# Chapter 3. Functions, sequences and relations 함수, 수열, 관계

## 3.1 Functions

***Definition* 3.1.1** Let $X$ and $Y$ be sets. A *function* (함수) from $X$ to $Y$, $f: X \to Y$ is a subset of the Cartesian product $X \times Y$, such that for each $x \in X$, there exists exactly one $y \in Y$ with $(x, y) \in f$.

(1) A function $f: X \to Y$ is a rule which assigns for every $x \in X$ unique $y \in Y$: *i.e.*, $\forall x \in X$, $\exists$ unique $y \in Y$, such that $(x, y) \in f$.

(2) The set $X$ is called the *domain* (정의역) of $f$ and the set $Y$ is called the *co-domain* (공변역) of $f$.

(3) When $(x, y) \in f$, $y$ is called the *image* of $x$.

(4) The set

(3.1) $\{y \mid (x, y) \in f\}$

is called the *range* (치역) of $f$: The range is the set of all images.

(5) A function is called by an *operator* if the domain and co-domain are the same.

*Example* 3.1 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.

← This function can be described as

$$f(1) = a, \; f(2) = b, \; f(3) = a$$

← We can depict this function $f: X \to Y$ as shown Fig. 3.1. An arrow from $j$ to $x$ means that we assign the element $x$ in the co-domain to the element $j$ in the domain. Such a picture is known as the *arrow diagram*.

← For an arrow diagram to be a function, there should be exactly one arrow from each element in the domain.

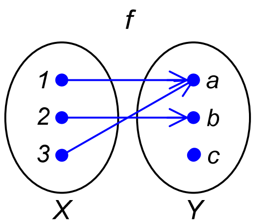


Figure 3.1 Arrow diagram of the function defined in Example 3.1

*Example* 3.2 (a) Set $\{(1, a), (2, a), (3, b)\}$ with $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$

(b) Set $\{(1, a), (2, b), (3, c), (1, b)\}$ with $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$

(c) Function $f(x) = x^2$ with $X = Y = R$: $f = \{(x, x^2) \mid x \in R\}$

***Definition*** If $x \in Z$ and $y \in N$, we define $x(mod\ y)$ to be the *remainder* (나머지), when $x$ is divided by $y$: *i.e.*, $x = ky + x(mod\ y)$, $0 \leq x(mod\ y) < y$, for some integer $k$.

*Example* 3.3 (a) $6\ (mod\ 2)$, $5\ (mod\ 3)$, $-3\ (mod\ 4)$

(b) What day of the week will it be 365 days from Wednesday?

*Example* 3.4 (*Pseudo-random numbers*: 의사난수) Linear congruential method (선형합동법) requires 4 integers: the modulus $m$, the multiplier $a$, the increment $c$, and the seed $s$ such that

$$2 \le a < m\,,\ 0 \le c < m,\ 0 \le s < m$$

Starting from the initial seed, $x_0 = s$, we generate a sequence of pseudo-random numbers by

$$x_n = (ax_{n-1} + c\,)(mod\ m)$$

When $m = 11, a = 7, c = 5$, and $s = 3$, we have a pseudo-random sequence defined by

$$x_n = (7x_{n-1} + 5\,)(mod\ 11),\ \text{with}\ x_0 = 3$$

and it is

$$3, 4, 0, 5, 7, 10, 9, 2, 8, 6, 3, \ldots$$

(1) A commonly used pseudo-random sequence makes use of $m = 2^{31} - 1$ (prime) and $a = 7^5$ with $c = 0$.

(2) One major problem in this method is that it is periodic with period-$m$.

***Definition*** The *floor* (바닥) of $x$, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to $x$. The *ceiling* (천장) of $x$, denoted $\lceil x \rceil$, is the least integer greater than or equal to $x$.

