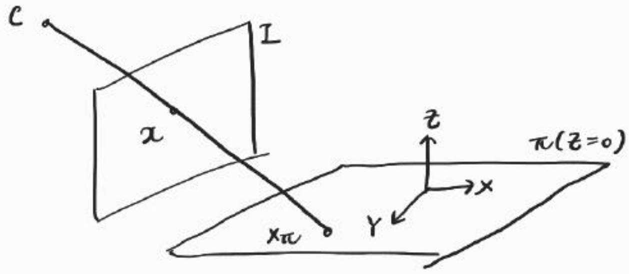
• Action of projective camera on planes



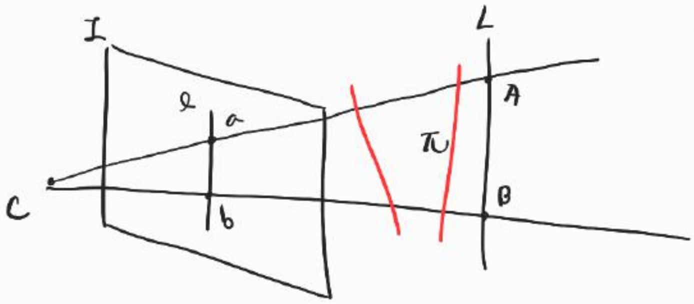
$$x = PX = [p_1\ p_2\ p_3\ p_4]\begin{pmatrix} X \\ Y \\ 0 \\ 1 \end{pmatrix} = [p_1\ p_2\ p_4]\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

Homography matrix (full rank 3)

P^2(XY-plane) $\mapsto$ P^2(image)

H calculated using 4 point correspondences $\because$ mapping b/w $x_\pi = \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$ and x is planar homography.
(plane-to-plane)

• Action of projective camera on lines



◦ Forward projection

3D-space line (L) $\mapsto$ image line (l)

line l = intersection of π and image plane

Since $L = X(\mu) = A + \mu B$,

$l = x(\mu) = PX(\mu) = PA + P\mu B = a + \mu b$

$\Rightarrow$ l is line joining a and b

◦ Backward projection

Impossible to recover L directly from l

↳ Every line in π (e.g. red lines) projects to l

Set of points mapping to line l is the plane $\pi = P^T l$

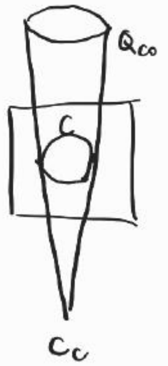
$\because$ if point x lies on l, $x^T l = 0$

Since $x = PX$, $x^T l = (PX)^T l = X^T \underbrace{P^T l}_{=\pi} = 0$

$\therefore$ $X^T \pi = 0$ → point X in 3-space lies on π

• Action of Projective camera on conics

◦ Back projection of conics



$Q_{co} = P^T C P$ where $Q_{co} \in \mathbb{R}^{4\times4}$, $C \in \mathbb{R}^{3\times3}$

↳ Conic C back-projects to <u>degenerate</u> quadric (cone) Q_{co}

$\therefore$ Q_{co} isn't full rank

$\because$ $x^T C x = 0$ and $x = PX$

$\Rightarrow$ $(PX)^T C (PX) = X^T (P^T C P) X = 0$ $\therefore$ $Q_{co} = P^T C P$ Eq(1)

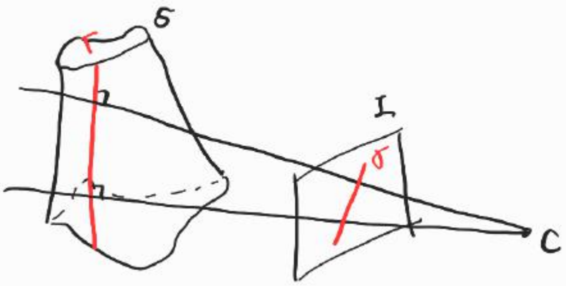
↳ camera center is null-vector of quadric i.e. $Q_{co} C_c = 0$ Eq(2) with Eq(1) and Eq(2), $Q_{co}C = (P^T C P) C = P^T C \underbrace{(PC)}_{=0} = 0$

e.g. if $P = K[I|0]$ (canonical frame : camera = world frame ∵ Extrinsic $= I$)

$$Q_{co} = P^T C P = \begin{bmatrix} K^T \\ 0^T \end{bmatrix} C [K|0] = \begin{bmatrix} K^T C K & 0 \\ 0^T & 0 \end{bmatrix}$$

