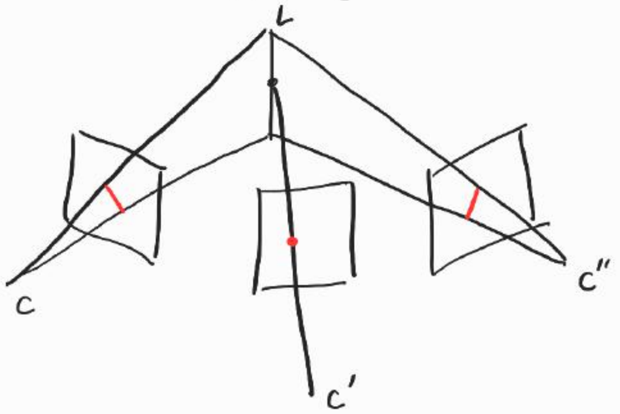
- Three-view geometry



Trifocal tensor : Extension of fundamental matrix in three views

* Can use correspondences of mixture of points/lines

e.g. three corresponding lines ($l - l' - l''$)

Back-projected planes ($\pi - \pi' - \pi''$) must intersect at unique 3D line L

↳ Such relation/constraint used to define trifocal tensor

* In two-views, two back-projected plane must intersect at line

∴ No unique relation can be derived

↓ w/ additional third view

Not generally meet at single line → geometric constraint exists

- Geometric basis of trifocal tensor

Let $P = [I \mid 0]$ (canonical), $P' = [A \mid a_4]$, $P'' = [B \mid b_4]$ where $A, B \in \mathbb{R}^{3\times3}$

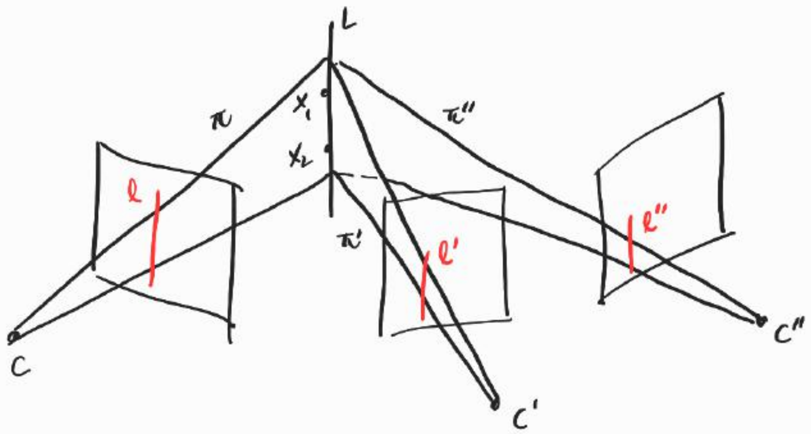
Then, $e' = P'C = [A \mid a_4][0,0,0,1]^T = a_4$

$e'' = P''C = [B \mid b_4][0,0,0,1]^T = b_4$

Also, $\pi = P^T l = \begin{pmatrix} l \\ 0 \end{pmatrix}$, $\pi' = P'^T l' = \begin{pmatrix} A^T l' \\ a_4^T l' \end{pmatrix}$, $\pi'' = P''^T l'' = \begin{pmatrix} B^T l'' \\ b_4^T l'' \end{pmatrix}$

↳ π, π', π'' are linearly dependent ∵ intersect at line

$M = [\pi, \pi', \pi'']$ are algebraic intersection constraint w/ $\text{rank}(M) = 2$



Points on L : $X = \alpha X_1 + \beta X_2$

↓ Lie on all three planes

$\pi^T X = \pi'^T X = \pi''^T X = 0$

$M^T X = 0$

↓ ∵ M has 2d null space (# of columns − rank)

$M^T X_0 = 0$, $M^T X_1 = 0$

Let $M = [m_1, m_2, m_3] = \begin{bmatrix} l & A^T l' & B^T l'' \\ 0 & a_4^T l' & b_4^T l'' \end{bmatrix}$

