Chapter 4. System of ODEs. Phase plane

System of Differential Equations

- ODE with multiple dependent variables and equations 연립상미분방정식
  - $y_1' = a_{11}y_1 + a_{12}y_2$, $y_2' = a_{21}y_1 + a_{22}y_2$:
    - $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
    - $\boldsymbol{y}' = A\boldsymbol{y}$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$
      - Derivative of a matrix or vector with variable entries is obtained by term-wise differentiation.
  - System of n-equations;
    - $\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$, $\boldsymbol{y}' = A\boldsymbol{y}$, where $A \in (n \times m)$ and $\boldsymbol{y} \in (n \times 1)$

Eigenvalue and eigenvector

- Let A be an $(n \times n)$ square matrix. A scalar λ is called an eigenvalue of the matrix A, if there is a non-zero vector $\boldsymbol{x} \in (n \times 1)$, such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Such a vector $\boldsymbol{x}$ is called the eigenvector corresponding to λ.
  - λ is an eigenvalue of the square matrix $A \in (n \times n)$, if and only if $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has trivial solution (eigenvector $\boldsymbol{x}$ cannot be a zero vector)
  - Equivalently, $A - \lambda I$ is singular or $\mathrm{p}(\lambda) = \det(A - \lambda I) = 0$ (characteristic equation).
  - When all entries of the matrix A are real numbers, then the characteristic polynomial is a polynomial of λ of degree-n and has at most n-distinct roots (eigenvalues).
    - When $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $p(\lambda) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0$.
    - For $\lambda_1 = 3$, $A\boldsymbol{x}^{(1)} = \lambda_1 \boldsymbol{x}^{(1)}$ or $(A - \lambda_1 I)\boldsymbol{x}^{(1)} = 0$. Choose $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

System of ODEs: Engineering model 1

- Example 4.1.1: Mixing problem involving two tanks
  - Tank T_1 and T_2 contain initially 100 [gal] of water each.
  - In T_1, the water is pure, whereas 150 [lb] of fertilizer are dissolved in T_2.
  - By circulating liquid at a rate of 2 [gal/min] and stirring, the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?

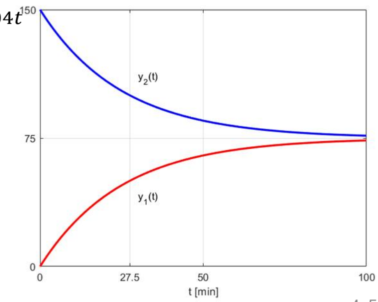
  1. Setting up model: $y_1' = \frac{2}{100}y_2 - \frac{2}{100}y_1$, $y_2' = \frac{2}{100}y_1 - \frac{2}{100}y_2$: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -.02 & .02 \\ .02 & -.02 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
  2. General solution: Guess $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ is a solution.
     - $\boldsymbol{y}' = \lambda \boldsymbol{x} e^{\lambda t} = A\boldsymbol{x}e^{\lambda t} \Rightarrow A\boldsymbol{x} = \lambda \boldsymbol{x}$ (eigenvalue/eigenvector problem)

System of ODEs: Engineering model 1

- Characteristic equation: $\det(A-\lambda I)\boldsymbol{x}=\mathbf{0}$, $p(\lambda)=\lambda(\lambda+.04)=0$
- When $\lambda_1=0$, $(A-\lambda_1 I)\boldsymbol{x}^{(1)}=\mathbf{0} \Rightarrow Ax_1=0$:
  - $\begin{bmatrix}-.02 & .02\\ .02 & -.02\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Choose $x_1=x_2=1. \Rightarrow \boldsymbol{x}^{(1)}=\begin{bmatrix}1\\ 1\end{bmatrix}$.
- When $\lambda_2=0$, $(A-\lambda_2 I)\boldsymbol{x}^{(2)}=\mathbf{0}$
  - $\begin{bmatrix}.02 & .02\\ .02 & .02\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Choose $x_1=1$ and $x_2=-1. \Rightarrow \boldsymbol{x}^{(2)}=\begin{bmatrix}1\\ -1\end{bmatrix}$.
- $\boldsymbol{y}(t)=c_1\boldsymbol{x}^{(1)}e^{\lambda_1 t}+c_2\boldsymbol{x}^{(2)}e^{\lambda_2 t}=c_1\begin{bmatrix}1\\ 1\end{bmatrix}+c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-0.04t}$

