Combinatorics: The Next Step

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Chapter 1

Introduction to Combinatorics

It is not surprising if students in junior section are not sure of what combinatorics exactly is. Formally, combinatorics is the field which is concerned with arrangements of the objects of a set into patterns which satisfy predefined rules. Students will probably encounter their first combinatorial problem in finding the number of possible arrangements of unique items. This class of problem is generally known as enumerative combinatorics. Enumerative combinatorics mainly deal with the number of possible arrangements, given certain conditions and restrictions that the objects involved are succumb to. Apart from the plethora of counting problems, there is another genre of combinatorial problem known as existence combinatorics, which involves proving the existence of certain configurations. The techniques involved in existence combinatorics can be very different from that of enumerative combinatorics. Unfortunately, these techniques are often neglected as students often focus on enumerative combinatorics which comes out commonly in the first round of SMO.

A combinatorics textbook is usually organised based on different techniques involved in combinatorics. In MO, we are often concerned with the following topics in combinatorics:

1. Permutations and combinations
2. Pigeonhole principle
3. Binomial coefficients and combinatorial identities
4. Principle of inclusion and exclusion
5. Recurrence relations and generating functions

However, it is not sufficient to just master the tools in combinatorics. Students have to be familiar with the arguments used in combinatorial problems and the necessary assumptions to make in proving such problems. We often require the use of the following techniques in solving combinatorial problems, especially in existence combinatorial problems:

1. Pigeonhole principle
2. Bijection principle

3. Fubini principle (Double Counting)
4. Extremal principle
5. Invariance, monovariance and colouring
6. Bounding
7. Use of models such as recurrence, incidence matrices and graphs

In junior section, students do not have to learn advanced tools in combinatorics as the problems involved are not very complicated. However, it is important for students to familiarise themselves with combinatorial arguments and assumptions since these strategies can be used in solving problems in the second round. Besides, some of these concepts are applicable in other fields of MO too.

Chapter 2

Enumerative Combinatorics

In junior section, students are only required to solve three classes of enumerative combinatorics problem. They are:

1. Permutations and combinations
2. Principle of inclusion and exclusion
3. Constructing recurrence relations

The problems involved are much less complicated than the problems in senior or open section. I will provide some examples to illustrate the tools listed above.

2.1 Permutations and Combinations

By now, students should be familiar with the usage of addition principle and multiplication principle in counting. The addition principle is used when the cases involved are pairwise disjoined while multiplication principle is used when the choice of subsequent object is dependent on the choice of previous objects.

The r-**permutations** of a set is defined by the ordered arrangements of any r objects from a set of n objects. We can denote the number of $r-permutations$ in a set of n elements as $P(n,r)$ or ${}^nP_r$. For example, the 2-permutations of the set $S = \{a,b,c\}$ are ab, ac, ba, bc, ca, cb. As such, ${}^3P_2 = 6$. To evaluate ${}^nP_r$, we have the following theorem:

Theorem 1. *For positive integers n and r where $r \leq n$, we have*

$$^nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

One also has to take note of whether the permutations are made linearly or in an enclosed manner. If the objects are placed about a k-sided regular polygon, we have to divide the total number of arrangements by k since the k sided polygon has rotational symmetry of order k. If the object are placed around a circle, we need to divide the number of arrangements by the number of objects.

The problem is slightly stickier if there are identical objects in the set. Suppose $n_1, n_2, \cdots n_k$ are positive integers such that $n_1 + n_2 + \cdots n_k = n$, and that object 1 is duplicated n_1 times, object 2 is duplicated n_2 times, and so on. We will need to account for identical permutations when we arrange these n objects. The formula for the number of arrangements is given by:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Consider the following problem:

Example 1. *Suppose I have 3 green marbles, 4 red marbles and 2 blue marbles in a black box. I will take one marble out at a time until 8 marbles are collected. What is the total number of possible sequences of marbles which can be made?*

To solve this problem, we need to consider the following three different cases:

1. 2 green marbles, 4 red marbles, 2 blue marbles are collected. The total number of 8-permutations is
$$\frac{8!}{2!4!2!} = 420;$$
2. 3 green marbles, 3 red marbles, 2 blue marbles are collected. The total number of 8-permutations is
$$\frac{8!}{3!3!2!} = 280;$$
3. 3 green marbles, 4 red marbles, 1 blue marble are collected. The total number of 8-permutations is
$$\frac{8!}{3!4!1!} = 560.$$

Hence, the total number of possible sequences is $420 + 280 + 560 = 1260$. Unfortunately, we have to list down all possible cases in choosing the 8 marbles. The problem will be more complicated if less marbles are chosen instead. This problem can only be simplified if the technique of generating functions is applied.

The notion of **combinations** arises when one chooses a number of objects without considering the arrangement of the objects. Here, we are concerned with the number of possible unordered subsets which can be chosen from a larger set. For example, there are three ways where we can choose 3 numbers from the set $\{1, 2, 3, 4\}$, namely 123, 134, 124 and 234. We often notate the number of r-combinations from n-objects as $\binom{n}{r}$ or ${}^nC_r$. I have even read books which uses the notation ${}^rC_n$, which is rather confusing in my opinion. We can use the following theorem to evaluate combinations:

Theorem 2. *For positive integers n and r where $r \leq n$,*

$$\binom{n}{r} = \frac{1}{r!}P(n,r) = \frac{n!}{r!(n-r)!}.$$

In theorem above, each n objects are distinct and we do not have to fear that the selected r objects may be repeated in another selection. If there are repeated objects, we may need to apply the principle of inclusion and exclusion to count the number of combinations. Instead, I would like to direct your attention to the case when each objects are repeated **infinitely**. Right now, we have n types of objects of infinite supply, and we are supposed to choose r objects out of the set of objects. We have the following theorem to count the number of such combinations:

Theorem 3. *The number of combinations to choose r objects from n types of objects, each with an infinite amount of objects, is*

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

The following examples below can be solved using the above theorem:

Example 2. *Ah Beng wants to order food for himself and his 9 other friends. He can choose to purchase chicken rice, mee rebus, pasta or sushi. How many different possible ways are available for him to purchase the food?*

Example 3. *Find the number of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$.*

Example 4. *A number is considered* superior *if all digits of the number are larger than or equal to the digit on its left. Find the total number of 10-digit* superior *numbers.*

There are two techniques which are extremely useful in solving questions involving permutations and combinations. The first technique is known as the **method of insertion.** It is often used when one type of object are required to be separated or inserted among other objects. Consider the following problem:

Example 5 (SMO(J)2009)**.** *The number of ways to arrange 5 boys and 6 girls in a row such that girls can be adjacent to other girls but boys cannot be adjacent to other boys is* $6! \times k$*. Find the value of k.*

Since there are less boys than girls, our strategy is to arrange the girls first and insert the boys into the empty spaces between the girls. There are 6! ways to arrange the girls and there are 5 empty spaces between the girls. There is also another 2 empty spaces which we should consider at the end of the line, so we have a total of 2 empty spaces. The five boys will occupy 5 of these empty spaces. Hence, the total number of arrangements by multiplication principle is:

$$6! \times \binom{7}{2} \times 5! = 6! \times 2520$$

which gives us the final answer 2520.

The other common technique is the **method of complementary set.** Instead of finding the number of arrangements that satisfy the condition imposed by the question, we find out the number of arrangements which does not satisfy the condition and then subtract it from the total number of possible arrangement. Sometimes, it is much faster to calculate the number of arrangements in the complementary set. The following problem illustrates this concept:

Example 6. *On a 4×4 chessboards, there are 4 rooks, each of a different colour, which occupies one space each. A rook can attack another rook if they are placed in the same row or column. What is the total number of configurations such that at least 2 rooks are attacking one another?*

Instead of finding the number of configurations where there are attacking rooks, we will find the number of configurations where the rooks are not attacking each other and subtract is from the total number of configurations. Since each row must contain only 1 rook, we have 4 possible spaces to place the rook on the first row, 3 remaining possible spaces to place the rook on the second row, and so on. Since the rooks are distinct, we have to permute the rooks for each configuration. This gives us $4!4!$ non-attacking arrangements. The final answer would be $16 \times 15 \times 14 \times 13 - 4!4! = 43104$.

