Math 309
Krantz

Spring, 2023
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FINAL EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 15 problems and is worth 200 points. You will have 2 hours to take this exam.

(12 points) **1.** Let $\mathcal{P}_3$ be the collection of all polynomials of degree 3 or less. Explain why this is a vector space.

If p, q are in $\mathcal{P}_3$ then $p+q \in \mathcal{P}_3$.
If $p \in \mathcal{P}_3$ and $c \in \mathbb{R}$ then $cp \in \mathcal{P}_3$.
Also $0 \in \mathcal{P}_3$.

(14 points) **2.** Give an explicit description of the null space of the matrix

$$A = \begin{bmatrix} 1 & -3 & -4 \\ 1 & 2 & 1 \\ 0 & -5 & -5 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free so $x_3 = \lambda$. $x_2 = -x_3 = -\lambda$

$x_1 = 3x_2 + 4x_3 = -3\lambda + 4\lambda = \lambda$, So null space is

$$\left\{ \begin{bmatrix} \lambda \\ -\lambda \\ \lambda \end{bmatrix} \right\} = \left\{ \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(14 points) **3.** Consider the vector space

$$\mathrm{Span}\{\mathbf{v}_1 \mathbf{v}_2 . \mathbf{v}_3\}$$

where

$$\mathbf{v}_1 = \langle 1 - 3, -4 \rangle,$$
$$\mathbf{v}_2 = \langle 1, 2, 1 \rangle,$$
$$\mathbf{v}_3 = \langle 0, -5, -5 \rangle.$$

What is the dimension of this space?

Note that $\underline{v}_3 = \underline{v}_1 - \underline{v}_2$. And $\underline{v}_1, \underline{v}_2$ are not multiples of each other. So they are linearly independent,

Hence $\{\underline{v}_1, \underline{v}_2\}$ is a basis for the space.

The space has dimension 2.

(14 points) **4.** Find a basis for the column space of

$$B = \begin{bmatrix} 0 & -3 & 5 \\ 1 & 2 & -1 \\ 9 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 1 & 2 & -1 \\ 9 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 9 & 0 & 0 \\ 0 & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -18 & 9 \\ 0 & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -18 & 9 \\ 0 & 0 & 7/2 \end{bmatrix}$$

There are three pivot columns. So

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space.

(14 points) **5.** Let

$$\mathbf{b}_1 = \langle 1, 0, 2 \rangle,$$
$$\mathbf{b}_2 = \langle 2, 0, 1 \rangle,$$
$$\mathbf{b}_3 = \langle 0, 1, 0 \rangle.$$

Write $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. What are the coordinates of the vector $\langle 1, 1, 1 \rangle$ with respect to $\mathcal{B}$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -3 & 0 & -1 \end{array}\right]$$

$x_3 = 1$, $x_2 = \frac{1}{3}$, $x_1 = -2x_2 + 1 = \frac{1}{3}$.

So coords. w.r.t. $\mathcal{B}$ are $\left(\frac{1}{3}, \frac{1}{3}, 1\right)$.

(14 points) **6.** Find all eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1).$

So $\lambda = -1, 3$

$\lambda = -1$: $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_2 = \lambda$, $x_1 = -\lambda$ eigenvector $\lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 3$: $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_2 = \lambda$, $x_1 = \lambda$ eigenvector $\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(14 points) **7.** Use any method to diagonalize the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

(14 points) **8.** What is the characteristic polynomial of the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}?$$

$$\det \begin{bmatrix} 3-\lambda & 0 & 1 \\ -1 & 1-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{bmatrix} = (3-\lambda)\cdot\left[(1-\lambda)^2 - 0\right] + 1\left[-1-2(1-\lambda)\right]$$

$$= (3-\lambda)(1-2\lambda+\lambda^2) - 1 - 2 + 2\lambda$$

$$= 3 - 6\lambda + 3\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 3 + 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 5\lambda$$

(14 points) **9.** What is the cosine of the angle between the vector $\langle 3, 1, 2\rangle$ and the $\langle 1, 4, 6\rangle$?

$$\cos\theta = \frac{\langle 3,1,2\rangle \cdot \langle 1,4,6\rangle}{\|\langle 3,1,2\rangle\| \|\langle 1,4,6\rangle\|}$$

$$= \frac{3+4+12}{\sqrt{9+1+4}\sqrt{1+16+36}}$$

$$= \frac{19}{\sqrt{14}\sqrt{53}}$$

(12 points) **10.** What is the projection of the vector $\mathbf{y} = \langle 2, 1, 4\rangle$ into the vector $\mathbf{u} = \langle 3, 3, 6\rangle$?

