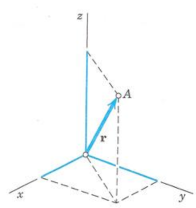
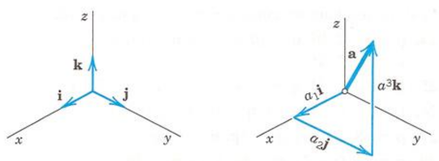
Chapter 9. Vector Differential Calculus

1. Inner Product
2. Cross Product
3. Derivative of Vector Functions
4. Curves
5. Gradient, Divergence, and Curl

Vectors in 2-D & 3-D Spaces



- Vector as an ordered-triple, $\boldsymbol{a} = (a_1, a_2, a_3)$
  - Components $\{a_1, a_2, a_3\}$
- Position vector, $\overrightarrow{OA} = (x, y, z)$
- Operations in vectors, $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$
  - Equality: $\boldsymbol{a} = \boldsymbol{b} \leftrightarrow a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$
  - Vector addition: $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
  - Scalar multiplication: $c\boldsymbol{a} = (ca_1, ca_2, ca_3)$, where c is a scalar (real number).
- Standard unit vectors
  - $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$, $\boldsymbol{k} = (0, 0, 1)$
  - $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$



Inner Product

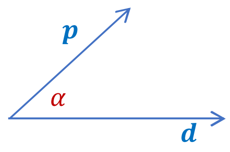
- Dot product 내적
  - $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Angle
  - $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cos\gamma$, where $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$ … length of vector, $\boldsymbol{a}$
    - † $\cos\gamma = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}| \cdot |\boldsymbol{b}|}$
  - Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ (perpendicular or right angle)
    - † $\boldsymbol{a} \perp \boldsymbol{b}$

Inner Product: Properties

- For any vectors, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and scalars k_1, k_2
  1. $(k_1 \boldsymbol{a} + k_2 \boldsymbol{b}) \cdot \boldsymbol{c} = k_1 \boldsymbol{a} \cdot \boldsymbol{c} + k_2 \boldsymbol{b} \cdot \boldsymbol{c}$ (linearity)
  2. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (commutative)
  3. $\boldsymbol{a} \cdot \boldsymbol{a} \geq 0$, and $\boldsymbol{a} \cdot \boldsymbol{a} = 0$ if and only if $\boldsymbol{a} = \boldsymbol{0}$. (positive-definite)
  4. $(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$ (distributive)
  5. $|\boldsymbol{a} \cdot \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$ (Cauchy-Schwarz inequality)
  6. $|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$ (triangle inequality)
  7. $|\boldsymbol{a} + \boldsymbol{b}|^2 + |\boldsymbol{a} - \boldsymbol{b}|^2 = 2(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)$ (parallelogram equality)

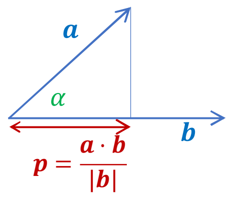
Inner Product

- Example 1 $\boldsymbol{a} = (1, 2, 0)$ and $\boldsymbol{b} = (3, -2, 1)$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$
  - $|\boldsymbol{a}| = |\boldsymbol{a} \cdot \boldsymbol{a}| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$
  - $\cos\gamma = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = -\frac{1}{\sqrt{70}}$
- Example 2 Work done by a force
  - Work done by $\boldsymbol{p}$ in the displacement $\boldsymbol{d}$
    - $W = \boldsymbol{p} \cdot \boldsymbol{d} = |\boldsymbol{p}||\boldsymbol{d}| \cos\alpha$
    - When $\boldsymbol{p} \perp \boldsymbol{d}$, $W = 0$.

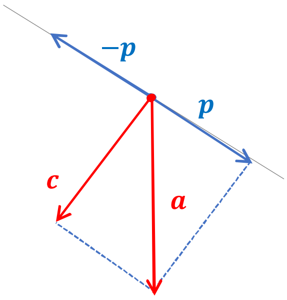


Projection

- Orthogonal projection
  - Projection of a vector $\boldsymbol{a}$ in the direction of a vector $\boldsymbol{b}$ ($\boldsymbol{b} \neq \boldsymbol{0}$)



- Example 3 Component of a force in a given direction
  - Weight, $\boldsymbol{a} = (0, -5000)$ (gravity force heading downward)
  - $\boldsymbol{a} = \boldsymbol{c} + \boldsymbol{p}$
    - $\boldsymbol{c}$ … force that the car exerts on the ramp
    - $\boldsymbol{p}$ … force parallel to the slope



