

C

Integer Programming: The Branch and Bound Method

The Branch and Bound Method

The branch and bound method is a solution approach that partitions the feasible solution space into smaller subsets of solutions.

The *branch and bound method* is not a solution technique specifically limited to integer programming problems. It is a *solution approach* that can be applied to a number of different types of problems. The branch and bound approach is based on the principle that the total set of feasible solutions can be partitioned into smaller subsets of solutions. These smaller subsets can then be evaluated systematically until the best solution is found. When the branch and bound approach is applied to an integer programming problem, it is used in conjunction with the normal noninteger solution approach. We will demonstrate the branch and bound method using the following example.

The owner of a machine shop is planning to expand by purchasing some new machines—presses and lathes. The owner has estimated that each press purchased will increase profit by \$100 per day and each lathe will increase profit by \$150 daily. The number of machines the owner can purchase is limited by the cost of the machines and the available floor space in the shop. The machine purchase prices and space requirements are as follows.

Machine	Required Floor Space (ft ²)	Purchase Price
Press	15	\$8,000
Lathe	30	4,000

The owner has a budget of \$40,000 for purchasing machines and 200 square feet of available floor space. The owner wants to know how many of each type of machine to purchase to maximize the daily increase in profit.

The linear programming model for an integer programming problem is formulated in exactly the same way as the linear programming examples in chapters 2 and 4 of the text. The only difference is that in this problem, the decision variables are restricted to integer values because the owner cannot purchase a fraction, or portion, of a machine. The linear programming model follows.

$$\begin{aligned}
 &\text{maximize } Z = \$100x_1 + 150x_2 \\
 &\text{subject to} \\
 &8,000x_1 + 4,000x_2 \leq \$40,000 \\
 &15x_1 + 30x_2 \leq 200 \text{ ft}^2 \\
 &x_1, x_2 \geq 0 \text{ and integer}
 \end{aligned}$$

where

x_1 = number of presses

x_2 = number of lathes

The decision variables in this model are restricted to whole machines. The fact that *both* decision variables, x_1 and x_2 , can assume any integer value greater than or equal to zero is what gives this model its designation as a total integer model.

A linear programming model solution with no integer restrictions is called a **relaxed** solution.

We begin the branch and bound method by solving the problem as a regular linear programming model without integer restrictions (i.e., the integer restrictions are *relaxed*). The linear programming model for the problem and the optimal relaxed solution is

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq \$40,000$$

$$15x_1 + 30x_2 \leq 200 \text{ ft}^2$$

$$x_1, x_2 \geq 0$$

and

$$x_1 = 2.22, x_2 = 5.56, \text{ and } Z = 1,055.56$$

The branch and bound method uses a tree diagram of **nodes** and **branches** to organize the solution partitioning.

The branch and bound method employs a diagram consisting of *nodes* and *branches* as a framework for the solution process. The first node of the branch and bound diagram, shown in Figure C-1 contains the relaxed linear programming solution shown earlier *and* the rounded-down solution.

Figure C-1

The initial node in the branch and bound diagram

UB = 1,055.56 ($x_1 = 2.22, x_2 = 5.56$)

LB = 950 ($x_1 = 2, x_2 = 5$)



The **optimal integer solution** will always be between the upper bound of the relaxed solution and a lower bound of the rounded-down integer solution.

Notice that this node has two designated bounds: an upper bound (UB) of \$1,055.56 and a lower bound (LB) of \$950. The lower bound is the Z value for the rounded-down solution, $x_1 = 2$ and $x_2 = 5$; the upper bound is the Z value for the relaxed solution, $x_1 = 2.22$ and $x_2 = 5.56$. The *optimal integer solution* will be between these two bounds.

Rounding down might result in a suboptimal solution. In other words, we are hoping that a Z value greater than \$950 might be possible. We are not concerned that a value *lower than* \$950 might be available. Thus, \$950 represents a *lower bound* for our solution. Alternatively, since $Z = \$1,055.56$ reflects an optimal solution point *on the solution space boundary*, a greater Z value cannot possibly be attained. Hence, $Z = \$1,055.56$ is the *upper bound* of our solution.

Branch on the variable with the solution value with the greatest fractional part.

