095946- ADVANCED ALGORITHMS AND PARALLEL PROGRAMMING

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- **Competitive Analysis**
  - Self-organizing lists
  - Move-to-front heuristic
  - Competitive analysis of MTF

List L of n elements

- **The operation ACCESS(x) costs $\text{rank}_L(x)$ = distance of x from the head of L.**
- **L can be reordered by transposing adjacent elements at a cost of 1.**

Self-organizing lists

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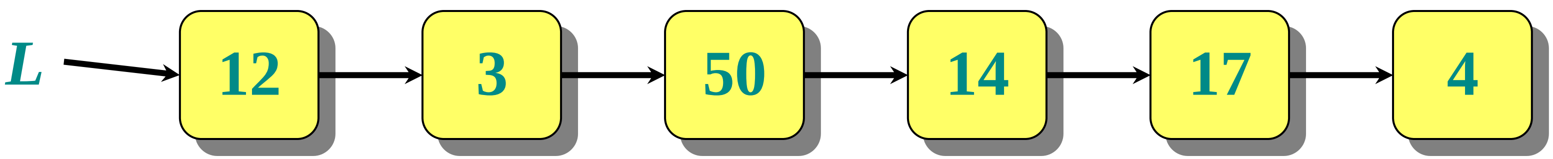

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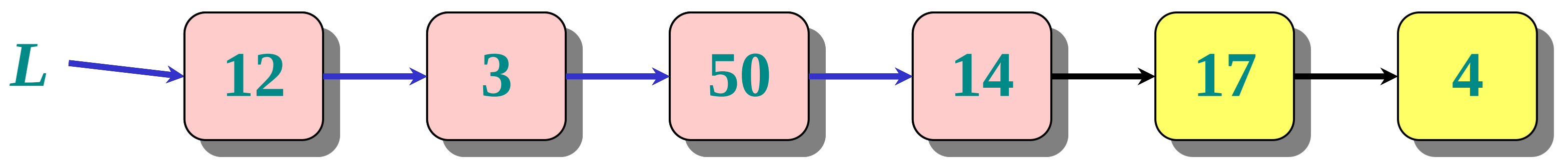

Accessing the element with key 14 costs 4.

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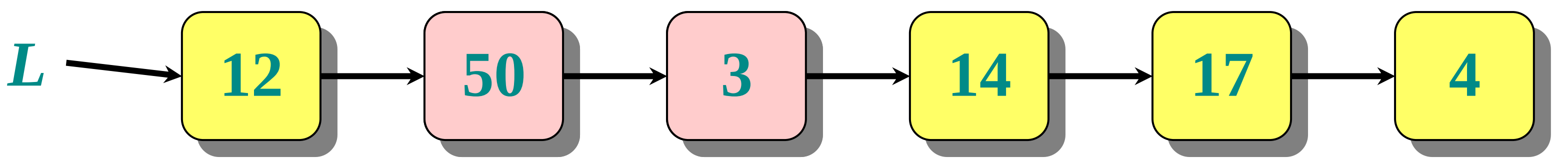
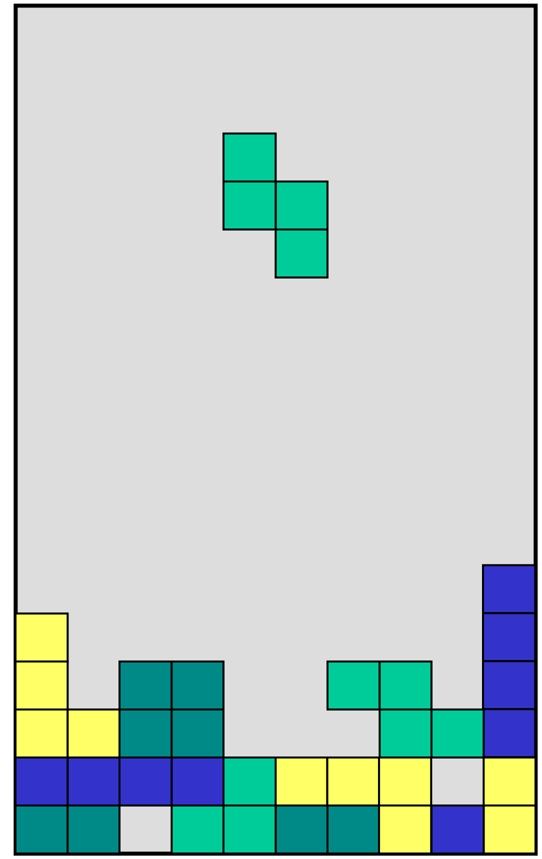


Transposing 3 and 50 costs 1.

