[13 - 7]

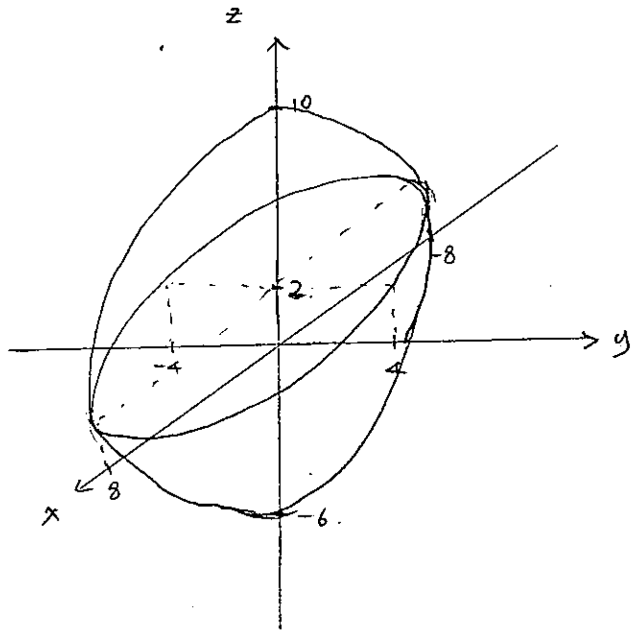
#2. $x^2 + 4y^2 + 16z^2 = 64z$

$\Rightarrow \quad x^2 + 4y^2 + 16z^2 - 64z = 0$

$x^2 + 4y^2 + (4z - 8)^2 = 64$

$\dfrac{x^2}{8^2} + \dfrac{y^2}{4^2} + \dfrac{(4z-8)^2}{8^2} = 1.$

$\therefore$ 타원면

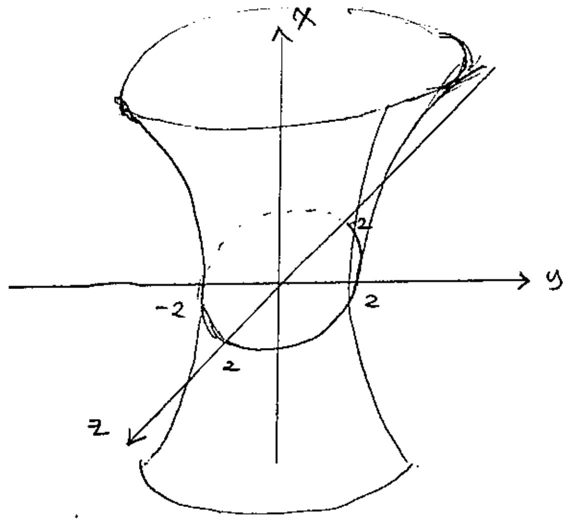


#10. $y^2 + z^2 = 4x^2 + 4$

$\Rightarrow \quad -4x^2 + y^2 + z^2 = 4$

$-\dfrac{x^2}{1} + \dfrac{y^2}{2^2} + \dfrac{z^2}{2^2} = 1$

$\therefore$ 단엽 쌍곡면



#14. $-x^2 - 4y^2 + 4z^2 - 2x + 8y + 8z - 5 = 0$

$\Rightarrow \quad -(x^2 + 2x) - 4(y^2 - 2y) + 4(z^2 + 2z) - 5 = 0$

$-(x+1)^2 - 4(y-1)^2 + 4(z+1)^2 - 4 = 0$

$-(x+1)^2 - 4(y-1)^2 + 4(z+1)^2 = 4$

$\therefore$ 쌍엽 쌍곡면

[14-2]

#2. $z = x^2y^2 + xy^4$

$\Rightarrow z_x = 2xy^2 + y^4 \quad , \quad z_y = 2x^2y + 4xy^3$

#10. $w = a^{bx + c/y}$

$\Rightarrow \ln w = (bx + \frac{c}{y}) \ln a$

$w_x = w \cdot (b \ln a) = a^{bx+\frac{c}{y}} (b \ln a)$

$w_y = w \cdot (-\frac{c}{y^2}) \ln a = a^{bx+\frac{c}{y}} (-\frac{c}{y^2}) \ln a$