∴ $\text{rank}(Q_{co}) = 3$

- Images of smooth surfaces



Central projection
(projective camera)

Image outline of smooth surface S

→ forward projection of surface points (tangent to rays)

→ backward projection of tangent planes to surface

- Contour generator Γ

  set of points X on S that are tangent to rays

  depends on position of C and S

- apparent contour (outline/profile) γ

  set of points x that are image of X

  depends on position of I

- Action of projective camera on quadrics
  - Forward projection of quadric

$C^* = P Q^* P^T$ i.e. defined by tangent of lines(C^*) and planes(Q^*)

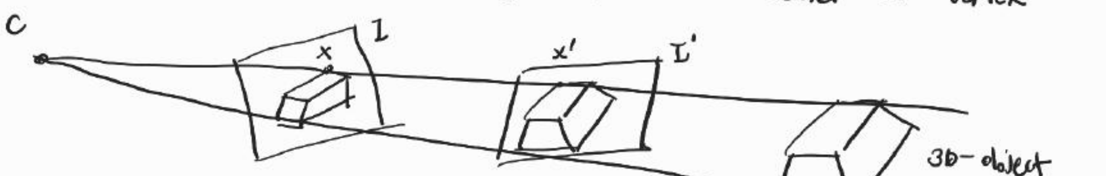
∵ $l^T C^* l = \pi^T Q^* \pi = 0$ and $l = P\pi \rightarrow \pi = P^T l$

$$\pi^T Q^* \pi = l^T \underbrace{P Q^* P^T}_{C^* = PQ^*P^T} l = l^T C^* l = 0$$

- Importance of camera center

  set of rays : defined by 3D object and camera center → image = rays intersecting p2 plane

  ↳ refered to cone of rays w/ camera center as vertex



$I' = HI$

$x' = Hx$

* Projection affected by R, C, f, I ...

Images (I, I') mapped from same camera center (C) can be defined as homography

$\Rightarrow$ Projective equivalent

Let $P = KR[I \mid -\tilde{C}]$, $P' = K'R'[I \mid -\tilde{C}]$ i.e. camera center is equal

$\therefore P' = (K'R')(KR)^{-1}P$

Similarly, $x' = P'X = (K'R')(KR)^{-1}PX = \underbrace{(K'R')(KR)^{-1}}_{\text{homography } H \ (P^2 \mapsto P^2)}x$

(i) Translation of image plane (δf)

↳ simple magnification effect

$x = PX = K[I \mid 0]X$

$x' = P'X = \underbrace{K'}_{\text{only } f \text{ differs}}[I \mid 0]X = K'(K^{-1}K)[I \mid 0]X = K'K^{-1}(K[I \mid 0]X) = K'K^{-1}x$

$\therefore x' = Hx$ where $H = K'K^{-1}$

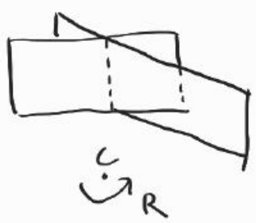
$H = K'K^{-1} = \begin{bmatrix} rI & (1-r)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix}$ where $\tilde{x}_0 = (\tilde{x}_0, \tilde{y}_0)^T$
$r = f'/f$ is magnification factor

$\therefore K^{-1} = \begin{bmatrix} rI & (1-r)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix} K = \begin{bmatrix} rI & (1-r)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} A & \tilde{x}_0 \\ 0^T & 1 \end{bmatrix}$ where $A = \mathrm{diag}(f, f, 1)$

$= \begin{bmatrix} rA & \tilde{x}_0 \\ 0^T & 1 \end{bmatrix} = K \begin{bmatrix} rI & \\ & 1 \end{bmatrix}$

$\Rightarrow$ effect of zooming by factor $r = f/f'$

(ii) Camera rotation



Rotation at center C w/o change in internal parameters

$x = PX = K[I \mid 0]X$ e.g. canonical case

$x' = P'X = K[R \mid 0]X = KR(K^{-1}K)I[R \mid 0]X = KRK^{-1}x$

$\therefore H = KRK^{-1}$ called conjugate rotation

↳ H, R shares equal eigenvalues up to scale $\{\mu, \mu e^{i\theta}, \mu e^{-i\theta}\}$