Now that the possible feasible solutions have been narrowed to values between the upper and lower bounds, we must test the solutions within these bounds to determine the best one. The first step in the branch and bound method is to create *two* solution subsets from the present relaxed solution. This is accomplished by observing the relaxed solution value for each variable,

$$x_1 = 2.22$$

$$x_2 = 5.56$$

and seeing which one is the farthest from the rounded-down integer value (i.e., which variable has the greatest fractional part). The .56 portion of 5.56 is the greatest fractional part; thus, x_2 will be the variable that we will “branch” on.

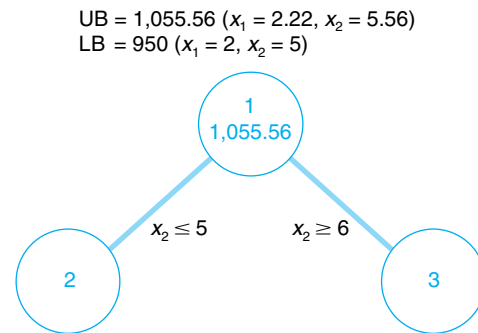
Create two constraints (or subsets) to eliminate the fractional part of the solution value

Because x_2 must be an integer value in the optimal solution, the following constraints can be developed.

$$\begin{aligned}x_2 &\leq 5 \\x_2 &\geq 6\end{aligned}$$

In other words, x_2 can be 0, 1, 2, 3, 4, 5, or 6, 7, 8, etc., but it cannot be a value between 5 and 6, such as 5.56. These two new constraints represent the two solution subsets for our solution approach. Each of these constraints will be added to our linear programming model, which will then be solved normally to determine a relaxed solution. This sequence of events is shown on the branch and bound diagram in Figure C-2. The solutions at nodes 2 and 3 will be the relaxed solutions obtained by solving our example model with the appropriate constraints added.

Figure C-2
Solution subsets x_2



First, the solution at node 2 is found by solving the following model with the constraint $x_2 \leq 5$ added.

$$\begin{aligned}\text{maximize } Z &= \$100x_1 + 150x_2 \\ \text{subject to} \\ 8,000x_1 + 4,000x_2 &\leq 40,000 \\ 15x_1 + 30x_2 &\leq 200 \\ x_2 &\leq 5 \\ x_1, x_2 &\geq 0\end{aligned}$$

The optimal solution for this model with integer restrictions relaxed (solved using the computer) is $x_1 = 2.5$, $x_2 = 5$, and $Z = 1,000$.

Next, the solution at node 3 is found by solving the model with $x_2 \geq 6$ added.

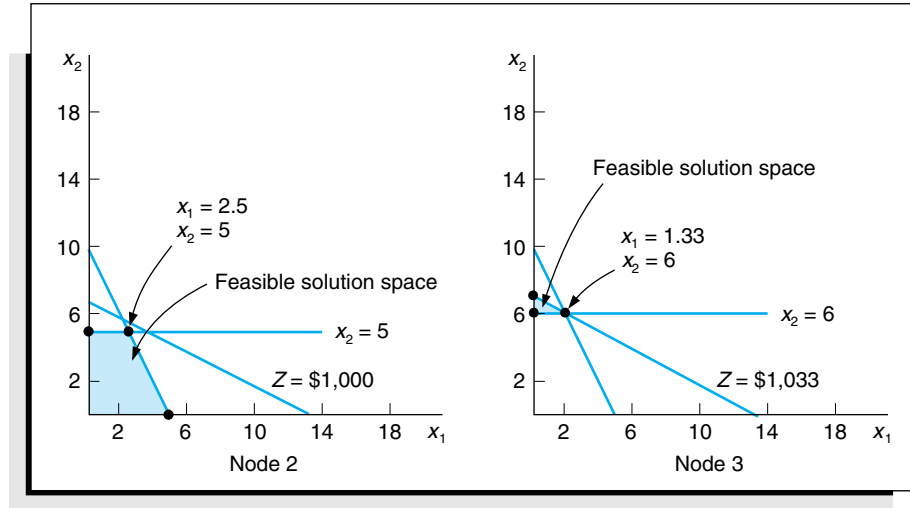
$$\begin{aligned}\text{maximize } Z &= \$100x_1 + 150x_2 \\ \text{subject to} \\ 8,000x_1 + 4,000x_2 &\leq 40,000 \\ 15x_1 + 30x_2 &\leq 200 \\ x_2 &\geq 6 \\ x_1, x_2 &\geq 0\end{aligned}$$

The optimal solution for this model with integer restrictions relaxed is $x_1 = 1.33$, $x_2 = 6$, and $Z = 1,033.33$.