*Example* 3.5 $\lfloor 8.3 \rfloor = 8$, $\lfloor -8.7 \rfloor = -9$, $\lfloor 4 \rfloor = 4$, $\lceil 1.4 \rceil = 2$, and $\lceil -2.5 \rceil = -2$

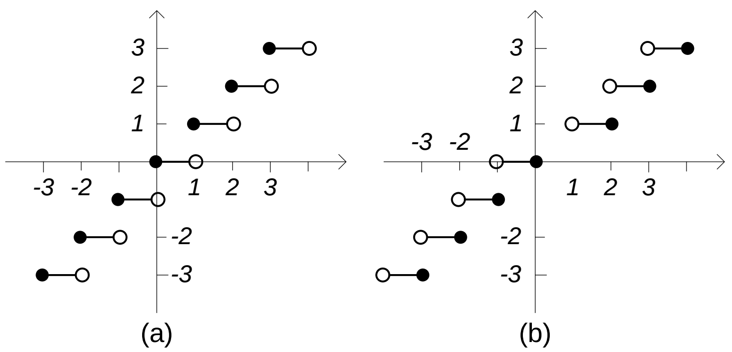


Figure 3.2 Graph of (a) floor and (b) ceiling functions

**Theorem 3.1** (*Quotient-remainder theorem*) If $d \in Z^+$ (*divisor*) and $n \in Z$, $\exists!$ integers $q$ (*quotient* 몫) and $r$ (*remainder* 나머지) such that

(3.3) $$n = dq + r,\ 0 \le r < d$$

(1) Uniqueness is important: *i.e.* if $n = dq_1 + r_1$ and $n = dq_2 + r_2$ with $0 \le r_1, r_2 < d$, then $q_1 = q_2$ and $r_1 = r_2$.

(2) $\left\lfloor \frac{n}{d} \right\rfloor = q$ and $n\ (mod\ d) = r$

## Types of functions

***Definition*** (a) A function $f$ from $X$ to $Y$ is said to be 1-1 (*one-to-one*, *injective*, 단사), if for each $y \in Y$, $\exists! x \in X$ with $f(x) = y$.

(b) If $f$ is a function from $X$ to $Y$ and range of $f$ is $Y$ (*i.e.*, the co-domain and the range are the same), then $f$ is said to be *onto* (*surjective*, 전사).

(c) A function that is both one-to-one and onto is called *bijection* (전단사).

Interpretation of function types

(1) one-to-one: $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$: *i.e.*

(3.4) $\forall x_1 \forall x_2 \big((f(x_1) = f(x_2)) \rightarrow x_1 = x_2\big)$

(2) onto: $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$: *i.e.*

(3.5) $\forall y \in Y\ \exists x \in X\ (f(x) = y)$

*Example* 3.6 (a) $f = \{(1,b),(2,c),(3,a)\}$ with $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$

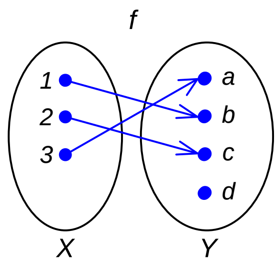


Figure 3.3 Arrow diagram of the function defined in Example 3.6(a)

(b) $f = \{(1,a),(2,b),(3,a)\}$ with $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$

(c) Function $f: N \rightarrow N$, $f(n) = 2n + 1$ is one-to-one.

(d) Function $f: Z^+ \rightarrow Z$, $f(n) = 2^n - n^2$ is *not* one-to-one.

Consider the negation of (3.4): *i.e.*

$$\sim\Big(\forall x_1 \forall x_2 \big((f(x_1) = f(x_2)) \rightarrow x_1 = x_2\big)\Big) \equiv \exists x_1\, \exists x_2 \sim\big((f(x_1) = f(x_2)) \rightarrow x_1 = x_2\big)$$

$$\equiv \exists x_1\, \exists x_2\, \big((f(x_1) = f(x_2)) \wedge\!\sim (x_1 = x_2)\big)$$

(e) $f = \{(1,a),(2,c),(3,b)\}$ with $X = \{1,2,3\}$ to $Y = \{a,b,c\}$

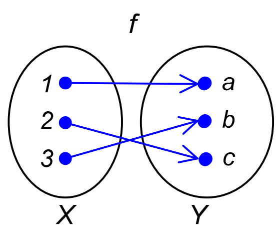


Figure 3.4 Arrow diagram of the function defined in Example 3.6(e)

(g) Function $f: R\backslash\{0\} \rightarrow R^+$ such that $f(x) = \frac{1}{x^2}$ is onto.

