7 장 연습문제 풀이

2006년 5월 23일

7.3 이상적분

피적분함수 $f(x)$ 가 적분구간 $[a, b]$ 에서 유계가 아니던가, 또는 a, b 의 값 중 적어도 하나가 무한일 때의 정적분.

1. (별지의 그림 1 참고)

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$

2. (별지의 그림 2 참고)

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$

3. (별지의 그림 3 참고)

$$\int_{-\infty}^\infty f(x)dx = \lim_{\substack{b\to\infty \\ a\to-\infty}}\int_a^b f(x)dx.$$

4. (별지의 그림 4 참고) $\lim\limits_{x\to a} f(x) = \pm\infty$ 일때

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx.$$

5. (별지의 그림 5 참고) $\lim_{x \to b} f(x) = \pm\infty$ 일때

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx.$$

6. (별지의 그림 6 참고) $a < c < b$ 인점 c 에서 $\lim_{x \to c} f(x) = \pm\infty$ 일때

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx.\end{aligned}$$

연습문제 풀이

1.

$$\begin{aligned}\int_1^\infty \frac{1}{(x+2)^3}dx &= \lim_{t \to \infty} \int_1^t \frac{1}{(x+2)^3}dx \\ &= \lim_{t \to \infty} \left[-\frac{1}{2}(x+2)^{-2}\right]_1^t \\ &= \lim_{t \to \infty} \left(-\frac{1}{2(t+2)^2} + \frac{1}{2} \cdot \frac{1}{9}\right) \\ &= \frac{1}{18}.\end{aligned}$$

2.

$$\int_1^\infty \frac{1}{\sqrt[3]{x}}dx = \lim_{t \to \infty} \int_1^t x^{-\frac{1}{3}}dx = \lim_{t \to \infty} [\frac{3}{2}x^{\frac{2}{3}}]_1^t = \infty$$

3.

$$\int_{-\infty}^1 e^x dx = \lim_{t \to -\infty} \int_t^1 e^x dx = \lim_{t \to -\infty} [e^x]_t^1 = \lim_{t \to -\infty} (e - e^t) = e$$

4.

$$\begin{aligned}\int_{-\infty}^{-1} \frac{1}{x^2}dx &= \lim_{t\to-\infty}\int_t^{-1}\frac{1}{x^2}dx\\ &= \lim_{t\to-\infty}\left[-\frac{1}{x}\right]_t^{-1}\\ &= \lim_{t\to-\infty}(1+\frac{1}{t})\\ &= 1\end{aligned}$$

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$$\begin{aligned}\int_0^{\infty}\frac{a^3}{a^2+x^2}dx = \lim_{t\to\infty}\int_0^t\frac{a^3}{a^2+x^2}dx = \lim_{t\to\infty}\left[a^2\tan^{-1}\frac{x}{a}\right]_0^t\\ = \lim_{t\to\infty}a^2\tan^{-1}\frac{t}{a} = \frac{\pi a^2}{2}\end{aligned}$$

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$$\int_0^{\infty} xe^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^a = 1$$

7.

$$\begin{aligned}\int_0^{\infty}\frac{1}{\sqrt{x^2+a^2}}dx &= \lim_{t\to\infty}\int_0^t\frac{1}{\sqrt{x^2+a^2}}dx\\ &= \lim_{t\to\infty}\left[\sinh^{-1}\frac{x}{a}\right]_0^t = \infty\end{aligned}$$

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$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = 2\int_0^{\infty}\frac{1}{(x^2+a^2)^2}dx &= 2\lim_{t\to\infty}\int_0^t\frac{1}{(x^2+a^2)^2}dx\\ &= 2\lim_{t\to\infty}\left[\frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a}+\frac{ax}{a^2+x^2}\right)\right]_0^t\\ &= 2\lim_{t\to\infty}\frac{1}{2a^3}\left(\tan^{-1}\frac{t}{a}+\frac{at}{a^2+t^2}\right)\\ &= \frac{\pi}{2a^3}\end{aligned}$$

$\int \frac{1}{(x^2+a^2)^2} dx$ 의계산

$x = a\tan\theta$ 라하면 $dx = a\sec^2\theta$, $x^2 + a^2 = a^2\sec^2\theta$ 이므로

$$\begin{aligned}
\int \frac{1}{(x^2+a^2)^2} dx &= \int \frac{a\sec^2\theta}{a^4\sec^4\theta} d\theta \\
&= \frac{1}{a^3}\int \cos^2\theta d\theta = \frac{1}{a^3}\int \frac{1+\cos 2\theta}{2} d\theta \\
&= \frac{1}{a^3}(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta) + C \\
&= \frac{1}{2a^3}(\theta + \sin\theta\cos\theta) + C
\end{aligned}$$

그런데, $\tan\theta = \dfrac{x}{a}$ 이므로

$$\begin{aligned}
\sin\theta = \frac{x}{\sqrt{a^2+x^2}}, &\quad \cos\theta = \frac{a}{\sqrt{a^2+x^2}} \\
\Longrightarrow \sin\theta\cos\theta &= \frac{ax}{a^2+x^2}
\end{aligned}$$

따라서 $\displaystyle\int \frac{1}{(x^2+a^2)^2} dx = \frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a} + \frac{ax}{a^2+x^2}\right) + C$

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$$\begin{aligned}
\int_1^\infty \ln x dx &= \lim_{t\to\infty}\int_1^t \ln x dx \\
&= \lim_{t\to\infty}[x\ln x - x]_1^t = \lim_{t\to\infty}(t\ln t - t + 1) \\
&= \infty - \infty(\text{ 발산 })
\end{aligned}$$

11.

