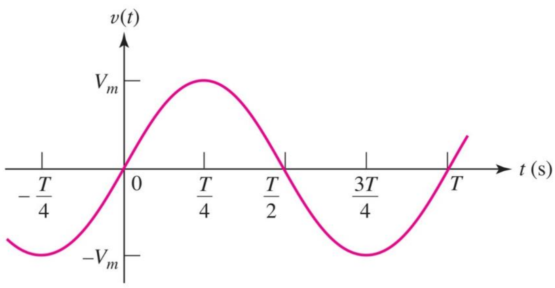
Chapter 10. Sinusoidal Steady-state Analysis

1. Forced response to sinusoidal source
2. Complex forcing function and Phasor
3. AC circuit analysis using Phasor
4. Phasor diagram

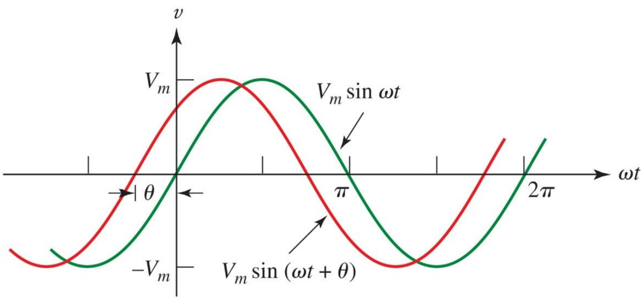
Sinusoids 정현파

- $v(t) = V_m \cdot \sin \omega t$



- Amplitude or magnitude, $V_m > 0$
- Angular frequency, $\omega = 2\pi f \, [\frac{rad}{sec}]$, frequency $f \, [Hz]$
- Periodic with period $T = 1/f$

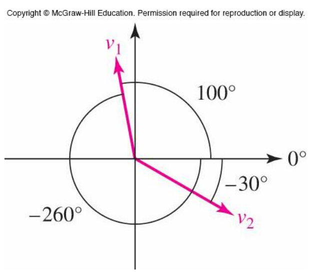
Phase of Sin Wave



- $v_1(t) = V_m \cdot \sin(\omega t + \theta)$
  - $v_1(t)$ leads $v_0(t) = V_m \cdot \sin \omega t$ by θ.
  - $v_0(t)$ lags $v_1(t)$ by θ.
  - $V_m \cdot \sin(\omega t + \theta_1)$ and $V_m \cdot \sin(\omega t + \theta_2)$ are in-phase (동상), if $\theta_1 = \theta_2$.
    - Out-of-phase, if $\theta_1 \neq \theta_2$
- Example: $v_1(t) = V_{m1} \cdot \cos(5t + 10°)$ and $v_2(t) = V_{m2} \cdot \sin(5t - 30°)$
  - $v_1(t)$ leads $v_2(t)$ by 130°.
  - $\sin \omega t = \cos(\omega t - 90°)$

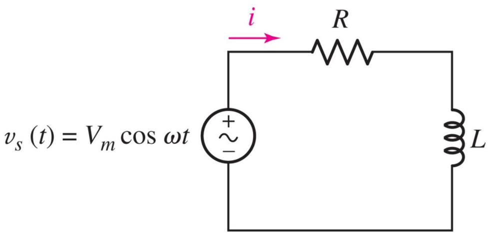
Phasor Diagram

- Given $v_1(t) = V_{m1} \cdot \cos(5t + 10°)$ and $v_2(t) = V_{m2} \cdot \sin(5t - 30°)$
  - Graphical representation of two sinusoids $v_1(t)$ and $v_2(t)$
    - Amplitude and phase information:
      - Vector notation, phasor, v_1 and v_2
    - Only possible when they have the same frequency.
    - Leading/lagging is determined by the angular difference between two vectors in CW direction.
      - v_2 leads v_1 by 130°.
      - v_1 lags v_2 by 130°.



