Extended Proofs for “Relay Pursuit for Multirobot Target Tracking on Tile Graphs”

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The document provides a detailed proof of properties of tile graphs, tile cover and minimal tile cover used in the submitted version of ICRA 2023 paper titled “Relay Pursuit for Multirobot Target Tracking on Tile Graphs”.

A. Minimal Tile Cover Computation

Given an environment, a Minimal Tile Cover (MTC) is a tile cover of minimum cardinality.

MTCP. *An Apriori Structure is defined as a set of sets $W = \{T_1, \ldots, T_u\}$ where $T_i \subseteq Q$ where $Q = \{c_1, \ldots, c_n\}$. Given a Apriori Structure W and corners Q, what is the minimum size of a disjoint cover F such that $x \in F \wedge x \subseteq T_p \wedge T_p \in W$ and $Q = \bigcup_{y \in F} y$?*

Theorem 1. *MTCP is NP-Hard.*

Set Cover. *Given a universe of elements $U = \{e_1, \ldots, e_m\}$ and a set of sets $\mathcal{S} = \{S_1, \ldots, S_t\}$ where $S_i \subseteq U$, what is the minimum size of a set $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{S}$ and $U = \bigcup_{j \in \mathcal{C}} j$?*

Proof. Minimal Tile Cover Problem (MTCP) can be shown to be NP-Hard by reducing an instance of Set Cover (a known NP-Hard problem [1]) to it. An instance of Set Cover can be represented as $\langle U, \mathcal{S} \rangle$ and an instance of MTCP can be represented as $\langle Q, W \rangle$. To map an instance of Set Cover to an instance of MTCP, $U \mapsto Q$ since both describe the universe of elements. The Apriori Structure W is also a set of sets similar to $\mathcal{S}$. So $\mathcal{S} \mapsto W$. This can be done in polynomial time. For the reverse mapping, when we get a MTCP G (set of disjoint sets), the sets in G can be traced back to its superset in W. These sets are a part of $\mathcal{S}$ and this operation doesn’t change the size of the cover and is also in polynomial time. Since, G covers all elements in Q and each of these elements can be mapped to a set in $\mathcal{S}$, we can guarantee that the resulting set cover is the minimum set cover. Hence, $\langle U, \mathcal{S} \rangle \mapsto \langle Q, W \rangle$. Since Set Cover is a known NP-Hard problem, MTCP has to be NP-Hard. ∎

Algorithm 3 uses a greedy strategy to generate a tile cover on an Apriori structure. Since higher levels of the Apriori structure combine more corners compared to lower levels, the algorithm greedily iterates from the top to the bottom of the Apriori structure. In Algorithm 3, U denotes the set of tiles that form the tile cover. At each level of the apriori structure, we add a set of disjoint tiles to U that maximizes its cardinality. This is done as follows. F denotes the set of tiles at a given level that does not have any corners already present in U (Line 5). Next, we construct a graph $G = (V, E)$ such that V is a set of tiles in F. An edge exists between two vertices if the corresponding tiles have a non-empty intersection (Line 9). Initially, all vertices are marked as *valid*. Next, vertices of G are scanned in the increasing order of their degree. At any stage, a vertex marked valid is added to U, and its neighbours are marked *invalid* before removing the vertex from G. This process continues till either G is empty or all vertices of G are marked invalid.

Algorithm 3 Algorithm for Tile Cover

Input: k Height of structure, L Set of items set till k
Output: U Tile Cover

1: **function** MINIMIZE TILES(L, k)
2: $U \leftarrow \phi$
3: **while** $k \geq 0$ **do**
4: **for** $i \in L_k$ **do**
5: **if** $\forall x \in i, \nexists y \in U \wedge x \notin y$ **then**
6: Add i to F
7: **end if**
8: **end for**
9: Create a graph $G = (V, E)$ s.t. $V = \{v | \forall v \in F\}$ and $E = \{(u, v) | u, v \in F \wedge u \cap v \neq \phi\}$.
10: $j \leftarrow 0$
11: **while** *valid* nodes remain in G **do**
12: Add nodes with degree j to U and set linked nodes to *invalid*
13: $j \leftarrow j + 1$
14: **end while**
15: $k \leftarrow k - 1$
16: **end while**
17: **return** U
18: **end function**

B. Tight bound on the number of guards and their speed

Let $O = \{o_1, \ldots, o_k\}$ be the number of observers such that $k < m$, where m is the cardinality of the minimum tile cover. Let us assume that there exists a speed ratio $r > 1$ for k observers to track an evader e. Let v_p and v_e be the speed of the observers and evader respectively such that $v_e = r v_p$. The polygon in Figure 1 has m fringes each separated by arbitrary distances. Each fringe induces a star tile. If $k = m$, the whole polygon would be covered. However $k < m$, so there aren’t

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enough tiles to cover all the fringes. Let t be the width of these fringes and w be the width of the hallways. Let the extensions of the left and right columns of the fringe be L_1 and L_2 respectively. The field of visibility induced by the corner at L_2 and hallway is dependent on the ratio $\frac{t}{w}$. If the ratio $\frac{t}{w}$ becomes very large, the field of visibility becomes very small such that it almost coincides with the line L_2. We show how the local properties of each fringe allows the evader to always have an escape strategy in the global setting if $v_p < v_e$. