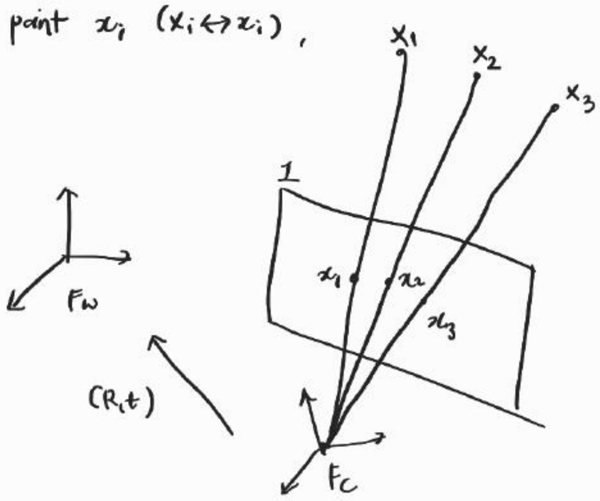
- Perspective pose Estimation

Given 3D points X_i and corresponding image point x_i $(X_i \leftrightarrow x_i)$,
find camera pose (R, t)



$\downarrow$ X_i in F_w, x_i in F_c

As rays intersect at C, it is called
Perspective-n-point (PnP) problem

- Uncalibrated camera case

$P = K[R|t]$ where K, R, t are all unknown

$P \in \mathbb{R}^{3\times4}$ w/ 12 unknowns

$\{X_i \leftrightarrow x_i\}$ $\forall i$, $\gamma_i x_i = PX_i$

$\therefore (\gamma_i x_i) \times (PX_i) = x_i \times (PX_i) = 0 \Rightarrow \begin{bmatrix} & A & \end{bmatrix} \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} = 0$

$A \in \mathbb{R}^{3\times12}$ $\quad$ $p \in \mathbb{R}^{12\times1}$

$\uparrow$ Ignore last row : linearly dependent
$\Rightarrow A \in \mathbb{R}^{2\times12}$

$\downarrow$ For n 3D-2D correspondences

$Ap = 0$ where $A \in \mathbb{R}^{2n\times12}$

Since $\text{rank}(A) = 11$ $(12 - \text{scale}(1))$, at least 6 2D-3D correspondences required

$\downarrow$ $SVD(A) = U\Sigma V^T$

p = last column of V

If collected data has noise (which is most cases), required $n > 6$ correspondences

(i) Minimize algebraic error

Naive method : $\min \|Ap\| \rightarrow p = 0$

↳ Add normalization constraint
- $\|p\| = 1$ (Same as homography / fundamental case)
- $\|\hat{p}_3\| = 1$ where $\hat{p}_3$ is first entries of last row (only applicable in $Ap=0$ case)

(ii) Minimize geometric error

Refine P after (i) using reprojection error

$\min_P \sum_i d(x_i, PX_i)^2$ where x_i is measured image point & PX_i is projected 3D point

Solved using non-linear optimization method

e.g. Gauss-Newton, Levenberg-Marquardt

- Data normalization

  - 2D points (x_i) : origin = centroid of points
    RMS distance from centroid = $\sqrt{2}$

$$\Rightarrow T = \begin{bmatrix} s & 0 & -sc_x \\ 0 & s & -sc_y \\ 0 & 0 & 1 \end{bmatrix}$$ where c : centroid, $\bar{d}$: mean distance from centroid, s : $\frac{\sqrt{2}}{\bar{d}}$

  - 3D points (X_i) : origin = centroid of points
    RMS distance from centroid = $\sqrt{3}$

$$\Rightarrow T = \begin{bmatrix} s & 0 & 0 & -sc_x \\ 0 & s & 0 & -sc_y \\ 0 & 0 & s & -sc_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ where c : centroid, $\bar{d}$: mean distance from centroid, s : $\frac{\sqrt{3}}{\bar{d}}$