Apply to last row:

$0 = \alpha(a_4^T l') + \beta(b_4^T l'')$

$\therefore \alpha = k(b_4^T l'')$, $\beta = -k(a_4^T l')$

Since M isn't full rank, $m_1 = \alpha m_2 + \beta m_3$

Up to homogenous scale factor $k=1$, first row becomes:

$$l = \alpha(A^T l') + \beta(B^T l'')$$

$$= (l''^T b_4)(A^T l') - (l'^T a_4)(B^T l'')$$

Since $l = [l_1, l_2, l_3]^T$;

$$l_i = l''^T (b_4 a_i^T) l' - l'^T (a_4 b_i^T) l''^T = l'^T \underbrace{(a_i b_4^T - a_4 b_i^T)}_{T_i \in \mathbb{R}^{3\times3}} l''$$

$\therefore$ Incidence relation holds:

$l_i = l'^T T_i l''$ for $i = 1,2,3$ $\rightarrow \{T_1, T_2, T_3\} \in \mathbb{R}^{3\times3\times3}$ is trifocal tensor

$$\boxed{l^T = l'^T [T_1, T_2, T_3] l''}$$

Similarly, if $P' = [I|0]$, $l'^T = l^T [T_i'] l''$

if $P'' = [I|0]$, $l''^T = l^T [T_i''] l'$

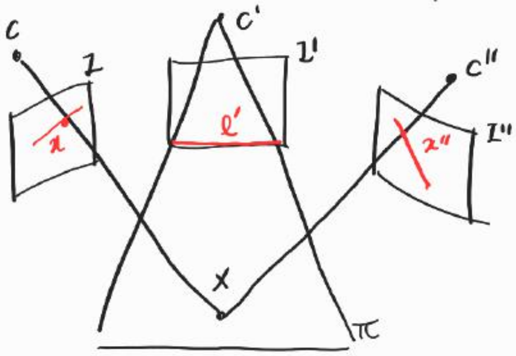
$[T_i], [T_i'], [T_i'']$ are distinct (have non-trivial relationships)

* Trifocal tensor has 18 dof

$\therefore$ P, P', P'' $(3\times(3\times4-1) = 33)$ $-$ projective ambiguity H $(4\times4-1=15) = 18$

- Homographies induced by plane

↳ fundamental geometric properties encoded by trifocal tensor



⇒ plane π back-projected from l' induces homography b/w I and I''

Also, can define homography b/w lines l and l'' w/o knowing 3D line L (similarly X).

↳ In this case, $x'' = H_{13}(l')x = [T_1^T, T_2^T, T_3^T]^T l' x$

Similarly, l'' defines homography b/w I_1 and I_2 : $x' = H_{12}(l'')x = [T_1, T_2, T_3] l'' x$

$\because$ homography b/w I and I'' : $x'' = Hx$, $\underline{l = H^T l''}$

line incidence relationships in three views : $l_i = l'^T T_i l''$ $\Big]$ $(l'^T T_i)^T = T_i^T l' = h_i$

$\therefore H_{13}(l') = [h_1, h_2, h_3]$ where $h_i = T_i^T l'$

Similarly, b/w I_1 and I_2, $l = H^T l' \rightarrow l^T = l'^T H$ $\quad \therefore H_{12}(l'') = [h_1, h_2, h_3]$ where $h_i = T_i l''$

- point and Line Incidence Relations

  - Line-Line-Line Correspondences

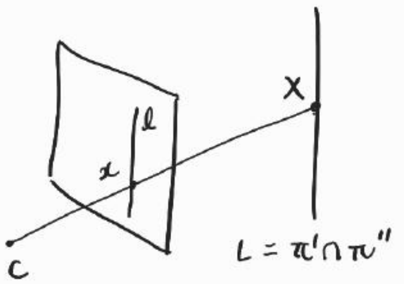
    $l^T = l'^T [T_1, T_2, T_3] l'' \rightarrow l^T \times l = 0$

