Unit 1.4 Applications

Numerical Analysis

EE/NTHU

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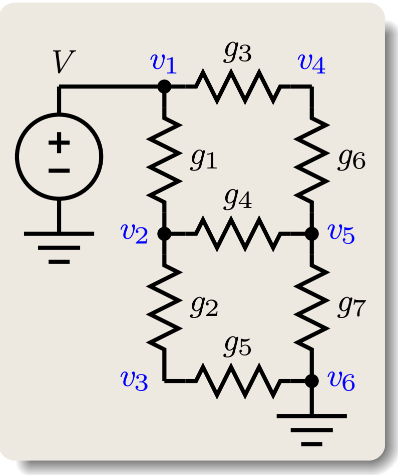
Simple Resistor Network

- Linear system solution methods have many applications.
- Two examples are given in this unit.
  - Resistor network,
  - Equivalent resistance of a 2-D conductor.

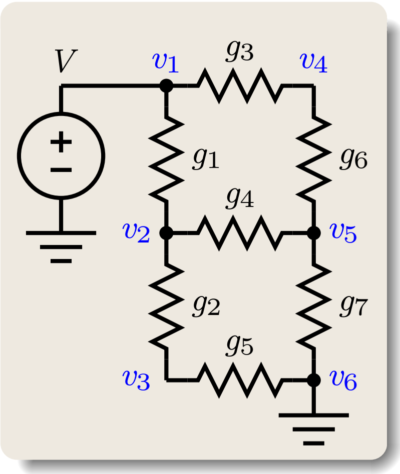


- A simple resistor network is shown.
- $g_i = \dfrac{1}{r_i}$ is the conductance.
- Need to find the equivalent resistance of the network.
  - In order to do that we need find the voltage on every node.
  - This can be done by applying the Kirchhoff current law.

$$\sum_k j_{ik} = 0, \qquad \text{for every node } i, \tag{1.4.1}$$

  where j_{ik} is the current from node i to node k.
  - There are 6 node voltages to be solved for and 6 equations.
  - Two boundary conditions: $v_1 = V$ and $v_6 = 0$.

Simple Resistor Network – Equation Formulation



- Apply Kirchhoff Current Law and boundary conditions:

$$
\begin{aligned}
\mathsf{n}_1 &: \quad v_1 = V, \\
\mathsf{n}_2 &: \quad g_1(v_2 - v_1) + g_2(v_2 - v_3) + g_4(v_2 - v_5) = 0, \\
\mathsf{n}_3 &: \quad g_2(v_3 - v_2) + g_5(v_3 - v_6) = 0, \\
\mathsf{n}_4 &: \quad g_3(v_4 - v_1) + g_6(v_4 - v_5) = 0, \\
\mathsf{n}_5 &: \quad g_6(v_5 - v_4) + g_4(v_5 - v_2) + g_7(v_5 - v_6) = 0, \\
\mathsf{n}_6 &: \quad v_6 = 0.
\end{aligned}
$$

- The linear system is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-g_1 & g_1+g_2+g_4 & -g_2 & 0 & -g_4 & 0 \\
0 & -g_2 & g_2+g_5 & 0 & 0 & -g_5 \\
-g_3 & 0 & 0 & g_3+g_6 & -g_6 & 0 \\
0 & -g_4 & 0 & -g_6 & g_4+g_6+g_7 & -g_7 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}
=
\begin{bmatrix} V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Simple Resistor Network – Equation Formulation, II

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-g_1 & g_1+g_2+g_4 & -g_2 & 0 & -g_4 & 0 \\
0 & -g_2 & g_2+g_5 & 0 & 0 & -g_5 \\
-g_3 & 0 & 0 & g_3+g_6 & -g_6 & 0 \\
0 & -g_4 & 0 & -g_6 & g_4+g_6+g_7 & -g_7 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}
=
\begin{bmatrix} V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

- The system matrix can be solved using LU decomposition and forward and backward substitutions.
  - Conductance and applied voltage need to be given.
- The matrix is diagonally dominant.
  - The matrix is nonsingular.
  - Unique solution exists.
- The matrix is, however, not symmetric.

Simple Resistor Network – Forming Symmetric Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_1+g_2+g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2+g_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_3+g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4+g_6+g_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} V \\ g_1 V \\ 0 \\ g_3 V \\ 0 \\ 0 \end{bmatrix}$$

- It is possible to transform the matrix into a symmetric form.
  - Move the nondiagonal entries of the first and the last columns to the right hand side.
- The matrix is now symmetric.
  - Cholesky factorization can be used for better solution efficiency.

