095946- ADVANCED ALGORITHMS AND PARALLEL PROGRAMMING

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so we have different costs in doing something and we have to ammortize that costs during all the execution time

- **Amortized Analysis** the goal is going to try to understand the average complexity / costs
  - Dynamic tables
  - Aggregate method
  - Accounting method
  - Potential method

How large should a hash table be?

Goal: **Make the table as small as possible, but large enough so that it won't overflow (or otherwise become inefficient).**

Problem: **What if we don't know the proper size in advance?**

Solution: ***Dynamic tables.***

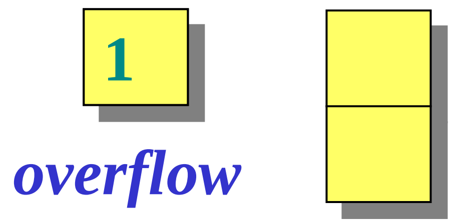
like for example how we do with arrays.

IDEA: **Whenever the table overflows, "grow" it by allocating (via** `malloc` **or** `new`**) a new, larger table. Move all items from the old table into the new one, and free the storage for the old table.**

by doubling the size. -> quando c'è un overflow duplichiamo la size.

Example of a dynamic table

1. INSERT
2. INSERT



i'm doubling the size.

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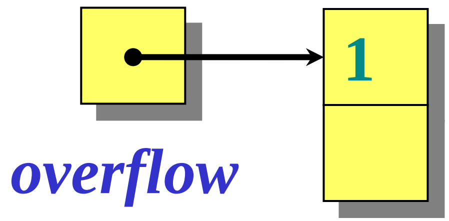

i'm coopying the first value

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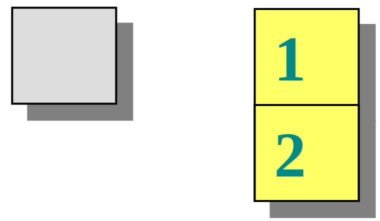

Example of a dynamic table

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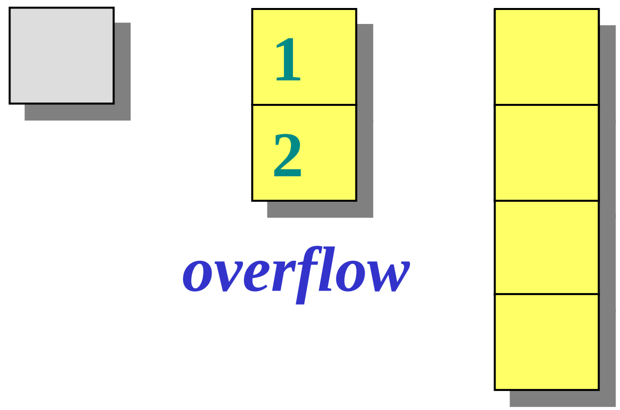
2. INSERT
3. INSERT



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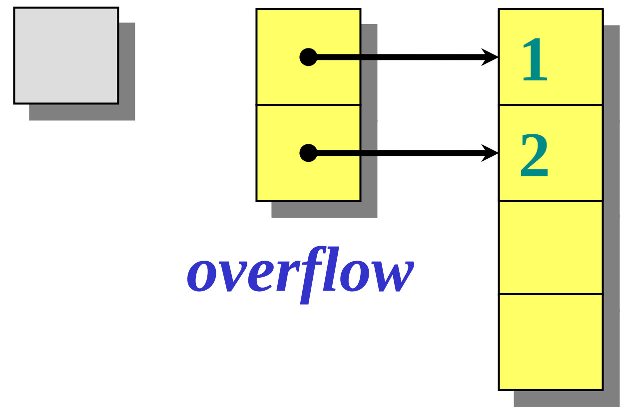

