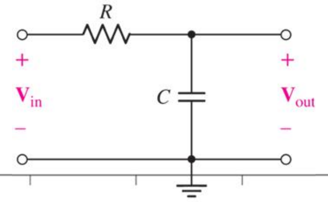
Chapter 15. Frequency Response

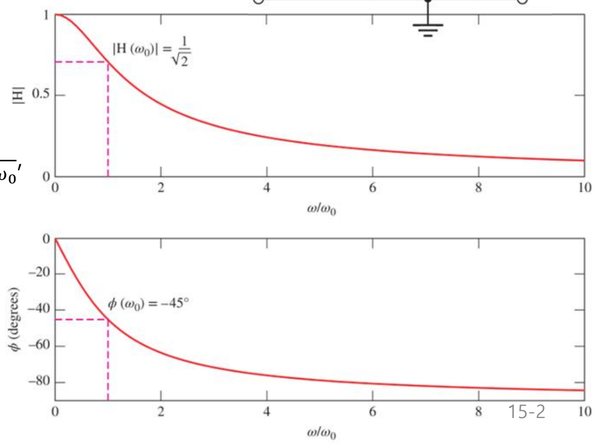
1. Transfer Function
2. Bode Diagram
3. Parallel Resonance
4. Bandwidth
5. Series Resonance

Transfer Function & Frequency Response

- Frequency response, $H(j\omega) = |H(j\omega)|\angle\phi(j\omega)$
- Example 15.1 RC circuit



  - $\mathbf{V}_{out} = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}\mathbf{V}_{in}$ (phasor form)
  - $V_{out}(j\omega) = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}V_{in}(j\omega)$
  - Frequency response, $H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1+j\omega/\omega_0}$,
    - † $\omega_0 = \frac{1}{RC}$ (natural frequency)
    - † $|H(j\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$ and
    - † $\angle H(j\omega) = \phi(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$

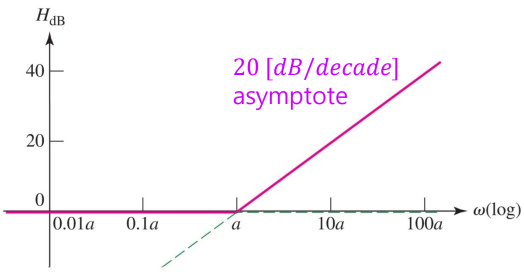


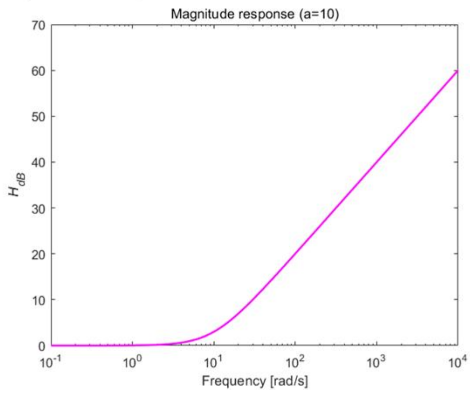
Bode Diagram

- Bode plot … Approximate visual plot of frequency response
  - Magnitude and phase plot on a logarithmic scale of frequency, ω
  - Magnitude, $H_{dB} = 20 \cdot \log_{10}|H(j\omega)|$
    - $|H(j\omega)| = 10^{H_{dB}/20}$
- dB values
  - $\log_{10} 1 = 0,\ \log_{10} 2 = 0.3,\ \log_{10} 10 = 1$
    - $20 \cdot \log_{10} 1 = 0,\ 20 \cdot \log_{10} 2 = 6,\ 20 \cdot \log_{10} 10 = 20,\ 20 \cdot \log_{10} 10^n = 20n\ [dB]$
    - $20 \cdot \log_{10} 0.1 = -20,\ 20 \cdot \log_{10} \frac{1}{\sqrt{2}} = -3\ [dB]$
  - When $|H(j\omega)|$ increases 2-times, H_{dB} increases 6 $[dB]$.
    - 10-fold increase on $|H(j\omega)| \Rightarrow 20\ [dB]$ increase in H_{dB}.
    - 10^n times increase on $|H(j\omega)| \Rightarrow 20n\ [dB]$ increase in H_{dB}.