On-line and off-line problems

Definition. **A sequence *S* of operations is provided one at a time. For each operation, an *on-line* algorithm *A* must execute the operation immediately without any knowledge of future operations (e.g., *Tetris*). An *off-line* algorithm may see the whole sequence *S* in advance.**



The game of Tetris

Goal: **Minimize the total cost $C_A(S)$.**

Worst-case analysis of self-organizing lists

An adversary always accesses the tail (nth) element of L. Then, for any on-line algorithm A, we have

$$C_A(S) = \Omega(|S| \cdot n)$$

in the worst case.

Average-case analysis of self-organizing lists

Suppose that element *x* is accessed with probability *p*(*x*). Then, we have

$$E[C_A(S)] = \sum_{x \in L} p(x) \cdot \mathrm{rank}_L(x) \qquad ,$$

which is minimized when *L* is sorted in decreasing order with respect to *p*.

Heuristic: Keep a count of the number of times each element is accessed, and maintain *L* in order of decreasing count.

The move-to-front heuristic

Practice: **Implementers discovered that the *move-to-front (MTF)* heuristic empirically yields good results.**

IDEA: **After accessing x, move x to the head of L using transposes:**

$$\text{cost} = 2 \cdot \text{rank}_L(x)\ .$$

The MTF heuristic responds well to locality in the access sequence S.

Competitive analysis

Definition. **An on-line algorithm A is *α-competitive* if there exists a constant k such that for any sequence S of operations,**

$$C_A(S) \leq \alpha \cdot C_{\mathrm{OPT}}(S) + k \,,$$

where OPT is the optimal off-line algorithm ("God's algorithm").

MTF is O(1)-competitive

Theorem. **MTF is 4-competitive for self-organizing lists.**

MTF is O(1)-competitive

Theorem. MTF is 4-competitive for self-organizing lists.

Proof. Let L_i be MTF's list after the ith access, and let L_i^* be OPT's list after the ith access.

Let c_i = MTF's cost for the ith operation
$= 2 \cdot \text{rank}_{L_{i-1}}(x)$ if it accesses x;
c_i^* = OPT's cost for the ith operation
$= \text{rank}_{L_{i-1}^*}(x) + t_i$,

where t_i is the number of transposes that OPT performs.

defines the current status of the 2 lists: so the list from the MTF method and the list of the optimal one

Potential function

Define the potential function $\Phi:\{L_i\} \to \mathbf{R}$ by

$$\begin{aligned}\Phi(L_i) &= 2 \cdot |\{(x, y) : x \prec_{L_i} y \text{ and } y \prec_{L_i^*} x\}| \\ &= 2 \cdot \#\, \textit{inversions}\,.\end{aligned}$$

list with the optimal

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Example.

$L_i \to$ E $\to$ C $\to$ A $\to$ D $\to$ B

$L_i{}^* \to$ C $\to$ A $\to$ B $\to$ D $\to$ E

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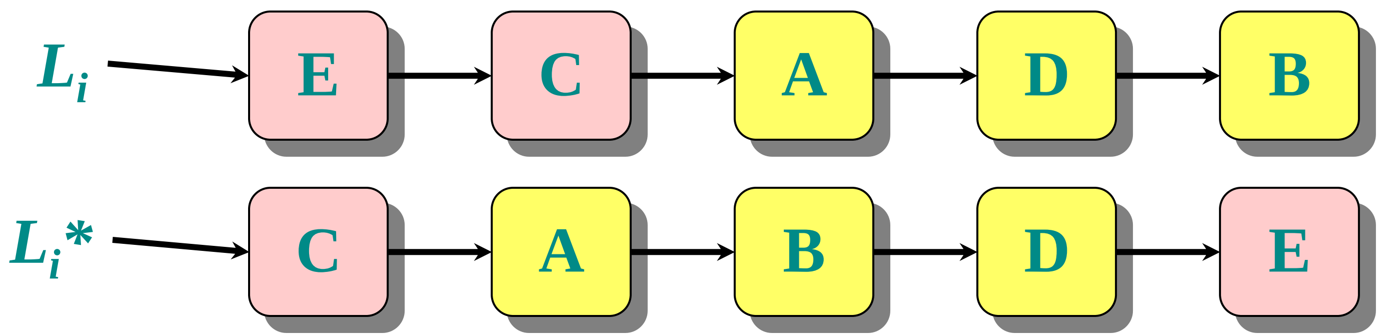

