Functional Equations

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1 Strategies

Here are some useful strategies that are helpful for functional equations.

Find clever values to plug in to your equation to give you a simpler equation: in particular, make things in your equation equal 0, 1, or cancel with each other.

Try to guess the answer to your functional equation; sometimes it will be obvious, but there may also be other solutions besides the most obvious one. Constant, linear, or quadratic functions are most common, but others happen sometimes.

Remember to check that the answers you get actually work!

Sometimes it helps to change the function you are looking at (e.g. if you want to prove $f(x) = cx$ it might be useful to look at the function $g(x) = f(x)/x$ and prove that g is constant.

If the domain of your function is $\mathbb{Z}^+$ (positive integers), $\mathbb{Z}$, or even $\mathbb{Q}$, consider using inductive strategies in your proof.

1.1 Potentially useful lemmas/intermediate steps when solving a functional equation

- Try to figure out the values of, $f(0)$, $f(1)$ or any other specific values of your function that seem useful.
- Prove that f is odd/even/find some other formula relating $f(-x)$ to $f(x)$.
- Prove that the function is injective (one-to-one).
- Prove that the function is surjective (onto).

1.2 WARNING!: Cauchy's functional equation.

If you have shown that $f(x + y) = f(x) + f(y)$ you CANNOT conclude that there is a constant k such that $f(x) = kx$ for all x. The functional equation $f(x + y) = f(x) + f(y)$ is known as Cauchy's functional equation, and (assuming the axiom of choice) there exist non-linear functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy this equation. The proof of this is beyond the scope of the lecture, and such pathological counterexamples never come up in olympiad problems.

However, you CAN deduce that conclusion IF one of the following is true: the domain of your function is $\mathbb{Z}$ or $\mathbb{Q}$, or if f is monotonic (increasing or decreasing). In such cases you can generally cite "Cauchy's functional equation," unless it trivializes the problem.

2 Examples

1 (South Africa 1997). Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m + f(n)) = f(m) + n.$$

2 (MOP '01). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{yf(x)}{xf(y)} = \frac{f(x) + y}{f(y) + x}$$

for all $x, y \in \mathbb{R}$ such that x, y, $f(y)$, and $f(y) + x$ are all nonzero.

3 (Czech-Slovak '97). Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

3 Problems

Notations: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^+$ denotes the set of positive integers.

4 (ELMO 2001). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(y) + f(xy) = (f(x) + f(y))^2.$$

for all $x, y \in \mathbb{R}$.

5 (USAMO 2002). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real x and y.

6 (Estonia '00). Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in \mathbb{Z}^+$.

7 (MOP '00). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = (f(x))^2 + y$$

for any real numbers x and y.

8 (IMO shortlist '02). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = 2x + f(f(y) - x).$$

9 (IMO '02)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

10 (APMO '02)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying:

1. There are only finitely many $s \in \mathbb{R}$ such that $f(s) = 0$,

   and

2. $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all $x, y \in \mathbb{R}$.

11 (Czech-Slovak-Polish '01)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x^2 + y) + f(f(x) - y) = 2f(f(x)) + 2y^2$$

for all $x, y \in \mathbb{R}$.

12 (Balkan '07)**.** Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

13 (MOP '04)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y).$$