H(DIV)-CONFORMING DISCONTINUOUS GALERKIN DISCRETIZATION OF THE NAVIER–STOKES EQUATIONS

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Abstract.

Key words.

AMS subject classifications.

1. Introduction.

2. The Navier–Stokes problem.

Consider the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and let the time interval of interest be given by $I = (0, t_N]$. Let $\nu \in \mathbb{R}^+$ be the kinematic viscosity and let $f : \Omega \times I \to \mathbb{R}^d$ be a given forcing term. The Navier–Stokes problem for the velocity field $u : \Omega \times I \to \mathbb{R}^d$ and kinematic pressure field $p : \Omega \times I \to \mathbb{R}$ are given by

$$\partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u + \nabla p = f \qquad \text{in } \Omega \times I, \tag{1a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \times I, \tag{1b}$$

$$u = g \qquad \text{on } \Gamma_D \times I, \tag{1c}$$

$$(u \otimes u - \nu \nabla u + p\mathbb{I}) \cdot n - \max(u \cdot n, 0) u = h \qquad \text{on } \Gamma_N \times I, \tag{1d}$$

where the boundary of Ω has been partitioned into a Dirichlet (Γ_D) and Neumann (Γ_N) boundary: $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Throughout we assume $\Gamma_D \neq \emptyset$. On Γ_D, $g : \Gamma_D \times I \to \mathbb{R}^d$ is given Dirichlet boundary data, while $h : \Gamma_N \times I \to \mathbb{R}^d$ is given Neumann boundary data. If $\Gamma_N = \emptyset$, i.e., $\partial\Omega = \Gamma_D$, then the Dirichlet boundary data g must satisfy the compatibility condition

$$0 = \int_{\partial\Omega} g \cdot n \, \mathrm{d}x, \tag{2}$$

and the pressure mean is set to zero.

3. The $H(\mathrm{div})$-conforming discontinuous Galerkin method.

In this section we present the $H(\mathrm{div})$-conforming discontinuous Galerkin method for the Navier–Stokes problem (1). For this, we use a symmetric IP-DG version of the discontinuous Galerkin method proposed in [4]. As discussed in [3], the use of an $H(\mathrm{div})$ conforming discontinuous Galerkin method for the Navier–Stokes equations, which results in an approximate velocity field that is pointwise divergence free, results in a scheme that is stable and locally conservative.

3.1. Notation.

Let $\mathcal{T}$ be a triangulation of mesh size h_K of the the domain Ω into simplices $\{K\}$. We denote the set of all interior and boundary facets of $\mathcal{T}$ by, respectively, $\mathcal{F}_I$ and $\mathcal{F}_B$. The set of all facets is denoted by $\mathcal{F}$. On the boundary of a cell, ∂K, the outward unit normal vector is denoted by n. On interior facets, the average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators are defined in the usual way, i.e, for a scalar q

$$\{\!\{q\}\!\} := \tfrac{1}{2}\left(q^+ + q^-\right), \qquad [\![qn]\!] := q^+ n^+ + q^- n^-, \tag{3}$$

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where we remark that $n^+ = -n^-$. On boundary facets we set $\{\!\{q\}\!\} := q$ and $[\![qn]\!] := qn$. Average and jump operators for vectors and tensors are defined similarly.

The finite element space for the pressure is defined as

$$Q_h^k := \left\{ q_h \in L^2(\mathcal{T}),\ q_h \in P_k(K)\ \forall K \in \mathcal{T} \right\}, \tag{4}$$

where $P_k(K)$ denotes the space of polynomials of degree $k > 0$ on element K. For the velocity approximation, we consider the Brezzi–Douglas–Marini (BDM) function spaces. We remark that the BDM spaces are $H(\mathrm{div})$-conforming function spaces [2]. Denote BDM spaces of order k by BDM_h^k. The pair $BDM_h^k \backslash Q_h^{k-1}$ forms an inf-sup stable finite element pair that furthermore has the desirable property that $\nabla \cdot BDM_h^k = Q_h^{k-1}$. For notational purposes, we set $V_h = BDM_h^k$ and $Q_h = Q_h^{k-1}$.

3.2. Weak formulation. At each discrete time step, to find a solution the Navier–Stokes problem (1), one typically requires the solution of the linearized Navier–Stokes equations (or Oseen equations) within some iterative process. As such, we describe in this section the weak formulation for the Oseen problem,

$$\partial_t u + \nabla \cdot (u \otimes w) - \nu \Delta u + \nabla p = f \qquad \text{in } \Omega \times I, \tag{5a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \times I, \tag{5b}$$

$$u = g \qquad \text{on } \Gamma_D \times I, \tag{5c}$$

$$(u \otimes w - \nu \nabla u + p\mathbb{I}) \cdot n - \max(w \cdot n, 0) u = h \qquad \text{on } \Gamma_N \times I, \tag{5d}$$

with $w : \Omega \times I \to \mathbb{R}^d$ a given convective divergence free velocity field.

We consider first the momentum equation (5a). Let $\sigma : \Omega \times I \to \mathbb{R}^{d\times d}$ be an auxiliary variable, then we may write (5a) as

$$\sigma = \nabla u \qquad \text{in } \Omega \times I, \tag{6a}$$

$$\partial_t u + \nabla \cdot (u \otimes w) - \nu \nabla \cdot \sigma + \nabla p = f \qquad \text{in } \Omega \times I. \tag{6b}$$

We consider approximations of σ in the finite element space W_h (which remains unspecified, as the auxiliary variable will be eliminated later on). Multiplying (6a) by a testfunction $\tau_h \in W_h$ and (6b) by a testfunction $v_h \in V_h$, integrating and summing over all elements of the triangulation, and integration by parts, we obtain the following equations for the approximate solution $(\sigma_h, u_h, p_h) \in W_h \times V_h \times Q_h$:

$$\sum_{K\in\mathcal{T}} \int_K \sigma_h : \tau_h \,\mathrm{d}x = -\sum_{K\in\mathcal{T}} \int_K u_h \cdot \nabla \cdot \tau_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} \overline{u}_h \tau_h \cdot n \,\mathrm{d}s, \tag{7a}$$

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K (v_h \cdot \partial_t u_h - u_h \otimes w : \nabla v_h + \nu \sigma_h : \nabla v_h - p_h \nabla \cdot v_h) \,\mathrm{d}x \\ &\quad + \sum_{K\in\mathcal{T}} \int_{\partial K} \left(\overline{u_h \otimes w - \nu \sigma_h + p_h \mathbb{I}} \right) : v_h \otimes n \,\mathrm{d}s = \sum_{K\in\mathcal{T}} \int_K f \cdot v_h \,\mathrm{d}x, \end{aligned} \tag{7b}$$

where $\overline{(\cdot)}$ are numerical fluxes, defined later. For now, we write $H = \left(\overline{u_h \otimes w - \nu \sigma_h + p_h \mathbb{I}} \right)$. Writing element boundary integrals in terms of facet integrals, i.e.,

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \overline{u}_h \tau_h \cdot n \,\mathrm{d}s = \sum_{F\in\mathcal{F}_I} \int_F \overline{u}_h \cdot [\![\tau_h \cdot n]\!] \,\mathrm{d}s + \sum_{F\in\mathcal{F}_B} \int_F \overline{u}_h \otimes n : \tau_h \,\mathrm{d}s, \tag{8a}$$

$$\sum_{K\in\mathcal{T}} \int_{\partial K} H : v_h \otimes n \,\mathrm{d}s = \sum_{F\in\mathcal{F}_I} \int_F [\![v_h \otimes n]\!] : H \,\mathrm{d}s + \sum_{F\in\mathcal{F}_B} \int_F v_h \otimes n : H \,\mathrm{d}s. \tag{8b}$$

Note furthermore that, integration by parts and writing element boundary integrals in terms of facet integrals, results in

$$
\begin{aligned}
(9)\qquad -\sum_{K\in\mathcal{T}}\int_K u_h\cdot\nabla\cdot\tau_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}}\int_K \nabla u_h:\tau_h\,\mathrm{d}x - \sum_{F\in\mathcal{F}_I}\int_F \{\!\{u_h\}\!\}\cdot[\![\tau_h\cdot n]\!]\,\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_I}\int_F [\![u_h\otimes n]\!]:\{\!\{\tau_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_B}\int_F u_h\otimes n:\tau_h\,\mathrm{d}s.
\end{aligned}
$$

