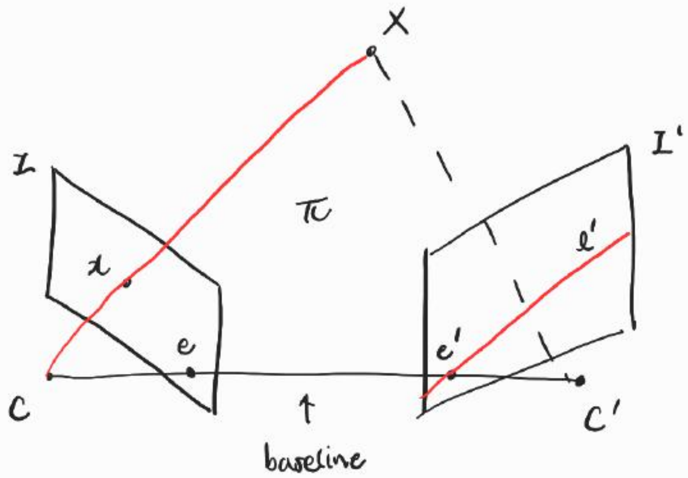
- The Epipolar geometry



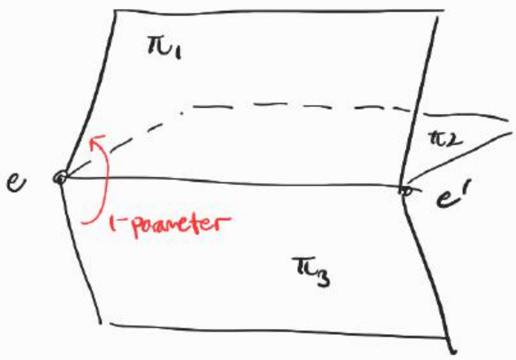
⇒ Constraints defined by two different views
s.t. eipoplar plane π spanned by ray $\overline{CX}$ and baseline

image (I) point x $\xrightarrow{\text{backprojects}}$ ray $\overline{CX}$ $\xrightarrow[I']{\text{projects}}$ epipolar line l'

x' must lie on l'

∴ Search space reduced from 2D(I') → 1D(l')

- Terminology
  - Epipole (e, e')
    ↳ intersection of baseline (join of C, C') and image plane (I, I')
    e : image projection of C' to I , e' : image projection of C to I'
    Vanishing point of baseline direction
  - Epipolar plane (π)
    ↳ plane defined by backprojected ray of x and baseline
    Any point correspondences (x, x') define π w/ one-parameter family revolving around fixed baseline



where π_i defined under (x_i, x_i') pair

  - Epipolar lines (l, l')
    ↳ intersection of epipoplar plane (π) and image plane (I, I')
    intersects epipole i.e. $l = x \times e$, $l' = x' \times e'$
    Each point correspondences (x, x') define unique l, l'

- The Fundamental Matrix

↳ linear mapping from point in one view to epipolar line in another view

$x \mapsto l'$ or $x' \mapsto l$

- Geometric derivation

2-step decomposition
1) Find homography mapping H_π $x \mapsto x'$: $x' = H_\pi x$
2) Calculate l' : line joining $e' = P'C$ and $x' = Hx$

$\therefore l' = e' \times x' = [e']_\times x' = \underbrace{[e']_\times H_\pi x}_{\text{Fundamental matrix } F}$

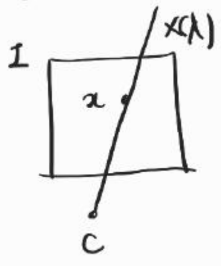
$F = [e']_\times H_\pi$
$\uparrow$ $\uparrow$ $\uparrow$
2 2 3 rank

$*$ $a \in R^{3\times1} \rightarrow [a]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in R^{3\times3}$

s.t. $\text{rank}([a]_\times) = 2$ $\because$ Last row is linearly dependent

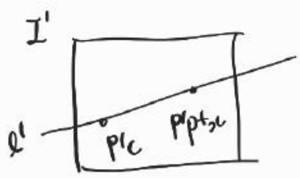
$*$ for homography, 3D points required to be at particular plane, but fundamental matrix x requires such requirement

- Algebraic derivation



back-projected ray : $X(\lambda) = P^+x + \lambda C$

$\lambda = 0$: $X(0) = P^+x$
$\lambda = \infty$: $X(\infty) \rightarrow C$
→ Project 2 points to I' :
$P'P^+x$, $P'C$



since $P'C = e'$ and $P'P^+x$ is projected point,

$l' = (P'C) \times (P'P^+x)$

$= \underbrace{[e']_\times (P'P^+) x}_{\text{Fundamental matrix } F}$

$\therefore F$
Geometric : $[e']_\times H_\pi$
Algebraic : $[e']_\times P'P^+$
→ $H_\pi = P'P^+$ (i.e. homography not requires 3D point X to derive)

* If $C = C'$, algebraic derivation fails

$\because\ e' = P'C = P'C' = 0 \ \rightarrow\ F = [e']_\times P'P^+ = 0 \times (P'P^+) = 0$

• Example

$P = K[I\,|\,0],\ P' = K'[R\,|\,t] \ \rightarrow$ then using $PP^+ = I,\ P^+ = [K^{-1}\ 0^T]^T \in \mathbb{R}^{4\times3}$ & $C = [0\ 1]^T$

$\therefore\ F = [e']_\times P'P^+ = [P'C]_\times P'P^+ = [K't]_\times K'RK^{-1} = \cdots = K'^{-T}RK^T[KR^Tt]_\times$

Also, $e = PC' = P\begin{pmatrix} -R^Tt \\ 1 \end{pmatrix} = KR^Tt$ * $P' = K'[R|t] = K'[R\,|\,\underbrace{-R\tilde{C}}_{=t}] \ \rightarrow\ t = -R\tilde{C},\ \tilde{C} = -R^Tt$

$e' = P'C = P'\begin{pmatrix} 0 \\ 1 \end{pmatrix} = K't$

$$F = [e']_\times P'P^+ = K'^{-T}RK^T[e]_\times$$

• Correspondence condition

$x'^T F x = 0$ for corresponding pair (x, x')

$\because\ l' = Fx \ \rightarrow\ x'^T l' = 0$ (as x' lies on l')

$\Rightarrow$ Able to derive F in homography-like way w/o camera matrices

• Properties

(P, P') camera w/ $F \ \leftrightarrow\ (P', P)$ camera w/ F^T $\ \because\ (x'^T F x)^T = x^T F^T x' = 0$

$l' = Fx \ \leftrightarrow\ l = F^T x'$ $\ \because\ x'^T(Fx) = x^T(F^T x') = 0$

$e'^T l' = (e'^T F)x = 0 \ \rightarrow\ e'^T F = 0$ e' is left null-space of F

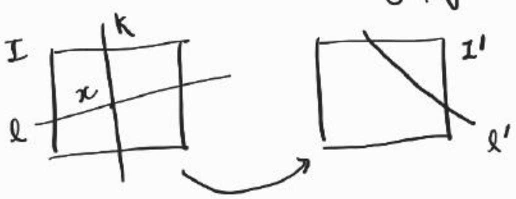
$e^T l = (e^T F^T)x' = 0 \ \rightarrow\ Fe = 0$ e is right ——

F has 7 dof : 9 elements − scale(1) − $\det(F) = 0$ (1)

F is not invertible $l = Fx \longrightarrow x = F^{-1}l$ doesn't exist!

For pair of (P, P'), F is uniquely defined

• The Epipolar Line Homography



$l' = F[k]_\times l$

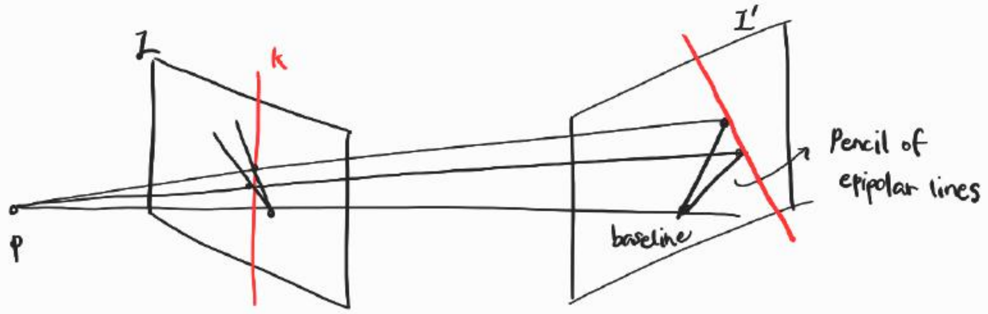
where k is any line not passing e

$\because$ Intersection of l and k : $k \times l = [k]_\times l = x$

$\downarrow$ since $l' = Fx$

$l' = \underbrace{F[k]_\times l}_{H \text{ mapping } \mathbb{P}^2 \mapsto \mathbb{P}^2}$

