LEARNING SEMINAR ON PRISMATIC COHOMOLOGY

1. Overview

Prismatic cohomology is a cohomology theory for algebraic varieties over p-adically complete rings that unifies various (integral) cohomology theories in the sense that they can be recovered as speical cases. In this seminar, we aim to introduce the notions of prismatic cohomology and its applications.

We will mainly follow Eilenberg lecture note by Bhatt [1] and *Prism and prismatic Cohomology* by Bhatt and Scholze [2]. The other useful references include [3].

2. Talks

2.1. **Overview talk.** Introductory talk.

2.2. **Delta rings.** The speaker will discuss the definition and examples of δ-rings , p-derivations and Frobenius lifts, the category of δ-rings and its properties (adjunction, free objects). The speaker should explain how Witt vectors induce an equivalence between the category of perfect rings of characteristic p and the category of p-adically complete perfect δ-rings.

2.3. **Distinguised elements and prisms.** The speaker will discuss the definition, examples, and properties of distinguished elements and prisms. The speaker should discuss derived completions. It is necessary to have a good theory of completions along an ideal to work effectively with prisms. Unfortunately, the rings (resp. modules) that we shall encounter are often non-Noetherian (resp. not finitely generated). In this setting, the classical theory of completion does not behave so well. This defect is remedied by the theory of derived completions. The speaker should explain how to define derived completions and their properties.

2.4. **Perfect prisms and perfectoid rings.** The speaker will discuss the notion of a perfect prism. Here we will pause briefly to explain the notion of a perfectoid ring. All of this will be done with a view towards establishing an equivalence of categories between the category of perfect prisms and the category of perfectoid rings. To go from perfectoid rings to perfect prisms, we will need Witt vectors. So, here we will take a minute to go through the basics of Witt vectors (e.g. construction, some basic examples to keep in mind, and basic properties that will be useful). The speaker should try to conclude with a discussion of the structure theorem for perfectoid rings.

2.5. **The prismatic site.** The speaker will define the prismatic site, and define the basic structure sheaves on this site. From here, we can easily define the so-called prismatic complex and the Hodge-Tate complex in separate derived categories, whose cohomologies we will call the prismatic/Hodge-Tate cohomologies. We then proceed with a little more on the prismatic site, such as defining the left adjoint to the forgetful functor from prisms over a prism (A, I) to δ-pairs over (A, I), called the *prismatic envelope*, and will likely reserve discussion of it for later when we get to the Hodge-Tate comparison theorems. We aim to conclude with a discussion of products in the prismatic site, and a few more remarks.

2.6. **The Hodge-Tate and crystalline comparison theorems.** The goal of this talk is to sketch the proof of the Hodge-Tate and crystalline comparison theorem. The speaker will introduce the divided power and relate this to the δ-rings. Then we can use this to compute the prismatic cohomology over a crystalline prism in terms of crystalline cohomology. The speaker will discuss the Hodge-Tate comparison in characteristic p case by Cartier isomorphism and the crystalline comparison. The general case follows from technical base-change arguments.

2.7. **Derived prismatic cohomology.** The speaker will introduce non-abelian derived functors and derived prismatic cohomology over a fixed base prism (A, I). This extends the prismatic cohomology functor from formally smooth A/I-algebras to arbitrary p-complete A/I-algebras. We shall use this extension later to formulate and prove the étale comparison theorem in full generality.

2.8. **Perfections in mixed characteristic.** The goal of this talk is to define perfection and perfectoidization of a p-complete A/I-algebra R. The speaker will discuss their basic properties, in particular, the independence of the base. The speaker will state and prove André's flatness lemma and conclude that for a semiperfectoid ring S, the canonical map from S to S_{perf} is surjective. This plays a key role in the almost purity theorem.

2.9. **The étale comparsion theorem.** The goal of this talk is to state the étale comparison theorem and sketch a proof. One can either follow Bhatt's note and use the necessary tools from the étale cohomology of adic spaces, or follow Kedlaya's approach using the arc topology and Artin-Schreier-Witt exact sequences to avoid any use of analytic geometry. The speaker will cover some applications including the dimension inequality between étale and de Rham cohomology.

2.10. **The q-de Rham complex.**

2.11. **q-crystalline cohomology.**

2.12. **Prismatic cohomology via THH.**

References

1. Bhargav Bhatt, *Eilenberg lectures on prismatic cohomology*, (2018).
2. Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), no. 3, 1135–1275. MR 4502597
3. Kiran Kedlaya, *Notes on prismatic cohomology*, (2021).