8장 연습문제 풀이

2006년 6월 11일

8.1

31. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $x + y = a$ (그림은 각자가)

풀이 우선 성망형이 x-축 사이의 면적을 구하기 위하여

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}} \Longrightarrow y^{\frac{1}{2}} = a^{\frac{1}{2}} - x^{\frac{1}{2}} \Longrightarrow y = a + x - 2\sqrt{ax}$$

이므로

$$\begin{aligned} A &= \int_0^a y dx = \int_0^a (a + x - 2\sqrt{ax}) dx \\ &= \left[ax + \frac{1}{2}x^2 - 2\sqrt{a}\frac{2}{3}x^{\frac{3}{2}} \right]_0^a \\ &= a^2 + \frac{1}{2}a^2 - \frac{4}{3}\sqrt{a}a^{\frac{3}{2}} \\ &= \frac{a^2}{6} \end{aligned}$$

$x + y = a$ 와 x-축 y-축사이의 도형은 면적이 $\frac{1}{2}a^2$ 인 삼각형이므로 $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ 과 $x + y = a$ 으로 둘러쌓인 부분의 넓이는 $\frac{1}{2}a^2 - \frac{1}{6}a^2 = \frac{1}{3}a^2$.

8.2

2. $r = a(1+\sin\theta)$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2} a^2 \int_0^{2\pi} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta \\ &= \frac{1}{2} a^2 \left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \frac{1}{2} a^2 (3\pi) = \frac{3\pi a^2}{2} \end{aligned}$$

3. $r = 2\cos 3\theta$. 대칭성을 이용함

$$\begin{aligned} A &= 6\int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 d\theta = 3\int_0^{\frac{\pi}{6}} 4\cos^2 3\theta d\theta \\ &= 12\left[\frac{\theta}{2} + \frac{1}{12}\sin 6\theta\right]_0^{\frac{\pi}{6}} \\ &= \pi \end{aligned}$$

4. $r = a(1-\sin\theta)$

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (1-\sin\theta)^2 d\theta \\ &= \frac{1}{2} a^2 \int_0^{2\pi} \left(1 - 2\sin\theta + \sin^2\theta\right) d\theta \\ &= \frac{1}{2} a^2 \left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \frac{1}{2} a^2 (3\pi) = \frac{3\pi a^2}{2} \end{aligned}$$

6. $r = 1 + \cos\theta$

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta \\ &= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \frac{3\pi}{2} \end{aligned}$$

7. $r^2 = 4\cos 2\theta$

$$\begin{aligned} A &= 4\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = 2\int_0^{\frac{\pi}{4}} 4\cos 2\theta d\theta \\ &= 8\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = 4 \end{aligned}$$

13. $r = a(1 - \cos\theta)$

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (1-\cos\theta)^2 d\theta \\ &= \frac{1}{2}a^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta \\ &= \frac{1}{2}a^2 \left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \frac{1}{2}a^2(3\pi) = \frac{3\pi a^2}{2} \end{aligned}$$

15. $r = a(2 + \cos\theta)$

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (4 + 4\cos\theta + \cos^2\theta) d\theta \\ &= \frac{1}{2}a^2 \left[4\theta + \sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{9\pi a^2}{2} \end{aligned}$$

18. $r_1 = 5\sin\theta$, $r_2 = 5\cos\theta$ 의 공통부분은 $\theta = \frac{\pi}{4}$ 에 대하여 대칭이므로

$$\begin{aligned} A &= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}r_2^2 d\theta \\ &= \int_0^{\frac{\pi}{4}} (5\cos\theta)^2 d\theta \\ &= 25\int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} d\theta \\ &= \frac{25}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{25}{2}(\frac{\pi}{4} + \frac{1}{2}) \end{aligned}$$

20. $r_1 = a$, $r_2^2 = 2a^2\cos 2\theta$ 의 공통부분은 대칭성을 이용. 우선 교점을 구하면

$$\begin{aligned} a^2 = 2a^2\cos 2\theta &\Longrightarrow \cos 2\theta = \frac{1}{2} \\ &\Longrightarrow 2\theta = \pm\frac{\pi}{3} \\ &\Longrightarrow \theta = \pm\frac{\pi}{6} \end{aligned}$$

우선 1사분면에서의 공통부분의 면적을 구하면

$$\begin{aligned} A &= \int_0^{\frac{\pi}{6}} \frac{1}{2}a^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} a^2\cos 2\theta d\theta \\ &= \left[\frac{1}{2}a^2\theta\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{2}a^2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \frac{1}{2}a^2\left(\frac{\pi}{6} + 1 - \sin\frac{\pi}{3}\right) \\ &= \frac{1}{2}a^2(\frac{\pi}{6} + 1 - \frac{\sqrt{3}}{4}) \end{aligned}$$

따라서 공통부분의 넓이는 $2a^2(\frac{\pi}{6} + 1 - \frac{\sqrt{3}}{4})$.

