

## Unit 7.3 Nonlinear System Solutions

Numerical Analysis

EE/NTHU

May 3, 2017

### Nonlinear Systems

- Newton's method is effective in solving nonlinear equations with only one variable.
- Need to have an effective method to solve nonlinear systems with more than one variable.
- Example. To find  $x$  and  $y$  that satisfy the following equations.

$$\begin{aligned} 2x^3 - y^2 - 1 &= 0 \\ xy^3 - y &= 0 \end{aligned} \quad (7.3.1)$$

- We can set  $\mathbf{x} = [x_1, x_2]$  and

$$\begin{aligned} F_1(\mathbf{x}) &= 2x_1^3 - x_2^2 - 1 \\ F_2(\mathbf{x}) &= x_1 x_2^3 - x_2 \end{aligned} \quad (7.3.2)$$

- Then the problem becomes to find  $\mathbf{x}^*$  such that

$$\mathbf{F}(\mathbf{x}^*) = \mathbf{0}. \quad (7.3.3)$$

- In this example,  $\mathbf{x} \in \mathbb{R}^2$ ,  $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , and is a 2-dimensional nonlinear system problem.

# Newton's Method in $N$ -Dimension

- Consider one of the equation,  $F_1(\mathbf{x})$ , around the solution point,  $\mathbf{x}^* = [x_1^*, x_2^*]$ ,

$$\begin{aligned}
 F_1(x_1^*, x_2^*) = & F_1(x_1, x_2) + (x_1^* - x_1) \frac{\partial F_1(x_1, x_2)}{\partial x_1} + (x_2^* - x_2) \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\
 & + \frac{(x_1^* - x_1)^2}{2} \frac{\partial^2 F_1(\xi_1, \xi_2)}{\partial x_1^2} + \frac{(x_2^* - x_2)^2}{2} \frac{\partial^2 F_1(\xi_1, \xi_2)}{\partial x_2^2} \\
 & + \frac{(x_1^* - x_1)(x_2^* - x_2)}{2} \frac{\partial^2 F_1(\xi_1, \xi_2)}{\partial x_1 \partial x_2}
 \end{aligned} \quad (7.3.4)$$

where  $(\xi_1, \xi_2)$  is in the neighborhood of  $(x_1^*, x_2^*)$ .

- Suppose  $|x_1^* - x_1| \ll 1$  and  $|x_2^* - x_2| \ll 1$  then

$$F_1(x_1^*, x_2^*) \approx F_1(x_1, x_2) + (x_1^* - x_1) \frac{\partial F_1(x_1, x_2)}{\partial x_1} + (x_2^* - x_2) \frac{\partial F_1(x_1, x_2)}{\partial x_2} \quad (7.3.5)$$

- And we have

$$(x_1^* - x_1) \frac{\partial F_1(x_1, x_2)}{\partial x_1} + (x_2^* - x_2) \frac{\partial F_1(x_1, x_2)}{\partial x_2} = -F_1(x_1, x_2). \quad (7.3.6)$$

- By the same process, we also have

$$(x_1^* - x_1) \frac{\partial F_2(x_1, x_2)}{\partial x_1} + (x_2^* - x_2) \frac{\partial F_2(x_1, x_2)}{\partial x_2} = -F_2(x_1, x_2). \quad (7.3.7)$$

## Newton's Method in $N$ -Dimension, II

- Combining Eqs. (7.3.6) and (7.3.7), and arrange in matrix form

$$\begin{bmatrix} \frac{\partial F_1(x_1, x_2)}{\partial x_1} & \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1, x_2)}{\partial x_1} & \frac{\partial F_2(x_1, x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1^* - x_1 \\ x_2^* - x_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1, x_2) \\ -F_2(x_1, x_2) \end{bmatrix} \quad (7.3.8)$$

- Or

$$\begin{bmatrix} x_1^* - x_1 \\ x_2^* - x_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1(x_1, x_2)}{\partial x_1} & \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1, x_2)}{\partial x_1} & \frac{\partial F_2(x_1, x_2)}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} -F_1(x_1, x_2) \\ -F_2(x_1, x_2) \end{bmatrix} \quad (7.3.9)$$

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1(x_1, x_2)}{\partial x_1} & \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1, x_2)}{\partial x_1} & \frac{\partial F_2(x_1, x_2)}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} \quad (7.3.10)$$

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial F_1(x_1, x_2)}{\partial x_1} & \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1, x_2)}{\partial x_1} & \frac{\partial F_2(x_1, x_2)}{\partial x_2} \end{bmatrix}^{-1} \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} \quad (7.3.11)$$

- The matrix is called Jacobian matrix and is defined as

$$\mathbf{J}_{\mathbf{F}}(\mathbf{x}) = \left[ \frac{\partial F_i}{\partial x_j} \right]. \quad (7.3.12)$$

It is an  $n \times n$  matrix for  $n$ -dimensional problems.

