Chapter 6. Counting methods 경우의 수: 순열과 조합

Example 6.1 (a) Lunch menu for a dinning hole consists of 2 appetizers, 3 main dishes, and 4 drinks. How many different lunches consist of 1 appetizer, 1 main dish, and 1 drinks?

(b) How many strings of length 4 can be formed using the letters ABCDE,

① if repetitions are not allowed.

② if repetitions are allowed.

③ if the string starts from B.

④ if the string does not start from B.

(c) How many ways can we select 2 books from different subjects among 5 distinct CS books, 3 distinct MATH books, and 2 distinct EE books?

(d) How many ways can we select a chairman, secretary, and treasure from a 6-person committee composed of A, B, C, D, E, and F.

(e) How many ways can we select a committee of 3 from a group of 10 distinct persons?

(f) How many strings can be formed using letters, MISSISSIPPI?

6.1 Basic principles

(1) Multiplication principle, 곱의 법칙

두 사건 A, B에서 $n(A) = m$, $n(B) = k$이면, A, B가 동시에 일어날 경우의 수는 $m \cdot k$이다.

If an activity consists of a sequence of (*independent,* 독립적인) choices in which there are n_1-ways in the 1^{st}-choice, n_2-ways in the 2^{nd}- choice, ..., and n_t-ways in choice -t, then the number of different activities is

(6.1) $\quad n_1 n_2 \cdots n_t$

Example 6.2 (a) How many 7-character license plates are possible if the first 3 characters must be letters, the last four must be digits $0 \sim 9$, and repeated characters are allowed?

(b) 문자 ABCDE 를 중복 없이 사용하여 4개의 문자로 구성된 string 을 생성할 때 몇 가지 string 을 생성할 수 있는가?

(c) 두 개의 주사위 A, B를 동시에 던졌을 때, 두 수의 합이 홀수가 나오는 경우의 수.

(d) n명을 3개의 (구별 가능한) 방에 채우는 방법의 수는 얼마인가?

각자가 독립적으로 방을 선택한다고 할 때, 첫 번째는 n개의 방 중에서 하나를 선택할 수 있고, 두 번째 역시 n개의 방 중에서 하나를 선택할 수 있다. 따라서, $3 \cdot 3 \cdot \cdots \cdot 3 = 3^n$의 방법이 있다.

예를 들어, 두 명, $\{a, b\}$인 경우를 고려해 보자. 각 방을 채우는 인원을 순서쌍으로 표현할 때, 방을 배정하는 방법을 나열해 보면,

$$\left\{\left(0,0,(a,b)\right)_1, (0,(a,b),0)_2, \left((a,b),0,0\right)_3, (0,a,b)_4, (0,b,a)_5, (a,0,b)_6, (b,a,0)_7, (a,b,0)_8, (b,a,0)_9\right\}$$

이며, 모두 9가지 방법이 있음을 알 수 있다.

(e) 집합 X가 n개의 원소를 갖는다고 할 때, $A \subseteq B \subseteq X$를 만족하는 부분집합의 순서쌍 (A, B)의 종류는 몇 가지인가?

주어진 순서쌍 (A, B)가 $A \subseteq B \subseteq X$를 만족한다면, X의 각 원소는 A, $B - A$, $X - B$ 중 하나의 집합에만 속한다. 따라서, 순서쌍 (A, B)의 종류의 수는 세 집합에 X의 원소를 할당하는 방법의 수와 같다. 따라서, 위의 예제에서와 같이 총 3^n의 종류가 있다.

먼저 $|X| = 2$인 경우를 고려해 보자. $|A| = n_1$, $|B - A| = n_2$, $|X - B| = n_3$라 할 때, $n_1 + n_2 + n_3 = 2$이며, 2개의 성분을 배열하는 방법은, $(n_1, n_2, n_3) = \{(0,0,2), (0,2,0), (2,0,0), \dots\}$이며, $n_1 = 0$인 부분 집합은 공집합을 의미한다.

(2) Addition principle, 합의 법칙

상호 배타적인 두 사건(mutually exclusive events: *i.e.*, there are no common outcomes) A, B에서 (즉, $A \cap B = \emptyset$) $n(A) = m$, $n(B) = k$이면, A 혹은 B가 일어날 경우의 수는 $m + k$이다:

(6.2) $\quad n(A \cup B) = n(A) + n(B)$

Example 6.3 (a) Given a set $S = \{-4, -2, 1, 3, 5, 6, 7, 8, 9, 10\}$, how many ways can we choose a negative number or an odd number from S?