EC and CE are in the reverse order

$$\Phi(L_i) = 2 \cdot |\{(\mathrm{E},\mathrm{C}), \ldots\}|$$

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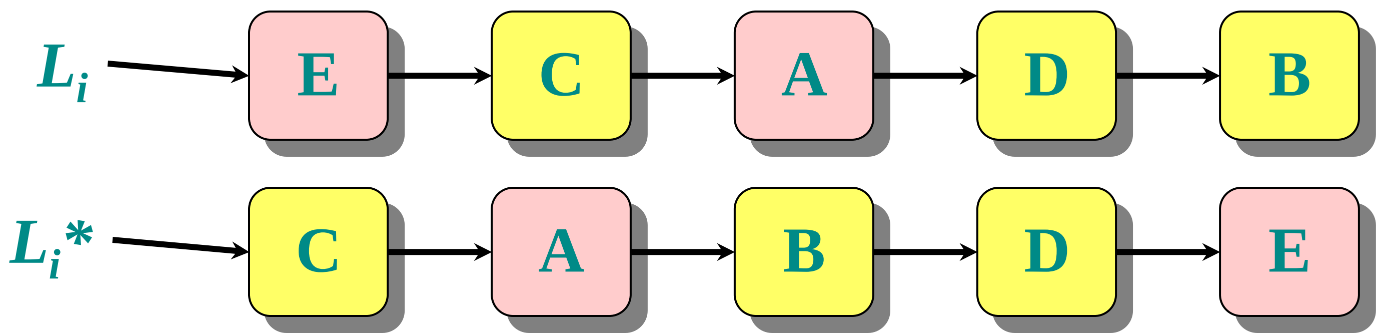

$$\Phi(L_i) = 2 \cdot |\{(\mathrm{E,C}), (\mathrm{E,A}), \ldots\}|$$

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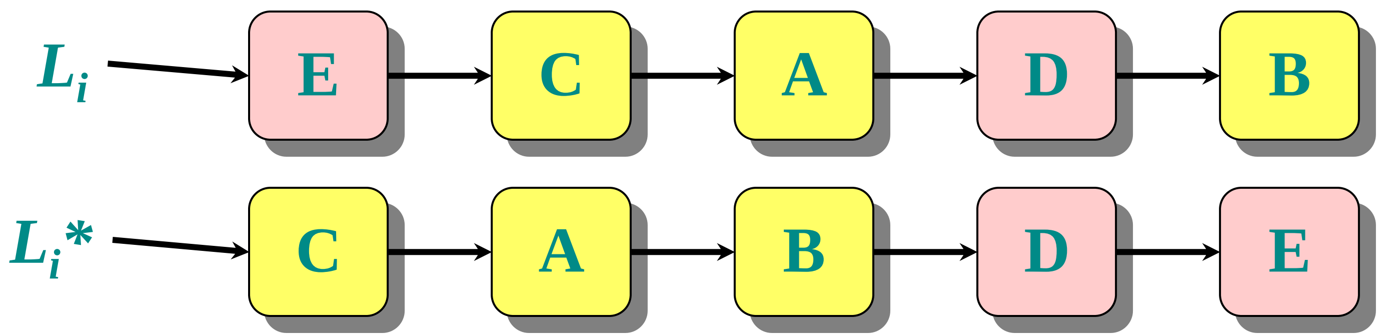

$$\Phi(L_i) = 2 \cdot |\{(E,C), (E,A), (E,D), \ldots\}|$$

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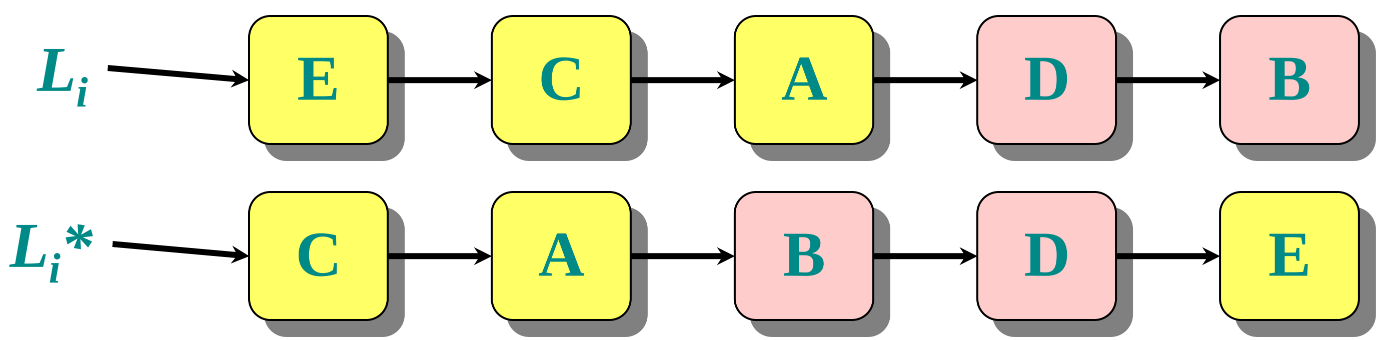