Asymptote 점근선

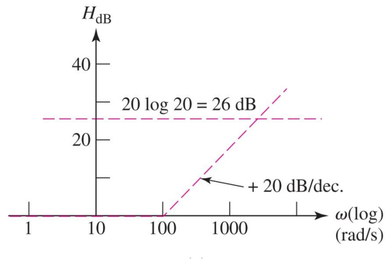
- Simple zero at $s = 0$
  - $H(s) = 1 + \frac{s}{a}$ or $H(j\omega) = 1 + \frac{j\omega}{a}$
  - $|H(j\omega)| = \sqrt{1 + \frac{\omega^2}{a^2}},\ H_{dB} = 20 \cdot \log_{10} \sqrt{1 + \frac{\omega^2}{a^2}}$
    - If $\omega \ll a$, $|H(j\omega)| \approx 20 \cdot \log_{10} 1 = 0$
    - If $\omega \gg a$, $|H(j\omega)| \approx 20 \cdot \log_{10} \frac{\omega}{a}$ …
      - asymptote with $20\ [dB/decade]$ or $6\ [dB/octave]$
    - When $\omega = a$,
      - $|H(j\omega)|\big|_{\omega=a} = \sqrt{2},\ H_{dB}\big|_{\omega=a} = 3\ [dB]$
      - corner, $3\ dB$, half-power frequency

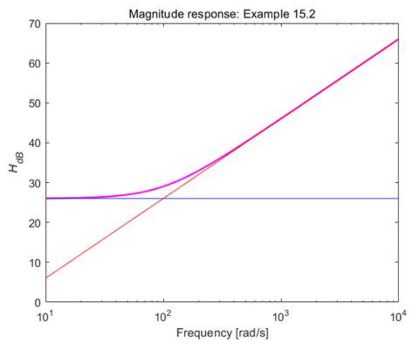




Asymptote 2

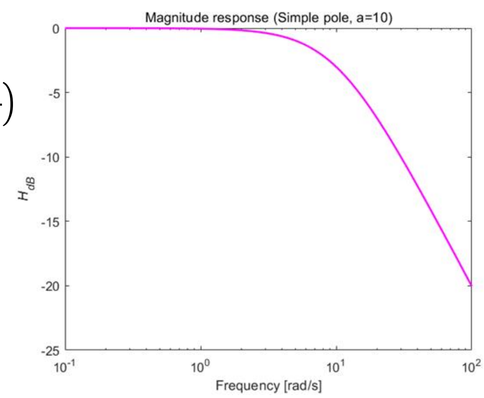
- Two simple zeros
  - ∨ $H(s) = K\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)$
  - ∨ $H_{dB} = 20 \cdot \log_{10}\left|K\left(1+\frac{j\omega}{\omega_1}\right)\left(1+\frac{j\omega}{\omega_1}\right)\right|$ or
  - ∨ $H_{dB} = 20 \cdot \log K + 20 \cdot \log\sqrt{1+\frac{\omega^2}{\omega_1^2}} + 20 \cdot \log\sqrt{1+\frac{\omega^2}{\omega_2^2}}$
- Example 15.2 $H(s) = Z_{in}(s) = 20 + 0.2s$
  - ∨ $H(s) = 20\left(1+\frac{s}{100}\right)$





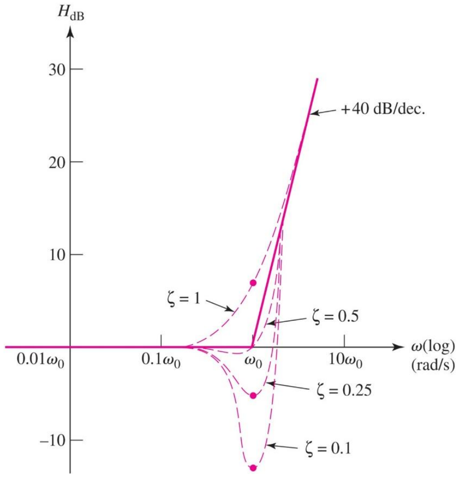
Asymptote 3

- Simple pole at $s = -a$
  - ∨ $H(s) = \frac{1}{1+s/a}$ or $H(j\omega) = \frac{1}{1+j\omega/a}$
  - ∨ $|H(j\omega)| = \left(1+\frac{\omega^2}{a^2}\right)^{-1/2}$
  - ∨ $H_{dB} = 20 \cdot \log_{10}\left(1+\frac{\omega^2}{a^2}\right)^{-\frac{1}{2}} = -10 \cdot \log_{10}\left(1+\frac{\omega^2}{a^2}\right)$
    - † If $\omega \ll a$, $|H(j\omega)| \approx 0$
    - † If $\omega \gg a$, $|H(j\omega)| \approx -20 \cdot \log_{10}\frac{\omega}{a}$
      - – asymptote with $-20\ [dB/decade]$
    - † When $\omega = a$, $H_{dB} = -3\ [dB]$



