5장 1절 연습문제 풀이

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1.

$$\begin{cases} x = 3 - t \\ y = t^2 - 2 \end{cases}$$

$t = 3 - x$ 이므로

$$y = (3-x)^2 - 2 = x^2 - 6x + 7.$$

3.

$$\begin{cases} x = \frac{1}{v+1} \\ y = \frac{v}{1-v^2} \end{cases}$$

$$x = \frac{1}{v+1} \Longrightarrow v = \frac{1}{x} - 1 \Longrightarrow$$
$$y = \frac{v}{1-v} \cdot \frac{1}{1+v} = x \cdot \frac{\frac{1}{x} - 1}{1 - (\frac{1}{x} - 1)} = \frac{x(1-x)}{2x-1}$$

5.

$$\begin{cases} x = \tan\theta \\ y = \tan 2\theta \end{cases}$$

$$y = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2x}{1-x^2}.$$

6.

$$\begin{cases} x = \sin\theta \\ y = \cos 2\theta \end{cases}$$

$$y = \cos 2\theta = 1 - 2\sin^2 2\theta = 1 - 2x^2.$$

7.

$$\begin{cases} x = \ln t \\ y = t^2 - 1 \end{cases}$$

$$x = \ln t \Longrightarrow t = e^x \Longrightarrow y = e^{2x} - 1.$$

8.

$$\begin{cases} x = e^v + e^{-v} \\ y = e^v - e^{-v} \end{cases}$$

$$x = 2\cosh v, \quad y = 2\sinh v$$

$$\cosh^2 v - \sinh^2 v = (\frac{x}{2})^2 - (\frac{y}{2})^2 = 1.$$

9. $x = \cos t$, $y = \sin t \Longrightarrow x^2 + y^2 = 1.$

10. $x = 2 + 4\sin t$, $y = 3 - 2\cos t$

$$\Longrightarrow \sin t = \frac{x-2}{4}, \quad \cos t = \frac{3-y}{2} \Longrightarrow (\frac{x-2}{4})^2 + (\frac{3-y}{2})^2 = 1.$$

11. $x = \sec t$, $y = \tan t$

$$\Longrightarrow \tan^2 t + 1 = \sec^2 t \Longrightarrow y^2 + 1 = x^2.$$

12. $x = \cosh t$, $y = 2\sinh t$

$$\cosh^2 t - \sinh^2 t = 1 \Longrightarrow x^2 - \frac{y^2}{2} = 1.$$

14. $x = 3 + 2\text{sech }t$, $y = 4 - 3\tanh t$

$$1 - \tanh^2 t = \text{sech }^2 t \Longrightarrow 1 - (\frac{4-y}{3})^2 = (\frac{x-3}{2})^2$$

17. $x^2 + 2xy + 4y^2 = 8x$

$$\begin{aligned} x = ty &\Longrightarrow x^2 + 2tx^2 + 4t^2x^2 = 8x \\ &\Longrightarrow x(x + 2tx + 4t^2x - 8) = 0 \\ &\Longrightarrow x = \frac{8}{1+2t+4t^2}, \quad y = \frac{8}{t(1+2t+4t^2)} \end{aligned}$$

19. $x = u^3 + 1$, $y = u^2 + 1$

$$\begin{aligned}
\frac{dx}{du} &= 3u^2, \quad \frac{dy}{du} = 2u \\
\Longrightarrow \frac{dy}{dx} &= \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{2u}{3u^2} = \frac{1}{3u} \\
\Longrightarrow \frac{dy'}{du} &= -\frac{2}{3u^2} \\
\Longrightarrow \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{du}}{\frac{dx}{du}} = \frac{-\frac{2}{3u^2}}{3u^2} = -\frac{2}{9u^4}
\end{aligned}$$

20. $x = \frac{1}{1-t}$, $y = \frac{t}{t-1}$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{(1-t)^2}, \quad \frac{dy}{dt} = \frac{t-1-t}{(t-1)^2} \\
\Longrightarrow \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t-1-t}{(t-1)^2}}{\frac{1}{(1-t)^2}} = -1 \\
\Longrightarrow \frac{dy'}{dt} &= 0 \\
\Longrightarrow \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = 0
\end{aligned}$$

21. $x = \frac{1}{1-t}$, $y = \frac{t}{t^2-1}$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{(1-t)^2}, \quad \frac{dy}{dt} = \frac{t^2-1-t(2t-1)}{(t^2-1)^2} = \frac{-t^2+t-1}{(t^2-1)^2} \\
\Longrightarrow \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-t^2+t-1}{(t^2-1)^2}}{\frac{1}{(1-t)^2}} = t^2 - t + 1 \\
\Longrightarrow \frac{dy'}{dt} &= 2t - 1 \\
\Longrightarrow \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{\frac{1}{(1-t)^2}} = -(2t-1)(t-1)^2.
\end{aligned}$$