Similarly, $l = F^T[k']_\times l'$



- Corresponding points of x lies on straight line ($\because$ linear mapping)
- $p - x - x'$ for all corresponding points meet at projective point p

  ↳ cross-ratio is invariant, correspondence b/w epipolar lines is 1D homography

- Special case : pure translation

$$P = K[I|0], \quad P' = K[I|t] \quad \rightarrow \quad F = [e']_\times K' R K^{-1} = [e']_\times \quad (F \text{ has rank} = 2)$$

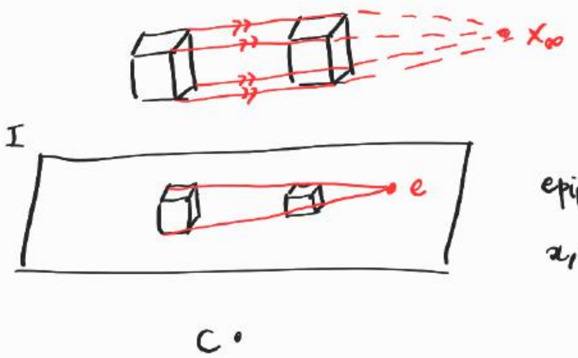
$$l' = Fx = [e']_\times x \quad \rightarrow \quad x'^T l' = \underline{x'^T [e]_\times x = 0}$$

↳ x', e, x are collinear

$\therefore$ In pure translation auto-epipolar (collinearity property) holds

- Alternative interpretations

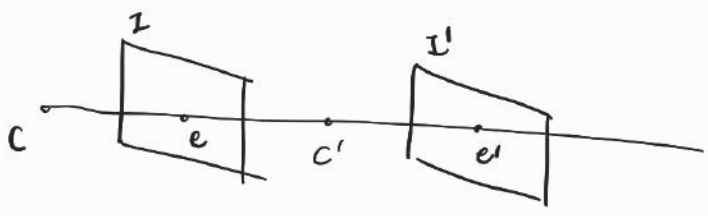
(i) fixed camera, world w/ $-t$ translation



epipole e is vanishing point ($e = PX_\infty$)

x, x', e are collinear

(ii) Camera center moves forward



e and e' remains unchanged

→ epipolar lines remains unchanged

BUT, position of $x \leftrightarrow x'$ changes

$\therefore$ 

$\because$ collinearity preserved

→

points moving radially outwards

- General motion

  ↳ Composition of pure rotation and translation

  for $P = K[I|0]$, $P' = K'[R|t]$ $\begin{cases} \text{Pure rotation : } H_\infty = K'RK^{-1} \\ \text{pure translation: } \tilde{F} = [e']_\times \end{cases}$

  $\therefore F = \tilde{F}H_\infty = [e']_\times K'RK^{-1}$, which matches previous algebraic derivation

- Retrieving camera matrices

  $F \xrightarrow[\text{retrieve}]{} P, P'$

  - Projective invariance : (P, P') and $(PH, P'H)$ share same F

    $\because$ As $P \to PH$ and $X \to H^{-1}X$, $PX \to (PH)(H^{-1}X) = PX$

    $\Rightarrow x \leftrightarrow x'$ are corresponding points $\begin{cases} \text{cameras } (P, P') \text{ w/ 3D point } X \\ \text{cameras } (PH, P'H) \text{ w/ 3D point } H^{-1}X \end{cases}$

  $\therefore$ Given $(P, P') \longrightarrow$ unique $F = [Pc']_\times P'P^+$

  Given $F \longrightarrow$ NOT unique: (P, P') or $(PH, P'H)$

- Canonical form of camera matrices

  if $P = [I|0]$ and $P' = [M|m]$, $F = [m]_\times M$

  $\because e' = P'C = [M|m][0,0,0,1]^T = m$

  $F = [e']_\times P'P^+ = [m]_\times [M|m] \begin{bmatrix} I_{3\times3} \\ 0_{1\times3} \end{bmatrix} = [m]_\times M$