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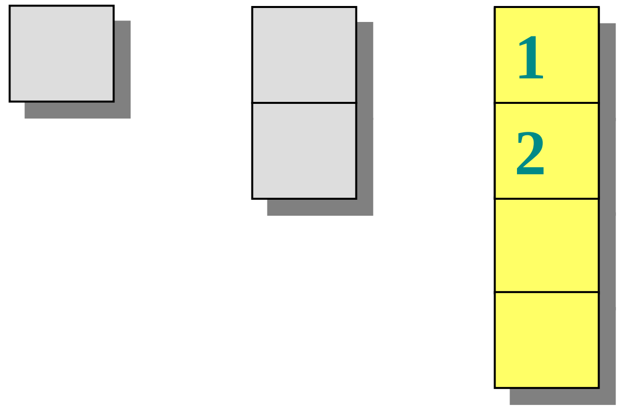

Example of a dynamic table

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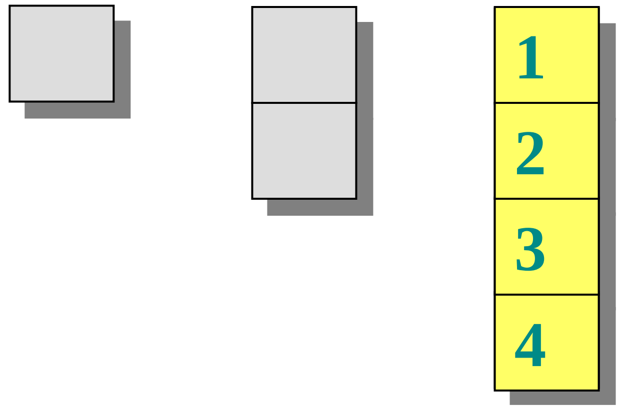

Example of a dynamic table

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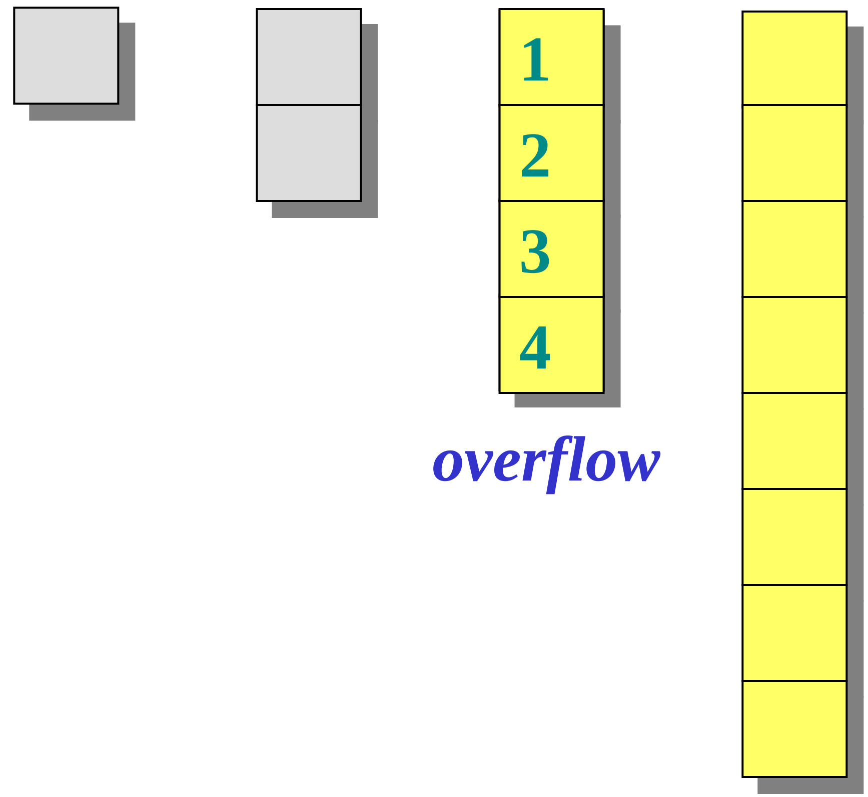
2. INSERT
3. INSERT
4. INSERT
5. INSERT



