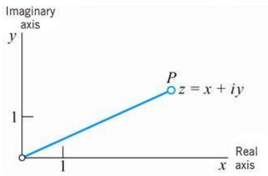
Chapter 13. Complex Numbers & Functions

Complex Numbers

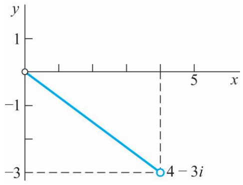
- Ordered pair (x, y) of real numbers x and y
  - $z = (x, y) = x + jy$
    - $x = Re\{z\}$, $y = Im\{z\}$, and $j = (0, 1)$
    - $j^2 = -1$
- Operations in complex numbers
  - Addition, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
  - Subtraction, $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$
  - Multiplication, $(x_1 + jy_1) \cdot (x_2 + jy_2) = x_1x_2 - y_1y_2, +j(x_1y_2 + x_2y_1)$
  - Division, $\frac{x_1+jy_1}{x_2+jy_2} = \frac{(x_1+jy_1)(x_2-jy_2)}{(x_2+jy_2)(x_2-jy_2)} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + j\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$

Complex Numbers: Geometric Representation

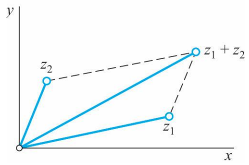
- Complex plane
  - Complex number as a point in the complex plane (Real-Imaginary axes)



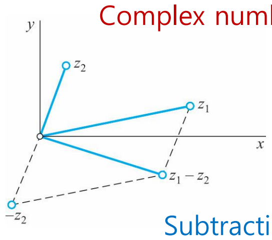
complex plane



Complex number in complex plane



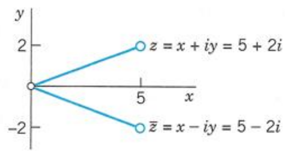
Addition of complex numbers



Subtraction of complex numbers

Complex Conjugate

- Given $z = x + jy$
  - Complex conjugate, $\bar{z} = z^* = x - jy$

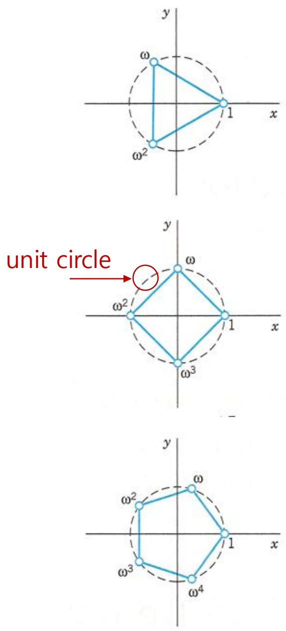


- Complex conjugate: Properties
  - $Re\{z\} = x = \frac{1}{2}(z + z^*)$, $Im\{z\} = y = \frac{1}{j2}(z - z^*)$
  - $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$
  - $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$
  - $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

Polar Form of Complex Numbers

- $z = x + jy$
  - Polar form, $z = r\angle\theta = re^{j\theta} = r(\cos\theta + j\sin\theta)$
    - † $r = \sqrt{x^2 + y^2} = |z|$ … magnitude, absolute value, or modulus of z
      - – $|z| = \sqrt{zz^*}$
    - † $\theta = \tan^{-1}\frac{y}{x} = \arg z$ … phase or argument of z
      - – Principal value, $\mathrm{Arg}(z)$, $-\pi < \mathrm{Arg}(z) \le \pi$
- Multiplication
  - Given $z_1 = r_1\angle\theta_1$, and $z_2 = r_2\angle\theta_2$, $z_1 \cdot z_2 = r_1 \cdot r_2\angle(\theta_1 + \theta_2)$
    - † $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
    - † $z_1 \cdot z_2 = r_1(\cos\theta_1 + j\sin\theta_1) \cdot r_2(\cos\theta_2 + j\sin\theta_2) = r_1 r_2(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$