#18. $z = \sin^{-1}(xy)$

$\Rightarrow z_x = \dfrac{y}{\sqrt{1-(xy)^2}} \quad , \quad z_y = \dfrac{x}{\sqrt{1-(xy)^2}}$

#22. $z = (xy)^{xy} + x^y + y^x$

$\Rightarrow$ (i) $u = (xy)^{xy}$

$\ln u = xy \ln xy$

$\frac{u_x}{u} = y \ln xy + y$

$u_x = (xy)^{xy} \cdot (y \ln xy + y)$

(ii) $u = x^y$

$\ln u = y \ln x$

$\frac{u_x}{u} = \frac{y}{x}$

$u_x = \dfrac{yx^y}{x}$

(iii) $u = y^x$

$\ln u = x \ln y$

$\frac{u_x}{u} = \ln y$

$u_x = y^x \ln y$

(i) $u = (xy)^{xy}$

$\ln u = xy \ln xy$

$\frac{u_y}{u} = x \ln xy + x$

$u_y = (xy)^{xy} \cdot (x \ln xy + x)$

(ii) $u = x^y$

$\ln u = y \ln x$

$\frac{u_y}{u} = \ln x$

$u_y = x^y \ln x$

(iii) $u = y^x$

$\ln u = x \ln y$

$\frac{u_y}{u} = \frac{x}{y}$

$u_y = \dfrac{xy^x}{y}$

$$\therefore z_x = (xy)^{xy}(y\ln xy + y) + \frac{yx^y}{x} + y^x \ln y$$

$$z_y = (xy)^{xy}(x\ln xy + x) + x^y \ln x + \frac{xy^x}{y}$$

#36. $f(x,y) = \dfrac{x}{x-y}$

$$\Rightarrow f_x(x,y) = \frac{(x-y)-x}{(x-y)^2} = \frac{-y}{(x-y)^2} \Rightarrow f_x(1,0) = 0$$

$$f_y(x,y) = \frac{+x}{(x-y)^2} \Rightarrow f_y(1,0) = +1.$$

#42. $f(u,v) = e^{uv}\sec\dfrac{u}{v}$

$$\Rightarrow f_u = ve^{uv}\sec\frac{u}{v} + \frac{1}{v}e^{uv}\sec\frac{u}{v}\tan\frac{u}{v}$$

$$f_v = ue^{uv}\sec\frac{u}{v} - \frac{u}{v^2}e^{uv}\sec\frac{u}{v}\tan\frac{u}{v}$$

#14.3

#6. $z = \ln\sqrt{x^3+y^4}$

$$z_x = \frac{\frac{3x^2}{2\sqrt{x^3+y^4}}}{\sqrt{x^3+y^4}} = \frac{3x^2}{2(x^3+y^4)}$$

$$z_y = \frac{\frac{4y^3}{2\sqrt{x^3+y^4}}}{\sqrt{x^3+y^4}} = \frac{4y^3}{2(x^3+y^4)} = \frac{2y^3}{x^3+y^4}$$

$$z_{xx} = \frac{3}{2}\cdot\frac{2x(x^3+y^4) - x^2(3x^2)}{(x^3+y^4)^2} = \frac{3(2xy^4 - x^4)}{2(x^3+y^4)^2}$$

$$z_{xy} = z_{yx} = -\frac{3}{2}\cdot\frac{4y^3x^2}{(x^3+y^4)^2} = -\frac{6x^2y^3}{(x^3+y^4)^2}$$

$$z_{yy} = \frac{6y^2(x^3+y^4) - 2y^3(4y^3)}{(x^3+y^4)^2} = \frac{6y^2x^3 - 2y^6}{(x^3+y^4)^2}$$

16. $z = \sin^{-1}(x+3y)$

$$z_x = \frac{1}{\sqrt{1-(x+3y)^2}} \qquad z_{xx} = \frac{x+3y}{[1-(x+3y)^2]^{\frac{3}{2}}}$$

$$z_y = \frac{3}{\sqrt{1-(x+3y)^2}} \qquad z_{yy} = \frac{9(x+3y)}{[1-(x+3y)^2]^{\frac{3}{2}}}$$

$$z_{xy} = \frac{3(x+3y)}{[1-(x+3y)^2]^{\frac{3}{2}}}$$

#26. $f(x,y,z) = \sin xyz$

$f_x = yz\cos xyz \qquad f_{xy} = z\cos xyz - xyz^2\sin xyz$

$f_{xyz} = (1 - x^2y^2z^2)\cos xyz - 3xyz\sin xyz$

$f_z = xy\cos xyz \qquad f_{zy} = x\cos xyz - x^2yz\sin xyz$

$f_{zyx} = (1 - x^2 y^2 z^2)\cos xyz - 3xyz\sin xyz$

#34. $f(x,y) = \sin^2(xy)$

$$f_x = 2y\sin(xy)\cos(xy) = y\sin(2xy)$$

$$f_{xy} = \sin(2xy) + 2xy\cos(2xy)$$

$$f_y = x\sin(2xy)$$

$$f_{yx} = \sin(2xy) + 2xy\cos(2xy)$$

#42. $z = \ln(x^2+3y^2)^4 = 4\ln(x^2+3y^2)$

$$\frac{\partial z}{\partial y} = \frac{24y}{x^2+3y^2} \qquad \frac{\partial^2 z}{\partial x\partial y} = \frac{-48xy}{(x^2+3y^2)^2}$$

$$\frac{\partial^3 z}{\partial x^2\partial y} = \frac{-48y(x^2+3y^2)^2 - 48xy\left[2(2x)(x^2+3y^2)\right]}{(x^2+3y^2)^4} = \frac{144(x^2y-y^3)}{(x^2+3y^2)^3}$$

$$\frac{\partial z}{\partial x} = \frac{8x}{x^2+3y^2} \qquad \frac{\partial^2 z}{\partial y\partial x} = \frac{-48xy}{(x^2+3y^2)^2}$$

$$\frac{\partial^3 z}{\partial x\partial y\partial x} = \frac{144(x^2y-y^3)}{(x^2+3y^2)^3}$$

$$\frac{\partial z}{\partial x} = \frac{8x}{x^2+3y^2} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{24y^2-8x^2}{(x^2+3y^2)^2}$$

$$\frac{\partial^3 z}{\partial y\partial x^2} = \frac{144(x^2y-y^3)}{(x^2+3y^2)^3}$$

#46. $z = \ln(2x+2y) + \tan(2x-2y)$

$$z_x = \frac{2}{2x+2y} + 2\sec^2(2x-2y)$$

$$z_{xx} = \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\tan(2x-2y)$$

$$z_y = \frac{1}{(x+y)} - 2\sec^2(2x-2y)$$

$$z_{yy} = \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\tan(2x-2y)$$

#14.4

#2. $f(s,t) = s^2 e^{\sin t}$

$$dz = \frac{\partial z}{\partial s} ds + \frac{\partial z}{\partial t} dt$$

$$= 2s e^{\sin t} ds + s^2 \cdot \cos t\, e^{\sin t} dt$$

#9. $f(x,y) = \sinh \frac{x}{y}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{1}{y} \cosh \frac{x}{y} dx - \frac{x}{y^2} \cosh \frac{x}{y} dy$$

#13. $f(x,y) = x^2 y^3$; $x = 2$, $y = 1$, $\Delta x = 0.1$, $\Delta y = 0.05$

(i) 증분 $\Delta z = f(2.1,\ 1.05) - f(2,1)$

$$= (2.1)^2 \cdot (1.05)^3 - 2^2 \cdot 1^3 \fallingdotseq 1.105$$

(ii) 전미분 $dz = 2xy^3\, dx + 3x^2y^2\, dy$

$$= 2 \cdot 2 \cdot 1 \cdot 0.1 + 3 \cdot 4 \cdot 1 \cdot 0.05 = 1.$$

#29. $f(x,y,z) = \sinh(xe^{yz})$

$$\Rightarrow \sinh(xe^{yz}) e^{2yz} dx^2 + 2(\sinh(xe^{yz}) x z e^{2yx} + \cosh(xe^{yz}) z e^{yz})\, dx dy$$