- Overall algorithm

  1) Normalize x_i, X_i
  2) DLT : Solve $Ap = 0$ s.t $\min \|Ap\|$, $\|p\| = 1$
  3) Reprojection error : Refine P using $\min_P \sum_i d(x_i, PX_i)^2$
  4) Denormalize $P = T_{2D}^{-1} \hat{P} T_{3D}$

- Decomposition of P

$P = [p_1 \; p_2 \; p_3 \; p_4] = [M \mid -M\tilde{C}] = K[R \mid -R\tilde{C}]$

$M = KR$: K, R obtained by RQ decomposition of M

$p_4 = -M\tilde{C} \rightarrow \tilde{C} = -M^{-1}p_4$: C solved by M and last column of P

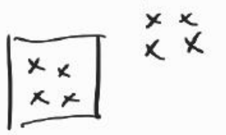
$t = -R\tilde{C}$: translation is up-to-scale

↳ solve depth λ using 3D point X

$X = P^{+}x + \lambda C$ where $P^{+} = P^T(PP^T)^{-1}$

Then $t \leftarrow \frac{t}{\|t\|} \cdot \lambda$ gives absolute scale of t

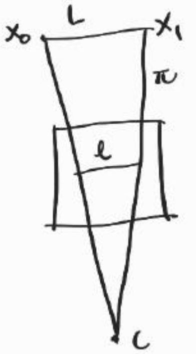
- Line correspondences



$\rightarrow R, t \quad \Rightarrow$

$\rightarrow R, t$

2D-3P (previously)   2D-3D of lines



$L \leftrightarrow l$ correspondence:

Line L represented by 2 points: X_0, X_1

Line l back-projected to plane $\pi = P^T l$

Since X_0, X_1 on π $\begin{cases} (l^T P) X_0 = 0 \\ (l^T P) X_1 = 0 \end{cases}$ $\therefore$ Each 3D-2D correspondence gives 2 linear equations

$Ap = 0$ where $A \in \mathbb{R}^{2n \times 12}$, $p \in \mathbb{R}^{12 \times 1}$

$\downarrow$

Solve $\underset{p}{\operatorname{argmin}} \|Ap\|$ s.t. $\|p\| = 1$ or $\|\hat{p}_3\| = 1$

$\Leftrightarrow$ $SVD(A) = U \Sigma V^T \rightarrow \operatorname{argmin}(p) =$ last column of V

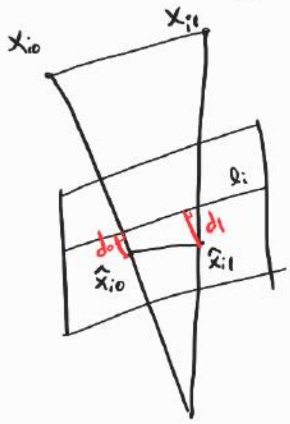
Algebraic error

Then, minimize geometric error:

$$\min_P \sum_i e(l_i, \underbrace{PX_{i0}}_{\hat{x}_{i0}}, \underbrace{PX_{i1}}_{\hat{x}_{i1}}) \quad \text{where } e = \frac{d_0 + d_1}{2(\|\hat{x}_{i0} - \hat{x}_{i1}\|)}, \quad d_j = \frac{|(x_{ij})^T l_i|}{\sqrt{l_{ix}^2 + l_{iy}^2}}$$

$\uparrow$

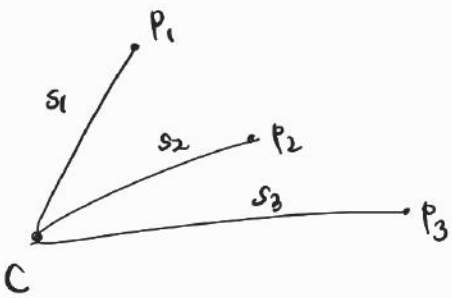
Normalize $\because$ inherit bias on longer line