3. Particular solution: Initial condition, $y(0)=\begin{bmatrix}0\\ 150\end{bmatrix}$

- $c_1\begin{bmatrix}1\\ 1\end{bmatrix}+c_2\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}0\\ 150\end{bmatrix} \Rightarrow c_1=75$ and $c_2=-75$
- $y_1(t)=75-75e^{-0.04t}$, $y_2(t)=75+75e^{-0.04t}$



System of ODEs: Engineering model 1

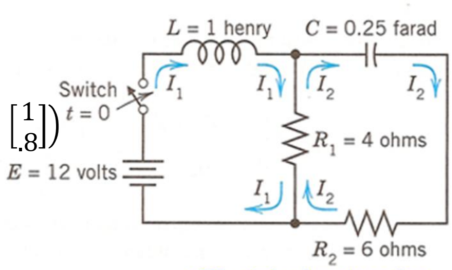
4. Diagonalization: Put $\boldsymbol{z}=P\boldsymbol{y}$, where $P=[\boldsymbol{x}^{(1)} \quad \boldsymbol{x}^{(2)}]$ (eigenvector matrix)

- $\boldsymbol{z}'=P\boldsymbol{y}'=PA\boldsymbol{y}=PAP^{-1}\boldsymbol{z}$, if P is invertible with $P^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.
- Matrix $D=PAP^{-1}$ is a diagonal matrix, $D=\begin{bmatrix}0 & 0\\ 0 & -.04\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$
- $\begin{bmatrix}z_1'\\ z_2'\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & -.04\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \Rightarrow z_1'=0$ and $z_2'=-0.04z_2$
  - $z_1(t)=c_1$ (const) and $z_2(t)=c_2e^{-.04t}$
- $\boldsymbol{y}=P^{-1}\boldsymbol{z}$
  - $y_1(t)=\frac{1}{2}(z_1(t)+z_2(t))=\frac{c_1}{2}+\frac{c_2}{2}e^{-.04t}$ and
  - $y_1(t)=\frac{1}{2}(z_1(t)-z_2(t))=\frac{c_1}{2}-\frac{c_2}{2}e^{-.04t}$

System of ODEs: Engineering model 2

- Example 4.1.2: Electric circuit
  - Circuit had been in rest before switching: all currents and charges are zero at $t = 0$

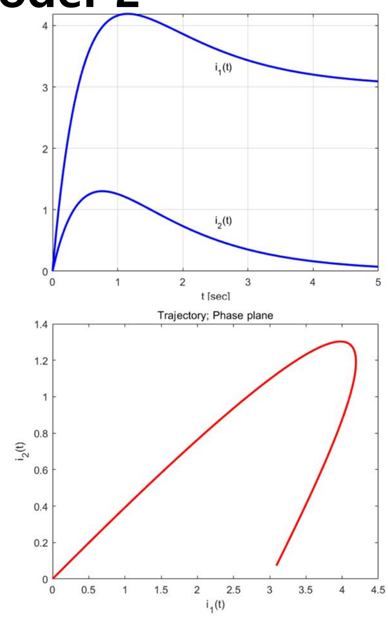
  1. Setting up model:
     - $i_1' = -4i_1 + 4i_2 + 12$, $i_2' = -1.6i_1 + 1.2i_2 + 4.8$
     - $\begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1.6 & 1.2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix}$, $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{g}$ … Non-homogeneous equation
  2. Homogeneous solution: $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$
     - $\boldsymbol{y}' = \lambda \boldsymbol{x} e^{\lambda t} = A\boldsymbol{x}e^{\lambda t} \Rightarrow A\boldsymbol{x} = \lambda \boldsymbol{x}$
     - Eigenpairs, $\left(\lambda_1 = -2, \boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$, $\left(\lambda_2 = -.8, \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ .8 \end{bmatrix}\right)$
     - $\boldsymbol{y}_h(t) = c_1 \boldsymbol{x}^{(1)} e^{-2t} + c_2 \boldsymbol{x}^{(2)} e^{-.8t}$