(h) Function $f: Z^+ \rightarrow Z^+$ such that $f(n) = 2n - 1$ is not onto.

Consider the negation of (3.5): *i.e.*

$$\sim\big(\forall y \in Y\ \exists x \in X\ (f(x) = y)\big) \equiv \exists y \in Y\ \forall x \in X, \sim(f(x) = y)$$

$$\equiv \exists y, \forall x\ (f(x) \neq y)$$

**Definition** (a) If a function $f$ from $X$ to $Y$ is bijection, the function $f^{-1} = \{(y,x) | (x,y) \in f\}$ is called $f$ *inverse* (역함수).

(b) Let $g$ be a function from $X$ to $Y$ and let $f$ be a function from $Y$ to $Z$. The *composition* (합성함수) of $f$ with $g$, denoted by $f \circ g$ is the function

(3.6) $(f \circ g)(x) = f(g(x))$

from $X$ to $Z$.

*Example* 3.7 (a) When $f = \{(1,a),(2,c),(3,b)\}$ with $X = \{1,2,3\}$ to $Y = \{a,b,c\}$,

$$f^{-1} = \{(a,1),(c,2),(3,b)\}$$

(b) The function $f(x) = 2^x$ from $R$ to $R^+$ is one-to-one and onto. Its inverse is $\{(y,x) | y \in R^+, x \in R\}$ with $f^{-1}(y) = x$ or $y = 2^x$. This implies that

$$f^{-1}(y) = x = \log_2 y$$

(c) Given $g = \{(1,a),(2,a),(3,c)\}$, a function from $X = \{1,2,3\}$ to $Y = \{a,b,c\}$ and $f = \{(a,y),(b,x),(c,z)\}$, a function from $Y$ to $Z = \{x,y,z\}$, the composition from $X$ to $Z$ is the function

$$f \circ g = \{(1,y),(2,y),(3,z)\}$$

**Definition** (a) A function from $X \times X$ to $X$ is called a *binary operator* (이항 연산자) on $X$.

(b) A function from $X$ to $X$ is called a *unary operator* (단항 연산자) on $X$.

*Example* 3.8 Function $f: Z^+ \to Z^+$ such that $f(x,y) = x + y$

## 3.2 Sequence 수열

A sequence is a function whose domain is a set of consecutive integers.

*Example* 3.9 (a) $\{F_n\}_{n=0}^{\infty} = \{1,1,2,3,5,8,\ldots\}$

(b) $C_n = 1 + 0.5(n-1),\ n \geq 1$

(c) $s_n = 2^n + 4 \cdot 3^n,\ n \geq 0$

$$s_n = 5s_{n-1} - 6s_{n-2},\ n \geq 2$$

**Definition** Let $\{s_n\}$ be a sequence defined for $n \geq m$ and let $n_1, n_2, \ldots$ be an increasing sequence whose values are in the set $\{m, m+1, \ldots\}$. We call the sequence $\{s_{n_k}\}$ a *subsequence* (부분수열) of $\{s_n\}$.

Some useful notations

Given a finite sequence $\{a_i\}_{i=m}^{n}$,

(3.7) $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n$

(3.8) $\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot \cdots \cdot a_n$

Sometimes the sum and product notations are modified to denote sums and products indexed over an arbitrary set of integers: *i.e.* if $S$ is a finite set of integers (*index set*) and $a$ is a sequence, $\sum_{i\in S} a_i$ and $\prod_{i\in S} a_i$ denote the sum and product of elements $\{a_i \mid i \in S\}$, respectively.

***Definition*** (a) A *string* (문자열) over a finite set $X$ is a finite sequence of elements from $X$.

(b) A string $\beta$ is a *substring* (부분 문자열) of the string $\alpha$ if there are strings $\gamma$ and $\delta$ with $\alpha = \gamma\beta\delta$.

Notes on string

(1)The string with no elements is called the *null string* and is denoted by $\lambda$.

(2) We let $X^*$ denote the set of all string over $X$, including the null string, and we let $X^+$ denote the set of all non-null strings over $X$.

