Chapter 2. 2^{nd}-order Linear ODEs

1. 2^{nd} order linear homogeneous ODEs
2. Linear homogeneous ODEs with constant coefficients
3. Mass-spring system
4. Euler-Cauchy Eqns.
5. Non-homogeneous ODEs

Homogeneous Linear ODEs of Second Order

- Linear ODEs of second order, standard form : $y'' + p(x)y' + q(x)y = r(x)$
  - ∨ Homogeneous, if $r(x) = 0$ (input or forcing function).
  - ∨ $p(x)$ and $q(x)$ … coefficients (계수)
- Examples:
  1. $y'' + 25y = e^x \cos x$ … linear homogeneous (선형 제차)
  2. $y'' + \frac{1}{x}y' + y = 0$ … linear homogeneous (선형 비제차)
  3. $y''y + (y')^2 = 0$ … non-linear

Homogeneous Linear ODEs of Second Order

- Theorem 1. (Superposition Principle: 중첩의 원리)
  - For a homogeneous linear ODE, any linear combination of two solutions on an open interval I, is again solution of the equation on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.
    - If $y_1(x)$ and $y_2(x)$ are solutions, so are $y_1(x) + y_2(x)$ and $ky_1(x)$.
    - The linear combination of $y_1(x)$ and $y_2(x)$, say $c_1y_1(x) + c_2y_2(x)$ is also a solution.
    - This theorem holds only for the homogeneous linear ODEs.
- Example 2.1.1 $y'' + y = 0$
  - Both $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are solutions, $\forall x \in R$.
  - $c_1 \cos x + c_2 \sin x$ is also a (general) solution.
  - For the non-homogeneous ODE, $y'' + y = 1$, $y_1(x) = 1 + \cos x$ and $y_2(x) = 1 + \sin x$ are solutions, not a linear combination of two.

Initial Value Problem

- $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = K_0$ and $y'(x_0) = K_1$
  - Two initial conditions determine the coefficients c_1 and c_2 in the general solution (일반해), which lead to the particular solution (특수해).
- Example 2.1.4 $y'' + y = 0$, $y(0) = 3$ and $y'(0) = -0.5$
  - General solution: $y = c_1 \cos x + c_2 \sin x$
    - Check both $\cos x$ and $\sin x$ are solutions (known as basis, 기저)
      - Two functions in the basis should be linearly independent each other.
    - Either $\cos x$ or $\sin x$ cannot satisfy two initial conditions.
  - Particular solution: $y(0) = c_1 = 3$ and $y'(0) = c_2 = -0.5 \Rightarrow y = 3\cos x - 0.5 \sin x$

2^{nd}-order Homogeneous Linear ODEs

- Definition

A general solution of an ODE on an open interval I is a solution $y = c_1 y_1 + c_2 y_2$, in which y_1 and y_1 are solutions of the equation on I that are linearly independent (*i.e.*, not proportional, $y_1 \neq k y_2$) and c_1, c_2 are arbitrary constants. There y_1, y_2 are called a basis of solutions of the equation on I.

A particular solution of the equation on I is obtained if we assign specific values to c_1 and c_2 in the general solution.

† Two functions y_1 and y_2 are called linearly independent on I, where they are defined, if $k_1 y_1(x) + k_2 y_2(x) = 0$ everywhere on I implies $k_1 = 0$ and $k_2 = 0$.

† If not (linearly dependent), then y_1 and y_2 are proportional (scalar multiple of the other).

- Definition

∨ A basis of solutions of the ODE on an open interval I is a pair of LID solutions.

2^{nd}-order Homogeneous Linear ODEs

- Reduction of order: Find a basis if one solution is known.

∨ Given the 2^{nd}-order homogeneous ODE, $y'' + p(x)y' + q(x)y = 0$, put $y_2 = uy_1$.