< Electrical network >

System of ODEs: Engineering model 2

3. Particular solution
   - Choose $\boldsymbol{y}_p = \boldsymbol{a}$ (const)
     - $\boldsymbol{y}_p' = \boldsymbol{0}$ and thus $A\boldsymbol{a} + \boldsymbol{g} = \boldsymbol{0} \Rightarrow \boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
4. General solution
   - $\boldsymbol{y} = \boldsymbol{y}_h + \boldsymbol{y}_p = c_1 \boldsymbol{x}^{(1)} e^{-2t} + c_2 \boldsymbol{x}^{(2)} e^{-.8t} + \boldsymbol{a}$
   - $i_1(t) = 2c_1 e^{-2t} + c_2 e^{-0.8t} + 3$, $i_2(t) = c_1 e^{-2t} + 0.8c_2 e^{-0.8t}$
5. Particular solution: Initial conditions, $i_1(0) = i_2(0) = 0$
   - $2c_1 + c_2 = -3$, $c_1 + 0.8c_2 = 0 \Rightarrow c_1 = -4$ and $c_2 = 5$



Conversion to System of ODEs

- Given an n^{th}-order ODE, $y^{(n)} = F\left(t, y, y', \cdots, y^{(n-1)}\right)$
  - Set $y_1 = y$, $y_2 = y'$, $y_3 = y''$, … , $y_n = y^{(n-1)}$ to get a system of ODEs
  - $\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ F(\cdot) \end{bmatrix}$
- Example 4.1.3: Mass-spring system, $y'' = -\frac{c}{m}y' - \frac{k}{m}y$
  - Given $y_1 = y$ and $y_1' = y_2$, $y_2' = y_1'' = -\frac{k}{m}y_1 - \frac{c}{m}y_2$
  - $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\boldsymbol{y}$, with characteristic equation, $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

Wronskian

- System of ODEs

(1) $y_1' = f_1(t, y_1, \cdots, y_n)$, $y_2' = f_2(t, y_1, \cdots, y_n)$, …, $y_n' = f_n(t, y_1, \cdots, y_n)$

with initial conditions, $y_1(t_0) = K_1$, $y_2(t_0) = K_2$, …, $y_n(t_0) = K_n$

- Theorem 1. Existence & uniqueness

  Let f_1, f_2, …, f_n be continuous functions having continuous partial derivatives $\frac{\partial f_1}{\partial y_1}, \cdots, \frac{\partial f_1}{\partial y_n}, \frac{\partial f_2}{\partial y_1}, \cdots, \frac{\partial f_n}{\partial y_n}$ in some domain R. Then the first-order system, $y' = f(t, y)$ in (1) has a solution on some interval, satisfying the initial condition , and this solution is unique.

Linear Systems

(3) $y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + g_1(t)$,

$y_2' = a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + g_2(t)$, …

$y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + g_n(t)$

- Homogeneous

  $\boldsymbol{y}' = A\boldsymbol{y}$, where $\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$,

- Non-homogeneous

  $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{g}$

System of ODEs: Basic Theory

- Theorem 2. Existence and uniqueness in linear system of ODEs
  - Let a_{jk}'s and g_j's be continuous functions of t on an open interval, $a < t < b$ containing the point $t = t_0$. Then the linear system has a solution $\boldsymbol{y}(t)$ on this interval satisfying initial conditions and this solution is unique.

- Theorem 3. Superposition principle
  - If $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ be two solutions of the homogeneous linear system on some interval, so is any linear combination, $\boldsymbol{y} = c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)}$.