Combining (7a), (8a) and (9) and taking $\tau_h = \nabla v_h$ results in

$$
\begin{aligned}
(10)\qquad \sum_{K\in\mathcal{T}}\int_K \sigma_h:\nabla v_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}}\int_K \nabla u_h:\nabla v_h\,\mathrm{d}x - \sum_{F\in\mathcal{F}_I}\int_F [\![u_h\otimes n]\!]:\{\!\{\nabla v_h\}\!\}\,\mathrm{d}s \\
&\quad + \sum_{F\in\mathcal{F}_I}\int_F \left(\overline{u}_h - \{\!\{u_h\}\!\}\right)\cdot[\![\nabla v_h\cdot n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_B}\int_F (\overline{u}_h - u_h)\otimes n:\nabla v_h\,\mathrm{d}s.
\end{aligned}
$$

If we furthermore define the numerical flux

$$
(11)\qquad \overline{u}_h = \begin{cases} \{\!\{u_h\}\!\} & \text{on } F\in\mathcal{F}_I, \\ g & \text{on } F\in\mathcal{F}_D, \\ u_h & \text{on } F\in\mathcal{F}_N, \end{cases}
$$

where $\mathcal{F}_B = \mathcal{F}_D\cup\mathcal{F}_N$, with $\mathcal{F}_D$ and $\mathcal{F}_N$ the sets of facets on which, respectively, Diriclet and Neumann boundary conditions are prescribed. We may then write (10) as

$$
\begin{aligned}
(12)\qquad \sum_{K\in\mathcal{T}}\int_K \sigma_h:\nabla v_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}}\int_K \nabla u_h:\nabla v_h\,\mathrm{d}x - \sum_{F\in\mathcal{F}_I}\int_F [\![u_h\otimes n]\!]:\{\!\{\nabla v_h\}\!\}\,\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_D}\int_F (u_h - g)\otimes n:\nabla v_h\,\mathrm{d}s.
\end{aligned}
$$

Combining now (7b), (8b), and (12), we obtain

$$
\begin{aligned}
(13)\qquad \sum_{K\in\mathcal{T}}\int_K f\cdot v_h\,\mathrm{d}x &= \sum_{K\in\mathcal{T}}\int_K \left(v_h\cdot\partial_t u_h - u_h\otimes w:\nabla v_h + \nu\nabla u_h:\nabla v_h - p_h\nabla\cdot v_h\right)\mathrm{d}x \\
&\quad + \sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!]:H\,\mathrm{d}s + \sum_{F\in\mathcal{F}_B}\int_F v_h\otimes n:H\,\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_I}\int_F \nu[\![u_h\otimes n]\!]:\{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F \nu\,(u_h - g)\otimes n:\nabla v_h\,\mathrm{d}s.
\end{aligned}
$$

Next consider the integrals involving H. We split up $H = \overline{u_h\otimes w} + \overline{p_h}\mathbb{I} - \nu\overline{\sigma_h}$. We consider first the convective term,

$$
\begin{aligned}
(14)\qquad \sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!]:\overline{u_h\otimes w}\,\mathrm{d}s &= \sum_{F\in\mathcal{F}_I}\int_F (v_h^+ - v_h^-)\cdot(\overline{u_h\otimes w}\cdot n)\,\mathrm{d}s \\
&= \sum_{F\in\mathcal{F}_I}\int_F (v_h^+ - v_h^-)\cdot\Big(w\cdot n\{\!\{u_h\}\!\} + \tfrac{1}{2}|w\cdot n|(u_h^+ - u_h^-)\Big)\,\mathrm{d}s.
\end{aligned}
$$

Similarly, on Dirichlet boundary facet integrals,

$$
\begin{aligned}
\sum_{F\in\mathcal{F}_D}\int_F v_h\otimes n : \overline{u_h\otimes w}\,\mathrm{d}s &= \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot(\overline{u_h\otimes w}\cdot n)\,\mathrm{d}s \\
&= \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot\big(\tfrac{1}{2}w\cdot n(u_h+g)+\tfrac{1}{2}|w\cdot n|(u_h-g)\big)\,\mathrm{d}s,
\end{aligned} \tag{15}
$$

while on Neumann boundary facet integrals,

$$
\sum_{F\in\mathcal{F}_N}\int_F v_h\otimes n : \overline{u_h\otimes w}\,\mathrm{d}s = \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot(\overline{u_h\otimes w}\cdot n)\,\mathrm{d}s = \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot(w\cdot n u_h)\,\mathrm{d}s. \tag{16}
$$

Next, consider the pressure term,

$$
\begin{aligned}
&\sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!] : \overline{p_h}\mathbb{I}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F v_h\otimes n : \overline{p_h}\mathbb{I}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F v_h\otimes n : \overline{p_h}\mathbb{I}\,\mathrm{d}s \\
&\quad= \sum_{F\in\mathcal{F}_I}\int_F (v_h^+ - v_h^-)\cdot n^+\overline{p_h}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot n\overline{p_h}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot n\overline{p_h}\,\mathrm{d}s \\
&\quad= \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot n p_h\,\mathrm{d}s,
\end{aligned} \tag{17}
$$

where we set $\overline{p_h} = p_h$ on $F\in\mathcal{F}_N$ and where the last equality is due to the single valuedness of $\overline{p_h}$ and $v_h\cdot n$ on interior facets, and since $v_h\cdot n = 0$ on Dirichlet boundary facets. Finally, consider the viscosity term,

$$
-\sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!] : \nu\overline{\sigma_h}\,\mathrm{d}s = -\sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!] : \left(\nu\{\!\{\nabla u_h\}\!\} - \frac{\nu\alpha}{h}(u_h^+ - u_h^-)\otimes n\right)\mathrm{d}s. \tag{18}
$$

On Dirichlet boundary facets,

$$
-\sum_{F\in\mathcal{F}_D}\int_F v_h\otimes n : \nu\overline{\sigma_h}\,\mathrm{d}s = -\sum_{F\in\mathcal{F}_D}\int_F v_h\otimes n : \left(\nu\nabla u_h - \frac{\nu\alpha}{h}(u_h - g)\otimes n\right)\mathrm{d}s, \tag{19}
$$

and on Neumann boundary facets,

$$
-\sum_{F\in\mathcal{F}_N}\int_F v_h\otimes n : \nu\overline{\sigma_h}\,\mathrm{d}s = -\sum_{F\in\mathcal{F}_N}\int_F v_h\otimes n : (\nu\nabla u_h)\,\mathrm{d}s. \tag{20}
$$

Combining now these expressions with (13),

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K f\cdot v_h\,\mathrm{d}x \\
&\quad= \sum_{K\in\mathcal{T}}\int_K \left(v_h\cdot\partial_t u_h - u_h\otimes w : \nabla v_h + \nu\nabla u_h : \nabla v_h - p_h\nabla\cdot v_h\right)\mathrm{d}x \\
&\quad- \sum_{F\in\mathcal{F}_I}\int_F \nu[\![u_h\otimes n]\!] : \{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I}\int_F \nu[\![v_h\otimes n]\!] : \{\!\{\nabla u_h\}\!\}\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D}\int_F \nu\,(u_h-g)\otimes n : \nabla v_h\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F \nu v_h\otimes n : \nabla u_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I}\int_F \frac{\nu\alpha}{h}[\![u_h\otimes n]\!] : [\![v_h\otimes n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \frac{\nu\alpha}{h}(u_h-g)\cdot v_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I}\int_F (v_h^+ - v_h^-)\cdot\Big(w\cdot n\{\!\{u_h\}\!\} + \tfrac{1}{2}|w\cdot n|(u_h^+ - u_h^-)\Big)\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot\big(\tfrac{1}{2}w\cdot n(u_h+g) + \tfrac{1}{2}|w\cdot n|(u_h-g)\big)\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_N}\int_F v_h\otimes n : (u_h\otimes w - \nu\nabla u_h + p_h\mathbb{I})\,\mathrm{d}s.
\end{aligned}
\tag{21}
$$

Using the Neumann boundary condition (5d) for the last integral of (21), we obtain the following discontinuous Galerkin weak formulation for the momentum equation (1a):

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K f\cdot v_h\,\mathrm{d}x \\
&\quad= \sum_{K\in\mathcal{T}}\int_K \left(v_h\cdot\partial_t u_h - u_h\otimes w : \nabla v_h + \nu\nabla u_h : \nabla v_h - p_h\nabla\cdot v_h\right)\mathrm{d}x \\
&\quad- \sum_{F\in\mathcal{F}_I}\int_F \nu[\![u_h\otimes n]\!] : \{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I}\int_F \nu[\![v_h\otimes n]\!] : \{\!\{\nabla u_h\}\!\}\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D}\int_F \nu\,(u_h-g)\otimes n : \nabla v_h\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F \nu v_h\otimes n : \nabla u_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I}\int_F \frac{\nu\alpha}{h}[\![u_h\otimes n]\!] : [\![v_h\otimes n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \frac{\nu\alpha}{h}(u_h-g)\cdot v_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I}\int_F (v_h^+ - v_h^-)\cdot\Big(w\cdot n\{\!\{u_h\}\!\} + \tfrac{1}{2}|w\cdot n|(u_h^+ - u_h^-)\Big)\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot\big(\tfrac{1}{2}w\cdot n(u_h+g) + \tfrac{1}{2}|w\cdot n|(u_h-g)\big)\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot\big(h + \max(w\cdot n, 0)u_h\big)\,\mathrm{d}s.
\end{aligned}
\tag{22}
$$

The DG weak formulation for the mass equation (1b) is simply given by

$$
\sum_{K\in\mathcal{T}}\int_K q_h\nabla\cdot u_h\,\mathrm{d}x = 0. \tag{23}
$$

3.3. Time discretization. To not have to solve a non-linear system at each time step, we use the unconditionally stable, second-order accurate in time, approach proposed in [9], in which a trapezoidal rule is used to discretize the equations and in which the convective velocity is approximated by a linear combination of u_h at previous time steps. We also consider a first order in time approximation in which the Backward Euler method is used to discretize in time and in which the convective velocity is approximation by u_h evaluated at the previous time level.