- For the 2-dimensional problem above, we have

$$\mathbf{J}_{\mathbf{F}}(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1(x_1, x_2)}{\partial x_1} & \frac{\partial F_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1, x_2)}{\partial x_1} & \frac{\partial F_2(x_1, x_2)}{\partial x_2} \end{bmatrix} \quad (7.3.13)$$

- Using Jacobian and matrix-vector notation, Eq. (7.3.11) can be rewritten as

$$\mathbf{x}^* = \mathbf{x} - \mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}). \quad (7.3.14)$$

- Thus, the  $n$ -dimensional Newton's iteration is

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^{(k)}) \cdot \mathbf{F}(\mathbf{x}^{(k)}). \quad (7.3.15)$$

- The Newton's method in solving  $n$ -dimensional nonlinear system still have order 2 convergence property.

## Newton's Method Example

- For the nonlinear system of Eq. (7.3.2) we have

$$F_1(\mathbf{x}) = 2x_1^3 - x_2^2 - 1$$

$$F_2(\mathbf{x}) = x_1x_2^3 - x_2$$

$$\partial F_1(\mathbf{x})/\partial x_1 = 6x_1^2$$

$$\partial F_1(\mathbf{x})/\partial x_2 = -2x_2$$

$$\partial F_2(\mathbf{x})/\partial x_1 = x_2^3$$

$$\partial F_2(\mathbf{x})/\partial x_2 = 3x_1x_2^2 - 1$$

- Thus, the Jacobian matrix is

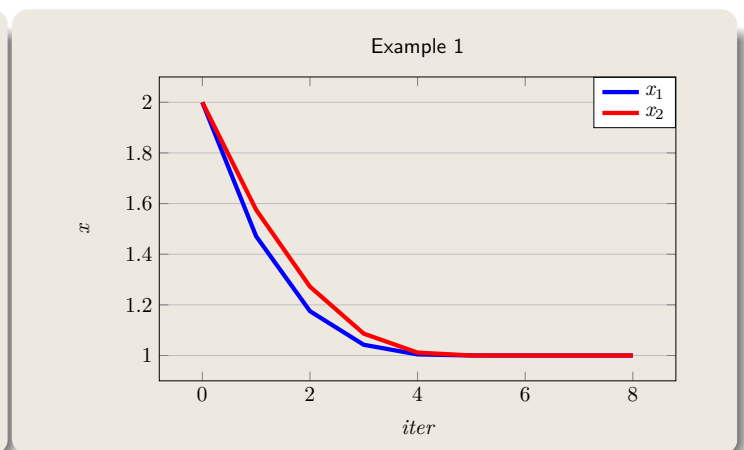
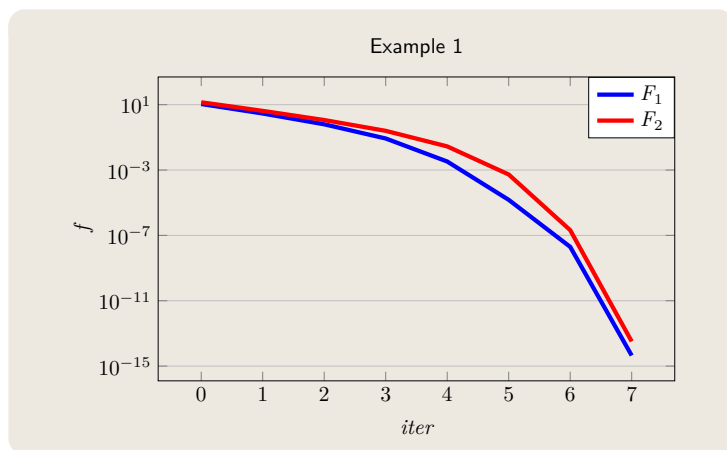
$$\mathbf{J}_{\mathbf{F}}(\mathbf{x}) = \begin{bmatrix} 6x_1^2 & -2x_2 \\ x_2^3 & 3x_1x_2^2 - 1 \end{bmatrix}$$

- Given an initial guess  $\mathbf{x}^{(0)}$ , the Newton's iteration to solve Eq. (7.3.1) is

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 6x_1^2 & -2x_2 \\ x_2^3 & 3x_1x_2^2 - 1 \end{bmatrix}^{-1} \begin{bmatrix} 2x_1^3 - x_2^2 - 1 \\ x_1x_2^3 - x_2 \end{bmatrix}$$

where the matrix inversion can be done using LU decomposition or any linear system solution method.

# Newton's Method Example, II



- Newton's method is effective in solving nonlinear systems
- The initial guess,  $\mathbf{x}^{(0)}$  is still an important issue that affects the convergence of the algorithm
  - Initial guess needs to be close to the solution  $\mathbf{x}^*$
  - In some applications, other techniques to find an approximated solution is employed first before using Newton's method for accurate solution.
  - In this example, the initial guess is  $\mathbf{x}^{(0)} = [2, 2]$ .

