- Removing projective distortions

Distortions = Affine + Similarity + Projective

↳ Projective: Removed if image of l_∞ specified

↳ Affine: Removed if image of circular points specified

- the line at infinity

l_∞ is fixed under projective transformation if H is affinity.

* if point transformation $x' = Hx$, then line transformation $l' = H^{-T} l$

$$\therefore \ l'_\infty = H_A^{-T} l_\infty = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty$$

⇒ identifying l_∞ allows recovery of affine properties (parallelism, length ratios)

↕ for general projective transformation $H_P = \begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$ where $v_1, v_2 \neq 0$

$$H_P x_\infty = x' \Rightarrow \begin{bmatrix} A & t \\ v^T & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{bmatrix} A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{bmatrix}$$

$\neq x_\infty$

May not be 0

$$H_P^{-T} l_\infty = l' \Rightarrow \begin{bmatrix} A & t \\ v^T & v \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} a_{21} v_2 - a_{22} v_1 \\ - \\ - \end{bmatrix}$$

$\neq l_\infty$

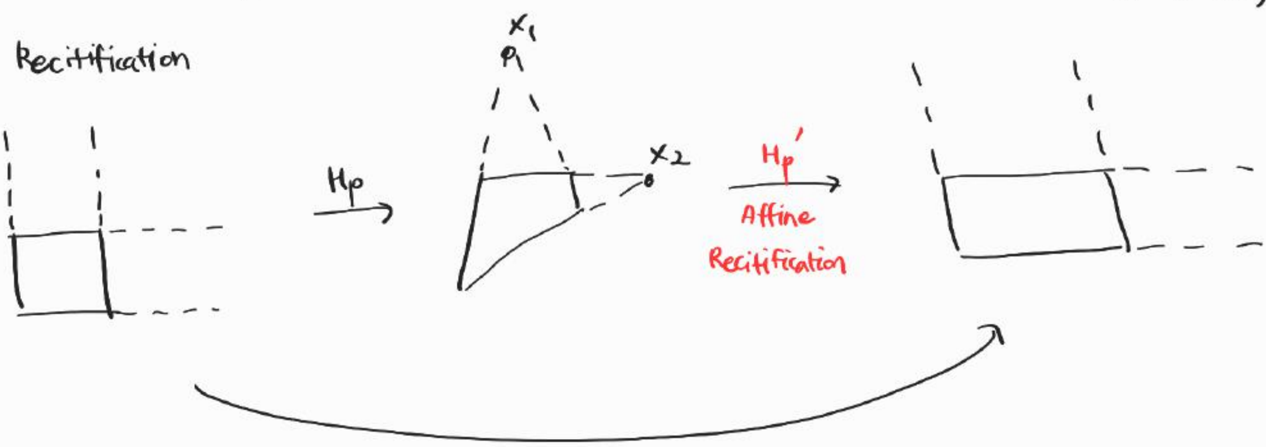
* Although $l_\infty \xrightarrow{H_A} l_\infty$, the order of points are not preserved

$$\therefore \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \\ 0 \end{pmatrix} \begin{matrix} x_1' \\ x_2' \\ \end{matrix}$$

> linear combination of x_1 and x_2
∴ order may change by A

if $A = kI = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, H_A will project points in same order $(x_1' = kx_1, \ x_2' = kx_2)$

- Affine Recitification



H_A : Parallel invariants preserved
orthogonality × preserved

$$l_\infty, x_\infty \xrightarrow{H_p} l = H_p^{-T} l_\infty,\ l \neq l_\infty \xrightarrow{H_p'} l_\infty = H_p'^{-T} l$$

$$l_\infty = H_A^{-T} l_\infty = (0\ 0\ 1)^T$$

- Finding H_p'

if $l = (l_1, l_2, l_3)^T$ where $l_3 \neq 0$ (l is finite line $\overline{x_1 x_2}$ after projection)

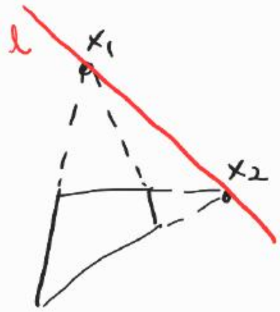
Eq.1 $l = H_p^{-T} l_\infty \Rightarrow l_\infty = H_p^T l = H_p^T (l_1, l_2, l_3)^T = (0, 0, 1)^T$

$$\therefore H_p^T = \begin{pmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{pmatrix}$$