Polar Form of Complex Numbers



- Division
  - Given $z_1 = r_1\angle\theta_1$, and $z_2 = r_2\angle\theta_2$, $\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$
    - † $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- De Moivre's formula
  - $z^n = \left(r(\cos\theta + j\sin\theta)\right)^n = r^n(\cos n\theta + j\sin n\theta)$
- Roots
  - n^{th}-root of z is a complex number satisfying $z = w^n$
    - † $w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + j\sin\frac{\theta+2k\pi}{n}\right)$, $k = 0, 1, \ldots, n-1$
  - n^{th}-root of unity: $\sqrt[n]{1} = \cos\frac{2k\pi}{n} + j\sin\frac{2k\pi}{n}$
    - † $\{1, w, \ldots, w^{n-1}\}$, where $w = \cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n}$

Some Topologies in Complex Plane

- Unit circle: $\{z: |z| = 1\}$
- Open circular disk: $\{z: |z - a| < \rho\}$
- Closed circular disk: $\{z: |z - a| \leq \rho\}$
- Neighborhood of a : An open circular disk, $\{z: |z - a| < \rho\}$
- Open annulus: $\{z: \rho_1 < |z - a| < \rho_2\}$
- Closed annulus: $\{z: \rho_1 \leq |z - a| \leq \rho_2\}$
- Upper half-plane: Set of all point $\{z = x + jy: Im\{z\} = y > 0\}$
- Lower half-plane: $\{z = x + jy: Im\{z\} = y < 0\}$
- Right half-plane: $\{z = x + jy: Re\{z\} = x > 0\}$
- Left half-plane: $\{z = x + jy: Re\{z\} = x < 0\}$

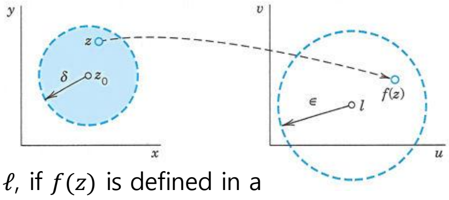
Some Topologies in Complex Plane

- Let S be a set of complex numbers.
  - S is open, if S has a neighborhood consisting entirely of points that belong to S.
  - Complement of S is the set of all points of the complex plane that do not belong to S.
    - S is closed, if its complement is open.
  - S is connected, if any two of its points can be joined by a broken line of points that belong to S.
  - A point z_0 is a boundary point of S, if every neighborhood of z_0 contains both points that belong to S and points that do not belong to S.
    - Boundary : Set of all boundary points.
- Domain is an open connected set, and region is the set consisting of a domain plus some or all of its boundary points.

Complex Functions

- $w = f(z) = u(x, y) + jv(u, v)$
  - A complex function is a rule that assigns to every $z \in S$ a unique complex number w.
    - S … domain of f
- Example 1. $w = f(z) = z^2 + 3z$
  - $w = (x + jy)^2 + 3(x + jy) = x^2 - y^2 + 3x + j(-2xy + 3y)$
    - $u(x, y) = x^2 - y^2 + 3x$ and $v(x, y) = -2xy + 3y$
  - When $z_0 = 1 + j3$, $f(z_0) = -5 + j15$
- Example 2. $w = f(z) = j2z + 6z^*$
  - $w = j2(x + jy) + 6(x + jy)^* = 6x - 2y + j(2x - 6y)$
  - $f\left(\frac{1}{2} + j4\right) = -5 - j23$

Complex Functions: Continuity



- Limit
  - A function $f(z)$ is said to have limit ℓ as $z \to z_0$, $\lim_{z \to z_0} f(z) = \ell$, if $f(z)$ is defined in a neighborhood of z_0 and $\forall \epsilon > 0$, $\exists \delta > 0$, such that, whenever $|z - z_0| < \delta$, $|f(z) - \ell| < \epsilon$.
    - z may approach to z_0 from any direction in the complex plane.
- Continuous
  - A function $f(z)$ is said to be continuous at z_0, if $f(z)$ is defined and $\lim_{z \to z_0} f(z) = f(z_0)$.
    - A function $f(z)$ is continuous in a domain S, if it is continuous $\forall z \in S$.
- Derivative
  - The derivative of a complex function $f(z)$ at z_0, is defined by $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, provided this limit exists.
    - $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$.