$y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + 2u'y_1' + uy_1''$

$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$, $u''y_1 + u'(py_1 + 2y_1') + u(y_1'' + py_1' + qy_1) = 0$

$u'' + \frac{py_1 + 2y_1'}{y_1}u' = 0$ or $v' + \frac{py_1 + 2y_1'}{y_1}v = 0$, where $v = u'$

$\frac{dv}{v} = -\left(\frac{2y_1'}{y_1} + p\right)dx \Rightarrow v = \frac{1}{y_1^2}e^{-\int p\,dx}$, $u = \int v\,dx$, and $y_2 = y_1 \int v\,dx$

(1) $\int \frac{y_1'}{y_1}dx = \ln y_1$
(2) $\int \frac{dx}{x-1}dx = \ln(x-1)$
(3) $e^{\ln(x-1)} = x - 1$

- Example: Given $(x^2 - x)y'' - xy' + y = 0$ and $y_1 = x$

∨ $p = -\frac{1}{x-1}$, $v = \frac{1}{x^2}\exp\left(\int \frac{dx}{x-1}\right) = \frac{1}{x} - \frac{1}{x^2}$, and $y_2 = x\left(\ln x + \frac{1}{x}\right) = x \cdot \ln x + 1$

Homogeneous Linear ODE with constant coefficients

- 2nd-order homogeneous linear ODE with constant coefficients 2계 제차 선형 상미분 방정식:
  - $y'' + p(x)y' + q(x)y = y'' + ay' + by = 0$ with a, b const.
  - Claim that y$(x) = e^{\lambda x}$ is a solution.
    - $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$, so that $\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0$
    - $\lambda^2 + a\lambda + b = 0$ … characteristic equation 특성 다항식, $\lambda_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$
      - Three possible solutions

        ① Two real roots: $\lambda_{1,2} \Rightarrow y(x) = c_1 e^{\lambda_1} + c_2 e^{\lambda_2}$

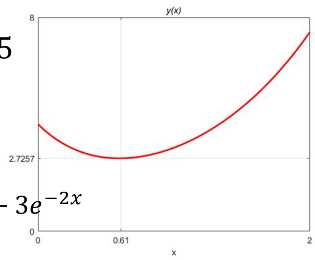
        ② One real double root, $\lambda \Rightarrow y(x) = (c_1 + c_2 x)e^{\lambda}$

        ③ Complex conjugate pair, $\lambda_{1,2} = -\frac{a}{2} \pm j\omega \Rightarrow y(x) = (A \cdot \cos\omega x + B \cdot \sin\omega x)e^{\lambda_2 x}$

Euler's relation
$e^{jx} = \cos x + j \cdot \sin x$

Homogeneous Linear ODE with constant coefficients

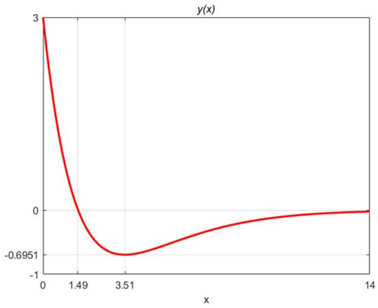
- Example 2.2.2 $y'' + y' - 2y = 0$, $y(0) = 4$, and $y'(0) = -5$
  - Char. Eq.: $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = -2$
  - General solution: $y(x) = c_1 e^{x} + c_2 e^{-2x}$
  - Particular solution: $c_1 + c_2 = 4$ and $c_1 - 2c_2 = -5 \Rightarrow y(x) = e^{x} + 3e^{-2x}$



- Double root case, $a^2 - 4b = 0$: $\lambda_{1,2} = -\frac{a}{2}$ (double)
  - $y_1(x) = e^{-ax/2}$ is a solution.
  - Using the 'reduction of order' process with $p = a$, $y_2(x) = y_1 \int v dx$ is also a solution, where $v = \frac{1}{y_1^2} e^{-\int p dx}$.
  - $v = \frac{1}{e^{-ax}} e^{-\int a dx} = 1$ and $y_2(x) = xe^{-ax/2}$

Homogeneous Linear ODE with constant coefficients

- Example 2.2.4 $y'' + y' + 0.25y = 0$, $y(0) = 3$, and $y'(0) = -3.5$
  - Char. Eq.: $\lambda^2 + \lambda + 0.25 = \left(\lambda + \frac{1}{2}\right)^2 = 0 \Rightarrow \lambda_{1,2} = -\frac{1}{2}$ (double)
  - General solution: $y(x) = (c_1 + c_2 x)e^{-x/2}$
  - Particular solution: $c_1 = 3$ and $-0.5c_1 + c_2 = 3.5 \Rightarrow y(x) = (3 - 2x)e^{-x/2}$