    $(l'^T [T_1, T_2, T_3] l'')[l]_\times = 0^T$ or $(l'^T [T_i] l'')[l]_\times = 0^T$

    $\because l'$ and l'' are symmetric, $(l''^T [T_i^T] l')[l]_\times = 0^T$

  - point-Line-Line Correspondences

    Line-line correspondence $\times$ gives enough constraint to derive R,t



$\Rightarrow$ By adding third view, point x should lie on line l

$$\left.\begin{aligned} x^T l &= \sum_i x^i l_i = 0 \\ l_i &= l'^T T_i l'' \end{aligned}\right] \quad \begin{aligned} &\sum_i x^i (l'^T T_i l'') = 0 \\ &l'^T \underbrace{\left(\sum_i x^i T_i\right)}_{3\times3 \text{ matrix}} l'' = 0 \end{aligned}$$

  - Point-Line-point Correspondences

    e.g. $x - l' - x''$: Homography b/w x and x'' induced by back-projected plane π'

    $x'' = H_{13}(l')x = [T_1^T l', T_2^T l', T_3^T l'] x = (\sum_i x^i T_i^T) l'$

    $\downarrow$ Removing homogenous scale factor

    $x''^T [x'']_\times = l'(\sum_i x^i T_i^T)[x'']_\times = 0^T$ $\quad$ * similar step can be done to $x - x' - l''$

  - point-point-point Correspondences

    $[x']_\times (\sum_i x^i T_i)[x'']_\times = 0$

    $\because l'(\sum_i x^i T_i^T)[x'']_\times = 0^T$ from previous relationships $(x - l' - x'')$

    Since x' is on l', thus for any point y' on l' : $l' = x' \times y' = [x']_\times y'$

    $\therefore l'(\sum_i x^i T_i^T)[x'']_\times = y'^T [x']_\times (\sum_i x^i T_i^T)[x'']_\times = 0^T$

    $\uparrow$ since this is always true for $\forall x_i$ on l_i, it is independent to y'

    $\Rightarrow [x']_\times (\sum_i x^i T_i^T)[x'']_\times = 0^T$

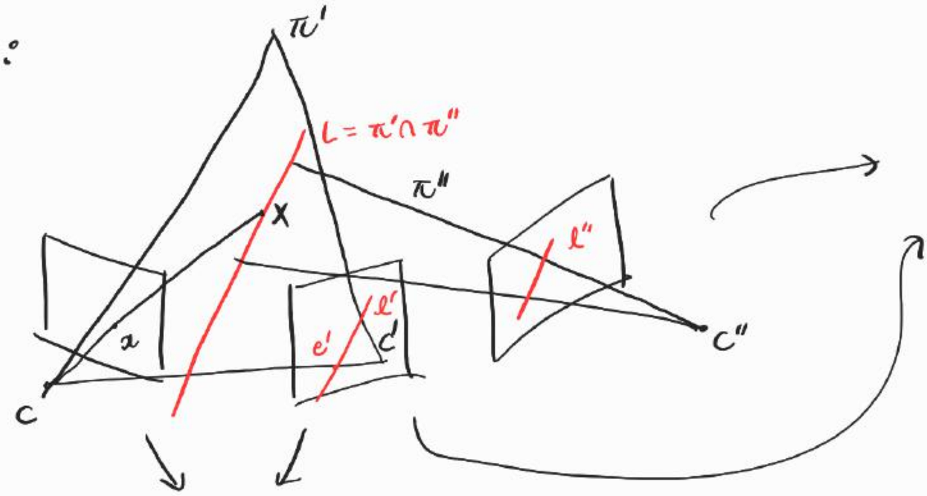
- Epipolar lines

if x is a point $\begin{cases} l' \text{ is corresponding epipolar lines} : l^T(\sum_i x^i T_i) = 0^T \\ l'' \text{ is corresponding epipolar lines} : (\sum_i x^i T_i) l'' = 0 \end{cases}$

left null vector

right null vector

$\because$



l'' back-projects to π'', and intersects w/ π' at L

$\Rightarrow$ Then X must intersect L

If X lies on plane π', ray CX lies on π'