22. $r^2 = 6\cos 2\theta$, $r = 2\sin\theta$.

교점을 구하면

$$6\cos 2\theta = 4\sin^2\theta \Longrightarrow 6(1-2\sin^2\theta) = 4\sin^2\theta$$
$$\Longrightarrow \sin^2\theta = \frac{6}{16} \Longrightarrow \sin\theta = \pm\frac{\sqrt{6}}{4}$$
$$\Longrightarrow \theta = \sin^{-1}\frac{\sqrt{6}}{4}(\text{ 1사 분면에서})$$

1사 분면에서의 넓이를 구하면

$$A = \int_0^{\sin^{-1}\frac{\sqrt{6}}{4}} \frac{1}{2}(2\sin\theta)^2 d\theta + \int_{\sin^{-1}\frac{\sqrt{6}}{4}}^{\frac{\pi}{4}} 3\cos 2\theta d\theta$$
$$= \int_0^{\sin^{-1}\frac{\sqrt{6}}{4}} (1-\cos 2\theta)d\theta + 3\int_{\sin^{-1}\frac{\sqrt{6}}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta$$
$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}\frac{\sqrt{6}}{4}} + \frac{3}{2}[\sin 2\theta]_{\sin^{-1}\frac{\sqrt{6}}{4}}^{\frac{\pi}{4}}$$
$$= \sin^{-1}\frac{\sqrt{6}}{4} - \sin(\sin^{-1}\frac{\sqrt{6}}{4})\cos(\sin^{-1}\frac{\sqrt{6}}{4}) + \frac{3}{2} - 3\sin(\sin^{-1}\frac{\sqrt{6}}{4})\cos(\sin^{-1}\frac{\sqrt{6}}{4})$$
$$= \frac{3}{2} + \sin^{-1}\frac{\sqrt{6}}{4} - 4(\frac{\sqrt{6}}{4})\cos(\sin^{-1}\frac{\sqrt{6}}{4})$$

여기서

$$\cos(\sin^{-1}\frac{\sqrt{6}}{4}) = \sqrt{1-\sin^2(\sin^{-1}\frac{\sqrt{6}}{4})}$$
$$= \sqrt{1-\frac{6}{16}} = \frac{\sqrt{10}}{4}$$

이므로

$$A = \frac{3}{2} + \sin^{-1}\frac{\sqrt{6}}{4} - 4(\frac{\sqrt{6}}{4})\frac{\sqrt{10}}{4} = \frac{3}{2} + \sin^{-1}\frac{\sqrt{6}}{4} - \frac{\sqrt{60}}{4}$$

이고 전체넓이는 이것의 2 배이다.

24. $r = 2(1+\cos\theta)$, $r = 1$. 교점을 구하면

$$2(1+\cos\theta) = 1 \Longrightarrow \cos\theta = -\frac{1}{2} \Longrightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$
\begin{aligned}
A &= 2\int_0^{\frac{2\pi}{3}} \left(\frac{1}{2}(2+2\cos\theta)^2 - \frac{1}{2}\right) d\theta \\
&= \int_0^{\frac{2\pi}{3}} \left(1 - 4 - 8\cos\theta - 4\cos^2\theta\right) d\theta \\
&= \left[-3\theta - 8\cos\theta - 2\theta - \sin 2\theta\right]_0^{\frac{2\pi}{3}} \\
&= \frac{10\pi}{3} - (4 + \frac{\sqrt{3}}{2})
\end{aligned}
$$

25. $r^2 = 8\cos 2\theta,\ r = 2$

교점을 구하면

$$8\cos 2\theta = 4 \Longrightarrow \cos 2\theta = \frac{1}{2} \Longrightarrow \theta = \pm\frac{\pi}{6}$$

$$
\begin{aligned}
A &= 4\int_0^{\frac{\pi}{6}} \left(\frac{1}{2}(8\cos 2\theta) - \frac{1}{2}2^2\right) d\theta \\
&= 2\int_0^{\frac{\pi}{6}} (8\cos 2\theta - 4) d\theta \\
&= 8\left[\sin 2\theta - \theta\right]_0^{\frac{\pi}{6}} \\
&= 8(\frac{\sqrt{3}}{2} - \frac{\pi}{6})
\end{aligned}
$$