- Projective Ambiguity of cameras given F

  Let (P, P') and $(\tilde{P}, \tilde{P}')$ are two pairs of cameras sharing same F, then there exists $H_{4\times4}$ s.t. $\tilde{P} = PH$ and $\tilde{P}' = P'H$

  $\because$ Suppose using canonical form : $P = \tilde{P} = [I|0]$, $P' = [A|a]$, $\tilde{P}' = [\tilde{A}|\tilde{a}]$

  then, $F = [a]_\times A = [\tilde{a}]_\times \tilde{A}$

  <u>Lemma</u> : If rank 2 matrix $F = [a]_\times A = [\tilde{a}]_\times \tilde{A}$, then $\tilde{a} = ka$, $\tilde{A} = k^{-1}(A + av^T)$

  for non-zero constant k and 3-vector v

  $\because a^T F = a^T [a]_\times A = (a \times a)^T A = 0$, $\tilde{a}^T F = 0$

  ↳ $a, \tilde{a}$ are both left nullspace of $F \Rightarrow \tilde{a} = ka$

$[a]_\times A = [\tilde{a}]_\times \tilde{A} = [ka]_\times \tilde{A} = [a]_\times k\tilde{A} \rightarrow [a]_\times (k\tilde{A} - A) = 0$

$\hookrightarrow k\tilde{A} - A = av^T \rightarrow A = k^{-1}(A + av^T)$

$\Rightarrow P' = [A|a]$, $\tilde{P}' = [\tilde{A}|\tilde{a}] = [k^{-1}(A+av^T) | ka]$, $P = \tilde{P} = [I|0]$

Let $H = \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}v^T & k \end{bmatrix}$, then $PH = k^{-1}[I|0] = k^{-1}\tilde{P}$ $\therefore \tilde{P} = PH$ up-to-scale

$P'H = [k^{-1}(A+av^T) | ka] = [\tilde{A}|\tilde{a}] = \tilde{P}'$

$\therefore (P, P')$ and $(\tilde{P}, \tilde{P}')$ are projectively related by H if they leads same F

- Decomposition of F Matrix

$F \rightarrow (P, P')$ iff $P'^T F P$ is skew-symmetric

$\because x'^T F x = 0 \rightarrow (P'X)^T F (PX) = 0 \rightarrow X^T \underline{P'^T F P} X = 0$

$\therefore P'^T F P = F'$ is skew-symmetric

$\Rightarrow$ F can be decomposed as $P = [I|0]$, $P' = [[e']_\times F | e']$

$\because P'^T F P = [[e']_\times F | e']^T F [I|0]$

$= \begin{bmatrix} F^T [e']_\times F & 0 \\ e'^T F & 0 \end{bmatrix} = \begin{bmatrix} F^T [e']_\times F & 0 \\ 0 & 0 \end{bmatrix}$ and $F^T [e']_\times F$ is skew-symmetric

- Essential Matrix

Use for known intrinsics : $x \leftrightarrow x'$ (F) to $K^{-1}x \leftrightarrow K^{-1}x'$ (E) at normalized camera coordinates

$x'^T F x = x'^T K'^{-T} E K^{-1} x = \hat{x}'^T E \hat{x} = \hat{x}'^T [t]_\times R \hat{x} = 0$

$\boxed{F = K'^{-T} E K^{-1}, \quad E = [t]_\times R}$

as P is canonical frame

$\because F = [e']_\times P' P^+$, $P = K[I|0]$, $P' = K'[R|t]$ $\rightarrow$ then $P^+ = \begin{bmatrix} K^{-1} \\ 0_{1\times 3} \end{bmatrix}$, $C = \begin{bmatrix} 0_{3\times 1} \\ 1 \end{bmatrix}$

$\downarrow$

$F = [e']_\times P' P^+ = [P'C]_\times P' P^+ = [K't]_\times K' R K^{-1} = K'^{-T} \underbrace{[t]_\times R}_{=E} K^{-1}$

- Properties
  - 5 dof : $R(3) + t(3)$ − scale ambiguity (1)
  - Singular values : $\mathrm{diag}(\sigma_1, \sigma_2, 0)$ where $\sigma_1 = \sigma_2$ & $\mathrm{rank}(E) = 2$