System of ODEs: Basic Theory

- Basis and general solution
  - Basis or fundamental system of solutions of the homogeneous system on some interval J is the set of linearly independent solutions, $\boldsymbol{y}^{(1)}$, $\boldsymbol{y}^{(2)}$, … , $\boldsymbol{y}^{(n)}$ of the homogeneous system on the interval.
  - General solution of the homogeneous system is a linear combination of basis:

$$\boldsymbol{y} = c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)} + \cdots + c_n\boldsymbol{y}^{(n)}$$

- Wronskian of basis

$$W\left(\boldsymbol{y}^{(1)},\cdots,\boldsymbol{y}^{(n)}\right) = \det(Y),\ \text{where } Y = \begin{bmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{bmatrix}$$ … Fundamental matrix

System of ODEs with constant coefficients

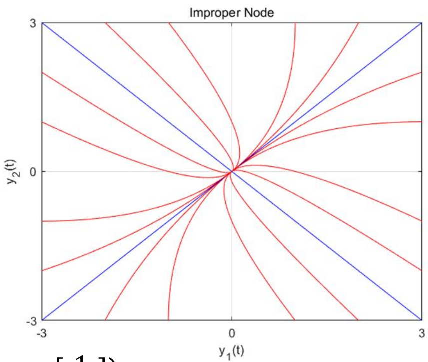
- Homogeneous linear system with constant coefficients
  - $\boldsymbol{y}' = A\boldsymbol{y}$, where $A \in (n \times n)$ with constant elements
  - As usual, we claim that $\boldsymbol{y} = \boldsymbol{x}e^{\lambda t}$ is a solution:
  - $\boldsymbol{y}' = \lambda\boldsymbol{x}e^{\lambda t} = A\boldsymbol{x}e^{\lambda t} \Rightarrow A\boldsymbol{x} = \lambda\boldsymbol{x}$ (Eigenvalue problem)
- Theorem 1. General solution
  - If the constant matrix A in the homogeneous linear system has a linearly independent set of n eigenvectors, then the corresponding solutions, $\boldsymbol{y}^{(1)}$, $\boldsymbol{y}^{(2)}$, …, $\boldsymbol{y}^{(n)}$ form a basis of solutions, and the corresponding general solution is given by

$$\boldsymbol{y} = c_1\boldsymbol{y}^{(1)} + c_2\boldsymbol{y}^{(2)} + \cdots + c_n\boldsymbol{y}^{(n)}$$

Phase plane method

- Homogeneous linear system with constant coefficients (autonomous system)
  - ∨ Solution of $\boldsymbol{y}' = A\boldsymbol{y}$, where $A \in (2 \times 2)$, is given by $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.
  - ∨ We can graph solutions as two curves on the t-axis, one for each components of $\boldsymbol{y}(t)$.
  - ∨ We also can graph as a single curve in the $y_1 y_2$-plane.
    - † Parametric representation with parameter t
  - ∨ Trajectory (or path) is a single curve in the $y_1 y_2$-plane (phase plane)
  - ∨ Phase plane provides a qualitative method in the sense that we can obtain general qualitative information on solutions without actually solving the ODE.

Phase plane method: Example



- Example 4.3.1 Trajectories in the phase plane
  - ∨ $\boldsymbol{y}' = A\boldsymbol{y}$, with $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$
    - † Characteristic equation: $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8$
    - † Eigenpairs: $\left(\lambda_1 = -2, \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(\lambda_2 = -3, \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$
    - † General solution: $\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$

Two real eigenvalues with the same sign

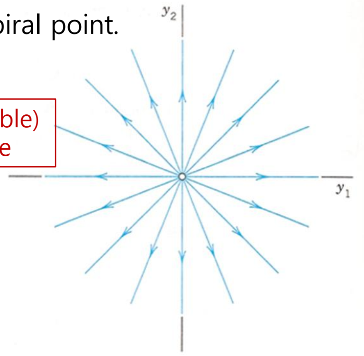
- Critical points:
  - ∨ $\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$ … Point such that $\frac{dy_2}{dy_1} = \frac{0}{0}$ (undefined)

All trajectories, except for two, have the same limiting direction of the tangent.