Forced Response to Sine Sources

- Steady-state response of RL series circuit to the sinusoidal source
  - $v_s(t) = V_m \cdot \cos\omega t,\ -\infty < t < \infty$



  - KVL: $L\frac{di}{dt} + Ri = v_s(t)$
  - Choose $i(t) = A \cdot \cos\omega t + B \cdot \sin\omega t$
    - † Method of undetermined coefficient, 미정 계수법
  - $i(t) = \frac{RV_m}{R^2+\omega^2L^2}\cos\omega t + \frac{\omega LV_m}{R^2+\omega^2L^2}\sin\omega t$

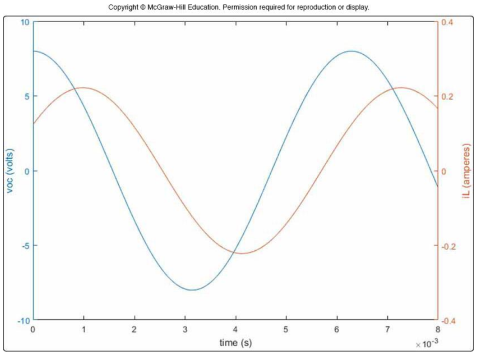
Forced Response to Sine Sources

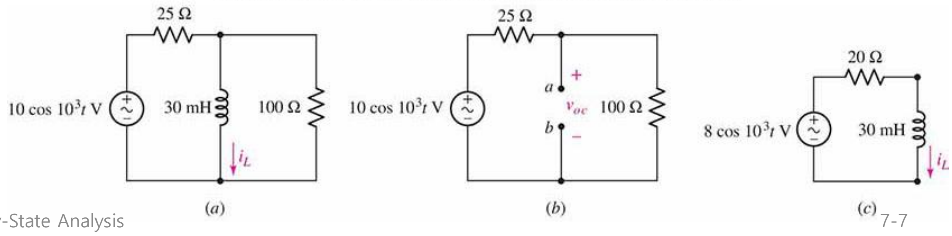
- If $i(t) = a \cdot \cos\omega t + b \cdot \sin\omega t$,
  - $i(t) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}cos\,\omega t + \frac{b}{\sqrt{a^2+b^2}}sin\,\omega t\right) = \sqrt{a^2+b^2}\,cos(\omega t - \theta)$
    - † where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
- In the RL series circuit,
  - $i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$
    - † Current $i(t)$ lags the source voltage $v_s(t)$ by $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$, $0° < \theta < 90°$.

In a linear circuit with sinusoidal source, every circuit variables (currents as well as voltages) are sinusoids with the same angular frequency as the source.

Steady-State Response

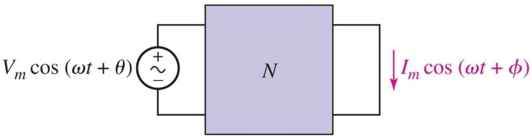
- Example 10.1 Find $i_L(t)$.
  - Thevenin equivalent seen from terminals $a-b$ (or from the inductor):
  - $v_{oc} = v_s(t) \cdot \frac{100}{100+25} = 8 \cdot \cos 10^3 t$ and $R_{th} = 25||100 = 20\ [\Omega]$
  - $i_L(t) = \frac{8}{\sqrt{20^2+10^6 \cdot 9\times10^{-4}}} \cdot \cos(10^3 t - \tan^{-1}\frac{30}{20})$
    - † $i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$





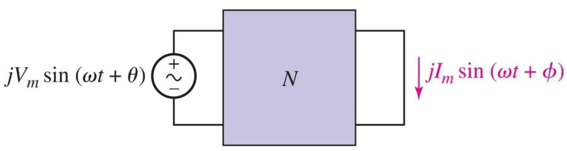
Complex Forcing Function

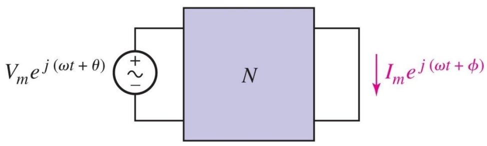
- $v_s(t) = V_m \cdot e^{j(\omega t+\theta)} = V_m \cdot \cos(\omega t + \theta) + jV_m \cdot \sin(\omega t + \theta)$



Assume response $i(t) = I_m \cdot \cos(\omega t + \phi)$ when $v_s(t) = V_m \cdot \cos(\omega t + \theta)$

- $V_m \cdot \sin(\omega t + \theta) = V_m \cdot \cos(\omega t + \theta - 90°) \rightarrow I_m \cdot \cos(\omega t + \phi - 90°) = I_m \cdot \sin(\omega t + \phi)$





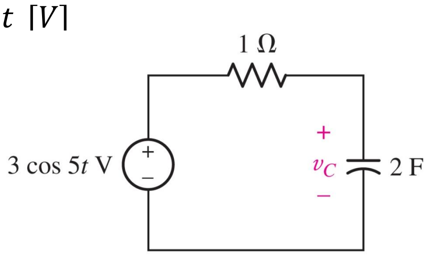
Real part of complex response is generated by the real part of the complex forcing function, and so is the imaginary part.