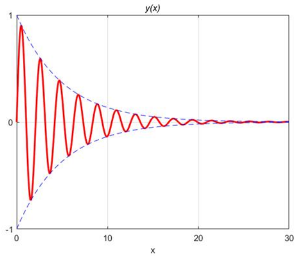


Homogeneous Linear ODE with constant coefficients

- Complex conjugate case, $a^2 - 4b < 0$: $\lambda_{1,2} = -\frac{a}{2} \pm j\omega$, $\omega^2 = b - \frac{a^2}{4}$
  - $y(x) = c_1 e^{-ax/2} e^{j\omega x} + c_1^* e^{-ax/2} e^{-j\omega x} = (A \cdot \cos\omega x + B \cdot \sin\omega x)e^{-ax/2}$
    - † $c_1(\cos\omega x + j\sin\omega x) + c_1^*(\cos\omega x - j\sin\omega x) = (c_1 + c_1^*)\cos\omega x + j(c_1 - c_1^*)\sin\omega x$

Euler's relation
$e^{jx} = \cos x + j \cdot \sin x$

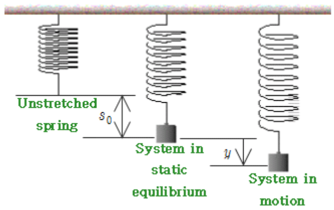
- Example 2.2.5 $y'' + .4y' + 9.04y = 0$, $y(0) = 0$, and $y'(0) = 3$
  - Char. Eq.: $\lambda_{1,2} = -.2 \pm j3$
  - $y(x) = (A\cos 3x + B\sin 3x)e^{-0.2x}$
  - $A = 0$ and $B = 1 \Rightarrow y(x) = e^{-0.2x}\sin 3x$



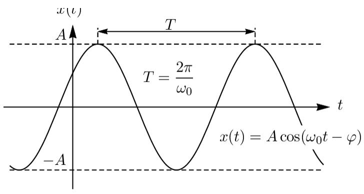
Differential Operators

- Operator: A transformation that transforms a function into another function.
  - An operator is a function whose domain and co-domain are the same.
  - Operational calculus: The technique and application of operators.
  - Differential Operator, D: An operator which transforms a (differentiable) function into its derivative: $Dy = \frac{dy}{dx} = y'$.
  - Identity Operator, I: $Iy = y$
  - Second-order differential operator:
    - † $L = P(D) = D^2 + aD + bI$, so that $Ly = P(D)y = y'' + ay' + by$

Mass-spring system



- Physical information
  - Newton's second law: $F = ma$
  - Hook's law: $F_1 = -ky$
    - † Negative sign indicates F_1 points upward, against the displacement.
    - † We choose the downward direction as the positive direction.
- Model



  - System in static equilibrium: $F_0 = -ks_0$, k … spring constant

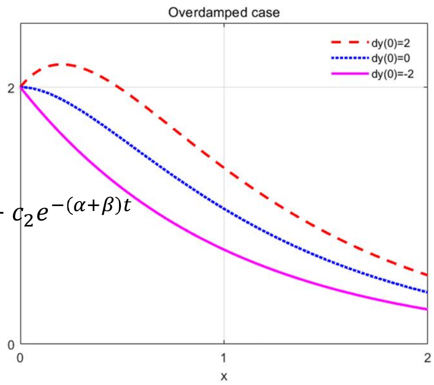
    $F_0 + W = -ks_0 + mg$
  - System in motion

    $my'' + ky = 0$ (un-damped system), $y(t) = A\cos\omega_0 t + B\sin\omega_0 t$, $\omega_0 = \sqrt{\frac{k}{m}}$

Mass-spring system

- Example 2.4.1 $y'' + \frac{k}{m}y = 0$, $\sqrt{\frac{k}{m}} = 9$, $y(0) = 0.16$, and $y'(0) = 0$
  - $y(t) = A\cos 3t + B\sin 3t$
  - $A = 0.16$ and $B = 0 \Rightarrow y(t) = 0.16\cos 3t$ [m]
- Damping (friction) force: $F_2 = -cy'$, c … damping constant
  - Negative, since the damping force acts against the motion.
    - When $y' > 0$ (downward motion), F_2 points upward.
  - $my'' + cy' + ky = 0$,
  - Char. equation, $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$, $\lambda_{1,2} = \alpha \pm \beta = \frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$