Simple Resistor Network – System Formulation

- Formulation of the linear system for a resistor network

Algorithm 1.4.1. System Equation for a Resistor Network

Let the unknown vector be all node voltages, v_i, $i = 1, \ldots, n$,
create an $n \times n$ matrix $\mathbf{A}$ and an n-vector $\mathbf{b}$ and initialize both to $\mathbf{0}$.
for each node i not connecting to voltage sources,

for each resistor, with conductance $g_k = \dfrac{1}{r_k}$, connecting node i and j,

$$\begin{aligned} \mathbf{A}_{ii} &= \mathbf{A}_{ii} + g_k, \\ \mathbf{A}_{ij} &= \mathbf{A}_{ij} - g_k. \end{aligned}$$

for each node i connecting to a fixed voltage V_i,

$$\begin{aligned} \mathbf{A}_{ii} &= 1, \\ \mathbf{A}_{ij} &= 0, \qquad j \neq i, \\ \mathbf{b}_i &= V_i. \end{aligned}$$

Simple Resistor Network – Symmetric System Formulation

Algorithm 1.4.2. Symmetric System Equation for a Resistor Network

Let the unknown vector be all node voltages, v_i, $i = 1, \ldots, n$,
create an $n \times n$ matrix $\mathbf{A}$ and an n-vector $\mathbf{b}$ and initialize both to $\mathbf{0}$.
for each node i not connecting to voltage sources,

for each resistor, with conductance $g_k = \dfrac{1}{r_k}$, connecting node i and j,

$$\begin{aligned}\mathbf{A}_{ii} &= \mathbf{A}_{ii} + g_k,\\ \mathbf{A}_{ij} &= \mathbf{A}_{ij} - g_k.\end{aligned}$$

for each node i connecting to a fixed voltage V_i,

$$\begin{aligned}\mathbf{A}_{ii} &= 1,\\ \mathbf{A}_{ij} &= 0, \qquad j \neq i,\\ \mathbf{b}_i &= V_i,\end{aligned}$$

for each resistor, with conductance $g_k = \dfrac{1}{r_k}$, connecting node i and j,

$$\begin{aligned}\mathbf{A}_{ji} &= 0,\\ \mathbf{b}_i &= \mathbf{b}_i + g_i V_i.\end{aligned}$$

Simple Resistor Network, System Formulation, II

- The preceding algorithms generate system equations for general resistor network.
- The resulting linear system can be solved to get a unique solution if
  - At least one of the nodes is grounded ($V_i = 0$),
  - At least one of the non-grounded nodes is connecting to a fixed voltage.
- Multiple grounded nodes or multiple voltage supply nodes can still be solved to get a unique solution.
- Computation complexity $\mathcal{O}(mn)$, n is the number of nodes and m is the maximum number resistors connecting to a single node.
  - Each node needs to be processed
    - And the each resistor connecting to the node processed
- Data structures
  - Each node has a linked list for the resistors connecting to the node,
  - A linked list for the nodes connecting to a fixed voltage, including grounded nodes.

System Formulation, Stamping Approach

Algorithm 1.4.3. System Equation Using Stamps

Let the unknown vector be all node voltages, v_i, $i = 1, \dots, n$,
create an $n \times n$ matrix $\mathbf{A}$ and an n-vector $\mathbf{b}$ and initialize both to $\mathbf{0}$.
For each resistor, with conductance $g_k = \dfrac{1}{r_k}$, connecting node i and j,

if both nodes i and j are connecting to fixed voltages, V_i and V_j, respectively, then

$$\mathbf{A}_{ii} = 1, \mathbf{b}_i = V_i, \mathbf{A}_{jj} = 1, \mathbf{b}_j = V_j,$$

else if node i is connecting to a fixed voltage V_i

$$\mathbf{A}_{ii} = 1, \mathbf{b}_i = \mathbf{b}_i + V_i,$$

else

$$\mathbf{A}_{ii} = \mathbf{A}_{ii} + g_k, \mathbf{A}_{ij} = \mathbf{A}_{ij} - g_k,$$

if node j is connecting to a fixed voltage V_j

$$\mathbf{A}_{jj} = 1, \mathbf{b}_j = \mathbf{b}_j + V_j,$$

else

$$\mathbf{A}_{jj} = \mathbf{A}_{jj} + g_k, \mathbf{A}_{ji} = \mathbf{A}_{ji} - g_k.$$

Stamping Approach, II

- In the stamping approach a resistor with conductance g_k connecting nodes i and j, with neither node connecting to a fixed voltage, modifies four matrix entries

$$\begin{aligned} \mathbf{A}_{ii} &= \mathbf{A}_{ii} + g_k, & \mathbf{A}_{ij} &= \mathbf{A}_{ij} - g_k, \\ \mathbf{A}_{ji} &= \mathbf{A}_{ji} - g_k, & \mathbf{A}_{jj} &= \mathbf{A}_{jj} + g_k. \end{aligned}$$