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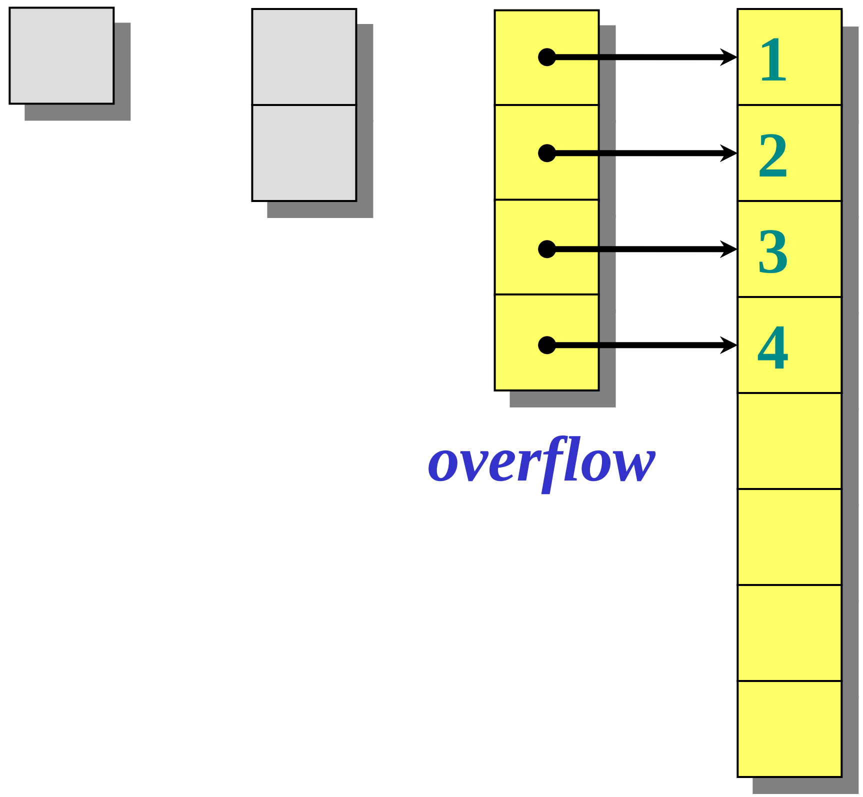