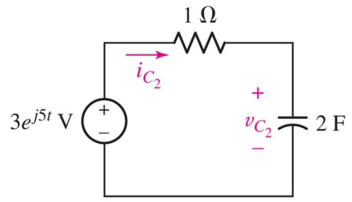
Steady-State Response to Complex Source

- Consider a series RL circuit with $v_s(t) = V_m \cdot e^{j\omega t}$
  - $L\frac{di}{dt} + Ri = V_m \cdot e^{j\omega t}$
  - Choose $i(t) = I_m \cdot e^{j(\omega t+\phi)}$ $(= I_m e^{j\phi} \cdot e^{j\omega t})$
    - $j\omega L I_m e^{j(\omega t+\phi)} + R I_m e^{j(\omega t+\phi)} = V_m \cdot e^{j\omega t}$
    - $j\omega L I_m e^{j\phi} + R I_m e^{j\phi} = V_m \Rightarrow I_m e^{j\phi}(j\omega L + R) = V_m$
    - $I_m e^{j\phi} = \frac{V_m}{R+j\omega L} = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \exp\left\{-j\tan^{-1}\left(\frac{\omega L}{R}\right)\right\} = I_m \angle\phi$ (polar notation: phasor)
    - $I_m = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}$ and $\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$
  - Response to the source $v_s(t) = V_m \cdot \cos\omega t$ is then
    - $i(t) = I_m \cdot \cos(\omega t + \phi) = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cdot \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$

Steady-State Response

- Example 10.2 Find $v_c(t)$, when $v_s(t) = 3 \cdot \cos 5t$ $[V]$
  - Circuit equation:
    - $2\frac{dv_c}{dt} = v_s(t) - v_c(t)$
  - Equation with exponential source
    - $2\frac{dv_{ce}}{dt} = 3e^{j5t} - v_{ce}(t)$, where $v_{se}(t) = 3e^{j5t}$
  - Let $v_{ce}(t) = V_m e^{j5t}$ (much easier to solve)
    - $V_m = \frac{3}{1+j10} = \frac{3}{\sqrt{1+100^2}} \angle -\tan^{-1} 10$
  - $v_c(t) = Re\{v_{ce}(t)\} = 29.85\cos(5t - 84.3°)$





Phasor

In a linear circuit, every circuit variables with sinusoidal input is sinusoid with the same angular frequency as the source. Thus, they can be represented by their magnitude and phase only, known as phasor.

- The term $e^{j\omega t}$ is common to all circuit variables:
  - Guaranteed in the method of undetermined coefficients
- Phasor

sinusoidal function	complex function	phasor
$v(t) = V_m \cos \omega t$	$v_c(t) = V_m e^{j\omega t}$	$\mathbf{V} = V_m \angle 0°$
$i(t) = I_m \cos(\omega t + \phi)$	$i_c(t) = I_m e^{j(\omega t + \phi)}$	$\mathbf{I} = I_m \angle \phi$

  - $i(t) = I_m \cos(\omega t + \phi) = Re\{I_m e^{j(\omega t+\phi)}\} = Re\{\mathbf{I} e^{j\omega t}\}$
  - Phasor representation of a voltage (or current) is a frequency-domain representation.

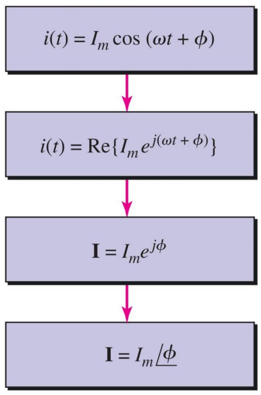
Phasor

- Example 10.3
  - $v_a(t) = 100\cos(400t - 30°)$, $\boldsymbol{V}_a = 100\angle -30°$
  - $v_b(t) = -5\sin(580t - 110°) = -5\cos(580t - 200°) = 5\cos(580t - 20°)$

    $\boldsymbol{V}_b = 5\angle -20°$
  - $v_c(t) = 3\cos(4t - 30°) + 4\sin(4t - 100°) = 3\cos(4t - 30°) + 4\cos(4t + 170°)$

