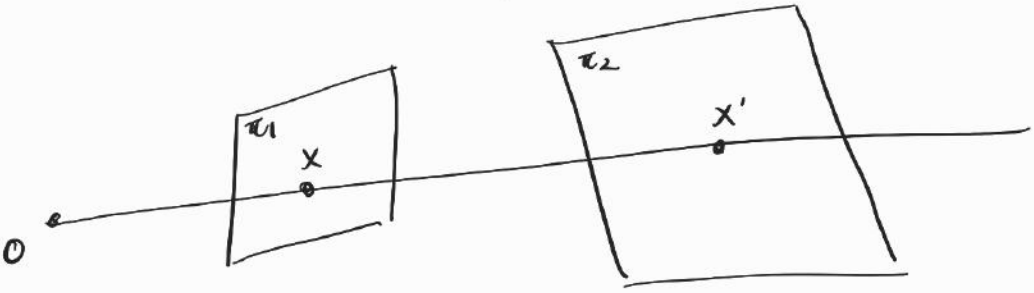
- Planar projective transformation

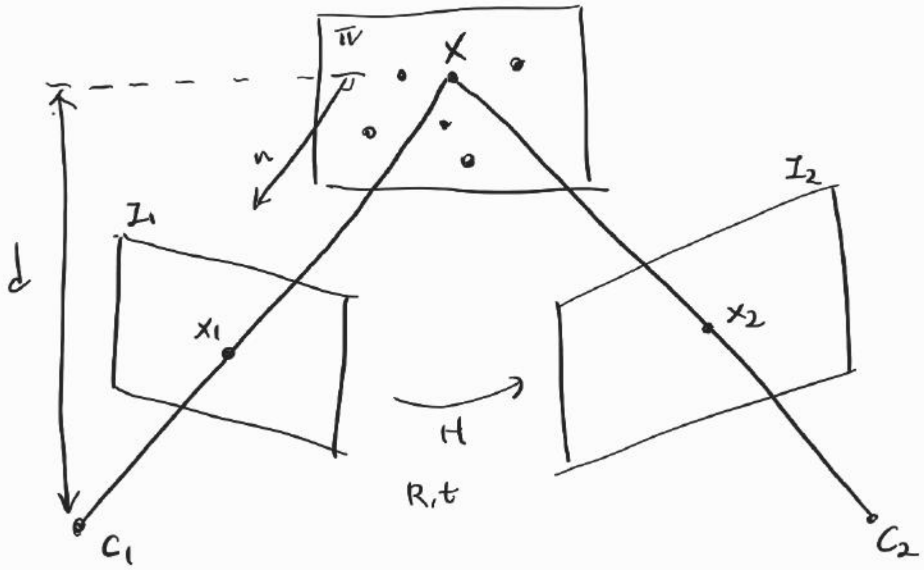
mapping points in one plane ($x \in P^2$) to points in another plane ($x' \in P^2$)

$x' = Hx$: linear mapping of homogenous coordinates → also known homography



- Existence of projective Homography

1) Planar scene : 3D points are coplanar



e.g. two-view case C_1, C_2

Given 3D point X, it is expressed as X_1, X_2 in C_1, C_2 frame

Eq 1 $\quad X_2 = RX_1 + t$

X_1, X_2 are intersection of ray from camera center (C_1, C_2) and image plane (I_1, I_2)

$N = [n_1, n_2, n_3]^T$ is unit normal vector representing π &

Perpendicular distance b/w C_1 and π is d

⇒ then equation of plane π is rewritten $\quad N^T X_1 = n_1 X + n_2 Y + n_3 Z = d$

