Numerical Analysis
Homework 5. Conjugate Gradient Methods

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1. Objective

In this assignment, I will apply conjugate gradient method to solve different size resistor networks, and observe the convergence behavior and the CPU time.

2. Approach

Formulate the "conjugate gradient method".

Algorithm. Conjugate Gradient Method

Input: initial guess $\mathbf{x}^{(0)}$ to the diagonal dominant matrix **A**, right-hand side vector **b**, converge criterion.

Output: solution when maxIteration is reached or the error between the solution of LU decomposition method and CG method is smaller than 10^{-7}.

$k = 0$;

$p^{(0)} = r^{(0)} = b - Ax^{(0)}$;

while (k < maxIteration) **and** (error > 10^{-7}) **do**

$$\alpha_k = \frac{(p^{(k)})^T r^{(k)}}{(p^{(k)})^T A p^{(k)}}$$

$$x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$$

$$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}$$

$$\beta_k = \frac{(p^{(k)})^T A r^{(k+1)}}{(p^{(k)})^T A p^{(k)}}$$

$$p^{(k+1)} = r^{(k+1)} - \beta_k p^{(k)}$$

$$error = \sqrt{\frac{(r^{(k+1)})^T r^{(k+1)}}{n}}$$

$$k = k + 1$$

end

We suppose that A is a symmetric positive definite (SPD) matrix. For general linear problems $Ax = b$, where A does not have to be a SPD matrix, but whenever A is invertible,

$$A^T Ax = A^T b,$$

$A^T A$ is a SPD matrix. Therefore, if A satisfies this property, we can apply CG method to solve this system.

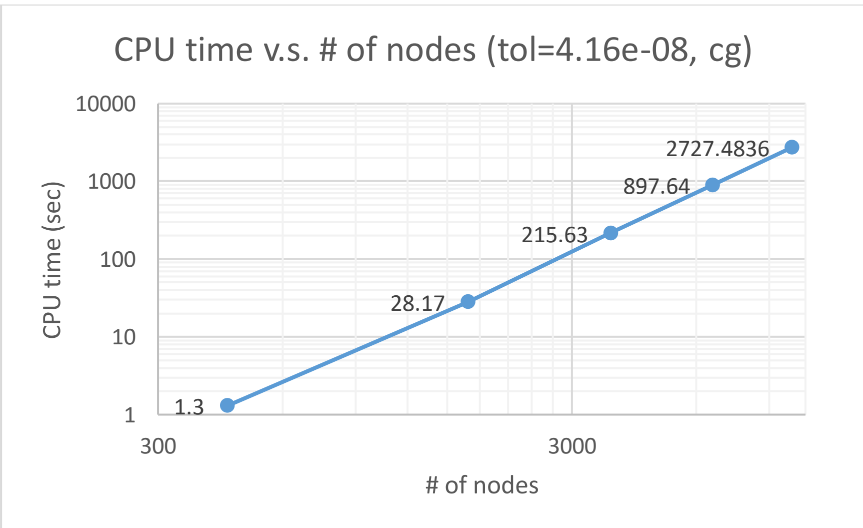
3. Computational Complexity

- Only one matrix-vector multiplication is needed for each iteration → $O(n^2)$
- Since the conjugate gradient method takes at most *n* iterations, the overall complexity is $O(n^3)$.
- In case of sparse matrix for each iteartion → $O(NZ)$, where NZ is the number of nonzero entries in the matrix.
- Sparse matrix overall complexity → $O(n * NZ)$.

4. Results

- **CPU time**

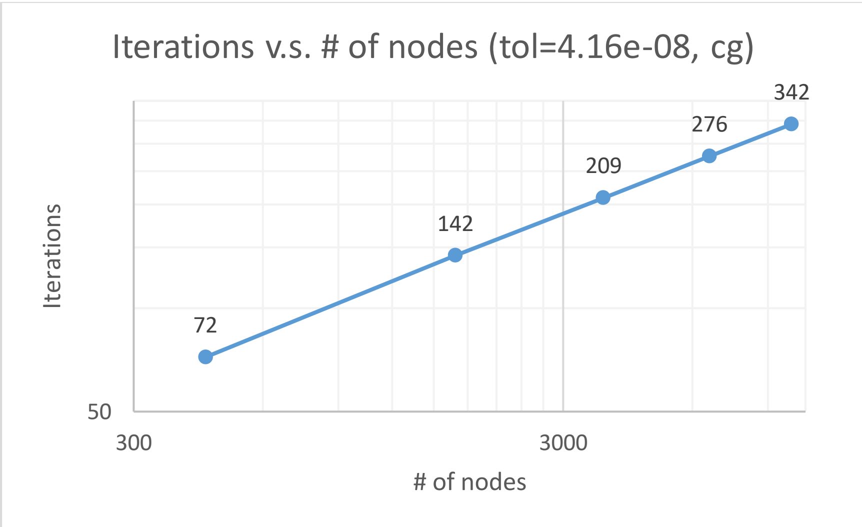
# of nodes	CPU time (sec)
441	1.3
1681	28.17
3721	215.63
6561	897.64
10201	2727.4836



- This matches the computational complexity.

- **Iteration**

# of nodes	Iterations
441	72
1681	142
3721	209
6561	276
10201	342



- This matches the computational complexity.

5. Conclusion

Conjugate Gradient method is way faster than the Jacobi, Gauss-Seidel methods.