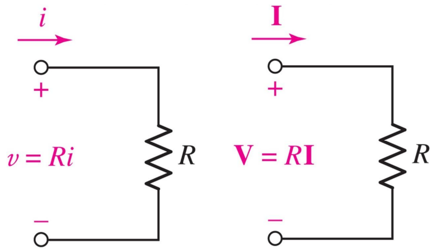
    $\boldsymbol{V}_c = 3\angle -30° + 5\angle 170°$

    $\boldsymbol{V}_c = (2.5981 - j1.5) + (-4.9240 + j.8682) = -2.326 - j.6318 = 2.4102\angle -164.8044°$

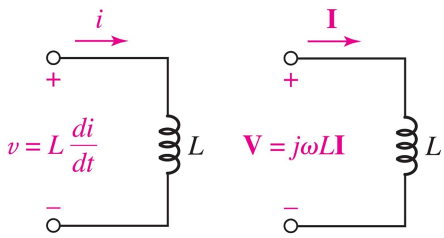


Phasor: Resistor

- Phasor in resistor
  - $v(t) = Ri(t)$
  - When $i(t) = I_m e^{j(\omega t+\theta)}$, $v(t) = RI_m e^{j(\omega t+\theta)}$
  - $\mathbf{V} = R\mathbf{I}$, where $\mathbf{I} = I_m e^{j\theta}$
  - $R = \frac{\mathbf{V}}{\mathbf{I}}$



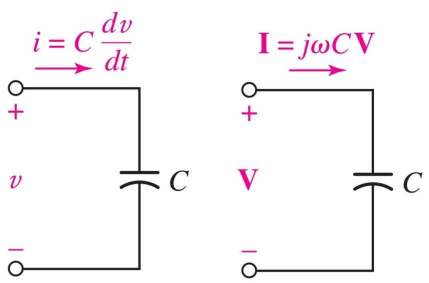
Phasor: Inductor



- Phasor in inductor
  - $v(t) = L\frac{di(t)}{dt}$
  - When $i(t) = I_m e^{j(\omega t+\phi)}$, $v(t) = LI_m \cdot j\omega \cdot e^{j(\omega t+\phi)} = j\omega L I_m e^{j(\omega t+\phi)}$
  - $\mathbf{V} = j\omega L\mathbf{I}$, where $\mathbf{I} = I_m e^{j\phi}$
  - $Z_L = \frac{\mathbf{V}}{\mathbf{I}} = j\omega L$
    - † Differentiation in time becomes multiplication in phasor form.

Phasor: Capacitor

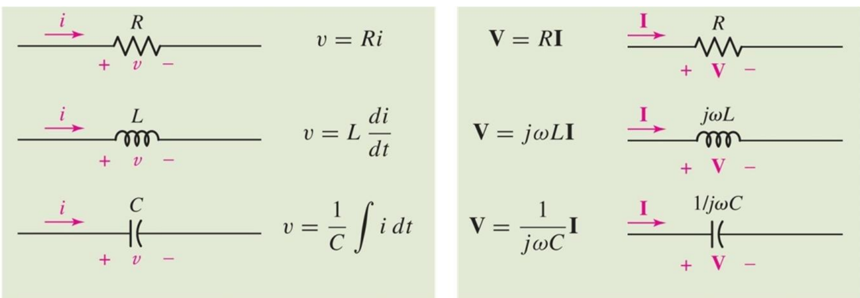
- Phasor in capacitor
  - ∨ $i(t) = C\frac{dv(t)}{dt}$
  - ∨ When $v(t) = V_m e^{j(\omega t+\theta)}$, $i(t) = j\omega C V_m e^{j(\omega t+\theta)}$
  - ∨ $\boldsymbol{I} = j\omega C\boldsymbol{V}$, where $\mathbf{V} = V_m e^{j\theta}$
  - ∨ $Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$



Phasor: Voltage/current relationship

Time Domain

Phasor (frequency domain)

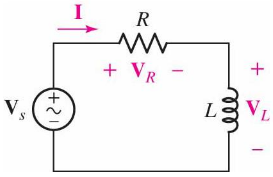


Calculus (hard but real)

Algebra (easy but complex)