(b) In how many ways can a number be chosen from 1 to 22 such that it is a multiple of 3 or 8?

(c) 5종류의 컴퓨터 서적, 3종류의 수학 서적, 그리고 2 종류의 예술 서적에서, 서로 다른 주제의 책 2권을 선택하는 방법은 몇 가지인가?

- 종류별 선택방법: {컴퓨터, 수학}, {컴퓨터, 예술}, {수학, 예술}

(d) A, B, ..., F 까지 6 인으로 구성된 위원회에서 의장, 간사, 총무를 선택하고자 한다.

① 선출 방법의 총 수

② A 나 B 가 반드시 의장이 되어야 하는 경우, 선출 방법의 총 수

{A 가 의장인 경우} or {B 가 의장인 경우}

③ E 가 반드시 하나의 직책을 맡아야 하는 경우, 선출 방법의 총 수

{E 가 의장인 경우} or {E 가 간사인 경우} or {E 가 총무인 경우}

④ D 와 F 가 모두 하나의 직책을 맡아야 하는 경우, 선출 방법의 총 수

(3) Inclusion-exclusion principle

Consider 2-finite sets X and Y which are disjoint: *i.e.* $X \cap Y \neq \emptyset$. Then,

(6.3) $\quad |X \cup Y| = |X| + |Y| - |X \cap Y|$

Example 6.4 A, B, ..., F 까지 6 인으로 구성된 위원회에서 의장, 간사, 총무를 선택하고자 한다. A 나 B, 혹은 둘 다 선출되는 방법의 총 수.

- X: A 가 선출되는 경우의 집합, Y: B 가 선출되는 경우의 집합

- $|X \cup Y|$

6.2 Permutations and combinations 순열과 조합

Definition A *permutation* (순열) of n distinct elements x_1, ..., x_n is an ordering of these n elements.
서로 다른 원소들을 순서를 고려하여 배열하는 것을 순열이라 한다.

Theorem 6.1 There are $n!$ permutations of n elements.

Definition An r-*permutation* of n distinct elements x_1, ..., x_n is an ordering of an r element subset of $\{x_1, x_2, \ldots, x_n\}$. The number of r-permutations of a set of n distinct elements is denoted by nPr or $P(n, r)$. (서로 다른 n개의 원소들 중에서 r개를 선택하는 순열의 수)

Theorem 6.2 The number of r-permutations of a set of n distinct elements is

(6.4) $$nPr = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

같은 것을 포함하는 순열의 경우에는, 즉 n개 중에서 같은 것이 각각 p개, q개, ..., r개씩 있을 경우, n개를 배열하여 만들 수 있는 순열의 수는

$$\frac{n!}{p!q!\cdots r!},\ p+q+\cdots+r=n$$

이다.

Definition Given a set $X = \{x_1, x_2, \cdots, x_n\}$ containing n distinct elements,

(a) An r-combination (조합) of X is an unordered selection of r-elements of X (*i.e.* r-element subset of X).

(b) The number of r-combinations of a set of n distinct elements is denoted by nCr, $\binom{n}{r}$, or $C(n, r)$.

Theorem 6.3 The number of r-combinations of a set of n distinct elements is

(6.5) $$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

Example 6.5 (a) 문자 A~F 의 순열 중, 부분 문자열 DEF 를 포함하는 경우의 수는?

(b) DEF 세 문자가 연속해 있지만 그 순서에는 제한이 없는 문자 A~F 의 순열은 몇 가지인가?

(c) A~F, 6명이 원탁에 둘러앉을 수 있는 방법은 몇 가지인가? 단, 옆 자리에 앉은 사람이 동일하게 배치되는 방법을 하나의 방법으로 가정한다.

(d) 10인으로 구성된 위원회에서 의장, 간사, 총무를 선택하는 방법은 몇 가지인가?

(e) 7명의 서로 다른 화성인과 5명의 서로 다른 목성인이 줄을 설 때, 목성인끼리는 서로 인접하지 않도록 줄을 서는 방법은 몇 가지인가?