Orthogonal Vectors

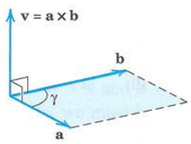
- Example 4 Orthogonal/orthonormal basis
  - The set of 3 vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ in 3-D space is an orthonormal basis, if they are unit vectors and pairwise orthogonal.
    - $\forall \boldsymbol{v}$, $\boldsymbol{v} = \ell_1 \boldsymbol{a} + \ell_2 \boldsymbol{b} + \ell_3 \boldsymbol{c}$, where $\ell_1 = \boldsymbol{a} \cdot \boldsymbol{v}$, $\ell_2 = \boldsymbol{b} \cdot \boldsymbol{v}$, and $\ell_3 = \boldsymbol{c} \cdot \boldsymbol{v}$.
  - Standard unit vectors, $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an orthonormal basis.

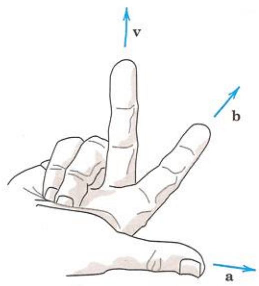
Normal Vectors

- Example 5 Normal vector of a line (in 2-D space)
  - Straight line L_1 through point $P{:}\,(1,3)$ and normal to the line $L_2{:}\, x - 2y + 2 = 0$.
    - Put $L_1{:}\, a_1 x + a_2 y = c$
    - Normal vectors: $\boldsymbol{n}_1 = (a_1, a_2)$ and $\boldsymbol{n}_2 = (1, -2)$
    - $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0 \Rightarrow a_1 - 2a_2 = 0$ and $a_1 + 3a_2 = c$
    - Infinitely many solutions: Choose $a_2 = 2$. $a_1 = 2$ and $c = 5$: $x + 2y = 5$
- Example 6 Normal vector of a plane (in 3-D space)
  - Unit vector perpendicular to the plane, $4x + 2y + 4z + 7 = 0$
    - $\boldsymbol{a} = (4, 2, 4)$ and $\boldsymbol{n} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ … unit normal vector

Cross Product

- When $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$,
  - $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, if $\boldsymbol{a} = k\boldsymbol{b}$ for some scalar k (same direction).
  - $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$
    - † Length: $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\gamma$
    - † Direction: perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (right-handed rule)





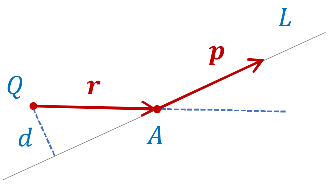
Cross Product

- Example 1 $\boldsymbol{a} = (1, 1, 0)$ and $\boldsymbol{b} = (3, 0, 0)$. Find $\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b}$.
  - $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix}$
- Example 2 Standard basis vectors
  - $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
  - $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$, $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$, $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$

Cross Product: Properties

- For any vectors, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and scalar ℓ
  1. $(\ell\boldsymbol{a}) \times \boldsymbol{b} = \ell(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (\ell\boldsymbol{b})$
  2. $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$ (distributive)
     $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{a} \times \boldsymbol{c})$
  3. $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ (not commutative)
  4. $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ (not associative)

Cross Product



- Example 3 Moment of force
  - Moment m of a force $\boldsymbol{p}$ about a point Q: $m = |\boldsymbol{p}|d$
    - † d … distance between Q and the line of action L of $\boldsymbol{p}$
  - Let $\boldsymbol{r} = \overrightarrow{QA}$. Then $d = r \cdot \sin\gamma$ and $m = |\boldsymbol{r}||\boldsymbol{p}|\sin\gamma = |\boldsymbol{r} \times \boldsymbol{p}|$.
  - $\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{p}$ … moment vector of $\boldsymbol{p}$ about Q.
- Example 5 Velocity of a rotating arm
  - $\boldsymbol{\omega}$ … direction (axis or rotation) and length (angular speed)
  - $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$
    - † $\boldsymbol{r} = \overrightarrow{OP}$ … position vector of P, P is a point in B with a speed ωd.
      - $d = |r|\sin\gamma$ … distance from axis to P
      - $\omega d = |\boldsymbol{\omega}||\boldsymbol{r}|\sin\gamma = |\boldsymbol{\omega} \times \boldsymbol{r}|$

Scalar Triple Product

- Given vectors, $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$, and $\boldsymbol{c} = (c_1, c_2, c_3)$,
  - $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
  - $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$
    - Geometrical meaning: Absolute value $|(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})|$ is the volume of the parallelepiped with $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ as edge vectors.
    - Linear independence: Three vectors in $\mathbb{R}^3$ are linearly independent if and only if their scalar triple product is not zero.