2.2 Principle of Inclusion and Exclusion

The addition principle provides a method for counting when each different cases are mutually exclusive. What if some of the cases are not mutually exclusive? Consider the following problem:

Example 7. *Find the number of integers between 0 and 10000 inclusive which are divisible by 2, 3 or 5.*

Aside from 0, We know that the number of integers divisible by 2 is $\lfloor \frac{10000}{2} \rfloor$, the number of integers divisible by 3 is $\lfloor \frac{10000}{3} \rfloor$, the number of integers divisible by 5 is $\lfloor \frac{10000}{5} \rfloor$. However, we have counted the number of integers which are divisble by 6, 10 and 15 twice, and counted the number of integers which are divisible by 30 thrice. To rectify the situation, we need to subtract the quantities $\lfloor \frac{10000}{6} \rfloor$, $\lfloor \frac{10000}{10} \rfloor$, $\lfloor \frac{10000}{15} \rfloor$ from the original sum. Upon subtraction, we have removed the integers which are divisible by 30. Hence, we need to add the quantity $\lfloor \frac{10000}{30} \rfloor$ back. Our final solution is

$$1 + \lfloor \frac{10000}{2} \rfloor + \lfloor \frac{10000}{3} \rfloor + \lfloor \frac{10000}{5} \rfloor - \lfloor \frac{10000}{6} \rfloor - \lfloor \frac{10000}{10} \rfloor - \lfloor \frac{10000}{15} \rfloor + \lfloor \frac{10000}{30} \rfloor = 7335$$

In the above problem, there are three different cases (whether the numbers are divisibly by 2, 3, or 5). There may be problems with more cases than the above problem, but the approach will be similar. If you are fluent with set notations, the principle of inclusion and exclusion may be stated formally as follows:

Theorem 4 (Principle of Inclusion and Exclusion)**.** *For any n finite sets $A_1, A_2, \cdots A_n$, we have*

$$|A_1+A_2+\cdots+A_n| = \sum_{i=1}^{n} |A_i| + \sum_{i<j}^{n} |A_i \cap A_j| + \sum_{i<j<k}^{n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{q+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

The theorem above allows us to break a problem into cases that we can enumerate, and then combine these case using the theorem. The application of this theorem often requires some creativity. For example:

Example 8. *Find the number of non-negative integer solutions to the equation* $x_1+x_2+x_3 = 15$ *where* $x_1 \le 5, x_2 \le 6, x_3 \le 7$.

We are not able to deal with these upper bounds by using conventional means. However, we can find out the number of solutions which does not satisfy the conditions above. Let us denote the property $x_1 \ge 6$ as $P(1)$, property $x_2 \ge 7$ as $P(2)$ and property $x_3 \ge 8$ as $P(3)$. We will use the principle of inclusion and exclusion to find out the number of solutions which satisfy at least one of $P(1), P(2)$ or $P(3)$.

We will use the method introduced in example 3 to help us find out the number of solutions in each cases. For $P(1)$, since $x_1 \ge 5$, we can subtract 6 from the right hand side of the equation and place the 6 "objects" into x_1 first, leaving 9 more objects behind. This will give us $\binom{9+3-1}{2}$ solutions. By principle of inclusion and exclusion, the total number of solutions which satisfy at least one of the above conditions amounts to:

$$\binom{11}{2} + \binom{10}{2} + \binom{9}{2} - \binom{4}{2} - \binom{3}{2} - \binom{2}{2}$$

The total number of solutions without any conditions is $\binom{17}{2}$. Hence, the number of solutions which fits the condition in the question is

$$\binom{17}{2} - \left(\binom{11}{2} + \binom{10}{2} + \binom{9}{2} - \binom{4}{2} - \binom{3}{2} - \binom{2}{2}\right) = 10$$

There's a generalisation of principle of inclusion and exclusion which allows us to find out the exact number of cases for specific number of conditions specified. Interested students may want to research on that.

2.3 Recurrence Relations

Problems which involve recurrence relations are characterised by number of arrangements which are dependent the number of arrangements with smaller objects. In particular, suppose a_n is the number of arrangements for n objects. We need to construct a relationship between a_n and its previous terms. For example, the Fibonacci sequence is a recurrence relation which is based on the rules $a_0 = 1, a_1 = 1, a_n = a_{n-1} + a_{n-2}$.

Hence, we need to construct an equation describing the recurrence relation, and then find out a_n based on its previous terms. In junior section, the n given in the question will not be too big so that it is possible to evaluate a_n just by counting from the initial terms. As we progress, students are expected to be able to derive the general formula based on the initial condition and the recurrence relation.

Here's a simple problem which illustrates this approach.

Example 9. *A student can choose to take one step or two steps at a time while climbing up a ladder. Suppose the ladder has 10 steps. In how many ways can the student climb to the top of the ladder?*

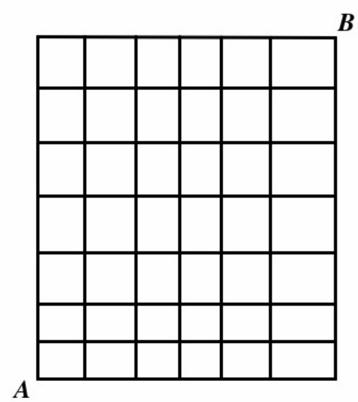
Let us define the number of different ways to climb the ladder with n steps as a_n. Before arriving at the final step, the student can either take one step or take two steps at a time. Hence, we need to find the sum of a_9 and a_8. In other words, $a_{10} = a_9 + a_8$. The same can be said for a_9 and a_8. Since we know that $a_1 = 1$ and $a_2 = 2$, we can find out the values of a_n up to $n = 10$. We have $a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, a_7 = 21, a_8 = 34, a_9 = 55$ and finally $a_{10} = 89$.

2.4 Problem Set

Permutations and Combinations

1. (AIME 1993) Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

2. Compute the total number of shortest path from point A to point B in the diagram below.



3. 6 boys and 5 girls are to be seated around a table. Find the number of ways that this can be done in each of the following cases:

   (a) There are no restriction;

   (b) No 2 girls are adjacent;

   (c) All girls form a single block;

   (d) A particular girl cannot be adjacent to either two particular boys B_1 and B_2.

4. (SMO(J) 2011 Modified) How many ways are there to put 7 distinct apples into 4 identical packages so that each package has at least one apple?

5. (AIME 1983) Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen - all choices of three being equally likely - and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

6. (SMO(S) 2010) Find the number of ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

7. (AIME 1984) A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{p}{q}$ in lowest terms be the probability that no two birch trees are next to one another. Find $p+q$.

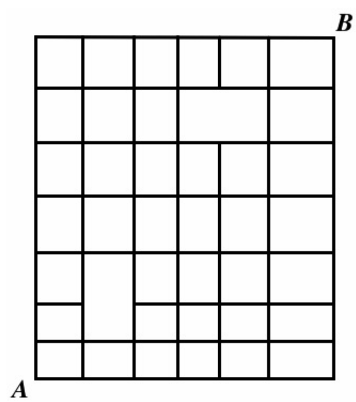
8. (AIME 1998) Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

Principle of Inclusion and Exclusion

1. (SMO(J) 2008) 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.

2. How many arrangements of $a, a, a, b, b, b, c, c, c$ are there such that no two consecutive letters are the same?

3. Compute the total number of shortest path from point A to point B in the diagram below.



4. (Putnam 1983) How many positive integers n are there such that n is a divisor of at least one of the numbers $10^{40}.20^{30}$?

Recurrence Relations

1. (AIME 1990) A fair coin is to be tossed 10 times. Let $\frac{i}{j}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i+j$.

2. (AIME 1985) Let A, B, C and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

3. (SMO(J) 2009) Using digits 0, 1, 2, 3 and 4, find the number of 13-digit sequences that can be written so that the difference between any two consecutive digits is 1. Example of such sequences are 0123432123432, 2323432321234 and 3210101234323.

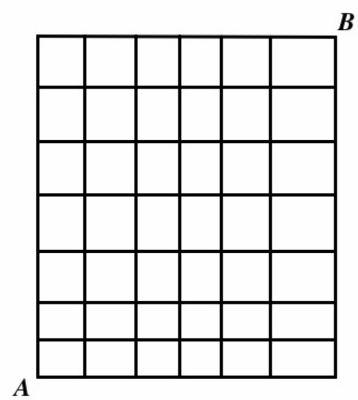
2.5 Solutions

Permutations and Combinations

1. (AIME 1993) Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

   Solution. We denote the two subsets as m and n. Each element will either be in subset m, subset n or in both subsets. Since there are 3 possibilities for each subsets, there will be 3^6 possibilities for 6 elements. However, since order does not matter, each case will be counted twice except the case when both m and n contains all 6 elements. Hence the total number of ways to select the subsets is $\dfrac{3^6 - 1}{2} = 365$.

2. Compute the total number of shortest path from point A to point B in the diagram below.

   Solution. The shortest path from point A to point B will contain 13 segments, of which 7 segments are vertical while 6 segments are horizontal. Hence, this problem is equivalent to finding the number of ways to arrange the sequence of vertical segments and horizontal segments, which amounts to $\dbinom{13}{6} = 1716$.