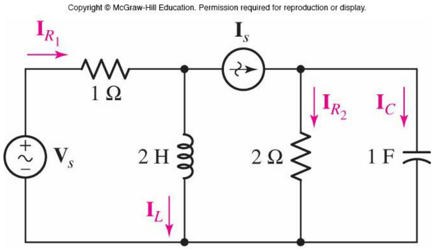
KVL & KCL

- KVL: $\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{0}$ and KCL: $\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{0}$
- Example: RL series circuit
  - $v_R(t) + v_L(t) = v_s(t)$
  - $\mathbf{V}_R + \mathbf{V}_L = \mathbf{V}_S$
  - $R\mathbf{I} + j\omega L\mathbf{I} = \mathbf{V}_s$
  - $I = \frac{\mathbf{V}_s}{R+j\omega L} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \angle - \tan^{-1}\left(\frac{\omega L}{R}\right)$



Circuit Analysis with Phasor

- Example 10.5 Find $i_s(t)$, when $\omega = 2[rad/s]$ and $\mathbf{I}_C = 2\angle - 28°$.
  - $\mathbf{V}_c = \frac{1}{j\omega C}\mathbf{I}_c = 1\, e^{-j62°}$ (exponents are usually written in [rad]).
  - $\mathbf{I}_{R2} = \frac{1}{2}\mathbf{V}_c$
  - $\mathbf{I}_s = \mathbf{I}_{R2} + \mathbf{I}_c = \frac{1}{2}(\cos(-62°) + j\sin(-62°)) + 2(\cos(-28°) + j\sin(-28°))$
    $\mathbf{I}_s = (.2347 - j.4415) + (1.7659 - j.9389) = 2.0 - j1.3804 = 2.4307\angle - 34.6053°$
  - $i_s(t) = 2.4307\cos(2t - 34.6053°)$



Impedance

- Impedance, $\mathbf{Z} = \mathbf{V}/\mathbf{I}$
  - $\mathbf{V} = R\mathbf{I}$, $\mathbf{V} = j\omega L\mathbf{I}$, $\mathbf{V} = \frac{1}{j\omega C}$: $\mathbf{Z}_R = R$, $\mathbf{Z}_L = j\omega L$, and $\mathbf{Z}_C = \frac{1}{j\omega C}$
    - † Impedance is the equivalent of resistance in phasor (frequency domain).
    - † Impedance is a complex number.
    - † Impedance in series/parallel can be combined as resistance.
- Example
  - Series connection of a inductor ($L = 5\ [mH]$) and a capacitor ($C = 100\ [\mu F]$) at $\omega = 10^4 [rad/s]$
  - $\mathbf{Z}_S = \mathbf{Z}_L + \mathbf{Z}_C = j(10^4)(5 \times 10^{-3}) + \frac{1}{j(10^4)(100\times10^{-6})} = j49\ [\Omega]$
  - If parallel, $\mathbf{Z}_p = \mathbf{Z}_L||\mathbf{Z}_C = \mathbf{Z}_L|| = \frac{\mathbf{Z}_L\mathbf{Z}_C}{\mathbf{Z}_L+\mathbf{Z}_C} = \frac{50}{j49} [\Omega]$

Admittance & Reactance

- Admittance, $\mathbf{Y} = \frac{1}{\mathbf{Z}}$
  - $\boldsymbol{Y}_R = \frac{1}{R}$, $\boldsymbol{Y}_L = \frac{1}{j\omega L}$, and $\boldsymbol{Y}_C = j\omega C$
- If $\mathbf{Z} = R + jX$ (rectangular form)
  - R … resistance
  - X … reactance $[\Omega]$
- If $\mathbf{Y} = G + jB$
  - G … conductance
  - B … susceptance $[S, Siemen]$

Combination of Impedance

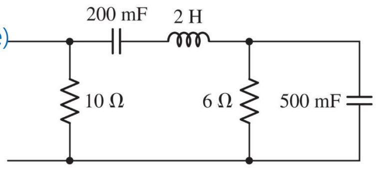
- Example 10.6 $\omega = 5\ [rad/s]$ radian (not degree)
  - $6||(-j0.4) = \frac{-j2.4}{6-j0.4} = \frac{-j2.4}{6.0133\angle -0.0666} = 0.3991\angle -1.5042$

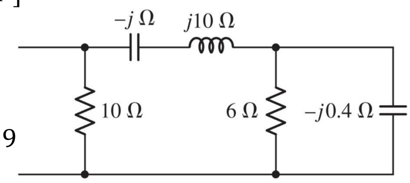
    or $.02655 - j.3982$
  - Series connection of $2\ [H]$, $200\ [mF]$ and $6\ [\Omega]||500[mF]$