(3) The *length* of a string $\alpha$ is the number of elements in $\alpha$ and is denoted by $|\alpha|$.

(4) If $\alpha$ and $\beta$ are two strings, the string which consists of $\alpha$ followed by $\beta$, written as $\alpha\beta$, is called the *concatenation* (연접) of $\alpha$ and $\beta$.

(5) A substring of a string $\alpha$ is obtained by selecting some or all consecutive elements of $\alpha$.

*Example* 3.10 (a) "Let's read Rolling Stones."

(b) $101111$

(c) When $X = \{a, b\}$, $X^* = \{\lambda, a, b, ab, ba, \ldots, abab, \ldots, b^{20}a^5ba, \ldots\}$

(d) Let $X = \{a, b, c\}$ and define

$$f(\alpha, \beta) = \alpha\beta$$

where $\alpha$ and $\beta$ are strings over $X$, then $f$ is a binary operator on $X^*$.

(e) Let $X = \{a, b\}$. If $\alpha \in X^*$, let $\alpha^R$ denote $\alpha$ written in reverse order: *e.g.* if $\alpha = abb$, then $\alpha^R = bba$. Define a function from $X^*$ to $X^*$ as

$$f(\alpha) = \alpha^R$$

Then, $f$ is bijection.

(f) Let $X = \{a, b\}$. Define a function from $X^* \times X^*$ to $X^*$ as

$$f(\alpha, \beta) = \alpha\beta$$

Then, $f$ is onto, but not one-to-one. What if we define $g(\alpha) = \alpha ab$?

$\leftarrow$ Concatenating a string $\alpha$ with the null string $\lambda$ does not change $\alpha$.

## 3.3 Relation 관계

***Definition*** A (binary) *relation* $R$ ((이항) 관계) from a set $X$ to a set $Y$ is a subset of $X \times Y$. If $(x, y) \in R$, we denote $xRy$ and say that $x$ is related to $y$. If $X = Y$, we call $R$ be a (binary) relation on $X$.

(1) A function $f: X \to Y$ is a special type of relation from $X$ to $Y$: $\forall x \in X, \exists!\, y \in Y\ ((x, y) \in f)$.

(2) $(x, y) \in R\ \equiv xRy$

(3) As in the function, $X$ is the domain and $Y$ is called the range.

*Example* 3.11 (a) Given $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$, define a relation from $X$ to $Y$ by

$(x, y) \in R$, if $x$ divides $y$ ($x$가 $y$를 나누어 떨어진다).

Then, we obtain $R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$. This relation can be described by a table as shown below.

Table 3-1 Table notation for the relation defined in Example 3.11(a)

| $X$ | $Y$ |
|---|---|
| 2 | 4 |
| 2 | 6 |
| 3 | 3 |
| 3 | 6 |
| 4 | 4 |

(b) Let $R$ be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$, if $x \le y$, $x, y \in X$. Then

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

A binary relation on a set $X$ can be depicted by a *diagraph* (directed graph, 방향그래프). To draw the diagraph of a relation on a set $X$, we first draw *vertices* (*node*, 정점/꼭지점) to represent the elements of the set $X$. Next, if the element $(x, y)$ is in the relation, we draw an *arrow* (called a *directed edge* or *branch*: 간선) from $x$ to $y$. Figure 3.5 shows a diagraph for the relation defined on Example 3.11(b). Notice that an element of the form $(x, x)$ in a relation corresponds to a directed edge from $x$ to $x$. Such an edge is called a *loop*.

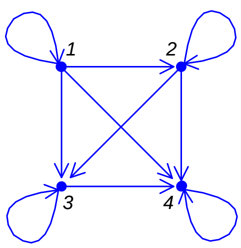


Figure 3.5 Diagraph of the relation in Example 3.11(b)

<u>Properties of binary relations</u>

(1) A relation $R$ on a set $X$ is *reflective* (반사적), if

(3.9) $\qquad (x, x) \in R,\ \forall x \in X$

← Every vertex has a loop.
← Not reflexive, if $\exists x \in X$, such that $(x,x) \notin R$

