EE407002 Numerical Analysis

Homework 6. Matrix Condition Numbers

Due: April 14, 2016

There are two parts in this homework. This first part analyzes the termination conditions for the power method that finds the largest eigenvalue of an $n \times n$ square matrix $\mathbf{A}$. And, the second part uses the termination condition that you select for both power method and inverse power method to find the matrix condition numbers of several resistor network problems.

To study the termination condition, the matrix for the resistor network needs to be modified such that the largest eigenvalue is not 1, $\lambda_1 \neq 1$. This can be done by modifying the voltage stamps. The original equation for the node voltage, v_k, connecting to a grounded voltage source of value V is

$$v_k = V, \tag{6.1}$$

This creates a row in the matrix $\mathbf{A}$ with 1 on the diagonal and 0's for all the off-diagonal entries, thus, an eigenvalue of 1 exists for the matrix $\mathbf{A}$. This equation can be easily modified to be

$$\alpha \cdot v_k = \alpha \cdot V, \tag{6.2}$$

In this case, the diagonal is set to α and the corresponding right hand side entry is set to $\alpha \cdot V$. Using this stamp, the system solution is still correct but the eigenvalue of the matrix is changed to α. In this homework we'll set $\alpha = 0.003$ such that the largest non-one eigenvalue can be found using the power method.

The core computation of the power method is the following equations:

$$\mathbf{q}^{(k+1)} = \frac{\mathbf{A}\mathbf{q}^{(k)}}{\|\mathbf{A}\mathbf{q}^{(k)}\|_2} \tag{6.3}$$

$$\nu^{(k+1)} = (\mathbf{q}^{(k)})^T \mathbf{A}\mathbf{q}^{(k)} \tag{6.4}$$

It was mentioned in the class that there are more than one way to terminate the iterations. In this homework, we will study four different termination conditions.

$$\epsilon_1 = |\nu^{(k+1)} - \nu^{(k)}|, \tag{6.5}$$

$$\epsilon_2 = \|\mathbf{q}^{(k+1)} - \mathbf{q}^{(k)}\|_2, \tag{6.6}$$

$$\epsilon_3 = \|\mathbf{r}^{(k+1)}\|_2, \tag{6.7}$$

$$\epsilon_4 = \frac{\|\mathbf{r}^{(k+1)}\|_2}{|(\mathbf{w}^{(k)})^T \mathbf{q}^{(k)}|}, \tag{6.8}$$

where we assume $\mathbf{A}$ is an $n \times n$ diagnosable matrix, $\mathbf{q}^{(0)}$ is a nonzero n-vector, and

$$\mathbf{r}^{(k)} = \mathbf{A}\mathbf{q}^{(k)} - \nu^{(k)}\mathbf{q}^{(k)}, \tag{6.9}$$

$$\mathbf{w}^{(k)} = \frac{(\mathbf{q}^{(k)})^T \mathbf{A}}{\|(\mathbf{q}^{(k)})^T \mathbf{A}\|_2}. \tag{6.10}$$

ϵ_1 is simply the difference of the calculated eigenvalues, ϵ_2 is the difference of the calculated eigenvectors, ϵ_3 is the magnitude of the residue and ϵ_4 is the condition suggested in the class handout.

1. For the first part of this home work, please use the matrix solving the 20×20 resistor network as the test case to test the four different termination conditions. Implement the power method to find the largest eigenvalue. Please record ϵ_1, ϵ_2, ϵ_3, and ϵ_4 at each iteration. Then plot out these four quantities vs. the iteration number and state which of the four termination conditions would you prefer to use.

Using the termination condition you selected to implement the following non-member functions:

```
int EVpwr(MAT &A,VEC &q0,double &lambda,double tol,int maxiter);
int EViPwr(MAT &A,VEC &q0,double &lambda,double tol,int maxiter);
int EViPwrShft(MAT &A,VEC &q0,double &lambda,double mu,double tol,int maxiter);
```

They are the *power method*, the *inverse power method* and the *inverse power method with shifting*. The arguments are matrix `A`, of which the eigenvalues are to be found; initial guess for eigenvector `q0`; eigenvalue `lambda`; tolerance for iteration termination `tol`, and the maximum number of iterations allowed `maxiter`. For the inverse power method with shifting, an additional argument, `mu`, is added. Use these functions to find the matrix condition numbers for the following resistor networks. The iterations should terminate using `tol`$=10^{-9}$. Please record the CPU time as well.

Please note that the matrix condition number is defined as

$$\kappa_2(\mathbf{A}) = \frac{\lambda_1}{\lambda_n}. \tag{6.11}$$

Again, λ_1 is the largest non-1 eigenvalue and λ_n is the smallest eigenvalue.

2. Please construct the 2×2 resistor network matrix as Q1 of `hw03` and find its condition number.

3. Please construct the 4×4 resistor network matrix as Q2 of `hw03` and find its condition number.

4. Please construct the 10×10 resistor network matrix as Q3 of `hw03` and find its condition number.

5. Please construct the 20×20 resistor network matrix as Q4 of `hw03` and find its condition number.

6. Please construct the 40×40 resistor network matrix as Q5 of `hw03` and find its condition number.

7. Please construct the 50×50 resistor network matrix as Q6 of `hw03` and find its condition number.

8. Please state your observations on the answers you found for the six questions above.

Notes.

1. For this homework you need to turn in a set of C++ source codes. That includes `hw06.cpp`, which finds all six matrix condition numbers. `MAT.h`, the new header file, `MAT.cpp`, which implements the necessary power methods, and the `VEC.h` and `VEC.cpp`. The compiled program should be executed in the same way as `hw04`. The command line argument is to be used to specified the number of resistor per side. Thus, for question 2 one needs to type the following command to run.

   ```
   $ ./a.out 2
   ```

   And the program outputs the matrix condition number and the iteration numbers.

2. A `pdf` file is also needed. Please name this file `hw06a.pdf`.

3. Submit your files on EE workstations. Please use the following command to submit your homework 6.

   ```
   $ ~ee407002/bin/submit hw06 hw06a.pdf hw06.cpp MAT.h MAT.cpp VEC.h VEC.cpp
   ```

   where `hw06` indicates homework 6.

4. Your report should be clearly written such that I can understand it. The writing, including English grammar, is part of the grading criteria.

5. All three functions should be implemented. But for questions 2 to 7 you are free to use any function to find λ_1, λ_n and κ_2.