Chapter 14. s-Domain Circuit Analysis

Complex Frequency

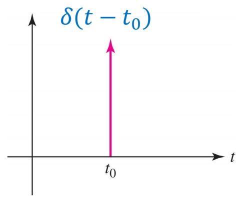
- Exponentially damped sinusoid
  - $v(t) = V_m e^{\sigma t} \cos(\omega t + \theta) = \mathrm{Re}\{V_m e^{j\theta} e^{st}\}$, where $s = \sigma + j\omega$ (complex frequency)
    - DC, if $\sigma = \omega = 0$
    - Sinusoidal, if $\sigma = 0$
    - Exponential, if $\omega = 0$
  - Complex function, $v(t) = Ke^{st}$, where both K and s are complex.
- Sinusoid, $v(t) = V_m \cos(\omega t + \theta)$
  - $\cos(\omega t + \theta) = \frac{1}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right)$
  - $v(t) = \frac{1}{2} V_m \left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$, where $s_{1,2} = \pm j\omega$

Laplace Transform

- Two-sided Laplace Transform
  - $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ $\quad f(t) \overset{\mathcal{L}}{\leftrightarrow} F(s)$, $F(s) = \mathcal{L}\{f(t)\}$, $f(t) = \mathcal{L}^{-1}\{F(s)\}$
    - † Frequency domain representation of time-domain waveform $f(t)$
- Unilateral (one-sided) Laplace Transform
  - $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$ $\quad f(t) = \frac{1}{j2\pi}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{st}ds$ (complex contour integration)
    - † We are interested in only functions that are defined for $t \geq 0$.

- Example 14.1 $f(t) = 2 \cdot u(t-3)$
  - $F(s) = 2\int_{3}^{\infty} e^{-st}dt = -\frac{2}{s}e^{-st}\Big|_{t=3}^{\infty} = \frac{2}{s}e^{-3s}$

Laplace Transform



- Unit-step function, $u(t)$
  - $u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}$ ($Re(s) > 0$)
- Unit-impulse function, $\delta(t)$
  - $\delta(t) \overset{\mathcal{L}}{\leftrightarrow} 1$
    - † $f(t)\delta(t) = f(0)\delta(t)$ and $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$ (sifting property)
    - † $\int_{0^-}^{0^+} \delta(t)dt = 1$
- Exponential function, $e^{-\alpha t}u(t)$
  - $\mathcal{L}\{e^{-\alpha t}u(t)\} \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}$

Notes on Laplace Transform

- Linearity:
  - If $F_1(s) = \mathcal{L}\{f_1(t)\}$ and $F_2(s) = \mathcal{L}\{f_2(t)\}$, $aF_1(s) + bF_2(s) = \mathcal{L}\{af_1(t) + bf_2(t)\}$

- Example 14.2
  - $G(s) = \frac{7}{s} - \frac{31}{s+17} \Rightarrow g(t) = (7 - 31e^{-17t})u(t)$
    - † $u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}$ and $e^{-\alpha t}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+\alpha}$
  - $F(s) = 2 \cdot \frac{s+2}{s} = 2 + \frac{4}{s} \Rightarrow f(t) = 2\delta(t) + 4u(t)$

Partial Fraction Expansion

- Rational Laplace Transform, $V(s) = \frac{N(s)}{D(s)}$
  - Both $N(s)$ and $D(s)$ are polynomial of s.
  - Proper, when $\deg(N(s)) < \deg(D(s))$
  - Pole: $D(s) = 0$, zero: $N(s) = 0$

- Simple poles
  - $V(s) = \frac{1}{(s+\alpha)(s+\beta)} = \frac{A}{s+\alpha} + \frac{B}{s+\beta}$
    - † $A = (s+\alpha)V(s)|_{s\leftarrow -\alpha} = \frac{1}{\beta-\alpha}$ and $B = (s+\beta)V(s)|_{s\leftarrow -\beta} = -A$
    - † $v(t) = \frac{1}{\alpha-\beta}\left(e^{-\alpha t} - e^{-\beta t}\right)u(t)$