8.3

3. $y = \sqrt{a^2 - x^2},\ -a \leq x \leq a$

극방정식으로 바꾸면 $r = a,\ 0 \leq \theta \leq \pi$ 이므로

$$
\begin{aligned}
s &= \int_0^{\pi} \sqrt{r^2 + (r')^2} d\theta \\
&= \int_0^{\pi} \sqrt{a^2} d\theta = \int_0^{\pi} a d\theta \\
&= [a\theta]_0^{\pi} = \pi a
\end{aligned}
$$

4. $y = x\sqrt{x} = x^{\frac{3}{2}}$, $0 \le x \le 1$.

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1+(y')^2}dx \\
&= \int_0^1 \sqrt{1+\frac{9}{4}x}dx \\
&= \left[\frac{2}{3}\cdot\frac{4}{9}(1+\frac{9}{4}x)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{8}{27}(\frac{13}{8}\sqrt{13}-1)
\end{aligned}
$$

6. $6y = x^3 + 3x^{-1}$, $1 \le x \le 3$

$$
\begin{aligned}
y' &= \frac{1}{6}(3x^2 - 3x^{-2}) \\
1+(y')^2 &= 1 + \frac{1}{36}(3x^2 - 3x^{-2})^2 \\
&= 1 + \frac{1}{4}(x^4 - 2 + x^{-4} \\
&= \frac{1}{4}(x^4 + 2 + x^{-4} \\
&= \frac{1}{4}(x^2 + x^{-2})^2
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_1^3 \sqrt{1+(y')^2}dx \\
&= \int_1^3 \sqrt{\frac{1}{4}(x^2+x^{-2})^2}dx \\
&= \frac{1}{2}\int_1^3 (x^2 + x^{-2})dx \\
&= \frac{1}{2}\left[\frac{1}{3}x^3 - x^{-1}\right]_1^3 \\
&= \frac{1}{2}(\frac{27}{3} - \frac{1}{3} - \frac{1}{3} + 1) = \frac{13}{3}
\end{aligned}
$$

8. $y = \ln(x^2 - 1)$, $2 \le x \le 3$.

$$
y' = \frac{2x}{x^2-1}
$$

$$
\begin{aligned}
1+(y')^2 &= 1+(\frac{2x}{x^2-1})^2 \\
&= \frac{x^4-2x^2+1+4x^2}{(x^2-1)^2} = \frac{(x^2+1)^2}{(x^2-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_2^3 \sqrt{1+(y')^2}dx \\
&= \int_2^3 \frac{x^2+1}{x^2-1}dx \\
&= \int_2^3 \frac{x^2-1+2}{x^2-1}dx \\
&= \int_2^3 \left(1-\frac{2}{1-x^2}\right)dx \\
&= \left[x-2\tanh^{-1}x\right]_2^3 \\
&= 3-2\tanh^{-1}3-2+2\tanh^{-1}2 \\
&= 1-2(\tanh^{-1}3-\tanh^{-1}2)
\end{aligned}
$$

12. $x=\ln\sin y$, $\frac{\pi}{3}\le y\le\frac{2\pi}{3}$.

$$
x' = \frac{\cos y}{\sin y} = \cot y
$$

$$
\begin{aligned}
s &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1+(x')^2}dy \\
&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1+\cot^2 y}dy \\
&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\csc^2 y}dy \\
&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \csc y dy \\
&= [\ln(\csc y-\cot y)]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\
&= \ln 3
\end{aligned}
$$

13. $3x = (y-3)\sqrt{3}$, $0 \le y \le 3$

$$x = \frac{1}{3}(y^{\frac{3}{2}} - 3y^{\frac{1}{2}}) = \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{1}{2}}$$

$$\begin{aligned} x' &= \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}} \\ 1 + (x')^2 &= 1 + (\frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{2}})^2 \\ &= \frac{1}{4}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 \end{aligned}$$

$$\begin{aligned} s &= \int_0^3 \sqrt{1 + (x')^2} dy \\ &= \int_0^3 \sqrt{\frac{1}{4}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 dy} \\ &= \int_0^3 \frac{1}{2}(y^{\frac{1}{2}} + y^{-\frac{1}{2}}) dy \\ &= \frac{1}{2}\left[\frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}}\right]_0^3 \\ &= \frac{1}{2}\left(\frac{2}{3}3^{\frac{3}{2}} + 2\sqrt{3}\right) \\ &= 2\sqrt{3} \end{aligned}$$

