Derivative of multi-variable function

- Gradient

For multi-variable scalar function $f: \mathbb{R}^n \mapsto \mathbb{R}$,

Gradient of f $\quad \nabla f = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^{1\times n}$

- Jacobian Matrix

For multi-variable vector function $f: \mathbb{R}^n \mapsto \mathbb{R}^m$,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

$\therefore$ $f: \mathbb{R}^n \mapsto \mathbb{R}$

$f' = \nabla f$

$f'' = H$

$f: \mathbb{R}^n \mapsto \mathbb{R}^m$

$f' = J$

i-th row vector of J = ∇f_i

- Hessian matrix

For multi-variable scalar function $f: \mathbb{R}^n \mapsto \mathbb{R}$,

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Laplacian

For multi-variable scalar function $f: \mathbb{R}^n \mapsto \mathbb{R}$,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \in \mathbb{R}^{1\times n}$$

- Taylor expansion

Approximate $f(x)$ to polynomial function at local area $x=a$

$$f(x)\big|_{x=a} = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$$

If $f(\cdot)$ is multi-variable scalar function,

$$f(x)\big|_{x=a} = f(a) + \nabla f(x-a) + \frac{1}{2!}(x-a)^T H (x-a) + \cdots$$