These solutions with $x_2 \leq 5$ and $x_2 \geq 6$ reflect the partitioning of the original relaxed model into two subsets formed by the addition of the two constraints. The resulting solution sets are shown in the graphs in Figure C-3.

Figure C-3

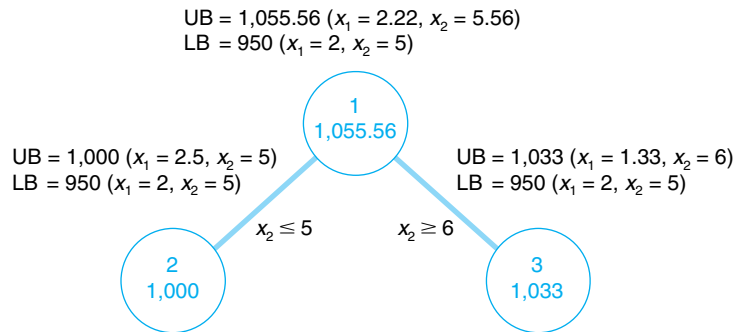
Feasible solution spaces for nodes 2 and 3



Notice that in the node 2 graph in Figure C-3, the solution point $x_1 = 2.5$, $x_2 = 5$ results in a maximum Z value of \$1,000, which is the upper bound for this node. Next, notice that in the node 3 graph, the solution point $x_1 = 1.33$, $x_2 = 6$ results in a maximum Z value of \$1,033. Thus, \$1,033 is the upper bound for node 3. The lower bound at each of these nodes is the maximum *integer* solution. Since neither of these relaxed solutions is totally integer, the lower bound remains \$950, the integer solution value already obtained at node 1 for the rounded-down integer solution. The diagram in Figure C-4 reflects the addition of the upper and lower bounds at each node.

Figure C-4

Branch and bound diagram with upper and lower bounds at nodes 2 and 3



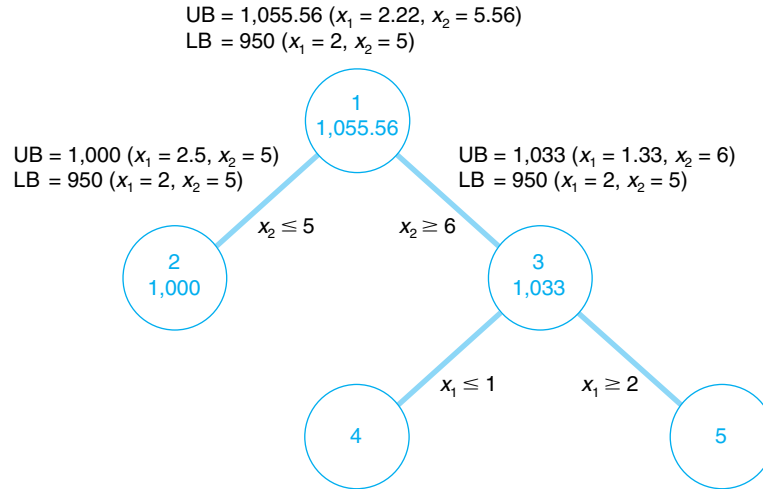
Since we do not have an optimal and feasible integer solution yet, we must continue to branch (i.e., partition) the model, from either node 2 or node 3. A look at Figure C-4 reveals that if we branch from node 2, the maximum value that can possibly be achieved is \$1,000 (the upper bound). However, if we branch from node 3, a higher maximum value of \$1,033 is possible. Thus, we will branch from node 3. In general, *always branch from the node with the maximum upper bound*.

Now the steps for branching previously followed at node 1 are repeated at node 3. First, the variable that has the value with the greatest fractional part is selected. Because x_2 has an integer value, x_1 , with a fractional part of .33, is the only variable we can select. Thus, two new constraints are developed from x_1 ,

$$\begin{aligned}x_1 &\leq 1 \\x_1 &\geq 2\end{aligned}$$

This process creates the new branch and bound diagram shown in Figure C-5.

