- 3D projective geometry ($P^3$)

↳ generalizations of $P^2$ space e.g. $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in P^2 \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in P^3$

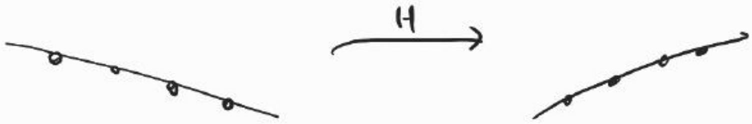
additional properties e.g. two lines may not intersect at 3D projective space

- Points in $P^3$

$\begin{cases} \text{homogenous}: X = (X_1, X_2, X_3, X_4)^T \quad X_4 \neq 0 \quad (\text{if } X_4 = 0\text{, points of infinity}) \\ \text{inhomogenous}: X = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}\right)^T \end{cases}$

$X' = HX$ : 4x4 linear transformation matrix ∴ 15 dof

mapping H is a collineation (as $P2$) : lines (collinear points) map to lines



∴ preserves incidence relations e.g. intersection of line and plane

- planes in $P^3$

$\begin{cases} \text{homogenous}: \pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0 \quad / \quad \pi^T X = 0 \\ \text{inhomogenous}: \pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0 \end{cases}$

plane $\pi$ : $(\pi_1, \pi_2, \pi_3, \pi_4)$

∴ Dot product of plane $\pi$ and point $X$ equals zero implies $X$ is in $\pi$

(anagolous to $l^T x = 0$ in $P2$)

+ plane and point are duality (interchangable in 3D)

3dof : $\{\pi_1 : \pi_2 : \pi_3 : \pi_4\}$ ratio

$(\pi_1, \pi_2, \pi_3)^T$ defines normal vector

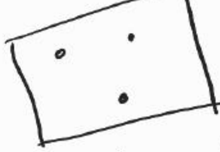
$\Rightarrow n \cdot X + d = 0$ where $X = (X, Y, Z)^T$, $\pi_4 = d$

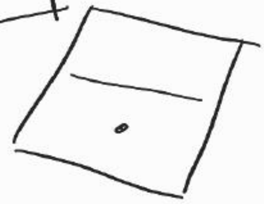
↳ $\frac{d}{\|n\|}$ is distance of plane from origin

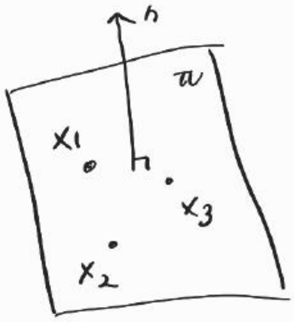
$X' = HX \rightarrow \pi' = H^{-T}\pi$ for plane transformation (proof: use $\pi^T X = 0$)

- Join and Incidence Relations

(i) Plane is uniquely defined $\left[\begin{array}{l}\text{not collinear three points} \\ \text{not incident one line and one point}\end{array}\right.$







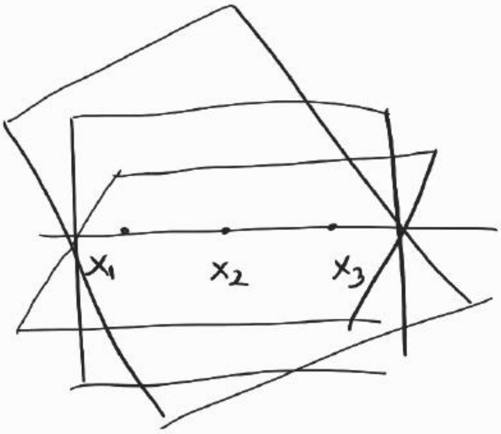

since $\pi^T X_1 = \pi^T X_2 = \pi^T X_3 = 0$

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \pi = 0$$

obtained uniquely (up-to-scale)
1D right null space
($\because Ax = 0,\ A \in R^{m \times n} \rightarrow \dim(\text{right null space}) = n - \text{rank}(A)$)

↑ 3x4 matrix where each incidence relations are linearly independent

$\therefore$ rank 3 matrix




if $X_1, X_2, X_3$ are collinear, $[X_1\ X_2\ X_3]^T$ has rank = 2

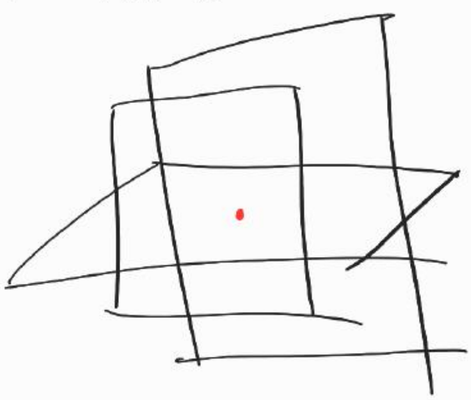
$\therefore$ 2D right null space

(ii) three distinct planes intersect at unique point

since $\pi_1^T X = \pi_2^T X = \pi_3^T X = 0$, $\begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} X = 0$