Let $y^{n+\theta} = (1-\theta)y^n + \theta y^{n+1}$, and let $u_h^\star = \frac{3}{2}u_h^n - \frac{1}{2}u_h^{n-1}$ if $\theta = 1/2$ and $u_h^\star = u_h^n$ if $\theta = 1$. The time discrete weak formulation is given by: Find $(u_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$ the following holds:

$$
\begin{aligned}
0 = &\sum_{K\in\mathcal{T}} \int_K \frac{1}{\Delta t}(u_h^{n+1} - u_h^n)\cdot v_h \,\mathrm{d}x \\
&+ \sum_{K\in\mathcal{T}} \int_K \Big(-u_h^{n+\theta} \otimes u_h^\star : \nabla v_h + \nu \nabla u_h^{n+\theta} : \nabla v_h - p_h^{n+\theta}\nabla\cdot v_h\Big)\,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I} \int_F \nu [\![u_h^{n+\theta}\otimes n]\!] : \{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I} \int_F \nu [\![v_h \otimes n]\!] : \{\!\{\nabla u_h^{n+\theta}\}\!\}\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D} \int_F \nu \Big(u_h^{n+\theta} - g^{n+\theta}\Big)\otimes n : \nabla v_h \,\mathrm{d}s - \sum_{F\in\mathcal{F}_D} \int_F \nu v_h \otimes n : \nabla u_h^{n+\theta}\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I} \int_F \frac{\nu\alpha}{h} [\![u_h^{n+\theta}\otimes n]\!] : [\![v_h\otimes n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \frac{\nu\alpha}{h}(u_h^{n+\theta} - g^{n+\theta})\cdot v_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I} \int_F (v_h^+ - v_h^-)\cdot \Big(u_h^\star\cdot n \{\!\{u_h^{n+\theta}\}\!\} + \tfrac{1}{2}|u_h^\star\cdot n|((u_h^{n+\theta})^+ - (u_h^{n+\theta})^-)\Big)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D} \int_F v_h\cdot \Big(\tfrac{1}{2}u_h^\star\cdot n(u_h^{n+\theta} + g^{n+\theta}) + \tfrac{1}{2}|u_h^\star\cdot n|(u_h^{n+\theta} - g^{n+\theta})\Big)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N} \int_F v_h\cdot \Big(h^{n+\theta} + \max(u_h^\star\cdot n, 0)u_h^{n+\theta}\Big)\,\mathrm{d}s - \sum_{K\in\mathcal{T}} \int_K f^{n+\theta}\cdot v_h\,\mathrm{d}x,
\end{aligned}
\tag{24}
$$

and

$$
\sum_{K\in\mathcal{T}} \int_K q_h \nabla\cdot u_h^{n+\theta}\,\mathrm{d}x = 0. \tag{25}
$$

4. The convection-diffusion equation. The Navier–Stokes problem (1) will be coupled to the convection-diffusion equation, so that we discuss its discontinuous Galerkin weak form next.

The convection-diffusion problem is given by

$$
\begin{aligned}
\partial_t\rho + \nabla\cdot(\rho u) - \nabla\cdot\epsilon\nabla\rho &= 0 && \text{in } \Omega\times I, && \text{(26a)}\\
\rho &= \psi && \text{on } \Gamma_D\times I, && \text{(26b)}\\
-\tfrac{1}{2}u\cdot n\rho + \tfrac{1}{2}|u\cdot n|\rho + \epsilon\nabla\rho\cdot n &= \phi && \text{on } \Gamma_N\times I, && \text{(26c)}
\end{aligned}
$$

where $\rho : \Omega\times I \to \mathbb{R}$ is the density, ϵ a visosity and ψ and ϕ given boundary data.

Consider the following finite element space for the density

$$
U_h := \Big\{u_h \in L^2(\mathcal{T}),\ u_h \in P_k(K)\ \forall K\in\mathcal{T}\Big\}. \tag{27}
$$

The DG weak formulation of (26) is a scalar version of the DG weak formulation for the Navier–Stokes problem (with $p_h = 0$). Starting with (21), the DG weak formulation of (26) is given by

$$
\begin{aligned}
0 = &\sum_{K\in\mathcal{T}}\int_K \left(w_h\partial_t\rho_h - \rho_h u_h\cdot\nabla w_h + \epsilon\nabla\rho_h\cdot\nabla w_h\right)\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I}\int_F \epsilon[\![\rho_h n]\!]\cdot\{\!\{\nabla w_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I}\int_F \epsilon[\![w_h n]\!]\cdot\{\!\{\nabla\rho_h\}\!\}\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D}\int_F \epsilon\left(\rho_h-\psi\right)n\cdot\nabla w_h\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F \epsilon w_h n\cdot\nabla\rho_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I}\int_F \frac{\epsilon\alpha}{h}[\![\rho_h]\!][\![w_h]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \frac{\epsilon\alpha}{h}(\rho_h-\psi)w_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I}\int_F (w_h^+ - w_h^-)\left(u_h\cdot n\{\!\{\rho_h\}\!\} + \tfrac{1}{2}|u_h\cdot n|(\rho_h^+-\rho_h^-)\right)\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F w_h\left(\tfrac{1}{2}u_h\cdot n(\rho_h+\psi) + \tfrac{1}{2}|u_h\cdot n|(\rho_h-\psi)\right)\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N}\int_F w_h\left(\tfrac{1}{2}\rho_h(u_h\cdot n + |u_h\cdot n|) - \phi\right)\mathrm{d}s,
\end{aligned}
\tag{28}
$$

where we substituted the Neumann boundary condition (26c) into the last integral.

5. Coupling of Navier–Stokes and convection-diffusion: time-stepping.

In this section we consider the time-stepping algorithm when coupling the Navier–Stokes problem (1) to the convection-diffusion problem (26). In particular, consider the following problem:

$$\partial_t u + \nabla\cdot(u\otimes u) - \nu\Delta u + \nabla p = \rho e_3, \tag{29a}$$

$$\nabla\cdot u = 0, \tag{29b}$$

$$\partial_t\rho + \nabla\cdot(\rho u) - \epsilon\Delta\rho = 0, \tag{29c}$$

where e_3 is the vertical direction. Following [7], we consider the following time-stepping approach. Let $u_h^\star = \frac{3}{2}u_h^n - \frac{1}{2}u_h^{n-1}$ (if $\theta = 1/2$) or $u_h^\star = u_h^n$ (if $\theta = 1$), then

$$\frac{u_h^{n+1}-u_h^n}{\Delta t} + \nabla\cdot(u_h^{n+\theta}\otimes u_h^\star) - \nu\Delta u_h^{n+\theta} + \nabla p_h^{n+\theta} = \rho_h^{n+\theta}e_3, \tag{30a}$$

$$\nabla\cdot u_h^{n+\theta} = 0, \tag{30b}$$

$$\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t} + \nabla\cdot(\rho_h^{n+\theta}u_h^\star) - \epsilon\Delta\rho_h^{n+\theta} = 0. \tag{30c}$$

Note that by having linearized the convection field in both (30a) and (30c), by introducing $u_h^\star$, we have uncoupled the problem: we first compute θ_h^{n+1} from (30c), after which we compute u_h^{n+1} and p_h^{n+1} by solving (30a)–(30b).

To discretize (30), we use the weak formulations for the Navier–Stokes problem (24)–(25) and the convection-diffusion problem (28).

6. Compatibility. An important property of the $H(\mathrm{div})$ conforming discontinuous Galerkin discretization discussed in section 3, is its compatibility with discontinuous Galerkin discretizations of the convection-diffusion equation (26a). The importance of compatible flow and transport schemes, to prevent loss of accuracy and global conservation, was shown in [5]. They defined compatible schemes to satisfy the following two properties:

1. The scheme should be zeroth-order accurate: Let the initial and boundary data be equal to a constant ρ_c. If we replace u by its numerical approximation u_h in (26a), the scheme (28) must reproduce the constant solution $\rho_h \equiv \rho_c$ for all time.
2. The scheme must be globally conservative when u is replaced by its numerical approximation u_h in (26a).