(f) A~E, 5명으로 구성된 학과 학생회에서 학과장을 면담하고자 한다. 3명만이 면담에 참여할 수 있을 때, 면담에 참여할 3명을 구성하는 방법은 몇 가지인가?

(g) 10인으로 구성된 위원회에서 3명의 간부를 선택하는 방법은 몇 가지인가?

(h) 5명의 여성과 6명의 남성으로 구성된 집단에서 2명의 여성과 3명의 남성으로 구성된 위원회를 구성하는 방법은 몇 가지인가?

(i) 8-bit 이진수가 정확하게 4개의 1을 갖는 경우는 몇 가지인가?

(j) 주머니에 크기가 서로 다른 3개의 빨간 공과 4개의 흰 공이 들어 있을 때,

① 주머니에서 3개의 공을 뽑는 경우의 수

② A 나 B 가 빨간 공 2개와 흰 공 3개를 뽑는 경우의 수

6.3 Generalized permutations and combinations

Permutation with repetition

Theorem 6.4 (Permutation with repeated elements) Suppose that a sequence S of n-items has n_1 identical objects of type-1, n_2 identical objects of type-2, ..., n_t identical objects of type-t. Then, the number of ordering of S is

(6.6) $$P(n; n_1, n_2, \dots, n_t) = \frac{n!}{n_1! n_2! \cdots n_t!}, \text{ with } n_1 + n_2 + \cdots + n_t = n$$

Example 6.6 With 3 a's and 2 b's, we can write 5 letter words, $\{aaabb, aabab, abaab, baaab, aabba, ababa, abbaa, babaa, bbaaa\}$. If we can distinguish the different copies of a letter with subscripts (*e.g.*, $a_1a_2a_3b_1b_2$), then there would be $5!$-permutations. That is, we can generate each permutation of these 5-elements by choosing

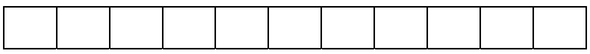
(1) position of each kind of letter, ($P(5;3,2)$ ways)

(2) subscripts to place on the 3 a's, ($3!$ ways)

(3) subscripts to place on the b's ($2!$ ways).

Total $P(5;3,2) \times 3! \times 2! = 5!$

Example 6.7 1개의 M, 4개의 I, 4개의 S, 그리고 2개의 P로 구성할 수 있는 string 의 총 수는? 이 문제는 11개의 방에 4가지 종류의 글자를 배치하는 것과 동일하다.



(1) $C(11,2)$-ways for choosing rooms for P,

(2) $C(9,4)$-ways for choosing rooms for S,

(3) $C(5,4)$-ways for choosing rooms for I, and remaining room for M.

Total $P(11; 2, 4, 4, 1) = C(11, 2)C(9,4)C(5,4) = \frac{11!}{2!4!4!1!}$

Combination with repetition

Suppose that X is a set containing t-objects. Any r-selections from X, where each object can be chosen more than once, is called a "combination of t-objects taken r at a time with repetition".

Example 6.8 The combination of letters a, b, c, d taken 3 at a time with repetition are: $aaa, aab, aac, aad, abb, abc, abd, acc, acd, add, bbb, bbc, bbd, bcc, bcd, bdd, ccc, ccd, cdd, ddd$.

Two combinations with repetition are considered identical, if they have the same elements repeated the same number of times, regardless of their order (*e.g.*, $aac = caa$). Note that the followings are equivalent:

(1) The number of combinations of t-objects taken r at a time with repetition.

(2) The number of ways r-identical objects that can be distributed among t distinct rooms.

(3) The number of non-negative integer solutions of the equation:

$x_1 + x_2 + \ldots + x_t = r$

Example 6.9 Assume that we have 3 different (empty) milk containers and 7 liter of milk and that we can measure with a one-liter measuring cup. In how many ways we can distribute the milk among the three containers?

Let $\{x_1, x_2, x_3\}$ be the volume (in liter) of milk to put in containers, 1, 2, and 3, respectively.

(1) $x_1 + x_2 + x_3 = 7$

(2) Let's represent the solution $x_1 = 2,\ x_2 = 1,\ x_3 = 4$, or $2 + 1 + 4$, as $|| + | + ||||$.