Complex Functions: Differentiable

- Differential rule
  - $(cf)' = cf'$, $(f+g)' = f' + g'$, $(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, $(z^n)' = nz^{n-1}$
  - If $f(z)$ is differentiable at $z = z_0$, then it is continuous at z_0.
- Example 3. $f(z) = z^2$
  - $f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z \cdot \Delta z + \Delta z^2}{\Delta z} = 2z$ ← differentiable
- Example 4. $f(z) = z^* = x - jy$, $\frac{(z+\Delta z)^* - z^*}{\Delta z} = \frac{\Delta z^*}{\Delta z} = \frac{\Delta x - j\Delta y}{\Delta x + j\Delta y}$
  - $\lim_{\Delta x \to 0} \left(\lim_{\Delta y \to 0} \frac{\Delta x - j\Delta y}{\Delta x + j\Delta y} \right) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$
  - $\lim_{\Delta y \to 0} \left(\lim_{\Delta x \to 0} \frac{\Delta x - j\Delta y}{\Delta x + j\Delta y} \right) = -\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta y} = -1$ ← not differentiable !!!

Analytic Function

- Defn. Analytic

  A function $f(z)$ is said to be analytic at a domain D, if $f(z)$ is defined and differentiable at all points in D.
  - A function $f(z)$ is said to be analytic at $z = z_0$, if $f(z)$ is analytic in a neighborhood of z_0.
- Example 5. $f(z) = z^k$
  - Monomial function is analytic in the entire complex plane.
  - $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ ... polynomial function is also analytic.
  - $f(z) = \frac{g(z)}{h(z)}$ ... rational function is analytic except at the point where $h(z) = 0$.

Cauchy-Riemann Equations

- Theorem 1. CRE

  Let $f(z) = u(x, y) + jv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + jy$ and differentiable at z. Then, the 1st order partial derivatives of u and v exist and satisfy the CRE,

  $u_x = v_y$, $u_y = -v_x$

  - If $f(z)$ is analytic in a domain D, partial derivatives exist and satisfy the CRE for all $z \in D$.
  - Given $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y)+jv(x+\Delta x, y+\Delta y)-(u(x,y)+jv(x,y))}{\Delta x + j\Delta y}$,
  - $\lim_{\Delta x \to 0}\left(\lim_{\Delta y \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y)+jv(x+\Delta x, y)-(u(x,y)+jv(x,y))}{\Delta x} = u_x + jv_x$
  - $\lim_{\Delta y \to 0}\left(\lim_{\Delta x \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x, y+\Delta y)+jv(x, y+\Delta y)-(u(x,y)+jv(x,y))}{j\Delta y} = v_y - ju_y$

Cauchy-Riemann Equations

- Theorem 2. CRE

  If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ have continuous 1st partial derivatives that satisfy the CRE in some domain D, then, the complex function $f(z) = u(x, y) + jv(x, y)$ is analytic in a domain D.

- Examples
  - $f(z) = z^2 \Rightarrow u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$; $u_x = 2x = v_y$ and $u_y = -2y = -v_x \Rightarrow$ analytic
  - $f(z) = z^* \Rightarrow u(x, y) = x$, $v(x, y) = -y$; $u_x = 1 \neq v_y \Rightarrow$ not analytic
  - $f(z) = e^x(\cos y + j \sin y) \Rightarrow u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$; $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x \Rightarrow$ analytic

Cauchy-Riemann Equations

- Example 3. Constant magnitude analytic function
  - $f(z)$ analytic in a domain D and $|f(z)| = c$ (const.) $\forall z \in D$
    - † $|f(z)|^2 = u^2 + v^2 = c^2$, $uu_x + vv_x = 0$ and $uu_y + vv_y = 0 \Rightarrow uu_x - vu_y = 0$ and $uu_y - vu_x = 0$
    - † $(u^2 + v^2)u_x = 0$ and $(u^2 + v^2)u_y = 0 \Rightarrow u_x = u_y = 0 \Rightarrow u(x, y)$ const.
- CRE in polar form
  - $f(z) = u(r, \theta) + jv(r, \theta)$, $z = r\angle\theta$
  - $u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$