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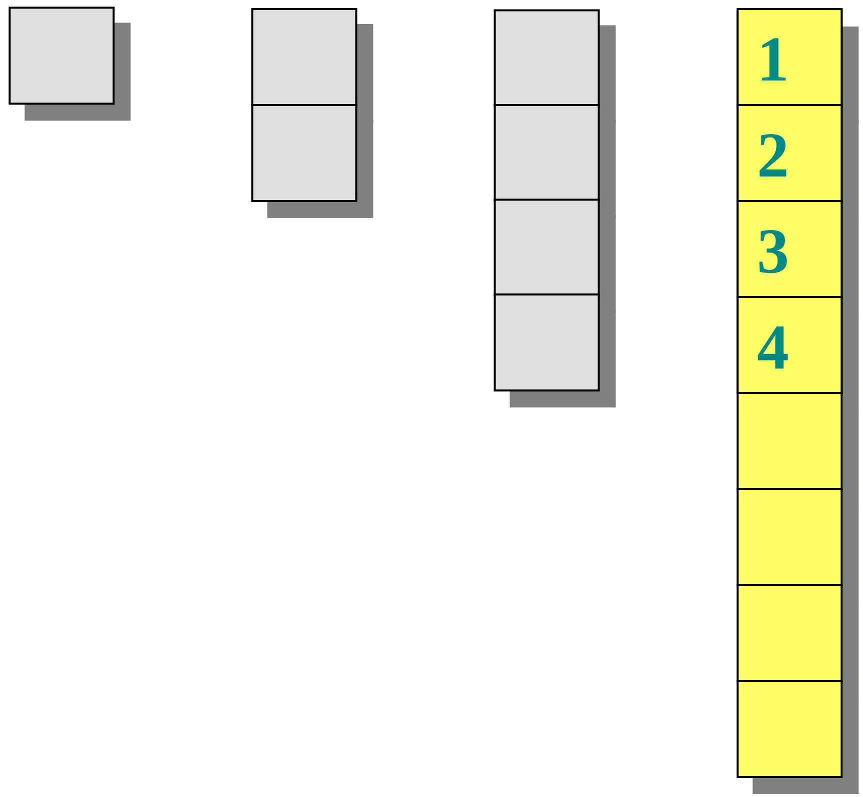
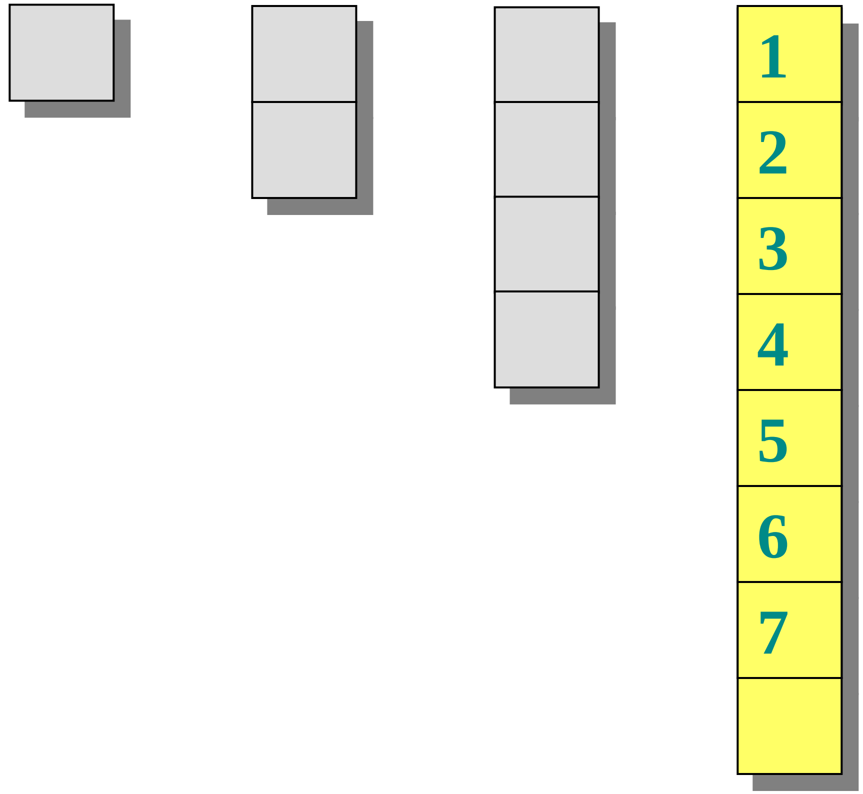


Example of a dynamic table

1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT
6. INSERT
7. INSERT



Worst-case analysis

because i'm allocating a new size and i'm copying each time i'm doubling

Consider a sequence of n insertions. The worst-case time to execute one insertion is $\Theta(n)$. Therefore, the worst-case time for n insertions is $n \cdot \Theta(n) = \Theta(n^2)$.

WRONG! In fact, the worst-case cost for n insertions is only $\Theta(n) \ll \Theta(n^2)$. but the asymptotic complexity is theta of n not n^2.

Let's see why.

Tighter analysis

in that case we have to reallocate.

Let $c_i =$ the cost of the i th insertion

$$= \begin{cases} i & \text{if } i-1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

(basta che lo provi a mano e capisci)

(altrimenti non sto riallocando quindi il costo è solo quello di inserire il nuovo valore. -> cost=1)

this is i-th insertion

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1

cost for each iteration.

Tighter analysis

Let c_i = the cost of the i th insertion

$$= \begin{cases} i & \text{if } i-1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
cost just for inserting so the total il n	1	1	1	1	1	1	1	1	1	1
c_i		1	2		4				8	
cost for copying after a realloc.										

Tighter analysis (continued)

Cost of *n* insertions

$$= \sum_{i=1}^{n} c_i$$

cost= sum of single costs

this is 2n. so the total is 3n

for just inserting n elements
(la riga di tutti 1 sopra)

$$\le n + \sum_{j=0}^{\lfloor \lg(n-1) \rfloor} 2^j$$

this is the cost when i'm reallocating

$$\le 3n$$

$$= \Theta(n)$$

•

Thus, the average cost of each dynamic-table operation is $\Theta(n)/n = \Theta(1)$.

the average is simply the total cost divided by n iterations.