(3) Each possible solution is an arrangement of 7 strokes and 2 plus signs, so the number of arrangements is $P(9; 7,2) = 9!/7!\,2!$ (Here, 9 for strokes plus plus-sign, 7 for strokes and 2 for plus-signs).

Example 6.10 There are 3-types of books: CS, Physics, and History. How many ways are there to choose 6-books with repetition?

Let $\{x_1, x_2, x_3\}$ be the number of CS, Physics, and History books, respectively.

(1) $x_1 + x_2 + x_3 = 6$

(2) If we choose $x_1 = 2,\ x_2 = 1,\ x_3 = 3$, then it can be represented as $|| + | + |||$.

(3) The number of choices is $P(8; 6,2) = 8!/6!\,2!$.

Theorem 6.5 The combination of t-objects taken r at a time with repetition is

(6.7) $$\binom{r + t - 1}{t - 1} = \binom{r + t - 1}{r}$$

Example 6.10 (cont'd) Given $r = 6$, and $t = 3$, we get $C(6 + 3 - 1, 3 - 1) = C(8,2)$.

Example 6.11 We have 3-types of fruits: apple, orange, and pears. How many ways can we select 4 fruits (with repetition allowed)?

Possible choices are:

4 with 1-type only: $4a, 4o, 4p$ ($C(3,1)$-ways)

3 with 1-type and 1 with other-type: $3a1o, 3a1p, 3o1a, 3o1p, 3p1a, 3p1o$ ($C(3,2) \cdot 2$-ways)

2 with 1-type and 2 with other-type: $2a2o, 2a2p, 2o2p$ ($C(3,2)$-ways)

2 with 1-type, 2 with different types: $2a1o1p, 2o1a1p, 2p1a1p$ ($C(3,2)$-ways)

Total $C(3,1) + C(3,1) \cdot 2 + C(3,2) + C(3,2) = C(4 + 3 - 1, 3 - 1) = 15$

6.4 Algorithms for generating permutations and combinations

Problem statement

We have n audio files, each t_1, t_2, ..., t_n seconds. We want to choose maximum number of files with constraints

(6.8) $\sum_{j=1}^{k} t_k \leq C$

Equivalently, we want to choose a subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$ such that the condition (6.8) holds and also maximizes k.

A direct and brute-force method is to examine all possible subsets and choose the subset which has maximum elements. This approach would require $O(2^n)$ operations. Also, we need an algorithm which will generate all possible combinations (corresponding to subsets) of the n-element set.

Example 6.12 (a) Salesman problem visiting n-cities with minimum driving time.

(b) A set of audio files with running times, $\{7, 15, 24, 32, 9, 17, 27, 18\}$. An audio CD can accommodate audio files up to 60-minutes.

(c) A set X consists of 6 numbers. We want find a collection of these numbers which sum up to 100.

(d) A research lab has 60 engineers and each engineer has more than 1 specialties. We want to set a group of 12 engineers with 25 combinations of specialties.

In order to optimize the search for the optimal solution, the algorithm should list permutations and combinations in *lexicographic order* (사전적 순서): *i.e.* generalized ordinary dictionary order. Given two distinct words, to determine whether one precedes the other in the dictionary, we compare letters in words. There are two possibilities:

(a) The words have different lengths, and each letter in the shorter word is identical to the corresponding letter in the longer word.

(b) The words have the same or different lengths, and at some position, the letters in the words differ.

If (a) holds, the shorter precedes the longer: *e.g.* “dog” precedes “doghouse.” If (b) holds, we locate the leftmost position p at which the letters differ. The order of the words is determined by the order of the letters at position p: *e.g.* “gladi**a**tor” precedes “gladi**o**lus.”

Definition Let $\alpha = s_1 s_2 \cdots s_p$ and $\beta = t_1 t_2 \cdots t_q$ be strings over $S = \{1, 2, \dots, n\}$. We say that α is lexicographically less than β and write $\alpha < \beta$ (*i.e.*, 앞선다), if either

(a) $p < q$ and $s_i = t_i$, for $i = 1, \dots, p$ or

(b) for some i, $s_i \neq t_i$, and for the smallest such i, we have $s_i < t_i$.

