5장 2절 연습문제 풀이

2006년 4월 14일

1. $x = a\cosh\frac{x}{a}$

$$
\begin{aligned}
y' &= \sinh\frac{x}{a}, \qquad y'' = \frac{1}{a}\cosh\frac{x}{a} \\
\kappa &= \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \\
&= \frac{\frac{1}{a}\cosh\frac{x}{a}}{[1+(\sinh\frac{x}{a})^2]^{\frac{3}{2}}} \\
&= \frac{\frac{1}{a}\cosh\frac{x}{a}}{\cosh^3\frac{x}{a}} = \frac{1}{a}\mathrm{sech}^{\,2}\frac{x}{a}
\end{aligned}
$$

2. $y = \ln(\cos x)$

$$
\begin{aligned}
y' &= \frac{-\sin x}{\cos x} = \tan x, \quad y'' = -\sec^2 x \\
\kappa &= \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \\
&= \frac{|-\sec^2 x|}{[1+(-\tan x)^2]^{\frac{3}{2}}} \\
&= \left|\frac{\sec^2 x}{\sec^3 x}\right| = |\cos x|
\end{aligned}
$$

3. $y = e^{2x}$

$$y' = 2e^{2x}, \quad y'' = 4e^{4x}$$
$$\kappa = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} = \frac{4e^{4x}}{[1+(2e^{2x})^2]^{\frac{3}{2}}}$$

4. $x = a\cos^3\theta,\ y = a\sin^3 t$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$
$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$
$$\Longrightarrow \frac{dy'}{dt} = -\sec^2 t$$
$$y'' = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\sec^2 t}{-3a\cos^2 t\sin t} = \frac{1}{3a}\sec^4 t\csc t$$
$$\kappa = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}}$$
$$= \frac{|\frac{1}{3a}\sec^4 t\csc t|}{[1+(-\tan t)^2]^{\frac{3}{2}}} = |\frac{1}{3a}\sec t\csc t| = |\frac{2}{3a}\csc 2t|$$

7. $x = \ln\sec y$

$$x' = \frac{\sec y\tan y}{\sec y} = \tan y$$
$$x'' = \sec^2 y$$
$$\kappa = \frac{|x''|}{[1+(x')^2]^{\frac{3}{2}}}$$
$$= \frac{|\sec^2 y|}{[1+(\tan y)^2]^{\frac{3}{2}}} = |\cos y|$$

5. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

$$\begin{aligned}
\frac{dx}{d\theta} &= a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta \\
\frac{dy}{d\theta} &= a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta \\
y' = \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta \\
\frac{dy'}{d\theta} &= \sec^2\theta \\
y'' = \frac{\frac{dy'}{d\theta}}{\frac{dx}{d\theta}} &= \frac{\sec^2\theta}{a\theta\cos\theta} = \frac{1}{a\theta}\sec^3\theta \\
\kappa &= \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \\
&= \frac{|\frac{1}{a\theta}\sec^3\theta|}{[1+(\tan\theta)^2]^{\frac{3}{2}}} = |\frac{1}{a\theta}|
\end{aligned}$$

6. $x = a(\theta - \sin\theta) = f(\theta)$, $y = a(1 - \cos\theta = g(\theta)$

$$\begin{aligned}
f' &= a(1-\cos\theta), & f'' &= a\sin\theta \\
g' &= a\sin\theta, & g'' &= a\cos\theta \\
\kappa &= \frac{|f'g'' - f''g'|}{[(f')^2 + (g')^2]^{\frac{3}{2}}} \\
&= \frac{a^2(1-\cos\theta)\cos\theta - a^2\sin\theta\sin\theta}{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}} \\
&= \frac{a^2(\cos\theta - 1)}{[2a^2(1-\cos\theta)]^{\frac{3}{2}}} = \frac{1}{2\sqrt{2}a\sqrt{1-\cos\theta}}
\end{aligned}$$

9. $x = \frac{y^4}{4} + \frac{1}{8y^2}$

$$\begin{aligned} x' &= y^3 - \frac{1}{4y^3} \\ x'' &= 3y^2 + \frac{3}{4y^4} \\ \kappa &= \frac{|x''|}{[1+(x')^2]^{\frac{3}{2}}} \\ &= \frac{|3y^2 + \frac{3}{4y^4}|}{[1+(y^3 - \frac{1}{4y^3})^2]^{\frac{3}{2}}} \\ &= \frac{48y^5}{(4y^6+1)^2} \end{aligned}$$