## Newton's Method in $N$ -Dimension, IV

### Theorem 7.3.1.

Let  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  function in a convex open set  $D$  of  $\mathbb{R}^n$  that contains  $\mathbf{x}^*$ . Suppose that  $\mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^*)$  exists and that there are positive constants  $R$ ,  $C$  and  $L$ , such that  $\|\mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^*)\| \leq C$  and

$$\|\mathbf{J}_{\mathbf{F}}(\mathbf{x}) - \mathbf{J}_{\mathbf{F}}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in B(\mathbf{x}^*, R)$$

with the consistent vector and matrix norms,  $\|\cdot\|$ . Then, there is an  $r > 0$  such that for any  $\mathbf{x}^{(0)} \in B(\mathbf{x}^*, r)$ , the Newton's iteration converges to  $\mathbf{x}^*$  with

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\| \leq CL\|\mathbf{x}^{(k)} - \mathbf{x}^*\|^2, \quad (7.3.16)$$

where  $B(\mathbf{x}, r)$  is an open ball in  $\mathbb{R}^n$  centered at  $\mathbf{x}$  with radius  $r$ .

- If  $\|\mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^*)\|$  and  $\|\mathbf{J}_{\mathbf{F}}(\mathbf{x}^*)\|$  are bounded, then the Newton's method is convergent with order 2.



- Newton's iteration is

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^{(k)}) \cdot \mathbf{F}(\mathbf{x}^{(k)}). \quad (7.3.17)$$

- It can also be written as

$$\begin{aligned} \mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)}) \delta \mathbf{x}^{(k)} &= -\mathbf{F}(\mathbf{x}^{(k)}), \\ \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \delta \mathbf{x}^{(k)}. \end{aligned} \quad (7.3.18)$$

- For each iteration
  - $\mathbf{F}(\mathbf{x}^{(k)})$  needs to be evaluated :  $n$  function evaluations
  - $\mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)})$  needs to be computed :  $n \times n$  function evaluations
    - Forming the linear system as Eq. (7.3.17) is sometimes known as **linearizing** the nonlinear system
    - Linearizing the nonlinear system is dominated by forming the Jacobian matrix.
  - $\mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}^{(k)}) \cdot \mathbf{F}(\mathbf{x}^{(k)})$  needs to be computed
    - LU decomposition takes  $\mathcal{O}(n^3)$  operations
    - Forward and backward substitutions takes  $\mathcal{O}(n^2)$  operations
  - Overall computational complexity maybe dominated by LU decomposition  $\mathcal{O}(n^3)$ .
- Solving a large nonlinear system could be very time consuming.

## Improving Nonlinear System Solution Time

- Solving nonlinear system could be slow
  - Linearizing the nonlinear system
    - Dominated by Jacobian matrix formation time
  - Linear system solution
    - Dominated by LU decomposition
- Some techniques have been developed to speed nonlinear system solution time
  - Cyclic update of Jacobian matrix
  - Difference approximations of the Jacobian matrix
  - Inexact solution of the linear system

# Cyclic Updates of Jacobian Matrix

- Typical Newton's method

## Algorithm 7.3.2. Newton's Method for $N$ -Dimensional Problems.

Given  $\mathbf{x}^{(0)}$  and a small  $\epsilon \geq 0$ , let  
 $k = 0$ ,  $err^{(0)} = 1 + \epsilon$ ,  
while ( $err^{(k)} > \epsilon$ ) {  
    evaluate  $\mathbf{F}(\mathbf{x}^{(k)})$ ,  
    evaluate  $\mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)})$ ,  
    solve  $\mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)}) \delta \mathbf{x} = -\mathbf{F}(\mathbf{x}^{(k)})$ ,  
     $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \delta \mathbf{x}$ ,  
     $k = k + 1$ ,  
     $err^{(k)} = \|\mathbf{F}(\mathbf{x}^{(k)})\|$ ,  
}

- In the above algorithm, the iteration error is essentially the residue of the nonlinear system.
- It is also possible to use  $\|\delta \mathbf{x}\|$  as the iteration error.
- Both methods have been adopted in real applications.

# Cyclic Updates of Jacobian Matrix, II

- With cyclic update of Jacobian matrix

## Algorithm 7.3.3. Cyclic Jacobian Updates

Given  $\mathbf{x}^{(0)}$  and a small  $\epsilon \geq 0$ , let  
 $k = 0$ ,  $err^{(0)} = 1 + \epsilon$ ,  
while ( $err^{(k)} > \epsilon$ ) {  
    evaluate  $\mathbf{F}(\mathbf{x}^{(k)})$ ,  
    if ( $k \% p = 0$ ) evaluate  $\mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)})$ ,  
    solve  $\mathbf{J}_{\mathbf{F}}(\mathbf{x}) \delta \mathbf{x} = -\mathbf{F}(\mathbf{x}^{(k)})$ ,  
     $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \delta \mathbf{x}$ ,  
     $k = k + 1$ ,  
     $err^{(k)} = \|\mathbf{F}(\mathbf{x}^{(k)})\|$ ,  
}

- The most expensive operation in linearizing the nonlinear system is not repeated at every iteration.
  - CPU time for each iteration is improved.
  - Convergent rate might be affected.
  - Careful selection of  $p$  can be critical.

# Difference Approximation of the Jacobian Matrix

- In case that explicit Jacobian matrix is difficult to evaluate, then the Jacobian can approximate numerically

$$(\mathbf{J}_h^{(k)})_j = \frac{\mathbf{F}(\mathbf{x}^{(k)} + h_j^{(k)} \mathbf{e}_j) - \mathbf{F}(\mathbf{x}^{(k)})}{h_j^{(k)}}. \quad (7.3.19)$$

where  $\mathbf{e}_j$  is the  $j$ -th unit vector of the space  $\mathbb{R}^n$  and  $h_j^{(k)} > 0$  is a small increment at iteration  $k$ .