Also, l' is epipolar line $(e' \times x')$

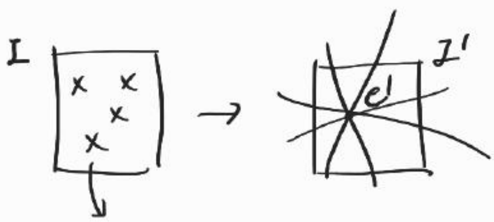
$\therefore$ 3-way intersection of $x - l' - l''$ $\Rightarrow$ $l'^T \sum_i x^i T_i l'' = 0$

$l'^T \sum_i x^i T_i = 0$ $\because$ l'' is any line w/o constraint

If l'' is epipolar line (l' is any line), $\sum_i x^i T_i l'' = 0$

- Epipole

↳ intersection of epipolar lines (projection of different back-projected rays)



Convinient choice of x : $[1,0,0]^T$, $[0,1,0]^T$, $[0,0,1]^T$

$\because$ $\sum_i x^i T_i$: T_1 T_2 T_3

$\therefore$ e' : intersection of epipolar lines $l'^T(\sum_i x^i T_i) = 0^T$ → left null-vectors

e'' : intersection of epipolar lines $(\sum_i x^i T_i) l'' = 0$ → right null-vectors

- Algebraic properties of T_i

Recall $T_i = a_i b_4^T - a_4 b_i^T$ where $P' = [A|a_4]$, $P'' = [B|b_4]$ (page 2)

$= a_i e''^T - e' b_i^T$

$\rightarrow$ $\text{rank}(T_i) = 2$ $\because$ sum of two outer products

Also, right null-vector of T_i : $l_i'' = e'' \times b_i$ for $x = (1,0,0)^T, (0,1,0)^T, (0,0,1)^T$

$\because$ $l'' = e'' \times x''$

$= (P''C) \times (P''P^+ x)$ — $P^+ \in R^{4\times3}$

$= e'' \times [b_1, b_2, b_3, b_4] \begin{bmatrix} I_{3\times3} \\ 0_{1\times3} \end{bmatrix} x = e'' \times [b_1, b_2, b_3] x$

$\therefore$ $\underline{l_i'' = e'' \times b_i}$

Since e'' is intersection of epipolar lines, $\underline{e''^T [l_1'', l_2'', l_3''] = 0}$

Similarly, $\underline{l_i' = e' \times a_i}$ and $\underline{e'^T [l_1', l_2', l_3'] = 0}$

For general x, sum of matrices $M(x) = \sum_i x^i T_i$ has rank $= 2$ $\because$ linear combination of rank 2 T_i matrices

$\downarrow$ then, can find epipolar line

$M(x) l''^T = 0$, $l'^T M(x) = 0^T$

- Extracting Fundamental Matrices

Recall $x' = ([T_1, T_2, T_3] l'') x$ shows induced homography b/w I and I'

Then epipolar line l' is the join of x' and e'

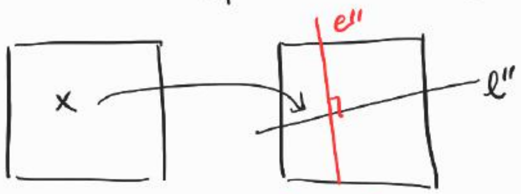
↳ $l' = \underbrace{[e']_\times ([T_1, T_2, T_3] l'')}_{\text{Fundamental Matrix } F_{21}} x$ holds for any vector l'' except degenerate condition

$F_{21} = 0$

l'' is degenerate when l'' is at null-space of any T_i i.e. $T_i l'' = 0$

↳ Good choice of l'' to avoid degeneracy $= e''$ $\rightarrow$ $\perp$ right null-space of T_i