$$+ 2(\sinh(xe^{yz}) xy e^{2yz} + \cosh(xe^{xy}) y e^{yz})\, dx dz + (\sinh(xe^{yz}) x^2 z^2 e^{2yx}$$

$$+ \cosh(xe^{yz}) x z^2 e^{yz})\, dy^2 + 2(\sinh(xe^{yx}) x^2 y z e^{2yz}$$

$$+ \cosh(xe^{yz}) x e^{yz} + \cosh(xe^{yz}) xyz e^{yz})\, dy dz$$

$$+ (\sinh(xe^{yz}) x^2 y^2 e^{2yz} + \cosh(xe^{yz}) x y^2 e^{yz})\, dz^2$$

#31. $f(x,y) = \sqrt{x^2+y^2}$; $(1,1)$

$$\sqrt{x^2+y^2} \approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= \sqrt{2} + \frac{x_\circ}{\sqrt{x_\circ^2+y_\circ^2}}(x-1) + \frac{y_\circ}{\sqrt{x_\circ^2+y_\circ^2}}(y-1)$$

$$= \sqrt{2} + \frac{\sqrt{2}}{2}(x-1) + \frac{\sqrt{2}}{2}(y-1) = \frac{\sqrt{2}}{2}(x+y)$$

#14.5

#4. $u = \ln\sqrt{\frac{x-y}{x+y}}$, $x = \sec t$, $y = \tan t$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{y}{x^2-y^2}\cdot \sec t\cdot\tan t + \frac{-x}{x^2-y^2}\cdot\sec^2 t$$

$$= \frac{1}{x^2-y^2}\cdot\sec t\cdot\left(y\tan t - x\sec t\right)$$

#8. $u = \tan(x^2+y^2)$, $x = 3\theta$, $y = e^\theta$

$$\Rightarrow \frac{du}{d\theta} = \frac{\partial u}{\partial x}\cdot\frac{dx}{d\theta} + \frac{\partial u}{\partial y}\cdot\frac{dy}{d\theta}$$

$$= 2x\sec^2(x^2+y^2)\cdot 3 + 2y\sec^2(x^2+y^2)\cdot e^\theta$$

$$= 2\sec^2(x^2+y^2)\{3x + ye^\theta\}$$

#12. $u = e^{\sin\frac{y}{x}}$, $x = \sqrt{t}$. $y = \frac{1}{t}$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= -\frac{1}{x^2}\cdot\cos\frac{y}{x}\cdot e^{\sin\frac{y}{x}}\cdot\frac{1}{2\sqrt{t}} + \frac{1}{x}\cos\frac{y}{x}\,e^{\sin\frac{y}{x}}\cdot\frac{-1}{t^2}$$

$$= \frac{1}{x}\cos\frac{y}{x}\,e^{\sin\frac{y}{x}}\left(-\frac{1}{2x\sqrt{t}} - \frac{1}{t^2}\right)$$

#24. $u = xy + \ln(x+y)$, $y = \sqrt{1+x^2}$

$$\Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= y + \frac{1}{x+y} + \left(x + \frac{1}{x+y}\right) \cdot \frac{x}{\sqrt{1+x^2}}$$

#32. $z = x^2 - xy - y^2$, $x = s+t$, $y = 2st$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x - y) + (-x - 2y) \cdot 2t = 2x(1-t) - y(1+4t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= (2x-y) + (-x-2y)\,2s = 2x(1-s) - y(1+4s)$$

#14. 6

#1. $2x^3 + 3x^2y - y^3 = 1$.