3. 6 boys and 5 girls are to be seated around a table. Find the number of ways that this can be done in each of the following cases:

   (a) There are no restriction;

   (b) No 2 girls are adjacent;

   (c) All girls form a single block;

   (d) A particular girl cannot be adjacent to either two particular boys B_1 and B_2.

   Solution. (a) The number of ways is $\dfrac{11!}{11} = 3628800$.

(b) Suppose we label the chairs from 1 to 11. We will girl A to sit at chair 1. This effectively changes the problem into a linear arrangement problem. We need to slot the 6 guys in between the 5 girls. There are 5 possible spaces for the guys to slot in, and out of this 5 spaces there must be a slot with 2 guys. Hence, the total number of possible arrangements is $4! \times \binom{5}{1} \times 6! = 86400$.

(c) We treat the block of girls as a single person. The number of ways is $\frac{7!5!}{7} = 86400$.

(d) We will count the number of arrangements when the girl is adjacent to either B_1 or B_2, which amounts to $4 \times 9! - 2 \times 8! = 1370880$. We subtract this number from the total number of arrangements which will give us the answer of $3628800 - 1370880 = 2096649$.

4. (SMO(J) 2011 Modified) How many ways are there to put 7 distinct apples into 4 identical packages so that each package has at least one apple?

Solution. There are three possible configurations of placing the apples, namely $(1, 1, 1, 4)$, $(1, 1, 2, 3), (1, 2, 2, 2)$.
In the first case, there are $\binom{7}{4} = 35$ ways to do so.
In the second case, there are $\binom{7}{3} \times \binom{4}{2} = 210$ ways to do so.
In the third case, there are $\binom{7}{2} \times \binom{5}{2} \times \binom{3}{2} = 105$ ways to do so.
Hence the total number of ways is $35 + 105 + 210 = 350$.

5. (AIME 1983) Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen - all choices of three being equally likely - and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

Solution. We will find the probability when the knights are not seated next to each other. Suppose we have already chosen the first knight. We will break this problem into two cases:

(i) If the second knight chosen is seated one seat away from the first knight chosen, the total number of seats where the third knight can be selected without being adjacent to the first two knights is 20, and hence the probability of this case is $\frac{2}{24} \times \frac{20}{23} = \frac{5}{69}$.

(ii) If the second knight chosen is more than one seat away from the first knight chosen, the total number of seats where the third knight can be selected without being adjacent to the first two knights is 19, and hence the probability of this case of $\frac{20}{24} \times \frac{19}{23} = \frac{95}{138}$.

The probability of these two cases is $\frac{5}{69} + \frac{95}{138} = \frac{35}{46}$. Hence the probability of these two cases not occurring is $\frac{11}{46}$. The final answer is 57.

6. (SMO(S) 2010) Find the number of ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

   Solution. We first arrange the 5 red balls and place a blue ball in between the five red balls. We are left with 9 blue balls and we can place then between the red balls or at the ends of the straight line. Hence, the total number of arrangements is $\binom{9+6-1}{9} = 2002$.

7. (AIME 1984) A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{p}{q}$ in lowest terms be the probability that no two birch trees are next to one another. Find $p+q$.

   Solution. We first arrange the maple trees and oak trees. There are 8 spaces in between the trees and at the edge of the line, of which we will plant at most 1 birch tree in the spaces. There are $\binom{8}{5} = 56$ ways to choose the spaces. Hence the probability will be $\frac{\binom{8}{5} \times 7! \times 5!}{12!} = \frac{7}{99}$. Hence $p+q = 106$.

8. (AIME 1998) Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

   Solution. We write each variables as $x_1 = 2y_1+1, x_2 = 2y_2+1, x_3 = 2y_3+1, x_4 = 2y_4+1$ where y is a non-negative integer to remove the restriction that the solution to the equation must be an odd number. Hence, the equation simplifies into $y_1+y_2+y_3+y_4 = 47$, which yields $\binom{47+4-1}{4-1} = 19600$ solutions. The final answer would be 196.

Principle of Inclusion and Exclusion

1. (SMO(J) 2008) 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.

   Solution. Let us define $P(B), P(W) and P(R)$ as the property that black balls, white balls and red balls are in consecutive block respectively. By the Principle of Inclusion and Exclusion, we have:

$$\begin{aligned}
&|P(B \cup W \cup R)| \\
&= |P(B)| + |P(W)| + |P(R)| - |P(B \cap W)| - |P(B \cap R)| - |P(R \cap W)| + |P(B \cap R \cap W)| \\
&= 2\frac{7!}{4!}2! + \frac{9!}{4!}4! - 2\frac{6!}{4!} - \frac{4!}{2!} + 3! \\
&= 774
\end{aligned}$$

Hence the total number of ways such that balls of the same colour do not appear in a consecutive block is $\frac{10!}{4!4!2!} - 774 = 2376$.

2. How many arrangements of $a, a, a, b, b, b, c, c, c$ are there such that no two consecutive letters are the same?

Solution. Similar to the above problem, let us define $P(a), P(b) and P(c)$ as the property that there are two or more consecutive letters of a, b and c respectively. By the Principle of Inclusion and Exclusion, we have:

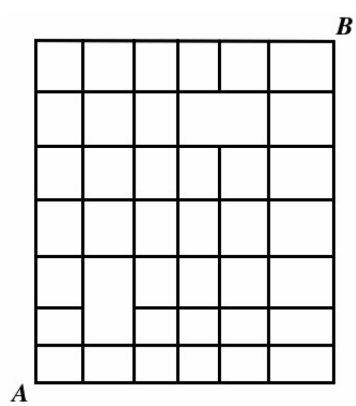
$$|P(a)| = \frac{8!}{3!3!} - \frac{7!}{3!3!} = 980$$
$$|P(a \cap b)| = \frac{7!}{3!} - 2\frac{6!}{3!} + \frac{5!}{3!} = 620$$
$$|P(a \cap b \cap c)| = 6! - 3 \cdot 5! + 3 \cdot 4! - 3! = 426$$

Hence, by Principle of Inclusion and Exclusion again,

$$\begin{aligned}
&|P(a \cup b \cup c)| \\
&= |P(a)| + |P(b)| + |P(c)| - |P(a \cap b)| - |P(b \cap c)| - |P(a \cap c)| + |P(a \cap b \cap c)| \\
&= 3 \times 980 - 3 \times 620 + 426 \\
&= 1506
\end{aligned}$$

Hence the final answer is $\frac{9!}{3!3!3!} - 1506 = 174$. I admit I was being evil when I included this question :)

3. Compute the total number of shortest path from point A to point B in the diagram below.



Solution. We call the missing segment close to A segment a, the segment closer to B segment b. Let $P(a)$ be the property that the path crosses a while $P(b)$ be the property that the path crosses b. The total number number of paths which crosses either a or b or both is given by:

$$\binom{3}{2}\binom{9}{4} + \binom{9}{4}\binom{3}{2} - \binom{3}{2}\binom{5}{2}\binom{3}{2} = 666$$

Hence the number of paths that do not cross either segments is given by:

$$\binom{13}{6} - 666 = 1716 - 666 = 1050$$

4. (Putnam 1983) How many positive integers n are there such that n is a divisor of at least one of the numbers $10^{40}.20^{30}$?

   Solution. Note that $10^{40} = 2^{40}5^{40}$ and $20^{30} = 2^{60}5^{30}$. 10^{40} has 41^2 factors while 20^{30} has 61×31 factors. Their greatest common divisor, $2^{40}5^{30}$, has 41×31 factors. Hence, the total number of such integers is $41^2 + 61 \times 31 - 41 \times 31 = 2301$.

Recurrence Relations

1. (AIME 1990) A fair coin is to be tossed 10 times. Let $\frac{i}{j}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

   Solution. Let H_n be the number of possible tossing sequences such that heads never occur on n consecutive toss. We note that for any sequence of length $n-1$, we may flip another tails to obtain a valid sequence of n tosses. For any sequence of length $n-2$, we may flip a "heads, tails" sequence to obtain another valid sequence of n tosses. Hence we have $H_n = H_{n-1} + H_{n-2}$. Since $H_1 = 2$ and $H_2 = 3$, we can derive that $H_{10} = 144$ and hence the probability would be $\dfrac{144}{2^{10}} = \dfrac{9}{64}$. Hence $i + j = 73$.

2. (AIME 1985) Let A, B, C and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

   Solution. Let P_n be the probability that the bug is at vertex A after travelling n metres. Before arriving at A after travelling n metres, the bug must be at another point. At this other points, there is one in three chance that the bug will choose to travel to A. Hence, we have $P_n = \dfrac{1}{3}(1 - P_{n-1})$. We note that $P_1 = 0$ and we can iterate this function 6 more times to obtain $P_7 = \dfrac{182}{729}$. Hence $n = 182$.