Vector Functions

- Vector function is a function whose values are vectors;
  - $\boldsymbol{v} = \boldsymbol{v}(P) = \big(v_1(P), v_2(P), v_3(P)\big)$
  - A vector function defines a vector field in a domain of interest.
    - *e.g.*, field of tangent vectors of a curve, normal vectors of a surface, and velocity field of a rotating body
- Example 2 Vector field of rotation
  - Vector field with velocity vectors $\boldsymbol{v}(P)$ of a rotating body B.
  - $\boldsymbol{v}(x, y, z) = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix}$
    - $\boldsymbol{r} = (x, y, z)$ … position vector of a point $P \in B$

Vector Calculus

- Convergence
  - An infinite sequence of vectors $\boldsymbol{a}_n$ is said to converge, if $\exists \boldsymbol{a}$, such that $\lim_{n\to\infty} |\boldsymbol{a}_n - \boldsymbol{a}| = \boldsymbol{0}$.
    - † Component-wise convergence
  - A vector function $\boldsymbol{v}(t)$ of a real variable t is said to have the limit ℓ as $t \to t_0$, if it is defined in some neighborhood of t_0 and $\lim_{t\to t_0} |\boldsymbol{v}(t) - \ell| = \boldsymbol{0}$.
- Continuous
  - A vector function $\boldsymbol{v}(t)$ is said be continuous at $t = t_0$, if it is defined in some neighborhood of t_0 (including t_0) and $\lim_{t\to t_0} \boldsymbol{v}(t) = \boldsymbol{v}(\boldsymbol{t}_0)$.
    - † If $\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$, $\boldsymbol{v}(t)$ is continuous at $t = t_0$, if and only if each component is continuous at $t = t_0$.

Derivative of a Vector Function

- Differentiable
  - A vector function $\boldsymbol{v}(t)$ is said to be differentiable at a point t, if the limit exists:

    $\boldsymbol{v}'(t) = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t)}{\Delta t}$
    - † The vector $\boldsymbol{v}'(t)$ is called the derivative of $\boldsymbol{v}(t)$.
    - † If $\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$, $\boldsymbol{v}'(t) = (v_1'(t), v_2'(t), v_3'(t))$
- Derivative: Properties
  1. $(c\boldsymbol{v})' = c\boldsymbol{v}'$, for some scalar c
  2. $(\boldsymbol{u} + \boldsymbol{v})' = \boldsymbol{u}' + \boldsymbol{v}'$
  3. $(\boldsymbol{u} \cdot \boldsymbol{v})' = \boldsymbol{u}' \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}'$ … chain rule
  4. $(\boldsymbol{u} \times \boldsymbol{v})' = \boldsymbol{u}' \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{v}'$
  5. $(\boldsymbol{u}\,\boldsymbol{v}\,\boldsymbol{w})' = (\boldsymbol{u}'\,\boldsymbol{v}\,\boldsymbol{w}) + (\boldsymbol{u}\,\boldsymbol{v}'\,\boldsymbol{w}) + (\boldsymbol{u}\,\boldsymbol{v}\,\boldsymbol{w}')$

Partial Derivative

- If $\boldsymbol{v}(t) = \big(v_1(t_1, t_2), v_2(t_1, t_2), v_3(t_1, t_2)\big)$, partial derivatives
  - $\frac{\partial \boldsymbol{v}}{\partial t_1} = \left(\frac{\partial v_1}{\partial t_1}, \frac{\partial v_2}{\partial t_1}, \frac{\partial v_2}{\partial t_1}\right)$
  - $\frac{\partial^2 \boldsymbol{v}}{\partial t_1 \partial t_2} = \left(\frac{\partial^2 v_1}{\partial t_1 \partial t_2}, \frac{\partial^2 v_2}{\partial t_1 \partial t_2}, \frac{\partial^2 v_2}{\partial t_1 \partial t_2}\right)$
- Example 4 For a vector function $\boldsymbol{v}(t)$ with $|\boldsymbol{v}(t)| = c$ (const.)
  - $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = c^2$
  - $(\boldsymbol{v} \cdot \boldsymbol{v})' = 2\boldsymbol{v} \cdot \boldsymbol{v}' = \frac{dc^2}{dt} = 0 \Rightarrow$ either $\boldsymbol{v} = 0$ or $\boldsymbol{v} \perp \boldsymbol{v}'$
- Example 5 $\boldsymbol{r}(t_1, t_2) = (a \cdot \cos t_1, a \cdot \sin t_1, t_2)$
  - $\frac{\partial \boldsymbol{r}}{\partial t_1} = (-a \cos t_1, a \sin t_1, 0)$, $\frac{\partial \boldsymbol{r}}{\partial t_2} = (0, 0, 1)$

