Chapter 15. Power & Taylor Series

1. Sequence & Series
2. Power Series
3. Taylor & Maclaurin Series
4. Uniform Convergence

Sequences 수열

- Sequence: list of numbers, $\{z_n\}$
  - Convergence: A series $\{z_n\}$ converges to a limit c, if $\forall\epsilon > 0$, $\exists N > 0$, such that whenever $n > N$, $|z_n - c| < \epsilon$.
    - † Divergent 발산
- Examples
  - $z_n = \frac{i^n}{n} \to 0$,
  - $z_n = i^n$ ...divergent
  - $z_n = (1+i)^n$ ...divergent
  - $z_n = x_n + jy_n = \left(1 - \frac{1}{n^2}\right) + j\left(2 + \frac{4}{n}\right) \to 1 + j2$ ... component-wise (real & imaginary part) convergent

Series 급수

- $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + z_3 + \ \ldots$
  - Partial sum 부분합, $s_n = \sum_{m=1}^{n} z_m = z_1 + z_2 + \cdots + z_n$
  - Remainder, $R_n = \sum_{m=n+1}^{\infty} z_m = z_{n+1} + z_{n+2} + \cdots$
  - Convergence: A series converges if the sequence of partial sums $\{s_n\}$ converges.
    - † $s_n \to s = \sum_{m=1}^{\infty} z_m \Rightarrow R_n \to 0$

- Theorem 2. Real & imaginary-part convergence

  A series $\sum_{m=1}^{\infty} z_m$ with $z_m = x_m + jy_m$ converges to $s = u + jv$, if and only if $\sum_{m=1}^{\infty} x_m$ converges to u and $\sum_{m=1}^{\infty} y_m$ converges to v.

Test of Convergence

- Theorem 3. Divergence

  If a series $\sum_{m=1}^{\infty} z_m$ converges, then $\lim_{m \to \infty} z_m = 0$ (necessary condition, but not sufficient condition, for convergence).
  - † Harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, diverges ($z_m \to 0$).

- Theorem 4. Cauchy's convergence principle

  A series $\sum_{m=1}^{\infty} z_m$ converges, if and only if $\forall \epsilon > 0$, we can find $N > 0$, such that $|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon$, for every $n > N$ and $p = 1, 2, \ldots$.

  - A series $\sum_{m=1}^{\infty} z_m$ is called absolutely convergent, if $\sum_{m=1}^{\infty} |z_m|$ is convergent.
    - † If a series is absolutely convergent, then it is convergent.
    - † Series, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converges, but not absolutely convergent.

Test of Convergence

- Theorem 5. Comparison test

  If a series $\sum_{m=1}^{\infty} z_m$ is given and we can find a convergent series $\sum_{m=1}^{\infty} b_m$, $b_m \geq 0$, such that $|z_1| < b_1$, $|z_2| < b_2$, ..., then the series $\lim_{m\to\infty} z_m$ is absolutely convergent (and thus convergent).

- Theorem 6. Geometric series

  A series $\sum_{m=1}^{\infty} q^m = 1 + q + q^2 + \cdots$ converges to $\frac{1}{1-q}$, if $|q| < 1$ and diverges if $|q| > 1$.

  † Partial sum, $s_n = \sum_{m=1}^{n} q^m = \frac{1-q^{n+1}}{1-q}$

- Theorem 7. Ration test

  Consider a series $\sum_{m=1}^{\infty} z_m$, $z_m \neq 0$. If $\exists N > 0$, such that whenever $n > N$, we have

  $\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1$ (fixed q)

  then the series converges absolutely. If $\left|\frac{z_{n+1}}{z_n}\right| \geq 1$, $\forall n > N$, the series diverges.

Test of Convergence

- Theorem 6. Ratio test

  If a series $\sum_{m=1}^{\infty} z_m$, $z_m \neq 0$ is such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, then

  (a) If $L < 1$, the series converges absolutely.

  (b) If $L > 1$, the series diverges.

  (c) If $L = 1$, no conclusion.