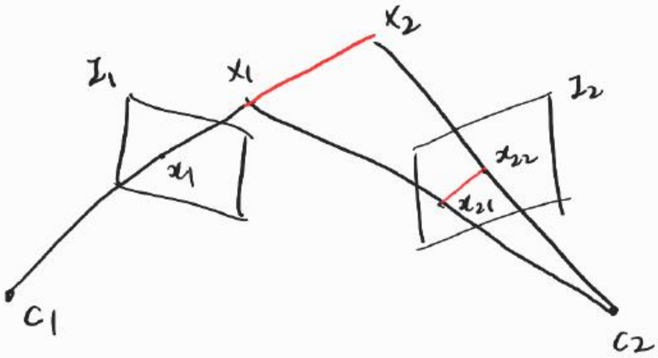
steps : take 4 point correspondences of two image view $\rightarrow$ calculate H

→ Calculate eigenvalue of H → Derive θ

(iii) Moving camera center (motion parallex)

fixed camera center (case (i), (ii)) : No info. of 3d space structure

only $x' = Hx$



ΔC ($C_1 \to C_2$)

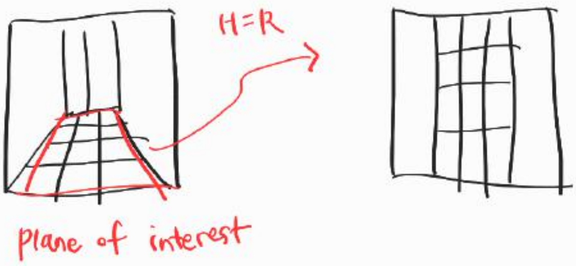
Image points now also depends on 3D point

($X_1 \to X_2$ ↔ $x_{21} \to x_{22}$ in I_2)

- Applications and Examples (fixed camera center)

Ex(1) Synthetic views

warping planar homography from plane of interest



- Algorithm

1) Compute H from quadrilateral (source) to rectangle (target) w/ known aspect ratio

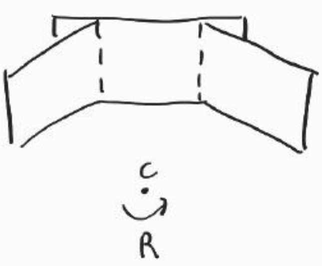
↳ usually take 4 corners as point correspondences ($X' = HX$)

2) projectively warp source image

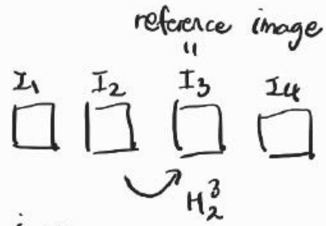
↳ lookup table $x = H^{-1}x'$ so that all pixels of target image contains a value

Ex(2) planar panoramic Mosaicing

set of images that camera rotates at center → Panorama image stitching



- Algorithm



1) choose a reference image
2) Calculate H that maps one image to reference image
3) warp non-reference images using H, augment non-overlapping regions of reference image

* If no point correspondences w/ ref. image, use property

e.g. $H_1^3 = H_1^2 \cdot H_2^3$ $\because$ linear mapping

- Usage of calibration $K_{3\times3}$

(i) Direction of rays

In E^3 space, points on the ray : $\tilde{X} = \lambda d$ (inhomogenous form)

Points $\tilde{X}$ map into $x = K[I|0]\underbrace{(\lambda d^T, 1)^T}_{X} = Kd$

$\therefore$ $d = K^{-1}x$ $\rightarrow$ direction of ray obtained from image point (x unit vector)

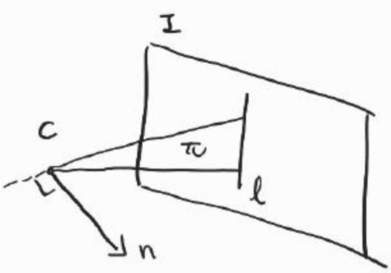
(ii) Angle b/w two rays

$$\cos\theta = \frac{d_1^T d_2}{\|d_1\|\|d_2\|} = \frac{(K^{-1}x_1)^T(K^{-1}x_2)}{\|K^{-1}x_1\|\|K^{-1}x_2\|} = \frac{x_1^T(K^{-T}K^{-1})x_2}{\sqrt{x_1^T(K^{-T}K^{-1})x_1}\sqrt{x_2^T(K^{-T}K^{-1})x_2}}$$