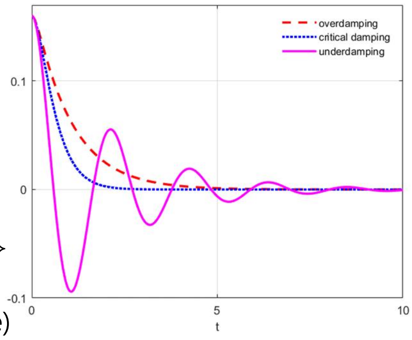
Mass-spring system



- Case I. Overdamping
  - $c^2 > 4mk$, two distinct real roots: $y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$
- Case II. Critical damping
  - $c^2 = 4mk$, double roots: $y(t) = (c_1 + c_2 t)e^{-\alpha t}$
- Case III. Underdamping
  - $c^2 < 4mk$, complex-conjugate roots
  - $\lambda_{1,2} = \frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \alpha \pm j\omega_0$
  - $y(t) = e^{-\alpha t}(A\cos\omega_0 t + B\sin\omega t) = Ce^{-\alpha t}\cos(\omega_0 t - \delta)$, $C^2 = A^2 + B^2$ and $\delta = \tan^{-1} B/A$

Mass-spring system

- Example 2.4.2 $my'' + cy' + ky = 0$, $y(0) = 0.16$, $y'(0) = 0$, $k = 90$, and $m = 10$.



① $c = 100\ [kg/s^2]$: $y'' + 10y' + 9y = 0$, $\lambda_{1,2} = -9, -1$

$y(t) = c_1 e^{-9t} + c_2 e^{-t}$, $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0 \Rightarrow$
$y(t) = -.02e^{-9t} + .18e^{-t}$

② $c = 60\ [kg/s^2]$: $y'' + 6y' + 9y = 0$, $\lambda_{1,2} = -3$ (double)

$y(t) = (c_1 + c_2 t)e^{-3t}$, $c_1 = 0.16$, $c_2 - 3c_1 = 0 \Rightarrow$
$y(t) = (.16 + .48t)e^{-3t}$

③ $c = 10\ [kg/s^2]$: $y'' + y' + 9y = 0$, $\lambda_{1,2} = -.5 \pm j2.96$

$y(t) = e^{-.5t}(A\cos 2.96t + B\sin 2.96t)$, $A = 0.16$,
$-.5A + 2.96B = 0 \Rightarrow y(t) = e^{-.5t}(.16\cos 2.96t +$
$.027\sin 2.96t) = .162e^{-.5t}\cos(2.96t - 0.17)$

Euler-Cauchy Equations

- $x^2y'' + axy' + by = 0$, a, b const.
  - Put $y = x^m$. $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$: $m(m-1)x^m + amx^m + bx^m = 0$
  - Auxiliary eq.: $m^2 + (a-1)m + b = 0 \Rightarrow m_{1,2} = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(1-a)^2 - b}$
    - † Two real roots, $y = c_1x^{m_1} + c_2x^{m_2}$
    - † Double root, $y = (c_1 + c_2 \ln x)x^m$

Existence and Uniqueness of Solutions: Wronskian

● **Theorem 1** Existence and Uniqueness Theorem for Initial Value Problem

(1) $y'' + p(x)y' + q(x)y = 0,\ y(x_0) = K_0$ and $y'(x_0) = K_1$

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and $x_0 \in I$, then the initial value problem has a unique solution on the interval I.

● **Theorem 2 Linear Dependence and Independence of Solutions**

Let $p(x)$ and $q(x)$ be continuous on an open interval I. Then two solutions y_1 and y_2 of (1) on I are linearly dependent on I, if and only if their "Wronskian"

(6) $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$ $\quad x = x_0 \quad W \equiv 0$

$x_1 \quad y_1, \quad y_2$

is 0 at some $x_0 \in I$. Furthermore, if $W = 0$ at an x_0, then $W = 0$ on I.

Existence and Uniqueness of Solutions: Wronskian

● **Theorem 1** Existence and Uniqueness Theorem for Initial Value Problem

(1) $y'' + p(x)y' + q(x)y = 0,\ y(x_0) = K_0$ and $y'(x_0) = K_1$

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and $x_0 \in I$, then the initial value problem has a unique solution on the interval I.