$\because$ If l'' is epipolar line, it is right null-space of T_i as $\left(\sum_i x^i T_i\right) l'' = 0$



optimal solution

$\because$ large F_{21}

Since e'' lies on l'', $e''^T l'' = e'' \cdot l'' = 0$

$\therefore$ $e'' \perp l''$ (= right null-space of all T_i)

$\therefore F_{21} = [e]_\times [T_1, T_2, T_3] e''$

, and similarly $F_{31} = [e'']_\times [T_1^T, T_2^T, T_3^T] e'$

- Retrieving Camera Matrices

Since trifocal tensor expresses relationships of image entities, independent to projective transformations $\rightarrow \therefore P, P', P''$ derived under projective ambiguity

Let $P = [I|0]$ (canonical frame)

$\downarrow$ As $F_{21} = [a]_\times A$ if $P' = [A|a]$ & $F_{21} = [e]_\times [T_1, T_2, T_3] e''$

$[a]_\times A = [e]_\times [T_1, T_2, T_3] e''$ $\quad \therefore P' = [[T_1, T_2, T_3] e'' \,|\, e']$

Similarly, can we derive $P'' = [b]_\times B = [[T_1^T, T_2^T, T_3^T] e' \,|\, e'']$? $\rightarrow$ No! $\because P''$ depends on $\{P, P'\}$

↳ By triangulation of $\{x, x'\}$, 3D point X is already determined $\therefore P''$ has No projective ambiguity

Recall F can decomposed into $\begin{cases} P = [I|0],\ P' = [A|a] \\ \tilde{P} = [I|0],\ \tilde{P}' = [A + av^T | \lambda a] \end{cases}$ for vector v and scalar λ

$\therefore P' = [[T_1, T_2, T_3] e'' \,|\, e'] \longrightarrow \tilde{P}' = [[T_1, T_2, T_3] e'' + e' v^T \,|\, \lambda e']$

Because of projective ambiguity, free to choose $P' = [[T_1, T_2, T_3] e'' \,|\, e']$, thus $a_i = T_i e''$

$\downarrow$ then P'' can be defined uniquely (up-to-scale)

$a_i = T_i e''$ & $T_i = a_i e''^T - e' b_i^T \rightarrow T_i = T_i e'' e''^T - e' b_i^T \rightarrow e' b_i^T = T_i (e'' e''^T - I)$

Scale can be chosen s.t. $\|e'\| = e'^T e' = 1$

By multiplying e'^T & transpose on both sides, $b_i = (e'' e''^T - I) T_i^T e'$

$\therefore P'' = [B | b_4] = [(e'' e''^T - I)[T_1^T, T_2^T, T_3^T] e' \,|\, e'']$

- Summary

Given trifocal tensor $[T_1, T_2, T_3]$, 1) Calculate epipole e', e''

2) Calculate Fundamental matrix F_{21}, F_{31}

3) Calculate P', P''

- Trifocal Tensor and Tensor notation

row i, column j of matrix A : a^i_j $\quad$ * row, column are also called contravariant, covariant

↳ Image point $x = (x^1, x^2, x^3)^T$, Image line $l = (l_1, l_2, l_3)$

$\uparrow$ superscript $\qquad \uparrow$ subscript e.g. $x' = Ax$ is equivalent to $x'^i = \sum_j a^i_j x^j = a^i_1 x^1 + a^i_2 x^2 + \cdots + a^i_J x^J$

OR $= a^i_j x^j$ for simplicity

If subscript-superscript matches, Σ of matching identity occurs

If not matching, this is scalar product

- Rewriting trifocal tensor

$T \in R^{3\times3\times3}$, $T_i \in R^{3\times3}$

$T_i = a_i b_4^T - a_4 b_i^T \rightarrow T_i^{jk} = a_i^j b_4^k - a_4^j b_i^k$ where jk and row, column of matrix T_i

e.g. $l^T = l'^T [T_1, T_2, T_3] l'' \rightarrow l_i = l'_j l''_k T_i^{jk} = \sum_{jk} l'_j T_i^{jk} l''_k = l'_j T_i^{jk} l''_k$