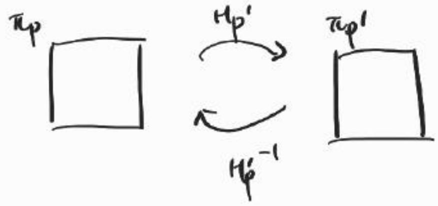
Eq.2 $$H_A = H_p' H_p \Rightarrow H_p' = H_A H_p^{-1} = H_A \begin{pmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{pmatrix}^{-T}$$

H_A can be any affine transformation $\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$

- Algorithms



1) select two pair of parallel lines at projected plane $\Rightarrow x_1, x_2$
2) Get $l = (l_1, l_2, l_3)^T$ by $l = x_1 \times x_2$
3) Find H_p' using l



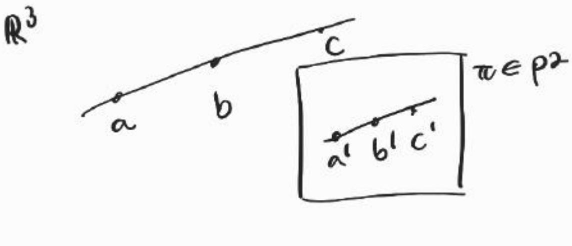
* $\pi_p' = H_p'(\pi_p)$ is not one-to-one mapping $\therefore$ Holes may appear at π_p'

$\Downarrow$

use $\pi_p = H_p'^{-1}(\pi_p')$ for every pixel of π_p'

$\Rightarrow$ Affine recitification H_p'
- preserves parallelism and ratio of lines
- NOT preserves angles (orthogonality)

- Computing vanishing point from length ratio
  - ↳ use affine invariants - length ratio to find line/point of infinity



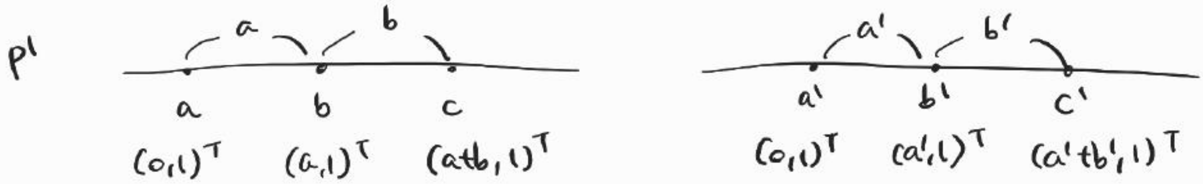
$\overline{abc}$ projected to $\overline{a'b'c'}$

- Algorithms

  if $d(a,b) : d(b,c) = a : b$ is known,

  1) Measure $d(a',b') = d(b',c') = a' : b'$

  2) Represent two lines into P^1 space w/ homogenous coordinates



  3) Calculate $H_{2\times2}$ 1D projective transformation matrix

$$\begin{bmatrix} b' \\ c' \end{bmatrix} = H_{2\times2} \begin{bmatrix} b \\ c \end{bmatrix} \rightarrow \text{4 equations, 4 unknowns}$$

  4) Calculate image of point at infinity

$$x_\infty = H_{2\times2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Circular points

  ↳ two points on l_∞ that are fixed position after similarity transformation

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \text{pair of complex conjugate ideal points}$$

* I, J are fixed under projective transformation H iff H is similarity

$$I' = H_S I = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = se^{-i\theta} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \simeq I$$

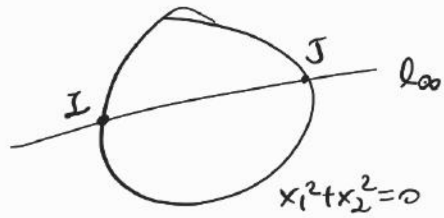
  ↳ I, J are intersection of circle and l_∞

  for general conic equation of circle $(a=c, b=0)$

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

  $x_3 = 0$ ∵ points need to intersect w/ l_∞

  then, $x_1^2 + x_2^2 = 0 \rightarrow (x_1 - ix_2)(x_1 + ix_2) = 0$



$\therefore$ $x_1 = I = (1, i, 0)^T$ and $x_2 = J = (1, -i, 0)^T$

- Dual of circular points

$C^*_\infty = IJ^T + JI^T$ (same equation to degenerate conics)