3. (SMO(J) 2009) Using digits 0, 1, 2, 3 and 4, find the number of 13-digit sequences that can be written so that the difference between any two consecutive digits is 1. Example of such sequences are 0123432123432, 2323432321234 and 3210101234323.

   Solution. We need to define several sequences to solve this problem. Let A_n, B_n and C_n be the sequences which satisfy the condition that end with 0 or 4, 1 or 3, and

2 respectively. Based on the definition of the equations, we can obtain the following relationships:

$$A_n = B_{n-1}$$
$$B_n = A_{n-1} + 2C_{n-1}$$
$$C_n = B_{n-1}$$

From this system of equations, we can substitute the first equation and third equation into the second equation to obtain $B_n = 3B_{n-2}$. Since $B_1 = 2$ and $B_2 = 4$, we obtain $A_{13} + B_{13} + C_{13} = B_{13} + 2B_{12} = 2 \times 3^6 + 2 \times 4 \times 3^5 = 3402$.

Chapter 3

Existence Combinatorics

We do find several problems which involves existence combinatorics in SMO(J) round 2. The concepts behind these proofs are not hard to grasp, but it will be very challenging for students to derive these concepts without being exposed to them beforehand. As a recap, here are some principles that students have to grasp in order to solve common combinatorial problems:

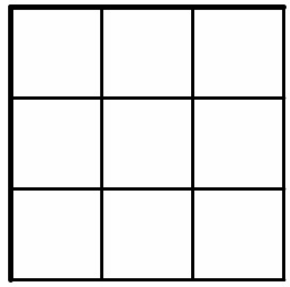
1. Pigeonhole principle
2. Bijection principle
3. Fubini's principle (Double Counting)
4. Extremal principle
5. Invariance, monovariance and colouring
6. Bounding

3.1 Pigeonhole Principle

Also known as the Dirichlet Box Principle, the Pigeonhole Principle should be a principle that students should already be familiar with. The idea behind this principle is straightforward: Given n boxes and r objects, there will be at least one box with at least $\lfloor \frac{n}{r} \rfloor + 1$ objects. This simple idea goes a long way in proving many complicated problems, including many problems found in the International Mathematics Olympiad. In applying this principle, it is important for students to identify the "boxes" that they need to construct to satisfy the principle.Try and construct the boxes for the following problem:

Example 10. *Given a 3×3 square, prove that within any set of 10 points in the square, there will be at least two points whose distance apart is no more than $\sqrt{2}$.*

The quantity $\sqrt{2}$ gives us a hint on the type of boxes that we have to construct to solve this problem. Since $\sqrt{2}$ is the diagonal of a square with side length 1, we will divide the big square into nine smaller squares with side length 1, as shown in the diagram below:

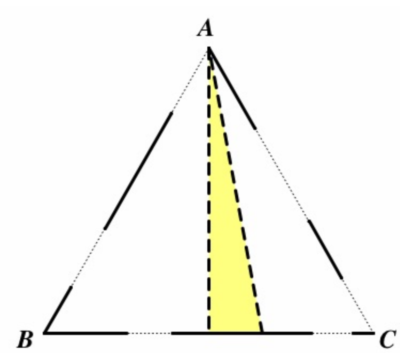


By Pigeonhole Principle, at least one of the small boxes above will contain at least 2 points. Since any two points within a 1×1 square have a distance of at most $\sqrt{2}$ the proposition is proven. Note that any other shapes of boxes will not allow us to prove the proposition.

Since this principle is very straightforward, questions which involve the Pigeonhole Principle are expected to be more challenging so that the application of Pigeonhole Principle will not be very obvious. The following problem from IMO does not require advanced technique, but students must be very keen to spot the application of the Pigeonhole Principle.

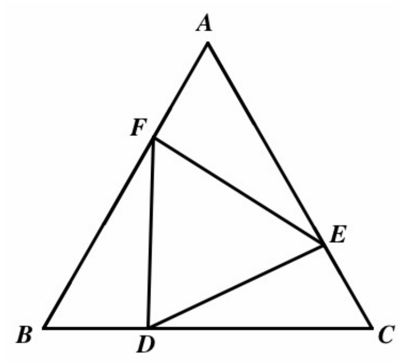
Example 11 (IMO 1983 P4)**.** *Let $\triangle ABC$ be an equilateral triangle and ϵ be the set of all points contained in the 3 segments AB, BC, CA. Show that, for every partition of ϵ into two disjoint subsets, at least one of the 2 subsets contains the vertices of a right-angled triangle.*

To understand this problem further, let us call the two subsets the black set and the white set. Here's a diagram which describes a way to divide the points into two sets, and we see that a right-angled triangle can be formed in one of the subsets:



In this problem, it is rather hard to identify the boxes and the objects to use the Pigeonhole Principle. Perhaps students may not even know that they are supposed to use the Pigeonhole Principle to solve this problem! Let us appreciate the following proof before discussing the motivation behind the proof.

Suppose that there is a way to partition the points such that each set of points does not contain the vertices of a right-angled triangle. We locate points D, E, F on BC, AC, AB respectively such that $AF : FB = 1 : 2, BD : DC = 1 : 2, CE : EA = 1 : 2$, as shown in the diagram below:



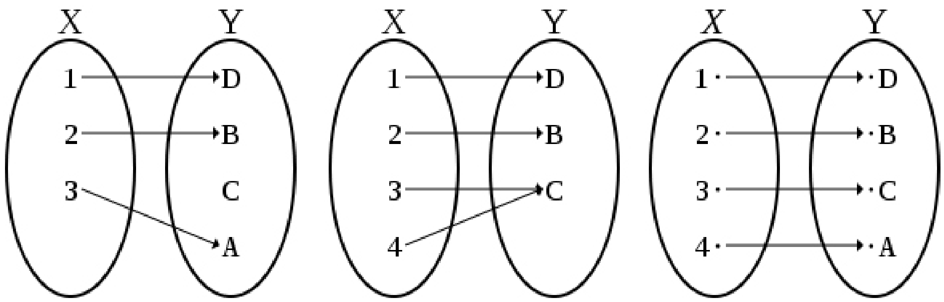
Our objects will be the points D, E and F. We are going to divide these three points into two boxes, which represents the two colours. Hence, there will be at least two points with the same colour among D, E and F. Without loss of generality, let us assume that D and E are black colour. We note that $FD \perp BC$ since $triangle BFD$ is a 30-60-90 triangle. As such, all points on BC except D must be coloured white. However, point E will form a right-angled triangle with its perpendicular foot on BC and another white point on BC and this leads to a contradiction. Hence regardless of the way we divide the points one of the partition will contain vertices of a right-angled triangle.

The motivation behind this proof is to select three points such that each two points are perpendicular to a corresponding side of the triangle. Since the triangle is equilateral, this encourages us to think about the 30-60-90 triangle and construct points D, E, F as shown above. Only then we can see the usefulness of Pigeonhole Principle in this problem.

3.2 Bijection Principle

The concept of injection, surjection and bijection is rather important in the field of mathematics, in particular set theory (which is the foundation of mathematics). Formally, suppose $f : X \mapsto Y$ is a function which maps an element from set A to set B. If $f(a) = f(b)$ implies that $a = b$, then the function is **injective.** In layman terms, this means that every element in set Y can only be traced back to one unique element in set X. An injective function is known as a one-to-one function. On the other hand, a **surjective** function is one whereby every element in Y can be traced back to at least one element in X. In other words, the function spans all elements in Y. The word "surjective" is synonymous with the word "onto". Finally, a **bijective** function is a function that is both injective and surjective. A bijective function fulfils the condition of one-to-one correspondence (take note that it is different from an injective function).

The diagram below illustrates the concept of the aforementioned properties. The diagram to the left shows a mapping which is injective but not surjective, the diagram in the middle shows a mapping which is surjective but not injective, and the diagram to the right shows a bijective mapping.



Establishing a bijection is important in combinatorics due to the bijection principle. The bijection principle states that if $f : A \mapsto B$ is a bijective function, then the number of elements in A is the same as the number of elements in B. The injection principle is sometimes useful too. If $f : A \mapsto B$ is a injective function, then the number of elements in A is less than that of B. One can understand why these principles work just by scrutinising the diagrams above.