Geometric error

- Calibrated camera case

In general, 2-stage approach $\begin{cases} \text{Compute depths } s_i \text{ of three 3D points } P_i \\ \text{solve } R, t \text{ using absolute orientation algo.} \end{cases}$



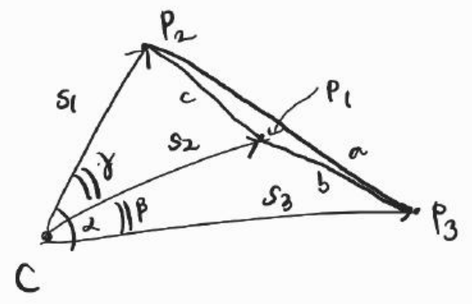
$\rightarrow$ $(P_1^w, P_2^w, P_3^w) \overset{R,t}{\leftrightarrow} (P_1^c, P_2^c, P_3^c)$

world frame $\quad$ camera frame

Each correspondence $P_i^w \leftrightarrow P_i^c$ gives 2 constraints

$\therefore$ Minimum 3 correspondences required to solve R(3 dof), t(3 dof)

- The Grunet (1841) Solution – P3P (Minimal 3 points)



Let P_i are inhomogenous 3D points i.e. $P_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$

Then, length b/w points are calculated as:

$a = \|P_2 - P_3\|$, $b = \|P_1 - P_3\|$, $c = \|P_1 - P_2\|$

Let q_i are perspective projective points of P_i i.e. $q_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$

Since K is known, if $c_x = c_y = 0$, $u_i = f\frac{X_i}{Z_i}$, $v_i = f\frac{Y_i}{Z_i}$

Then, unit vectors pointing from C to P_i are: $j_i = \frac{1}{\sqrt{u_i^2 + v_i^2 + f^2}} \begin{pmatrix} u_i \\ v_i \\ f \end{pmatrix}$

Derive
- angle b/w j vectors : $\cos\alpha = j_2 \cdot j_3$, $\cos\beta = j_1 \cdot j_3$, $\cos\gamma = j_1 \cdot j_2$
- distance of P from C : $P_i = s j_i$

↓ using law of cosines,

$$\left.\begin{aligned} s_2^2 + s_3^2 - 2s_2 s_3 \cos\alpha &= a^2 \\ s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta &= b^2 \\ s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma &= c^2 \end{aligned}\right]$$

three unknowns : s_1, s_2, s_3

Let $s_2 = u s_1$, $s_3 = v s_2$ → three unknowns: s_1, u, v

After organizing equations, 4-degree univariate polynomial is obtained:

$$A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$$ w/ known coefficients A_i

↓ use companion matrix

Eigen-value of $\begin{bmatrix} 0 & 0 & 0 & -A_4/A_0 \\ 1 & 0 & 0 & -A_3/A_0 \\ 0 & 1 & 0 & -A_2/A_0 \\ 0 & 0 & 1 & -A_1/A_0 \end{bmatrix}$ are roots of equation

∴ u, v calculated, thus s_1, s_2, s_3 can be obtained

- Absolute orientation

Since $P_i' = s_i j_i$, 3D points in camera frame P_i' can be obtained

objective : Find R, t s.t. $\underset{\text{camera frame}}{(P_1', P_2', P_3')} \xrightarrow{R,t} \underset{\text{world frame}}{(P_1, P_2, P_3)}$

$$\Rightarrow \underset{R,t}{\operatorname{argmin}} \sum_i \| p_i - (Rp'_i + t) \|$$

- Steps

1) Remove translation

$r_i = p_i - \bar{p}$, $r'_i = p'_i - \bar{p}'$ where $\bar{p}, \bar{p}'$ are centroid w.r.t. world/camera frame