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x} dx &= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x} dx \quad (\ln x \text{ 를 치환}) \\
&= \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^t \\
&= \lim_{t\to\infty}\frac{1}{2}(\ln t)^2 \\
&= \infty \quad (\text{ 발산})
\end{aligned}$$

12.

$$\begin{aligned}
\int_0^\infty \tan^{-1} x dx &= \lim_{t\to\infty} \int_0^t \tan^{-1} x dx \\
&= \lim_{t\to\infty} \left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) \right]_0^t \\
&= \lim_{t\to\infty} \left(t\tan^{-1} t - \frac{1}{2}\ln(1+t^2) \right) = \infty
\end{aligned}$$

$\int \tan^{-1} x dx$ 계산은 6장 5절 예제 1번 풀이 참고.

15.

$$\begin{aligned}
\int_{-\infty}^\infty \frac{1}{x^2+2x+2} dx &= \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b \frac{1}{x^2+2x+2} dx \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b \frac{1}{1+(x+1)^2} dx \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left[\tan^{-1}(x+1) \right]_a^b \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left\{ \tan^{-1}(b+1) - \tan^{-1}(a+1) \right\} \\
&= \frac{\pi}{2} + \frac{\pi}{2}
\end{aligned}$$

18.

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\tan^{-1} x}{x^2+1} dx &= \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b \frac{\tan^{-1} x}{x^2+1} dx \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left[\frac{1}{2}\tan^{-1} x \right]_a^b \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left(\frac{1}{2}\tan^{-1} b - \frac{1}{2}\tan^{-1} b \right) \\
&= \frac{1}{2}(\frac{\pi}{2} + \frac{\pi}{2}) = \frac{\pi}{2}
\end{aligned}$$

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$$\begin{aligned}
\int_a^{2a} \frac{1}{\sqrt{x-a}}dx &= \lim_{t\to a}\int_t^{2a} \frac{1}{\sqrt{x-a}}dx = \lim_{t\to a}\int_t^{2a} (x-a)^{-\frac{1}{2}}dx \\
&= \lim_{t\to a}\left[2(x-a)^{\frac{1}{2}}\right]_t^{2a} \\
&= \lim_{t\to a}\left(2\sqrt{a} - 2(t-a)^{\frac{1}{2}}\right) = 2\sqrt{a}
\end{aligned}$$

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$$\begin{aligned}
\int_a^{2a} \frac{1}{(x-a)^{\frac{3}{2}}}dx &= \lim_{t\to a}\int_t^{2a} \frac{1}{(x-a)^{\frac{3}{2}}}dx = \lim_{t\to a}\int_t^{2a} (x-a)^{-\frac{3}{2}}dx \\
&= \lim_{t\to a}\left[-2(x-a)^{-\frac{1}{2}}\right]_t^{2a} \\
&= \lim_{t\to a}\left(-2\frac{1}{\sqrt{a}} + 2(t-a)^{-\frac{1}{2}}\right) \\
&= \lim_{t\to a}\left(-2\frac{1}{\sqrt{a}} + \frac{2}{\sqrt{(t-a)}}\right) = \infty
\end{aligned}$$

21.

$$\begin{aligned}
\int_0^1 \ln x dx &= \lim_{t\to 0}\int_t^1 \ln x dx \\
&= \lim_{t\to 0}\left[x\ln x - x\right]_t^1 \\
&= \lim_{t\to 0}\left(-1 - t\ln t - t\right) = -1
\end{aligned}$$

22.

$$\begin{aligned}
\int_0^1 \frac{\ln x}{x}dx &= \lim_{t\to 0}\int_t^1 \frac{\ln x}{x}dx \\
&= \lim_{t\to 0}\left[\frac{1}{2}(\ln x)^2\right]_t^1 \\
&= \lim_{t\to 0}\left(-\frac{1}{2}(\ln t)^2\right) = \infty
\end{aligned}$$

23. 주의: $\tan\frac{\pi}{2}=\infty$

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\tan x dx &= \lim_{t\to\frac{\pi}{2}}\int_0^t \tan x dx\\ &= \lim_{t\to\frac{\pi}{2}}\int_0^t \frac{\sin x}{\cos x}dx\\ &= \lim_{t\to\frac{\pi}{2}}[-\ln\cos x]_0^t\\ &= \lim_{t\to\frac{\pi}{2}}(-\ln\cos t)\\ &= \infty \quad (\text{ 발산})\end{aligned}$$