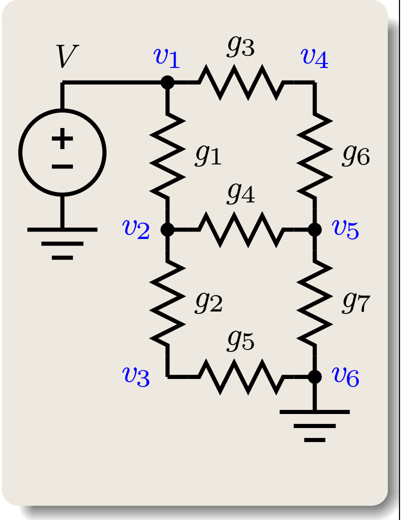
  This is the stamp a the resistor.
- Computational complexity for stamping all the resistor is $\mathcal{O}(N_R)$, where N_R is the number of resistors.
- Stamps can also be added for a voltage source.
- For voltage source with a fixed value V_k connecting nodes i and j, an extra unknown for the current of the voltage source, i_k, is added, and the boundary condition is treated as an extra equation of the system.

$$\begin{aligned} v_i - v_j &= V_k, \\ i_k + \sum_{r_{ij}} g_{ij}(v_i - v_j) &= 0, \\ -i_k + \sum_{r_{ji}} g_{ji}(v_j - v_i) &= 0. \end{aligned}$$

  where the second equation is the Kirchhoff current law at node i, and the third equation is the Kirchhoff current law at node j.

Stamping Approach, III

- Thus, the voltage stamps for a voltage source connecting nodes i and j, and the current equation k are



$$\mathbf{A}_{ki} = 1, \qquad \mathbf{A}_{kj} = -1, \qquad \mathbf{b}_k = V_k,$$
$$\mathbf{A}_{ik} = 1,$$
$$\mathbf{A}_{jk} = -1.$$

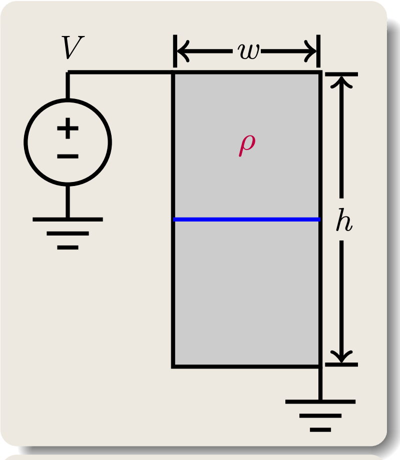
  - Note that voltage stamps involve right hand side vector.
- Example
- Unknown vector $= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & i_V \end{bmatrix}$
- System equation

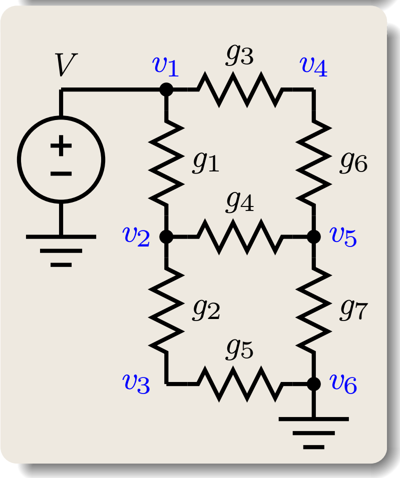
$$\begin{bmatrix} g_1+g_3 & -g_1 & 0 & -g_3 & 0 & 1 \\ -g_1 & g_1+g_2+g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2+g_5 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & g_3+g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4+g_6+g_7 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i_V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V \end{bmatrix}$$

Stamping Approach, IV

- This approach of creating a linear system of equations for the resistor network is known as modified nodal approach.
- This approach has been adopted in some SPICE programs.
- In this approach, the ground node voltage is not an unknown, but the current of each voltage source is.
- The resulting system matrix can still be symmetric.
- But it is not diagonal dominant.
- LU decomposition is still an effective approach to solving the system.
  - Many fill-ins created for in the last column and last row (borders).
- But, Cholesky decomposition may not be applicable.