Phase plane method: Example

- Types of critical points
  - Improper node, proper node, saddle point, center, and spiral point.
- Example 4.3.2 Proper node
  - $\boldsymbol{y}' = A\boldsymbol{y}$, with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
    - † $p(\lambda) = \lambda^2 - 2\lambda + 1$
    - † Eigenpairs: $\left(\lambda_1 = 1, \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
    - † General solution: $\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^t$
      - $c_1 y_1 = c_2 y_2$

One (double) eigenvalue



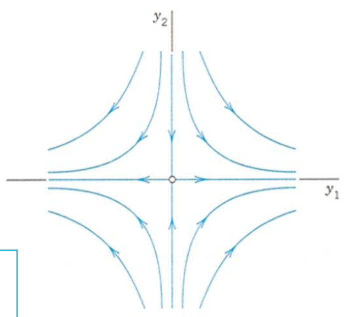
< Trajectories (Proper node) >

Every trajectory has a definite limiting direction and for any given direction $\boldsymbol{d}$ at P_0, there is a trajectory having $\boldsymbol{d}$ as its limiting direction.

Phase plane method: Example

- Example 4.3.3 Saddle point
  - $\boldsymbol{y}' = A\boldsymbol{y}$, with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
    - † $p(\lambda) = (\lambda + 1)(\lambda - 1)$
    - † Eigenpairs: $\left(\lambda_1 = 1, \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\left(\lambda_2 = -1,, \boldsymbol{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
    - † General solution: $\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$
      - $y_1(t) = c_1 e^t$ and $y_2(t) = c_2 e^{-t} \Rightarrow y_1 y_2 = const.$

Two real eigenvalues with different sign



There are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

< Trajectories (Saddle Point) >

Phase plane method: Example

- Example 4.3.4 Center

Pure imaginary conjugate eigenvalues

∨ $\boldsymbol{y}' = A\boldsymbol{y}$, with $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

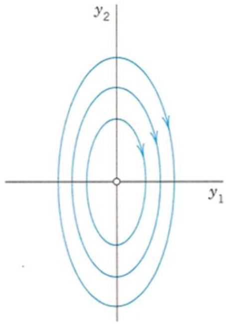
† $p(\lambda) = \lambda^2 + 4$

† Eigenpairs: $\left(\lambda_1 = j2, \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ j2 \end{bmatrix}\right)$, $\left(\lambda_2 = -j2, , \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -j2 \end{bmatrix}\right)$

† General solution: $\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ j2 \end{bmatrix} e^{j2t} + c_2 \begin{bmatrix} 0 \\ -j2 \end{bmatrix} e^{-j2t}$

– $y_1' = y_2$ and $y_2' = -4y_1 \Rightarrow 4y_1y_1' = -y_2y_2'$

– $2y_1^2 + \frac{1}{2}y_2^2 = const.$



< Trajectories (Center) >

Critical point P_0 is enclosed by infinitely many closed trajectories.

Phase plane method: Example

- Example 4.3.5 Spiral point

Complex conjugate eigenvalues

∨ $\boldsymbol{y}' = A\boldsymbol{y}$, with $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

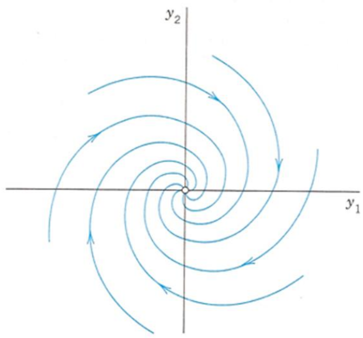
† $p(\lambda) = \lambda^2 + 2\lambda + 2$

† $\left(\lambda_1 = -1 + j, \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ j \end{bmatrix}\right)$, $\left(\lambda_2 = -1 - j, , \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ -j \end{bmatrix}\right)$

† General solution: $y = c_1 \begin{bmatrix} 1 \\ j \end{bmatrix} e^{(-1+j)t} + c_2 \begin{bmatrix} 0 \\ -j \end{bmatrix} e^{(-1-j)t}$