2) Compute rotation matrix

Let $M = \sum_{i=1}^{n} r'_i r_i^T$ i.e. sum of outer product

Then, $R = M(M^T M)^{-\frac{1}{2}} = M\underline{Q^{-\frac{1}{2}}}$

↳ can be easily computed using eigendecomposition

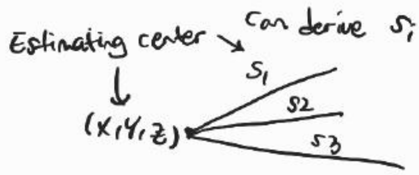
If $Q = V \Sigma U^T$ where $\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$,

$Q^{-\frac{1}{2}} = V \,\mathrm{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}\right) V^T$

3) Compute translation vector

$t = \bar{p}' - R\bar{p}$

- Degenerate solution



To solve for s_1, s_2, s_3:

$$\left.\begin{aligned} s_2^2 + s_3^2 - 2s_2 s_3 \cos\alpha &= a^2 \\ s_1^2 + s_3^2 - 2s_1 s_3 \cos\beta &= b^2 \\ s_1^2 + s_2^2 - 2s_1 s_2 \cos\gamma &= c^2 \end{aligned}\right] \xrightarrow{\text{rewritten}} \begin{aligned} f_1(x,y,z) &= 0 \\ f_2(x,y,z) &= 0 \\ f_3(x,y,z) &= 0 \end{aligned}$$

where $(x,y,z)^T$ is the camera center

WLOG, for n unknowns and equations,

$f_i(x_1, x_2, \cdots, x_n) = 0$ where $i = 1, 2, \cdots, n$

For non-degenerate / stable solution, small first-order change of unknown $\left(\frac{\partial f_i}{\partial x_j}\right)$ shouldn't affect the solution ($dx_j = 0$)

$$J \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\frac{\partial f_i}{\partial t} = \frac{\partial f_i}{\partial x_1}\frac{\partial x_1}{\partial t} + \cdots + \frac{\partial f_i}{\partial x_n}\frac{\partial x_n}{\partial t}$

→ Total derivative : $\partial f_i = \frac{\partial f_i}{\partial x_1} dx_1 + \cdots + \frac{\partial f_i}{\partial x_n} dx_n$

↳ must have no non-trivial solution

For P3P algorithm, the form is :

$$J\,dx = df$$

$$J\,dx = 0$$

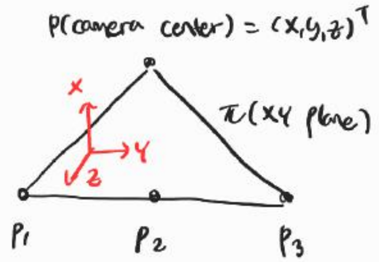
$$M_{3\times3}\,dx = \frac{1}{s_1 s_2 s_3} AB \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} df_1 \\ df_2 \\ df_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{Form of homogerous linear equation } Mx = 0$$

$\downarrow$

When M is rank-deficient , $dx \neq 0$, which is degenerate condition

2 cases

- P_1, P_2, P_3 are collinear



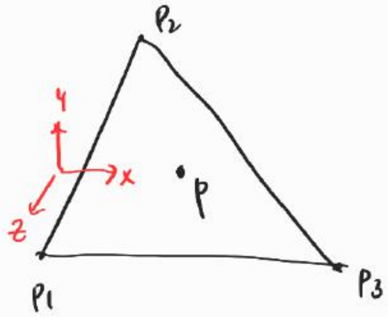
$$A = \begin{pmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{pmatrix}$$

$\uparrow$ This column is 0 $\because z = z_1 = z_2 = z_3$

$\therefore$ A is rank-deficient $\rightarrow$ M is rank deficient

$\rightarrow$ dx has non-trivial solution

- Camera center and (P_1, P_2, P_3) plane are coplanar



$$A = \begin{pmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{pmatrix}$$

$\uparrow$ This column is 0 $\because z = z_1 = z_2 = z$

$\therefore$ A is rank-deficient $\rightarrow$ M is rank deficient

$\rightarrow$ dx has non-trivial solution

- Problem of P3P

  4^{th} degree polynomial $\rightarrow$ 4 possible solutions

  $\therefore 4^{th}$ point is required to get a unique solution

  i.e. project P_4 using 4 possible cases , select w/ smallest reprojection distance