- $(\mathbf{J}_h^{(k)})_j$  is a column vector and for each  $j$  the subtraction and division need to be carried out  $n$  times, and the overall Jacobian still needs  $n + 1$   $\mathbf{F}(\mathbf{x})$  (vector functions) evaluations or  $n \times (n + 1)$  scalar function evaluations.
- The small increment  $h$  needs to be small for accurate partial derivative calculations.
  - But, if  $h$  is too small then computer round-off error might increase.
  - Selecting proper  $h$  is important.
  - For the first few iterations, Jacobian is known to be inaccurate and thus larger  $h$  can be used.
  - When  $\mathbf{x}^{(k)} \rightarrow \mathbf{x}^*$ , then more accurate Jacobian can improve convergence rate. Smaller  $h$  is preferred.

## Inexact Solution of the Linear System

- In case that the LU decomposition time dominates the total solution time, faster but less accurate linear solution methods might be adopted.
- Iterative solution methods may need large number of iterations for accurate solution of the linear system.
- But the in the early phase of Newton's iteration, the solution needs not be very accurate.
- Each iteration of the linear iterative solution method improves the solution accuracy.
- Thus, one can perform linear iterative method for a fixed number of iterations or with a lower accuracy tolerance in the early phase of Newton's iterations.
- Note that the total number of function evaluations might be increased.
- Thus, this technique is valuable if linearizing the nonlinear system takes small portion of the CPU time.

# Bairstow's Method

- Polynomial's quadratic factors can be found by Lin's quadratic method.
- But, Lin's method has linear convergence rate.
- Newton's method can be applied to speed up the convergence.
  - Bairstow's method.
- Recall that the  $n$  degree polynomial  $P_n(x)$  is to be factorized as

$$\begin{aligned}P_n(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\&= (x^2 + px + q)(b_{n-2} x^{n-2} + b_{n-3} x^{n-3} + \cdots + b_1 x + b_0) + Rx + S.\end{aligned}$$

Where  $b_{n-2}, b_{n-3}, \dots, b_1, b_0, R, S$  have been shown to be

$$\begin{aligned}b_{n-2} &= a_n \\b_{n-3} &= a_{n-1} - pb_{n-2} \\b_{n-4} &= a_{n-2} - pb_{n-3} - qb_{n-2} \\&\vdots\end{aligned}$$

$$b_0 = a_2 - pb_1 - qb_2$$

$$R = a_1 - pb_0 - qb_1$$

$$S = a_0 - qb_0$$

(7.3.20)

## Bairstow's Method, II

- And we seek  $p$  and  $q$  such that

$$R = 0$$

$$S = 0$$

Or

$$R = a_1 - pb_0 - qb_1 = 0$$

$$S = a_0 - qb_0 = 0$$

- To apply Newton's method, we need to find  $\frac{\partial R}{\partial p}, \frac{\partial R}{\partial q}, \frac{\partial S}{\partial p}, \frac{\partial S}{\partial q}$ , to form the iterations.

$$\begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} = \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix} - \begin{bmatrix} \frac{\partial R}{\partial p} & \frac{\partial R}{\partial q} \\ \frac{\partial S}{\partial p} & \frac{\partial S}{\partial q} \end{bmatrix}^{-1} \begin{bmatrix} R(p^{(k)}, q^{(k)}) \\ S(p^{(k)}, q^{(k)}) \end{bmatrix} \quad (7.3.21)$$



- From Eq. (7.3.20) we have

$$\frac{\partial b_{n-2}}{\partial p} = 0$$

$$\frac{\partial b_{n-3}}{\partial p} = -b_{n-2} - p \frac{\partial b_{n-2}}{\partial p}$$

$$\frac{\partial b_{n-4}}{\partial p} = -b_{n-3} - p \frac{\partial b_{n-3}}{\partial p} - q \frac{\partial b_{n-2}}{\partial p}$$

...

$$\frac{\partial b_0}{\partial p} = -b_1 - p \frac{\partial b_1}{\partial p} - q \frac{\partial b_2}{\partial p}$$

$$\frac{\partial R}{\partial p} = -b_0 - p \frac{\partial b_0}{\partial p} - q \frac{\partial b_1}{\partial p} \quad (7.3.22)$$

$$\frac{\partial S}{\partial p} = -q \frac{\partial b_0}{\partial p} \quad (7.3.23)$$

$$\frac{\partial b_{n-2}}{\partial q} = 0$$

$$\frac{\partial b_{n-3}}{\partial q} = -p \frac{\partial b_{n-2}}{\partial q}$$

$$\frac{\partial b_{n-4}}{\partial q} = -p \frac{\partial b_{n-3}}{\partial q} - b_{n-2} - q \frac{\partial b_{n-2}}{\partial q}$$

...