The following lemma shows that the flow and transport schemes presented here are compatible.

Lemma 1 (Compatible flow and transport schemes). *The $H(\mathrm{div})$ conforming discontinuous Galerkin method discussed for the Navier–Stokes problem (24)–(25), is compatible with the discontinuous Galerkin method for transport (28).*

Proof. Global conservation is guaranteed by using a discontinuous Galerkin method for the conservative form of the transport equation, hence we only check whether the scheme is able to preserve the constant solution. For this, we consider Backward Euler time stepping for compactness of notation, but note that the proof is easily generalized to time stepping by the theta-method. We first note, that since $u_h \in BDM_h^k$ and $\nabla \cdot BDM_h^k = Q_h^{k-1}$, (25) implies that $\nabla \cdot u_h = 0$ pointwise on a cell K. Furthermore, since $u_h \in BDM_h^k$, $u_h \cdot n$ is single valued on cell facets. Assuming that $\rho_h^n = \rho_c$, with ρ_c a constant, then substituting in (28)

$$
\begin{aligned}
(31)\quad \sum_{K\in\mathcal{T}} \int_K w_h \frac{\rho_h^{n+1} - \rho_c}{\Delta t}\,\mathrm{d}x = {}& \sum_{K\in\mathcal{T}} \int_K \rho_h^{n+1} u_h \cdot \nabla w_h \,\mathrm{d}x \\
& - \sum_{F\in\mathcal{F}_I} \int_F (w_h^+ - w_h^-)\rho_h^{n+1} u_h \cdot n \,\mathrm{d}s - \sum_{F\in\mathcal{F}_D} \int_F w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s \\
& - \sum_{F\in\mathcal{F}_N} \int_F w_h \left(\tfrac{1}{2}\rho_h^{n+1}(u_h \cdot n + |u_h \cdot n|) - \phi\right)\mathrm{d}s,
\end{aligned}
$$

where we remark that $\phi = 0$ on outflow boundaries ($u_h \cdot n > 0$) and $\phi = \rho_c u_h \cdot n$ on inflow boundaries ($u_h \cdot n \le 0$). Integrating by parts the first term on the right hand side,

$$
\begin{aligned}
(32)\quad \sum_{K\in\mathcal{T}} \int_K \rho_h^{n+1} u_h \cdot \nabla w_h \,\mathrm{d}x &= - \sum_{K\in\mathcal{T}} \int_K w_h \rho_h^{n+1} \nabla \cdot u_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s \\
&= - \sum_{K\in\mathcal{T}} \int_K w_h \rho_h^{n+1} \nabla \cdot u_h \,\mathrm{d}x \\
&\quad + \sum_{F\in\mathcal{F}_I} \int_F \rho_h^{n+1} \left([\![u_h]\!] \{\!\{w_h\}\!\} + \{\!\{u_h\}\!\} \cdot [\![w_h]\!] \right) \mathrm{d}s \\
&\quad + \sum_{F\in\mathcal{F}_D} \int_F w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s + \sum_{F\in\mathcal{F}_N} \int_F w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s.
\end{aligned}
$$

Since $u_h \cdot n$ is single valued on cell facets and since $\nabla \cdot u_h = 0$ pointwise on cells, this simplifies to

$$
\begin{aligned}
\sum_{K\in\mathcal{T}} \int_K \rho_h^{n+1} u_h \cdot \nabla w_h \,\mathrm{d}x = \sum_{F\in\mathcal{F}_I} \int_F (w_h^+ - w_h^-)\rho_h^{n+1} u_h \cdot n \,\mathrm{d}s \\
+ \sum_{F\in\mathcal{F}_D} \int_F w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s + \sum_{F\in\mathcal{F}_N} \int_F w_h \rho_h^{n+1} u_h \cdot n \,\mathrm{d}s.
\end{aligned} \tag{33}
$$

Substituting into (31),

$$
\sum_{K\in\mathcal{T}} \int_K w_h \frac{\rho_h^{n+1} - \rho_c}{\Delta t} \,\mathrm{d}x = \sum_{F\in\mathcal{F}_N} \int_F w_h \left[\tfrac{1}{2}\rho_h^{n+1}\left(u_h \cdot n - |u_h \cdot n|\right) - \phi\right] \mathrm{d}s. \tag{34}
$$

Noting that on outflow boundaries $u_h \cdot n - |u_h \cdot n| = 0$ and $\phi = 0$, and that on inflow boundaries $\frac{1}{2}\rho_h^{n+1}\left(u_h \cdot n - |u_h \cdot n|\right) = \rho_h^{n+1} u_h \cdot n = \phi$, we obtain

$$
\sum_{K\in\mathcal{T}} \int_K w_h \frac{\rho_h^{n+1} - \rho_c}{\Delta t} \,\mathrm{d}x = \sum_{F\in\mathcal{F}_N} \int_F w_h u_h \cdot n \left(\rho_h^{n+1} - \rho_c\right) \mathrm{d}s. \tag{35}
$$

We see that $\rho_h^{n+1} = \rho_c$ satisfies (35) independent of u_h. Since ρ_h^{n+1} is unique, it follows that the method is always zeroth-order accurate, and hence, the flow and transport schemes are compatible. ∎

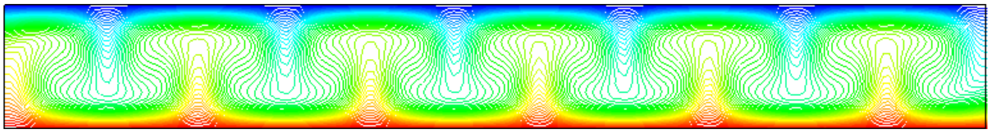
7. Numerical examples.

All simulations have been performed using the FEniCS finite element library [1, 8].

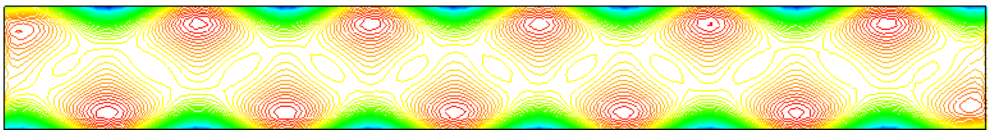
7.1. Vertically heated horizontal cavity.

In this first test case we consider vertial heating, see [7, Section 4.2]. For this test case we take in (29) $\nu = \sqrt{\mathrm{Pr}/\mathrm{Ra}}$ and $\epsilon = 1/\sqrt{\mathrm{Pr}\,\mathrm{Ra}}$, where Pr is the Prandtl number and Ra is the Rayleigh number. We take $\mathrm{Pr} = 7.1$ and $\mathrm{Ra} = 1.5 \cdot 10^4$. As computational domain we take $\Omega = [0, 8] \times [0, 1]$ and a $160 \times 20 \times 2$ uniform mesh of triangles. We compute the solution over the time interval $I = [0, 500]$ for which we set the time step to $\Delta t = 0.01$. As boundary conditions we impose $u = 0$ on $\partial\Omega$, $\epsilon\nabla\rho \cdot n = 0$ on $\Gamma_N := \{x \in \partial\Omega | x_1 = 0 \text{ or } x_1 = 8\}$ and $\rho = \left(\frac{1}{2} - x_2\right)\left(1 - \exp(-10t)\right)$ on $\Gamma_D = \partial\Omega\backslash\Gamma_N$. As initial conditions we impose $u = 0$ and $\rho = 0$. We consider linear polynomial approximations for the velocity and temperature fields, and a constant polynomial approximation for the pressure. For the time stepping, we consider both the $\theta = 1$ and $\theta = 1/2$ cases.

As discussed in [7, Section 4.2], this test case computes the transition from an unstable equilibrium flow to a stable configuration of convection rolls. We plot the quasi-steady solution in Figure 1 which shows the convective roll pattern as expected. The evolution of the kinetic energy is shown in Figure 2. We observe the same behaviour as described by [7], namely, that at about $t = 100$ the temperature ρ_h is a linear function of the vertical coordinate and the velocity u_h is close to zero. As illustrated by the evolution of the kinetic energy, this solution is not stable: soon after $t = 100$ the transition to the quasi-steady solution (convection rolls) follows. Furthermore, the numerical method is less diffusive using the second order time stepping method. The evolution of the divergence of the velocity field is also shown in Figure 2. It is clear that the divergence remains small, i.e., $\mathcal{O}(10^{-13})$. *SR: We need to evaluate $\nabla \cdot u_h = 0$ at t_{n+1}, not at $t_{n+1/2}$, then it's okay.*



(a) Isotherms.