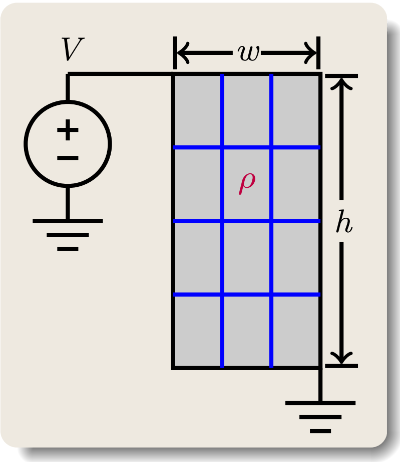
Effective Resistance of Conductor





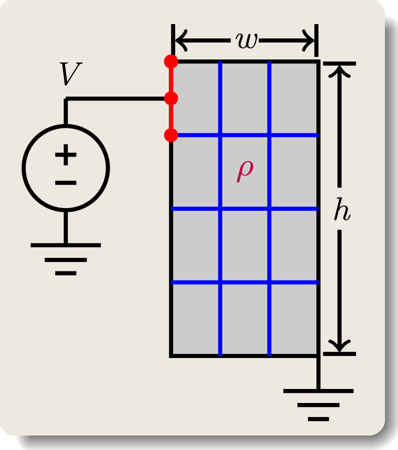
- Given a piece of conductor with resistivity ρ then it can be approximated by a resistor network.
- If uniform current is flown from left to right in conductor, then the resistance is
$$r = \frac{\rho w}{h} \tag{1.4.2}$$
- Since the current flow is not uniform, we need to discretize the conductor into small pieces to get approximated solution.
- The example shown bisects the conductor into two pieces.
- It can be shown that
$$r_1 = r_2 = r_6 = r_7 = \rho \frac{h}{w}$$
$$r_3 = r_5 = \rho \frac{4w}{h} \qquad r_4 = \rho \frac{2w}{h}$$
- And, $g_i = \dfrac{1}{r_i}$, as usual.
- Then the linear system can be solved to find the solution.
- The equivalent resistance of the configuration given is
$$R_{eq} = \frac{V}{i_V}. \tag{1.4.3}$$

Effective Resistance of Conductor, II



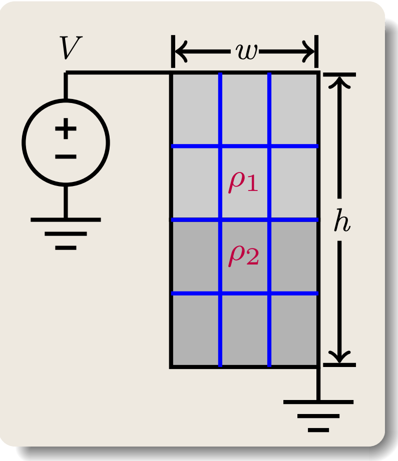
- The preceding example bisects the conductor into two pieces, with the total number of grid points equals to 6.
  - The number of grid points in x-direction, N_x, is 2.
  - And the number of grid points in y-direction, N_y, is 3.
- The equivalent resistance calculated in this case is not very accurate.
- It is straightforward to increase the number of grid points to increase the accuracy.
  - The larger number of grid points, the solution accuracy is usually better.
  - But the solution time and storage space are getting larger.
  - The number of variables increases with $N_x \times N_y$.
  - The time to solve the system is then $\mathcal{O}(N_x^3 \times N_y^3)$, if full matrix LU decomposition method is employed.
  - Sparse matrix techniques can very helpful in reducing the linear system solution time.

Effective Resistance of Conductor, III



- For general problems, the boundary condition can be different.
- As shown, the voltage is applied to two grid points.
- One approach to solve this problem is to have two voltage sources connecting to each grid point individually.
  - Extra current variables are needed.
  - Total current supplied are the sum of these currents.
  - Formulation of the linear system using resistor stamp method is straightforward.
- The other approach is to treat these two grid points as a single variable.
  - Only one voltage source is needed.
  - Smaller matrix can be obtained.
  - Formulation of the matrix is a little more complicated.
- Either method can deliver good solution.
- Multiple voltage source or multiple ground nodes can also be formulated and solved.

Effective Resistance of Conductor, IV



- The conductor may consist of different materials with different resistivity, as shown.
- In this case, the resistor stamping function needs to be able to calculate the resistance (and conductance) based on the material of the region.
- Once that is done, the solution method is identical to the preceding discussions.
- It is also possible that a void ($R = \infty$) is placed in the conductor.
  - In this case, the conductance is 0.
  - If a grid point is completely inside the void region, this grid point should be eliminated.

Summary

- Resistor network
  - General resistor network can be solved
- Resistor stamping
  - Systematic formulation of the linear system
- Equivalent resistance of a conductor
  - Can solve general interconnect resistance calculation.
- Linear system solution has many applications
  - Try to apply this solution method in your own research.