$$\frac{\partial b_0}{\partial q} = -p \frac{\partial b_1}{\partial q} - b_2 - q \frac{\partial b_2}{\partial q}$$

$$\frac{\partial R}{\partial q} = -p \frac{\partial b_0}{\partial q} - b_1 - q \frac{\partial b_1}{\partial q} \quad (7.3.24)$$

$$\frac{\partial S}{\partial q} = -b_0 - q \frac{\partial b_0}{\partial q} \quad (7.3.25)$$

# Bairstow's Method, IV

- Let  $c_j = \frac{\partial b_j}{\partial p}$  and  $d_j = \frac{\partial b_j}{\partial q}$ , then we have

$$c_{n-2} = 0$$

$$c_{n-3} = -b_{n-2} - p c_{n-2}$$

$$c_{n-4} = -b_{n-3} - p c_{n-3} - q c_{n-2}$$

...

$$c_0 = -b_1 - p c_1 - q c_2$$

$$\frac{\partial R}{\partial p} = -b_0 - p c_0 - q c_1 \quad (7.3.26)$$

$$\frac{\partial S}{\partial p} = -q c_0 \quad (7.3.27)$$

$$d_{n-2} = 0$$

$$d_{n-3} = -p d_{n-2}$$

$$d_{n-4} = -p d_{n-3} - b_{n-2} - q d_{n-2}$$

...

$$d_0 = -p d_1 - b_2 - q d_2$$

$$\frac{\partial R}{\partial q} = -p d_0 - b_1 - q d_1 \quad (7.3.28)$$

$$\frac{\partial S}{\partial q} = -b_0 - q d_0 \quad (7.3.29)$$

- Thus, to find a quadratic factor  $x^2 + px + q$  of an  $n$  degree polynomial,

$$P_n(x) = \sum_{k=0}^n a_k x^k, \text{ we have the following algorithm}$$

# Bairstow's Method, V

## Algorithm 7.3.4. Bairstow's Method

Given  $p^{(0)}$ ,  $q^{(0)}$ , and integer  $maxiter$  and a small number  $\epsilon$ ,  
let  $err = 1 + \epsilon$ ,  $k = 0$

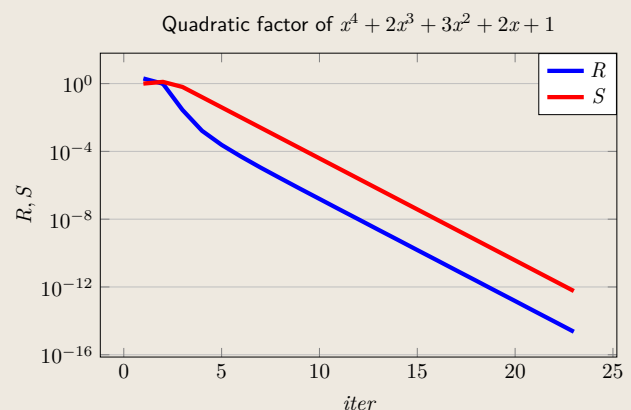
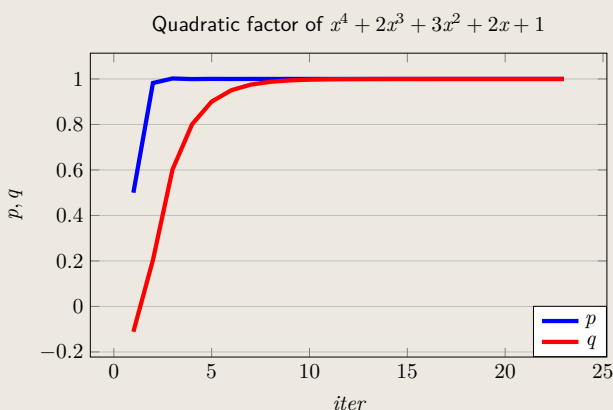
while ( $err \geq \epsilon$ ) {

$$\begin{aligned} b_{n-2} &= a_n, & b_{n-3} &= a_{n-1} - p^{(k)} b_{n-2}, \\ b_j &= a_{j+2} - p^{(k)} b_{j+1} - q^{(k)} b_{j+2}, & j &= n-4, \dots, 0, \\ R &= a_1 - p^{(k)} b_0 - q^{(k)} b_1, & S &= a_0 - q^{(k)} b_0, \\ c_{n-2} &= 0, & c_{n-3} &= -b_{n-2} - p^{(k)} c_{n-2}, \\ c_j &= -b_{j+1} - p^{(k)} c_{j+1} - q^{(k)} c_{j+2}, & j &= n-4, \dots, 0, \\ \frac{\partial R}{\partial p} &= -b_0 - p^{(k)} c_0 - q^{(k)} c_1, & \frac{\partial S}{\partial p} &= -q^{(k)} c_0, \\ d_{n-2} &= 0, & d_{n-3} &= -p^{(k)} d_{n-2}, \\ d_j &= -p^{(k)} d_{j+1} - b_{j+2} - q^{(k)} d_{j+2}, & j &= n-4, \dots, 0, \\ \frac{\partial R}{\partial q} &= -p^{(k)} d_0 - b_1 - q^{(k)} d_1, & \frac{\partial S}{\partial q} &= -b_0 - q^{(k)} d_0, \\ \begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} &= \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix} - \begin{bmatrix} \partial R / \partial p & \partial R / \partial q \\ \partial S / \partial p & \partial S / \partial q \end{bmatrix}^{-1} \begin{bmatrix} R \\ S \end{bmatrix}, \\ k &= k + 1, & err &= \max(|R|, |S|), \end{aligned}$$

}

## Bairstow's Method, VI

- Example: find the quadratic factor,  $x^2 + px + q$  of  $P_4 = x^4 + 2x^3 + 3x^2 + 2x + 1$  with initial guess  $p^{(0)} = 0$ ,  $q^{(0)} = 0$ .