Partial Fraction Expansion

- Double poles
  - $V(s) = \frac{1}{(s+\alpha)^2(s+\beta)} = \frac{A}{(s+\alpha)^2} + \frac{B}{s+\alpha} + \frac{C}{s+\beta}$
    - $A = (s+\alpha)^2 V(s)|_{s\leftarrow -\alpha}$ and $C = (s+\beta)V(s)|_{s\leftarrow -\beta}$
    - $(s+\alpha)^2 V(s) = A + B(s+\alpha) + C\frac{(s+\alpha)^2}{s+\beta} \Rightarrow \frac{d}{ds}\left((s+\alpha)^2 V(s)\right) = B + C\frac{(s+\alpha)(s+2\beta-\alpha)}{(s+\beta)^2}$
    - $v(t) = \left(Ate^{-\alpha t} + Be^{-\alpha t} + Ce^{-\beta t}\right)u(t)$
      - $te^{-\alpha t}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{(s+\alpha)^2}$

Partial Fraction Expansion

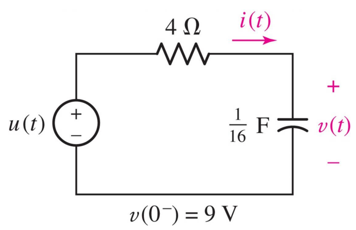
- Example 14.4
  - $P(s) = \frac{7s+5}{s^2+s} = \frac{7s+5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$
    - $A = sP(s)|_{s\leftarrow 0} = \left.\frac{7s+5}{s+1}\right|_{s\leftarrow 0} = 5$ and $B = (s+1)P(s)|_{s\leftarrow -1} = \left.\frac{7s+5}{s}\right|_{s\leftarrow -1} = 2$
    - $p(t) = (5 + 2e^{-t})u(t)$
  - $V(s) = \frac{2}{s(s+6)^2} = \frac{A}{(s+6)^2} + \frac{B}{s+6} + \frac{C}{s}$
    - $A = (s+6)^2 V(s)|_{s\leftarrow -6} = -\frac{1}{3}$ and $C = sV(s)|_{s\leftarrow 0} = \frac{1}{18}$
    - $B = \left.\frac{d}{ds}\left((s+6)^2 V(s)\right)\right|_{s\leftarrow -6} = -\left.\frac{2}{s^2}\right|_{s\leftarrow -6} = -\frac{1}{18}$
    - $v(t) = \left(-\frac{1}{3}te^{-6t} - \frac{1}{18}e^{-6t} + \frac{1}{18}\right)u(t)$

Laplace Transform: Properties

- Time differentiation, $\frac{dv}{dt} \overset{\mathcal{L}}{\leftrightarrow} sV(s) - v(0^-)$
  - $\mathcal{L}\left\{\frac{dv}{dt}\right\} = \int_{0^-}^{\infty} \frac{dv}{dt} e^{-st} dt = v(t)e^{-st}|_{t=0^-}^{\infty} - \frac{1}{s}\int_{0^-}^{\infty} v(t)e^{-st} dt$
  - $\mathcal{L}\left\{\frac{d^2v}{dt^2}\right\} = s^2 V(s) - sv(0^-) - v'(0^-)$
  - Differential operator, s
- Time integration, $\int_{0^-}^{t} v(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s} V(s)$
  - $\mathcal{L}\left\{\int_{0^-}^{t} v(\tau)d\tau\right\} = \left(\int_{0^-}^{t} v(\tau)d\tau\right) \cdot \left(-\frac{1}{s}e^{-st}\right)\Big|_{t=0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} v(t)e^{-st} dt$
    - † $\frac{d}{dt}\left(\int_{0^-}^{t} v(\tau)d\tau\right) = v(t)$ … Leibniz integral rule
  - Integral operator, $1/s$