Example 6.13 *lexicographic order*, (a) $\alpha = 132$, $\beta = 1324$, and $S = \{1, 2, 3, 4\}$: $\alpha < \beta$

(b) $\alpha = 13246$, $\beta = 1342$, and $S = \{1, 2, 3, 4, 5, 6\}$: $\alpha < \beta$

(c) $\alpha = 1324$, $\beta = 1342$, and $S = \{1, 2, 3, 4\}$: $\alpha < \beta$

(d) $\alpha = 13542$, $\beta = 21354$, and $S = \{1, 2, 3, 4, 5\}$: $\alpha < \beta$

Consider the problem of listing all r-combinations of $S = \{1, 2, \ldots, n\}$, $r < n$. We want to list the r-combination $\{x_1, \ldots, x_r\}$ as the string s_1, s_2, ..., s_r, where $s_1 < s_2 < \cdots < s_r$ and $\{x_1, x_2, \ldots, x_r\} = \{s_1, s_2, \ldots, s_r\}$. For example, 3-combination of $\{6, 2, 4\}$ will be listed as 246. Likewise, if we list the r-combinations of $\{1, 2, \ldots, n\}$ in lexicographic order, the first string is $12 \cdots r$, whereas the last string is $(n - r + 1) \cdots n$.

Example 6.14 Given $S = \{1, 2, \ldots, 7\}$

(a) All 5-combination of S: $C(7,5)$ different combinations, $\{12345, 12346, \ldots, 34567\}$

(b) String that follows 13467 in 5-combination of S: $\{13567, 14567, 23456, \ldots\}$

(c) String that follows 2367 in 4-combination of S: $\{2456, 2457, 2467, \ldots\}$

We know the first r-combination in the list. Given a string $\alpha = s_1 s_2 \cdots s_r$, which represents the r-combination $\{s_1, s_2, \ldots, s_r\}$, to find the next string $\beta = t_1 t_2 \cdots t_r$, we find the rightmost element s_m that is not at its maximum value. Then,

$$t_i = s_i, \text{ for } i = 1, \ldots, m - 1.$$

The element t_m is equal to $s_m + 1$. For the remainder of the string β we have

$$t_{m+1} \cdots t_r = (s_m + 2)(s_m + 3) \cdots$$

Algorithm 6.1: List all r-combinations of $\{1, 2, \ldots, n\}$

```
input: r, n
output: List of all r-combinations in increasing lexicographic order
combination(r, n) {
    for i = 1 to r
        s_i = i
    println(s_1, ..., s_r)          // First string
    for i = 2 to C(n, r) {
        m = r
        y = n                       // maximum value
        while (s_m == y) {          // find the rightmost element not at its maximum value
            m = m - 1
            y = y - 1
        }
        s_m = s_m + 1
        for j = m + 1 to r
            s_j = s_{j-1} + 1
        println(s_1, ..., s_r)
    }
}
```

6.7 Binomial coefficients

Theorem 6.6 If a and b are real numbers and n is a positive integer, then

(6.9) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Theorem 6.7 If n is a positive integer, then

(6.10) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, for $1 \leq k \leq n$

***Corollaries to Theorem* 6.7.6**

(6.11) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(6.12) $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$

6.8 Pigeonhole principle

If n pigeonholes are occupied by m pigeons and $m > n$, then at least one pigeonhole is occupied by more than one pigeon. The pigeonhole principle is used for proving that a certain situation must actually occur.

Example 6.15 (a) In any given set of 13 people at least two of them have their birthday during the same month.

(b) Let S be a set of eleven 2-digit numbers. Claim that S must have two elements whose digits have the same difference.

If $S = \{10, 14, 19, 22, 26, 28, 49, 53, 70, 90, 93\}$, the digits of the numbers 28 and 93 have the same difference: $8 - 2 = 6, 9 - 3 = 6$.

The digits of a two-digit number can have 10 possible differences (from 0 to 9). So, in a list of 11 numbers, there must be two with the same difference.

(c) Assume that we choose three different digits from 1 to 9 and write all permutations of those digits. Claim that, among the 3-digit numbers written that way, there are two whose difference is a multiple of 500.

There are $9 \cdot 8 \cdot 7 = 504$ permutations of three digits. On the other hand, if we divide the 504 numbers by 500 we can get only 500 possible remainders, so at least two numbers give the same remainder, and their difference must be a multiple of 500.