$$\mathrm{proj}_{\underline{u}}\,\underline{y} = \frac{\underline{y}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\,\underline{u} = \frac{\langle 2,1,4\rangle\cdot\langle 3,3,6\rangle}{\langle 3,3,6\rangle\cdot\langle 3,3,6\rangle}\langle 3,3,6\rangle$$

$$= \frac{6+3+24}{9+9+36}\langle 3,3,6\rangle = \frac{33}{54}\langle 3,3,6\rangle$$

$$= \frac{11}{18}\langle 3,3,6\rangle = \left\langle \frac{33}{18}, \frac{33}{18}, \frac{66}{18}\right\rangle$$

(12 points) **11.** What is the distance of the point $(1, 1, 1)$ from the 2-dimensional subspace

$$W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$$

with

$$\mathbf{u}_1 = \langle 1, 0, 2\rangle \qquad \mathbf{u}_2 = \langle 2, 0, 1\rangle\,?$$

$$\mathrm{proj}_W \langle 1,1,1\rangle = \frac{\langle 1,1,1\rangle\cdot\langle 1,0,2\rangle}{\langle 1,0,2\rangle\cdot\langle 1,0,2\rangle}\langle 1,0,2\rangle + \frac{\langle 1,1,1\rangle\cdot\langle 2,0,1\rangle}{\langle 2,0,1\rangle\cdot\langle 2,0,1\rangle}\langle 2,0,1\rangle$$

$$= \frac{3}{5}\langle 1,0,2\rangle + \frac{3}{5}\langle 2,0,1\rangle = \left\langle \frac{9}{5}, 0, \frac{9}{5}\right\rangle$$

$$\left\| \langle 1,1,1\rangle - \left\langle \frac{9}{5}, 0, \frac{9}{5}\right\rangle \right\| = \sqrt{\left(-\frac{4}{5}\right)^2 + 1^2 + \left(-\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{16}{25} + \frac{25}{25} + \frac{16}{25}} = \sqrt{\frac{57}{25}} = \frac{\sqrt{57}}{5}.$$

(12 points) **12.** Are any of the vectors $\langle 2, 1, 4\rangle$, $\langle -1, -2, 1\rangle$, and $\langle 1, 0, 1\rangle$ perpendicular to any of the others?

$\langle 2,1,4\rangle \cdot \langle -1,-2,1\rangle = -2-2+4 = 0$ so these are $\perp$

$\langle 2,1,4\rangle \cdot \langle 1,0,1\rangle = 2+0+4 = 6 \neq 0$ so not $\perp$

$\langle -1,-2,1\rangle \cdot \langle 1,0,1\rangle = -1+0+1 = 0$ so these are $\perp$

(14 points) **13.** What is a basis for the subspace of $\mathbf{R}^3$ spanned by $\langle 3, 1, 2\rangle$, $\langle 1, 4, 2\rangle$, and $\langle 2, -3, 0\rangle$?

$\langle 3,1,2\rangle - \langle 1,4,2\rangle = \langle 2,-3,0\rangle$.

The first two vectors are linearly independent because they are not multiples.

So $\langle 3,1,2\rangle, \langle 1,4,2\rangle$ is a basis.

(14 points) **14.** Use the Gram-Schmidt orthogonalization method to produce an orthogonal basis for the subspace of $\mathbf{R}^3$ spanned by $\langle 2,4,1\rangle$ and $\langle 2,3,6\rangle$.

$$\underline{v}_1 = \langle 2,4,1\rangle$$

$$\underline{v}_2 = \langle 2,3,6\rangle - \frac{\langle 2,3,6\rangle\cdot\langle 2,4,1\rangle}{\langle 2,4,1\rangle\cdot\langle 2,4,1\rangle}\cdot\langle 2,4,1\rangle$$

$$= \langle 2,3,6\rangle - \frac{4+12+6}{4+16+1}\langle 2,4,1\rangle$$

$$= \langle 2,3,6\rangle - \frac{22}{21}\langle 2,4,1\rangle$$

$$= \langle \frac{-2}{21}, \frac{-25}{21}, \frac{104}{21}\rangle$$

(12 points) **15.** Produce a vector of length 1 that is orthogonal to the vector $\langle 3,1,2\rangle$ in the space $\mathbf{R}^3$.

$\langle -1,1,1\rangle$ is $\perp$ to $\langle 3,1,2\rangle$

because $\langle -1,1,1\rangle\cdot\langle 3,1,2\rangle = -3+1+2 = 0.$

$\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$ is a unit vector

in that direction.