Actually, the bijection principle is more useful in enumerative combinatorics than existence combinatorics. In applying this principle, we aim to establish a mapping from a quantity that we are required to count to another quantity which is easier to count. If we establish that the mapping is bijective, we can simply count the latter quantity and equate it to the former quantity. Many SMO round 1 problems utilise this property, such as the following problem:

Example 12 (SMO(S) 2008)**.** *Find the number of 11-digit positive integers such that the digits from the left to right are non-decreasing. (For example, 12345678999, 55555555555, 23345557889)*

This problem is very similar to example 4 in the previous section. It is very challenging for us to count the number of integers which satisfy the condition directly, since we need to account for different factors such as the leading digit, the number of repetitions, etc. We need to map it to another quantity which is more convenient for us to count. Usually, it can be helpful to map problems which involves integers with special conditions imposed on its digits into a binary sequence. We are not trying to convert these numbers into its binary numbers - that will be very troublesome. Instead, we want to match each number here with a suitable binary sequence such that the mapping is bijective and the number of binary sequences is easy to count.

These integers can be matched with a binary sequence in this manner: we let '0' indicate the number of repetitions and '1' indicate the increase in value of the digit. A number like 11123466789 will be represented as 0001010101100101010 and a number like 33456678888 will be represented as 1100101010010100001. We note that the binary sequence always contain 11 '0's and 8 '1's, which is logical since there are 11 digits and $1 + 8 = 9$. Is this mapping a bijection? To prove that it is a bijection we need to show that it is both injective and surjective. It is surjective because all binary sequence with 11 '0's and 8 '1's can definitely be written as a non-decreasing 11-digit integer sequence and no two unique 11-digit integers can be mapped to the same binary sequence. The problem now simplifies into counting the number of binary sequences, which is easy since it equates to $\binom{19}{8} = 75582$.

3.3 Fubini's Principle

Fubini's principle, also known as double counting, is often used to prove equality between two quantities. It is one of the strongest tools to prove many combinatorial identities, often in an elegant manner. I would recommend students to consider using this strategy in many other geometry problems and functional equations.

Here's a classical example to demonstrate the Fubini's principle:

Example 13. *Prove that* $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

I will provide two solutions to this problem, both utilising Fubini's principle. First, we consider the coefficient of x^n in the expansion of $(1+x)^{2n}$. There are two methods that we can use in order to evaluate the coefficient. First, we can use the binomial theorem to determine the coefficient of x^n. Students who are familiar with the binomial theorem or the Pascal's triangle should be able to identify that this coefficient equals to $\binom{2n}{n}$.

On the other hand, I can also find out the coefficient of x^n by expanding $(1+x)^n(1+x)^n$. We note that:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x + \binom{n}{n}x^n.$$

When we multiply the two terms together, the term x^n is formed by the product of 1 and x^n, x and x^{n-1}, x^2 and x^{n-2}, etc. Hence, the coefficient of x^n is equal to:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

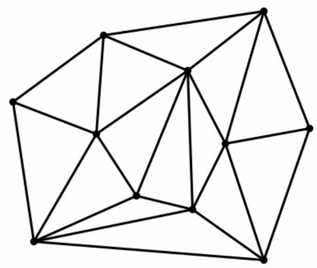
By equating this with $\binom{2n}{n}$ above, we have proved our first combinatorial identity.

Here's another creative solution which does not makes use of algebra so solve this problem. Suppose we want to select n people from $2n$ people to represent the class in a competition. The total number of possible way is obviously $\binom{2n}{n}$. On the other hand, we can split the class into two groups, each with n people. Then, we select n people from both groups in the following manner: first select zero people from the first group and n people from the second group, then select 1 people from the first group and $n-1$ people from the second group, and so on. Since this is another method to select n people, this method of counting should also produce $\binom{2n}{n}$ ways to select the students. Hence we have:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Fubini's principle often produces proofs which are very straightforward and does not require advance algebra to solve a problem. Still, student's have to be very creative in selecting the two methods to count a certain quantity. Consider the application of Fubini's principle in the following problem:

Example 14. *Given that there are m points selected within a convex n sides polygon. Connect these points with the vertices of the polygon such that no two line segments intersect, and that all polygons formed within the polygon are triangles. Express the number of triangles in terms of m and n.*



The above diagram shows an example of splitting a convex hexagon using 5 points, which in turn creates 14 triangles. To apply Fubini's principle, we need to find a quantity which has to be counted twice, and then express this quantity in two different expressions involving m and n. The most convenient quantity which we can quantify is the total sum of angles within the polygon. This can be done in two ways: using 180° multiplied by the number of triangles formed, and using the total angle within a polygon and the number of points to find the sum of angles. In the first approach, suppose the number of triangles is x. The sum of angles will be $180° \times x$. On the other hand, the sum of interior angles of a polygon is $180° \times (n-2)$ and the m interior points will contribute angles with a sum of $360° \times m$. By equating the two quantities and rearranging, we obtain the final result $x = 2m + n - 2$.

3.4 Extremal Principle

The extremal principle is one of the trickiest technique in combinatorics because it is challenging to identify. This principle is very universal since it can be applied in almost every field of MO. Students have to practice problems involving extremal principle in order to be adept in using this technique.

The idea behind extremal principle is to assume that there is an extreme example which fulfils the condition given in the problem. Then, based on the condition of the problem, we construct another example which is more extreme than the assumed example. This creates a contradiction and we can disprove the existence of such example through this contradiction. The following problem illustrates this concept:

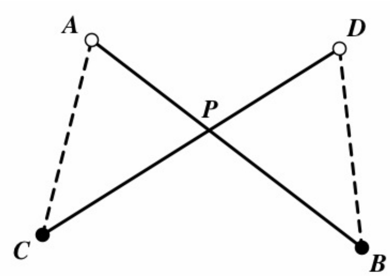
Example 15. *Every participant in a tournament plays against all other participants in the tournament exactly once. There is no draw in this tournament. After the tournament, each player creates a list which contains the names of the players who were beaten by him and the names of player who were beaten by the players beaten by him. Prove that there is one person in the tournament who has a list of the names of all other players.*

To use the extremal principle, we need to consider the person with a certain extreme quality. Naturally, we would want to focus our attention to the participant who has won the most opponents. Let's call him A. Now we suppose that A does not have a list which contains the names of all other players. There will be a player whose name will not be in A's list, which suggest that this person beats A. We call him B. Since B is not on the list, this suggest that B has beaten all the players whom A has beaten. Since B also defeated A, B has defeated more opponents than A, but this contradicts with our initial assumption that A is the person who has defeated the most amount of opponents. As such, A must have a list which contains the names of all other players.

The extremal principle can be quite helpful in solving problems involving combinatorial geometry. Consider the following problem:

Example 16. *Given $2n$ points in a plane with no three points collinear, with n red points and n blue points. Prove that we can connect them using disjoint line segments such that each line segments contain exactly 1 red point and 1 blue point and no two line segments intersect.*

This time, the extremal quantity that we consider will be the sum of lengths of all the line segments drawn between the points. Since the total possible ways to construct the line segments is finite, there must be one configuration such that the sum of lengths of all segments is minimum. Now, suppose that two line segments intersect in this configuration, as shown in the diagram below:



By using triangle inequality, we note that $AC + BD > PA + PB + PC + PD = AB + CD$. Hence, if we were to connect AC and BD instead of AB and CD, we will be able to attain a configuration with a smaller sum of segments than the previous configuration. This will lead to a contradiction because there is a more extreme case than the supposedly extreme configuration, and hence the configuration with the smallest sum of lengths of all segments must satisfy the condition of the problem.

3.5 Invariance, Monovariance and Colouring

Invariance is defined as a certain quantity or quality which does not change after certain transformations. Monovariance, on the other hand, is defined as a certain quantity or quality which changes in a direction after similar transformations. Identifying the invariant or the monovariant quantity can be critical in solving a problem. Here's a combinatorial problem which makes use of a monovariant quantity:

Example 17. *In the Singapore Parliament, each member of parliament has at most 3 enemies. Prove that the parliament can be divided into 2 groups of people, such that each member has at most 1 enemy in his or her own house.*

What is the quantity that is monovariant under a certain transformation? In solving this problem, we need to find out both the quantity that is monovariant and the transformation which changes the quantity. It is not quite obvious if this is the first time that you have encountered such a problem. In this case, our monovariant quantity is the sum of the number of enemies for each of the members in the parliament. Let us define this quantity as N. Note that this quantity must be nonnegative.

Now, we need to find out the appropriate transformation. If we are able to find a transformation such that always let N decrease until we obtain a negative number, we will obtain a contradiction. We assume that the proposition is false, that is at least one member will have more than 1 enemy in his group. Let us investigate the effect of placing this member into the other group. Upon this transformation, the quantity N will decrease by at least 2. Hence, if there is always a member who has more than 1 enemy in his group, the quantity N will keep on decreasing until in becomes a negative number. This will give us a contradiction, and hence there must be one point such that each member will only have at most 1 enemy in his own house.