● **Theorem 2** Let $p(x)$ and $q(x)$ be continuous on an open interval I. Then two solutions y_1 and y_2 of (1) on I are linearly dependent on I, if and only if their "Wronskian"

(6) $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

is 0 at some $x_0 \in I$. Furthermore, if $W = 0$ at an x_0, then $W = 0$ on I.

▼ If $\exists x_1 \neq 0$ in I, such that $W \neq 0$, then y_1 and y_2 are linearly independent on I.

Non-homogeneous ODEs

(1) $y'' + p(x)y' + q(x)y = r(x)$, $r(x) \neq 0$

● **Definition** General Solution, Particular Solution

- A general solution of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $y(x) = y_h(x) + y_p(x)$,

where $y_h(x) = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE.

- A particular solution of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_1 in y_h.

Method of Undetermined Coefficients

- Steps in finding solutions of $y'' + p(x)y' + q(x)y = r(x)$
  - Find the homogeneous solution, $y_h(x)$, such that $y_h'' + p(x)y_h' + q(x)y_h = 0$.
  - Find any solution y_p and add up.
- Method of underdetermined coefficient 미정 계수법
  - $y'' + ay' + by = r(x)$, a, b const.
  - When $r(x)$ is of special form, its derivatives are similar to $r(x)$ itself. Thus, we choose a form of y_p similar to $r(x)$, but with unknown coefficients to be determined.

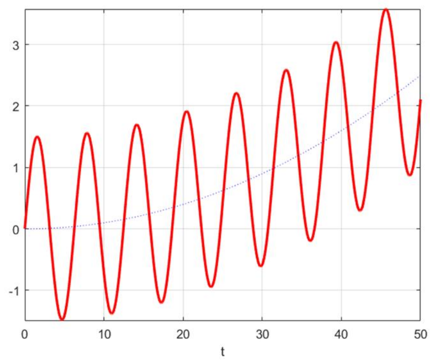
Method of Undetermined Coefficients

- Basic rule: If $r(x)$ is one of the function in Table, choose y_p in the same row and determine its undetermined coefficients by substituting y_p and its derivatives into the ODE.
- Modification rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE, multiply your choice of by x (or by x^2, if this solution corresponding to a double root of the characteristic equation of the homogeneous ODE).
- Sum rule: If $r(x)$ is a sum of functions in the first column of Table, choose for y_p the sum of the functions in the corresponding lines of the second column.

$r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n, $n = 0, 1, \ldots$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ or $k\sin\omega x$	$K_1\cos\omega x + K_2\sin\omega x$
$ke^{\alpha x}\cos\omega x$ or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K_1\cos\omega x + K_2\sin\omega x)$

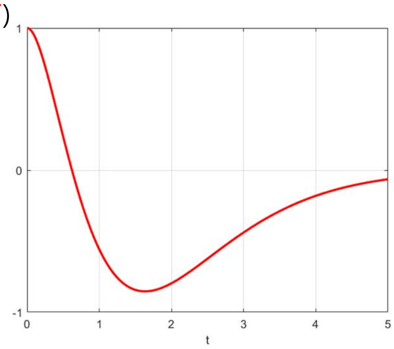
Method of Undetermined Coefficients

- Example 2.7.1 $y'' + y = .001x^2$, $y(0) = 0$, $y'(0) = 1.5$
  - $y_h = A\cos x + B\sin x$
  - Choose $y_p = K_2x^2 + K_1x + K_0 \Rightarrow (2K_2) + (K_2x^2 + K_1x + K_0) = .001x^2 \Rightarrow K_2 = .001$, $K_1 = 0$, and $K_0 = -.002$
  - $y = y_h + y_p = (A\cos x + B\sin x) + (.001x^2 - .002)$
  - Initial conditions: $A = .002$ and $B = 1.5$