(2) *symmetric* (대칭적), if

(3.10) $\forall x, y \in X$, if $(x,y) \in R$, then $(y,x) \in R$

← Every directed edge is bi-directional.
← Not symmetric, if $\exists x\, \exists y \sim\big((x,y) \in R \rightarrow (y,x) \in R\big) \equiv \exists x\, \exists y\, \Big(\big((x,y) \in R\big) \wedge \big((y,x) \notin R\big)\Big)$

(3) *anti-symmetric* (비대칭적), if

(3.11) $\forall x, y \in X$, if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$

← An edge is bi-directional, only if it is a loop.
← Equivalently, if $x \neq y$, then $(x,y) \notin R$ or $(y,x) \notin R$.
← Not anti-symmetric, if $\exists x\, \exists y\, \big[\big((x,y) \in R\big) \wedge \big((y,x) \in R\big) \wedge (x \neq y)\big]$: *i.e.* if $\exists x\, \exists y$ such that both $(x,y)$ and $(y,x)$ are in $R$.

(4) *transitive* (추이적), if

(3.12) $\forall x, y, z \in X$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

← Whenever there are directed edges (often called branch) from $x$ to $y$ and from $y$ to $z$, there must be a directed edge from $x$ to $z$.
← Not transitive, if $\exists x, y, z \in X$, $\big[\big((x,y) \in R\big) \wedge \big((y,z) \in R\big) \wedge \big((x,z) \notin R\big)\big]$

(5) *partial order* (반순서), if it is reflexive, anti-symmetric, and transitive.

← If a relation is partial order, we can order elements in the set.

*Example* 3.12 (a) $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$.

(b) Relation $R$ on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$, if $x \leq y$.

← To formally verify that this relation is transitive, we can list all pairs of the form $(x,y)$ and $(y,z)$ in $R$ to check that in every case $(x,z) \in R$ as shown in Table 3-2.

Table 3-2 List of pairs in Example 3.12(b)

| $(x,y)$ | $(y,z)$ | $(x,z)$ | $(x,y)$ | $(y,z)$ | $(x,z)$ |
|---|---|---|---|---|---|
| (1,1) | (1,1) | (1,1) | (2,2) | (2,2) | (2,2) |
| (1,1) | (1,2) | (1,2) | (2,2) | (2,3) | (2,3) |
| (1,1) | (1,3) | (1,3) | (2,2) | (2,4) | (2,4) |
| (1,1) | (1,4) | (1,4) | (2,3) | (3,3) | (2,3) |
| (1,2) | (2,2) | (1,2) | (2,3) | (3,4) | (2,4) |
| (1,2) | (2,3) | (1,3) | (2,4) | (4,4) | (2,4) |
| (1,2) | (2,4) | (1,4) | (3,3) | (3,3) | (3,3) |
| (1,3) | (3,3) | (1,3) | (3,3) | (3,4) | (3,4) |
| (1,3) | (3,4) | (1,4) | (3,4) | (4,4) | (3,4) |
| (1,4) | (4,4) | (1,4) | (4,4) | (4,4) | (4,4) |

Note that if $x = y$ or $y = z$, then the condition for transitive will be automatically satisfied. Suppose, for example, that $x = y$ and $(x, y)$ and $(y, z)$ are in $R$. Then, $(x, z) = (y, z) \in R$. These cases correspond to shaded area in the table. This leaves only 4 possible combinations to check if it is transitive.

(c) Relation $R$ on $X = N$ defined by $(x, y) \in R$, if $y\ (mod\ x) = 0$ (*i.e.* $x$ divides $y$).

Let $R$ be a partial order on a set $X$:

(1) This relation is denoted by $(x, y) \in R\ \equiv x \preccurlyeq y$ and it indicates the relation as an ordering of elements.

(2) If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$, we say that $x$ and $y$ are *comparable* (비교가능).

(3) If every pair of elements in $X$ is comparable, then we say $R$ be *total order* (전순서).