Laplace Transform: Circuit Analysis

- Example 14.6 RL series circuit, $L = 2\ [H]$, $R = 4\ [\Omega]$, and $v_s(t) = 3u(t)\ [V]$
  - $2\frac{di}{dt} + 4i = 3u(t)$
  - $2\left(sI(s) - i(0^-)\right) + 4I(s) = \frac{3}{s}$
  - $I(s) = \frac{1.5}{s(s+2)} + \frac{5}{s+2} = \frac{0.75}{s} + \frac{4.25}{s+2} \Rightarrow i(t) = (0.75 + 4.25e^{-2t})u(t)$
- Example 14.7 RC series circuit
  - $\frac{v(t)-u(t)}{4} + \frac{1}{16}\frac{dv}{dt} = 0$
  - $V(s) = \frac{4}{s(s+4)} + \frac{9}{s+4} = \frac{1}{s} + \frac{8}{s+4} \Rightarrow v(t) = (1 + 8e^{-4t})u(t)$

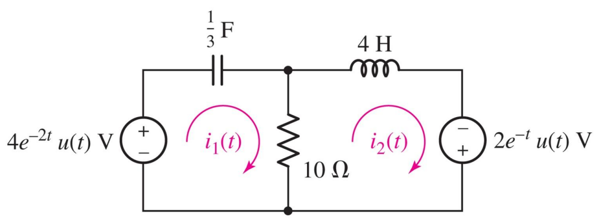


Laplace Transform: Properties

- Frequency shift
  - $\mathcal{L}\{e^{at}f(t)\} = \int_{0^-}^{\infty} e^{at}f(t)e^{-st}dt = F(s-a)$
  - $\mathcal{L}\{e^{j\omega t}u(t)\} = \frac{1}{s-j\omega}$, $\mathcal{L}\{\cos\omega t\, u(t)\} = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}\{\sin\omega t\, u(t)\} = \frac{\omega}{s^2+\omega^2}$
- Time shift
  - $\mathcal{L}\{f(t-t_0)u(t-t_0)\} = \int_{t_0}^{\infty} f(t-t_0)e^{-st}dt = e^{-st_0}F(s)\ (t_0 \geq 0)$
  - $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$
- Frequency derivative
  - $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$
    - † $\frac{d}{ds}\left(\int_{0^-}^{\infty} f(t)e^{-st}dt\right) = -\int_{0^-}^{\infty} tf(t)e^{-st}dt$
  - $\mathcal{L}\{te^{-\alpha t}u(t)\} = -\frac{d}{ds}\left(\frac{1}{s+\alpha}\right) = \frac{1}{(s+\alpha)^2}$

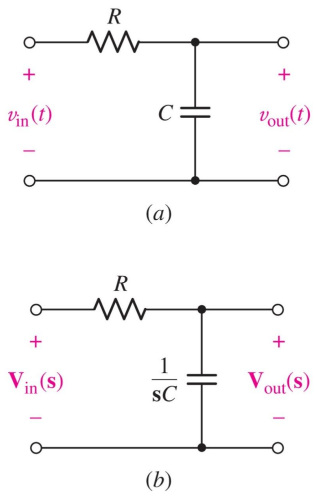
Laplace Transform: Mesh Analysis

- Example 14.13 Find mesh currents (zero initial conditions)
  - $3\int_{-\infty}^{t} i_1(\tau)d\tau + 10(i_1 - i_2) = 4e^{-2t}u(t)$ and $10(i_2 - i_1) + 4\frac{di_2}{dt} = 2e^{-t}u(t)$
  - $\frac{3}{s}I_1(s) + 10\big(I_1(s) - I_2(s)\big) = \frac{4}{s+2}$ and $10\big(I_2(s) - I_1(s)\big) + 4\big(sI_2(s) - i_2(0^-)\big) = \frac{2}{s+1}$
  - $\left(\frac{3}{s} + 10\right)I_1(s) - 10I_2(s) = \frac{4}{s+2}$ and $-10I_1(s) + (4s+10)I_2(s) = \frac{2}{s+1}$
  - $I_1(s) = \frac{2s(4s^2+19s+20)}{20s^4+66s^3+73s^2+57s+30}$ and $I_1(s) = \frac{30s^2+43s+6}{(s+2)(20s^3+26s^2+21s+15)}$