Asymptote 4

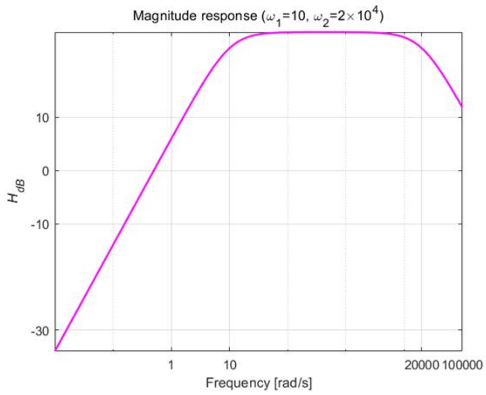
- Complex conjugate zeros
  - $H(s) = 1 + 2\zeta\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2$
    - When $0 \le \zeta < 1$ ($\zeta = \frac{\alpha}{\omega_a}$, damping factor), complex conjugate zeros.
  - $H_{dB} = 20 \cdot \log_{10}\left|1 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + \left(\frac{j\omega}{\omega_0}\right)^2\right|$
    - If $\omega \ll \omega_0$, $|H(j\omega)| \approx 0$
    - If $\omega \gg \omega_0$, $|H(j\omega)| \approx 40 \cdot \log_{10}\frac{\omega}{a}$
      - asymptote with 40 $[dB/decade]$
    - When $\omega = \omega_0$, $H_{dB} = 20 \cdot \log_{10}\left|2\zeta\left(\frac{\omega}{\omega_0}\right)\right|$

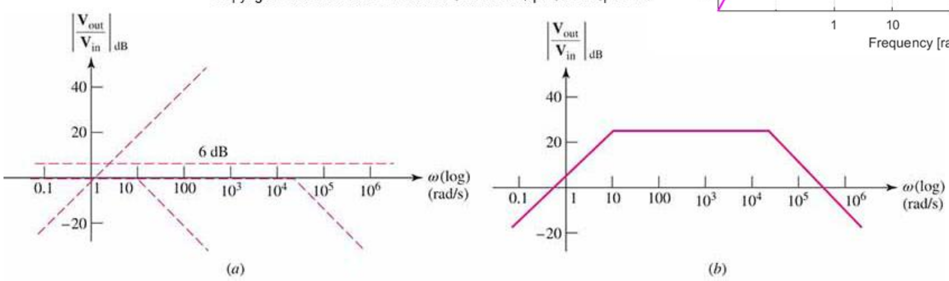


Corrections at corner freq.
(1) $\zeta = 1$, 6 $[dB]$
(2) $\zeta = 0.5$, 0 $[dB]$
(3) $\zeta = 0.1$, -14 $[dB]$

Asymptote 5

- Example 15.3
  - $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\frac{2s}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20000}\right)}$

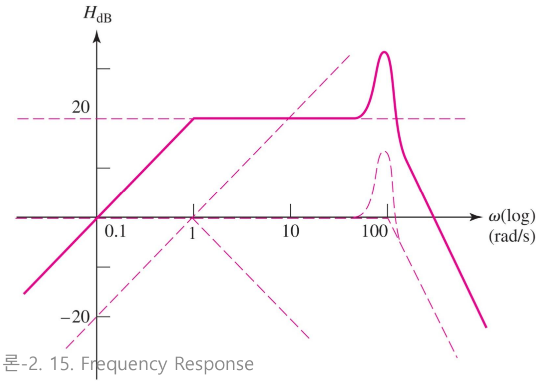




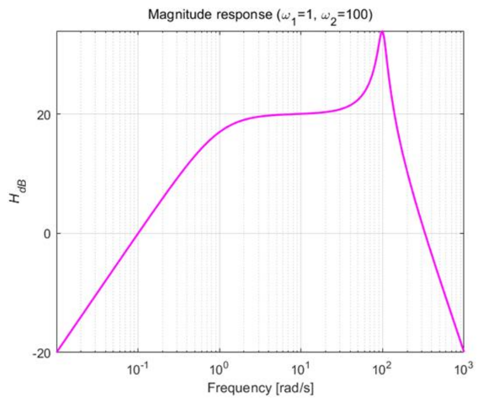
Asymptote 5

- Example 15.5
  - ∨ $H(s) = \frac{10s}{(1+s)(1+0.002s+0.0001s^2)} = \frac{10s}{\left(1+\frac{s}{\omega_1}\right)\left(1+2\zeta\frac{s}{\omega_2}+\frac{s^2}{\omega_2^2}\right)}$

