Please bring this handout to lecture (no extra handouts available). Please look over the ideas presented in the beginning of each chapter, and if you have time then please look at the problems.

Functional Equations
Lecturer - Andrei Jorza
Red Group

Functional equations are quite popular at the IMO and other international math olympiads but there is no generalized method for solving them. However, there are quite a few ideas that are very useful.

Ideas

1. Cauchy's functional equation is the following: $f : \mathcal{A} \longrightarrow \mathcal{A}$ so that $f(x+y) = f(x) + f(y)$ (this property is called additivity). For certain sets $\mathcal{A}$ and certain properties of f, the function turns out to be linear. And in any ring $\mathcal{A}$, the linear function satisfies Cauchy's equation. Here are a few examples:

   (a) If $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ then $f = f(1)x$.

   (b) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ and f is increasing, continuous or positive on the set $(0, \infty)$ then $f = f(1)x$.

   (c) If $f(x^2) = f^2(x)$ then $f = f(1)x$ because then f is positive on the positives.

   (d) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ then $f(q) = f(1)q$ for all rational numbers q.

2. When you get to a functional equation of type, e.g. $f(x^3) = f^3(x)$ or $f(x^3) = x^2 f(x)$ a good trick is to replace x by $x + y$. If we have additivity then this is very useful. Remember USAMO 2002 problem 4?

3. In functional equations try to get to relations of the type $f \circ f = x$ (called unipotent) or $f \circ f = f$ (called idempotent).

4. In the case of idempotent functions we can see that for any $x \in \mathrm{Im} f$ we have that $f(x) = x$. In this case the surjectivity of the function is important.

5. For unipotence a good trick is setting x to be $f(x)$.

6. When you get to the relation $f^2(x) = x^2$ do not conclude that $f = x$ or $f = -x$. Rather, this implies that $f(x) = x$ for $x \in \mathcal{E}$ and $f(x) = -x$ for $x \in \mathbb{R} \setminus \mathcal{E}$. From here on assume there are x, y so that $f(x) = x, f(y) = -y$.

7. In some problems, where the functional equation involves x and y if we set $y = x$ then we get an equation of type $f(g(x)) = g(x)$. The problem will be solved if we can prove that f has at most one fixed point.

Problems

Real Functions

1. Find all $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that for all real x, y we have $f(xf(x) + f(y)) = f^2(x) + y$. (BMO 2000)

2. Find all $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that for all real x, y we have $f(x^2 + f(y)) = f^2(x) + y$. (IMO 1992)

3. Find $f : (0, \infty) \longrightarrow (0, \infty)$ so that for all positive real x, y we have $f(xf(y)) = yf(x)$ and when x tends to ∞ then $f(x)$ tends to 0. (IMO 1983)

4. Find $f : (-1, \infty) \longrightarrow (-1, \infty)$ so that for all real x and y in $(-1, \infty)$ we have

$$f(x + f(y) + xf(y)) = y + f(x) + yf(x)$$

and $f(x)/x$ is increasing on each of $(-1, 0)$ and $(0, \infty)$. (IMO 1994)

5. Find the function $f : [0, \infty) \longrightarrow [0, \infty)$ so that $f(2) = 0$ and $f(x) \neq 0$ for $x \in [0, 2)$. Also for all nonnegative reals x, y we have $f(xf(y))f(y) = f(x + y)$. (IMO 1986)

Integer Functions

Integer functional equations ususally have different methods of approach than the real ones. We still have same ideas that we want to use, such as idempotence and unipotence. However, sometimes we need to rely on the special properties of the integers (and rationals).

1. One very important property is the uniqueness of decomposition into prime factors. This property also extends to rationals if we allow negative powers as well.

2. Another quite important method used is writing numbers in bases different from base 10. Most often this different base is base 2.

Problems

1. Find a bijection $f : \mathbb{N} \longrightarrow \mathbb{N}$ so that for all natural m and n we have

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$$

(Shortlist 1996)

2. Prove that there exists a function $f : \mathbb{Q}_+^* \longrightarrow \mathbb{Q}_+^*$ so that for all positive rationals x and y we have

$$f(xf(y)) = \frac{f(x)}{y}$$

(IMO 1993)

3. Prove that there are no functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ so that we have $(f \circ f)(x) = x + 1987$ for all natural numbers x. (IMO 1987)

4. Let S be the set of non-negative integers. Find all functions $f : S \longrightarrow S$ so that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in S$. (IMO 1996)

5. A function $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ satisfies the properties $f(1) = 1, f(3) = 3$ and

$$\begin{aligned} f(2n) &= f(n) \\ f(4n+1) &= 2f(2n+1) - f(n) \\ f(4n+3) &= 3f(2n+1) - 2f(n) \end{aligned}$$

Find the numbers of $n \leq 1988$ so that $f(n) = n$. (IMO 1988)

Homework

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that for any two real numbers x and y we have $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$. Find f.

   Hint: Use the properties of the Cauchy functional equations.

2. Prove that there is no bijection from the positive integers to the nonnegative integers so that for all positive integers m, n we have

   $$f(mn) = f(m) + f(n) + 3f(m)f(n)$$

   (BMO 1991)

   Hint: Look at problem 1 from Integer Functions.

3. Let $f : [0, 1) \cap \mathbb{Q} \longrightarrow [0, 1) \cap \mathbb{Q}$ so that if $x < 1/2$ then $f(x) = f(2x)/4$ and if $x \geq 1/2$ then $f(x) = (3 + f(2x - 1))/4$.

   Hint: Look at problem 5 from Integer Functions

4. Let $a \in \mathbb{R}$. Find all $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that $f(0) = 1/2$ and for all real x, y we have $f(x + y) = f(x)f(a - y) + f(a - x)f(y)$. (BMO 1987)

   Hint: Prove that the function is a constant one.

Extra Problems (Harder)

1. Find all $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that for all real x, y we have

   $$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

   (IMO 1999)

2. Find the smallest value of $f(1998)$ when $f : \mathbb{N} \longrightarrow \mathbb{N}$ and for all natural numbers m, n we have

   $$f(n^2 f(m)) = mf^2(n)$$

   (Shortlist 1998)

3. Let $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ so that for all positive integers m, n we have

   $$f(f(m) + f(n)) = m + n$$

   Find all possible values of $f(1988)$. (Shortlist 1988)