Figure C-5
Solution subsets for x_1



Next, the relaxed linear programming model with the new constraints added must be solved at nodes 4 and 5. (However, do not forget that the model is not the original, but the original with the constraint previously added, $x_2 \geq 6$.) Consider the node 4 model first.

$$\text{maximum } Z = 100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1 \leq 1$$

$$x_1, x_2 \geq 0$$

The optimal solution for this model with integer restrictions relaxed is $x_1 = 1, x_2 = 6.17$, and $Z = 1,025$.

Next, consider the node 5 model.

$$\text{maximize } Z = 100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

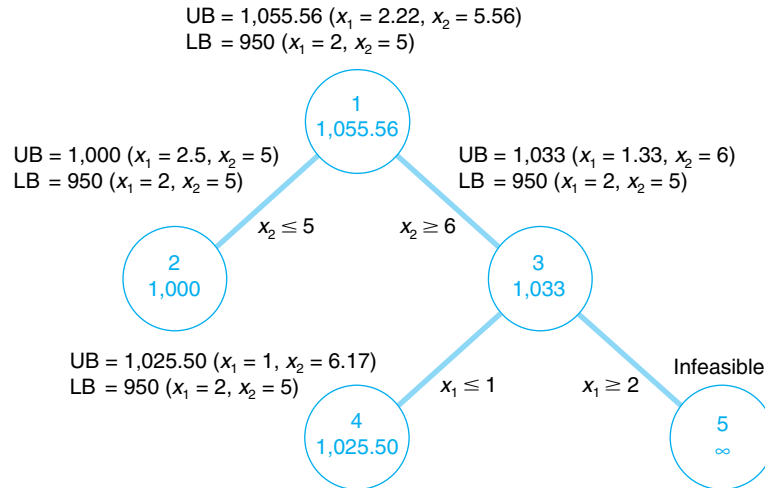
$$x_1 \geq 2$$

$$x_1, x_2 \geq 0$$

However, there is no feasible solution for this model. Therefore, no solution exists at node 5, and we have only to evaluate the solution at node 4. The branch and bound diagram reflecting these results is shown in Figure C-6.

Figure C-6

Branch and bound diagram
with upper and lower bounds at
nodes 4 and 5



The branch and bound diagram in Figure C-6 indicates that we still have not reached an optimal integer solution; thus, we must repeat the branching steps followed earlier. Since a solution does not exist at node 5, there is no comparison between the upper bounds at nodes 4 and 5. Comparing nodes 2 and 4, we must branch from node 4 because it has the greater upper bound. Next, since x_1 has an integer value, x_2 , with a fractional part of .17, is selected by default. The two new constraints developed from x_2 are

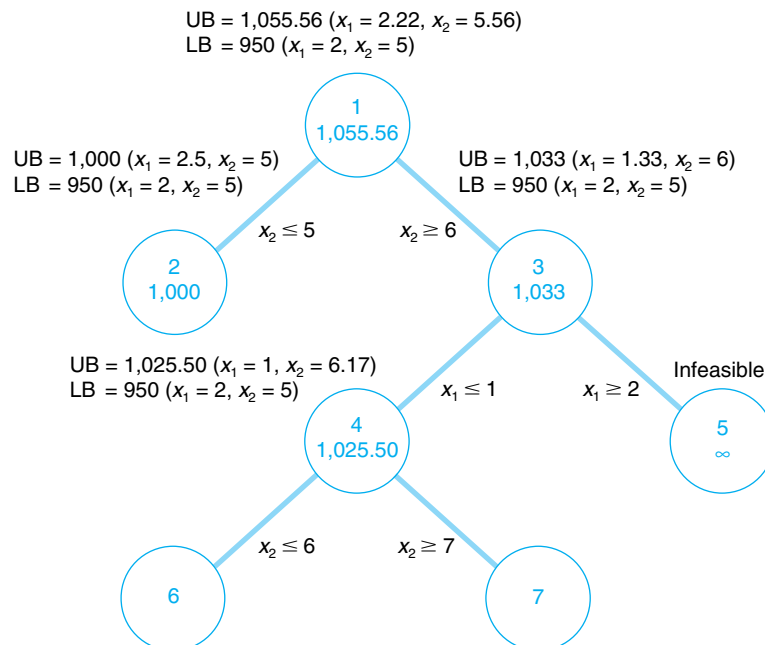
$$\begin{aligned} x_2 &\leq 6 \\ x_2 &\geq 7 \end{aligned}$$

This creates the new branch and bound diagram in Figure C-7.