(b) Pressure.

(c) Velocity field.

Fig. 1: Isotherms, pressure and the velocity field at $t = 500$ for the vertically heated horizontal cavity problem with $\mathrm{Pr} = 7.1$ and $\mathrm{Ra} = 1.5 \cdot 10^4$ on the domain $\Omega = [0, 8] \times [0, 1]$.

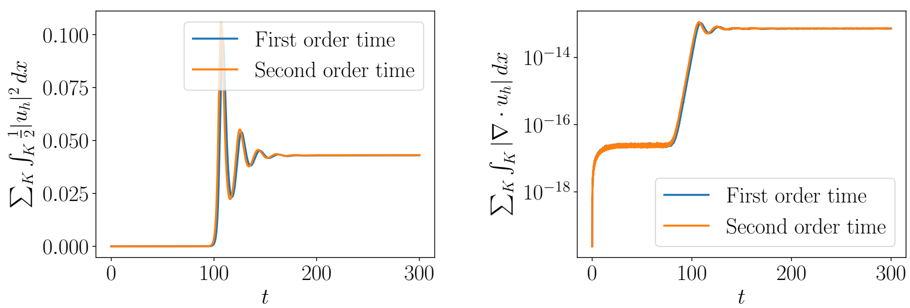


Fig. 2: Evolution of the kinetic energy and the divergence for the vertically heated horizontal cavity problem with $\mathrm{Pr} = 7.1$ and $\mathrm{Ra} = 1.5 \cdot 10^4$.

7.2. Internal wave generation over topography. Next we consider a (modified) test case from [10], namely internal wave generation over topography. We consider the Navier–Stokes equations in dimensional form,

$$\rho_0 \partial_t u + \rho_0 \nabla \cdot (u \otimes u) - \mu \Delta u = -\nabla p + \rho f, \tag{36a}$$

$$\nabla \cdot u = 0, \tag{36b}$$

$$\partial_t \rho + \nabla \cdot (\rho u) = \kappa \Delta \rho, \tag{36c}$$

where ρ is the fluid density, ρ_0 is the reference value for density, $\mu > 0$ is the dynamic viscosity, $\kappa > 0$ is a constant diffusivity, and ρf is a buoyancy forcing term acting downward. Writing the density and pressure in terms of a background hydrostatic component ($\bar{\rho}(z)$ and $\bar{p}(z)$) and perturbations to the hydrostatic component (ρ' and p'), i.e.,

$$\rho = \bar{\rho} + \rho', \qquad p = \bar{p} + p' \tag{37}$$

we write (36a) as

$$\begin{aligned} \partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u &= -\frac{1}{\rho_0} \left(\partial_z \bar{p} + g \bar{\rho} \right) e_z - \frac{1}{\rho_0} \nabla p' - \frac{g}{\rho_0} \rho' e_z \\ &= -\frac{1}{\rho_0} \nabla p' - \frac{g}{\rho_0} \rho' e_z, \end{aligned} \tag{38}$$

where $e_z = (0, 1)$ and where $\nu > 0$ is the kinematic viscosity. Define now the buoyancy $b = -g\rho'/\rho_0$, and for notational purposes, set $p := p'/\rho_0$. The momentum equation becomes

$$\partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u + \nabla p = b e_z. \tag{39}$$

Multiplying now the convection-diffusion equation (36c) by $-g/\rho_0$ and using $\rho = \bar{\rho} + \rho'$,

$$-\frac{g}{\rho_0} \left[\partial_t \rho' + \nabla \cdot (\bar{\rho} u) + \nabla \cdot (\rho' u) \right] = -\frac{g}{\rho_0} \left[\kappa \Delta \rho' \right], \tag{40}$$

where we note that $\Delta \bar{\rho} = 0$. Furthermore, note that $\nabla \cdot (\bar{\rho} u) = u \cdot \nabla \bar{\rho} = u_z \partial_z \bar{\rho}$. But $\partial_z \bar{\rho} = -\rho_0 N^2 / g$ so that we obtain

$$\partial_t b + \nabla \cdot (b u) = -u_z N^2 + \kappa \Delta b. \tag{41}$$

To conclude, for this test case we solve the following equations:

$$\partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u + \nabla p = b e_z, \tag{42a}$$

$$\nabla \cdot u = 0, \tag{42b}$$

$$\partial_t b + \nabla \cdot (b u) = -u_z N^2 + \kappa \Delta b. \tag{42c}$$

As initial conditions we set $u = (0, 0)$ and $b = 0$. As boundary conditions we impose $\nabla b \cdot n = 0$ on all boundaries. We impose a no-slip boundary condition $u \cdot n = 0$ as well as $\nabla u_\tau \cdot n = 0$, where u_τ is the component of u in the tangent plane on the top and bottom walls. On the inflow we impose $u_1 = u_m \sin(\omega_{\text{tide}} t)$ and $u_2 = 0$, where ω_{tide} is the tidal frequency. We set the following parameters:

$$\begin{gathered} u_m = 1\,\text{cm}\,\text{s}^{-1}, \quad \nu = 10^{-9}, \quad g = 9.81\,\text{m}\,\text{s}^{-2}, \quad \kappa = 10^{-9}, \\ N = 10^{-3}\,\text{s}^{-1}, \quad \omega_{\text{tide}} = 2\pi/44712\text{s}^{-1}. \end{gathered}$$

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Appendix A. Adding skew symmetric terms. The weak formulation of the Navier–Stokes equations, given in (22) is derived under the assumption that $\nabla \cdot u_h = 0$ pointwise and that $u_h \cdot n$ is continuous across faces. Although the latter is true by construction of the BDM function space, the former follows by solving (23). It depends on the tolerance of the linear solver to what accuracy this equation is solved. As such $\nabla \cdot u_h$ is at best machine precision, but it will be larger in general. As such, the weak formulation of the Navier–Stokes equations in subsection 3.2 may not be energy stable. To avoid instabilities forming, we need to modify the convective terms. For this, we follow [6].

A.1. A modification of the tri-linear form. Consider the convective term in the Navier–Stokes equations. For solenoidal w it holds that

$$\nabla \cdot (u \otimes w) = \nabla \cdot (u \otimes w) - \tfrac{1}{2}(\nabla \cdot w)u = w \cdot \nabla u + \tfrac{1}{2}(\nabla \cdot w)u. \tag{43}$$

Multiplying (43) by a testfunction $v_h \in V_h$, integrating and summing over all elements of the triangulation, and integration by parts of the first term, we obtain the following weak form for the convective term:

$$\begin{aligned} t_h(w_h, u_h, v_h) &= \sum_{K\in\mathcal{T}} \int_K \big((w_h \cdot \nabla u_h) \cdot v_h + \tfrac{1}{2}(\nabla \cdot w_h) u_h \cdot v_h\big) \,\mathrm{d}x \\ &= -\sum_{K\in\mathcal{T}} \int_K u_h \cdot \big(\nabla \cdot (v_h \otimes w_h)\big) \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} v_h \otimes n : \overline{u_h \otimes w_h} \,\mathrm{d}s \\ &\quad + \sum_{K\in\mathcal{T}} \int_K \tfrac{1}{2}(\nabla \cdot w_h) u_h \cdot v_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_I} \int_F \tfrac{1}{2} [\![w_h \cdot n]\!] \{\!\{ u_h \cdot v_h \}\!\} \,\mathrm{d}s \\ &\quad - \sum_{F\in\mathcal{F}_B} \int_F \tfrac{1}{2}\,(w_h \cdot n)\,(u_h \cdot v_h) \,\mathrm{d}s, \end{aligned} \tag{44}$$

where the last two terms on the right hand side were added in [6] to ensure stability and convergence. We note, however, that for $w_h \in V_h$, $w_h \cdot n$ is continuous across faces. Therefore,

$$
(45) \quad -\sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}[\![w_h\cdot n]\!]\{\!\{u_h\cdot v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_B}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s = \\
-\sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s.
$$

Next, consider the second term on the right hand side in (44). We write this integral over cell boundaries as an integral over facets:

$$
(46) \quad \sum_{K\in\mathcal{T}}\int_{\partial K} v_h\otimes n : \overline{u_h\otimes w_h}\,\mathrm{d}s = \sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!] : \overline{u_h\otimes w_h}\,\mathrm{d}s \\
+ \sum_{F\in\mathcal{F}_B}\int_F v_h\otimes n : \overline{u_h\otimes w_h}\,\mathrm{d}s.
$$

