Derivation of Taylor Series

- fundamental theorem of calculus

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$\int_a^x 1 \cdot f'(t)\,dt \qquad u = t + c \quad v = f'(t)$$
$$u' = 1 \quad v' = f''(t)$$

Let $c = -x$,

$$\int_a^x 1 \cdot f'(t)\,dt = uv\Big|_a^x - \int_a^x uv'\,dt = (t-x)f'(t)\Big|_a^x - \int_a^x (t-x)f''(t)\,dt$$

$$= (a-x)f'(a) - \left(\frac{1}{2}(t-x)f''(t)\Big|_a^x - \int_a^x \frac{1}{2}(t-x)^2 f'''(t)\,dt \right)$$

...

$$= \sum_{n=1}^{\infty} \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

$$f(x) - f(a) = \sum_{n=1}^{\infty} \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

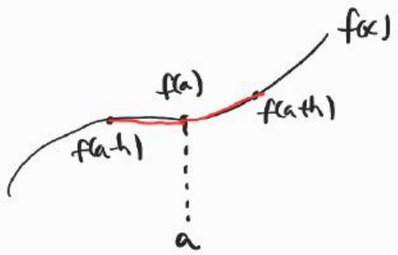
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

⇒ Advantage : Able to approximate any function into polynomial function

e.g. $y = e^x$ at $x = 0$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n e^0 = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Approximate of a function at a point (vs. fourier transform : interval, sinusodial function approx.)



use $x = a$ function value $f(a)$

& nearby points $x = a+h$, $a-h$ relationships (= derivative)

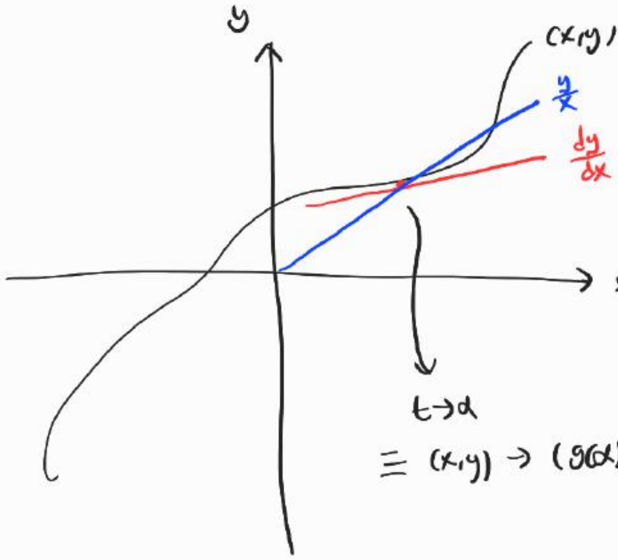
⇒ to approx. function nearby $x = a$

* Also applicable to multivariate function

Geometric meaning of L'Hopital's Rule

if $\lim_{t \to \alpha} \frac{f'(t)}{g'(t)} = L$, and $\lim_{t \to \alpha} f(t) = \lim_{t \to \alpha} g(t) = \pm\infty$ or $\lim_{t \to \alpha} f(t) = \lim_{t \to \alpha} g(t) = 0$

, then $\lim_{t \to \alpha} \frac{f(t)}{g(t)} = L$



$(x, y) = (g(t), f(t))$ Parametric equation

$\frac{f'(t)}{g'(t)} = \frac{dy}{dx}$, $\frac{f(t)}{g(t)} = \frac{y}{x}$

$\downarrow$ instantaneous rate of change

$\downarrow$ average rate of change

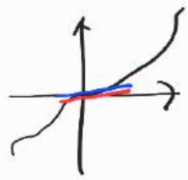
$t \to \alpha$

$\equiv (x, y) \to (g(\alpha), f(\alpha))$

(i) $\frac{0}{0}$ form

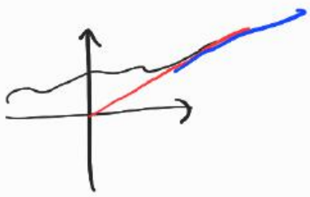
$t \to \alpha$, $f(t), g(t) \to 0$ $\Leftrightarrow$ $(x, y) \to (0, 0)$

$\therefore$ At $(0,0)$, if $\frac{dy}{dx} = L$, then $\frac{y}{x} \to L$



(ii) $\frac{\infty}{\infty}$ form

At (∞, ∞) , if $\frac{dy}{dx} = L$, then $\frac{y}{x} \to L$

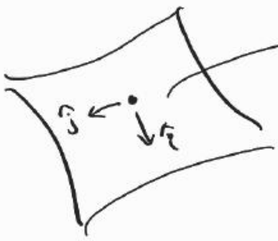


$\int x^{dx} - 1 = ?$

Gradient of Scalar field

- Partial derivative

$f(x,y) = x^2 + xy + y^2$ → derivative of f represents slope



→ derivative of two perpendicular directions (x and y)