10. $x = 5\cosh t,\ y = 3\sinh t$

$$\begin{aligned} \frac{dx}{dt} &= 5\sinh t, \quad \frac{dy}{dt} = 3\cosh t \\ y' &= \frac{3\cosh t}{5\sinh t} = \frac{3}{5}\coth t \\ \frac{dy'}{dt} &= -\frac{3}{5}\coth t\text{csch } t \\ y'' &= \frac{-\frac{3}{5}\coth t\text{csch } t}{5\sinh t} = -\frac{3}{25}\coth t\text{csch}^2 t \\ \kappa &= \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \\ &= \frac{|-\frac{3}{25}\coth t\text{csch}^2 t|}{[1+(\frac{3}{5}\coth t)^2]^{\frac{3}{2}}} \end{aligned}$$

11. $3y = x^2$; $(1, \frac{1}{3})$

$$y' = \frac{2}{3}x$$
$$y'' = \frac{2}{3}$$
$$x = 1 \Longrightarrow y' = \frac{2}{3}, y'' = \frac{2}{3}$$
$$\kappa = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = \frac{\frac{2}{3}}{(1 + \frac{4}{9})^{\frac{3}{2}}} = \frac{18}{13\sqrt{13}}$$
$$R = \frac{1}{\kappa} = \frac{13\sqrt{13}}{18}$$

12. $y = x - x^2$; $(0, 0)$

$$y' = 1 - 2x, \quad y'' = -2$$
$$x = 0 \Longrightarrow y' = 1, y'' = 0$$
$$\kappa = 0 \Longrightarrow R = \infty$$

13. $y^2 = x^3$; $(\frac{1}{4}, \frac{1}{8})$

$$2yy' = 3x^2 \Longrightarrow y' = \frac{3x^2}{2y}$$
$$2(y')^2 + 2yy'' = 6x$$
$$x = \frac{1}{4}, y = \frac{1}{8} \Longrightarrow y' = \frac{3}{4}, y'' = \frac{6}{4}$$
$$\kappa = \frac{\frac{6}{4}}{[1 + (\frac{3}{4})^2]^{\frac{3}{2}}} = \frac{96}{125}$$
$$R = \frac{1}{\kappa} = \frac{125}{96}$$

15. $y = \sin x$; $(\frac{\pi}{2}, 1)$

$$y' = \cos x, \quad y'' = -\sin x$$
$$x = \frac{\pi}{2} \Longrightarrow y' = 0, \ y'' = -1$$
$$\kappa = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = 1 \Longrightarrow R = 1$$

16. $y = \tan \frac{x}{4}$; $(\pi, 1)$

$$y' = \frac{1}{4}, \quad y'' = \frac{1}{8} \sec \tfrac{x}{4} \tan \frac{x}{4}$$
$$x = \pi \Longrightarrow y' = \frac{1}{2}, \quad y'' = \frac{1}{4}$$
$$\kappa = \frac{\frac{1}{4}}{(1 + \frac{1}{4})^{\frac{3}{2}}} = \frac{2}{5\sqrt{5}}$$
$$R = \frac{5\sqrt{5}}{2}$$

17. $y = e^x$; $(0, 1)$

$$y' = e^x, \quad y'' = e^x$$
$$x = 0 \Longrightarrow y' = 1, \quad y'' = 1$$
$$\kappa = \frac{1}{2\sqrt{2}} \Longrightarrow R = 2\sqrt{2}$$

18. $y = \ln \sin x$; $(\frac{\pi}{2}, 0)$

$$y' = \frac{\cos x}{\sin x} = \cot x, \quad y'' = -\csc^2 x$$
$$x = \frac{\pi}{2} \Longrightarrow y' = 0, \quad y'' = -1$$
$$\kappa = 1 \Longrightarrow R = 1$$