In the above problem, the transformation is the action of placing the member with more than one enemy into another group, and the monovariant quantity is the sum of the number of enemies of all members. Often, the monovariant quantity is the sum of certain property in the question. The concept of invariance or monovariance can often be found in blackboard problems, where some numbers are being replaced by several other numbers. Consider the problem below:

Example 18. *There are three numbers a, b, c written on a blackboard. One can choose two numbers on the blackboard and decrease the two numbers by 1, and increase the third number which is not selected by 1. Suppose we cannot decrease the number until it becomes negative. After a finite amount of operations, there will be one number that is left while the other two numbers decrease to 0. Determine which number will be left.*

In this problem, the transformation is the operation that is stated in the problem. It is harder to figure out the invariant or monovariant quality. In solving this type of problem, it is often useful to consider the parity of the quantities. Consider how the parity of a, b, c changes after a transformation. We note that all three numbers will have their parity changed after a transformation. If there is a number that has a different parity than the other two numbers, it will retain this property after each transformation. This is the invariant quality. At the final state, two numbers will be zero, and hence the number which intially has the different parity will be the number which is left on the blackboard. One can also show that it is not possible to determine which number will be left if all three numbers have the same parity.

Colouring proofs are a special type of problems which assigns colours to certain spaces and that these spaces cannot be covered by certain units as described by the problem. The following problem is a classical problem which utilises this technique:

Example 19. *Given a 8×8 chessboard where the two corners which are diagonally opposite each other are removed. Prove that it is impossible to cover the chessboard using 31 2×1 dominoes.*

To solve this problem, we will colour each spaces using black and white colour such that each adjacent spaces have alternate colours. We note that there will either be 30 white spaces and 32 black spaces, or vice versa. However, a domino must cover 1 black and 1 white space each. Hence, it is not possible to cover different number of black and white spaces.

3.6 Bounding

This is a technique which is borrowed from the topic of algebraic inequalities. The idea behind bounding is to limit the possible quantity of the question within a certain bound. This can be done using Pigeonhole principle or other methods.Then, we prove that the bound can be achieved using a smart construction. The idea may seem abstract but hopefully the following example clears your doubt:

Example 20. *Select k elements from $1, 2, \cdots, 50$ such that these numbers are not pairwise coprime. Find the maximum value of k.*

First, we prove that k cannot be larger than 25. If there are 26 elements selected, there will be two numbers which are consecutive and hence they must be coprime. As such we have $k \leq 25$. Indeed, the set $\{2, 4, \cdots, 50\}$ satisfies the condition and has 25 elements, and hence the maximum value is indeed 25.

Here, we see that $k \leq 25$ is the initial bound which we set for the quantity which we want to maximise. Then, we try and prove that $k = 25$ is achievable using an example. Sometimes, you may want to construct and example first before bounding the quantity. This may inspire you to find a smart guess for the bound of the question. Pigeonhole principle is often useful in bounding problems. In the above example, we can actually construct 25 boxes where each box contains consecutive integers. Then we can use Pigeonhole principle to show that the maximum value is indeed 25.

3.7 Problem Set

Pigeonhole Principle

1. What is the minimum number of elements required to select from the set $S = \{1, 2, 3, \cdots, 100\}$ such that there will be at least an element which is an integer multiple of another element selected?

2. (Putnam 1978) Let $S = \{1, 4, 7, 10, 13, 16 \cdots, 100\}$. Let T be a 20 element subset of S. Show that we can find two distinct elements in T with sum 104.

3. (AMM 1958) Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to one another.

4. A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but no more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

Bijection Principle

1. Suppose $S = \{1, 2, 3, \cdots, 14\}$, select three numbers a_1, a_2, a_3 from S such that $a_1 \le a_2 \le a_3$ and $a_2 - a_1 \ge 3$, $a_3 - a_2 \ge 3$. How many ways are there to select the three numbers a_1, a_2, and a_3?

2. How many ways are there to select 6 numbers from the set $S = \{1, 2, 3, \cdots, 49\}$ such that at least two numbers are adjacent to each other?

3. (SMO(O) 2010) All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g. 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on). If the 2010^{th} number in this sequence is denoted by p, find the value of $\lfloor \frac{p}{10} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

4. (IMO 1989 P6) A permutation $x_1 x_2 \cdots x_{2n}$ of the set $\{1, 2, \cdots, 2n\}$, where $n \in \mathbb{N}$, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \cdots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

Fubini's Principle

1. Prove the following identity:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

2. Prove the Vandermonde's identity:

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0} = \binom{m+n}{r}.$$

3. Given n numbers $x_1, x_2, \cdots x_n$, suppose each number is either 1 or -1 and that $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$, prove that n is divisible by 4.

Extremal Principle

1. In a gathering, every two people are either mutual acquaintances or complete strangers to one another. Suppose that if two people in the gathering has the same number of acquaintances, then both of them do not have any common acquaintances. Prove that if a person has at least 2012 acquaintances, then there will be a person who has exactly 2012 acquaintances.

2. Prove that the equation $x^4 + y^4 = z^2$ has no integer solutions.

3. (USATST 2005) Find all finite set of points S in the plane with the following property: for any three distinct points A, B, C in S, there is a fourth point D in S such that A, B, C and D are the vertices of a parallelogram (in some order).

4. In an island *Tigerpore*, there are several cities and each city is connected by exactly 1 one-way street with one another. Prove that there is a city such that all other cities can either reach this city through a one-way street, or can reach this city through exactly one other city.

Invariance, Monovariance and Colouring

1. A dragon has 100 heads. A strange knight can cut off 15, 17, 20 or 5 heads respectively with one blow of his sword. However, the dragon has mystical regenerative powers, and it will grow back 24, 2, 14 or 17 heads respectively in each cases. If all heads are blown off, the dragon dies. WIll the dragon ever die?

2. (St. Petersburg 2003) Several positive integers are written on a board. One can erase any two distinct numbers and write their greatest common divisor and lowest common multiple instead. Prove that eventually the numbers will stop changing.

3. $2n$ ambassadors are invited to a banquet. Every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.

4. Is it possible to pack a $10 \times 10 \times 10$ box with 250 $1 \times 1 \times 4$ bricks completely?

Bounding

1. (SMO(J) 2009) 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

2. (SMO(J) 2006 P3) Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum amount of phone calls between pairs of people required for everyone to know everything?

3. Let $M = \{1, 2, \cdots, 2012\}$, A is a subset of M such that if $x \in A$, then $15x \notin A$. Find the maximum number of elements in A.

4. (China MO 1996 P4) 8 singers take part in a festival. The organiser wants to plan m concerts. For every concert there are 4 singers who go on stage, with the restriction that the times of which every two singers go on stage in a concert are all equal. Find a schedule that minimises m.

3.8 Solutions

Pigeonhole Principle

1. What is the minimum number of elements required to select from the set $S = \{1, 2, 3, \cdots, 100\}$ such that there will be at least an element which is an integer multiple of another element selected?

   Solution. It is possible for us to choose 50 elements that there wont be any element which is an integer multiple of another element selected, since this can be done by choosing the elements from 51 to 100. We shall prove that it is not possible to do so with 51 elements.

   Suppose the set $\{a_1, a_2, \cdots a_{51}\}$ satisfy the condition. We can write each element $a_i = 2^k b_i$ such that b_i is odd. We note that there are only 50 odd numbers from 1 to 100. By Pigeonhole Principle, there will be two numbers, call then a_m and a_n, such that $a_m = 2^p b$ and $a_n = 2^q b$ for the same odd number b. Hence, either a_m is a multiple of a_n or vice versa. As such, the set with 51 elements cannot satisfy the condition and this implies that the solution is 51.

2. (Putnam 1978) Let $S = \{1, 4, 7, 10, 13, 16 \cdots, 100\}$. Let T be a 20 element subset of S. Show that we can find two distinct elements in T with sum 104.

   Solution. We create 18 boxes as follows: $\{1\}, \{4, 100\}, \{7, 97\}, \cdots, \{49, 55\}, \{52\}$. When 20 elements are chosen, there will be at least two elements which are in the same box. Hence, these elements will add up to 104 and the proposition is proven.

3. (AMM 1958) Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to one another.

   Solution. This problem belongs to a general class of problem known as Ramsey problems. We assume that the proposition is false. Consider a person called A. A must be either acquaintances or strangers with at least three other people by Pigeonhole Principle. Without loss of generality, let us assume that A and B, C, D are mutual acquaintances. If any two people among B, C, D are mutual acquaintances, these 2 people and A are all mutual acquaintances and this contradicts with our assumption. Hence, B, C, D are mutual strangers but this also contradicts with our assumption. Hence our assumption is wrong and there must be some three of them who are either mutual acquaintances or complete strangers to one another.