- CRE in polar form (for reference only)
  - $u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$
  - $r = \sqrt{x^2 + y^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$
  - $\theta = \tan^{-1}\frac{y}{x} \Rightarrow \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = -\frac{\sin\theta}{r}$, $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{\cos\theta}{r}$ $\boxed{\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx}}$
  - $u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\, u_r - \frac{\sin\theta}{r}u_\theta$, $u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\, u_r + \frac{\cos\theta}{r}u_\theta$
  - $v_x = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\, v_r - \frac{\sin\theta}{r}v_\theta$, $v_y = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\, v_r + \frac{\cos\theta}{r}v_\theta$
  - $u_x = v_y \Rightarrow \cos\theta\, u_r - \frac{\sin\theta}{r}u_\theta = \sin\theta\, v_r + \frac{\cos\theta}{r}v_\theta$ ... (1)
  - $u_y = -v_x \Rightarrow \sin\theta\, u_r + \frac{\cos\theta}{r}u_\theta = -\cos\theta\, v_r + \frac{\sin\theta}{r}v_\theta$ ... (2)
  - $(1) \times \cos\theta + (2) \times \sin\theta \Rightarrow u_r = \frac{1}{r}v_\theta$
  - $(1) \times \sin\theta + (2) \times (-\cos\theta) \Rightarrow -\frac{1}{r}u_\theta = v_r$

Laplace Equation

- Theorem 3. Laplace equation

  Let $f(z) = u(x, y) + jv(x, y)$ be analytic in a domain D, then both u and v satisfy the Laplace equation:

  $\nabla^2 u = u_{xx} + u_{yy} = 0$ and $\nabla^2 v = v_{xx} + v_{yy} = 0$

  and have continuous 2nd partial derivatives in D.

  - CRE: $u_x = v_y$, $u_y = -v_x$, $v_{xy} = v_{yx}$
  - Solutions of Laplace equation having continuous 2nd order partial derivatives are called the harmonic functions.
  - Real and imaginary parts of an analytic function are harmonic functions.

- Examples 4. $u(x, y) = x^2 - y^2 - y$
  - $u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$, $u_{yy} = -2 \Rightarrow \nabla^2 u = u_{xx} + u_{yy} = 0 \Rightarrow$ harmonic
  - $v_y = u_x = 2x \Rightarrow v(x, y) = 2xy + h(x)$, $v_x = 2y + h'(x) = -u_y$; $h(x) = c$ (const)
  - $v(x, y) = 2xy + c$ is a harmonic conjugate of $u(x, y)$, and $f(z) = u(x, y) + jv(x)$ is analytic.

Exponential Functions

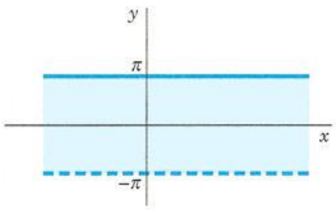
- $e^z = \exp(z) = e^x(\cos y + j \sin y)$
  - If $z = x$ (real), $e^z = e^x$
  - e^z is entire: i.e., analytic for all z in the complex plane.
  - $(e^z)' = e^z$

    † $(e^z)' = \frac{\partial}{\partial x}(e^x \cos y) + j\frac{\partial}{\partial x}(e^x \sin y) = e^x(\cos y + j \sin y) = e^z$
  - $e^{z_1+z_2} = e^{z_1} + e^{z_2}$
  - $e^z = e^x e^{jy} \Rightarrow e^{jy} = \cos y + j \sin y$ … Euler identity
  - In polar form, $z = r(\cos\theta + j\sin\theta) = re^{j\theta}$

When $f = u + jv$,
$f' = u_x + jv_x = v_y - ju_y$

Exponential Functions

- $e^z = \exp(z) = e^x(\cos y + j\sin y)$
  - $e^{j2\pi} = 1,\ e^{\pm j\pi} = -1,\ e^{j\pi/2} = j,\ e^{-j\pi/2} = -j$
  - $|e^z| = e^x$
  - $|e^{jy}| = 1$
  - $e^z \neq 0,\ \forall z$
  - $e^{z+j2\pi} = e^z$ … periodic with period $j2\pi$