    $-j + j10 + .0265 - j.3982 = .02655 + j8.602$
  - Parallel connection to $10\ [\Omega]$

    $(.02655 + j8.602)||10 = \frac{.2655+j86.02}{10.02655+j8.602} = 4.255 + j4.929$

    or $6.511\angle 49.20°[\Omega]$

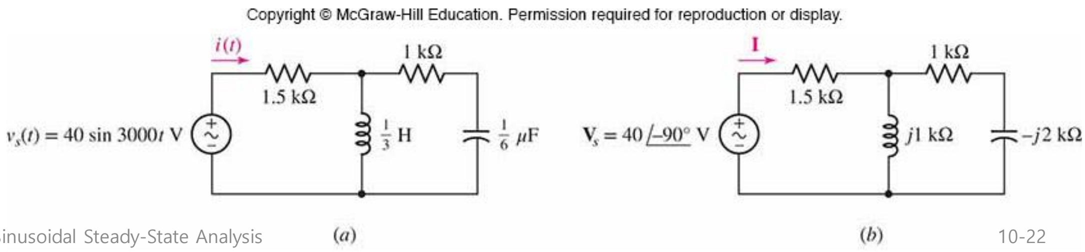




Circuit Analysis with Phasor

- Example 10.7 Find $i(t)$.
  - $\mathbf{Z}_{eq} = 1.5 + (1 - j2)||j = 1.5 + \frac{2+j}{1-j} = 1.5 + \frac{1+j3}{2} = 2 + j1.5 = 2.5e^{j36.87°}$,
  - $\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_{eq}} = 16e^{-j126.9°}$
  - $i(t) = 16\cos(3000t - 126.9°)\ [mA]$

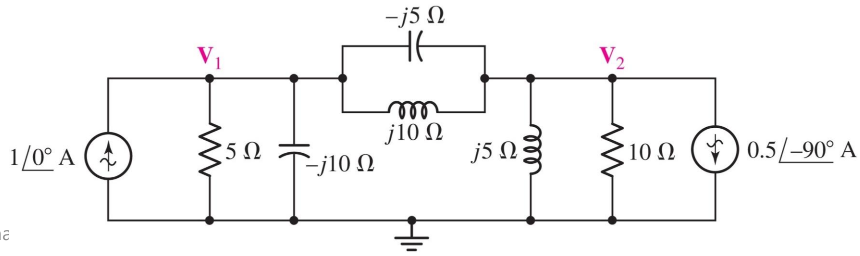
    † Impedance seen from the source is inductive (Positive phase in $\mathbf{Z}_{eq}$).



Nodal Analysis with Phasor

- Example 10.8 Find $v_1(t)$.
  - Phasor node voltages $\mathbf{V}_1$ and $\mathbf{V}_2$.

    $\frac{\mathbf{V}_1}{5} + \frac{\mathbf{V}_1}{-j10} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{-j5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j10} = 1, \frac{\mathbf{V}_2 - \mathbf{V}_1}{-j5} + \frac{\mathbf{V}_2 - \mathbf{V}_1}{j10} + \frac{\mathbf{V}_2}{j5} + \frac{\mathbf{V}_2}{10} = j0.5$
  - $(0.2 + j0.2)\mathbf{V}_1 - j0.1\mathbf{V}_2 = 1$ and $-j0.1\mathbf{V}_1 + (0.1 - j0.1)\mathbf{V}_2 = j0.5$
  - $\mathbf{V}_1 = 1 - j2 = 2.24\angle -63.4°$ and $\mathbf{V}_2 = -2 + j4 = 4.47\angle 116.6°$ $[V]$
  - $v_1(t) = 2.24\cos(\omega t - 63.4°)$ and $v_2(t) = 4.47\cos(\omega t + 116.6°)$



Mesh Analysis with Phasor

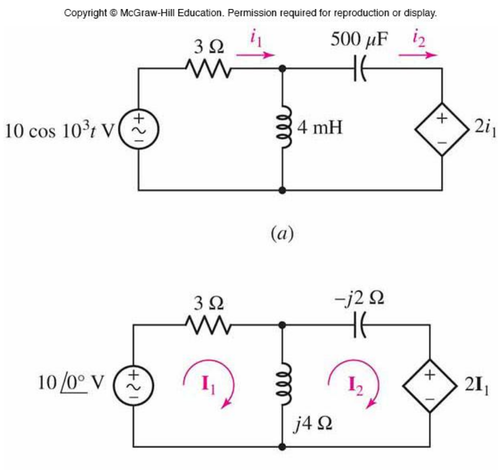
- Example 10.9 Find $i_1(t)$.
  - Phasor mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$.