• Linear 4-point algorithm

Recall for 3 point correspondences:

$$\begin{aligned} s_2^2 + s_3^2 - 2s_2s_3\cos\alpha &= a^2 \\ s_1^2 + s_3^2 - 2s_1s_3\cos\beta &= b^2 \\ s_1^2 + s_2^2 - 2s_1s_2\cos\gamma &= c^2 \end{aligned} \longrightarrow \begin{aligned} f_{12}(s_1,s_2) &= 0 \\ f_{13}(s_1,s_3) &= 0 \\ f_{23}(s_2,s_3) &= 0 \end{aligned} \xrightarrow[s_3 = vs_1]{s_2 = us_1} \text{4th-degree polynomial } g(s_1^2) = g(x) = 0$$

$\Downarrow$ w/ additional 4th point

$s_{12}(s_1,s_2)=0$, $s_{13}(s_1,s_3)=0$, $s_{14}(s_1,s_4)=0$

$s_{23}(s_2,s_3)=0$, $s_{24}(s_2,s_4)=0$, $s_{34}(s_3,s_4)=0$

Overconstrained system of 6 equations (4C2)

Naive approach: Take subset of 3 equations → 6C3 # of g → find common solution

↳ problem: ↑ computational cost, × Common solution b/c of noise, × profit from data redundancy

Better approach proposed by Quan and Lan [TPAMI '99]

Take three 4th-order polynomial: $g(x), g'(x), g''(x)$ where $x = s_1^2$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a'_1 & a'_2 & a'_3 & a'_4 & a'_5 \\ a''_1 & a''_2 & a''_3 & a''_4 & a''_5 \end{pmatrix} \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$At = 0$ matrix form

→ solve for t

Since $A \in \mathbb{R}^{3\times5}$, max rank$(A) = 3$ ∴ $t \in \mathbb{R}^5$ cannot be directly at A's 3d-span space

↳ Solution: Take $SVD(A) = U_{3\times5}\,\mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, 0, 0)\,V_{3\times5}^T$

Then, null-space of A is spanned by v_4, v_5

∴ $t(\lambda, \rho) = \lambda v_4 + \rho v_5$ where $\lambda, \rho \in \mathbb{R}$ Eq(1)

Non-linear constraints of $t = (x^4, x^3, x^2, x^1, x^0)^T$

: $t_i t_j = t_k t_l$ for $i+j = k+l$ Eq(2)

Substituting Eq(1) into Eq(2):

$b_1\lambda^2 + b_2\lambda\rho + b_3\rho^2 = 0$

where b_1, b_2, b_3 are coefficients defined by v_4, v_5 (i,j,k,l)

Since there are 7 possible combinations for (i,j,k,l):

$$\begin{pmatrix} b_1 & b_2 & b_3 \\ b'_1 & b'_2 & b'_3 \\ & \vdots & \\ b_1^{(6)} & b_2^{(6)} & b_3^{(6)} \end{pmatrix} \begin{pmatrix} \lambda^2 \\ \lambda\rho \\ \rho^2 \end{pmatrix} = By = 0$$

→ solve for y : overdetermined system

$SVD(B) = U\Sigma V^T$, then y = right column vector of V

$\therefore \lambda/\rho = y_0/y_1$ or y_1/y_2 Eq(3)

Also, $t = (1, x, x^2, x^3, x^4)^T$

$= \lambda(v_4^0, v_4^1, v_4^2 + v_4^3 + v_4^4)^T + \rho(v_5^0, v_5^1, v_5^2, v_5^3, v_5^4)^T$

$\therefore 1 = \lambda v_4^0 + \rho v_5^0$ Eq(4)

Can solve λ, ρ using Eq(3) and Eq(4)

↳ * Can use $\frac{\lambda}{\rho} = \frac{1}{2}\left(\frac{y_0}{y_1} + \frac{y_1}{y_2}\right)$ i.e. average

Since $x = t_1/t_0$ or t_2/t_1 or t_3/t_2 or t_4/t_3, take average of four solutions

($\because$ they are usually similar values)

$\therefore x = s_1^2 \rightarrow s_1 = \sqrt{x}$ (depth $s_i \geq 0$)

s_2, s_3, s_4 can be derived using $f_{12}(s_1, s_2) = 0$ etc.