$\uparrow$ rank 2 degenerate line conic w/ I, J

In Euclidean geometry, $C^*_\infty = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}(1 \ -i \ 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}(1 \ i \ 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Fixed under similarity transformations

$$C^*_\infty = H_S C^*_\infty H_S^T$$

- properties

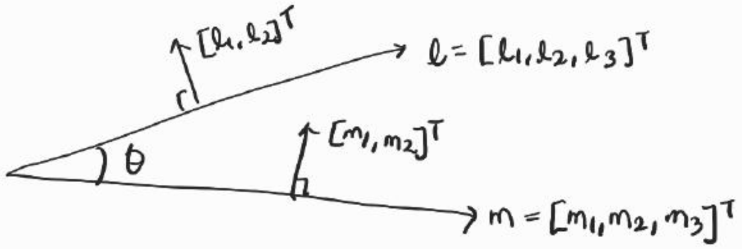
1) 4 dof : 3x3 symmetric matrix (5) − $\det(C^*_\infty) = 0$ (1) $= 4$
   $\because$ rank deficient

2) l_∞ is null vector of C^*_∞

   Since I, J are on l_∞, $I^T l_\infty = J^T l_\infty = 0$

   $\therefore$ $C^*_\infty l_\infty = (IJ^T + JI^T) l_\infty = I(J^T l_\infty) + J(I^T l_\infty) = 0$

- Angles of Projective plane



Angle b/w two lines

= Angle b/w two lines' normals

= θ

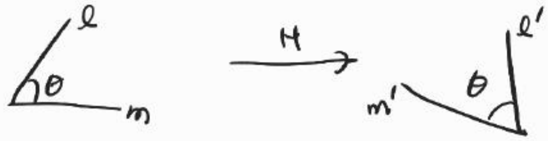
$$\cos\theta = \frac{l \cdot m}{\|l\|\|m\|} = \frac{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}^T \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

$\Rightarrow$ BUT this is only defined under Euclidean Geometry

NOT defined under Projective transformation (e.g. $l' = H^{-T} l$, $m' = H^{-T} m$)

Solution : Identify conic C^*_∞ on the projective plane

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$



this expression is invariant under projective transformation

$\because$ if $x' = Hx$, $l' = H^{-T} l$ and $C^{*\prime} = H C^* H^T$

then numerator transforms $l^T C_\infty^* m \xrightarrow{H} (l^T H^{-1})(H C_\infty^* H^T)(H^{-T} m) = l^T C_\infty^* m$

similarly, denominator is invariant under projective transformation H

↳ Also, if $l^T C_\infty^* m = 0$, $\cos\theta = 0$ $\quad \therefore l \perp m$

- Metric recitification using C_∞^*

If conic C_∞^* identified on projective plane, projective distortion may recitified up to similarity (i.e. ambiguity in similarity)

$\because$ If $x' = Hx$, then $C_\infty^{*\prime} = (H_P H_A H_S) C_\infty^* (H_P H_A H_S)^T$

$$= (H_P H_A)(H_S C_\infty^* H_S^T)(H_A^T H_P^T)$$

$$= (H_P H_A) C_\infty^* (H_A^T H_P^T) \quad \because C_\infty^* \text{ fixed under similarity}$$

$$= \begin{bmatrix} KK^T & KK^T v \\ v^T KK^T & v^T KK^T v \end{bmatrix}$$

$\therefore$ Projective image of C_∞^* has K, v element, but not similarity component

- Calculating $H(K, v)$ using SVD

$$C_\infty^{*\prime} = U C_\infty^* U^T = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

