2장 5절 연습문제 풀이

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1. $\sin^{-1}(0) = 0$, $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$,

$$\begin{aligned} \sec^{-1}(-1) = X &\Longrightarrow \sec X = -1 \Longrightarrow \cos X = -1 \\ &\Longrightarrow X = \pi \Longrightarrow \sec^{-1}(-1) = \pi. \end{aligned}$$

2. $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$, $\cot^{-1}(1) = \frac{\pi}{4}$, $\csc^{-1}(-2) = -\frac{\pi}{6}$.

5. $a > 0$ 일때 $\sin^{-1}(-a) = -\sin^{-1} a$ 임을 증명하여라.

증명.

$$\begin{aligned} \sin^{-1}(-a) = A &\Longrightarrow \sin A = -a \\ &\Longrightarrow -\sin A = a \Longrightarrow \sin(-A) = a \\ &\Longrightarrow -A = \sin a \\ &\Longrightarrow A = -\sin^{-1}(a). \end{aligned}$$

7. $\cos^{-1}(\frac{3}{5}) + \cos^{-1}(\frac{4}{5})$ 를 간단히하여라.

풀이 $\cos^{-1}(\frac{3}{5}) = A$, $\cos^{-1}(\frac{4}{5}) = B$ 라하면, $\cos A = \frac{3}{5}$, $\cos B = \frac{4}{5}$ 이고, A 와 B 는 1,2 사분면에 있으므로 $\sin A$ 와 $\sin B$ 는 양수이다. 따라서

$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5},$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

이므로

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= \frac{3}{5}\frac{4}{5} - \frac{4}{5}\frac{3}{5} = 0\\ \Longrightarrow A+B &= \frac{\pi}{2}.\end{aligned}$$

9. $\tan^{-1}\frac{1}{2} = A$, $\tan^{-1}(-\frac{1}{3}) = B$ 라하면

$$\tan A = \frac{1}{2}, \quad \tan B = -\frac{1}{3}$$

이므로

$$\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B}\\ &= \frac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2}\frac{1}{3}}\\ &= \frac{6}{7}\\ \Longrightarrow A+B &= \tan^{-1}(\frac{1}{7}).\end{aligned}$$

11. $\sin^{-1}(\frac{12}{13}) = A$, $\sin^{-1}(-\frac{5}{13}) = B$ 라하면, $\sin A = \frac{12}{13}$, $\sin B = -\frac{5}{13}$. 여기서 A, B 는 $-\frac{\pi}{2} \le A, B \le \frac{\pi}{2}$ 이므로 $\cos A$ 와 $\cos B$ 는 양수이다.

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$
$$\cos B = \sqrt{1-\sin^2 B} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$$

이므로

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ &= \frac{12}{13}\frac{12}{13} - \frac{5}{13}\frac{5}{13}\\ &= \frac{119}{169}\\ \Longrightarrow A+B &= \sin^{-1}(\frac{119}{169}).\end{aligned}$$

13. $a > 0$ 이면 $\csc^{-1}(a) = \sin^{-1}(\frac{1}{a})$ 임을 증명하여라.

증명.

$$\begin{aligned}\csc^{-1} a = A &\Longrightarrow \csc A = a \\ &\Longrightarrow \sin A = \frac{1}{a} \\ &\Longrightarrow A = \sin^{-1}(\frac{1}{a}) \\ &\Longrightarrow \csc^{-1}(a) = \sin^{-1}(\frac{1}{a}).\end{aligned}$$

나머지 두개도 같은 방식으로... 그래서 생략.

19.

$$y = \tan^{-1} 3x \Longrightarrow y' = \frac{3}{1+9x^2}$$

20.

$$\begin{aligned}y = \sec^{-1} \frac{1}{4}x \Longrightarrow y' &= \frac{\frac{1}{4}}{\frac{1}{4}x\sqrt{(\frac{1}{4}x)^2 - 1}} \\ &= \frac{4}{x\sqrt{x^2-16}}.\end{aligned}$$

21.

$$y = \cos^{-1}(1-x) \Longrightarrow y' = -\frac{-1}{\sqrt{1-(1-x)^2}} = \frac{1}{\sqrt{2x-x^2}}.$$

22.

$$y = \sin^{-1}(\frac{2}{x}) \Longrightarrow y' = \frac{-\frac{2}{x^2}}{\sqrt{1-(\frac{2}{x})^2}} = \frac{-\frac{2}{x^2}}{\frac{\sqrt{x^2-4}}{x}} = -\frac{2}{x\sqrt{x^2-4}}.$$

23.

$$y = \csc^{-1} \sqrt{x} \Longrightarrow y' = -\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}\sqrt{x-1}} = -\frac{1}{2x\sqrt{x-1}}.$$

24.

$$y = \cot^{-1} \sqrt{x^2-2x} \Longrightarrow y' = -\frac{-\frac{2x-1}{2\sqrt{x^2-2x}}}{1+(x^2-2x)} = -\frac{2x-1}{2(x-1)^2\sqrt{x^2-2x}}.$$

25.

$$y = (\sin^{-1} 4x)^2 \Longrightarrow y' = 2\sin^{-1} 4x \cdot \frac{4}{\sqrt{1-16x^2}} = \frac{8\sin^{-1} 4x}{\sqrt{1-16x^2}}$$

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$$y = x\tan^{-1}\frac{1}{2}x \Longrightarrow y' = \tan^{-1}\frac{x}{2} + \frac{\frac{x}{2}}{1+(\frac{x}{2})^2} = \tan^{-1}\frac{x}{2} + \frac{2x}{4+x^2}$$

27.