† We can not distinguish y and $y + 2\pi$ in e^z: Fundamental region, $-\pi \leq y \leq \pi$

- Example 1. $e^z = w$
  - $e^{1.4-j0.6} = e^{1.4}(\cos 0.6 - j\sin 0.6)$
  - Solve $e^z = 3 + j4$: $|e^z| = e^x = 5,\ x = \ln 5 = 1.6094,\ y = \tan^{-1}\frac{4}{3} = 0.9273$

Trigonometric Functions

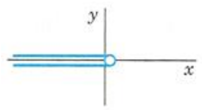
- $\cos z = \frac{1}{2}\left(e^{jz} + e^{-jz}\right),\ \sin z = \frac{1}{j2}\left(e^{jz} - e^{-jz}\right)$
  - $\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z}.\ \sec z = \frac{1}{\cos z},\ \csc z = \frac{1}{\sin z}$
  - Both $\cos z$ and $\sin z$ are entire functions.
  - $\tan z$ and $\sec z$ are analytic except at the point $\cos z = 0$.
  - $(\cos z)' = -\sin z,\ (\sin z)' = \cos z$, and $(\tan z)' = \sec^2 z$
  - Euler identity, $e^{jz} = \cos z + j\sin z$
- Example 2. $\cos z = 5$
  - $\frac{1}{2}\left(e^{jz} + e^{-jz}\right) = 5,\ e^{j2z} - 10e^{jz} + 1 = 0,\ e^{jz} = 9.899, 0.101 = e^{y-jx}$
  - $e^y = 9.899, 0.101$ and $x = 2n\pi$

Logarithm Function

- Natural logarithm, $w = \ln z$, $z \neq 0 \leftrightarrow e^w = e^{u+jv} = z$
  - $e^w = e^{u+jv} = re^{j\theta} = z \Rightarrow r = |e^w| = e^u$ and $v = \theta$
    - $u = \ln r$
  - $\ln z = \ln r + j\theta$, $r = |z| > 0$ and $\theta = \arg z$
    - Infinitely many valued function: $\ln z = \ln r + j(\theta + 2k\pi)$, $k \in \mathcal{N}$
  - $\operatorname{Ln} z = \ln|z| + j\operatorname{Arg}(z)$, $-\pi \leq \operatorname{Arg}(z) \leq \pi$ … Principal value
    - $\ln z = \operatorname{Ln} z + j2n\pi$
  - Properties
    - $\ln(z_1 z_2) = \ln z_1 + \ln z_2$, $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$, $\operatorname{Ln}(z_1 z_2) \neq \operatorname{Ln} z_1 + \operatorname{Ln} z_2$
    - $\operatorname{Ln}(1) = 0$, $\operatorname{Ln}(-1) = j\pi$
    - $e^{\ln z} = z$, $\ln(e^z) = z + j2n\pi$, $n = 0, 1, \ldots$

Logarithm Function

- $(\ln z)' = \frac{1}{z}$, $z \neq 0$ or z is not negative real
  - For every $n = 0, \pm 1, \pm 2, \ldots$, $\ln z = \operatorname{Ln} z + j2n\pi$ defines a function, which is analytic except at $z = 0$ and on the negative real axis (branch cut).



- General power
  - $z^c = e^{c \cdot \ln z}$, c complex and $z \neq 0$
  - If $c = \frac{1}{n}$, $n = 2, 3, \ldots$, $z^c = \sqrt[n]{z} = e^{\frac{1}{n}\ln z}$
    - When $z = 1$, $\sqrt[n]{1} = e^{j2\pi/n}$
  - $a^z = e^{z \cdot \ln a}$
  - $j^j = e^{j \cdot \ln j} = \exp\left\{j\left(\frac{j\pi}{2} \pm j2n\pi\right)\right\} = \exp\left\{-\frac{\pi}{2} \mp 2n\pi\right\}$
  - $(1+j)^{2-j} = e^{(2-j)\cdot\ln(1+j)} = \exp\left\{(2-j)\left(\ln\sqrt{2} + j\frac{1}{4}\pi \pm j2n\pi\right)\right\}$