Curves

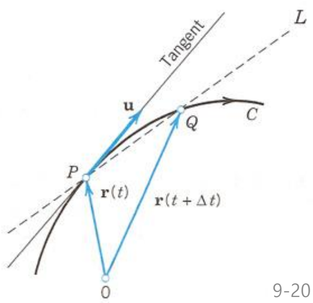
- Differential geometry
  - Application of vector calculus to study problems in geometry
  - It plays an important role in mechanics.
- Parametric representation of a curve or path C
  - $\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$, where t … parameter

Curves: Examples

- Circle in xy-plane: $C: x^2 + y^2 = 4$
  - † $\boldsymbol{r}(t) = (2\cos t, 2\sin t, 0)$, where $0 \le t < 2\pi$ (note on the orientation of curve)
- Ellipse in xy-plane: $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$
  - † $\boldsymbol{r}(t) = (a\cos t, b\sin t, 0)$, where $0 \le t < 2\pi$
- Line L through a point A in the direction of $\boldsymbol{b}$ (direction vector)
  - † $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{b} = (a_1 + tb_1, a_2 + tb_2, a_3 + tb_3)$, where $-\infty < t < \infty$ and $\boldsymbol{a} = \overrightarrow{OA}$
- Circular helix
  - † $\boldsymbol{r}(t) = (a\cos t, a\sin t, ct)$, where $0 \le t < 2\pi$
    - – Circle in xy-place

Tangent to a Curve

- Simple curve
  - A curve without any intersection
  - An arc of a curve is the portion between any two points of the curve.
- Tangent to a simple curve C at a point P in C
  - The limiting position of a straight line L through P and Q, as $P \to Q$ along C
  - $\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}$
  - Unit tangent vector, $\boldsymbol{u} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$



Arc Length

- Length of a curve
  - $\ell = \int_a^b \sqrt{\boldsymbol{r}' \cdot \boldsymbol{r}'}\,dt$, where $\boldsymbol{r}' = \frac{d\boldsymbol{r}}{dt}$
- Arc length of a curve
  - $s(t) = \int_a^t \sqrt{\boldsymbol{r}' \cdot \boldsymbol{r}'}\,d\tau$
- Line element
  - $\left(\frac{ds}{dt}\right)^2 = \frac{d\boldsymbol{r}}{dt} \cdot \frac{d\boldsymbol{r}}{dt} = |\boldsymbol{r}'(t)|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$
  - $ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = dx^2 + dy^2 + dz^2$
    - † $\boldsymbol{r}'(s) = \frac{d\boldsymbol{r}}{ds} = \frac{d\boldsymbol{r}}{dt} \cdot \frac{dt}{ds} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \boldsymbol{u}(s)$

1. Independent variable s (arc length)
2. $|\boldsymbol{r}'(s)| = 1$

Curves in Mechanics

1. For $\boldsymbol{v}(t)$ with $|\boldsymbol{v}| = c$, either $\boldsymbol{v}' \perp \boldsymbol{v}$ or $\boldsymbol{v} = \boldsymbol{0}$.
2. $|\boldsymbol{u}(s)| = 1$ and $\boldsymbol{u}(s) \neq \boldsymbol{0} \Rightarrow \frac{d\boldsymbol{u}}{ds} \perp \boldsymbol{u}(s)$