$l_i = l'^T T_i l'' = H_i l'' \rightarrow l_i = l'_j l''_k T_i^{jk} = l''_k (l'_j T_i^{jk}) = l''_k h_i^k$ where $h_i^k = l'_j T_i^{jk}$

row k, col i of H

$x'' = Hx \rightarrow x''^k = h_i^k x^i$

- Tensor $\epsilon_{rst}, \epsilon^{rst}$

for $r,s,t = 1, \dots, 3$, $\epsilon_{rst} = \begin{cases} 0 & \text{if } r,s,t \text{ aren't distinct} \\ 1 & \text{if } r,s,t \text{ have even permutation} \\ -1 & \text{if } r,s,t \text{ have odd permutation} \end{cases}$

e.g. $\epsilon_{122} = 0$

e.g. $\epsilon_{312} = 1$ $(123 \rightarrow 132 \rightarrow 312)$

e.g. $\epsilon_{132} = -1$ $(123 \rightarrow 132)$

Then, $a \times b$ can be represented as:

$c_i = (a \times b)_i = \epsilon_{ijk} a^j b^k$ and $([a]_\times)_{ik} = \epsilon_{ijk} a^j$ / $([v]_\times)^{ik} = \epsilon^{ijk} v_j$

row i, column k — column i, row j

$\Rightarrow$ Incidence relations of 3 views can be expressed as $\epsilon_{rst}, \epsilon^{rst}$

e.g. Line-line-line correspondence

$l^T = l'^T [T_1, T_2, T_3] l'' \rightarrow (l_r \epsilon^{ris}) l'_j l''_k T_i^{jk} = 0^s \in R^{3\times1}$

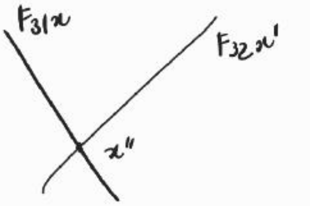
- Point transfer using F

Given $F_{21}, F_{31}, F_{32}, x, x'$, find x''

↳ x'' = intersection of epipolar lines $F_{31}x$ and $F_{32}x' = (F_{31}x) \times (F_{32}x')$

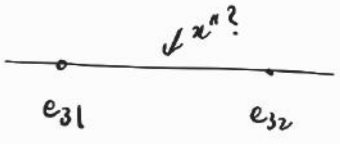
* This process is called epipolar transfer

- Degenerate case



$\longrightarrow$

if x'' lies on a line joining e_{31}, e_{32},



cannot find x''

↓ Avoid degeneracy using trifocal tensor

for point-line-point correspondence, $x''^k = x^i l'_j T_i^{jk}$ isn't degenerate

if l' is epipolar line of x, $x^i l'_j T_i^{jk} = 0$ $\therefore l'$ is at null-space of $\sum x^i T_i$ → degeneracy

∴ Select l' s.t. perpendicular to epipolar line l'_e and passing through x

$l' = (l_2, -l_1, -x_1 l_2 + x_2 l_1)^T$ → $x''^k = x^i l'_j T_i^{jk}$

- Computation of trifocal tensor - linear method

$At = 0$ where $t \in \mathbb{R}^{27\times 1}$ → Solve t using least squares problem of ≥ 26 equations

$\min_t \|At\|$ s.t. $\|t\| = 1$ → Use SVD(A) to find t

- Correspondences ⇔ # of equations

	# Eqs	
• 3 points : $x^i x'^j x''^k \varepsilon_{jqs} \varepsilon_{krt} T_i^{qr} = 0_{st}$	4	s,t,w = 1,2 ∵ rank$(T_i) = 2$ only 4 linearly independent eqs
• 2 points, 1 line : $x^i x'^j l''_r \varepsilon_{jqs} T_i^{qr} = 0_s$	2	
• 1 point, 2 lines : $x^i l'_q l''_r T_i^{qr} = 0$	1	
• 3 lines : $l_p l'_q l''_r \varepsilon^{piw} T_i^{qr} = 0^w$	2	

