta)^{\frac{5}{2}}\right]_0^{\pi} \\
&= \frac{32\pi a^2}{5}
\end{aligned}$$

31. $x = e^{\theta}\cos\theta,\ \ y = e^{\theta}\sin\theta,\ \ 0 \le \theta \le \frac{\pi}{2}$. $y = -1$ 중심회전. (별지 그림참고)

$$
\begin{aligned}
S &= 2\pi\int_0^{\frac{\pi}{2}} (y+1)\sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}} (e^{\theta}\sin\theta + 1)\sqrt{(e^{\theta}\cos\theta - e^{\theta}\sin\theta)^2 + (e^{\theta}\sin\theta + e^{\theta}\cos\theta)^2}d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}} (e^{\theta}\sin\theta + 1)\sqrt{2e^{2\theta}}d\theta \\
&= 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} (e^{\theta}\sin\theta + 1)e^{\theta}d\theta = 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} e^{\theta}d\theta + 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta \\
&= 2\sqrt{2}\pi\left[e^{\theta}\right]_0^{\frac{\pi}{2}} + 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta \\
&= 2\sqrt{2}\pi(e^{\frac{\pi}{2}} - 1) + 2\sqrt{2}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta
\end{aligned}
$$

여기서, $\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta$ 은 부분적분을 2회 하여 구한다.

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta
&\underset{\begin{pmatrix} u = e^{2\theta} & v' = \sin\theta \\ u' = 2e^{2\theta} & v = -\cos\theta \end{pmatrix}}{=} \left[-e^{2\theta}\cos\theta\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2\theta}\cos\theta d\theta \\
&\underset{\begin{pmatrix} u = e^{2\theta} & v' = \cos\theta \\ u' = 2e^{2\theta} & v = \sin\theta \end{pmatrix}}{=} 1 + 2\left[e^{2\theta}\sin\theta\right]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta \\
&= 1 + 2e^{\pi} - 4\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta
\end{aligned}
$$

따라서

$$5\int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta = 1 + 2e^{\pi} \Longrightarrow \int_0^{\frac{\pi}{2}} e^{2\theta}\sin\theta d\theta = \frac{1+2e^{\pi}}{5}$$

이되어,

$$S = 2\sqrt{2}\pi(e^{\frac{\pi}{2}} - 1 + \frac{1+2e^{\pi}}{5}).$$