The relaxed linear programming model with the new constraints added must be solved at nodes 6 and 7. Consider the node 6 model first.

Figure C-7

Solution subsets for x_2



maximize $Z = 100x_1 + 150x_2$
subject to
 $8,000x_1 + 4,000x_2 \leq 40,000$
 $15x_1 + 30x_2 \leq 200$
 $x_2 \geq 6$
 $x_1 \leq 1$
 $x_2 \leq 6$
 $x_1, x_2 \geq 0$

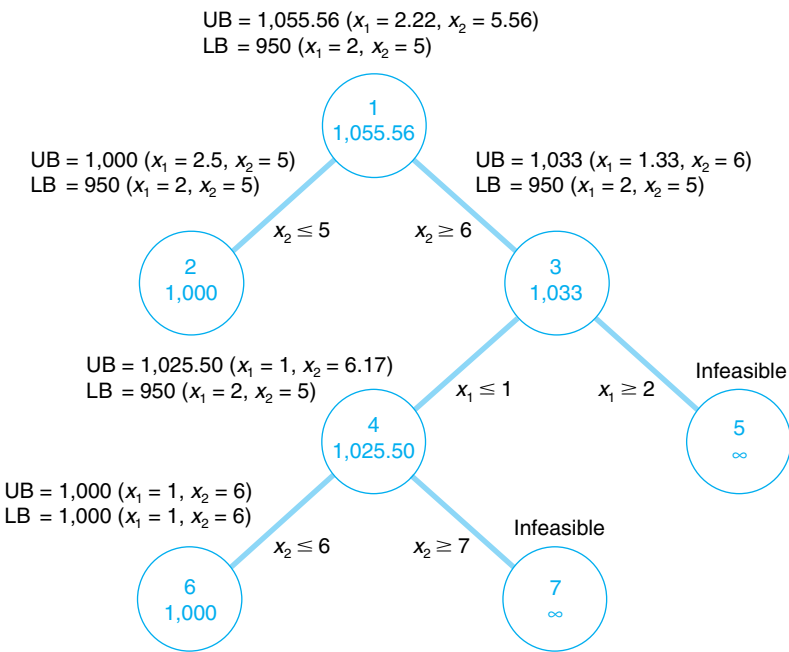
The optimal solution for this relaxed linear programming model is $x_1 = 1, x_2 = 6$, and $Z = 1,000$.
Next, consider the node 7 model.

maximize $Z = 100x_1 + 150x_2$
subject to
 $8,000x_1 + 4,000x_2 \leq 40,000$
 $15x_1 + 30x_2 \leq 200$
 $x_2 \geq 6$
 $x_1 \leq 1$
 $x_2 \geq 7$
 $x_1, x_2 \geq 0$

An optimal integer solution is reached when a feasible integer solution is achieved at a node that has an upper bound greater than or equal to the upper bound at any other ending node.

However, the solution to this model is infeasible and no solution exists at node 7. The branch and bound diagram reflecting these results is shown in Figure C-8. This version of the branch and bound diagram indicates that the optimal integer solution, $x_1 = 1, x_2 = 6$, has been reached at node 6. The value of 1,000 at node 6 is the maximum, or upper bound, integer value that can be obtained. It is also the recomputed lower bound because it is the maximum integer solution achieved to this point. Thus, it is not possible to achieve any higher value by further branching from node 6. A comparison of the node 6 solution with

Figure C-8
The branch and bound diagram with optimal solution at node 6



those at nodes 2, 5, and 7 shows that a better solution is not possible. The upper bound at node 2 is 1,000, which is the same as that obtained at node 6; thus, node 2 can result in no improvement. The solutions at nodes 5 and 7 are infeasible (and thus further branching will result in only infeasible solutions). By the process of elimination, the integer solution at node 6 is optimal.

In general, the optimal integer solution is reached when a feasible integer solution is generated at a node and the upper bound at that node is greater than or equal to the upper bound at any other *ending* node (i.e., a node at the end of a branch).

In the context of the original example, this solution indicates that if the machine shop owner purchases one press and six lathes, a daily increase in profit of \$1,000 will result.