$\therefore$ Calibrated camera work as directional sensor / 2d protractor

$\rightarrow$ find angle b/w any two image points

(iii) Normal of back-projected plane



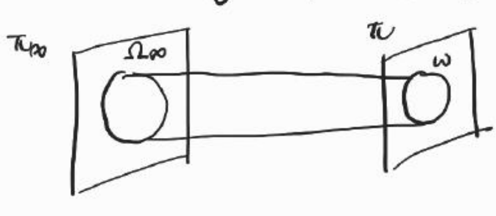
$n = K^T l$

$\because$ Points x on l back-projects to $d = K^{-1}x$

$d \perp n$ $\Rightarrow$ $d^T n = (K^{-1}x)^T n = x^T(K^{-T}n) = 0$

Since $x^T l = 0$, $l = K^{-T}n$ $\therefore$ $n = K^T l$

- (iv) The image of absolute conic



ω : image of absolute conic (IAC)

Not actually visible at real plane ($\because$ all imaginary points)

points on π_∞ : $X_\infty = (d^T, 0)^T$ w/ general camera $P = KR[I \mid -\tilde{C}]$,

$x = PX_\infty = KR[I \mid -\tilde{C}] \begin{pmatrix} d \\ 0 \end{pmatrix} = \underline{KR}d$ ∴ independent of c/t

↳ P2 → P2 homography H

ω (IAC) $= (KK^T)^{-1} = K^{-T}K^{-1}$

∵ for $x' = Hx$, $C' = H^{-T}CH^{-1}$

since Ω_∞ is $C = I$ at π_∞, $\omega = H^{-T}CH^{-1} = (KR)^{-T}I(KR)^{-1} = K^{-T}RR^{-1}K^{-1} = (KK^T)^{-1}$

↳ from equation (ii)

$$\cos\theta = \frac{x_1^T \omega x_2}{\sqrt{x_1^T\omega x_1}\sqrt{x_2^T\omega x_2}}$$

→ unchanged under projective transformation

$x_1^T \omega x_2 \xrightarrow{H} x_1'^T \omega' x_2'$

$= (x_1^T H^T)(H^{-T}\omega H^{-1})(Hx_2) = x_1^T \omega x_2$

↓

if $x_1 \perp x_2$, $x_1^T \omega x_2 = 0$

* Dual image of absolute conic (DIAC)

$C^* = C^{-1}$ when C is full rank (non-degenerate)

Since $\omega = K^{-T}K^{-1}$, since K is full rank, ω is always full rank

∴ ω^* (DIAC) $= \omega^{-1} = KK^T$

↳ this is dual (line) conic, although not visible in real plane

Also, ω^* = image of Q^*_∞ (dual absolute conic) $= PQ^*_\infty P^T$

☆ Given ω^*, K can be uniquely derived using cholesky decomposition

cholesky(ω^*) = UL = KK^T

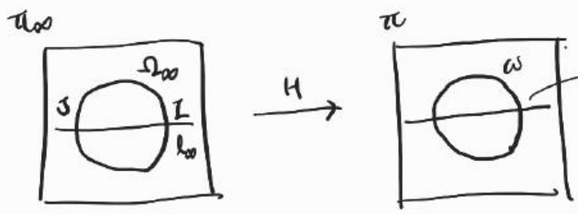
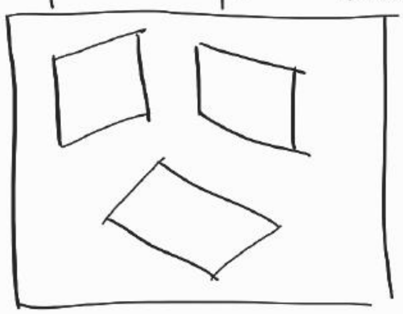


Image circular points

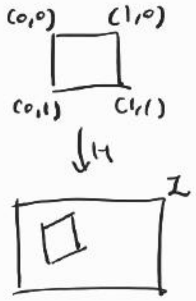
$= \omega \cap (\pi$ vanishing line)