    † $\zeta = 0.1$, $\omega_1 = 1$, and $\omega_2 = 100$

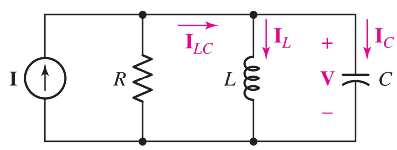


Corrections at $\omega_2 = 100$
$\zeta = 0.2 \Rightarrow 14\ [dB]$



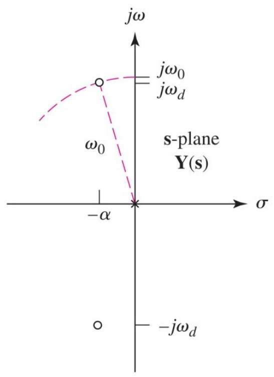
Parallel Resonance

- The network frequency response can be chosen to reject some frequency components of the forcing function, or to emphasize others.
  - ∨ Turning circuits for radio transmitters/receivers.
- A network is in resonance (공진), when voltage and current at input terminals are in phase.
  - ∨ Input impedance of the network is purely resistive.
- RLC parallel circuit
  - ∨ Admittance seen from the input terminal
  - ∨ $Y = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$
  - ∨ Voltage and current are in-phase, if $\omega C - \frac{1}{\omega L} = 0$: $\omega_0 = \frac{1}{\sqrt{LC}}$ (resonant frequency)



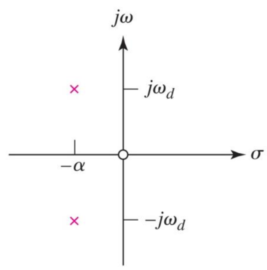
Parallel Resonance

- RLC parallel circuit
  - In s-domain, $Y(s) = \frac{1}{R} + \frac{1}{Ls} + Cs = C\frac{s^2+\frac{s}{RC}+\frac{1}{LC}}{s}$
  - $N(s) = s^2 + \frac{1}{RC}s + \frac{1}{LC} = (s+\alpha-j\omega_d)(s+\alpha+j\omega_d)$
    - $\alpha = \frac{1}{2RC}$ (exponential damping coefficient, 지수 감쇄계수), $\omega_0 = \frac{1}{\sqrt{LC}}$ (resonant frequency) and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ (natural resonant frequency, 고유 공진 주파수)
    - $Y(s)$ has a pole at $s=0$ and zeros at $s = \alpha \pm j\omega_d$.

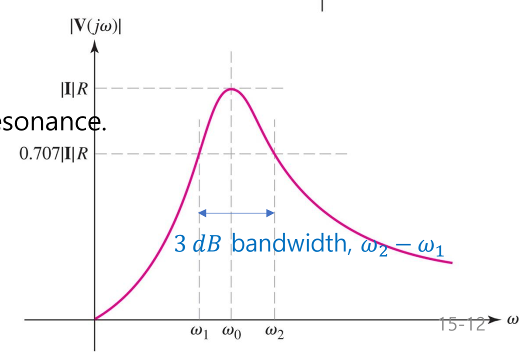


Parallel Resonance: Impedance

Pole-zero pattern of $Z(s)$

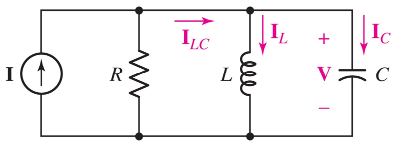


- Impedance, $Z(s) = \frac{1}{Y(s)} = \frac{s}{C(s+\alpha-j\omega_d)(s+\alpha+j\omega_d)}$
  - $Z(j\omega) = Z(s)|_{s\leftarrow j\omega} = \frac{j\omega}{C(\alpha+j(\omega-\omega_d))(\alpha+j(\omega+\omega_d))} = \frac{1}{C}\cdot\frac{j\omega}{\omega_0^2-\omega^2+j2\alpha\omega}$
  - $|Z(j\omega)|^2 = \frac{1}{C^2}\cdot\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\alpha^2\omega^2}$
    - $|Z(j\omega)|^2$ attains its maximum at $\omega = \omega_0$
    - $|Z(j\omega_0)| = \frac{1}{2\alpha C} = R$
      - Impedance has its maximum value at resonance.