$$\Phi(L_i) = 2 \cdot |\{(E,C), (E,A), (E,D), (E,B), (D,B)\}|$$

#inversions = 5 => phi(Li) = 2*5

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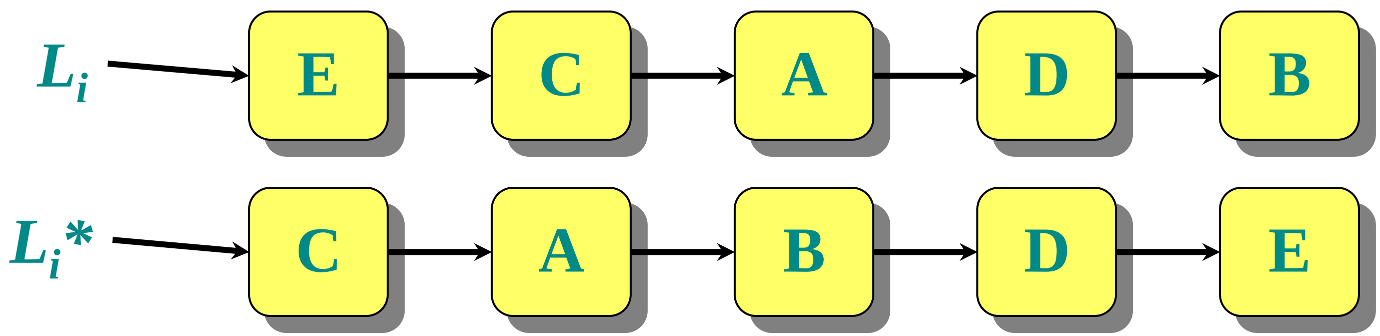

$$\Phi(L_i) = 2 \cdot |\{(E,C), (E,A), (E,D), (E,B), (D,B)\}|$$
$$= 10\ .$$

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Note that

- $\Phi(L_i) \geq 0$ **for** $i = 0, 1, \ldots,$
- $\Phi(L_0) = 0$ **if MTF and OPT start with the same list.** (because there are no inversions)

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How much does Φ change from 1 transpose?

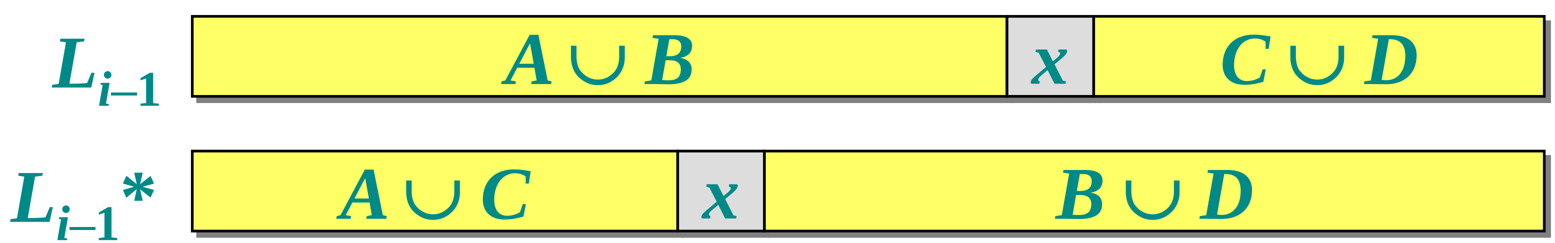
- **A transpose creates/destroys 1 inversion.**
- $\Delta\Phi = \pm 2$. because i'm reducing or adding an inversion.

What happens on an access?

Suppose that operation i accesses element x, and define

$$A = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\},$$
$$B = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\},$$
$$C = \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\},$$
$$D = \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\}.$$

items of A AND B



sto prendendo in considerazione la comparazione con la lista ottimale.

ok now we do the transformation

What happens on an access?

L_{i-1}: | $A \cup B$ | x | $C \cup D$ |

$r = \text{rank}_{L_{i-1}}(x)$

cost of acessing X in the list Li-1

L_{i-1}^*: | $A \cup C$ | x | $B \cup D$ |

$r^* = \text{rank}_{L_{i-1}^*}(x)$

We have $r = |A| + |B| + 1$ **and** $r^* = |A| + |C| + 1$.

cost of accessing X

position in list opt.