• Example : A simple Calibration Device



If three planes aren't parallel, able to find K (calibration)

- Calibration algorithm



1) For each square, compute H that maps 4 corners to image points

   * Circular points are invariant to similarity transformation

     ∴ Absolute scale of rectangle / corner is X important

2) Compute imaged circular points using $H = [h_1, h_2, h_3]$

   $H \cdot$ Circular points at $\pi_\infty) = H(1, \pm i, 0)^T = h_1 \pm i h_2$

3) since imaged circular points lies on IAC :

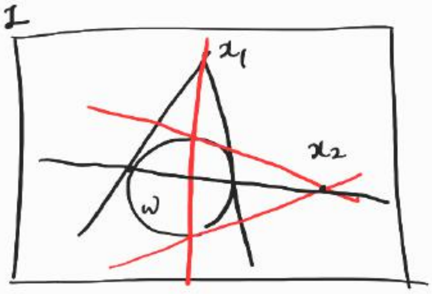
   $(h_1 \pm i h_2)^T \omega (h_1 \pm i h_2) = 0$ $\left[\begin{array}{l} h_1^T \omega h_2 = 0 \\ h_1^T \omega h_1 = h_2^T \omega h_2 \end{array} \right]$ 1 pair of circular points give 2 constraints

   linear r.t. ω

   ω has 6 unknowns (5 dof), requires three unique homography / square

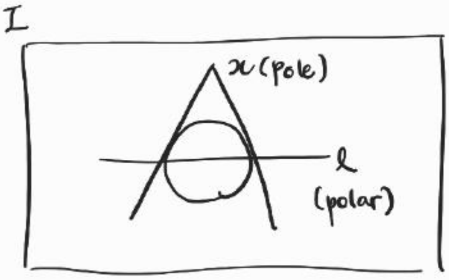
4) Compute K using cholesky factorization of $\omega = (KK^T)^{-1}$

- Orthogonality and ω



x_1, x_2 back-project to orthogonal rays if they are conjugate r.t. ω $\rightarrow x_1^T \omega x_2 = 0$

$\downarrow$ x_2 lies on l

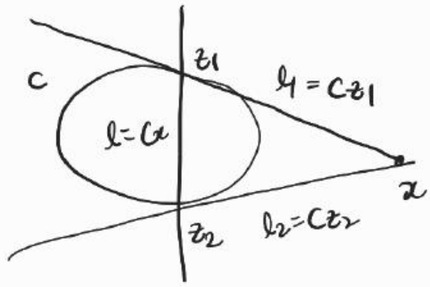


Back-projection of x (ray) and l (plane) is orthogonal if they are pole-polar $\rightarrow l = \omega x$

- pole-polar relationship

  if $l = Cx$, $\left[\begin{array}{l} x^T C x = 0 \text{ : } l \text{ is tangent line} \\ x^T C x \neq 0 \text{ : } l \text{ is polar line (pass two tangent points)} \end{array} \right.$

∵

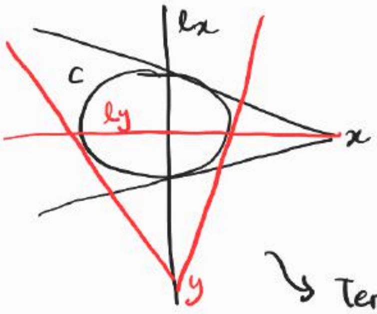


$x = l_1 \times l_2 = (Cz_1) \times (Cz_2) = \det(C) \underbrace{(C^{-1})^T}_{C \text{ is symmetric } = C^{-1}} \underbrace{(z_1 \times z_2)}_{= l}$

∴ $x = \det(C) C^{-1} l = k C^{-1} l$

$l = Cx$

- Conjugate points



As y lies on $l_x = Cx$, $y^T l_x = y^T C x = 0$

$\therefore y^T C x = 0$ means x, y are conjugate w.r.t. C

* $y^T C x = x^T C y = 0$ (symmetric)

↳ Terminology : $(x, l_x), (y, l_y)$ are (pole, polar) pair
x, y are conjugate

* Dual conjugacy for lines : lines l, m are conjugate if $l^T C^* m = 0$

- Vanishing points
  ↳ Intersection of image plane w/ ray parallel to world line passing camera center
  depends only on ray direction → Parallel lines share single vanishing point
  Join of any 2 vanishing points → vanishing line ($l = v_1 \times v_2$)