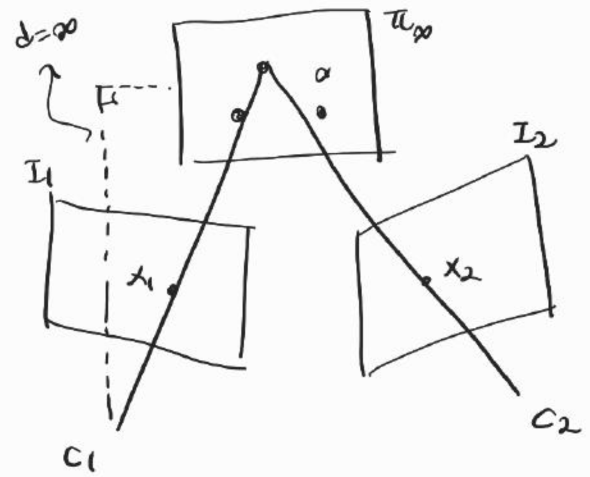
Eq 2 $\quad \dfrac{N^T X_1}{d} = 1$

Substituting Eq 2 into Eq 1, $\quad X_2 = \left(R + \dfrac{tN^T}{d}\right) X_1$

Since $X_1 = \lambda_1 x_1$, $X_2 = \lambda_2 x_2$, $\frac{\lambda_2}{\lambda_1} x_2 = \left(R + \frac{tN^T}{d}\right) x_1$

$\boxed{\text{Homography relationships : } \lambda x_2 = H x_1}$ or $x_2 = H x_1$ up to scale

2) plane at infinity : Scene is very far from camera



$$H = R + \frac{tN^T}{d}$$

$$\Rightarrow H_\infty = \lim_{d \to \infty} \left(R + \frac{tN^T}{d}\right) = R$$

$\therefore$ Same effect as pure rotation

i.e. $t = 0$

- 2D Homography

Given $x \leftrightarrow x'$ points correspondences of 2 images

$\rightarrow$ Compute 2D homography H s.t. $x' = Hx$

Requires 4 point correspondences $\because$ Each (x, x') pair give 2 constraints

& H has 8 dof (9 entries − 1 scale)

This is minimal / approximate solution

Image sensors have a noise $\therefore$ Requires more that 4 correspondences

, then solve least-squares of overdetermined system $Ax = b$

- Direct Linear Transform (DLT) Algorithm

↳ simple linear algorithm determining H given 4 point correspondences

$$Hx_i = x_i' \rightarrow x_i' \times x_i' = x_i' \times Hx_i = 0$$

cross product can be expressed in

$$\begin{pmatrix} x_i' \\ y_i' \\ w_i' \end{pmatrix} \quad \begin{pmatrix} h^{1T} x_i \\ h^{2T} y_i \\ h^{3T} w_i \end{pmatrix}$$

linear form

$$\begin{bmatrix} \text{———} \\ \text{———} \\ \text{Linearly dependent row} \end{bmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0$$

$\therefore A_i h = 0$ where $A_i \in R^{2\times 9}$, $h \in R^{9\times 1}$ for each point

$\Rightarrow$ Requires 4 correspondences to solve H

* for $x_i = [x_i, y_i, w_i]^T$, w_i is normally chosen as 1 $\rightarrow \left[\frac{x_i}{w_i}, \frac{y_i}{w_i}, 1\right]^T$

- Least-squares problem

Exact solution $Ah=0$ doesn't exist b/c of noise

$\downarrow$ LS problem

such constraint to avoid trivial solution $h=0$

$\underset{h}{\operatorname{argmin}} \|Ah\|$ s.t. $\|h\|=1$ $\Rightarrow$ find right null space of A

- Singular Value Decomposition (SVD)

$svd(A) = U\Sigma V^T$ where $A \in R^{2i\times 9}$

$U \in R^{2i\times 2i}$ is ColA's orthonormal basis

Singular value analysis

$\Sigma \in R^{2i\times 9}$ w/ singular values at diag terms

$V \in R^{9\times 9}$ is RowA's orthonormal basis

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}$$

$\sigma_1 > \sigma_2 > \cdots \sigma_n$

if $rank(A) = 8$ (h's dof),

A is not corrupted by noise = exact solution exists

else (which is most case)

corrupted by noise $\sigma_{n-r}, \cdots \sigma_n \neq 0$

As $A = U\Sigma V^T \Rightarrow AV = U\Sigma$, $\|Av_i\|$ is min when σ_i is smallest singular value($=\sigma_n$)

∴ Solution of least squares problem $Ah = 0$ → $h = V_n$

- DLT Summary

objective : Given 4+ $x_i \leftrightarrow x_i'$ 2D-2D point correspondences,
find homography matrix H s.t. $x' = Hx$