***Definition*** (a) Let $R$ be a relation from $X$ to $Y$. The inverse of $R$, denoted by $R^{-1}$, is the relation from $Y$ to $X$, defined by

(3.13) $\quad R^{-1} = \{(y, x) |\ (x, y) \in R\}$

(b) Let $R_1$ be a relation from $X$ to $Y$ and $R_2$ be a relation from $Y$ to $Z$. The composition of $R_1$ and $R_2$, denoted by $R_1 \circ R_2$, is the relation from $X$ to $Z$, defined by

(3.14) $\quad R_1 \circ R_2 = \{(x, z) |\ (x, y) \in R_1\ and\ (y, z) \in R_2\}$

## 3.4 Equivalence relation

**Theorem 3.2** Let $\mathcal{S}$ be a partition of a set $X$. Define $xRy$ to mean that, for some set $S \in \mathcal{S}$, both $x$ and $y$ belong to $S$. Then, $R$ is reflexive, symmetric, and transitive.

$\leftarrow$ If $S \in \mathcal{S}$, we can regard the members of $S$ as equivalent in the sense of the relation $R$.

***Definition*** A relation $R$ that is reflexive, symmetric, and transitive on a set $X$ is called an *equivalence relation* (동치관계) on $X$.

*Example* 3.13 (a) Suppose that we have a set $X$ of $10$ balls, each of which is either red, blue, or green. If we divide the balls into sets $R$, $B$, and $G$ according to color, the family $\{R, B, G\}$ is a partition of $X$. If we define the relation to be the same color, then this relation is an equivalence relation.

(b) Let $\mathcal{S} = \{\{1,3,5\},\{2,6\},\{4\}\}$ be a partition of the set $X = \{1,2,3,4,5,6\}$. The relation $R$ on $X$ given by Theorem 3.2 is given by

$$R = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,6),(6,2),(6,6),(4,4)\}$$

**Theorem 3.3** Let $R$ be an equivalent relation on a set $X$. For each $a \in X$, let

(3.15) $\quad [a] = \{x \in X \mid xRa\}$

*i.e.* $[a]$ is the set of all elements in $X$ that are related to $a$. Then,

(3.16) $\quad \mathcal{S} = \{[a] \mid a \in X\}$

is a partition of $X$.

← The sets $[a]$ are called the *equivalence classes* of $X$ given by the relation $R$.

*Example* 3.14 Let $X = \{1, 2, \ldots, 10\}$. Define $xRy$ to mean that $3$ divides $x - y$: *i.e.* $(x - y)\ (mod\ 3) = 0$.

(1) $R$ is an equivalence relation on $X$.

(2) There are 3 equivalent classes: $[1]$, $[2]$, and $[3]$.

## 3.5 Relation matrix 관계 행렬

$X$에서 $Y$로의 관계 $R$을 표시하는 방법 중에 행렬 표현법이 있다. 특히, 행렬 표현법은 컴퓨터에서 관계를 분석할 때 유용하게 사용될 수 있다. 관계행렬에서는 $X$의 원소들을 행에, $Y$의 원소들을 열에 배치한다. $xRy$이면 $x$행 $y$열에 $1$로, 그렇지 않으면 $0$으로 행렬 값을 지정한다.

*Example* 3.15 (a) Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d\}$, and $R = \{(1, b), (1, d), (2, c), (3, b), (3, c), (4, a)\}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(b) Let $X = \{2, 3, 4\}$, $Y = \{5, 6, 7, 8\}$, and $xRy$, if $x$ divides $y$.

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(c) Let $X = \{a, b, c, d\}$ and binary relation $R = \{(a, b), (b, b), (c, c), (d, d), (b, c), (c, b)\}$.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

**<u>Notes on relation matrix</u>**

(1) 집합 $X$에서 정의되는 이항관계 $R$에 대응하는 행렬은 항상 정방행렬(square matrix)이다.

(2) 이항 관계를 표시하는 행렬 $A$에서, "$A$의 모든 주 대각성분(main diagonal entries)이 $1$ 이다"는 "관계 $R$이 반사적이다"의 필요충분조건이다.

(3) "행렬 $A$가 대칭적(symmetric: $a_{ij} = a_{ji}$)이다"는 "관계 $R$이 대칭적이다"의 필요충분조건이다.

(4) The matrix $A$ is anti-symmetric (*i.e.*, if $a_{ij} = 1$, then $a_{ji} = 0$), if and only if the relation $R$ is anti-symmetric.