- Algebraic interpretation

$X(\lambda) = A + \lambda D$ where $D = (d^T, 0)^T$

$x(\lambda) = PX(\lambda) = K[I|0]X(\lambda) = K[I|0]A + \lambda K[I|0]D$
$= a + \lambda K d$

Vanishing point is at :

$v = \lim_{\lambda \to \infty} x(\lambda) = \lim_{\lambda \to \infty} a + \lambda K d = \lim_{\lambda \to \infty} \frac{a}{\lambda} + Kd = Kd$

$\Rightarrow v$ depends on direction d, not position A

- Geometric interpretation
  Image of intersection b/w π_∞ and set of parallel lines w/ direction d

  $v = PX_\infty = K[I|0]\begin{pmatrix} d \\ 0 \end{pmatrix} = Kd$

  * parallel image lines might not be parallel in 3-space
  e.g. any intersecting lines at principal plane ($z=0$) imaged as parallel lines
  $\because$ intersecting point is X_∞

e.g. rotation estimation from vanishing points

$P = K[I|0]$ $P' = K[R|t]$

$v = PX_\infty = K[I|0](d^T\ 0)^T = Kd$ $\therefore d = K^{-1}v$

$v' = P'X_\infty = K[R|t](d^T\ 0)^T = KRd = Kd'$ $\therefore d' = K^{-1}v'$

since K, v, v' is known, d, d' can be calculated → Derive R from $d' = Rd$

* $d = \begin{pmatrix} dx \\ dy \\ 1 \end{pmatrix} = K^{-1}v$ gives 2 constraints & R has 3 dof

⇒ it requires at least 2 (v, v') pairs to find R

e.g. angle b/w 2 scene lines

$$\cos\theta = \frac{v_1^T \omega v_2}{\sqrt{v_1^T \omega v_1}\sqrt{v_2^T \omega v_2}}$$

- chicken-and-egg problem

set of imaged parallel scene lines $\xrightarrow{\text{intersection of lines}}$ vanishing points

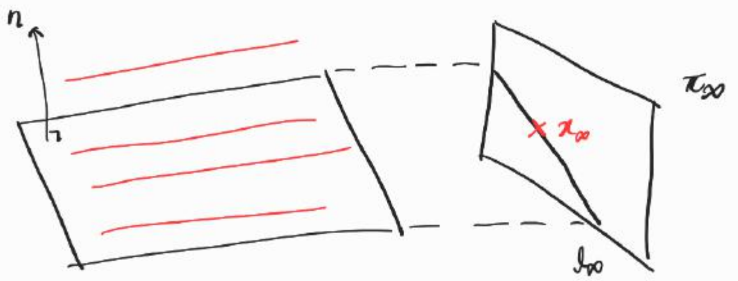
vanishing points $\xrightarrow{\text{lines passing } v}$ set of imaged parallel scene lines

⇒ Both are unknown!

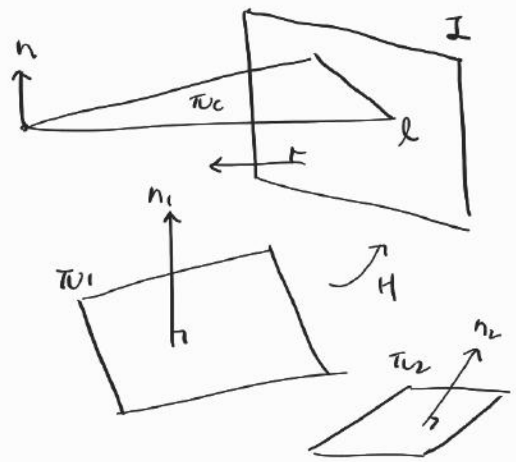
& detected lines inaccuracy due to noise

- Vanishing Lines
  - ↳ Image of line of infinity, which is the intersection of plane of infinity and set of parallel planes in 3-space.
  - ↳ OR, intersection of image plane and plane parallel to scene plane that contains camera center

∴ Dependent on camera center & plane orientation



- Usages (w/ known K)



π (plane w/ vanishing line l)

: $n = K^T l$

π_1 and I are related as plane-to-plane homography

: metrically rectify from n_1 to fronto-parallel scene

Determine angle b/w 2 scene planes

: since $n_1 = K^T l_1$, $n_2 = K^T l_2$.