    $3\mathbf{I}_1 + j4(\mathbf{I}_1 - \mathbf{I}_2) = 10$ or $(3 + j4)\mathbf{I}_1 - j4\mathbf{I}_2 = 10$

    $3\mathbf{I}_1 + j4(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 + 2\mathbf{I}_1 = 0$ or

    $(2 - j4)\mathbf{I}_1 + j2\mathbf{I}_2 = 0$
  - $\mathbf{I}_1 = \frac{14+j8}{13} = 1.24\angle 29.7°$ and $\mathbf{I}_2 = \frac{20+j30}{13} = 2.77\angle 56.3°$ $[V]$
  - $i_1(t) = 1.24\cos(\omega t + 29.7°)$ and

    $i_2(t) = 2.77\cos(\omega t + 56.3°)$

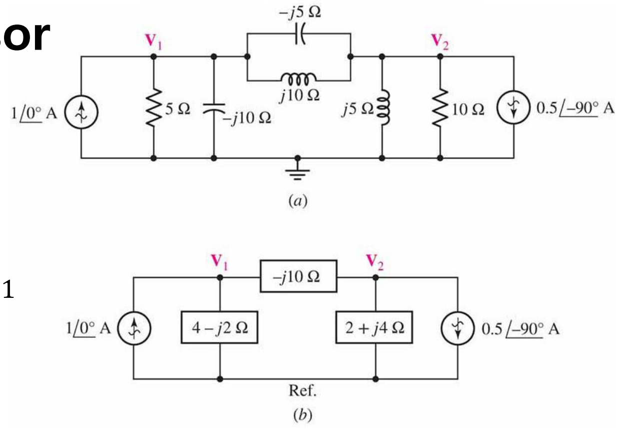


(a)

(b)

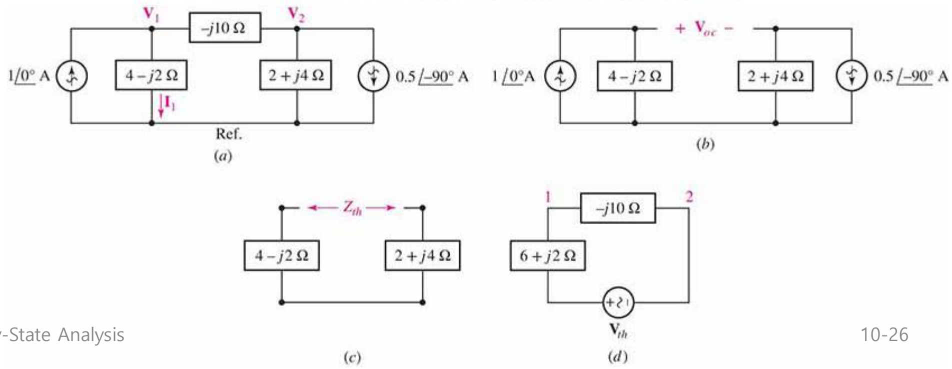
Superposition with Phasor

- Example 10.10 Find $v_1(t)$.
  - ∨ $\mathbf{V}_1 = \mathbf{V}_{1L} + \mathbf{V}_{1R}$.
  - ∨ $\mathbf{V}_{1L} = \frac{(4-j2)\cdot(-j10+2+j4)}{(4-j2)+(-j10+2+j4)} 1\angle 0° = 2 - j2$
  - ∨ $\mathbf{V}_{1R} = \frac{2+j4}{(4-j2)+(-j10+2+j4)} (-.5\angle -90°) = -1$
  - ∨ $\mathbf{V}_1 = 1 - j2\ [V]$
  - ∨ $v_1(t) = 2.24\cos(\omega t - 63.4°)$