- Curve $\boldsymbol{r}(t)$ or C: Path of moving body
  - Velocity, $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$
    - † $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds} \cdot \frac{ds}{dt} = \boldsymbol{u}(s)\frac{ds}{dt}$
  - Acceleration, $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$
    - † $\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left(\boldsymbol{u}(s)\frac{ds}{dt}\right) = \left(\frac{d\boldsymbol{u}}{ds} \cdot \frac{ds}{dt}\right) \cdot \frac{ds}{dt} + \boldsymbol{u}(s)\frac{d^2s}{dt^2} = \frac{d\boldsymbol{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \boldsymbol{u}(s)\frac{d^2s}{dt^2}$
    - † $\boldsymbol{a}(t) = \boldsymbol{a}_{tan} + \boldsymbol{a}_{norm}$
      - $\boldsymbol{a}_{tan} = \boldsymbol{u}(s)\frac{d^2s}{dt^2} = \frac{\boldsymbol{a} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\boldsymbol{v}$ … tangential acceleration vector
      - $\boldsymbol{a}_{norm} = \frac{d\boldsymbol{u}}{ds}\left(\frac{ds}{dt}\right)^2$ … normal acceleration vector ($\boldsymbol{a}_{norm} \perp \boldsymbol{a}_{tan}$)

Curves in Mechanics

- Example Centripetal Acceleration
  - A body moving along the path C: $\boldsymbol{r}(t) = (R\cos\omega t, R\sin\omega t)$ … circle of radius R
  - $\boldsymbol{v} = \boldsymbol{r}' = (-\omega R\sin\omega t, \omega R\cos\omega t)$
    - $\boldsymbol{v}$ is tangent to C.
    - $|\boldsymbol{v}| = |\boldsymbol{r}'| = \sqrt{\boldsymbol{r}'\cdot\boldsymbol{r}'} = \omega R$ … constant speed, where ω … angular speed
  - $\boldsymbol{a} = \boldsymbol{v}' = (-\omega^2 R\cos\omega t, -\omega^2 R\sin\omega t) = -\omega^2 \boldsymbol{r}$
    - Acceleration toward the center, centripetal acceleration.

Gradient

- For a scalar function f
  - $grad(f) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
    - $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$ … differential operator
  - Directional derivative of $f(x, y, z)$ at a point P in the direction of $\boldsymbol{b}$:
    - $D_{\boldsymbol{b}} f = \frac{df}{ds} = \lim_{s\to 0}\frac{f(Q)-f(P)}{s}$
      - Q … a variable point on the straight line L in the direction of $\boldsymbol{b}$
      - $|s|$ … distance between P and Q
    - When L: $\boldsymbol{r}(s) = (x(s), y(s), z(s)) = \boldsymbol{p}_0 + s\boldsymbol{b}$, where $|\boldsymbol{b}| = 1$ and $\boldsymbol{p}_0$ is the position vector for the point P,
    - $D_{\boldsymbol{b}} f = \frac{df}{ds} = \frac{df}{dx}x' + \frac{df}{dy}y' + \frac{df}{dz}z' = \boldsymbol{b}\cdot grad(f)$

$\boxed{\boldsymbol{r}'(s) = \boldsymbol{u}(s) = \boldsymbol{b}}$

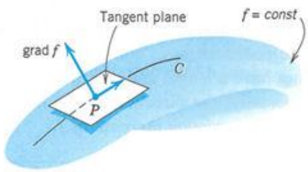
Gradient

- Example 1 $f(x,y,z) = 2x^2 + 3y^2 + z^2$, $P: (2,1,3)$, and $\boldsymbol{a} = (1,0,-2)$
  - $grad(f) = (4x, 6y, 2z)$, $grad\big(f(P)\big) = (8,6,6)$
  - $D_{\boldsymbol{a}}f = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot grad(f) = \frac{1}{\sqrt{5}}(1,0,-2) \cdot (8,6,6) = -1.789$
- Theorem 1. Maximum increase

  Let $f(P) = f(x,y,z)$ be a scalar function having continuous partial derivatives in some domain B. Then $grad(f)$ exists in B and it has the direction of maximum increase of f at B.
  - $D_{\boldsymbol{b}}f = \boldsymbol{b} \cdot grad(f) = |\boldsymbol{b}||grad(f)| \cos\gamma = |grad(f)| \cos\gamma$
  - $D_{\boldsymbol{b}}f$ attains its maximum $|grad(f)|$, when $\gamma = 0 \Rightarrow \boldsymbol{b} = k \cdot grad(f)$ (same direction)

Gradient: Surface Normal Vector

- Level surface S of f: a surface represented by $f(x,y,z) = c$ (const).
  - Tangent plane of S at P … A plane formed by the tangent vectors of all curves on S passing through P.
  - Surface normal to S at P … The straight line through P perpendicular to the tangent plane.
  - Surface normal vector of S at P … A vector in the direction of the surface normal.
  - $f(x,y,z) = c \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = grad(f) \cdot \boldsymbol{r}' = 0$
    - † $grad(f)$ is orthogonal to all vectors $\boldsymbol{r}'$ in tangent plane.
    - † $grad(f)$ is a normal vector of S at P.

