Extended Proofs for "Relay Pursuit for Multirobot Target Tracking on Tile Graphs"

Shashwata Mandal[1] and Sourabh Bhattacharya[1,2]

The document provides a detailed proof of properties of tile graphs, tile cover and minimal tile cover used in the submitted version of ICRA 2023 paper titled "Relay Pursuit for Multirobot Target Tracking on Tile Graphs".

A. *Minimal Tile Cover Computation*

Given an environment, a Minimal Tile Cover (MTC) is a tile cover of minimum cardinality.

MTCP. *An Apriori Structure is defined as a set of sets $W = \{T_1, ..., T_u\}$ where $T_i \subseteq Q$ where $Q = \{c_1, \ldots, c_n\}$. Given a Apriori Structure W and corners Q, what is the minimum size of a disjoint cover F such that $x \in F \wedge x \subseteq T_p \wedge T_p \in W$ and $Q = \bigcup_{y \in F} y$?*

Theorem 1. *MTCP is NP-Hard.*

Set Cover. *Given a universe of elements $U = \{e_1, \ldots, e_m\}$ and a set of sets $\mathcal{S} = \{S_1, \ldots, S_t\}$ where $S_i \subseteq U$, what is the minimum size of a set $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{S}$ and $U = \bigcup_{j \in \mathcal{C}} j$?*

Proof. Minimal Tile Cover Problem (MTCP) can be shown to be NP-Hard by reducing an instance of Set Cover (a known NP-Hard problem [1]) to it. An instance of Set Cover can be represented as $\langle U, \mathcal{S} \rangle$ and an instance of MTCP can be represented as $\langle Q, W \rangle$. To map an instance of Set Cover to an instance of MTCP, $U \mapsto Q$ since both describe the universe of elements. The Apriori Structure W is also a set of sets similar to $\mathcal{S}$. So $\mathcal{S} \mapsto W$. This can be done in polynomial time. For the reverse mapping, when we get a MTCP G (set of disjoint sets), the sets in G can be traced back to its superset in W. These sets are a part of $\mathcal{S}$ and this operation doesn't change the size of the cover and is also in polynomial time. Since, G covers all elements in Q and each of these elements can be mapped to a set in $\mathcal{S}$, we can guarantee that the resulting set cover is the minimum set cover. Hence, $\langle U, \mathcal{S} \rangle \mapsto \langle Q, W \rangle$. Since Set Cover is a known NP-Hard problem, MTCP has to be NP-Hard. ■

Algorithm 3 uses a greedy strategy to generate a tile cover on an Apriori structure. Since higher levels of the Apriori structure combine more corners compared to lower levels, the algorithm greedily iterates from the top to the bottom of the Apriori structure. In Algorithm 3, U denotes the set of tiles that form the tile cover. At each level of the apriori structure, we add a set of disjoint tiles to U that maximizes its cardinality. This is done as follows. F denotes the set of tiles at a given level that does not have any corners already present in U (Line 5). Next, we construct a graph $G = (V, E)$ such that V is a set of tiles in F. An edge exists between two vertices if the corresponding tiles have a non-empty intersection (Line 9). Initially, all vertices are marked as *valid*. Next, vertices of G are scanned in the increasing order of their degree. At any stage, a vertex marked valid is added to U, and its neighbours are marked *invalid* before removing the vertex from G. This process continues till either G is empty or all vertices of G are marked invalid.

Algorithm 3 Algorithm for Tile Cover

Input: k Height of structure, L Set of items set till k
Output: U Tile Cover

1: **function** MINIMIZE TILES(L, k)
2: $U \leftarrow \phi$
3: **while** $k \geq 0$ **do**
4: **for** $i \in L_k$ **do**
5: **if** $\forall x \in i, \nexists y \in U \wedge x \notin y$ **then**
6: Add i to F
7: **end if**
8: **end for**
9: Create a graph $G = (V, E)$ s.t. $V = \{v | \forall v \in F\}$ and $E = \{(u, v) | u, v \in F \wedge u \cap v \neq \phi\}$.
10: $j \leftarrow 0$
11: **while** *valid* nodes remain in G **do**
12: Add nodes with degree j to U and set linked nodes to *invalid*
13: $j \leftarrow j + 1$
14: **end while**
15: $k \leftarrow k - 1$
16: **end while**
17: **return** U
18: **end function**

B. *Tight bound on the number of guards and their speed*

Refer to Figure 1. Let $O = \{o_1, ..., o_k\}$ be the number of observers such that $k < m$, where m is the cardinality of the minimum tile cover. Let us assume that there exists a speed ratio $r > 1$ for k observers to track an evader e. Let v_p and v_e be the speed of the observers and evader respectively such that $v_e = r v_p$. The polygon in Figure 1 has m fringes each separated by arbitrary distances. Each fringe induces a star tile. If $k = m$, the whole polygon would be covered. However

[1]Department of Computer Science, Iowa State University, Ames, IA 50010, USA, [2]Department of Mechanical Engineering, Iowa State University, Ames, IA 50010, USA smandal@iastate.edu, sbhattac@iastate.edu

$k < m$, so there aren't enough tiles to cover all the fringes. Let the width of these fringes be some very really small finite value. Let us consider the following trajectory - the evader starts at the left-most fringe and travels to the right-most fringe trying to evade at each fringe. Let the fringes be labeled 1 to m. We can see that a guard needs to be placed at the tile associated with the fringe to prevent the escape of the guard. If a guard is covering a fringe, since the guard is slower than the evader, the same guard cannot keep up with the evader till it tries to cover the neighboring fringe. If the guard is currently at the j numbered fringe, we can guarantee that j guards have already been used since the evader would have escaped. So the number of remaining guards is $k - t$ which is less than the remaining number of fringes. So at the k-th fringe, the evader is guaranteed to escape. Hence, we can say the evader cannot be tracked for any guard speed less than v_e no matter what the policy.

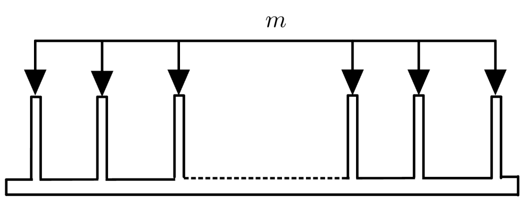

Fig. 1: A "comb" environment

References

[1] R. M. Karp, *Reducibility among Combinatorial Problems*. Boston, MA: Springer US, 1972, pp. 85–103. [Online]. Available: https://doi.org/10.1007/978-1-4684-2001-2$_9$