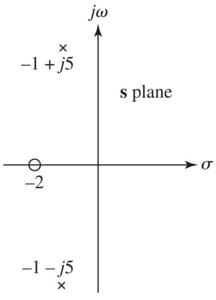
Transfer Function

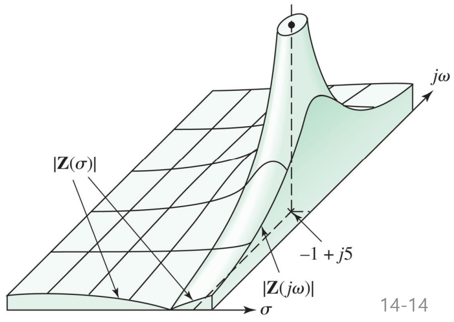
- $H(s) = \frac{V_{out}}{V_{in}}$ … Ratio of output to input.
  - $\frac{v_{in}-v_{out}}{R} = C\frac{dv_{out}}{dt}$
  - $(RCs+1)V_{out}(s) = V_{in}(s)$
    - † Assuming zero initial condition
  - $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+sRC}$
    - † Once we know the transfer function of a circuit, we can easily find the output for any other input.
    - † Transfer function itself provides a valuable information on the circuit.
    - † Pole at $s = -\frac{1}{RC}$



Plot of $F(s)$: s-plane

- $F(s)$ is a complex-valued function of complex variable.
  - Plot of $|F(s)|$ in s-plane reveals properties of $F(s)$.
- Pole-zero constellations
  - $Z(s) = \frac{s+2}{s^2+2s+16} = \frac{s+2}{(s+1)^2+5^2}$
    - † Poles at $s_{1,2} = -1 \pm j5$ and zero at $s = -2$





Convolution

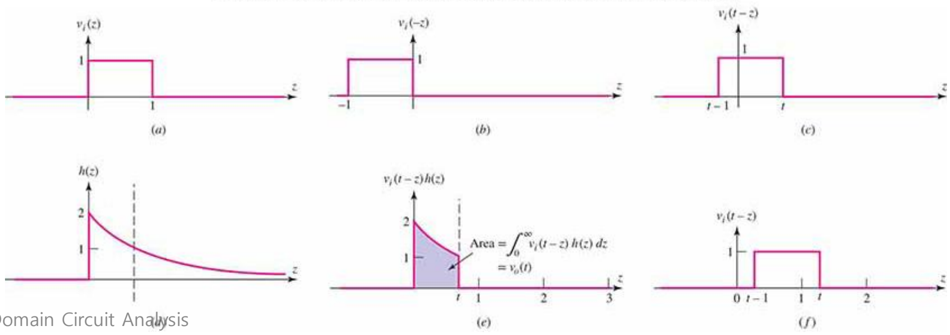
- Impulse response
  - $h(t) = \mathcal{L}^{-1}\{H(s)\}$, where $H(s)$ … transfer (system) function of circuit
  - For an input $x(t)$, corresponding output $y(t)$ is given by
    - $Y(s) = H(s)X(s)$
    - $y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$ … convolution

Independent variable in the integrand is τ.

- Convolution
  - $x(t-\tau) = x\big(-(\tau-t)\big)$ … folding followed by shift-t
  - For a fixed time, t
    - Fold & shift (by t) to get $x(t-\tau)$
    - Multiple to $h(\tau)$, and then integrate.

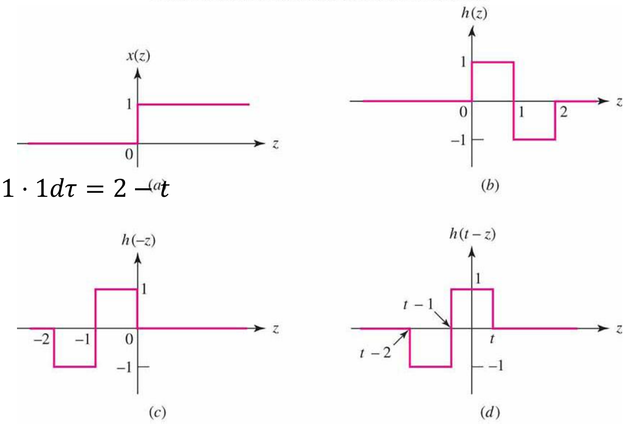
Convolution: Graphical Approach

- Input, $x(t) = u(t) - u(t-1)$ and impulse response, $h(t) = 2e^{-t}u(t)$.
  - Support interval of $x(\tau)$ … $(0, 1)$
    - $x(t-\tau) \neq 0$, for $0 < t-\tau < 1$ or $t-1 < \tau < t$
  - Support interval of $h(\tau)$ … $(0, \infty)$