- Examples. $\sum_{m=1}^{\infty} z_m$

  ∨ $z_n = \frac{1}{n}$, $\left|\frac{z_{n+1}}{z_n}\right| = \frac{n}{n+1} < 1$ and $L = 1$ (divergent)

  ∨ $z_n = \frac{1}{n^2}$, $\left|\frac{z_{n+1}}{z_n}\right| = \frac{n^2}{(n+1)^2} < 1$ and $L = 1$ (convergent) ... p-series

  ∨ $z_n = \frac{(100+j75)^n}{n!}$, $\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+j75|}{n+1} = \frac{125}{n+1}$ and $L = 0$ (convergent)

Power Series

- $\sum_{n=0}^{\infty} a_n (z - z_0)^n$
  - Coefficient, c_n, and center, z_0 ← Convergence depends on z.
- Examples
  - $\sum_{n=0}^{\infty} z^n$: Converges, if $|z| < 1$
  - $\sum_{n=0}^{\infty} \frac{z^n}{n!}$: $\left|\frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n}\right| = \left|\frac{z}{n+1}\right| \to 0$, Converges for every z.
- Theorem 1. Convergence of a power series

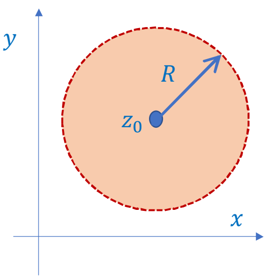
  (a) Every power series converges at the center z_0.

  (b) If a power series converges at a point $z = z_1 \neq z_0$, it converges absolutely for all $\{z : |z - z_0| < |z_1 - z_0|\}$.

  (c) If a power series diverges at a point $z = z_2$, it converges absolutely for all $\{z : |z - z_0| > |z_1 - z_0|\}$.

ROC, Radius of Convergence

- Radius of convergence
  - The power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all z in $\{z : |z - z_0| < R\}$.
  - Radius R of circle (region) of convergence
    - † $R = \infty \Rightarrow$ Series converges for all z.
    - † $R = 0 \Rightarrow$ Series converges only at the center z_0.



- Examples. $R = 1$, convergence on the boundary
  - $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ ... converges for all $|z| = 1$.
  - $\sum_{n=0}^{\infty} \frac{z^n}{n}$ ... converges at $z = -1$, diverges at $z = 1$
  - $\sum_{n=0}^{\infty} z^n$ ... diverges for all $|z| = 1$.

ROC

- Theorem 2. ROC

  Suppose that the sequence $\left|\frac{a_{n+1}}{a_n}\right|$, $n = 1, 2, \cdots$, converges with limit L^*. If $L^* = 0$, then $R = \infty$. If $L^* > 0$, then

  $R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ (Cauchy-Hadamard formula)

  † If $\left|\frac{a_{n+1}}{a_n}\right|$ diverges, then $R = 0$.

- Examples
  - $\sum_{n=0}^{\infty}\frac{z^n}{n^2}$, $a_n = \frac{1}{n^2}$, $R = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = 1$
  - $\sum_{n=0}^{\infty}\frac{z^n}{n}$, $a_n = \frac{1}{n}$, $R = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1$
  - $\sum_{n=0}^{\infty} z^n$, $a_n = 1$, $R = 1$
  - $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(z - j3)^n$, $a_n = \frac{(2n)!}{(n!)^2}$, $R = \lim_{n\to\infty}\left|\frac{(2n)!}{(n!)^2}\cdot\frac{((n+1)!)^2}{(2n+2)!}\right| = \frac{1}{4}$

Functions & Power Series

- A function can be represented by a power series.
  - $f(z) = \sum_{n=0}^{\infty} a_n z^n$, for $|z| < R$ and $R > 0$
  - Unique
- Theorem 1. Continuity

  If a function $f(z)$ can be represented by a power series with $R > 0$, then $f(z)$ is continuous at $z = 0$ (center).

- Theorem 2. Uniqueness

  Let two power series, $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, both be convergent for $|z| < R$ ($R > 0$) and $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \{z: |z| < R\}$. Then, these series are identical.

  † If a function $f(z)$ is represented by a power series, then it is unique.

Operations in Power Series

- Term-wise operations in power series
  - Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with ROC, R_1, and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ with ROC, R_2
  - Addition & subtraction
    - $f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$ with ROC, $R_1 \cap R_2$
  - Multiplication
    - $f(z) \cdot g(z) = \sum_{n=0}^{\infty} c_n z^n$ with ROC, $R_1 \cap R_2$ … Cauchy product
    - where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$
  - Differentiation
    - Derived series of a power series, $f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1}$, has the same ROC as the original power series.
  - Integration
    - Power series, $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$, has the same ROC as the original power series.