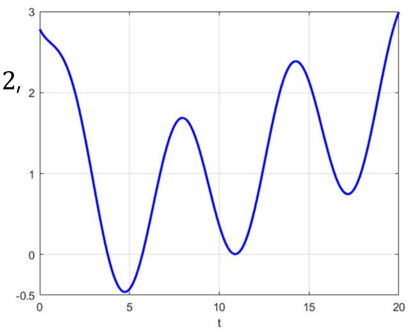
Method of Undetermined Coefficients

- Example 2.7.2 $y'' + 3y' + 2.25y = -10e^{-1.5x}$, $y(0) = 1$, $y'(0) = 0$
  - $y_h = (c_1 + c_2 x)e^{-1.5x}$
  - First choice for $y_p = Ce^{-1.5x}$ based on $r(x) = -10e^{-1.5x}$. Yet, it has a same form as one component of y_h.
  - Modification rule: Choose $y_p = Cx^2e^{-1.5x}$ (not $Cxe^{-1.5x}$)
  - $y_p' = 2Cxe^{-1.5x} - 1.5Cx^2e^{-1.5x}$ and
  - $y_p'' = C(2 - 3x - ex + 2.25x^2)e^{-1.5x}$
  - $y_p'' + 3y_p' + 2.25y_p = -10e^{-1.5x} \Rightarrow C = -5$
  - $y = y_h + y_p = (c_1 + c_2 x)e^{-1.5x} - 5x^2e^{-1.5x}$
  - Initial conditions: $c_1 = 1$ and $c_2 = 1.5$



Method of Undetermined Coefficients

- Example 2.7.3 $y'' + 2y' + .75y = 2\cos x - .25\sin x + .09x$, $y(0) = 2.78$, $y'(0) = -0.43$
  - $y_h = c_1 e^{-0.5x} + c_2 e^{-1.5x}$
  - Choose $y_p = K_1 x + K_0 + A\cos x + B\sin x$
  - $y_p' = K_1 - A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$
  - $y_p'' + 3y_p' + 2.25y_p = -10e^{-1.5x} \Rightarrow A = 0,\ B = 1,\ K_1 = 0.12,$ and $K_2 = -0.32$
  - $y = y_h + y_p = c_1 e^{-0.5x} + c_2 e^{-1.5x} + 0.12x - 0.32 + \sin x$
  - Initial conditions: $c_1 = 3.1$ and $c_2 = 0$



Forced Oscillation: Resonance

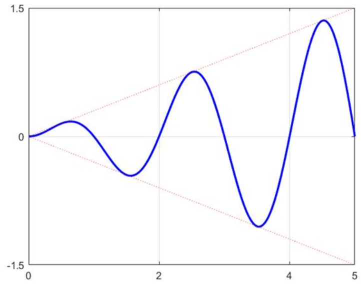
- Free vs. forced motions : Motions caused solely by internal forces vs. external force
  - $my'' + cy' + ky = r(t)$, where $y(t)$ … displacement (or output, response), $r(t)$ … input (or driving force)
- Forced motion with periodic external force
  - $r(t) = F_0 \cos \omega t$
  - Choose $y_p(t) = a \cos \omega t + b \sin \omega t$
    - $y' = -a\omega \sin \omega t + b\omega \cos \omega t,\ y'' = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$
  - $m(-a\omega^2 \cos \omega t - b\omega^2 \sin \omega t) + c(-a\omega \sin \omega t + b\omega \cos \omega t) + k(a \cos \omega t + b \sin \omega t) = F_0 \cos \omega t$
  - $\big((k - m\omega^2)a + \omega cb\big) \cos \omega t + (-\omega ca + (k - m\omega^2)b) \sin \omega t = F_0 \cos \omega t$

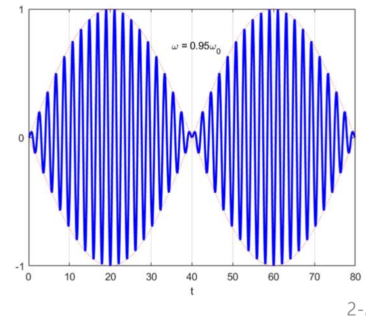
Forced Oscillation: Resonance

  - $(k - m\omega^2)a + \omega cb = F_0$ and $-\omega ca + (k - m\omega^2)b = 0$
  - $a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$ and $b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$, where $\omega_0^2 = \frac{k}{m}$ (or $k = m\omega_0^2$)
  - $y(t) = y_h(t) + y_p(t)$
- Case 1: Undamped forced oscillations: Resonance
  - When $c = 0$, $a = \frac{F_0}{m(\omega_0^2 - \omega^2)^2}$ and $b = 0$, so that
  - $y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)^2} \cos \omega t = \frac{F_0}{k\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)} \cos \omega t$, assuming that $\omega \neq \omega_0$.
    - Frequency of the forcing function ω vs. natural frequency ω_0
  - $y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{k\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)} \cos \omega t$
    - superposition of two harmonic oscillations