$f_x, f_y = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

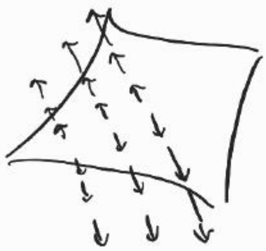
- Gradient

$\text{gradient}(f) = \nabla f = f_x \hat{i} + f_y \hat{j} = \frac{\partial}{\partial x} f(x,y)\, \hat{i} + \frac{\partial}{\partial y} f(x,y)\, \hat{j}$

↳ input : scalar (x, y), output : vector ($\hat{i}, \hat{j}$ component)

∴ gradient is an operator that forms vector field from scalar function

- Gradient of Scalar plane



e.g. $\frac{\partial}{\partial y} f(x,y)\, \hat{j}$

steepest slope of $f(x,y)$ at each point

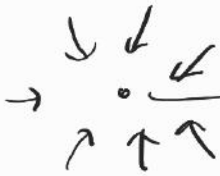
- Gradient of Scalar field

$T(x,y,z)$ → $\nabla T = T_x \hat{i} + T_y \hat{j} + T_z \hat{k}$ = direction to steepest (largest) change in T

Divergence of vector field

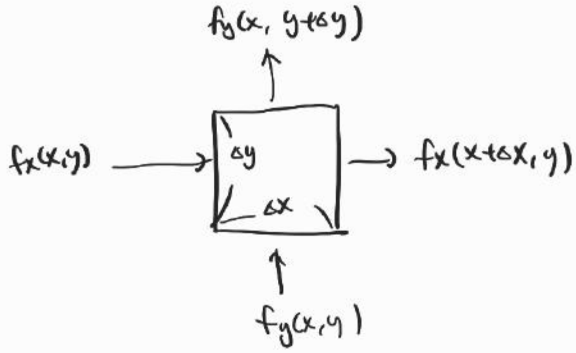
- Vector field : Each point in Euclidean space has a vector
- Divergence : operator that measures the extent whether vector field spreads out / converges at local region around (x,y)

  $\simeq$ in physics, amount of flow per unit volume



where vector field / flow converge

for infinitesimal region, we only consider difference b/w outflow and inflow of two direction



Divergence of infinitesimal region

= outflow - inflow of x and y direction

if $f(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$,

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x, y) - P(x,y)}{\Delta x} = \frac{\partial P}{\partial x}$$

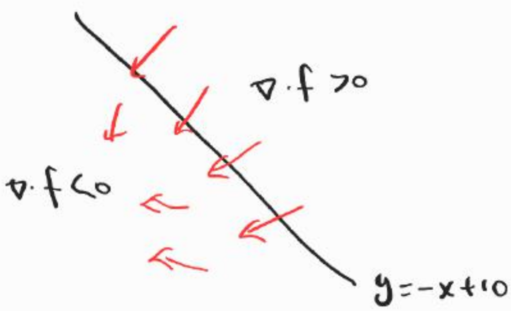
$$\lim_{\Delta y \to 0} \frac{P(x, y+\Delta y) - P(x,y)}{\Delta y} = \frac{\partial Q}{\partial y}$$

$$\therefore \nabla \cdot f = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Dot product of $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $f = (P, Q)$

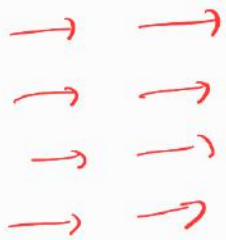
e.g. $f(x,y) = (x-2)(x-8)\hat{i} + (y-2)(y-8)\hat{j}$

$\nabla \cdot f = 2x - 10 + 2y - 10 = 2x + 2y - 20$



$f(x,y) = 2x\hat{i}$

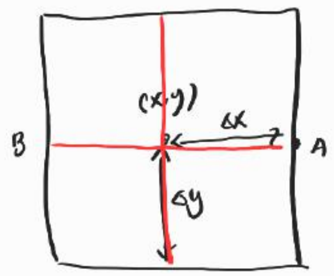
$\nabla \cdot f = 2$



Curl of vector field

⇒ measure of rotation of small region of a vector field

small region $f(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$



Extent of rotation of a small region

= rotation imposed by &

$A: f(x+\Delta x, y) = P(x+\Delta x, y)\hat{i} + Q(x+\Delta x, y)\hat{j}$

$B: f(x-\Delta x, y) = P(x-\Delta x, y)\hat{i} + Q(x-\Delta x, y)\hat{j}$