Parallel Resonance: Impedance

- In RLC parallel circuit
  - $\mathbf{I}_{L,0} = \frac{\mathbf{V}_0}{j\omega_0 L} = \frac{R\mathbf{I}}{j} \cdot \frac{\sqrt{LC}}{L} = -jR\sqrt{\frac{C}{L}}\mathbf{I}$
  - $\mathbf{I}_{C,0} = j\omega_0 C\mathbf{V}_0 = j\frac{C}{\sqrt{LC}} \cdot R\mathbf{I} = jR\sqrt{\frac{C}{L}}\mathbf{I}$
    - $\mathbf{I}_{LC,0} = \mathbf{I}_{L,0} + \mathbf{I}_{C,0} = 0$

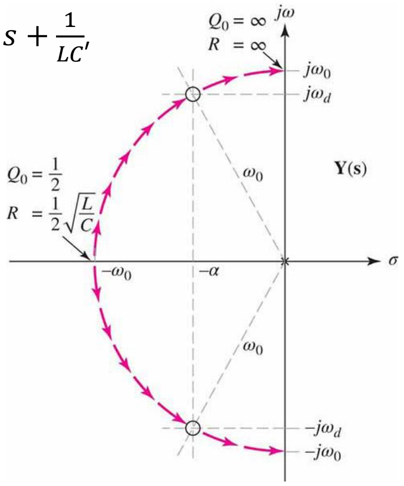


Quality Factor 품질 인자

- 3 dB bandwidth of the magnitude response, $|H(j\omega)|$
  - $Q = 2\pi \cdot \frac{\text{maximum energy stored}}{\text{total energy lost per period}}$
  - In RLC parallel circuit, $Q = 2\pi \cdot \frac{\max(w_L(t)+w_C(t))}{P_R T}$
    - Current source, $i(t) = I_m \cdot \cos\omega_0 t$
    - Voltage at resonance, $v(t) = Ri(t) = RI_m \cos\omega_0 t$
    - $w(t) = w_C(t) + w_L(t) = \frac{1}{2}I_m^2 R^2 C = w_{max}$
      - $w_C(t) = \frac{1}{2}Cv^2 = \frac{1}{2}I_m^2 R^2 C \cdot \cos^2\omega_0 t$, $w_L(t) = \frac{1}{2}Li_L^2 = \frac{1}{2}I_m^2 R^2 C \sin^2\omega_0 t$
    - $P_R = \frac{1}{2}I_m^2 R$
    - $Q_0 = 2\pi \frac{\frac{1}{2}I_m^2 R^2 C}{\frac{1}{2}I_m^2 R} = \omega_0 RC = R\sqrt{\frac{C}{L}}$

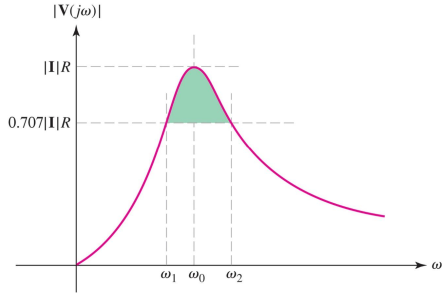
Quality Factor Q_0 & Damping Factor ζ

- $D(s) = s^2 + 2\alpha s + \omega_0^2 = s^2 + 2\zeta\omega_0 s + \omega_0^2 = s^2 + \frac{1}{RC}s + \frac{1}{LC}$,
  - $Y(s) = \frac{C}{s}D(s)$
  - $\zeta = \frac{\alpha}{\omega_0} = \frac{1}{2Q_0}$
  - $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}$