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$$\begin{aligned}\int_0^1 \frac{1}{1-x^2}dx &= \lim_{t\to1}\int_0^t \frac{1}{1-x^2}dx\\ &= \lim_{t\to1}[\tanh^{-1}x]_0^t\\ &= \tanh^{-1}1\end{aligned}$$

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$$\int_{-a}^{a}\frac{1}{(x-a)^{\frac{2}{3}}}dx = \lim_{t\to a}\int_{-a}^{t}\frac{1}{(x-a)^{\frac{2}{3}}}dx = \lim_{t\to a}\int_{-a}^{t}(x-a)^{-\frac{2}{3}}dx$$

27.

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{4x-x^2}}dx &= \lim_{\substack{b\to4\\a\to0}}\int_a^b \frac{1}{\sqrt{4x-x^2}}dx\\ &= \lim_{\substack{b\to4\\a\to0}}\int_a^b \frac{1}{\sqrt{2^2-(x-2)^2}}dx\\ &= \lim_{\substack{b\to4\\a\to0}}\left[\sin^{-1}\frac{x-2}{2}\right]_a^b\\ &= \lim_{\substack{b\to4\\a\to0}}\left(\sin^{-1}\frac{b-2}{2}-\sin^{-1}\frac{a-2}{2}\right)\\ &= \pi\end{aligned}$$

28.

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}} dx = \lim_{\substack{t_2 \to a \\ t_1 \to -a}} \int_{t_1}^{t_2} \frac{1}{\sqrt{a^2 - x^2}} dx$$

33. 주의 $x = 2 \Longrightarrow \frac{1}{\sqrt[3]{x-2}} = \infty$ 그림을 그려볼 것!!!

$$\begin{aligned}
\int_0^{\infty} \frac{1}{\sqrt[3]{x-2}} dx = \int_0^{2} \frac{1}{\sqrt[3]{x-2}} dx + \int_2^{\infty} \frac{1}{\sqrt[3]{x-2}} dx \\
&= \lim_{t \to \infty} \int_0^t (x-2)^{-\frac{1}{3}} dx + \lim_{\substack{b \to \infty \\ a \to 2}} \int_a^b (x-2)^{-\frac{1}{3}} dx \\
&= \lim_{t \to \infty} \left[\frac{3}{2}(x-2)^{\frac{2}{3}} \right]_0^t + \lim_{\substack{b \to \infty \\ a \to 2}} \left[\frac{3}{2}(x-2)^{\frac{2}{3}} \right]_a^b \\
&= \lim_{t \to \infty} \left\{ \frac{3}{2}(t-2)^{\frac{2}{3}} - \frac{3}{2}(-2)^{\frac{2}{3}} \right\} + \lim_{\substack{b \to \infty \\ a \to 2}} \left\{ \frac{3}{2}(b-2)^{\frac{2}{3}} - \frac{3}{2}(a-2)^{\frac{2}{3}} \right\} \\
&= \infty \quad (\text{ 발산})
\end{aligned}$$

37.

$$\begin{aligned}
\int_{-\infty}^{\infty} x \sin x^2 dx &= \lim_{\substack{b \to \infty \\ a \to -\infty}} \int_a^b x \sin x^2 dx \quad (\ x^2 = t \Longrightarrow 2xdx = dt) \\
&= \lim_{\substack{b \to \infty \\ a \to -\infty}} \int_{x=a}^{x=b} \frac{1}{2} \sin t dt \\
&= \lim_{\substack{b \to \infty \\ a \to -\infty}} \left[-\frac{1}{2} \cos x^2 \right]_a^b \\
&= \lim_{\substack{b \to \infty \\ a \to -\infty}} \left\{ -\frac{1}{2} \cos b^2 + \frac{1}{2} \cos a^2 \right\} \quad (\text{ 발산})
\end{aligned}$$

39. 주의 $x = 1 \Longrightarrow \sqrt{\frac{1+x}{1-x}} = \infty$. 이상적분을 하기전에 먼저 부정적분

$$
\begin{aligned}
\int \sqrt{\frac{1+x}{1-x}}dx &= \int \sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}dx \\
&= \int \frac{1+x}{\sqrt{1-x^2}}dx \\
&= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx \\
&= \sin^{-1} x - \sqrt{1-x^2} + C
\end{aligned}
$$

따라서

$$
\begin{aligned}
\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}dx &= \lim_{t\to 1} \int_{-1}^{t} \sqrt{\frac{1+x}{1-x}}dx \\
&= \lim_{t\to 1} \left[\sin^{-1} x - \sqrt{1-x^2}\right]_{-1}^{t} \\
&= \lim_{t\to 1} \left\{\sin^{-1} t - \sqrt{1-t^2} - \sin^{-1}(-1)\right\} \\
&= \frac{\pi}{2} + \frac{\pi}{2} = \pi
\end{aligned}
$$