$$y = \cos^{-1}(\sin x) \Longrightarrow y' = -\frac{\cos x}{\sqrt{1-\sin^2 x}} = -\frac{\cos x}{\cos x} = -1$$

28.

$$y = \tan^{-1}\sqrt{\frac{3x-4}{4}} \Longrightarrow y' = \frac{\frac{\frac{3}{4}}{2\sqrt{\frac{3x-4}{4}}}}{1+\frac{3x-4}{4}} = \frac{\frac{3}{4\sqrt{3x-4}}}{\frac{3x}{4}} = \frac{1}{x\sqrt{3x-4}}$$

29.

$$y = \cot^{-1}\sqrt{x^2-1} + \sec^{-1} x$$
$$\Longrightarrow y' = -\frac{\frac{x}{x^2-1}}{1+x^2-1} + \frac{1}{x\sqrt{x^2-1}} = -\frac{1}{x\sqrt{x^2-1}} + \frac{1}{x\sqrt{x^2-1}} = 0$$

30.

$$y = \sin^{-1}(2\sqrt{x-x^3}) \Longrightarrow y' = \frac{\frac{1-3x^2}{\sqrt{x-x^3}}}{\sqrt{1-4(x-x^3)}} = \frac{1-3x^2}{\sqrt{x-x^3}\sqrt{4x^3-4x+1}}$$

31.

$$y = \sin^{-1}\frac{x}{\sqrt{x^2+a^2}} \Longrightarrow$$
$$y' = \frac{(\frac{x}{\sqrt{x^2+a^2}})'}{\sqrt{1-\frac{x^2}{x^2+a^2}}} = \frac{\frac{\sqrt{x^2+a^2}-\frac{x^2}{\sqrt{x^2+a^2}}}{x^2+a^2}}{\sqrt{\frac{a^2}{x^2+a^2}}} = \frac{\frac{\frac{a^2}{\sqrt{x^2+a^2}}}{x^2+a^2}}{\sqrt{\frac{a^2}{x^2+a^2}}} = \frac{a}{x^2+a^2}$$

32.

$$y = x\sqrt{x^2+a^2} + a^2 \sin^{-1}\frac{a}{x} \Longrightarrow$$

$$y' = \sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}} + a^2\frac{\frac{1}{a}}{\sqrt{1-(\frac{1}{a})^2}} = \frac{a^2-2x^2}{\sqrt{a^2-x^2}} + \frac{a}{\frac{\sqrt{a^-x^2}}{a}}$$

$$= \frac{2(a^2-x^2)}{\sqrt{a^2-x^2}} = 2\sqrt{a^2-x^2}$$

33.

$$y = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{a}{x} \Longrightarrow$$

$$y' = \frac{\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}}}{a^2-x^2} + \frac{\frac{a}{x^2}}{\sqrt{1-(\frac{a}{x})^2}} = \frac{a^2}{(x^2-a^2)^{2/3}} + \frac{\frac{a}{x^2}}{\sqrt{\frac{x^2-a^2}{x^2}}}$$

$$= \frac{a^3}{x(a^2-x^2)^{2/3}}$$

35.

$$y = \tan^{-1}\frac{x}{a} + \tan^{-1}\frac{a}{x} \Longrightarrow$$

$$y' = \frac{\frac{1}{a}}{1+(\frac{x}{a})^2} + \frac{-\frac{a}{x^2}}{1+(\frac{a}{x})^2} = \frac{\frac{1}{a}}{\frac{a^2+x^2}{a^2}} - \frac{\frac{a}{x^2}}{\frac{a^2+x^2}{x^2}} = \frac{a}{a^2+x^2} - \frac{a}{a^2+x^2} = 0$$

39.

$$y = x\cos^{-1}x \Longrightarrow y' = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$$

$$x = -\frac{1}{2} \Longrightarrow y' = \cos^{-1}(-\frac{1}{2}) + \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = \frac{2\pi}{3} + \frac{\sqrt{3}}{3} = \frac{2\pi+\sqrt{3}}{3}.$$

41.

$$y = x^2\sec^{-1}\sqrt{x} \Longrightarrow y' = 2x\sec^{-1}\sqrt{x} + \frac{x^2\frac{1}{2\sqrt{x}}}{\sqrt{x}\sqrt{x-1}} = 2x\sec^{-1}\sqrt{x} + \frac{x}{2\sqrt{x-1}}$$

$$x = 2 \Longrightarrow y' = 4\sec^{-1}2 + 1 = \frac{4\pi}{3} + 1.$$

43.

$$y = \frac{1}{x}\tan^{-1}\frac{1}{x} \Longrightarrow y' = -\frac{1}{x^2}\tan^{-1}\frac{1}{x} - \frac{1}{x(x^2+1)}$$

$$x = -1 \Longrightarrow y' = -\tan^{-1}(-1) + \frac{1}{2} = \frac{\pi}{4} + \frac{1}{2}.$$

45.

$$\begin{aligned}
\sqrt{x^2-y^2} + \sin^{-1}\left(\frac{y}{x}\right) = 0 &\Longrightarrow \frac{x - yy'}{\sqrt{x^2-y^2}} + \frac{\frac{y'x-y}{x^2}}{\sqrt{1-(\frac{y}{x})^2}} = 0 \\
&\Longrightarrow \frac{x - yy'}{\sqrt{x^2-y^2}} + \frac{y'x-y}{x\sqrt{x^2-y^2}} = 0 \\
&\Longrightarrow \frac{x^2 - xyy' + y'x - y}{x\sqrt{x^2-y^2}} = 0 \\
&\Longrightarrow x^2 - y + (x - xy)y' = 0 \\
&\Longrightarrow y' = \frac{x^2-y}{xy-x}
\end{aligned}$$