Constant L and C: constant ω_0

3-dB Bandwidth

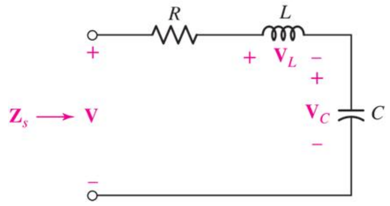


- 3 dB bandwidth, $B = \omega_1 - \omega_2$
  - $\omega_{1,2}$ … half-power frequency: $\left|V(j\omega_{1,2})\right|^2 = \frac{1}{2}|V(j\omega_0)|^2$
  - In RLC parallel circuit,
    - $Y(j\omega) = \frac{1}{R}\left(1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)$
    - $Y(j\omega_0) = \frac{1}{R}$ and $\left|Y(j\omega_{1,2})\right| = \frac{\sqrt{2}}{R}$
      - It happens only if the imaginary part is ± 1.
      - $Q_0\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = 1$ or $Q_0\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = -1 \Rightarrow \omega_{1,2} = \omega_0\left(\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} \mp \frac{1}{2Q_0}\right)$
      - $B = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$

High-Q Circuit

- High Q_0 ⇒ Narrow B/W and high frequency selectivity
- When $Q_0 > 5$,
  - ∨ $\alpha = \frac{\omega_0}{2Q_0} = \frac{1}{2}B$ and $\omega_d = \omega_0\sqrt{1-\left(\frac{1}{2Q_0}\right)^2} \approx \omega_0$
  - ∨ Zeros of $Y(s)$, $s_{1,2} = -\alpha \pm j\omega_d \approx -\frac{1}{2}B \pm j\omega_0$
  - ∨ $\omega_{1,2} = \omega_0\left(\sqrt{1+\left(\frac{1}{2Q_0}\right)^2} \mp \frac{1}{2Q_0}\right) \approx \omega_0\left(1 \mp \frac{1}{2Q_0}\right) = \omega_0 \mp \frac{1}{2}B$
    - † $|\omega_1 - \omega_0| = |\omega_2 - \omega_0| = \frac{B}{2}$ … symmetrical half-power frequency around ω_0
    - † $\omega_0 \approx \frac{1}{2}(\omega_1 + \omega_2)$ … arithmetic mean

Series RLC Resonance



- RLC series circuit with $v_s(t) = \cos\omega t$
  - ∨ $Ri + L\frac{di}{dt} + \frac{1}{c}\int^t i(\tau)d\tau = v_s(t)$, $\left(R + Ls + \frac{1}{Cs}\right)I(s) = V_s(s)$
  - ∨ $Z(s) = \frac{V_s(s)}{I(s)} = R + Ls + \frac{1}{Cs}$ or $Z(j\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$
    - † Resonant frequency, $\omega_0 = \frac{1}{\sqrt{LC}}$
    - † Series current at resonance, $i(t) = \frac{1}{R}\cos\omega_0 t$
      - − $w_C(t) = \frac{1}{2}Cv_C^2(t) = \frac{L}{2R^2}\sin^2\omega_0 t$, $w_L(t) = \frac{1}{2}Li^2(t) = \frac{L}{2R^2}\cos^2\omega_0 t$
      - − $w(t) = w_C(t) + w_L(t) = \frac{L}{2R^2}$
      - − $P_R T = \frac{1}{2}I_m^2 R \cdot T = \frac{1}{2R}\cdot\frac{1}{f_0}$

Series RLC Resonance

∨ $Q_0 = 2\pi \frac{L}{2R^2} \cdot 2f_0 R = \omega_0 \frac{L}{R}$

† $Q_0 = \omega_0 RC$ in RLC parallel circuit

† Half-power frequency, $\omega_{1,2} = \omega_0 \left(\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} \mp \frac{1}{2Q_0} \right)$

† $B = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$

∨ At resonance,

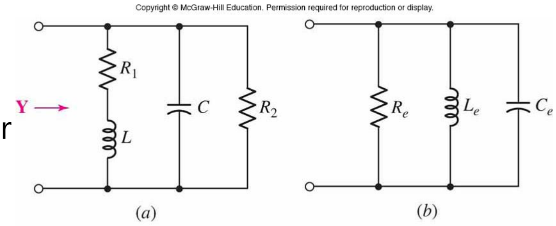
† $V_L(j\omega_0) = j\omega_0 L \cdot \frac{\mathbf{V}}{R} = jQ_0\mathbf{V}$

† $V_C(j\omega_0) = \frac{1}{j\omega_0 C} \cdot \frac{\mathbf{V}}{R} = -jQ_0\mathbf{V}$