– $y_1' = -y_1 + y_2$ and $y_2' = -y_1 - y_2 \Rightarrow y_1y_1' + y_2y_2' = -(y_1^2 + y_2^2)$

– $\frac{1}{2}(r^2)' = -r^2$ $(r^2 = y_1^2 + y_2^2) \Rightarrow r \cdot r' = -r^2$ and $r = ce^{-t}$

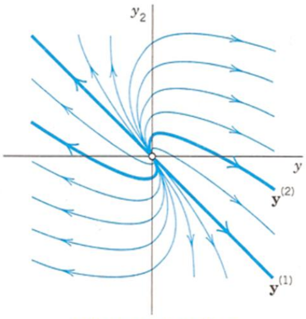


< Trajectories (Spiral point) >

Trajectories are spiral, approaching P_0 as $t \to \infty$.

Phase plane method: Example

- Example 4.3.6 Degenerate node
  - $\mathbf{y}' = A\mathbf{y}$, with $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$
    - † $p(\lambda) = \lambda^2 - 6\lambda + 9$ (double root)
    - † Eigenpairs: $\left(\lambda_1 = 3, \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ (only one LID eigenvector)
    - † General solution: $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$



< Degenerate node >

Criteria for Critical Points: Stability

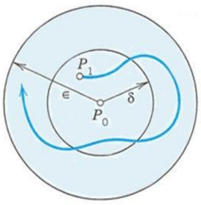
- Given the coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
  - $p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + |A| = \lambda^2 - p\lambda + q$
  - $p = a_{11} + a_{22} = \lambda_1 + \lambda_2$, $q = |A| = a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2$, $\Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$

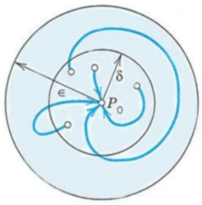
Name	p	q	Δ	Comments on λ_1, λ_2
Node		$q > 0$	$\Delta \geq 0$	Real, same sign
Saddle Point		$q < 0$		Real, opposite sign
Center	$p = 0$	$q > 0$		Pure imaginary
Spiral Point	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

Criteria for Critical Points: Stability

- Definition: Stable, unstable, and attractive
  - A critical point P_0 is called
    - † Stable, if all trajectories that are close to P_0 (within the δ-disc) at some instant remain close to P_0 (within the ϵ-disc) for all future time.
    - † Unstable, if it is not stable.
    - † Stable and attractive (asymptotically stable), if P_0 is stable and every trajectory approaches P_0 as $t \to \infty$.

Stable critical point





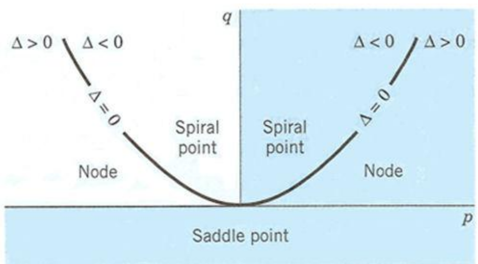
Asymptotically stable critical point

Criteria for Critical Points: Stability

- Stability criteria for critical points

Stability type	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$
Asymptotically stable	$p < 0$	$q > 0$
Stable	$p \le 0$	$q > 0$
Unstable	$p > 0$ or $q < 0$	

Stability chart, $\Delta = p^2 - 4q$



Non-linear Systems

- Qualitative method
  - Method of obtaining qualitative information on solutions without actually solving a system.
  - These method is particularly valuable for systems whose solution by analytic methods is difficult or impossible.
- Linearization
  - Non-linear system: $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$; $y_1' = f_1(y_1, y_2)$, $y_2' = f_2(y_1, y_2)$
  - Convert into a linear system: $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y})$;
    - † $y_1' = a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2)$, $y_2' = a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2)$