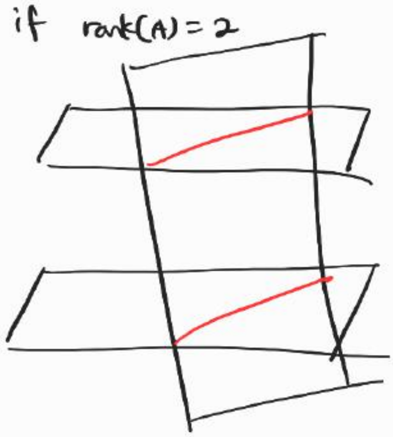
(A)

if rank(A) = 3



intersect at unique point

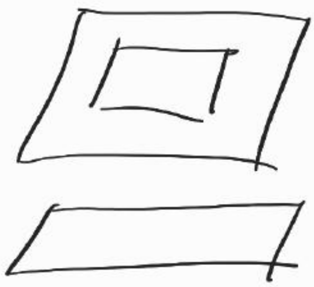
if rank(A) = 2



Some planes are parallel

$\therefore$ intersection of line / line pairs

if rank(A) = 1



All planes are parallel

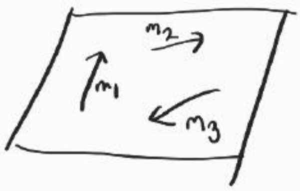
- Parameterized points on a plane

if point $X$ is on plane $\pi$,

$$X = Mx = \underbrace{\begin{bmatrix} | & | & | \\ m_1 & m_2 & m_3 \\ | & | & | \end{bmatrix}}_{4\times3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times1} = x_1 m_1 + x_2 m_2 + x_3 m_3$$

(i.e. linear combination of $m_1, m_2, m_3$ & $x_1, x_2, x_3$ parameterize point $X$)

$\pi$

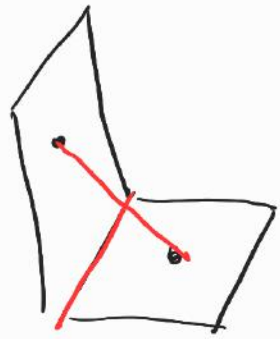


$\therefore \pi^T M = 0$ $\quad$ * $M$ is not unique

(iii) Two distinct planes intersect at unique line

- Lines in $P^3$

Defined by [ intersection of two planes / join of two points



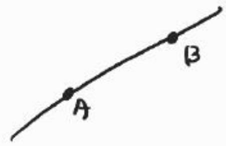
4 dof
(2 dof $(x, y)$ per point)

4 dof → 5d homogenous vector? Awkward representations in 3D space

- Rep1 Null space and span representation

if point $A, B$ are non-coincident, for $W \in \mathbb{R}^{2\times4}$

$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix}$, span of row space defines line connecting $A, B$



(i) span of $W^T$ is pencil of points (=line) $\mu A + \lambda B$

if $\mu A = A'$ and $\lambda B = B'$, $W' = \begin{bmatrix} A'^T \\ B'^T \end{bmatrix}$, then $\text{span}(W) = \text{span}(W')$

(ii) span of 2d null space ($Wx = 0$ → set of $x$) is pencil of planes

if $P, Q$ are basis of null space, then $WP = \begin{bmatrix} A^T \\ B^T \end{bmatrix} P = 0$

since $A^T P = B^T P = 0$ (incidence relationships), $P$ contains $A$ and $B$

similarly $A^T Q = B^T Q = 0$ $\quad \therefore \overline{AB}$ is plane intersection of $P$ and $Q$

→ $\mu P + \lambda Q$ are plane of pencil w/ $\overline{AB}$ axis

* Duality of line and plane

$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix}$

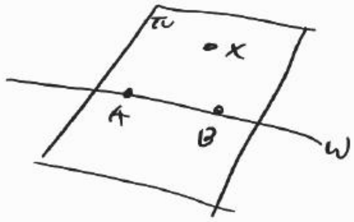
- span of $W^{*T}$ is pencil of planes $\mu P + \lambda Q$
- span of 2d null space of $W^*$ is pencil of points