23. $x = \frac{2}{1+v^2}$, $y = \frac{2}{v(1+v^2)}$

$$\frac{dx}{dv} = \frac{-4v}{(1+v^2)^2}, \quad \frac{dy}{dv} = \frac{-2(1+3v^2)}{v^2(1+v)^2}$$

$$\implies \frac{dy}{dx} = \frac{\frac{dy}{dtv}}{\frac{dx}{dv}} = \frac{\frac{-2(1+3v^2)}{v^2(1+v)^2}}{\frac{-4v}{(1+v^2)^2}} = \frac{2(1+3v^2)}{4v^3} = \frac{1+3v^2}{2v^3}$$

$$\implies \frac{dy'}{dv} = \frac{6v(2v^3) - (1+3v^2)(6v^2)}{4v^6} = \frac{-3(1-3v^2)}{2v^4}$$

$$\implies \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dv}}{\frac{dx}{dv}} = \frac{\frac{-3(1-3v^2)}{2v^4}}{\frac{-4v}{(1+v^2)^2}} = \frac{3(1-3v^2)(1+v^2)^2}{8v^5}$$

24. $x = \cos^3\theta$, $y = \sin^3\theta$

$$\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta, \quad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

$$\implies \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -\tan\theta$$

$$\implies \frac{dy'}{d\theta} = -\sec^2\theta$$

$$\implies \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3}\sec^4\theta\csc\theta$$

25. $x = \theta - \sin\theta$, $y = 1 - \cos\theta$

$$\frac{dx}{d\theta} = 1 - \cos\theta, \quad \frac{dy}{d\theta} = \sin\theta$$

$$\implies \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1-\cos\theta}$$

$$\implies \frac{dy'}{d\theta} = \frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2} = -\frac{1}{1-\cos\theta}$$

$$\implies \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\frac{1}{1-\cos\theta}}{1-\cos\theta} = -\frac{1}{(1-\cos\theta)^2}$$

26. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$

$$\begin{aligned}
\frac{dx}{dt} &= -\sin t + \sin t + t\cos t = t\cos t, \\
\frac{dy}{dt} &= \cos t - \cos t + t\sin t = t\sin t \\
\implies \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\sin t}{t\cos t} = \tan t \\
\implies \frac{dy'}{dt} &= \sec^2 t \\
\implies \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{t\cos t} = \frac{1}{t}\sec^3 t
\end{aligned}$$

27. $x = 1 - \ln t$, $y = t - \ln t$

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{1}{t}, \quad \frac{dy}{dt} = 1 - \frac{1}{t} \\
\implies \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - \frac{1}{t}}{-\frac{1}{t}} = -t + 1 \\
\implies \frac{dy'}{dt} &= -1 \\
\implies \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-1}{-\frac{1}{t}} = t
\end{aligned}$$

28. $x = e^t(\cos t - \sin t)$, $y = e^t(\cos t + \sin t)$

$$\begin{aligned}
\frac{dx}{dt} &= e^t(\cos t - \sin t) + e^t(-\sin t - \cos t) = -2e^t\sin t, \\
\frac{dy}{dt} &= e^t(\cos t + \sin t) + e^t(-\sin t + \cos t) = 2e^t\cos t \\
\implies \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^t\cos t}{-2e^t\sin t} = -\tan t \\
\implies \frac{dy'}{dt} &= -\sec^2 t \\
\implies \frac{d^2y}{dx^2} &= \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\sec^2 t}{-2e^t\sin t} = \frac{1}{2e^t}\sec^2 t\csc t
\end{aligned}$$

29. $x = t^2 - 2t$, $y = t^3 - 3t$; $t = 2$

$$\frac{dx}{dt} = 2t - 2, \quad \frac{dy}{dt} = 3t^2 - 3$$
$$\Longrightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t - 2}$$
$$t = 2 \Longrightarrow x = 0, y = 2, y' = \frac{2}{9}$$
$$\Longrightarrow \text{접선 EQ: } y - 2 = \frac{2}{9}(x - 0)$$
$$\text{법선 EQ: } y - 2 = -\frac{9}{2}(x - 0).$$

30. $x = \sin\theta$, $y = \tan\theta$; $\theta = \frac{\pi}{4}$

$$\frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\theta} = \sec^2\theta$$
$$\Longrightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sec^2\theta}{\cos\theta} = \cos^3\theta$$
$$\theta = \frac{\pi}{4} \Longrightarrow x = \frac{\sqrt{2}}{2}, y = 1, y' = \frac{\sqrt{2}}{4}$$
$$\Longrightarrow \text{접선 EQ: } y - 1 = \frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{2})$$
$$\text{법선 EQ: } y - 1 = -\frac{4}{\sqrt{2}}(x - \frac{\sqrt{2}}{2}).$$

31. $x = e^t$, $y = e^{-t}$; $t = 0$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -e^{-t}$$
$$\Longrightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t}}{e^t} = -1$$
$$t = 0 \Longrightarrow x = 1, y = 1, y' = -1$$
$$\Longrightarrow \text{접선 EQ: } y - 1 = -1(x - 1)$$
$$\text{법선 EQ: } y - 1 = (x - 1).$$

32. $x = t^2 - 1$, $y = t \ln t$; $t = e$

$$\begin{aligned}
&\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = \ln t + 1 \\
\Longrightarrow\ &\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln t + 1}{2t} \\
&t = e \Longrightarrow x = e^2 - 1, y = 2, y' = e \\
&\Longrightarrow \text{접선 EQ: } y - 2 = e(x - (e^2 - 1)) \\
&\qquad \text{법선 EQ: } y - 2 = -\frac{1}{e}(x - (e^2 - 1)).
\end{aligned}$$