Using an upwind flux for $\overline{u_h\otimes w_h}$, we find on interior facets

$$
(47) \quad \sum_{F\in\mathcal{F}_I}\int_F [\![v_h\otimes n]\!] : \overline{u_h\otimes w_h}\,\mathrm{d}s = \\
\sum_{F\in\mathcal{F}_I}\int_F \left(v_h^+ - v_h^-\right)\cdot\left(w_h\cdot n\{\!\{u_h\}\!\} + \tfrac{1}{2}|w_h\cdot n|(u_h^+ - u_h^-)\right)\mathrm{d}s,
$$

while on boundary facets we find

$$
(48) \quad \sum_{F\in\mathcal{F}_B}\int_F v_h\otimes n : \overline{u_h\otimes w_h}\,\mathrm{d}s = \sum_{F\in\mathcal{F}_N}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s \\
+ \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot\left(\tfrac{1}{2}w_h\cdot n(u_h+g) + \tfrac{1}{2}|w_h\cdot n|(u_h-g)\right)\mathrm{d}s.
$$

Integrating by parts now the first term on the right hand side of (44),

$$
(49) \quad -\sum_{K\in\mathcal{T}}\int_K u_h\cdot\left(\nabla\cdot(v_h\otimes w_h)\right)\mathrm{d}x = \sum_{K\in\mathcal{T}}\int_K (w_h\cdot\nabla u_h)\cdot v_h\,\mathrm{d}x \\
-\sum_{K\in\mathcal{T}}\int_{\partial K} u_h\otimes w_h : v_h\otimes n\,\mathrm{d}s.
$$

Writing the integral over cell boundaries as an integral over facets:

$$
(50) \quad -\sum_{K\in\mathcal{T}}\int_{\partial K} u_h\otimes w_h : v_h\otimes n\,\mathrm{d}s = \\
-\sum_{F\in\mathcal{F}_D}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s \\
-\sum_{F\in\mathcal{F}_I}\int_F \left(\{\!\{v_h\}\!\}w_h\cdot n[\![u_h]\!] + (v_h^+ - v_h^-)\cdot\{\!\{u_h\}\!\}w_h\cdot n\right)\mathrm{d}s,
$$

where we used that for $w_h \in V_h$, $w_h \cdot n$ is continuous across faces. Combining now (44)–(50):

$$
\begin{aligned}
t_h(w_h,u_h,v_h) = &\sum_{K\in\mathcal{T}}\int_K (w_h\cdot\nabla u_h)\cdot v_h \,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_D}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_I}\int_F \Big(\{\!\{v_h\}\!\} w_h\cdot n [\![u_h]\!] + (v_h^+ - v_h^-)\cdot\{\!\{u_h\}\!\} w_h\cdot n\Big)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I}\int_F \Big(v_h^+ - v_h^-\Big)\cdot\Big(w_h\cdot n\{\!\{u_h\}\!\} + \tfrac{1}{2}|w_h\cdot n|(u_h^+ - u_h^-)\Big)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N}\int_F (v_h\cdot u_h)(w_h\cdot n)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F v_h\cdot\big(\tfrac{1}{2}w_h\cdot n(u_h+g) + \tfrac{1}{2}|w_h\cdot n|(u_h-g)\big)\,\mathrm{d}s \\
&+ \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot w_h)u_h\cdot v_h\,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s,
\end{aligned}
\tag{51}
$$

which, after cancelling terms, results in

$$
\begin{aligned}
t_h(w_h,u_h,v_h) = &\sum_{K\in\mathcal{T}}\int_K (w_h\cdot\nabla u_h)\cdot v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot w_h)u_h\cdot v_h\,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I}\int_F w_h\cdot n[\![u_h]\!]\cdot\{\!\{v_h\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|[\![u_h]\!]\cdot[\![v_h]\!]\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,\big(-w_h\cdot n + |w_h\cdot n|\big)\,(u_h-g)\cdot v_h\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned}
\tag{52}
$$

A.2. A comparison of tri-linear forms. We compare the tri-linear form derived in subsection 3.2 with the modified tri-linear form derived in Appendix A.1.

Repeating the modified tri-linear form (52),

$$
\begin{aligned}
t_h(w_h,u_h,v_h) = &\sum_{K\in\mathcal{T}}\int_K (w_h\cdot\nabla u_h)\cdot v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot w_h)u_h\cdot v_h\,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I}\int_F w_h\cdot n[\![u_h]\!]\cdot\{\!\{v_h\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|[\![u_h]\!]\cdot[\![v_h]\!]\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,\big(-w_h\cdot n + |w_h\cdot n|\big)\,(u_h-g)\cdot v_h\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned}
\tag{53}
$$

We now assume that $\nabla \cdot w_h = 0$. Then

$$\frac{1}{2}(w_h \cdot \nabla u_h) \cdot v_h + \frac{1}{2}(\nabla \cdot w_h)(u_h \cdot v_h) = \frac{1}{2}\nabla \cdot ((u_h \otimes w_h) \cdot v) - \frac{1}{2}(u_h \otimes w_h) : \nabla v_h. \tag{54}$$

Adding $\frac{1}{2}(w_h \cdot \nabla u_h) \cdot v_h$ to both sides,

$$(w_h \cdot \nabla u_h) \cdot v_h + \frac{1}{2}(\nabla \cdot w_h)(u_h \cdot v_h) = \frac{1}{2}\nabla \cdot ((u_h \otimes w_h) \cdot v) - \frac{1}{2}(u_h \otimes w_h) : \nabla v_h + \frac{1}{2}(w_h \cdot \nabla u_h) \cdot v_h, \tag{55}$$

and using that $\frac{1}{2}(w_h \cdot \nabla u_h) \cdot v_h = \frac{1}{2} w_h \cdot \nabla (u_h \cdot v_h) - \frac{1}{2}(u_h \otimes w_h) : \nabla v_h$,

$$(w_h \cdot \nabla u_h) \cdot v_h + \frac{1}{2}(\nabla \cdot w_h)(u_h \cdot v_h) = \frac{1}{2}\nabla \cdot ((u_h \otimes w_h) \cdot v_h) - (u_h \otimes w_h) : \nabla v_h + \frac{1}{2} w_h \cdot \nabla (u_h \cdot v_h). \tag{56}$$

It follows that

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_K (w_h \cdot \nabla u_h) \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_K \frac{1}{2}(\nabla \cdot w_h) u_h \cdot v_h \,\mathrm{d}x = \\ - \sum_{K \in \mathcal{T}} \int_K (u_h \otimes w_h) : \nabla v_h \,\mathrm{d}x \\ + \sum_{K \in \mathcal{T}} \int_K \frac{1}{2} \nabla \cdot ((u_h \otimes w_h) \cdot v_h) \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_K \frac{1}{2} w_h \cdot \nabla (u_h \cdot v_h) \,\mathrm{d}x. \end{aligned} \tag{57}$$

Integrating by parts the last two terms on the right hand side:

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_K \frac{1}{2} \nabla \cdot ((u_h \otimes w_h) \cdot v_h) \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_K \frac{1}{2} w_h \cdot \nabla (u_h \cdot v_h) \,\mathrm{d}x = \\ \sum_{K \in \mathcal{T}} \int_{\partial K} (u_h \cdot v_h)(w_h \cdot n) \,\mathrm{d}x, \end{aligned} \tag{58}$$

where we used that $\nabla \cdot w_h = 0$. Writing the integral over cell boundaries as integrals over facets,

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_{\partial K} (u_h \cdot v_h)(w_h \cdot n) \,\mathrm{d}x = \sum_{F \in \mathcal{F}_I} \int_F w_h \cdot n (\llbracket v_h \rrbracket \cdot \{\!\{ u_h \}\!\} + \{\!\{ v_h \}\!\} \cdot \llbracket u_h \rrbracket) \,\mathrm{d}s \\ + \sum_{F \in \mathcal{F}_D} \int_F (v_h \cdot u_h) w_h \cdot n \,\mathrm{d}s + \sum_{F \in \mathcal{F}_N} \int_F (v_h \cdot u_h) w_h \cdot n \,\mathrm{d}s. \end{aligned} \tag{59}$$

Combined with (57), this results in

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_K (w_h \cdot \nabla u_h) \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_K \frac{1}{2}(\nabla \cdot w_h) u_h \cdot v_h \,\mathrm{d}x = \\ - \sum_{K \in \mathcal{T}} \int_K (u_h \otimes w_h) : \nabla v_h \,\mathrm{d}x + \sum_{F \in \mathcal{F}_I} \int_F w_h \cdot n (\llbracket v_h \rrbracket \cdot \{\!\{ u_h \}\!\} + \{\!\{ v_h \}\!\} \cdot \llbracket u_h \rrbracket) \,\mathrm{d}s \\ + \sum_{F \in \mathcal{F}_D} \int_F (v_h \cdot u_h) w_h \cdot n \,\mathrm{d}s + \sum_{F \in \mathcal{F}_N} \int_F (v_h \cdot u_h) w_h \cdot n \,\mathrm{d}s. \end{aligned} \tag{60}$$