- Bairstow's method uses Newton's iteration to find the quadratic factor,
- Order of convergence is 2.
- It is still a local convergent algorithm, but with much larger convergence window than Lin's quadratic method.
- Some formulas in reference books have very small convergence window.
- Our method is much more robust to find the quadratic factors.

- Newton's method, also known as Newton-Raphson method, to solve nonlinear system of equations, is very popular
- Examples:
  - Circuit simulators: SPICE, etc
  - Device simulators: Medici, etc
- In most applications, analytical derivatives are usually employed for the best convergence rate
  - When analytical derivative is not available, the difference scheme is adopted
- Newton's method converges quickly if the initial guess is close to the solution.
  - In SPICE, special algorithms have been developed to find good initial guess before employing Newton's algorithm for accurate solutions.
- You are actually capable of solving OP, DC and AC analysis in SPICE already

## Resistor Network Example

- Resistor voltage divider

$$R_i = R_{i0} + \kappa_i T_i, \quad i = 1, 2. \quad (7.3.30)$$

where  $R_{i0}$  is the resistance at room temp, and  $T_i$  is the temperature increase when the resistor is consuming power. Thus,

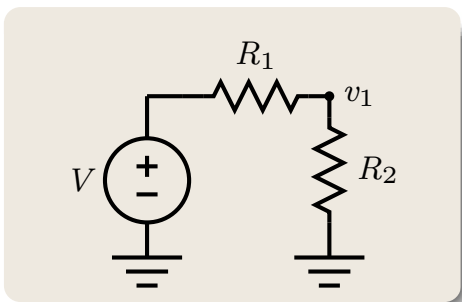
$$T_i = \beta_i I_i V_i = \frac{\beta_i V_i^2}{R_i}, \quad i = 1, 2. \quad (7.3.31)$$

And the current through the resistor is

$$I_i = \frac{V_i}{R_i} = \frac{V_i}{R_{i0} + \kappa_i T_i}, \quad i = 1, 2. \quad (7.3.32)$$

- Current continuity at node  $v_1$

$$\frac{v_1 - V}{R_{10} + \kappa_1 T_1} + \frac{v_1}{R_{20} + \kappa_2 T_2} = 0, \quad (7.3.33)$$



## Resistor Network Example, II

- And two resistor temperature increase

$$T_1 - \frac{\beta_1(v_1 - V)^2}{R_{10} + \kappa_1 T_1} = 0, \quad (7.3.34)$$

$$T_2 - \frac{\beta_2 v_1^2}{R_{20} + \kappa_2 T_2} = 0. \quad (7.3.35)$$

- Thus, the system has three variables,  $v_1$ ,  $T_1$  and  $T_2$  with three equations

$$F_1(v_1, T_1, T_2) = \frac{v_1 - V}{R_{10} + \kappa_1 T_1} + \frac{v_1}{R_{20} + \kappa_2 T_2} = 0, \quad (7.3.36)$$

$$F_2(v_1, T_1, T_2) = T_1 - \frac{\beta_1(v_1 - V)^2}{R_{10} + \kappa_1 T_1} = 0, \quad (7.3.37)$$

$$F_3(V_1, T_1, T_2) = T_2 - \frac{\beta_2 v_1^2}{R_{20} + \kappa_2 T_2} = 0. \quad (7.3.38)$$

## Resistor Network Example, III

- To find the Jacobian, we have

$$\begin{aligned} \frac{\partial F_1}{\partial v_1} &= \frac{1}{R_{10} + \kappa_1 T_1} + \frac{1}{R_{20} + \kappa_2 T_2}, & \frac{\partial F_2}{\partial T_2} &= 0, \\ \frac{\partial F_1}{\partial T_1} &= -\frac{\kappa_1(v_1 - V)}{(R_{10} + \kappa_1 T_1)^2}, & \frac{\partial F_3}{\partial v_1} &= -\frac{2\beta_2 v_1}{R_{20} + \kappa_2 T_2}, \\ \frac{\partial F_1}{\partial T_2} &= -\frac{\kappa_2 v_1}{(R_{20} + \kappa_2 T_2)^2}, & \frac{\partial F_3}{\partial T_1} &= 0, \\ \frac{\partial F_2}{\partial v_1} &= -\frac{2\beta_1(v_1 - V)}{R_{10} + \kappa_1 T_1}, & \frac{\partial F_3}{\partial T_2} &= 1 + \frac{\kappa_2 \beta_2 v_1^2}{(R_{20} + \kappa_2 T_2)^2}, \\ \frac{\partial F_2}{\partial T_1} &= 1 + \frac{\kappa_1 \beta_1(v_1 - V)^2}{(R_{10} + \kappa_1 T_1)^2}, \end{aligned}$$