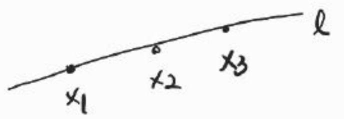
Algorithm : 1) Compute A_i for each point correspondences using cross product
2) Stack $A_i \in R^{2\times 9}$ making $A \in^{2n\times 9}$ for n point correspondences
3) Calculate SVD of A, $\min\|Ah\|$ is last column of V
4) Rearrange h to 3×3 matrix H

- Homography : Degeneracy

For $Ah = 0$, if A is not a full rank (e.g. $\text{rank}(A) < 8$), null-space of A ∄ exist

⇒ such status is called critical configuration / degeneracy

⇒ occurs when three of points are collinear



line $l = x_1 \times x_2$

then if $l^T x_3 = 0$, $\{x_1, x_2, x_3\}$ are collinear

- Normalization in DLT

for each pair (x, x'), $A = \begin{bmatrix} 0 & 0 & 0 & -x' & -y' & -1 & y'x & y'y & y' \\ x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \end{bmatrix} \in R^{2\times 9}$

order of 1 (e.g. x) and order of 2 (e.g. $-x'y$) elements are mixed ∴ Cause bad behavior in SVD

→ Solution : Data Normalization

Monte Carlo Simulation

if 5 points pair w/ gaussian noise w/ $\sigma = 0.1$ pixel use 100 trials to identify H :

sparse distribution of errors
↓
A is ill-conditioned

Unnormalized data

Normalized data

Data Normalization steps

- Find transformation of set of points by
  - (i) translation s.t. centroid of points is origin
  - (ii) scaling s.t. avg. distance from origin is $\sqrt{1^2+1^2} = \sqrt{2}$

$\ast$ Transformation is applied independently on $\{x_i\}_{i=1\dots n}$ and $\{x'_i\}_{i=1\dots n}$

$\therefore$ avg. point $= (1\ 1\ 1)^T$ after normalization & $\times$ magnitude difference

↳ This must not be considered optional

- Normalized DLT algorithm

1) Normalize points $\tilde{x}_i = T_{norm} x_i$, $\tilde{x}'_i = T'_{norm} x'_i$ $\quad (T_{norm} \neq T'_{norm})$

2) Apply DLT for $n \geq 4$ $\tilde{x}_i \Leftrightarrow \tilde{x}'_i$ to find H

3) Denormalize $H = (T'_{norm})^{-1} H\, T_{norm}$

$$T_{norm} = \begin{bmatrix} s & 0 & -sc_x \\ 0 & s & -sc_y \\ 0 & 0 & 1 \end{bmatrix}$$

where c = centroid of all data points

$\bar{d}$ = mean distance from centroid

$s = \dfrac{\sqrt{2}}{\bar{d}}$

- Different cost functions
  - Algebraic Distance

DLT algorithm : minimize $\|Ah\|$, thus define $\varepsilon = Ah$ as residual vector

for each $x_i \Leftrightarrow x'_i$, partial error vector $\varepsilon_i \in R^{2\times 1}$ is computed

$$d_{alg}(x'_i, Hx_i) = \|\varepsilon_i\|^2 = \left\| \begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \end{bmatrix} h \right\|^2$$ is algebraic distance

$\therefore$ Total algebraic error $= \sum_i d_{alg}(x'_i, Hx_i)^2 = \sum_i \|\varepsilon_i\|^2 = \|Ah\|^2 = \|\varepsilon\|^2$

(+) unique solution , computationally inexpensive

(−) quantity is not geometrically / statistically meaningful

$\Rightarrow$ Used for starting point of non-linear optimization to acquire approx. solution

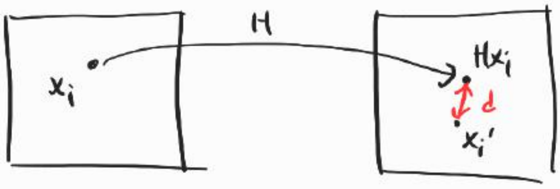
- Geometric Distance

Find difference b/w measured and estimated image coordinates

- Types

1) Transfer error

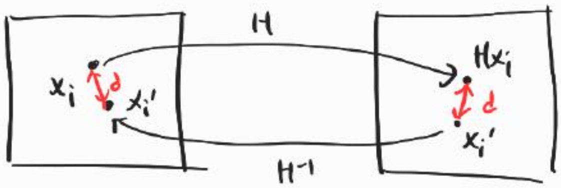
$\sum_i d(x_i', Hx_i)^2$ where $d(\cdot,\cdot)$ is an Euclidean distance