Combined now with (53),

$$
\begin{aligned}
&t_h(w_h,u_h,v_h) = \\
&\quad -\sum_{K\in\mathcal{T}}\int_K (u_h\otimes w_h):\nabla v_h \,\mathrm{d}x + \sum_{F\in\mathcal{F}_I}\int_F w_h\cdot n(\llbracket v_h\rrbracket\cdot\{\!\{u_h\}\!\} + \{\!\{v_h\}\!\}\cdot\llbracket u_h\rrbracket)\,\mathrm{d}s \\
&\quad + \sum_{F\in\mathcal{F}_D}\int_F (v_h\cdot u_h)w_h\cdot n\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F (v_h\cdot u_h)w_h\cdot n\,\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_I}\int_F w_h\cdot n\llbracket u_h\rrbracket\cdot\{\!\{v_h\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|\llbracket u_h\rrbracket\cdot\llbracket v_h\rrbracket\,\mathrm{d}s \\
&\quad + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(-w_h\cdot n + |w_h\cdot n|\right)(u_h-g)\cdot v_h\,\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned}
\tag{61}
$$

Cancelling terms,

$$
\begin{aligned}
t_h(w_h,u_h,v_h) = &-\sum_{K\in\mathcal{T}}\int_K u_h\otimes w_h:\nabla v_h\,\mathrm{d}x \\
&+ \sum_{F\in\mathcal{F}_I}\int_F (w_h\cdot n)\{\!\{u_h\}\!\}\cdot\llbracket v_h\rrbracket\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|\llbracket u_h\rrbracket\cdot\llbracket v_h\rrbracket\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(-w_h\cdot n + |w_h\cdot n|\right)(u_h-g)\cdot v_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned}
\tag{62}
$$

The tri-linear form derived in subsection 3.2 is given by

$$
\begin{aligned}
t_h^c(w_h,u_h,v_h) = &-\sum_{K\in\mathcal{T}}\int_K u_h\otimes w_h:\nabla v_h\,\mathrm{d}x \\
&+ \sum_{F\in\mathcal{F}_I}\int_F (w_h\cdot n)\{\!\{u_h\}\!\}\cdot\llbracket v_h\rrbracket\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|\llbracket u_h\rrbracket\cdot\llbracket v_h\rrbracket\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(-w_h\cdot n + |w_h\cdot n|\right)(u_h-g)\cdot v_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_D}\int_F (w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F (w_h\cdot n)(v_h\cdot u_h)\,\mathrm{d}s.
\end{aligned}
\tag{63}
$$

We note that

$$
\begin{aligned}
t_h(w_h,u_h,v_h) = {}& t_h^c(w_h,u_h,v_h) \\
&- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}(w_h\cdot n)(u_h\cdot v_h)\,\mathrm{d}s,
\end{aligned}
\tag{64}
$$

where the last two terms are exactly the terms (45) that were added to (44) due to $\nabla\cdot w_h \neq 0$ for stability and convergence.

We continue now with the trilinear form $t_h(w_h, u_h, v_h)$ given by (52). This means, that instead of (21), we consider now

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \int_K f\cdot v_h \,\mathrm{d}x \\
&\quad= \sum_{K\in\mathcal{T}} \int_K \left(v_h\cdot\partial_t u_h + \nu\nabla u_h : \nabla v_h - p_h \nabla\cdot v_h\right)\mathrm{d}x \\
&\quad+ \sum_{K\in\mathcal{T}} \int_K (w_h\cdot\nabla u_h)\cdot v_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_K \tfrac{1}{2}(\nabla\cdot w_h) u_h\cdot v_h \,\mathrm{d}x \\
&\quad- \sum_{F\in\mathcal{F}_I} \int_F \nu [\![u_h\otimes n]\!] : \{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I} \int_F \nu [\![v_h\otimes n]\!] : \{\!\{\nabla u_h\}\!\}\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D} \int_F \nu\,(u_h - g)\otimes n : \nabla v_h \,\mathrm{d}s - \sum_{F\in\mathcal{F}_D} \int_F \nu v_h\otimes n : \nabla u_h \,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I} \int_F \frac{\nu\alpha}{h} [\![u_h\otimes n]\!] : [\![v_h\otimes n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \frac{\nu\alpha}{h}(u_h - g)\cdot v_h \,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_I} \int_F w_h\cdot n [\![u_h]\!]\cdot\{\!\{v_h\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I} \int_F \tfrac{1}{2}|w_h\cdot n| [\![u_h]\!]\cdot[\![v_h]\!]\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2}\left(-w_h\cdot n + |w_h\cdot n|\right)(u_h - g)\cdot v_h \,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_N} \int_F v_h\otimes n : (u_h\otimes w - \nu\nabla u_h + p_h\mathbb{I})\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N} \int_F \tfrac{3}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned}
\tag{65}
$$

For the last two terms, we note that

$$
v_h\otimes n : u_h\otimes w - \tfrac{3}{2}\,(w_h\cdot n)\,(u_h\cdot v_h) = -\tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,. \tag{66}
$$

Using the Neumann boundary condition (5d) for the last integral of (65), we obtain the following discontinuous Galerkin weak formulation for the momentum equation (1a):

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K f\cdot v_h \,\mathrm{d}x \\
&\quad= \sum_{K\in\mathcal{T}}\int_K \left(v_h\cdot\partial_t u_h + \nu\nabla u_h : \nabla v_h - p_h\nabla\cdot v_h\right)\mathrm{d}x \\
&\quad+ \sum_{K\in\mathcal{T}}\int_K (w_h\cdot\nabla u_h)\cdot v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot w_h)u_h\cdot v_h\,\mathrm{d}x \\
&\quad- \sum_{F\in\mathcal{F}_I}\int_F \nu[\![u_h\otimes n]\!] : \{\!\{\nabla v_h\}\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_I}\int_F \nu[\![v_h\otimes n]\!] : \{\!\{\nabla u_h\}\!\}\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D}\int_F \nu\,(u_h-g)\otimes n : \nabla v_h\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F \nu v_h\otimes n : \nabla u_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_I}\int_F \frac{\nu\alpha}{h}[\![u_h\otimes n]\!] : [\![v_h\otimes n]\!]\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \frac{\nu\alpha}{h}(u_h-g)\cdot v_h\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_I}\int_F w_h\cdot n[\![u_h]\!]\cdot\{\!\{v_h\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|w_h\cdot n|[\![u_h]\!]\cdot[\![v_h]\!]\,\mathrm{d}s \\
&\quad- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(-w_h\cdot n + |w_h\cdot n|\right)(u_h-g)\cdot v_h\,\mathrm{d}s \\
&\quad+ \sum_{F\in\mathcal{F}_N}\int_F v_h\cdot\big(h + \max(w\cdot n,0)u_h\big)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{3}{2}\,(w_h\cdot n)\,(u_h\cdot v_h)\,\mathrm{d}s.
\end{aligned} \tag{67}
$$

A.3. Confirming energy stability. In this section we confirm energy stability of the weak formulation given in (67).

THEOREM 2. *If u_h solves (67) and (23) with homogeneous boundary conditions, then in the absence of forcing terms and for α large enough,*

$$\frac{d}{dt}\int_\Omega |u_h|^2\,\mathrm{d}x \le 0. \tag{68}$$

Proof. Setting $v_h = w_h = u_h$ in (67) gives

$$
\begin{aligned}
0 = &\sum_{K\in\mathcal{T}} \int_K \left(u_h \cdot \partial_t u_h + \nu \nabla u_h : \nabla u_h\right) \mathrm{d}x \\
&+ \sum_{K\in\mathcal{T}} \int_K (u_h \cdot \nabla u_h) \cdot u_h \, \mathrm{d}x + \sum_{K\in\mathcal{T}} \int_K \tfrac{1}{2} (\nabla \cdot u_h) u_h \cdot u_h \, \mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I} \int_F \nu [\![u_h \otimes n]\!] : \{\!\{\nabla u_h\}\!\} \, \mathrm{d}s - \sum_{F\in\mathcal{F}_I} \int_F \nu [\![u_h \otimes n]\!] : \{\!\{\nabla u_h\}\!\} \, \mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D} \int_F \nu u_h \otimes n : \nabla u_h \, \mathrm{d}s - \sum_{F\in\mathcal{F}_D} \int_F \nu u_h \otimes n : \nabla u_h \, \mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I} \int_F \frac{\nu\alpha}{h} [\![u_h \otimes n]\!] : [\![u_h \otimes n]\!] \, \mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \frac{\nu\alpha}{h} u_h \cdot u_h \, \mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_I} \int_F u_h \cdot n [\![u_h]\!] \cdot \{\!\{u_h\}\!\} \, \mathrm{d}s + \sum_{F\in\mathcal{F}_I} \int_F \tfrac{1}{2} |u_h \cdot n| [\![u_h]\!] \cdot [\![u_h]\!] \, \mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2} (u_h \cdot n)(u_h \cdot u_h) \, \mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2} \left(-u_h \cdot n + |u_h \cdot n|\right) u_h \cdot u_h \, \mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N} \int_F u_h \cdot u_h \max(u_h \cdot n, 0) \, \mathrm{d}s - \sum_{F\in\mathcal{F}_N} \int_F \tfrac{3}{2} (u_h \cdot n)(u_h \cdot u_h) \, \mathrm{d}s,
\end{aligned}
\tag{69}
$$