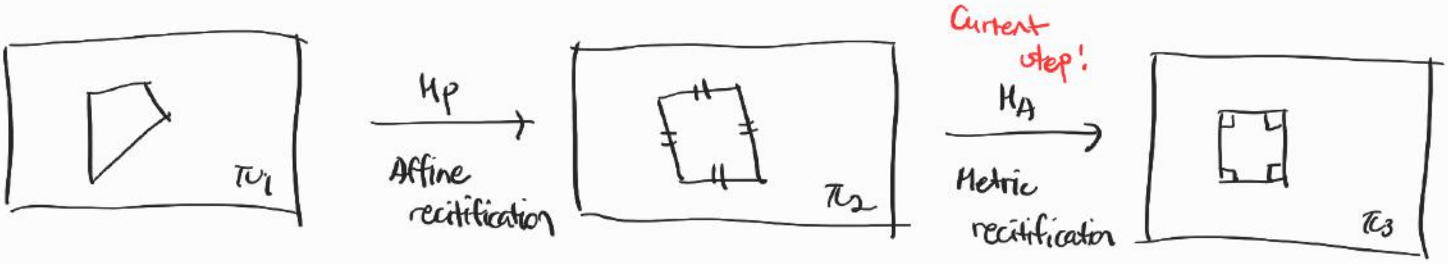
where $svd(C_\infty^*) = U \Sigma U^T$

$(U = V \because C_\infty^*$ is square matrix$)$

final diagonal element $= 0$ $\because$ rank $(C_\infty^{*\prime}) = 2$

then,

rectifying projectivity $H = U$ up to a similarity



⇒ remove projective distortion

⇒ remove affine distortion

$$C_\infty^{*'} = (H_P H_A H_S)\, C_\infty^* \,(H_P H_A H_S)^T = (H_P H_A)\, C_\infty^* \,(H_A^T H_P^T)$$

↓ Rearrange

$$\underline{H_P^{-1} C_\infty^{*'} H_P^{-T}} = H_A C_\infty^* H_A^T$$

$= C_\infty^{*''}$ which is image of conic C_∞^* after removal of Projective distortion H_P

- Steps

1) Select two pairs of orthogonal lines
2) Suppose lines l', m' are from affinely rectified image of orthogonal lines l, m at world plane
3) find $l' = [l_1', l_2', l_3']^T$, $m' = [m_1', m_2', m_3']^T$ using $l = x_1 \times x_2$ property
4) $l' C_\infty^{*''} m' = (l^T H_A^{-1}) H_A C_\infty^* H_A^T (H_A^{-T} m^T) = 0$
5) since $H_A = \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix}$, $l' \begin{bmatrix} KK^T & 0 \\ 0^T & 0 \end{bmatrix} m' = 0$
6) $S_{2\times2} = KK^T$ has 3 independent elements ∵ S is symmetric

   For each orthogonal pair, $(l_1'm_1',\ l_1'm_2' + l_2'm_1',\ l_2'm_2')(s_{11}, s_{12}, s_{22})^T = 0$

   Thus w/ two pairs, $As = 0$ where $A \in R^{2\times3}$, $s \in R^{3\times1}$

   ↑ two contraints (equations) & 3-1 (scale) = 2 unknowns

7) calculate null vector of A, which is s
8) since $s = KK^T$, calculate K (up to scale) using cholesky decomposition
9) since $H_A = \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix}$, H_A is obtained which is used to remove affine distortion (⇒ only similarity distortion is left in the image)

- Stratification

Convert 2-step process ($H_A \rightarrow H_P$) into single step ($H_P H_A$)

↳ done by identifying C_∞^* on perspective image

$$C_\infty^{*\prime} = (H_P H_A) C_\infty^* (H_A^T H_P^T) = \begin{bmatrix} KK^T & KK^T v \\ v^T KK^T & v^T KK^T v \end{bmatrix} \leftarrow \text{two unknowns : } K \text{ and } v$$

3 from K, 3 from v

Each orthogonal line pair l', m' gives constraint

$(- \; a \; -) \, c = 0$ where $a \in \mathbb{R}^{1\times 6}$, $c = (a\, b\, c\, d\, e\, f)^T$

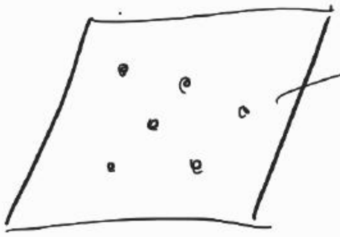
↑ 6D vector representation of $C_\infty^{*\prime}$

∴ 5 constraints (= orthogonal pairs) to solve $Ac = 0$

From c, obtain $C_\infty^{*\prime}$ ($= H_P H_A$)