(accedo tutto gli elementi
di A e poi di C)

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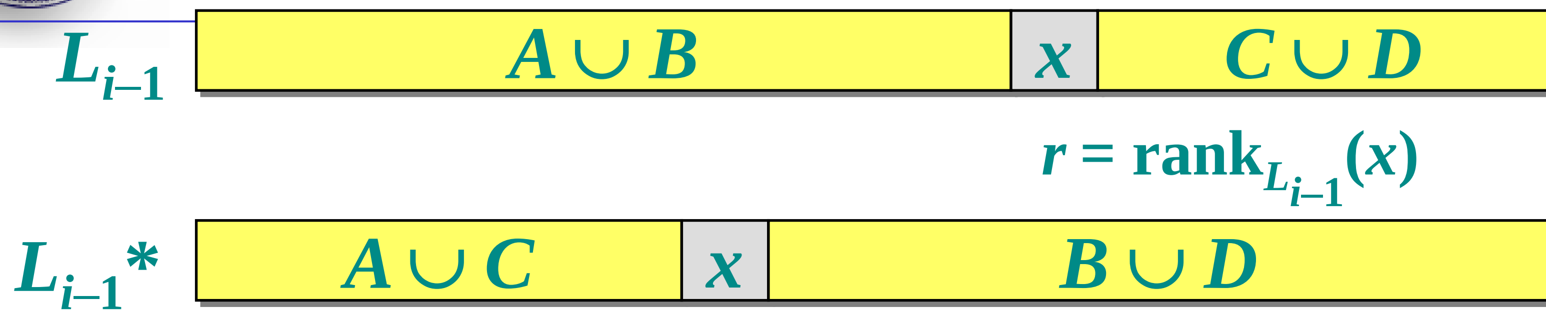

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We have $r = |A| + |B| + 1$ and $r^* = |A| + |C| + 1$.

When MTF moves x to the front, it creates $|A|$ inversions and destroys $|B|$ inversions. Each transpose by OPT creates ≤ 1 inversion. Thus, we have

this is the worst case

$$\Phi(L_i) - \Phi(L_{i-1}) \leq 2(|A| - |B| + t_i)\ .$$

Amortized cost

The amortized cost for the *i*th operation of MTF with respect to Φ is

$$\hat{c}_i = c_i + \Phi(L_i) - \Phi(L_{i-1})$$

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\hat{c}_i &= c_i + \Phi(L_i) - \Phi(L_{i-1}) \\
&\le 2r + 2(|A| - |B| + t_i) \\
&= 2r + 2(|A| - (r - 1 - |A|) + t_i)
\end{aligned}
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(since $r = |A| + |B| + 1$)

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(since $r^* = |A| + |C| + 1 \ge |A| + 1$)

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&\le 4(r^* + t_i) \\
&= 4c_i^*.
\end{aligned}
$$

The grand finale

Thus, we have

$$C \sum_{i=1}^{|S|} c_i \quad \mathrm{MTF}$$

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$$C\sum_{i=1}^{|S|} c_i \quad \text{MTF}$$

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$$\leq \left(\sum_{i=1}^{|S|} 4c_i *\right) + \Phi(L_0) - \Phi(L_{|S|})$$

The grand finale

Thus, we have

$$C_{\text{MTF}} \sum_{i=1}^{|S|} c_i$$

$$= \sum_{i=1}^{|S|} (\hat{c}_i + \Phi(L_{i-1}) - \Phi(L_i))$$

$$\leq \left(\sum_{i=1}^{|S|} 4c_i * \right) + \Phi(L_0) - \Phi(L_{|S|})$$

$$\leq 4 \cdot C_{\text{OPT}}$$

,

since $\Phi(L_0) = 0$ and $\Phi(L_{|S|}) \geq 0$.

Addendum

If we count transpositions that move x toward the front as "free" (models splicing x in and out of L in constant time), then MTF is 2-competitive.

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What if $L_0 \neq L_0^*$?

- **Then, $\Phi(L_0)$ might be $\Theta(n^2)$ in the worst case.**
- **Thus, $C_{MTF}(S) \leq 4 \cdot C_{OPT}(S) + \Theta(n^2)$, which is still 4-competitive, since n^2 is constant as $|S| \to \infty$.**

Acknowledge

Based on Introduction to Algorithms CLRS

Material adapted from Erik D. Demaine and Charles E. Leiserson slides

http://www.cs.cmu.edu/~sleator/papers/amortized-efficiency.pdf