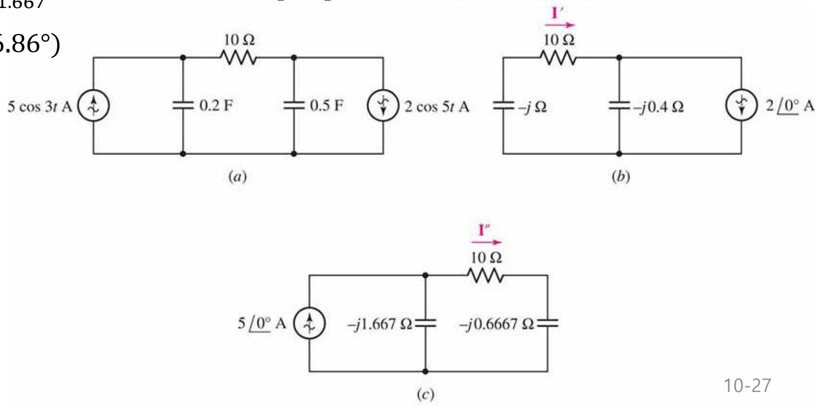
Thevenin with Phasor

- Example 10.11 Find the Thevenin equivalent circuit seen from $-j10$.
  - ∨ $\mathbf{V}_{oc} = (1\angle 0°)\cdot(4-j2) - (-.5\angle -90°)(2+j4)[V]$
  - ∨ $\mathbf{Z}_{th} = (4-j2) + (2+j4) = 6 + j2$, $\mathbf{I}_{12} = \frac{6-j3}{(6+j2)-j10} = 0.6 + j0.3[\Omega]$
  - ∨ $\mathbf{I}_1 = 1 - (0.6 + j0.3)\ [A]$

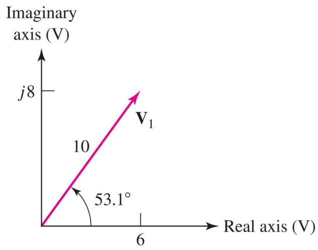


Thevenin with Phasor

- Example 10.12 Two sources with different frequencies.
  - $\mathbf{I}' = (2\angle 0°) \cdot \frac{-j0.4}{10-j-j0.4} = 79.23\angle -82.03°\ [mA]$, $i'(t) = 79.23\cos(5t - 82.03°)$
  - $\mathbf{I}'' = (5\angle 0°) \cdot \frac{-j1.667}{10-j0.6667-j1.667} = 811.7\angle -76.86°\ [mA]$,
  - $i''(t) = 811.7\cos(3t - 76.86°)$

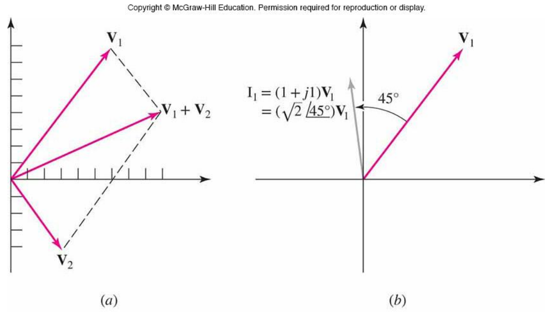


Phasor Diagram



- Complex phasor as a vector in the complex plane
  - $\mathbf{V}_1 = 6 + j8 = 10\angle 53.1°\ [V]$
    - † $v(t)$ … projection onto the real axis
  - Phasor addition, $\mathbf{V}_1 + \mathbf{V}_2$ = vector addition
  - vi-relation as phasors

    $\mathbf{I}_1 = \mathbf{Y}\mathbf{V}_1$, with $\mathbf{I}_1 = (1+j)\mathbf{V}_1 = \left(\sqrt{2}\angle 53.1°\right)\mathbf{V}_1$



Phasor Diagram: Parallel RLC Circuit

- Example:
  - Given $\mathbf{V} = 1\angle 0° \; [V]$
  - $\mathbf{I}_R = 0.2\angle 0° \; [A]$
  - $\mathbf{I}_L = 0.1\angle -90° \; [A]$
  - $\mathbf{I}_C = 0.3\angle 90° \; [A]$
    - $\mathbf{I}_x = \mathbf{I}_R + \mathbf{I}_L = 0.2 - j0.1 = 0.224\angle -26.6° \; [A]$
    - $\mathbf{I}_S = \mathbf{I}_x + \mathbf{I}_C = 0.283\angle 45° \; [A]$
    - $\mathbf{I}_S$ leads $\mathbf{I}_S$ by 45°, $\mathbf{I}_C$ by −45°, and $\mathbf{I}_x$ by 71.6°.
    - Angles are relative to the voltage phasor.