$\therefore W^* W^T = W W^{*T} = 0$

- plane = point + line

Null-space of $M = \begin{bmatrix} W \\ X^T \end{bmatrix}$ defines a plane containing line $W$ and point $X$

$\because M = \begin{bmatrix} W \\ X^T \end{bmatrix} \equiv \begin{bmatrix} A^T \\ B^T \\ X^T \end{bmatrix} \Rightarrow M\pi = 0$

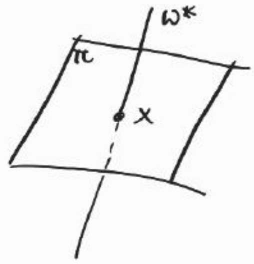



$\updownarrow$ Duality

- point = plane and line intersection

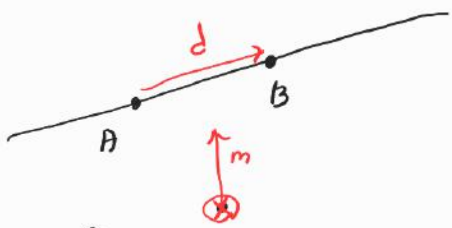
Null-space of $M = \begin{bmatrix} W^* \\ \pi^T \end{bmatrix}$ defines an intersection point $X$

$\because M = \begin{bmatrix} W^* \\ \pi^T \end{bmatrix} \equiv \begin{bmatrix} P^T \\ Q^T \\ \pi^T \end{bmatrix} \Rightarrow MX = 0$




- Rep2 plucker Line Coordinates

$$L = \{ \underbrace{l_{12}, l_{13}, l_{14}}_{\substack{d = B - A \\ \text{(direction)}}}, \underbrace{l_{23}, l_{42}, l_{34}}_{\substack{m = A \times B \\ \text{(momentum)}}} \}$$ 6 non-zero elements




if $L = \overline{AB}$, $\hat{L} = \overline{\hat{A}\hat{B}}$ are coplanar,

$\det(A, B, \hat{A}, \hat{B}) = d^T \hat{m} + m^T \hat{d} = 0$

$\downarrow$ $\because$ They have same expressions

$\hookrightarrow \because d$ or $\hat{d} \perp m$ or $\hat{m}$

$\updownarrow$ duality

if $L$ (intersection of plane $P, Q$) and $\hat{L}$ intersects,

$\det(P, Q, \hat{P}, \hat{Q}) = 0$

if $L$ (intersection of plane $P, Q$) and $\hat{L}$ (join of points $A, B$) intersects,

$P^T A Q^T B - Q^T A P^T B = 0$

- Quadrics and dual Quadrics
  ↳ surface in $P^3$ defined by $X^TQX = 0$ where $Q$ is 4x4 symmetric matrix
  (holds similar property of conic in $P2$ e.g. $x^TCx = 0$)
  if quadric surface $Q$ is not distinguished, it is refered as quadric $Q$

- properties
  1) 9 dof ∴ 4x4 symmetric (10) − scale (1)
  2) 9 points in general define a conic
  3) if $Q$ is singular ($\text{rank}(Q) < 4$), quadric is degenerate
     (can be defined by fewer points)
  4) intersection of plane $\pi$ and quadric $Q$ is conic $C$
     As point $X$ in $\pi$ can be parameterized $X = \pi x$,
     $X^TQX = x^TM^TQMx = 0$ ∴ $C = M^TQM$
  5) for point transformation $X' = HX$, quadric transforms $Q' = H^{-T}QH^{-1}$
     ∵ $X'^TQ'X' = (X^TH^T)Q'(HX) = X^T(H^TQ'H)X = X^TQX = 0$
     $H^TQ'H = Q \Rightarrow Q' = H^{-T}QH^{-1}$
  6) Dual of quadric is also quadric
     Dual quadric is defined by plane (vs. dual conic defined by line)
     $\pi^TQ^*\pi = 0$ where $Q^*$ is adjoint $Q$ or $Q^{-1}$ (if invertible)
  7) for point transformation $X' = HX$, dual quadric transforms $Q^{*\prime} = HQ^*H^T$

- Types of quadric
  Ellipsoid, Hyperboloid, Elliptic paraboloid, Hyperbolic paraboloid : $\text{Rank}(Q) = 4$
  Elliptic Cone : $\text{Rank}(Q) = 1$ ∴ degenerate

- 3D Hierarchy of transformations
  (i) projective $\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$ 15 dof
  (ii) Affine $\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$ 12 dof
  3D rigid body motion
  $R \in SO(3)$
  ↓
  (iii) Similarity $\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$ 7 dof
  (iv) Euclidean $\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$ 6 dof