Non-linear Systems: Qualitative method

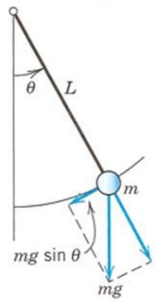
- Theorem 1. Linearization
  - If f_1 and f_2 are continuous and have continuous partial derivatives in a neighborhood of the critical point $(0,0)$, and if $\det(A) \neq 0$, then the kind and stability of the critical point of non-linear systems are the same as those of the linearized system

    $\boldsymbol{y}' = A\boldsymbol{y}$ or $y_1' = a_{11}y_1 + a_{12}y_2$, $y_2' = a_{21}y_1 + a_{22}y_2$

    - † Exceptions occur if A has equal or pure imaginary eigenvalues; then the nonlinear system may have the same kind of critical points as linearized system or a spiral point.

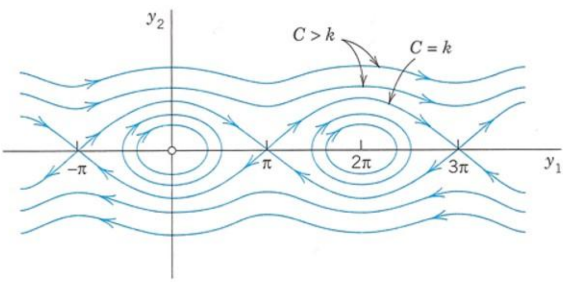
Non-linear Systems: Example

- Example 4.5.1 Free undamped pendulum
  - Model: $mL\theta'' + mg \cdot \sin\theta = 0 \Rightarrow \theta'' + k \cdot \sin\theta = 0$, where $k = g/L$.
    - † θ … angular displacement (CCW direction from the equil. position)
  - Critical points
    - † Set $y_1 = \theta$ to get $y_1' = y_2$, $y_2' = -k \cdot \sin y_1$
    - † $y_1' = y_2' = 0 \Rightarrow (0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, …
  - Linearization at $(0,0)$, $(\pm 2\pi, 0)$, …
    - † MacLaurin series, $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \cong x$, near $x = 0$
    - † $y_1' = y_2$, $y_2' = -ky_1 \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}$, $p = \mathrm{trace}(A) = 0$, $q = \det(A) = k$, and $\Delta < 0$
    - † $(0,0)$ is a center (always stable)



Non-linear Systems: Example

- Linearization at $(\pm\pi, 0)$, $(\pm 3\pi, 0)$, …
  - † Set $y_1 = \theta - \pi$ to get $y_1' = y_2$, $y_2' = -k \cdot \sin(y_1 + \pi) = k \cdot \sin y_1$
  - † MacLaurin series, $\sin\theta = \sin(y_1 + \pi) = -\sin y_1 \cong -y_1$, near $y_1 = \pm\pi$
  - † $y_1' = y_2$, $y_2' = ky_1 \Rightarrow A = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}$, $p = \mathrm{trace}(A) = 0$, $q = \det(A) = -k$, and $\Delta = p^2 - 4q = 4k > 0$
  - † $(\pm\pi, 0)$ are saddle points.



Non-homogeneous Linear Systems

- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{g},\ \boldsymbol{g} \neq \boldsymbol{0}$
  - General solution: $\boldsymbol{y} = \boldsymbol{y}^{(h)} + \boldsymbol{y}^{(p)}$
    - Homogeneous solution, $\boldsymbol{y}^{(h)}$ and particular solution, $\boldsymbol{y}^{(p)}$
- Example 4.6.1 Method of undetermined coefficients
  - $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ and $\boldsymbol{g} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$
  - $\boldsymbol{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$
  - Choose $\boldsymbol{y}^{(p)} = (\boldsymbol{u}t + \boldsymbol{v})e^{-2t}$
    - $-2\boldsymbol{u} = A\boldsymbol{u} \Rightarrow \boldsymbol{u} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ($a \neq 0$, const)
    - $\boldsymbol{u} - 2\boldsymbol{v} = A\boldsymbol{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \Rightarrow a = 2$ and $\boldsymbol{v} = \begin{bmatrix} k \\ k+4 \end{bmatrix}$. Choose $k = 0$