25. $y = \sin^{-1} x \Longrightarrow x = \sin y$

$$x' = \cos y, \quad x'' = -\sin y$$
$$\kappa = \frac{|x''|}{[1 + (x')^2]^{\frac{3}{2}}}$$
$$= \frac{|-\sin y|}{[1 + (\cos y)^2]^{\frac{3}{2}}} \Longrightarrow R = \frac{[1 + (\cos y)^2]^{\frac{3}{2}}}{|\sin y|}$$

20. $x = t^2 - 1,\ y = \frac{1}{3}t^3 - 1,\ t = 2$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2$$
$$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2}{2t} = \frac{t}{2}$$
$$\frac{dy'}{dt} = \frac{1}{2}$$
$$y'' = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2}}{2t}$$
$$t = 2 \Longrightarrow y' = 1, \quad y'' = \frac{1}{4}$$
$$\kappa = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = \frac{\frac{1}{4}}{2\sqrt{2}} = \frac{1}{8\sqrt{2}}$$
$$R = 8\sqrt{2}$$

29. 6번 풀이의 역수

30. 4번 풀이의 역수

31. $y = 4x - x^2$

$$y' = 4 - 2x, \quad y'' = -1$$
$$\kappa = \frac{1}{[1 + (4 - 2x)^2]^{\frac{3}{2}}}$$

이므로 $x = 2$ 일때 곡률은 극대가 된다.

32. $y = \sin x$

$$\kappa = \frac{|\sin x|}{[1 + \cos^2 x]^{\frac{3}{2}}}$$

이므로 $x = \frac{\pi}{2}$ 일 때 곡률은 극대가 된다.

45. $r = \theta^2;\ \theta = \frac{3}{2}$

$$
\begin{aligned}
r' &= 2\theta, \qquad r'' = 2 \\
\theta &= \frac{3}{2} \Longrightarrow r = \frac{9}{4}, \quad r' = 3, \quad r'' = 3 \\
\kappa &= \frac{|r^2 + 2(r')^2 - rr''|}{[r^2 + (r')^2]^{\frac{3}{2}}} \\
&= \frac{|(\frac{9}{4})^2 + 2(3)^2 - (\frac{9}{4})3|}{[(\frac{9}{4})^2 + (3)^2]^{\frac{3}{2}}} = \frac{44}{125} \\
R &= \frac{125}{44}
\end{aligned}
$$

46. $r = 4 + 3\sin\theta;\ \theta = 0$

$$
\begin{aligned}
r' &= 3\cos\theta, \qquad r'' = -3\sin\theta \\
\theta &= 0 \Longrightarrow r = 4, \quad r' = 3, \quad r'' = 0 \\
\kappa &= \frac{|4^2 + 2(3)^2 - 4 \cdot 0|}{[4^2 + (3)^2]^{\frac{3}{2}}} \\
&= \frac{44}{125} \\
R &= \frac{125}{44}
\end{aligned}
$$

47. $r = \tan\theta;\ \theta = \frac{3\pi}{4}$

$$
\begin{aligned}
r' &= \sec^2\theta, \qquad r'' = 2\sec^2\theta\tan\theta \\
\theta &= \frac{3\pi}{4} \Longrightarrow r = -1, \quad r' = 2, \quad r'' = -4 \\
\kappa &= \frac{|(-1)^2 + 2(2)^2 - 4|}{[(-1)^2 + (2)^2]^{\frac{3}{2}}} \\
&= \frac{7}{5\sqrt{5}} \\
R &= \frac{5\sqrt{5}}{7}
\end{aligned}
$$

49. $r^2 = \sin 2\theta;\ \theta = \frac{\pi}{4}$

$$
\begin{aligned}
2rr' &= 2\cos 2\theta \Longrightarrow r' = \frac{2\cos 2\theta}{r} \\
2(r')^2 + 2r(r'') &= -4\sin 2\theta \\
\theta = \frac{\pi}{4} &\Longrightarrow r = 1, \quad r' = 0, \quad r'' = -2 \\
\kappa &= \frac{|r^2 + 2(r')^2 - rr''|}{[r^2 + (r')^2]^{\frac{3}{2}}} \\
&= \frac{|1^2 + 2|}{[1^2]^{\frac{3}{2}}} = 3 \\
R &= \frac{1}{3}
\end{aligned}
$$