2) Systemic transfer error

⇒ Minimize error on both images

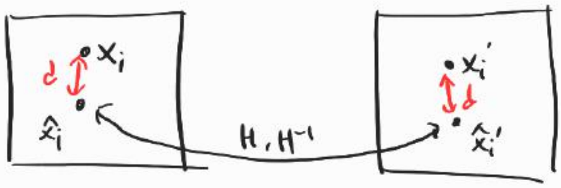
$\sum_i d(x_i, H^{-1}x_i')^2 + d(x_i', Hx_i)^2$



3) Reprojection error

⇒ find homography $\hat{H}$ s.t. $\hat{x}_i$ and $\hat{x}_i'$ are perfectly matched

$\sum_i d(x_i, \hat{x}_i)^2 + d(x_i', \hat{x}_i')^2$ for $\forall i$, $\hat{x}_i' = H\hat{x}_i$



Most accurate method ∵ not only optimizing H, but also $\hat{x}_i, \hat{x}_i'$

• Sampson Error

⊖ of geometric error : computationally complex ∵ optimize all $H, \hat{x}, \hat{x}'$

∴ Sampson error : complexity / accuracy in between algebraic and geometric error

• Methods

Let $C_H(x) = 0$ is cost function $Ah = 0$ given homography H and input $x = (x, y, x', y')$

↑ $x_i \leftrightarrow x_i'$ pair

$C_H(\hat{x}) = 0$ for desired point $\hat{x}$ s.t. $\delta_x = \hat{x} - x$

Then, by Taylor approximation,

$$C_H(x + \delta_x) \simeq C_H(x) + \frac{\delta C_H}{\delta x}(\hat{x} - x) = C_H(x) + J\delta_x = 0$$

∴ $J\delta_x = -\varepsilon$ where $\varepsilon = C_H(x)$

Apply right pseudo-inverse

↳ Redefined problem : $\min \|\delta x\|^2$ s.t. $J\delta x = -\varepsilon$

$$\delta x = -J^{+}\varepsilon = -J^{T}(JJ^{T})^{-1}\varepsilon$$

then, Sampson error is defined as $\|\delta x\|^2 = \delta x^T \delta x = \varepsilon^T (JJ^T)^{-1}\varepsilon$

⇒ for 2D homography estimation problem,

measurements $x = (x, y, 1)^T$, $x' = (x', y', 1)^T$ → input $X = (x, y, x', y')$

then, algebraic error vector $A_i h = \varepsilon = C_H(X) \in \mathbb{R}^{2\times 1}$

Jacobian $J = \left[\frac{\delta \varepsilon}{\delta x}, \frac{\delta \varepsilon}{\delta y}, \frac{\delta \varepsilon}{\delta x'}, \frac{\delta \varepsilon}{\delta y'}\right] = \frac{\delta C_H(X)}{\delta X} \in \mathbb{R}^{2\times 4}$

- Iterative Minimization

Geometric / Sampson errors are usually minimized using squared Mahalanobis distance

$$\underset{p}{\operatorname{argmin}} \|x - f(p)\|_{\Sigma}^2 = \underset{p}{\operatorname{argmin}} (x - f(p))^T \Sigma^{-1} (x - f(p))$$

where $x \in \mathbb{R}^N$ is measurement vector ($\Sigma = I$ if all measurement w/ equal weight)

$p \in \mathbb{R}^M$ is optimized parameter → These 3 components need to be defined

$f : \mathbb{R}^M \mapsto \mathbb{R}^N$ is mapping function

Such problem is unconstrained continuous optimization problem

↳ solved by Gauss-Newton / Levenberg-Marquardt

Case 1 : Error in one image (fixed x_i w/ n pairs)

- X : $2n$ inhomogenous points x_i'
- P : homography h
- f : $h \mapsto (Hx_1, Hx_2, \cdots, Hx_n)$