4. A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but no more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

   Solution. This is a very classical problem which can be solved using Pigeonhole Principle cleverly. We define the sequence a_n as the number of games played from the first day to the n^{th} inclusive. The following inequality must hold:

   $$1 \leq a_1 < a_2 < a_3 < \cdots a_{77} \leq 132 \Rightarrow 22 \leq a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \leq 153.$$

So we let $a_1, a_2, \cdots, a_{77}, a_1+21, a_2+21, \cdots a_{77}+21$ be 154 different items to be placed in 153 different boxes. By Pigeonhole Principle, at least two numbers above will be equal. Suppose $a_i = a_j + 21$. This means that the chessmaster played 21 games on the days $j+1, j+2, \cdots, i$, which proves the proposition of the problem.

Bijection Principle

1. Suppose $S = \{1, 2, 3, \cdots, 14\}$, select three numbers a_1, a_2, a_3 from S such that $a_1 \leq a_2 \leq a_3$ and $a_2 - a_1 \geq 3$, $a_3 - a_2 \geq 3$. How many ways are there to select the three numbers a_1, a_2, and a_3?

   Solution. For the three numbers that we select, we can perform a transformation from $\{a_1, a_2, a_3\}$ into $\{a_1, a_2 - 2, a_3 - 4\}$. The latter set is actually a three element subset from the set $\{1, 2, 3, \cdots 10\}$. This transformation is both injective and surjective and hence the number of subsets $\{a_1, a_2, a_3\}$ which we can select is $\binom{10}{3} = 120$.

2. How many ways are there to select 6 numbers from the set $S = \{1, 2, 3, \cdots, 49\}$ such that at least two numbers are adjacent to each other?

   Solution. We shall count the number of ways to select 6 numbers such that all numbers are not adjacent to each other. We will establish a bijection between the sequence of 6 numbers selected from this set such that no two numbers are adjacent and another set which is easier to count. We consider the transformation from $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ which satisfy the condition to $\{a_1, a_2 - 1, a_3 - 2, a_4 - 3, a_5 - 4, a_6 - 5\}$. The latter set is a subset of $\{1, 2, 3, \cdots 44\}$. We notice that this transformation is bijective and hence the number of ways to select the 6 numbers is the same as the number of 6 element subset of $\{1, 2, 3, \cdots, 44\}$. Hence the solution is $\binom{49}{6} - \binom{44}{6} = 6924764$.

3. (SMO(O) 2010) All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g. 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on). If the 2010^{th} number in this sequence is denoted by p, find the value of $\lfloor \frac{p}{10} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

   Solution. Brute force, use the approach similar to example 4 for each leading digit. You deserve compliments if you obtain the answer 86422 on the first try.

4. (IMO 1989 P6) A permutation $x_1 x_2 \cdots x_{2n}$ of the set $\{1, 2, \cdots, 2n\}$, where $n \in \mathbb{N}$, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \cdots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

   Solution. We will use the injection principle to solve this problem. By injection principle, if $f : A \mapsto B$ is an injective mapping, then $|B| \geq |A|$. We will let A be the set of sequences without property P and B be the set of sequences with property P. Now we need to define an appropriate transformation from A to B such that the transformation is injective but not surjective. For the sake of convenience, we shall call the pair of numbers n and $k+n$ partners of each other.

Now suppose $\alpha = x_1x_2\cdots x_{2n}$ is an element in A. α does not have property P. Suppose the partner of x_1 is x_r where $r \in \{3, 4, \cdots, 2n\}$. We can transform α into a sequence with property P by placing x_1 to the left of x_r. Let us define this transformation as $f(\alpha)$ where $f : A \mapsto B$. Obviously, this function is injective. We also note that $f(\alpha)$ are sequences that only contain exactly one adjacent pair of partners, but B still contain other sequences with more than one adjacent pair of partners. This shows that the function is not surjective. Hence by injection principle there are more permutations with property P than without.

Fubini's Principle

1. Prove the following identity:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Solution. Let us consider the number of ways to choose r objects from n distinct objects where $0 \le r \le n$. For each r, there are $\binom{n}{r}$ ways to choose r objects. As r ranges from 0 to n, we have $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$ total ways to choose the objects. On the other hand, each object is either chosen or not chosen. By multiplication principle, there are 2^n possibilities for n objects where each object can be chosen or not. Since the two quantities are equal, we obtain the above identity.

2. Prove the Vandermonde's identity:

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0} = \binom{m+n}{r}.$$

Solution. Given a cartesian plane, consider a rectangular grid with height r and width $m+n-r$ where the bottom left corner is placed at the origin. We shall count the number of paths to travel from the bottom left corner to the top right corner. On one hand, this can be easily counted using bijection principle, which gives us the answer of

$$\binom{m+n-r+r}{r} = \binom{m+n}{r}.$$

On the other hand, we can count the total number of paths which passes through the points $(m,0), (m-1,1,), (m-2,2), \cdots (m-r,r)$ respectively and take the sum of these quantities. The number of paths that passes through the point $(m-i,i)$ where $i \in \{0, 1, \cdots r\}$ is given by:

$$\binom{m-i+i}{i}\binom{m+n-r-(m-i)+(r-i)}{r-i} = \binom{m}{i}\binom{n}{r-i}.$$

By taking the sum of these quantities, we have:

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0} = \binom{m+n}{r}.$$

3. Given n numbers $x_1, x_2, \cdots x_n$, suppose each number is either 1 or -1 and that $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$, prove that n is divisible by 4.

   Solution. First, we prove that n is even. Among $x_1x_2, x_2x_3, \cdots x_{n-1}x_n, x_nx_1$, there is an equal number of 1s and -1s. As such, there must be an even number of terms. Now we suppose $n = 2k$ and hence there are k terms which are -1 among $x_1x_2, x_2x_3, \cdots x_nx_1$. We will count the quantity $(x_1x_2) \cdot (x_2x_3) \cdot (x_3x_4) \cdots (x_{n-1}x_n) \cdot (x_nx_1)$ in two different ways. Since there are k terms which are -1, we have $(x_1x_2) \cdot (x_2x_3) \cdot (x_3x_4) \cdots (x_{n-1}x_n) \cdot (x_nx_1) = (-1)^k$. On the other hand, $x_1^2x_2^2 \cdots x_n^2 = 1$. We have $(-1)^k = 1$ and hence k is an even number. As such, n is divisible by 4.

Extremal Principle

1. In a gathering, every two people are either mutual acquaintances or complete strangers to one another. Suppose that if two people in the gathering has the same number of acquaintances, then both of them do not have any common acquaintances. Prove that if a person has at least 2012 acquaintances, then there will be a person who has exactly 2012 acquaintances.

   Solution. Let A be the person with the most number of acquaintances. The people whom A knows cannot have the same number of acquaintances since A is their common acquaintance. Suppose A has k acquaintances where $k \geq 2012$. The *acquaintances* whom A knows must have $1, 2, \cdots k$ acquaintances each. Hence, one of them will have exactly 2012 acquaintances.

2. Prove that the equation $x^4 + y^4 = z^2$ has no integer solutions.

   Solution. We assume that there exist a number of non-trivial solutions i.e. solutions other than $(0, 0, 0)$. Among this number of non-trivial solution, there must be a solution set (x_0, y_0, z_0) such that $x_0 + y_0 + z_0$ is the smallest. If we take mod 16, we note that x_0, y_0, z_0 must all be even numbers. Also, z_0 is divisible by 4. Hence, $x_0 = 2x_1, y_0 = 2y_1, z_0 = 4z_1$ for some positive integers x_1, y_1, z_1. Substituting back into the equation, we have $16x_1^4 + 16y_1^4 = 16z_1^2$ and hence $x_1^4 + y_1^4 = z_1^2$. As such, (x_1, y_1, z_1) is another set of solutions. However, (x_1, y_1, z_1) has a smaller sum than (x_0, y_0, z_0). This contradicts with our initial assumption and hence the only solution is $(0, 0, 0)$.

3. (USATST 2005) Find all finite set of points S in the plane with the following property: for any three distinct points A, B, C in S, there is a fourth point D in S such that A, B, C and D are the vertices of a parallelogram (in some order).