$$\cos\theta = \frac{l_1^T \omega^* l_2}{\sqrt{l_1^T \omega^* l_1}\sqrt{l_2^T \omega^* l_2}}$$

- Computing vanishing lines
  1) Find v_1, v_2 from 2 set of parallel lines
  2) $l = v_1 \times v_2$

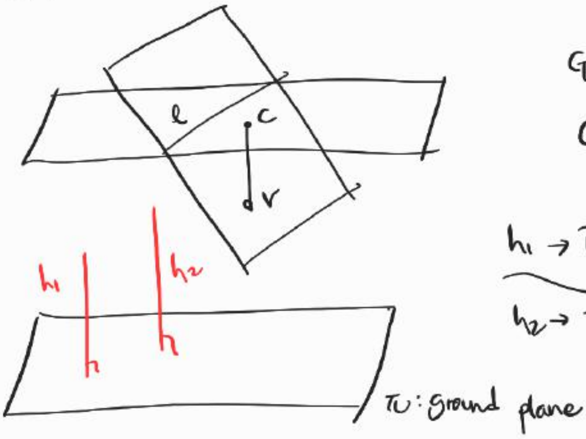
- Orthogonality relationships

  if v_1, v_2 have perpendicular lines, $v_1^T \omega v_2 = 0$

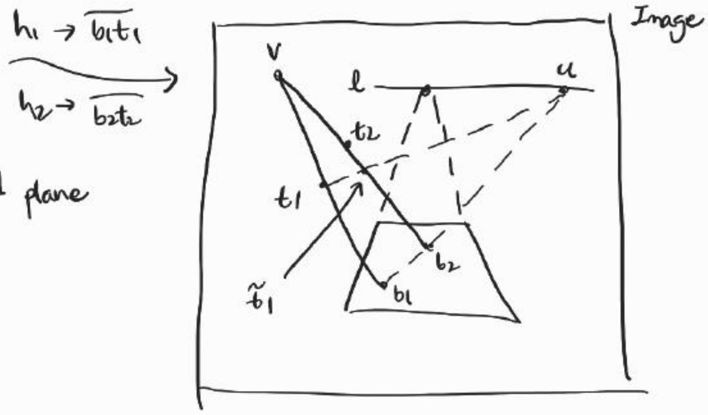
  if l, v have perpendicular plane/line, $l = \omega v$ or $v = \omega^* l$ ($\because \omega^{-1} = \omega^*$)

  if l_1, l_2 have perpendicular planes, $l_1^T \omega^* l_2 = 0$

- Affine 3D Measurements



Given vanishing point v and vanishing line l,
Compute ratio of vertical line segments h_2/h_1



- steps
  1) $u = (b_1 \times b_2) \times l$
  2) $\tilde{t}_1 = (t_1 \times u) \times (v \times b_2)$
  3) Define P^1 space: b_2 $(0,1)$, $\tilde{t}_1$, t_2, v $(v,1)$
  4) Compute $H_{2\times2}$ using $(0,1) \to (0,1)$ and $(v,1) \to (1,0)$ mapping
     (origin: $b_2 \to b_2$) ($v \to x_\infty$)

  $$\therefore H_{2\times2} = \begin{bmatrix} 1 & 0 \\ 1 & -v \end{bmatrix}$$

  5) $$\frac{h_2}{h_1} = \frac{H_{2\times2}(t_2, 1)^T}{H_{2\times2}(\tilde{t}_1, 1)^T}$$

- Determining K

- Linear constraints

  $v_1^T \omega v_2 = 0$: $v_1 \perp v_2$

  $l \times (\omega v) = 0$: $l \perp v$

  $h_1^T \omega h_2 = 0$, $h_1^T \omega h_1 = h_2^T \omega h_2$: known H

  $\omega_{12} = \omega_{21} = 0$: zero skew
  +
  $\omega_{11} = \omega_{22}$: Square pixels

- steps

1) Define $\omega \in R^{1\times 6}$ with 5 dof

2) stack above linear constraints $a^T\omega = 0$ $\rightarrow$ $A^T\omega = 0$ where $A \in R^{n\times 6}$

3) Solve ω, find K using cholesky decomposition : $SVD(\omega) = (KK^T)^{-1}$