Functional Equations

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1 Tips and Techniques

- Plug in special values for some or all of the variables: $x = 0, x = \pm 1, y = \pm x, y = f(x)$, and so on.
- Guess the answer! Usually there is one nontrivial function (or family of functions) satisfying the functional equation, such as $f(x) = x^2$ or $f(x) = cx$ for some $c \in \mathbb{R}$, but don't forget trivial constant solutions such as $f(x) = 1$ as well.
- Make substitutions like $g(x) = f(x + 1)$ or $g(x) = f(x) - f(0)$ to simplify the functional equation and/or its solutions.
- Consider special values of the function, such as $f(0), f(1)$, and $f(-1)$.
- Think about the values of the function on arithmetic progressions.
- Consider trying to prove that the function is injective (one-to-one) or surjective (onto), since both of these properties are helpful in simplifying expressions.
- Find a such that $f(a) = 0$; this usually (but not always!) means proving surjectivity first.
- Can you show that the function is monotone (increasing/decreasing)?
- Cauchy's functional equation $f(x + y) = f(x) + f(y)$ does not imply that $f(x) = cx$ for functions on $\mathbb{R}$ without any additional constraints, but it does if any one of the following is also true:
  - f is monotone on some interval
  - f is continuous at one point
  - f is bounded on some interval
  - f is actually only a function on $\mathbb{Q}$, not all of $\mathbb{R}$.

2 Problems

1. (APMC 97) Prove that there does not exist $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x + f(y)) = f(x)y$$

for all integers x, y.

2. (ISL 01/A4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$

for all x, y.

3. (ISL 03/A2) Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that
(i) $f(0) = 0, f(1) = 1$;
(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

4. (USAMO 00/1] Prove that there exists no function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$

for all x, y.

5. (Romania 05) Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$x(f(x+1) - f(x)) = f(x)$$

for all $x \in \mathbb{R}$ and

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

6. (ISL 02/A1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$$

for all real x, y.

7. (Romania TST 04) Find all one-to-one mappings $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n the following relation holds:

$$f(f(n)) \leq \frac{n + f(n)}{2}.$$

8. (ISL 04/N3) Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying

$$\left(f\left(m\right)^2 + f\left(n\right)\right) \mid \left(m^2 + n\right)^2$$

for any two positive integers m and n. Remark. The abbreviation $\mathbb{N}^*$ stands for the set of all positive integers: $\mathbb{N}^* = \{1, 2, 3, ...\}$.

9. (ISL 00/A3) Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x + g(y)\right) = x \cdot f(y) - y \cdot f(x) + g(x)$$

for all real x, y.

10. (China TST 03) Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}$ which satisfy $f(n+1) \geq f(n)$ for all $n \geq 1$ and $f(mn) = f(m)f(n)$ for all $(m, n) = 1$.

11. (IMO 99/6) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

12. (ISL 05/A2) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

$$f\left(x\right) f\left(y\right) = 2f\left(x + yf\left(x\right)\right)$$

for all positive real numbers x and y.