Forced Oscillation: Resonance

- When $\omega \to \omega_0$, $y_p(t) \to \infty$.
- Resonance, when $\omega = \omega_0$.
  - When $\omega = \omega_0$, $y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t$,
    - choose $y_p = t(a\cos\omega_0 t + b\sin\omega_0 t)$
    - $a = 0$ and $b = \frac{F_0}{2m\omega_0} \Rightarrow y_p(t) = \frac{F_0}{2m\omega_0} t \cdot \sin\omega_0 t$
- Beats, when $\omega \cong \omega_0$.
  - $y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t) = A\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$
    - $\omega + \omega_0 \gg \omega - \omega_0$

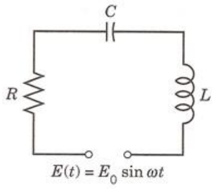




Forced Oscillation: Resonance

- Case 2: Damped forced oscillations
  - $y(t) = y_h(t) + y_p(t)$
  - $y_h(t) \to 0$ as $t \to \infty$ … transient solution
  - $y(t) \to y_p(t)$ as $t \to \infty$ … steady-state solution
  - When $c > 0$, $y_p(t) = a\cos\omega t + b\sin\omega t = D(\omega)\cos(\omega t - \eta)$, where
    - $a = F_0 \frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$ and $b = F_0\frac{\omega c}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$
    - $D(\omega) = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}}$ and $\eta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{\omega c}{m(\omega_0^2-\omega^2)}$
      - $D(\omega)$ attains its maximum value, $D_{max}(\omega_1) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2-c^2}}$, when $\omega_1^2 = \omega_0^2 - \frac{c^2}{2m^2}$

Electric Circuits



- RLC series circuit
  - $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = \frac{dE}{dt} = \frac{\omega E_0}{L} \cos \omega t$
  - Choose $i_p(t) = a \cos \omega t + b \sin \omega t$: $i_p' = -a\omega \sin \omega t + b\omega \cos \omega t$, $i_p'' = -a\omega^2 \cos \omega t + b\omega^2 \sin \omega t$
    - † $-Sa + Rb = E_0$, $-Ra - Sb = 0$, where $S = \omega L - \frac{1}{\omega C}$ (reactance)
    - † $a = -\frac{E_0 S}{R^2 + S^2}$, $b = \frac{E_0 R}{R^2 + S^2}$
  - $i_p(t) = -\frac{E_0 S}{R^2+S^2} \cos \omega t + \frac{E_0 R}{R^2+S^2} \sin \omega t = \frac{E_0}{\sqrt{R^2+S^2}} \sin(\omega t - \theta)$
  - Char. Equation: $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$, $\lambda_{1,2} = \alpha \pm \beta = \frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$

Electric Circuits

- Analogy of Electrical and Mechanical Quantities
  - Different physical systems may have the same mathematical model.

Electrical systems	**Mechanical systems**
Inductance, L	Mass, M
Resistance, R	Damping constant, c
Reciprocal of capacitance, $1/C$	Spring constant, k
Derivative of voltage source	Driving force
Current, $i(t)$	Displacement, $y(t)$

Solution by Variation of Parameters

- Given $y'' + p(x)y' + q(x)y = r(x)$,
  - ∨ When the "method of undetermined coefficient" fails.
  - ∨ Method of variation of parameters
  - ∨ $y_p(x) = -y_1 \int \frac{y_2 r}{W} dx \ + y_2 \int \frac{y_1 r}{W} dx$,
    - † where y_1, y_2 are basis of solutions for the homogeneous ODE
    - † $W = y_1 y_2' - y_2 y_1'$
- Example 2.10.1 $y'' + y = 1/\cos x$
  - ∨ $y_1 = \cos x$, $y_2 = \sin x$, so that $W = 1$.
  - ∨ $y_p = -\cos x \int \sin x \cdot \sec x \, dx + \sin x \int \cos x \cdot \sec x \, dx = \cos x \cdot \ln(\cos x) + x \cdot \sin x$