Functions & Power Series

- Theorem 5. Analytic functions

  A power series with ROC $R > 0$ represents an analytic function at every point inside its circle of convergence.
  - The derivatives of this function are obtained by differentiating the original series term-by-term.
  - All the series thus obtained have the same ROC as the original series and thus each of them represents an analytic function.

Taylor Series

- $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$
  - $a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{j2\pi} \oint_C \frac{f(\alpha)}{(\alpha - z_0)^{n+1}} d\alpha$
    - Integration CCW direction over a simple closed curve C that contains z_0 in its interior.
  - $f(z)$ is analytic in a domain containing C and every point inside C.
  - Maclaurin series, when $z_0 = 0$
  - Taylor formula,

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{1}{2!}(z - z_0)^2 f''(z_0) + \cdots + \frac{1}{n!}(z - z_0)^n f^{(n)}(z_0) + R_n(z)$$

$$R_n(z) = \frac{1}{j2\pi}(z - z_0)^{n+1} \oint_C \frac{f(\alpha)}{(\alpha - z_0)^{n+1}(\alpha - z)} d\alpha$$ ... remainder

Taylor Series

- Theorem 1. Taylor's theorem

  Let $f(z)$ be analytic in a domain D and let $z = z_0$ be any point in D. Then, there exists only one Taylor series with center z_0 representing $f(z)$. This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic.
    - Remainder, $R_n(z) = \frac{1}{j2\pi}(z - z_0)^{n+1} \oint_C \frac{f(\alpha)}{(\alpha - z_0)^{n+1}(\alpha - z)} d\alpha$
    - $|a_n| \le \frac{M}{r^n}$, where $M = \max|f(z)|$ on a circle $|z - z_0| = r$ in D.

- Singularity
  - On the circle of convergence, there exists at least one singular point of $f(z)$, a point $z = c$, where $f(z)$ is not analytic.
  - ROC R of the Taylor series is usually equal to the distance from z_0 to the nearest singular point of $f(z)$.

Taylor Series: Examples

- Power series *vs.* Taylor series
  - ∨ $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$
  - ∨ $f(z_0) = a_0$, $f'(z_0) = a_1$, $f''(z_0) = 2! \cdot a_1$, …, $f^{(n)}(z_0) = n! \cdot a_n$
- Examples
  - ∨ $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$ … geometric series
  - ∨ $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $|z| < \infty$ … exponential function
  - ∨ $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$, $|z| < \infty$
  - ∨ $\mathrm{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$, $|z| < 1$

Taylor Series: Examples

- Examples
  - ∨ Substitution: $f(z) = \frac{1}{1+z^2}$
    - † $f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \frac{1}{1-z}\Big|_{z \leftarrow -z^2} = \sum_{n=0}^{\infty} z^n|_{z \leftarrow -z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$, $|z| < 1$
  - ∨ Integration: $f(z) = \tan^{-1} z$
    - † $f'(z) = \frac{1}{1+z^2}$ and $f(0) = 0 \Rightarrow f'(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n}$
    - † $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$, $|z| < 1$
  - ∨ Variation of geometric series, $f(z) = \frac{1}{c-z}$, centered at $z = z_0$
    - † $\frac{1}{c-z} = \frac{1}{c-z_0-(z-z_0)} = \frac{1}{(c-z_0)\left(1-\frac{z-z_0}{c-z_0}\right)} = \frac{1}{c-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{c-z_0}\right)^n$

Binomial Series

- Negative binomial series
  - $\frac{1}{(1+z)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n = 1 - mz + \frac{1}{2!}m(m+1)z^2 - \frac{1}{3!}m(m+1)(m+2)z^3 + \cdots$
    - $\binom{-m}{n} = (-1)^n \binom{m+n-1}{n} = (-1)^n \frac{(m+n-1)!}{(m-1)!n!}$
  - $f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3}$, Maclaurin series centered at $z_0 = 1$
    - $f(z) = \frac{1}{(3+(z-1))^2} + \frac{2}{2-(z-1)} = \frac{1}{9} \cdot \frac{1}{\left(1+\frac{1}{3}(z-1)\right)^2} - \frac{1}{1-\frac{1}{2}(z-1)}$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-m}{n} \left(\frac{1}{3}(z-1)\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2}(z-1)\right)^n$$