Convolution

- Example 14.18 $x(t) = u(t)$ and $h(t) = u(t) - 2u(t-1) + u(t-2)$
  - ∨ $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
  - ∨ $t < 0,\ y(t) = 0$
  - ∨ $0 < t < 1,\ y(t) = \int_0^t 1 \cdot 1 d\tau = t$
  - ∨ $1 < t < 2,\ y(t) = \int_0^{t-1} 1 \cdot (-1) d\tau + \int_{t-1}^{t} 1 \cdot 1 d\tau = 2 - t$
  - ∨ $t > 2,\ y(t) = 0$



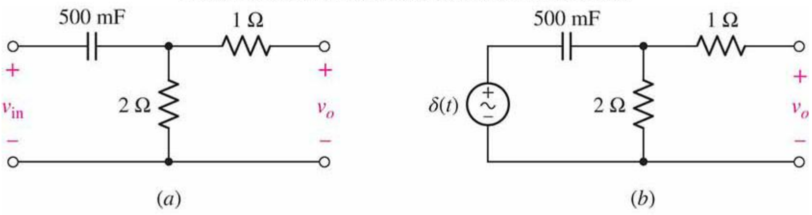
Convolution in s-domain

- Given $y(t) = h(t) * x(t)$
  - ∨ $\mathcal{L}\{h(t) * x(t)\} = \mathcal{L}\left\{\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right\} = \int_{0^-}^{\infty}\left(\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right)e^{-st}dt$
    - † Assume $h(t) = x(t) = 0$, for $t < 0$
  - ∨ $\mathcal{L}\{h(t) * x(t)\} = \int_{0^-}^{\infty} h(\tau)\left(\int_{0^-}^{\infty} x(t-\tau)e^{-s(t-\tau)}dt\right)e^{-s\tau}d\tau = H(s)X(s)$

- Example 14.19 $V(s) = \frac{1}{(s+\alpha)(s+\beta)}$
  - ∨ Let $V_1(s) = \frac{1}{s+\alpha}$ and $V_2(s) = \frac{1}{s+\beta}$
  - ∨ $v_1(t) = e^{-\alpha t}u(t)$ and $v_2(t) = e^{-\beta t}u(t)$
  - ∨ $v(t) = v_1(t) * v_2(t) = \int_{0^-}^{t} e^{-\alpha\tau} \cdot e^{-\beta(t-\tau)}dt = \frac{1}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right)u(t)$

Impulse Response & Transfer Function

- $h(t) \overset{\mathcal{L}}{\leftrightarrow} H(s) = \frac{Y(s)}{X(s)}$



- Example 14.20 $v_{in}(t) = 6e^{-t}u(t)[V]$
  - $C\frac{d}{dt}\left(v_{in}(t) - v_o(t)\right) = \frac{1}{R_2}v_o(t) \Rightarrow Cs\left(V_{in}(s) - V_o(s)\right) = \frac{1}{R_2}V_o(s)$
  - $\left(1 + \frac{1}{R_2Cs}\right)V_o(s) = V_{in}(s)$ or $V_o(s) = \frac{R_2Cs}{1+R_2Cs}V_{in}(s) = \frac{s}{s+1}V_{in}(s)$
  - When $v_{in}(t) = \delta(t)$ or $V_{in}(s) = 1$, $H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{s}{s+1}$ and $h(t) = \delta(t) - e^{-t}u(t)$
  - $V_o(s) = H(s)V_{in}(s) = \frac{s}{s+1} \cdot \frac{6}{s+1} = -\frac{6}{(s+1)^2} + \frac{6}{s+1} \Rightarrow v_o(t) = 6(1-t)e^{-t}u(t)$