- Decomposition of E Matrix

$E = [t]_\times R \rightarrow SVD(E) = U\Sigma V^T$

$\uparrow$

$E = \underbrace{(UZU^T)}_{\text{skew-symmetric}} \underbrace{(UXV^T)}_{\text{orthonormal}} = \underbrace{U(ZX)V^T}_{SVD(E)} \rightarrow$ find suitable Z and X

since E is up-to-scale and ignoring the sign

$$ZX \begin{cases} ZW = diag(1,1,0) \\ ZW^T = diag(-1,-1,0) \end{cases} \rightarrow Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$ Recovery step

$[t]_\times = UZU^T \rightarrow t = \pm U_3$ (third-column of U) as U is orthogonal, $[t]_\times$ is skew-symmetric

$R = UWV^T, UW^TU^T$ * right-hand coordinate ($det(R) > 0$)

if $det(R) < 0$, $R \leftarrow -R$

$\downarrow$

2 for t, 2 for R $\rightarrow$ 4 possible solutions for p':

3D point at front/behind two cameras

Select the case where most # of points appear in front of both cameras

- Linear 8-point Algorithm of F

$x'^T F x = 0 \rightarrow$ Let $x = (x, y, 1)^T$, $x' = (x', y', 1)^T$, $f \in R^9$ stacked version of F

Then, each point correspondances give 1 equation : $af = 0$ where $a \in R^{1 \times 9}$

$\therefore$ For n point correspondences, $Af = 0$ where $A \in R^{n \times 9}$ $\rightarrow$ least-square problem ($\because$ noise)

$\hookrightarrow$ requires $n \geq 8$ since $rank(A) = 8$

$SVD(A) = U\Sigma V^T \rightarrow f =$ last column of V

* Data normalization required like H estimation

- Singularity constraint of F

Least-square solution X ensures $rank(F) = 2$

if $rank(F) \neq 2$, epipole X exists $\because Fe = 0 \,/\, F^T e' = 0$ have X non-trivial solution

$\hookrightarrow$ epipolar line X intersects at single point

$\downarrow$ Method

Replace F into F' s.t. $\min_{F'} \|F - F'\|$ where $det(F') = 0$

$\hookrightarrow SVD(F) = U\Sigma V^T$ where $\Sigma = diag(r, s, t)$ w/ $r \geq s \geq t$

* For E, $\Sigma' = diag(\frac{r+s}{2}, \frac{r+s}{2}, 0)$

Convert $\Sigma \rightarrow \Sigma' = diag(r, s, 0)$ $\therefore F' = U\Sigma' V^T$

- Normalized 8-point algorithm of F

1) Normalization : $T = \begin{bmatrix} s & 0 & -sc_x \\ 0 & s & -sc_y \\ 0 & 0 & 1 \end{bmatrix}$ where $c =$ Centroid of data points

$\bar{d} =$ mean distance from c

$s = \frac{\sqrt{2}}{\bar{d}}$

$\hat{x}_i = Tx_i$, $\hat{x}'_i = T'x'_i$

2) Calculate $\hat{F}'$
- RANSAC
- Linear 8-point algorithm $\hat{x}_i \leftrightarrow \hat{x}_i'$
- singularity constraint

3) Denormalization : $F = T'^T \hat{F}' T$

- Normalized 8-point algorithm of E (K, K' known)

  1) Normalization : $\hat{x}_i = K^{-1} x_i$, $\hat{x}_i' = K'^{-1} x_i'$

  2) Calculate E
     - RANSAC
     - Linear 8-point algorithm $\hat{x}_i \leftrightarrow \hat{x}_i'$
     - singularity constraint

  3) Decomposition : $E \rightarrow R, t \rightarrow P = K[I|0]$, $P' = K'[R|t]$

- 3D Structure Computation

  Given $x \leftrightarrow x'$
  - uncalibrated $F \rightarrow (P, P')$ or $(PH, P'H)$
  - Calibrated $E \rightarrow R, t \rightarrow$ unique (P, P')

  find 3D point X

  - Linear Triangulation Method

    $x = PX$, $x' = P'X$

    $\downarrow$ Use $x \times x = x \times (PX) = 0$

    3 equations, in which 2 of them are linearly independent

    $\therefore$ For each $x \leftrightarrow x'$ pair,
    $$AX = \begin{bmatrix} xp^{3T} - p^{1T} \\ yp^{3T} - p^{2T} \\ x'p'^{3T} - p'^{1T} \\ y'p'^{3T} - p'^{2T} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = 0$$