   Solution. Since there is a finite number of points, there must be three points such that the three points form a triangle of the largest area. We call these three points A, B, C respectively. There must be a fourth point D which forms a parallelogram with triangle $\triangle ABC$. Also, there must be no points outside of parallelogram $ABCD$, otherwise it will form a triangle with larger area than $\triangle ABC$. Also, there must be no points within parallelogram $ABCD$. If there is a point within $ABCD$ which we shall denote as E, there must be points outside of $ABCD$ such that it forms a parallelogram with sides of $ABCD$ and point E. This contradicts with the fact there there must be no points outside of $ABCD$. Hence, there cannot be more than 4 points in the set and any set of points which form a parallelogram will satisfy the condition.

4. In an island *Tigerpore*, there are several cities and each city is connected by exactly 1 one-way street with one another. Prove that there is a city such that all other cities can either reach this city through the one-way street, or can reach this city through exactly one other city.

   Solution. Let M be the city with the most one-way street directed towards it. Suppose there are m one-way street directed towards this city. Let D be the set of cities which are connected by one-way streets towards M and R be the set of cities which are not in D (except M). If there are no cities in R, then the conclusion stands. Otherwise, if X is a city is R, there must be at least a one-way street linking X to a city in D. Otherwise, there will be at least $m+1$ one-way street directed towards X, which contradicts with the assumption that M has the most one-way street directed towards it. Hence all cities in R is connected to at least one city in D and the proposition in the question is proven.

Invariance, Monovariance and Colouring

1. A dragon has 100 heads. A strange knight can cut off 15, 17, 20 or 5 heads respectively with one blow of his sword. However, the dragon has mystical regenerative powers, and it will grow back 24, 2, 14 or 17 heads respectively in each cases. If all heads are blown off, the dragon dies. WIll the dragon ever die?

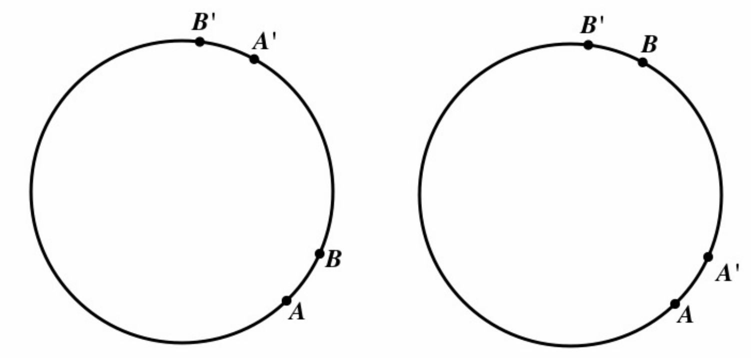
   Solution. Since the number of heads which grows back is equal to the number of heads being blown away mod 3, the remaining number of heads mod 3 is an invariant quantity. Since the dragon initially has 100 heads which is 1 mod three, it is impossible for the number of heads remaining to be divisibly by 3 and hence all heads cannot be blown away.

2. (St. Petersburg 2003) Several positive integers are written on a board. One can erase any two distinct numbers and write their greatest common divisor and lowest common multiple instead. Prove that eventually the numbers will stop changing.

   Solution. Let the greatest common divisor of all numbers be d. This is the invariant quantity regardless of the number of iterations. After any number of iterations, there will always be two numbers which have a greatest common divisor of exactly d. When these two numbers undergo the operation described in the problem, it produces the number d. When d operates with another number, it will return the same two numbers before the operation. We will isolate d (since it does not change any numbers after any operation) and consider the other elements. There will be a new greatest common divisor among the other numbers and the same process repeats itself. There will be a point where by new GCDs are being isolated until there is only one element left, and at that point all numbers will stop changing.

3. $2n$ ambassadors are invited to a banquet. Every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.

   Solution. Define M as the number of adjacent ambassadors who are enemies with each other. We need to find a transformation which is monovariant such that M decreases each time the transformation is performed. Suppose (A, B) are enemies while (A, A') and (B, B') are not enemies. B is sitting to the right of A and B' is sitting to the right of A'. Consider the transformation whereby the arc $A'B$ is inverted (everyone's seat within this arc is inverted). We obtain a new configuration shown in the diagram on the right. In this transformation, the value of M decrease by 1. We are left to show that this transformation is always possible under any circumstances i.e. there always exist a pair $A'B'$ some distance away from AB.

   Going counterclockwise to the right of A, there are at least n friends of A, The person there are n seats to the right of A's friends. These all seats cannot be all occupied by B's enemy since B has at most $n-1$ enemies. As such, there is always a pair $A'B'$ which allows for the above transformation.

4. Is it possible to pack a $10 \times 10 \times 10$ box with 250 $1 \times 1 \times 4$ bricks completely?

   Solution. We shall label the box in the following manner. We give each unit a coordinate (x, y, z) to indicate the row, column and height at which the unit is positioned, where $x, y, z \in \{1, 2, \cdots 10\}$. We will label each unit i where $i = x + y + z \pmod 4$. In total, there are 251 cells of colour 0. Since each brick covers four units of one colour each, when the box is filled the bricks must cover 250 units of each colour. This contradicts with the fact that there are 251 cells of colour 0 and hence it is not possible to do so.

Bounding

1. (SMO(J) 2009) 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

Solution. Let this minimum number be x. Suppose we let 1 indicate a "true" answer and "0" indicate a "false" answer. We can write the answers to the 10 questions as a binary sequence with 10 digits. We will create 512 boxes as follows: the n^{th} box will contain the numbers $2n-1$ and $2n$ written in base 2. Each boxes will contain binary sequences which only differ by 1 digit (i.e. has 9 identical answers). Hence, by Pigeonhole Principle, we need to choose at least 513 binary sequences such that at least 2 of the binary sequences have nine identical digits. This shows that $x \leq 513$.

Now we show that the case when 512 answer scripts can be chosen such that no two answer scripts have nine or more identical answers. We simply need to choose answer scripts with 0, 2, 4, 6, 8 and 10 "true" answers to ensure that no two answer scripts have nine. There can be $\binom{10}{0}+\binom{10}{2}+\binom{10}{4}+\binom{10}{6}+\binom{10}{8}+\binom{10}{10}=512$ possible scripts such that no two answer scripts are the same. As such, $x > 512$ and we must have $x = 513$.

2. (SMO(J) 2006 P3) Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum amount of phone calls between pairs of people required for everyone to know everything?

   Solution. Let this minimum number be x. Now, if everyone tells a person A everything they know, and then A tells everyone everything he knows, everyone will know everything. This requires $2n-2$ phone calls and hence $x \leq 2n-2$.

   Now, define M as the number of information which the person with the most information know. Notice that M can at most increase by 1 per phone call. Since M is initially 1 before any phone call, we need at least $n-1$ phone calls such that there will be at least one person who knows everything. However at this point, everyone else does not have all information yet, so it takes at least $n-1$ calls to everyone to supply them with information. As such, $x \geq 2n-2$. This implies that the minimum number of calls is indeed $2n-2$.

3. Let $M = \{1, 2, \cdots, 2012\}$, A is a subset of M such that if $x \in A$, then $15x \notin A$. Find the maximum number of elements in A.

   Solution. Let thus maximum number be k. We will try and construct A such that A has as many elements as possible. If $15x > 2012$, then x should be in A. This will include the numbers from 134 to 2012, which is a total of 1879 elements. On the other hand, if $15x < 134$, x should be in A too. This then includes the numbers from 1 to 8, which brings us to a total of 1887 elements. In this case, $A = \{1, 2, \cdots 8\} \cap \{134, 135, \cdots, 2012\}$. Hence, $k \geq 1887$.

   Note that there can only be one between x and $15x$ which can be included in A, where x ranges from 9 to 133. The maximum value of k is therefore bounded by $k \leq 2012 - (133 - 9 + 1) = 1887$. As such, we can conclude that $k = 1887$.

4. (China MO 1996 P4) 8 singers take part in a festival. The organiser wants to plan m concerts. For every concert there are 4 singers who go on stage, with the restriction that the times of which every two singers go on stage in a concert are all equal. Find a schedule that minimises m.

   Solution. Let S be the times of which every two singers go on stage in a concert. There are a total of $\binom{8}{2} = 28$ combinations of two singers. On the other hand, there are 6 pairs of two singers in every concert. Hence, we can set up the following equation:

   $$28S = 6m \Rightarrow m = \frac{14}{3}S \geq 14$$

   Now, we need to show that $m = 14$ is achievable (which is the hardest part of the question). The following schedule allows for 14 concerts to be held:

   $$(A, B, C, D), (A, B, E, F), (A, B, G, H), (C, D, E, F), (C, D, G, H),\\ (E, F, G, H), (A, C, E, G), (B, D, F, H), (A, C, F, H), (B, D, E, G),\\ (A, D, E, H), (B, C, F, G), (A, D, F, G), (B, C, E, H).$$