Amortized analysis

An *amortized analysis* is any strategy for analyzing a sequence of operations to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

Even though we're taking averages, however, probability is not involved!

- **An amortized analysis guarantees the average performance of each operation in the *worst case*.**

Types of amortized analyses

Three common amortization arguments:

- **the *aggregate* method,** idea to compute the cost for each insertion, doing the sum and dividing for the number of iterations
  (what we've seen so far)
- **the *accounting* method,**
- **the *potential* method.**

We've just seen an aggregate analysis.

The aggregate method, though simple, lacks the precision of the other two methods. In particular, the accounting and potential methods allow a specific *amortized cost* to be allocated to each operation.

Accounting method

- Charge *i* th operation a fictitious ***amortized cost*** $\hat{c}_i$, where **\$1** pays for **1** unit of work (*i.e.*, time).
- This fee is consumed to perform the operation.
- Any amount not immediately consumed is stored in the ***bank*** for use by subsequent operations.
- The bank balance must not go negative! We must ensure that

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i$$

for all *n*.
- Thus, the total amortized costs provide an upper bound on the total true costs.

Accounting analysis of dynamic tables

Charge an amortized cost of $\hat{c}_i$ = \$3 for the *i* th insertion.

- **\$1 pays for the immediate insertion.**
- **\$2 is stored for later table doubling.**

When the table doubles, \$1 pays to move a recent item, and \$1 pays to move an old item.

Example:

\$0	\$0	\$0	\$0	\$2	\$2	\$2	\$2

overflow

Accounting analysis of dynamic tables

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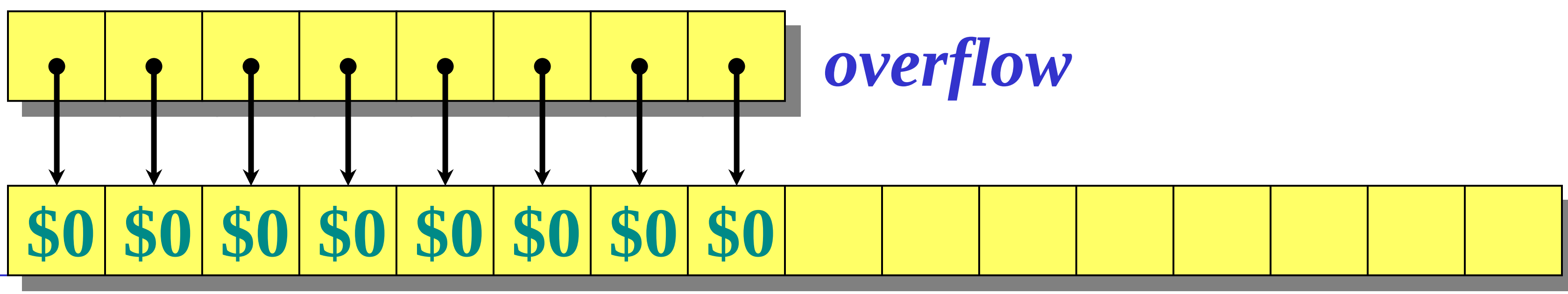

Accounting analysis of dynamic tables

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Example:

\$0	\$0	\$0	\$0	\$0	\$0	\$0	\$0	\$2	\$2	\$2					

Accounting analysis (continued)

Key invariant: **Bank balance never drops below 0. Thus, the sum of the amortized costs provides an upper bound on the sum of the true costs.**

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1*	2	2	4	2	4	6	8	2	4

***Okay, so I lied. The first operation costs only $2, not $3.**

Potential method

IDEA: View the bank account as the potential energy (*à la* physics) of the dynamic set.

Framework:

- Start with an initial data structure D_0.
- Operation i transforms D_{i-1} to D_i.
- The cost of operation i is c_i.
- Define a *potential function* $\Phi : \{D_i\} \to \mathbb{R}$, such that $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i.
- The *amortized cost* $\hat{c}_i$ with respect to Φ is defined to be $\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}$.

difference in the potential, between before and after.