- And, the Newton's iteration is

$$\begin{bmatrix} v_1^{(k+1)} \\ T_1^{(k+1)} \\ T_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} v_1^{(k)} \\ T_1^{(k)} \\ T_2^{(k)} \end{bmatrix} - \begin{bmatrix} \partial F_1 / \partial v_1 & \partial F_1 / \partial T_1 & \partial F_1 / \partial T_2 \\ \partial F_2 / \partial v_1 & \partial F_2 / \partial T_1 & \partial F_2 / \partial T_2 \\ \partial F_3 / \partial v_1 & \partial F_3 / \partial T_1 & \partial F_3 / \partial T_2 \end{bmatrix}^{-1} \begin{bmatrix} F_1(v_1^{(k)}, T_1^{(k)}, T_2^{(k)}) \\ F_2(v_1^{(k)}, T_1^{(k)}, T_2^{(k)}) \\ F_3(v_1^{(k)}, T_1^{(k)}, T_2^{(k)}) \end{bmatrix}.$$

# Resistor Network Example, IV

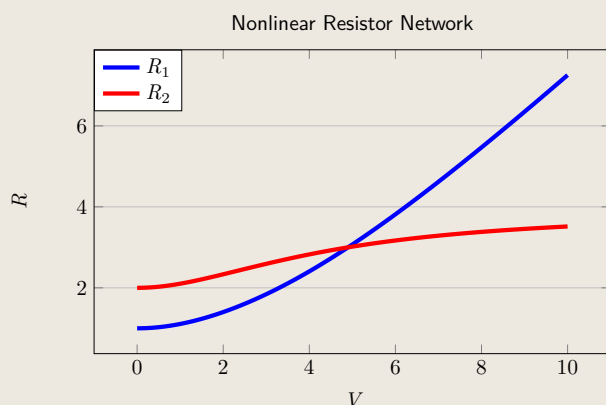
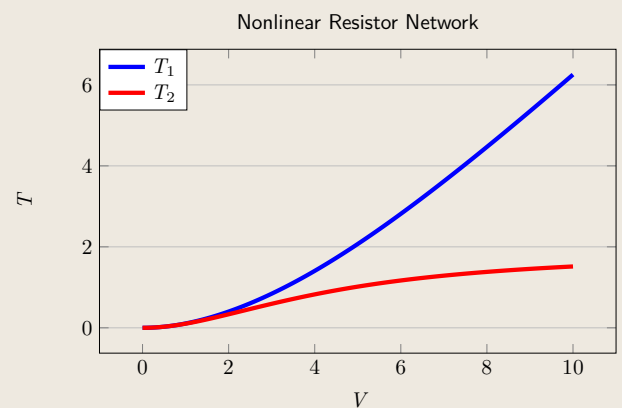
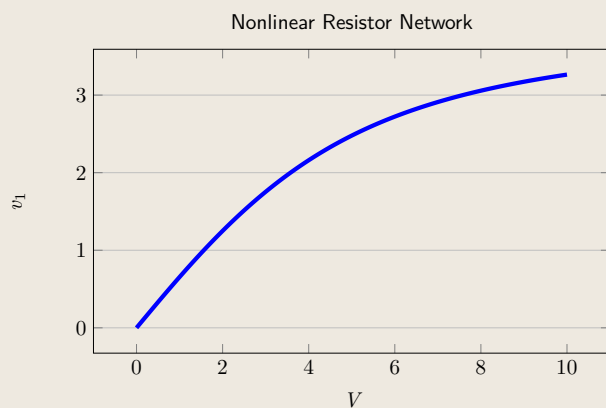
- Given any  $V$ , node voltage  $v_1$  and the temperature increases of the two resistors can be found.

- If we have

$$\begin{aligned} R_{10} &= 1, & \kappa_1 &= 1, & \beta_1 &= 1, \\ R_{20} &= 2, & \kappa_2 &= 1, & \beta_2 &= 0.5. \end{aligned}$$

- Then the voltage divider circuit can be solved given the supply voltage.
- One can increase the voltage supply and solve the circuit
  - SPICE DC analysis.

# Resistor Network Example, IV



- Voltage  $v_1$  is a nonlinear function of supply voltage  $V$
- Newton's method is effective in solving nonlinear systems
- In solving the circuit at a new  $V$ , the converged solution at the previous step should be used as the initial guess.



## Resistor Network Example, V

- The resistor network system can also be formulated as

$$\begin{aligned} v_0 &= V, \\ \frac{v_1 - v_0}{R_1} + \frac{v_1 - v_2}{R_2} &= 0, \\ v_2 &= 0, \\ T_1 - \frac{\beta_1(v_1 - v_0)^2}{R_1} &= 0, \\ T_2 - \frac{\beta_2(v_1 - v_2)^2}{R_2} &= 0, \end{aligned}$$

with the system unknowns  $\mathbf{x} = (v_0, v_1, v_2, T_1, T_2)$ , totally 5 variables. And Newton's iteration becomes

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (\mathbf{J}^{(k)})^{-1} \mathbf{F}$$

where  $\mathbf{F}$  are the five equations defined above and  $\mathbf{J}_{i,j} = \frac{\partial F_i}{\partial x_j}$ .