Apply absolute orientation to solve R, t

↳ uniquely determined if not degenerate

(i) $P_1 - P_4$ collinear

(ii) $P_1 - P_4$ & P coplanar

- Linear n-point Algorithm

e.g. 5-points : ${}_5C_2$ = ten $f_{ij}(s_i, s_j) = 0$

six 4th-degree polynomials :

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a'_1 & a'_2 & a'_3 & a'_4 & a'_5 \\ & & \vdots & & \\ a_1^{(5)} & a_2^{(5)} & a_3^{(5)} & a_4^{(5)} & a_5^{(5)} \end{pmatrix} \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $x = s_i^2$

$A_{6\times 5} t_5 = 0$

→ Sufficiently many equations to solve $At = 0$

$SVD(A_{6\times 5}) = U_{6\times 6} \Sigma_{6\times 5} V^T_{5\times 5}$

Since max rank$(A) = 4$, find $\min \|At\|$ s.t. $\|t\| = 1$

$\therefore t$ = right singular vector v_5 ($t_5 = \lambda v_5$)

Since first element of t is 1, $\lambda v_6^{(0)} = 1$ → find λ

Calculate $x = t_1/t_0$ or ⋯ t_4/t_3, then $\sigma_1 = \sqrt{x}$

⇓ For n points,

size of $A = \frac{(n-1)(n-2)}{2} \times 5$ → Take SVD(A) to find t

Problem: Expensive time complexity $O(n^3)$

- EPnP: Accurate $O(n)$ solution
  - Linear time complexity → tractable for large # of points
    - Express n point sets w/ 4 control points
    - ∴ problem to solve: depth values of n points (s_i) (PnP) → camera coordinates of control points (c_j^c) (EPnP)
  - 4 non-planar control points

    Known: 3D world points p_i^w $(i=1,2,\cdots,n)$

    Unknown: 4 control points c_j $(j=1,2,3,4)$

    Each referenced points expressed as weighted sum of control points

$$p_i^w = \sum_{j=1}^{4} \alpha_{ij} c_j^w \quad \text{where} \quad \sum_{j=1}^{4} \alpha_{ij} = 1 \quad / \quad p_i^c = \sum_{j=1}^{4} \alpha_{ij} c_j^c$$

(camera frame)

c_j^w: fixed value

α_{ij}: Different by reference point i, shared by world & camera frame

  - Computing c_j^w

    1) Select centroid as first control point

$$c_0^w = \frac{1}{n}\sum_{i=1}^{n} p_i^w$$

    2) Select principal axes of world points for other three control points (distribution of points)

    Calculated by SVD of covariance matrix

$$V = \frac{1}{n}\sum_{i=1}^{n}(p_i^w - c_0^w)(p_i^w - c_0^w)^T$$

$$SVD(V) = U\Sigma U^T$$

$$U = [u_1, u_2, u_3]^T = [c_1^w, c_2^w, c_3^w]^T$$

- Computing α_{ij}

$$p_i^w = \sum_{j=1}^{4} \alpha_{ij} c_j^w \rightarrow \begin{pmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ 1 \end{pmatrix} = \alpha_{i1} \begin{pmatrix} c_{1x} \\ c_{1y} \\ c_{1z} \\ 1 \end{pmatrix} + \alpha_{i2} \begin{pmatrix} c_{2x} \\ c_{2y} \\ c_{2z} \\ 1 \end{pmatrix} + \alpha_{i3} \begin{pmatrix} c_{3x} \\ c_{3y} \\ c_{3z} \\ 1 \end{pmatrix} + \alpha_{i4} \begin{pmatrix} c_{4x} \\ c_{4y} \\ c_{4z} \\ 1 \end{pmatrix}$$