The steps of the branch and bound method.

The steps of the branch and bound method for determining an optimal integer solution for a maximization model (with \leq constraints) can be summarized as follows.

1. Find the optimal solution to the linear programming model with the integer restrictions relaxed.
2. At node 1 let the relaxed solution be the upper bound and the *rounded-down* integer solution be the lower bound.
3. Select the variable with the greatest fractional part for branching. Create two new constraints for this variable reflecting the partitioned integer values. The result will be a new \leq constraint and a new \geq constraint.
4. Create two new nodes, one for the \geq constraint and one for the \leq constraint.
5. Solve the *relaxed* linear programming model with the new constraint added at each of these nodes.
6. The relaxed solution is the upper bound at each node, and the *existing* maximum integer solution (at any node) is the lower bound.
7. If the process produces a feasible integer solution with the greatest upper bound value of any ending node, the optimal integer solution has been reached. If a feasible integer solution does not emerge, branch from the node with the greatest upper bound.
8. Return to step 3.

The branch and bound method can be used for mixed integer problems, except only variables with integer restrictions are rounded down to achieve the initial lower bound and only integer variables are branched on.

For a minimization model, relaxed solutions are rounded up, and upper and lower bounds are reversed.

Solution of the Mixed Integer Model

Mixed integer linear programming problems can also be solved using the branch and bound method. The same basic steps that were applied to the total integer model in the previous section are used for a mixed integer model with only a few differences.

First, at node 1 only those variables with integer restrictions are rounded down to achieve the *lower bound*. Second, in determining which variable to branch from, we select the greatest fractional part from among only those variables that must be integer. All other steps remain the same. The optimal solution is reached when a feasible solution is generated at a node that has integer values for those variables requiring integers and that has reached the maximum upper bound of all ending nodes.

Solution of the 0–1 Integer Model

The 0–1 integer model can also be solved using the branch and bound method. First, the 0–1 restrictions for variables must be reflected as model constraints, $x_j \leq 1$. As an example, consider the following 0–1 integer model for selecting recreational facilities following from chapter 5 in the text.

A community council must decide which recreation facilities to construct in its community. Four new recreation facilities have been proposed—a swimming pool, a tennis

center, an athletic field, and a gymnasium. The council wants to construct facilities that will maximize the expected daily usage by the residents of the community subject to land and cost limitations. The expected daily usage and cost and land requirements for each facility follow.

Recreation Facility	Expected Usage (people/day)	Cost (\$)	Land Requirements (acres)
Swimming pool	300	35,000	4
Tennis center	90	10,000	2
Athletic field	400	25,000	7
Gymnasium	150	90,000	3

The community has a \$120,000 construction budget and 12 acres of land. Because the swimming pool and tennis center must be built on the same part of the land parcel, however, only one of these two facilities can be constructed. The council wants to know which of the recreation facilities to construct in order to maximize the expected daily usage. The model for this problem is formulated as follows.

$$\text{maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

subject to

$$35,000x_1 + 10,000x_2 + 25,000x_3 + 90,000x_4 \leq \$120,000 \text{ (capital budget)}$$

$$4x_1 + 2x_2 + 7x_3 + 3x_4 \leq 12 \text{ acres (space available)}$$

$$x_1 + x_2 \leq 1 \text{ facility}$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

where

Z = expected daily usage (people per day)

x_1 = construction of a swimming pool

x_2 = construction of a tennis center

x_3 = construction of an athletic field

x_4 = construction of a gymnasium

In this model, the decision variables can have a solution value of either *zero* or *one*. If a facility is not selected for construction, the decision variable representing it will have a value of zero. If a facility is selected, its decision variable will have a value of one.

The last constraint, $x_1 + x_2 \leq 1$, reflects the *contingency* that either the swimming pool (x_1) or the tennis center (x_2) can be constructed, but not both. In order for the sum of x_1 and x_2 to be less than or equal to one, either of the variables can have a value of one, or both variables can equal zero. This is also referred to as a *mutually exclusive constraint*.

To apply the branch and bound method, the following four constraints have to be added to the model in place of the single restriction $x_1, x_2, x_3, x_4 = 0 \text{ or } 1$.

$$x_1 \leq 1$$

$$x_2 \leq 1$$

$$x_3 \leq 1$$

$$x_4 \leq 1$$

The only other change in the normal branch and bound method is at step 3. Once the variable x_j with the greatest fractional part has been determined, the two new constraints

The branch and bound method can be used for 0–1 integer problems by adding “ ≤ 1 ” constraints for each variable.