where we added $\sum_{K\in\mathcal{T}} \int_K p_h \nabla \cdot u_h \, \mathrm{d}x = 0$, which follows from (23) by taking $q_h = -p_h$. Simplifying,

$$
\begin{aligned}
0 = &\sum_{K\in\mathcal{T}} \int_K \left(\tfrac{1}{2} \partial_t |u_h|^2 + \nu |\nabla u_h|^2\right) \mathrm{d}x \\
&+ \sum_{K\in\mathcal{T}} \int_K (u_h \cdot \nabla u_h) \cdot u_h \, \mathrm{d}x + \sum_{K\in\mathcal{T}} \int_K \tfrac{1}{2} (\nabla \cdot u_h) |u_h|^2 \, \mathrm{d}x \\
&- \sum_{F\in\mathcal{F}_I} \int_F 2\nu [\![u_h \otimes n]\!] : \{\!\{\nabla u_h\}\!\} \, \mathrm{d}s - \sum_{F\in\mathcal{F}_D} \int_F 2\nu u_h \otimes n : \nabla u_h \, \mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I} \int_F \frac{\nu\alpha}{h} |[\![u_h \otimes n]\!]|^2 \, \mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \frac{\nu\alpha}{h} |u_h|^2 \, \mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_I} \int_F u_h \cdot n [\![u_h]\!] \cdot \{\!\{u_h\}\!\} \, \mathrm{d}s + \sum_{F\in\mathcal{F}_I} \int_F \tfrac{1}{2} |u_h \cdot n| |[\![u_h]\!]|^2 \, \mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2} (u_h \cdot n) |u_h|^2 \, \mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2} \left(-u_h \cdot n + |u_h \cdot n|\right) |u_h|^2 \, \mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N} \int_F |u_h|^2 \max(u_h \cdot n, 0) \, \mathrm{d}s - \sum_{F\in\mathcal{F}_N} \int_F \tfrac{3}{2} (u_h \cdot n) |u_h|^2 \, \mathrm{d}s.
\end{aligned}
\tag{70}
$$

Next, using that $(u_h \cdot \nabla u_h) \cdot u_h = \frac{1}{2} \nabla \cdot ((u_h \otimes u_h) \cdot u_h) - \frac{1}{2} (\nabla \cdot u_h) |u_h|^2$, we note that the second and third integral on the right hand side combine to

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K (u_h\cdot\nabla u_h)\cdot u_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot u_h)|u_h|^2\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}\nabla\cdot((u_h\otimes u_h)\cdot u_h)\,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}}\int_{\partial K} \tfrac{1}{2}u_h\cdot((u_h\otimes u_h)\cdot n)\,\mathrm{d}s \\
&= \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}\left(\{\!\!\{u_h\}\!\!\}\cdot[\![u_h u_h\cdot n]\!] + [\![u_h\otimes n]\!]\{\!\!\{u_h\otimes u_h\}\!\!\}\right)\mathrm{d}s + \sum_{F\in\mathcal{F}_B}\int_F \tfrac{1}{2}(u_h\cdot n)|u_h|^2\,\mathrm{d}s.
\end{aligned}
\tag{71}
$$

Using that $u_h\cdot n$ is continuous across interior facets, we see that

$$
\tfrac{1}{2}\left(\{\!\!\{u_h\}\!\!\}\cdot[\![u_h u_h\cdot n]\!] + [\![u_h\otimes n]\!]\{\!\!\{u_h\otimes u_h\}\!\!\}\right) = [\![u_h]\!]\cdot\{\!\!\{u_h\}\!\!\}(u_h\cdot n),
\tag{72}
$$

so that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K (u_h\cdot\nabla u_h)\cdot u_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \tfrac{1}{2}(\nabla\cdot u_h)|u_h|^2\,\mathrm{d}x = \\
&\sum_{F\in\mathcal{F}_I}\int_F [\![u_h]\!]\cdot\{\!\!\{u_h\}\!\!\}(u_h\cdot n)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}(u_h\cdot n)|u_h|^2\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}(u_h\cdot n)|u_h|^2\,\mathrm{d}s.
\end{aligned}
\tag{73}
$$

Substituting (73) into (70),

$$
\begin{aligned}
0 &= \sum_{K\in\mathcal{T}}\int_K \left(\tfrac{1}{2}\partial_t|u_h|^2 + \nu|\nabla u_h|^2\right)\mathrm{d}x \\
&+ \sum_{F\in\mathcal{F}_I}\int_F [\![u_h]\!]\cdot\{\!\!\{u_h\}\!\!\}(u_h\cdot n)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}(u_h\cdot n)|u_h|^2\,\mathrm{d}s + \sum_{F\in\mathcal{F}_N}\int_F \tfrac{1}{2}(u_h\cdot n)|u_h|^2\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_I}\int_F 2\nu[\![u_h\otimes n]\!] : \{\!\!\{\nabla u_h\}\!\!\}\,\mathrm{d}s - \sum_{F\in\mathcal{F}_D}\int_F 2\nu u_h\otimes n : \nabla u_h\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_I}\int_F \frac{\nu\alpha}{h}|[\![u_h\otimes n]\!]|^2\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \frac{\nu\alpha}{h}|u_h|^2\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_I}\int_F u_h\cdot n[\![u_h]\!]\cdot\{\!\!\{u_h\}\!\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_I}\int_F \tfrac{1}{2}|u_h\cdot n||[\![u_h]\!]|^2\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(u_h\cdot n\right)|u_h|^2\,\mathrm{d}s + \sum_{F\in\mathcal{F}_D}\int_F \tfrac{1}{2}\left(-u_h\cdot n + |u_h\cdot n|\right)|u_h|^2\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_N}\int_F |u_h|^2\max(u_h\cdot n, 0)\,\mathrm{d}s - \sum_{F\in\mathcal{F}_N}\int_F \tfrac{3}{2}\left(u_h\cdot n\right)|u_h|^2\,\mathrm{d}s.
\end{aligned}
\tag{74}
$$

Cancelling terms,

$$
\begin{aligned}
0 = & \sum_{K\in\mathcal{T}} \int_K \left(\tfrac{1}{2}\partial_t |u_h|^2 + \nu|\nabla u_h|^2\right) \mathrm{d}x \\
& + \sum_{F\in\mathcal{F}_I} \int_F \left(\frac{\nu\alpha}{h} |[\![u_h \otimes n]\!]|^2 - 2\nu [\![u_h \otimes n]\!] : \{\!\{\nabla u_h\}\!\} \,\mathrm{d}s\right) \\
& + \sum_{F\in\mathcal{F}_D} \int_F \left(\frac{\nu\alpha}{h} |u_h|^2 - 2\nu u_h \otimes n : \nabla u_h\right) \mathrm{d}s \\
& + \sum_{F\in\mathcal{F}_I} \int_F \tfrac{1}{2} |u_h\cdot n| |[\![u_h]\!]|^2 \,\mathrm{d}s + \sum_{F\in\mathcal{F}_D} \int_F \tfrac{1}{2}\left(-u_h\cdot n + |u_h\cdot n|\right) |u_h|^2 \,\mathrm{d}s \\
& + \sum_{F\in\mathcal{F}_N} \int_F |u_h|^2 \left(\max(u_h\cdot n, 0) - (u_h\cdot n)\right) \mathrm{d}s.
\end{aligned} \tag{75}
$$

Note that on Γ_D, $\frac{1}{2}\left(-u_h\cdot n + |u_h\cdot n|\right) = |u_h\cdot n|$ if $u_h\cdot n < 0$ and $\frac{1}{2}\left(-u_h\cdot n + |u_h\cdot n|\right) = 0$ if $u_h\cdot n \geq 0$. Furthermore, on Γ_N, $(\max(u_h\cdot n, 0) - (u_h\cdot n)) = |u_h\cdot n|$ if $u_h\cdot n < 0$ and $(\max(u_h\cdot n, 0) - (u_h\cdot n)) = 0$ if $u_h\cdot n \geq 0$. For α large enough, also the second and third integrals on the right hand side are positive. We conclude, that since all terms are positive, the result (68) must hold. ∎