- The plane at infinity

$x_\infty = (0,1)^T \rightarrow l_\infty = (0,0,1)^T \rightarrow \pi_\infty = (0,0,0,1)^T \in P^3$



→ All points $= (x_1, x_2, x_3, 0)^T$

where (x_1, x_2, x_3) are direction from finite to infinite spaces

$\pi_\infty = (0,0,0,1)^T$

Identify affine properties (e.g. parallelism)

[Two planes are parallel iff intersecting line is at π_∞

A line is parallel to another line/plane iff intersecting point is at π_∞

Property of π_∞

π_∞ is fixed under affine transformation

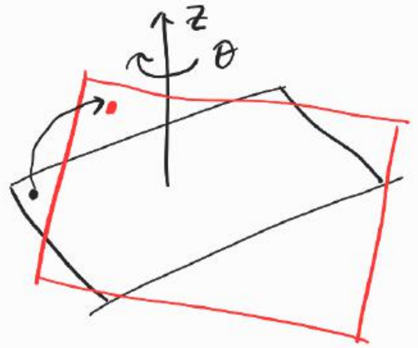
$$\pi_\infty' = H_A^{-T} \pi_\infty = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \pi_\infty$$

* Similar to l_∞, $\pi_\infty \xrightarrow{H_A} \pi_\infty$ X preserves order of points in π_∞

* particular affine transformations (eg. Euclidean) may have finite planes fixed after transformation, but π_∞ fixed for ANY affinity

Example : z-axis rotation

$$H_E = \begin{bmatrix} R & 0 \\ 0^T & 1 \end{bmatrix}$$



Planes orthogonal to z-axis are fixed under H_E

Algebrically, fixed planes of H are eigenvectors of H^T

$$H^{-T}v = \lambda v \iff H^{-T}\pi = \lambda\pi \quad (\pi \in \mathbb{R}^4)$$

for H_E, eigenvalues = diag$(e^{i\theta}, e^{-i\theta}, -1, 1)$ and eigenvectors E_1, E_2 complex

$E_3 = (0\ 0\ 1\ 0)^T \rightarrow$ plane w/ z-axis normal
$E_4 = (0\ 0\ 0\ 1)^T \rightarrow$ plane of infinity
] fixed under H_E + degenerate

$\therefore$ $\pi = \mu E_3 + \lambda E_4$ (span of E_3, E_4) are also fixed under H_E

Also, intersecting line b/w E_3 and E_4 is axis of pencil + fixed under H_E

$L^* = \begin{bmatrix} E_3^T \\ E_4^T \end{bmatrix}$ has null space basis $(1,0,0,0)^T$, $(0,0,0,1)^T$ > Point of infinity in π_∞

linear combination lies on l_∞

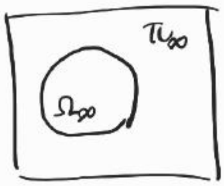
Applications

- uncalibrated two-view reconstructions → projective ambiguity

$\downarrow$ use π_∞

Affine ambiguity

- The Absolute Conic



= Point conic on π_∞

For metric frame $\pi_\infty = (0,0,0,1)^T$, two equations to define Ω_∞ :

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 = 0 \\ x_4 = 0 \end{cases}$$

if $x_4 = 0$ (i.e. direction of π_∞)

$$(x_1, x_2, x_3)\, I\, (x_1, x_2, x_3)^T = 0$$

$\uparrow$

$\therefore C = I$ is Ω_∞, and this is imaginary

* Ω_∞ requires 5 additional dof to specify metric properties in affine frame

Ω_∞ is fixed conic under projective transformation H iff H is similarity transformation

↳ $\because$ since Ω_∞ lies on π_∞, fixed under affine as $\pi_\infty \xrightarrow{H_A} \pi_\infty$

$$H_A = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$ and $\Omega_\infty = I$ at π_∞ $\Rightarrow A^{-T} I A = I$ (up to scale)