In implicit enumeration all feasible solutions are evaluated to see which is best.

developed from this variable are $x_j = 0$ and $x_j = 1$. These two new constraints will form the two branches at each node.

Another method for solving 0–1 integer problems is *implicit enumeration*. In implicit enumeration, obviously infeasible solutions are eliminated and the remaining solutions are evaluated (i.e., enumerated) to see which is the best. This approach will be demonstrated using our original 0–1 example model for selecting a recreational facility (i.e., without the $x_j \leq 1$ constraints).

The *complete enumeration* (i.e., the list of all possible solution sets) for this model is as follows.

<i>Solution</i>	x_1	x_2	x_3	x_4	<i>Feasibility</i>	<i>Z Value</i>
1	0	0	0	0	Feasible	0
2	1	0	0	0	Feasible	300
3	0	1	0	0	Feasible	90
4	0	0	1	0	Feasible	400
5	0	0	0	1	Feasible	150
6	1	1	0	0	Infeasible	∞
7	1	0	1	0	Feasible	700
8	1	0	0	1	Infeasible	∞
9	0	1	1	0	Feasible	490
10	0	1	0	1	Feasible	240
11	0	0	1	1	Feasible	550
12	1	1	1	0	Infeasible	∞
13	1	0	1	1	Infeasible	∞
14	1	1	0	1	Infeasible	∞
15	0	1	1	1	Infeasible	∞
16	1	1	1	1	Infeasible	∞

Solutions 6, 12, 14, and 16 can be immediately eliminated because they violate the third constraint, $x_1 + x_2 \leq 1$. Solutions 8, 13, and 15 can also be eliminated because they violate the other two constraints. This leaves eight possible solution sets (assuming that solution 1 —i.e., choosing none of the recreational facilities—can be eliminated) for consideration. After evaluating the objective function value of these eight solutions, we find the best solution to be 7, with $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$. Within the context of the example, this solution indicates that a swimming pool (x_1) and an athletic field (x_3) should be constructed and that these facilities will generate an expected usage of 700 people per day.

The process of eliminating infeasible solutions and then evaluating the feasible solutions to see which is best is the basic principle behind implicit enumeration. However, implicit enumeration is usually done more systematically, by evaluating solutions with branching diagrams like those used in the branch and bound method, rather than by sorting through a complete enumeration as in this previous example.

Problems

1. Consider the following linear programming model

$$\text{maximize } Z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 3x_1 + 4x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

- a. Solve this model using the branch and bound method.
 - b. Demonstrate the solution partitioning graphically.
2. Solve the following linear programming model using the branch and bound method.

$$\text{minimize } Z = 3x_1 + 6x_2$$

subject to

$$\begin{aligned} 7x_1 + 3x_2 &\geq 40 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

3. A tailor makes wool tweed sport coats and wool slacks. He is able to get a shipment of 150 square yards of wool cloth from Scotland each month to make coats and slacks, and he has 200 hours of his own labor to make them each month. A coat requires 3 square yards of wool and 10 hours to make, and a pair of pants requires 5 square yards of wool and 4 hours to make. He earns \$50 in profit from each coat he makes and \$40 from each pair of slacks. He wants to know how many coats and slacks to produce to maximize profit.
 - a. Formulate an integer linear programming model for this problem.
 - b. Determine the integer solution to this problem using the branch and bound method. Compare this solution with the solution without integer restrictions and indicate if the rounded-down solution would have been optimal.
4. A jeweler and her apprentice make silver pins and necklaces by hand. Each week they have 80 hours of labor and 36 ounces of silver available. It requires 8 hours of labor and 2 ounces of silver to make a pin, and 10 hours of labor and 6 ounces of silver to make a necklace. Each pin also contains a small gem of some kind. The demand for pins is no more than six per week. A pin earns the jeweler \$400 in profit, and a necklace earns \$100. The jeweler wants to know how many of each item to make each week in order to maximize profit.
 - a. Formulate an integer programming model for this problem.
 - b. Solve this model using the branch and bound method. Compare this solution with the solution without integer restrictions and indicate if the rounded-down solution would have been optimal.
5. A glassblower makes glass decanters and glass trays on a weekly basis. Each item requires 1 pound of glass, and the glassblower has 15 pounds of glass available every week. A glass decanter requires 4 hours of labor, a glass tray requires only 1 hour of labor, and the glassblower works 25 hours a week. The profit from a decanter is \$50, and the profit from a tray is \$10. The glassblower wants to determine the total number of decanters (x_1) and trays (x_2) that he needs to produce in order to maximize his profit.
 - a. Formulate an integer programming model for this problem.
 - b. Solve this model using the branch and bound method.
 - c. Demonstrate the solution partitioning graphically.
6. The Livewright Medical Supplies Company has a total of 12 salespeople it wants to assign to three regions—the South, the East, and the Midwest. A salesperson in the South earns \$600 in profit per month for the company, a salesperson in the East earns \$540, and a salesperson in the Midwest earns \$375. The southern region can have a maximum assignment of 5 salespeople. The company has a total of \$750 per day available for expenses for all 12 salespeople. A sales-