† $V_C(j\omega_0) + V_L(j\omega_0) = 0$

– $I_C(j\omega_0) + I_L(j\omega_0) = 0$, in RLC parallel circuit

Realistic Circuit Model



• Circuit with physical inductor & capacitor

∨ Resonance, when $Im\{Y(j\omega)\} = 0$

† $Y(j\omega) = \frac{1}{R_1 + j\omega L} + j\omega C + \frac{1}{R_2}$

– $Y(j\omega) = \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2} + j\omega C + \frac{1}{R_2} = \left(\frac{1}{R_2} + \frac{R_1}{R_1^2 + \omega^2 L^2}\right) + j\left(\omega C - \frac{\omega L}{R_1^2 + \omega^2 L^2}\right)$

† $Im\{Y(j\omega)\} = 0 \Rightarrow C = \frac{L}{R_1^2 + \omega^2 L^2}$ or $\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R_1}{L}\right)^2}$

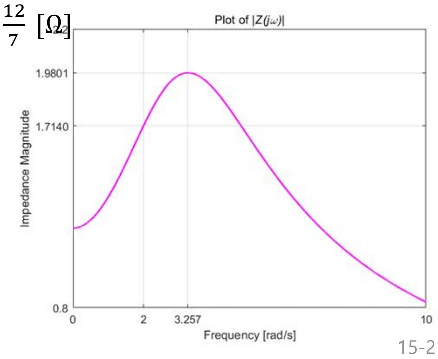
– $\omega_0 < \sqrt{\frac{1}{LC}}$

– Maximum magnitude of input impedance does not occur at ω_0.

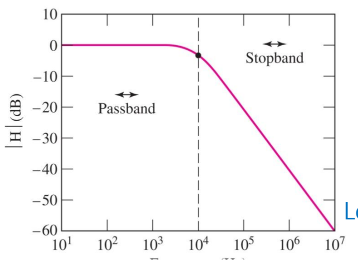
– Practical RLC circuit can be modeled with an ideal RLC circuit over a narrow frequency band (see Figure, part (b)).

Realistic Circuit Model

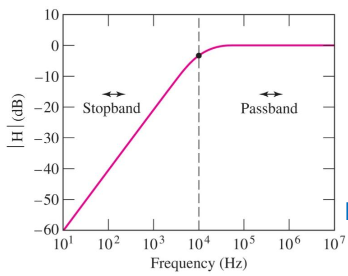
- Example 15.9 $R_1 = 2\ [\Omega]$, $L = 1\ [H]$, $C = 125\ [mF]$, and $R_2 = 3\ [\Omega]$
  - $Y(j\omega) = \frac{1}{R_1 + j\omega L} + j\omega C + \frac{1}{R_2}$ and $Z(j\omega) = \frac{1}{Y(j\omega)}$
  - $\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R_1}{L}\right)^2} = 2\ [rad/s]$
    - $Y(j\omega_0) = \frac{1}{2+j2} + j2 \cdot \frac{1}{8} + \frac{1}{3} = \frac{7}{12}\ [S]$ and $Z(j\omega_0) = \frac{12}{7}\ [\Omega]$
    - Maximum impedance magnitude at $\omega = 3.257$