15. $x = 9t^2$, $y = 9t^3 - 3t$, $0 \le t \le \frac{1}{\sqrt{3}}$.

$$\frac{dx}{dt} = 18t, \quad \frac{dy}{dt} = 27t^2 - 3$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 18^2t^2 + (27t^2 - 3)^2 \\ &= (27t^2 + 3)^2 \end{aligned}$$

$$
\begin{aligned}
s &= \int_0^{\frac{1}{\sqrt{3}}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^{\frac{1}{\sqrt{3}}} \sqrt{(27t^2+3)^2} dt \\
&= \int_0^{\frac{1}{\sqrt{3}}} (27t^2+3) dt \\
&= \left[9t^3 + 3t\right]_0^{\frac{1}{\sqrt{3}}} = 2\sqrt{3}
\end{aligned}
$$

17. $x = \frac{1}{3}t^2$, $y = \frac{1}{2}t^2$, $1 \le t \le 3$.

$$
\frac{dx}{dt} = \frac{2}{3}t, \quad \frac{dy}{dt} = t
$$

$$
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{4}{9}t^2 + t^2 = \frac{13}{9}t^2
$$

$$
\begin{aligned}
s &= \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_1^3 \sqrt{\frac{13}{9}t^2} dt \\
&= \frac{\sqrt{3}}{3} \int_1^3 t dt = \frac{\sqrt{3}}{3} \left[\frac{1}{2}t^2\right]_1^3 \\
&= \frac{4\sqrt{3}}{3}
\end{aligned}
$$

18. $x = \frac{1}{3}t^3 + \frac{1}{t}$, $y = 2t$, $1 \le t \le 3$

$$
\frac{dx}{dt} = t^2 - t^{-2}, \quad \frac{dy}{dt} = 2
$$

$$
\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^4 - 2 + t^{-4} + 4 \\
&= (t^2 + t^{-2})^2
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_1^3 \sqrt{(t^2 + t^{-2})^2} dt \\
&= \int_1^3 (t^2 + t^{-2}) dt \\
&= \left[\frac{1}{3}t^3 - \frac{1}{t}\right]_1^3 \\
&= 3 - \frac{1}{3} - \frac{1}{3} + 1 = \frac{10}{3}
\end{aligned}
$$

19. $x = \frac{1}{3}(2t+3)^{\frac{3}{2}}$, $y = \frac{1}{2}t^2 + t$, $0 \le t \le 3$.

$$\frac{dx}{dt} = (2t+3)^{\frac{1}{2}}, \quad \frac{dy}{dt} = t + 1$$

$$
\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 2t + 3 + (t+1)^2 \\
&= (t+2)^2
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^3 (t+2) dt \\
&= \left[\frac{1}{2}t^2 + 2t\right]_0^3 \\
&= \frac{9}{2} + 6 = \frac{21}{2}
\end{aligned}
$$

21. $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi$.

$$\frac{dx}{dt} = -3\cos^2 t \sin t, \quad \frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t$$
$$= 9\cos^2 \sin^2 t$$

$$s = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$
$$= 3\int_0^\pi \sqrt{\cos^2 t \sin^2 t} dt$$
$$= 3\int_0^\pi |\cos t \sin t| \, dt$$
$$= 6\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t dt$$
$$= 3\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}}$$
$$= 3(\frac{1}{2} + \frac{1}{2}) = 3$$

22. $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le 2\pi$

$$\frac{dx}{dt} = -3a\cos^2 t \sin t, \quad \frac{dy}{dt} = 3a\sin^2 t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^4 t \sin^2 t + 9a^2\sin^4 t \cos^2 t$$
$$= 9a^2\cos^2 \sin^2 t$$

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= 3a \int_0^{2\pi} \sqrt{\cos^2 t \sin^2 t} dt \\
&= 3a \int_0^{2\pi} |\cos t \sin t| \, dt \\
&= 12a \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2t dt \\
&= 12a \left[-\frac{1}{2} \cos 2t \right]_0^{\frac{\pi}{2}} \\
&= 6a(\frac{1}{2} + \frac{1}{2}) = 6a
\end{aligned}
$$