4 equations, 4 unknowns $\rightarrow$ can solve $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}$ for all points i

- Computing c_j^c

2D-3D correspondences given (known)

$$p_i^c = \sum_{j=1}^{4} \alpha_{ij} c_j^c, \quad \forall i \quad w_i \begin{bmatrix} u_i \\ 1 \end{bmatrix} = K p_i^c = K \sum_{j=1}^{4} \alpha_{ij} c_j^c$$

camera-to-image projection formula

Scalar projective parameters

Intrinsic (known)

Unknown

$$\forall i, \quad w_i \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & u_c \\ 0 & f_v & v_c \\ 0 & 0 & 1 \end{bmatrix} \sum_{j=1}^{4} \alpha_{ij} \begin{bmatrix} x_j^c \\ y_j^c \\ z_j^c \end{bmatrix}$$

] 2 independent equations

Only unknown in this formula

$\therefore$ 1 point correspondence gives 2 independent equations

$3 \times 4 = 12$ unknown parameters of control points

] n points stacked: $M_{2n \times 12} x_{12 \times 1} = 0$

$Mx = 0 \rightarrow x$ lies on null space of M

$x = \sum_{i=1}^{N} \beta_i v_i$ where v_i are right singular vectors of M

($\because$ noise, right singular values are unlikely be 0)

since $SVD(M) = SVD(M^T M)$ where $M^T M \in \mathbb{R}^{12 \times 12}$ (constant size)

Calculate $SVD(M^T M) = U \Sigma U^T$ which is most time consuming step $O(n)$

(i) $N = 1$

$x = \beta v$. Since $rank(M) = 11$, requires at least 6 point correspondences

$\downarrow$ solve β

Since $\forall i, j$, $\|c_i^w - c_j^w\|^2 = \|c_i^c - c_j^c\|^2$ (distance maintained although frame changes)

(known)

$= \|\beta v^{[i]} - \beta v^{[j]}\|^2$ where $v = [v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}]^T$

Can compute β w/ six pairs of $(v^{[i]}, v^{[j]})$

(ii) $N = 2$

$x = \beta_1 v_1 + \beta_2 v_2$. Since $\text{rank}(M) = 12 - 2 = 10$, requires $n \geq 5$ point correspondences

Calculate $\|c_i^w - c_j^w\|^2 = \|c_i^c - c_j^c\|^2 = \|(\beta_1 v_1^{[i]} + \beta_2 v_2^{[i]}) - (\beta_1 v_1^{[j]} - \beta_2 v_2^{[j]})\|^2$

$\downarrow$ Can be rewritten as

Overdetermined system $L\beta = \rho$ where $\beta = [\beta_1^2, \beta_1\beta_2, \beta_2^2]^T$ $\therefore \beta = L^+\rho$

(iii) $N = 3$

$x = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$. $\text{rank}(A) = 9$, $n \geq 5$

$L\beta = \rho$ where $\beta = [\beta_1^2, \beta_1\beta_2, \beta_1\beta_3, \beta_2^2, \beta_2\beta_3, \beta_3^2]^T$ $\therefore \beta = L^+\rho$

- 3 coplanar control points

  $\equiv$ Any control point is dependent to other 3 points

  $\therefore$ Only select 3 control points $\rightarrow$ size of $M = 2n \times 9$

  $\|c_i^w - c_j^w\|^2 = \|c_i^c - c_j^c\|^2$ $3C2 = 3$ constraints

  $\Rightarrow$ only solvable when $N = 1, 2$

- Calculate p_i^c

  $p_i^c = \sum_{j=1}^{4} \alpha_{ij} c_j^c$ $\rightarrow$ Then (R, t) calculated using absolute orientation b/w $p_i^c \leftrightarrow p_i^w$