Understanding potentials

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}_{\textit{potential difference } \Delta\Phi_i}$$

- **If $\Delta\Phi_i > 0$, then $\hat{c}_i > c_i$. Operation i stores work in the data structure for later use.**
- **If $\Delta\Phi_i < 0$, then $\hat{c}_i < c_i$. The data structure delivers up stored work to help pay for operation i.**

The amortized costs bound the true costs

The total amortized cost of *n* operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi(D_i) - \Phi(D_{i-1})\right)$$

Summing both sides.

The amortized costs bound the true costs

The total amortized cost of *n* operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

quindi faccio la diff tra i e i-1, poi alla prossima faccio la differenza tra i+1 e i e cosi via. => rimane solo Dn e D0 (serie telescopica)

$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

The series telescopes.

The amortized costs bound the true costs

The total amortized cost of *n* operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

$$\geq \sum_{i=1}^{n} c_i$$

since $\Phi(D_n) \geq 0$ and $\Phi(D_0) = 0$.

Potential analysis of table doubling

Define the potential of the table after the ith insertion by $\Phi(D_i) = 2i - 2^{\lceil \lg i \rceil}$. (Assume that $2^{\lceil \lg 0 \rceil} = 0$.)

Note:

- $\Phi(D_0) = 0$,
- $\Phi(D_i) \geq 0$ for all i.

Example:

sto alla sesta iterazione

•	•	•	•	•	•		

$\Phi = 2 \cdot 6 - 2^3 = 4$

(

$0	$0	$0	$0	$2	$2		

accounting method)

Calculation of amortized costs

The amortized cost of the *i* th insertion is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

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$$= \overbrace{\left\{\begin{array}{ll} i & \textbf{if } i-1 \textbf{ is an exact power of } 2, \\ 1 & \textbf{otherwise;} \end{array}\right\}}^{\text{Ci}}$$

$$+ \left(2i - 2^{\lceil \lg i \rceil}\right) - \left(2(i-1) - 2^{\lceil \lg (i-1) \rceil}\right)$$

Calculation of amortized costs

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$$= \left\{ \begin{array}{l} i \;\; \text{if } i-1 \text{ is an exact power of } 2, \\ 1 \text{ otherwise;} \end{array} \right\}$$

$$+ 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \quad .$$

Calculation

Case 1: $i - 1$ is an exact power of 2.

$$\hat{c}_i = i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil}$$

Calculation

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\begin{aligned}
\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\
&= i + 2 - 2(i - 1) + (i - 1)
\end{aligned}
$$

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&= 3
\end{aligned}
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Case 2: $i - 1$ is *not* an exact power of 2.

$$
\hat{c}_i = 1 + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil}
$$

Calculation

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Case 2: $i - 1$ is *not* an exact power of 2.

$$
\begin{aligned}
\hat{c}_i &= 1 + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\
&= 3 \qquad \text{(since } 2^{\lceil \lg i \rceil} = 2^{\lceil \lg (i-1) \rceil}\text{)}
\end{aligned}
$$

Calculation

Case 1: $i - 1$ is an exact power of 2.

$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i-1) + (i-1) \\ &= i + 2 - 2i + 2 + i - 1 \\ &= 3\end{aligned}$$

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Therefore, n insertions cost $\Theta(n)$ in the worst case.

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Therefore, n insertions cost $\Theta(n)$ in the worst case.

Exercise: Fix the bug in this analysis to show that the amortized cost of the first insertion is only 2.

Conclusions

- **Amortized costs can provide a clean abstraction of data-structure performance.**
- **Any of the analysis methods can be used when an amortized analysis is called for, but each method has some situations where it is arguably the simplest or most precise.**
- **Different schemes may work for assigning amortized costs in the accounting method, or potentials in the potential method, sometimes yielding radically different bounds.**

Acknowledge

- *Based on Introduction to Algorithms CLRS*
- Material adapted from Erik D. Demaine and Charles E. Leiserson slides