23. $x = \ln(\sec\theta + \tan\theta)$, $y = \ln(\csc\theta - \cot\theta)$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$.

$$
\frac{dx}{dt} = \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} = \sec\theta, \quad \frac{dy}{dt} = \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta - \cot\theta} = \csc\theta
$$

$$
\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \sec^2\theta + \csc^2\theta \\
&= \tan^2\theta + \cot^2\theta + 2 \\
&= (\tan\theta + \cot\theta)^2
\end{aligned}
$$

$$
\begin{aligned}
s &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\tan\theta + \cot\theta) d\theta \\
&= [-\ln\cos\theta + \ln\sin\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= [\ln\tan\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \ln\sqrt{3} - \ln\frac{1}{\sqrt{3}} = \ln 3
\end{aligned}
$$

24. $x = t - a\tanh\frac{t}{a}$, $y = a\mathrm{sech}\ \frac{t}{a}$, $-a \le t \le 2a$.

$$\frac{dx}{dt} = 1 - \mathrm{sech}^2\frac{t}{a} = \tanh^2\frac{t}{a},\ \ \frac{dy}{dt} = -\mathrm{sech}\ \frac{t}{a}\tanh\frac{t}{a}$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \tanh^4\frac{t}{a} + \mathrm{sech}^2\frac{t}{a}\tanh^2\frac{t}{a}\\ &= \tanh^2\frac{t}{a}(\tanh^2\frac{t}{a} + \mathrm{sech}^2\frac{t}{a}) = \tanh^2\frac{t}{a}\end{aligned}$$

$$\begin{aligned}s &= \int_{-a}^{2a}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta\\ &= \int_{-a}^{2a}\left|\tanh^t a\right|dt\\ &= -\int_{-a}^{0}\tanh\frac{t}{a}dt + \int_{0}^{2a}\tanh\frac{t}{a}dt \ \ = -\left[a\ln\cosh\frac{t}{a}\right]_0^{-a} + \left[a\ln\cosh\frac{t}{a}\right]_0^{2a}\\ &= a\ln(\cosh 2 - \cosh 1)\end{aligned}$$

28. 성망형 $x^{2/3} + y^{2/3} = a^{2/3}$ 의 둘레를 구하여라.

풀이. 성망형 $x^{2/3} + y^{2/3} = a^{2/3}$ 을 매개변수 방정식으로 쓰면 이문제는 22번 문제와 같은 문제임.

29. $r = a(1+\sin\theta)$ y-축대칭

$$\begin{aligned}
s &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2+(r')^2}d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2(1+\sin\theta)^2+a^2\cos^2\theta}d\theta \\
&= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1+\sin\theta)}d\theta \\
&= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1-\cos(\frac{\pi}{2}+\theta))}d\theta \\
&= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\sin^2\frac{1}{2}(\frac{\pi}{2}+\theta)}d\theta \\
&= 4a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\frac{1}{2}(\frac{\pi}{2}+\theta)d\theta \\
&= 4a\left[-\cos(\frac{\pi}{2}+\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 8a
\end{aligned}$$

30. $r = a(1-\sin\theta)$ y-축대칭

$$\begin{aligned}
s &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2+(r')^2}d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2(1-\sin\theta)^2+a^2\cos^2\theta}d\theta \\
&= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1-\sin\theta)}d\theta = 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1-\cos(\frac{\pi}{2}-\theta))}d\theta \\
&= 2a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\sin^2\frac{1}{2}(\frac{\pi}{2}-\theta)}d\theta = 4a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\frac{1}{2}(\frac{\pi}{2}-\theta)d\theta \\
&= 4a\left[\cos(\frac{\pi}{2}-\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 8a
\end{aligned}$$

31. $r = a(1+\cos\theta)$ x-축대칭

$$\begin{aligned}
s &= 2\int_0^{\pi} \sqrt{r^2+(r')^2}d\theta \\
&= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}d\theta = 2a\int_0^{\pi} \sqrt{2(1+\cos\theta)}d\theta \\
&= 2a\int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}d\theta \\
&= 4a\int_0^{\pi} \left|\cos\frac{\theta}{2}\right| d\theta \\
&= 4a\int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2}d\theta - 4a\int_{\frac{\pi}{2}}^{\pi} \cos\frac{\theta}{2}d\theta \\
&= 8a
\end{aligned}$$

32. 예제 5와 같은 문제임