Uniform Convergence

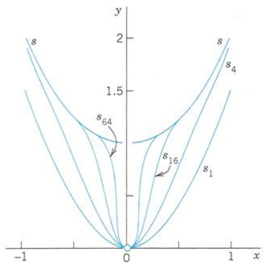
- Uniform convergence
  - A series with sum $s(z)$ is called uniformly convergent in a region G, if $\forall\, \epsilon > 0$, we can find $N(\epsilon) > 0$, independent of z, such that $|s(z) - s_n(z)| < \epsilon$, for all $n > N$ and for all $z \in G$.
- Theorem 1. Power series
  - A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, $R > 0$, is uniformly convergent in every circular disk, $|z - z_0| \leq r < R$.
- Properties of uniform convergence
  - If a series of continuous terms is uniformly convergent,
    - its sum is also continuous.
    - Term-wise integration is permissible.

Uniform Convergence

- Theorem 2. Continuity

  Let the series $\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$ be uniformly convergent in a region G. Let $F(z)$ be its sum. Then, if each term $f_m(z)$ is continuous at a point $z_1 \in G$, the function $F(z)$ is also continuous at z_1.

- Example 2. Series, $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots$ x real



  ∨ Partial sum, $s_n = x^2\left(1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n}\right)$

  † $s_n - \frac{1}{1+x^2}s_n = \frac{x^2}{1+x^2}s_n = x^2\left(1 - \frac{1}{(1+x^2)^{n+1}}\right) \Rightarrow s_n = 1 + x^2 - \frac{1}{(1+x^2)^n}$

  ∨ $s = \lim_{n\to\infty} s_n = \begin{cases} 1+x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$ ← discontinuous at $x = 0$

  † The convergence cannot be uniform in an interval containing $x = 0$.

Uniform Convergence: Term-wise integration

- Example 3. $\sum_{m=1}^{\infty} f_m(x)$

  ∨ $f_m(x) = u_m(x) - u_{m-1}(x)$, $u_m(x) = mxe^{-mx^2}$, $0 \le x \le 1$

  ∨ Partial sum, $s_n = f_1 + f_2 + \cdots + f_n = u_1 - u_0 + u_2 - u_1 + \cdots + u_n - u_{n-1} = u_n$

  ∨ Limit, $F(x) = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} u_n(x) = 0$, $0 \le x \le 1$

  † $\int_{x=0}^{1} F(x)dx = 0$

  ∨ Term-wise integration,

  $$\sum_{m=1}^{\infty}\left(\int_{x=0}^{1} f_m(x)dx\right) = \lim_{n\to\infty}\sum_{m=1}^{n}\left(\int_{x=0}^{1} f_m(x)dx\right) = \lim_{n\to\infty}\int_0^1\left(\sum_{m=1}^{n} f_m(x)\right)dx \ \ldots$$

  $$= \lim_{n\to\infty}\int_0^1 s_n(x)dx = \lim_{n\to\infty}\int_0^1 u_n(x)dx = \lim_{n\to\infty}\left(-\frac{1}{2}e^{-nx^2}\right)\Big|_{x=0}^{1} = \lim_{n\to\infty}\frac{1}{2}(1-e^{-n}) = \frac{1}{2}$$

  ∨ $\int_{x=0}^{1}\left(\sum_{m=1}^{\infty} f_m(x)\right)dx \neq \sum_{m=1}^{\infty}\left(\int_{x=0}^{1} f_m(x)dx\right)$

  ← Interchange between integration and infinite sum is not permissible.

Uniform Convergence: Term-wise Operations

- $F(z) = \sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$

- Theorem 3. Term-wise integration

  Let $F(z)$ be a uniformly convergent series in a region G. Let C be any path in G. Then, the series

  $$\sum_{m=0}^{\infty}\left(\int_C f_m(z)dz\right) = \int_C f_0(z)dz + \int_C f_1(z)dz + \cdots$$

  is convergent and has the sum $\int_C F(z)dz$.

  - We can exchange integration and infinite sum, only for uniformly convergent series.

- Theorem 2. Term-wise differentiation

  Let $F(z)$ be convergent in a region G and let $F(z)$ be its sum. If the series $f_0'(z) + f_1'(z) + \cdots$ converges uniformly in G and continuous in G, then

  $$F'(z) = f_0'(z) + f_1'(z) + \cdots, \ \forall z \in G$$