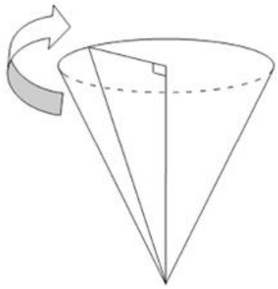


Surface Normal Vector

- Theorem 2. Gradient as surface normal vector

  Let $f(x,y,z)$ be a differential scalar function and let $f(x,y,z) = c$ (const.) represent a surface S. Then, if the gradient of f at a point P of S is not the zero vector, it is normal vector of S at P.

- Example 2 Cone $z^2 = 4(x^2+y^2)$, $P: (1,0,2)$
  - Level surface: $f(x,y,z) = 4(x^2+y^2) - z^2$
  - $grad(f) = (8x, 8y, -2z)$, $grad(f(P)) = (8, 0, -4) = \boldsymbol{g}$
  - $\boldsymbol{n} = \frac{\boldsymbol{g}}{|\boldsymbol{g}|} = \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)$



Vector Field from Gradient

- Gradient of a scalar function generates a vector field: Potentials.
  - Vector field, $\boldsymbol{v}(P) = grad\big(f(P)\big)$
    - † $f(t)$ is potential function of $\boldsymbol{v}(P)$.
    - † Vector field is conservative: Energy is conserved in a vector field.
- Theorem 3. Laplace Equation

  The force of attraction, $\boldsymbol{p} = -\frac{c}{r^3}\boldsymbol{r} = -c\left(\frac{x-x_0}{r^3}, \frac{x-x_0}{r^3}, \frac{x-x_0}{r^3}\right)$ between two particles at $P_0: (x_0, y_0, z_0)$ and $P: (x,y,z)$ has the potential $f(x,y,z) = \frac{c}{r}$.
    - † r … distance between P_0 and P, $r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$
    - † $\boldsymbol{p} = grad(f) = grad\left(\frac{c}{r}\right)$
    - † $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ … The potential f is a solution of Laplace equation.

Divergence

- Divergence of a vector function $\boldsymbol{v} = (v_1, v_2, v_3)$
  - $div(\boldsymbol{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$
    - $div(\boldsymbol{v}) = \nabla \cdot \boldsymbol{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (v_1, v_2, v_3)$
    - If $\boldsymbol{v} = grad(f)$,
      - $div(\boldsymbol{v}) = div\big(grad(f)\big) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$
    - *e.g.*, if $\boldsymbol{v} = (3xz, 2xy, -yz^2)$, $div(\boldsymbol{v}) = 3x + 2x - 2yz$

- Theorem 1. Invariance of divergence

  The divergence, $div(\boldsymbol{v})$, is a scalar function and its values are independent of the choice of coordinates.

Divergence

Divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

- Example 2. Flow of compressible fluid
  - Motion of a fluid in a region R (without source of sink in R)
    - Flow through a rectangular box B
    - Velocity vector of the motion, $\boldsymbol{v}(t)$
  - $div(\boldsymbol{u}) = div(\rho \boldsymbol{v}) = -\frac{\partial \rho}{\partial t}$ … continuity equation
    - ρ … density of fluid
    - For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and $div(\boldsymbol{v}) = 0$

Consider air as it is heated. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value.

Curl

- Curl of a vector function $\boldsymbol{v} = (v_1, v_2, v_3)$
  - ∨ $curl(\boldsymbol{v}) = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\boldsymbol{k}$
- Example 1 $\boldsymbol{v} = (yz, 3zx, z)$
  - ∨ $curl(\boldsymbol{v}) = (-3x, y, 2z)$
- Example 2 Rotation of rigid body
  - ∨ $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = (0, 0, \omega) \times (x, y, z) = (-\omega y, \omega x, 0)$
  - ∨ $curl(\boldsymbol{v}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = (0, 0, 2\omega) = 2\boldsymbol{\omega}$

Curl

- Theorem 1. Rotating rigid body

  The curl of the velocity field of a rotating rigid body is $curl(\boldsymbol{v}) = (0, 0, 2\omega) = 2\boldsymbol{\omega}$.

  - ∨ $curl(grad(f)) = \boldsymbol{0}$ ... gradient fields are irrotational.
  - ∨ $div(curl(\boldsymbol{v})) = 0$ ... divergence of the curl of a vector function is zero.