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)} = -\frac{6x^2+6xy}{3x^2-3y^2} = \frac{2x}{y-x}$$

#3. $\ln(x^2+y^2) = 2\tan^{-1}\frac{y}{x}$

$$\Rightarrow f_y(x,y) = \frac{2y}{x^2+y^2} - \frac{2x}{x^2+y^2} = \frac{2(y-x)}{x^2+y^2}$$

$$f_x(x,y) = \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2} = \frac{2(x+y)}{x^2+y^2}$$

$$\therefore \frac{dy}{dx} = -\frac{2(x+y)}{2(y-x)} = \frac{x+y}{x-y}$$

#13. $e^x + e^y + e^z = e^{x+y+z}$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{f_x(x.y.z)}{f_z(x.y.z)} = -\frac{e^x - e^{x+y+z}}{e^z - e^{x+y+z}}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x.y.z)}{f_z(x.y.z)} = -\frac{e^y - e^{x+y+z}}{e^z - e^{x+y+z}}$$

#15. $e^{xyz} + x^2 \sin yz = 0$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x(x.y.z)}{f_z(x.y.z)} = -\frac{yze^{xyz} + 2x\sin yz}{xye^{xyz} + yx^2\cos yz}$$

$$\frac{\partial z}{\partial y} = -\frac{xze^{xyz} + zx^2\cos yz}{xye^{xyz} + yx^2\cos yz} = -\frac{z}{y}$$

#14.7

#6. $\theta = \frac{5\pi}{4}$

$$\Rightarrow D_{\frac{5\pi}{4}} f(1.1) = f_x(1.1)\cos\frac{5\pi}{4} + f_y(1.1)\sin\frac{5\pi}{4}$$

$$= -4\frac{\sqrt{2}}{2} - 5\frac{\sqrt{2}}{2} = -\frac{9\sqrt{2}}{2}$$

#14. $T_x = 12x^2y - 4y^3 \quad T_y = 4x^3 - 12xy^2$

$$T_x\left(\tfrac{1}{2}.\tfrac{1}{2}\right) = \tfrac{1}{8}(12-4) = 1.$$

$$T_y\left(\tfrac{1}{2}.\tfrac{1}{2}\right) = \tfrac{1}{8}(4-12) = -1.$$

$$D_\theta T\left(\tfrac{1}{2}.\tfrac{1}{2}\right) = T_x\left(\tfrac{1}{2}.\tfrac{1}{2}\right)\cos\theta + T_y\left(\tfrac{1}{2}.\tfrac{1}{2}\right)\sin\theta$$

$$= \cos\theta - \sin\theta$$

$$D_\theta T\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 0 \Rightarrow \cos\theta = \sin\theta \Rightarrow \tan\theta = 1.$$

#22. $f(x,y,z) = \sin(xyz)$; $(1, \frac{\pi}{2}, \pi)$

$\Rightarrow \nabla f(x,y,z) = (yz\cos(xyz),\ xz\cos(xyz),\ xy\cos(xyz))$

$\nabla f(1, \frac{\pi}{2}, \pi) = \left(\frac{\pi^2}{2}\cos(\frac{\pi^2}{2}),\ \pi\cos(\frac{\pi^2}{2}),\ \frac{\pi}{2}\cos(\frac{\pi^2}{2})\right)$.

#28. $f(x,y) = 2x^2 + xy - y^2$; $(3,-2)$, $A = (1,-1)$

$\Rightarrow \nabla f(x,y) = (4x + y,\ x - 2y)$

$\nabla f(3,-2) = (10, 7)$

$u = \frac{A}{|A|} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$D_u f = \nabla f \cdot u = (10, 7)\cdot\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$

#36. $f(x,y,z) = x^3y^2z$; $(1,1,1)$, $A = (1,1,1)$

$\Rightarrow \nabla f(x,y,z) = (3x^2y^2z,\ 2x^3yz,\ x^3y^2)$

$\nabla f(1,1,1) = (3, 2, 1)$

$u = \frac{A}{|A|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$D_u f = \nabla f \cdot u = \frac{1}{\sqrt{3}}(3+2+1) = 2\sqrt{3}$

#42. $\nabla f(x,y,z) = (2x, 2y, -3)$

$\nabla f(1,3,5) = (2, 6, 5) \Rightarrow |\nabla f(1,3,5)| = \sqrt{65}$