person in the South has average expenses of \$80 per day, a salesperson in the East has average expenses of \$70 per day, and a salesperson in the Midwest has average daily expenses of \$50. The company wants to determine the number of salespeople to assign to each region to maximize profit.

- a. Formulate an integer programming model for this problem.
 - b. Solve this model using the branch and bound method.
7. Helen Holmes makes pottery by hand in her basement. She has 20 hours available each week to make bowls and vases. A bowl requires 3 hours of labor, and a vase requires 2 hours of labor. It requires 2 pounds of special clay to make a bowl and 5 pounds to produce a vase; she is able to acquire 35 pounds of clay per week. She sells her bowls for \$50 and her vases for \$40. She wants to know how many of each item to make each week in order to maximize her revenue.
 - a. Formulate an integer programming model for this problem.
 - b. Solve this model using the branch and bound method. Compare this solution with the solution with integer restrictions and indicate if the rounded-down solution would have been optimal.
 8. Lauren Moore has sold her business for \$500,000 and wants to invest in condominium units (which she intends to rent) and land (which she will lease to a farmer). She estimates that she will receive an annual return of \$8,000 for each condominium and \$6,000 for each acre of land. A condominium unit costs \$70,000, and land is \$30,000 per acre. A condominium will cost her \$1,000 per unit and an acre of land \$2,000 for maintenance and upkeep. Lauren wants to know how much to invest in condominiums and land in order to maximize her annual return.
 - a. Formulate a mixed integer programming model for this problem.
 - b. Solve this model using the branch and bound method.
 9. The owner of the Consolidated Machine Shop has \$10,000 available to purchase a lathe, a press, a grinder, or some combination thereof. The following 0–1 integer linear programming model has been developed for determining which of the three machines (lathe, x_1 ; press, x_2 ; grinder, x_3) should be purchased in order to maximize the annual profit.

$$\text{maximize } Z = 1,000x_1 + 700x_2 + 800x_3 \text{ (profit, \$)}$$

subject to

$$5,000x_1 + 6,000x_2 + 4,000x_3 \leq 10,000 \text{ (cost, \$)}$$

$$x_1, x_2, x_3 = 0 \text{ or } 1$$

Solve this model using the branch and bound method.

10. Solve the following mixed integer linear programming model using the branch and bound method.

$$\text{maximize } Z = 5x_1 + 6x_2 + 4x_3$$

subject to

$$5x_1 + 3x_2 + 6x_3 \leq 20$$

$$x_1 + 3x_2 \leq 12$$

$$x_1, x_3 \geq 0$$

$$x_2 \geq 0 \text{ and integer}$$

11. Solve problem 9 using the implicit enumeration method.
12. Consider the following linear programming model.

$$\text{maximize } Z = 20x_1 + 30x_2 + 10x_3 + 40x_4$$

subject to

$$2x_1 + 4x_2 + 3x_3 + 7x_4 \leq 10$$

$$10x_1 + 7x_2 + 20x_3 + 15x_4 \leq 40$$

$$x_1 + 10x_2 + x_3 \leq 10$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

- a. Solve this problem using the implicit enumeration method.
- b. What difficulties would be encountered with the implicit enumeration method if this problem were expanded to contain five or more variables and more constraints?