## Resistor Network Example, VI

- $\mathbf{J}$  can be derived as

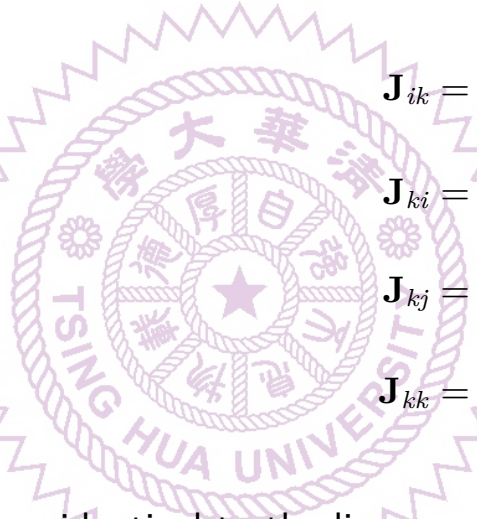
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} & \frac{v_1 - v_0}{R_1^2} \frac{\partial R_1}{\partial T_1} & \frac{v_1 - v_2}{R_2^2} \frac{\partial R_1}{\partial T_2} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{2\beta_1(v_1 - v_0)}{R_1} & -\frac{2\beta_1(v_1 - v_0)}{R_1} & 0 & 1 + \frac{\beta_1(v_1 - v_0)^2}{R_1^2} \frac{\partial R_1}{\partial T_1} & 0 \\ 0 & -\frac{2\beta_2(v_1 - v_2)}{R_2} & \frac{2\beta_2(v_1 - v_2)}{R_2} & 0 & 1 + \frac{\beta_1(v_1 - v_0)^2}{R_1^2} \frac{\partial R_2}{\partial T_2} \end{bmatrix}$$

- And the right-hand side,  $-\mathbf{F}$  is

$$\begin{bmatrix} -v_0 + V & -\frac{v_1 - v_0}{R_1} - \frac{v_1 - v_2}{R_2} & 0 & -T_1 + \frac{\beta_1(v_1 - v_0)^2}{R_1} & -T_2 + \frac{\beta_2(v_1 - v_2)^2}{R_2} \end{bmatrix}^T$$

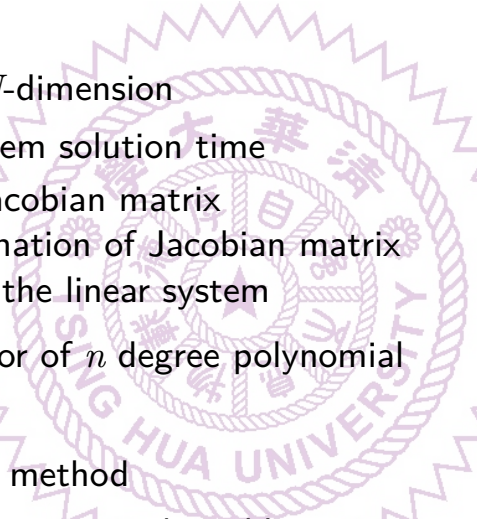
# Resistor Network Example, VII

- Assuming a nonlinear resistor,  $R_k$ , is connecting nodes  $i$  and  $j$ , then it contributes to the following stamps


$$\begin{aligned} \mathbf{J}_{ii+} &= \frac{1}{R_k}, & \mathbf{J}_{ik} &= \frac{v_i - v_j}{R_k^2} \frac{\partial R_k}{\partial T_k}, \\ \mathbf{J}_{ij-} &= \frac{1}{R_k}, & \mathbf{J}_{ki} &= -\frac{2\beta_k(v_i - v_j)}{R_k}, \\ \mathbf{J}_{ji-} &= \frac{1}{R_k}, & \mathbf{J}_{kj} &= \frac{2\beta_k(v_i - v_j)}{R_k}, \\ \mathbf{J}_{jj+} &= \frac{1}{R_k}, & \mathbf{J}_{kk} &= 1 + \frac{\beta_k(v_i - v_j)^2}{R_k^2} \frac{\partial R_k}{\partial T_k}, \end{aligned}$$

- The left-hand stamps are identical to the linear resistor case, while the right-hand stamps are newly added.
- If the network has  $n$  nodes and  $m$  resistors, then it has totally  $n + m$  unknowns.
- Using the stamping approach, the nonlinear system can be solved.
- General nonlinear network can be formulated similarly.

## Summary

- 
- Nonlinear systems
  - Newton's method in  $N$ -dimension
  - Improving nonlinear system solution time
    - Cyclic update of Jacobian matrix
    - Difference approximation of Jacobian matrix
    - Inexact solution of the linear system
  - Finding a quadratic factor of  $n$  degree polynomial
    - Bairstow's method
  - Application of Newton's method
  - Example: nonlinear resistor network problem
    - SPICE DC analysis