- Normalization

For each view [1) translation s.t. origin = centroid of points ; 2) Scaling s.t. RMS distance = $\sqrt{2}$ in avg.] H, H', H''

For lines, two endpoints are used for normalization

- Linear algorithm steps

Given any 3-view correspondences giving ≥ 26 equations.

e.g. $n \geq 7$ point correspondences ($7\times 4 \geq 26$), ≥ 13 line correspondences ($13\times 2 \geq 26$)

1) Find normalization matrix H, H', H'' → $\hat{x}^i = H^i_j x^j$, $\hat{l}_i = (H^{-1})^j_i l_j$

2) Compute $t \in \mathbb{R}^{27\times 1}$ by solving $At = 0$ → $\hat{T} \in \mathbb{R}^{3\times 3\times 3}$

3) Remove normalization $T_i^{jk} = H^r_i (H'^{-1})^j_s (H''^{-1})^k_t \hat{T}_r^{st}$

- Algebraic minimization algorithm

$\underset{t}{\operatorname{argmin}} \|At\|$ → X ensures rank$(t) = 2$

Similar to solving F, epipolar lines won't meet at single point (= epipole)

↓ enforcing constraints

Recall $(1,0,0)^T, (0,1,0)^T, (0,0,1)^T$ at first view's epipolar lines $\{v_1, v_2, v_3\}$ at third view is right null space of T_1, T_2, T_3

$\Rightarrow T_i v_i = 0$ → Two equations also face noise ∴ dependent to A,t

Also, e'' is intersection of $\{v_1, v_2, v_3\}$

$\Rightarrow e''^T [v_1, v_2, v_3] = 0$

- Retrieving the epipoles

$T_i v_i = 0 \rightarrow \underset{v_i}{\operatorname{argmin}} \|T_i v_i\|$

$e''^T [v_1, v_2, v_3] = 0 \rightarrow \underset{e''}{\operatorname{argmin}} \|V e''\|$ Compute using $SVD(T_i)$, $SVD(V)$

∴ For $P' = [A|a_4]$, $P'' = [B|b_4]$, $a_4^j = e'^j$ and $b_4^k = e''^k$ is known

$\rightarrow T_i^{jk} = a_i^j b_4^k - a_4^j b_i^k$: trifocal tensor can be written linearly $\underset{\substack{\uparrow \\ \text{known } a_4, b_4}}{t = E}\,a \leftarrow$ unknown a_i^j, b_i^k

↓ Algebraic error redefines

$\min_a \|AEa\|$ s.t. $\|Ea\| = 1$ w/ given choice of epipoles (a)

- Geometric distance algorithm

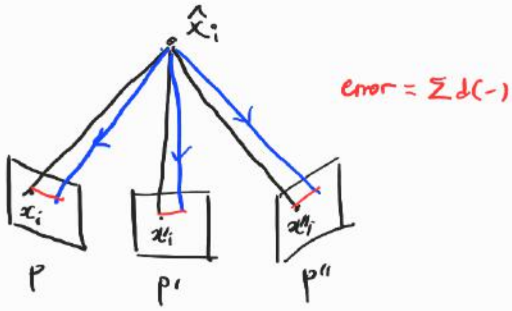
1) Initial T from algebraic error

Given $n \geq 7$ image point correspondences $\{x_i, x_i', x_i''\}$,

2) Retrieve P, P', P'' from T

3) Determine 3D point $\hat{X}_i$ using triangulation method

4) Calculate projection point : $\hat{x}_i = P\hat{X}_i$, $\hat{x}_i' = P'\hat{X}_i$, $\hat{x}_i'' = P''\hat{X}_i$



5) Minimize geometric cost : $\sum_i d(x_i - \hat{x}_i)^2 + d(x_i' - \hat{x}_i')^2 + d(x_i'' - \hat{x}_i'')^2$

↳ optimize 3n points + 12x2 P', P'' variables using non-linear optimization

e.g. Levenberg - Marquardt