$\therefore AA^T = I$, implies A is orthogonal/rotation matrix

$$H_A = \begin{bmatrix} A = \sigma R & t \\ 0^T & 1 \end{bmatrix} = H_S$$

- Properties

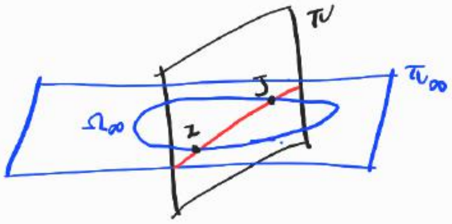
Ω_∞ shares properties of any conic

1) Ω_∞ is fixed as set under H_S, but not pointwise

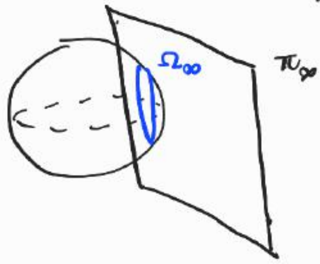
2) All circles intersect Ω_∞ in 2 points

If supporting plane of circle is π & π and π_∞ intersects at line l,

$$\begin{cases} \text{line } l \text{ intersects } \Omega_\infty \text{ at 2 points} \\ \text{These 2 points are circular points of } \pi \end{cases}$$

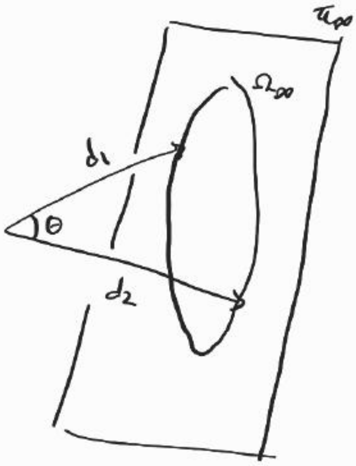


3) All spheres intersect π_∞ at Ω_∞



- Back to finding angles at projective plane with direction vectors $d_1, d_2 \in \mathbb{R}^3$,

$$\cos\theta = \frac{d_1^T \Omega_\infty d_2}{(d_1^T \Omega_\infty d_1)(d_2^T \Omega_\infty d_2)}$$



d_1, d_2 intersects w/ π_∞ containing $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

if $d_1 \perp d_2$, $\cos\theta = 0$ $\therefore d_1^T \Omega_\infty d_2 = 0$

- Application

$$\text{Affine ambiguity} \xrightarrow[\text{remove affine distortion}]{\pi_\infty,\, \Omega_\infty} \text{similarity ambiguity}$$

- Absolute dual conic

absolute conic $\Omega_\infty \xleftrightarrow{\text{duality}}$ degenerate dual conic Q^*_∞ = absolute dual/rim quadric

- Geometric : tangent planes of Ω_∞ (while absolute conic is defined by points)
- Algebraic : rank 3 4x4 homogenous matrix

$$Q^*_\infty = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$$

Q^*_∞ is degenerate conic - 8 dof : symmetric 4x4 (10) - scale (1) - det=0 (1)

Like Ω_∞, Q^*_∞ is fixed under similarity transformation

↳ ∵ Let $Q^*_\infty = H Q^*_\infty H^T$, then $\begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix} = \begin{bmatrix} A & t \\ v^T & k \end{bmatrix} \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} A & t \\ v^T & k \end{bmatrix} = \begin{bmatrix} AA^T & Av \\ v^TA^T & v^Tv \end{bmatrix}$

$\therefore AA^T = I \rightarrow A = sR$, $v = 0$ $\Rightarrow H = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix}$

π_∞ is null-vector of Q_∞^* i.e. $Q_\infty^* \pi_\infty = 0$

$$\begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

↳ this property also holds after transformation $x' = Hx$ ($Q_\infty^{*\prime} = H Q_\infty^* H^T$, $\pi_\infty' = H^{-T} \pi_\infty$)

$\because Q_\infty^{*\prime} \pi_\infty' = (H Q_\infty^* H^T)(H^{-T} \pi_\infty) = H(Q_\infty^* \pi_\infty) = 0$

☆☆ angle b/w two planes π_1, π_2 :

$$\cos\theta = \frac{\pi_1^T Q_\infty^* \pi_2}{\sqrt{(\pi_1^T Q_\infty^* \pi_1)(\pi_2^T Q_\infty^* \pi_2)}} \stackrel{\because}{=} \frac{n_1^T n_2^T}{\sqrt{(n_1^T n_1)(n_2^T n_2)}}$$

where
$\pi_1 = (n_1, d_1)^T$
$\pi_2 = (n_2, d_2)^T$
$Q_\infty^* = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$

this property holds after projective transformation