→ $\|X - f(h)\|^2 = \sum_i d(x_i', Hx_i)^2$

Case 2 : Systemic Transfer Error

- X : $4n$ inhomogenous points x_i, x_i'
- P : homography h
- f : $h \mapsto (H^{-1}x_1', \cdots, H^{-1}x_n', Hx_1, \cdots, Hx_n)$

→ $\|X - f(h)\|^2 = \sum_i d(x_i, H^{-1}x_i')^2 + d(x_i', Hx_i)^2$

Case 3 : Reprojection error

- X : $4n$ inhomogenous points x_i, x_i'
- P : $(h, \hat{x}_1, \cdots, \hat{x}_n)$
- f : $(h, \hat{x}_1, \cdots, \hat{x}_n) \mapsto (\hat{x}_1, \hat{x}_1', \cdots, \hat{x}_n, \hat{x}_n')$ where $\hat{x}_i' = \hat{H}\hat{x}_i$

→ $\|X - f(h)\|^2 = \sum_i d(x_i, \hat{x}_i)^2 + d(x_i', \hat{x}_i')^2$ for $\hat{x}_i' = \hat{H}\hat{x}_i$

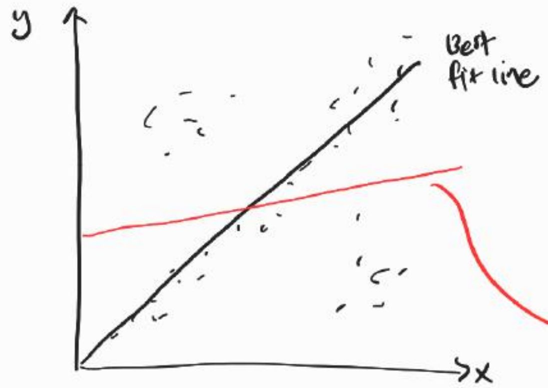
Case 4 : Sampson error

$$\begin{cases} x : (x, y, x', y')^T \\ p : h \\ f : \text{set as } \; x - f(h) = \delta_x \end{cases} \rightarrow \|x - h(x)\|^2 = \|\delta x\|^2 = \delta x^T \delta x = \varepsilon^T (JJ^T)^{-1} \varepsilon$$

- RANSAC (Random sample Consensus)

obstacles that hinders estimating parameters/optimization
- Measurement noise
- Data outliers

e.g. Line fitting $y = ax + b$



Estimating parameters (a, b) from n data points (x_i, y_i)

↓

Least squares problem $\underset{a,b}{\operatorname{argmin}} \sum_{i=1}^{n} \|y_i - (ax_i + b)\|^2$

Failed prediction b/c of outliers

→ Solution : RANSAC

- RANSAC steps

1) Select minimal subset of points (e.g. line-2, homography-4)
2) Hypothesize the model (e.g. line formed by 2 points)
3) Compute error of all data set (e.g. shortest distance to line)
4) Select points consistent w/ model (e.g. distance < threshold → inlier)
5) Repeat 2)-4) and record # of inliers/support set
6) Select model w/ highest # of inliers & re-estimate model w/ inliers

- RANSAC parameters
  - s : # of points → typically min # to define model
  - t : distance threshold → chosen empirically or set $t^2 = 3.84\sigma^2$
  - N : # of samples

↓ Statistical theorem

$$N = \frac{\log(1-p)}{\log(1-w^s)}$$

where p : Prob that at least one of random samples are free from outliers

w : prob that selected point is inlier

↓

* if $s \uparrow$, N increases exponentially $\therefore$ use min # for s

- Adaptive RANSAC

  In reality, w is unknown $\therefore$ (i) select worst case $w = 0.5$ (ii) adaptive w

  - Algorithm

    $N = \infty$, count $= 0$

    while $N >$ count

    1) choose sample and count inliers
    2) s = # inliers / # points
    3) $N = \log(1-p) / \log(1-w^s)$ w/ $p = 0.99$
    4) count += 1

- Robust 2D Homography Estimation

  1) Find feature/interest points w/ descriptors
  2) Match keypoints using descriptors] SIFT, SURF, ORB ...
  3) RANSAC robust estimation of H
  4) Re-estimate H using inliers