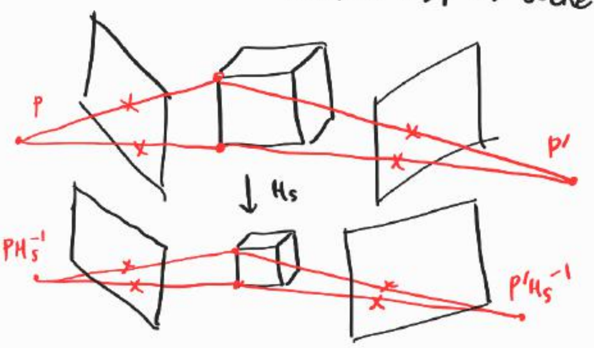
    overdetermined system

    least-squares

    $SVD(A) = U \Sigma V^T \rightarrow$ Solution : $A = V_4$ or V_4 / V_{44} to make last element 1

- Reconstruction (Similarity) Ambiguity

  with known calibration (E), 3D scene determined up to similarity



$$H_S = \begin{bmatrix} R & t \\ 0^T & \lambda \end{bmatrix}$$

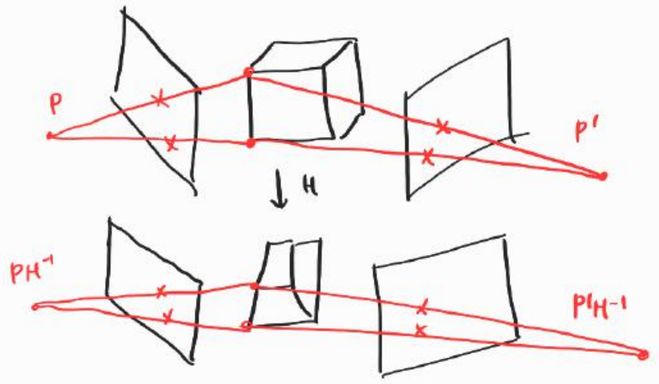
$PX_i = (PH_S^{-1})(H_S X_i) = x_i$ i.e. project to same image point

where $P = K[R_P | t_P]$

$PH_S^{-1} = K[R_P R^{-1} | t']$

- Reconstruction (Projective) Ambiguity

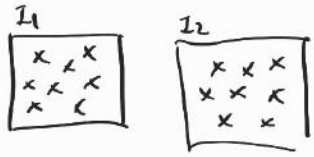
with unknown calibration (F), 3D scene determined up to Projectivity



$X\ (P, P') \longleftrightarrow HX\ (PH^{-1}, P'H^{-1})$

$\therefore PX = P'H^{-1}HX = x$

$\Rightarrow$



$\xrightarrow{\text{8-point}} F \xrightarrow{\text{decompose}} \underline{P, P'}$

projective ambiguity

- Stratified Reconstruction

Projective ambiguity $\xrightarrow[\text{Reconstruction}]{\text{Affine}}$ Affine ambiguity $\xrightarrow[\text{Reconstruction}]{\text{Metric}}$ similarity ambiguity

1) Affine Reconstruction

Find H s.t. $\pi \xrightarrow{H} \pi_\infty = (0,0,0,1)^T$

$H = \begin{bmatrix} I \mid 0 \\ \pi^T \end{bmatrix}$ where $\underline{\pi}$ is finite-like plane due to projective distortion, but actually plane of infinity

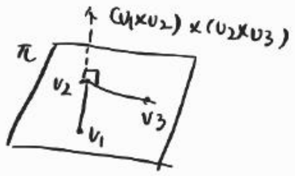
$\therefore H\pi = \pi_\infty$

Then, $X' = HX$ for all 3D points

Identity π using 3 set of known parallel lines :

If v_1, v_2, v_3 are intersection points of parallel lines

$\pi = (v_1 \times v_2) \times (v_2 \times v_3)$



2) Metric Reconstruction

Identify ω (Image of Absolute Conic)

$H = \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ where $AA^T = (M^T \omega M)^{-1} \rightarrow$ obtain A by cholesky decomposition

$P' = [M \mid m]$

$\because$ for known K', $P'_M = K'[R \mid t] = [K'R \mid K't]$

$$P'H^{-1} = [M \mid m] \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = [MA \mid m]$$

$\therefore MA = K'R$

$$MA(MA)^T = K'R(K'R)^T \longrightarrow A^TA = M^{-1}\underbrace{K'K'^T}_{\omega^* = \omega^{-1}}M^{-T}$$

$$= (M^T \omega M)^{-1}$$