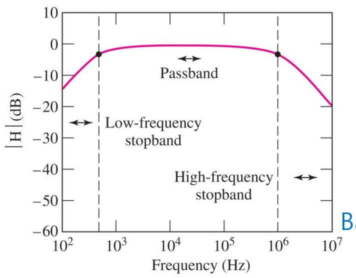
Basic Filter Design



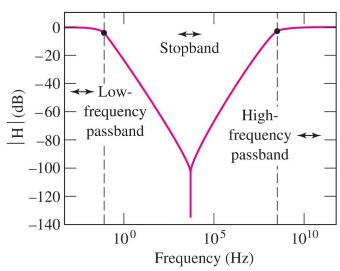
Low-pass filter



High-pass filter

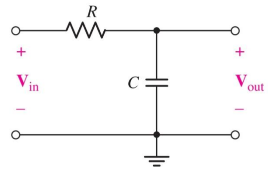


Band-pass filter

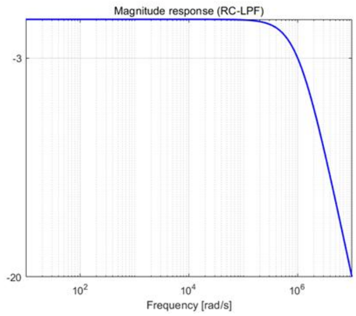


Band-stop filter

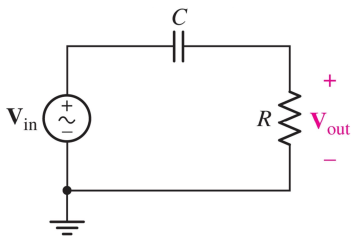
Low-pass Filter



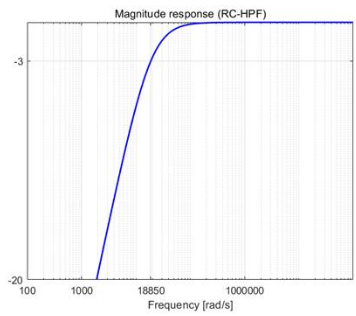
- Low-pass filter, $H(s) = \frac{1}{1+RCs}$
  - ∨ Corner frequency at $\omega = \frac{1}{RC}$
    - † At $\omega = 0$ (DC), capacitor acts like an open-circuit: $V_{out}(j\omega) = V_{in}(j\omega)$
    - † As $\omega \to \infty$, capacitor acts like a short-circuit: $Z_C = \frac{1}{j\omega C} \to 0$, $V_{out}(j\omega) = 0$
  - ∨ $R = 100\ [\Omega]$ and $C = 2\ [nF]$: $\omega = 10^6\ [rad/s]$



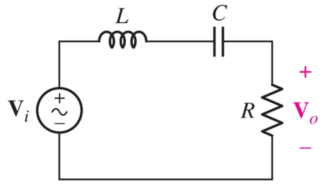
High-pass Filter



- High-pass filter, $H(s) = \frac{RCs}{1+RCs}$
  - ∨ Zero at $s = 0$ and pole at $s = -\frac{1}{RC}$
  - ∨ Corner frequency at $\omega_c = \frac{1}{RC}$
    - † At $\omega = 0$ (DC), capacitor acts like an open-circuit: $V_{out}(j\omega) = 0$
    - † As $\omega \to \infty$, capacitor acts like a short-circuit: $V_{out}(j\omega) = V_{in}(j\omega)$
  - ∨ $R = 4.7\ [k\Omega]$ and $C = 11.29\ [nF]$: $\omega_c = 18.85 \times 10^3\ [rad/s]$



Band-pass Filter



- $H(s) = \frac{V_o(s)}{V_i(s)} = \frac{RCs}{LCs^2+RCs+1}$ or $H(j\omega) = \frac{j\omega RC}{1-\omega^2 LC+j\omega RC}$
  - When $\omega = 0$, $V_o(j\omega) = 0$ and so is when $\omega \to \infty$.
  - Center frequency at $\omega = \frac{1}{\sqrt{LC}}$
  - At half-power frequencies, $|H(j\omega_c)| = \frac{\omega_c RC}{\sqrt{(1-\omega_c^2 LC)^2+\omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}}$
    - † $(1-\omega_c^2 LC)^2 = \omega_c^2 R^2 C^2 \Rightarrow 1-\omega_c^2 LC = \pm\omega_c RC$
      - – leading to two solutions $\omega_{H,L}$
      - – $1-\omega_L^2 LC = \omega_L RC$ and $1-\omega_H^2 LC = -\omega_H RC$
    - † $B = \omega_H - \omega_L = \frac{R}{L}$

Band-pass Filter

- Example 15.13 BPF with $B = 1\ [MHz]$ and $f_H = 1.1\ [MHz]$
    - † $f_L = f_H - B = 100\ [kHz]$,
    - † $\omega_L = 2\pi f_L = 628.3 \times 10^3\ [rad/s]$ and $\omega_H = 2\pi f_L = 6.912 \times 10^6\ [rad/s]$
    - † $B = 2\pi \times 10^6 [rad/s]$
  - $1-\omega_H^2 LC = -\omega_H RC \Rightarrow \omega_H^2 - B\omega_H = \frac{1}{LC}\ (B = \frac{R}{L})$
    - † $\frac{1}{LC} = 4.3464 \times 10^{12}$
  - Choose $L = 50\ [mH]$. Then, $R = BL = \pi \times 10^5\ [\Omega]$ and $C = 4.602 \times 10^{-12}\ [F]$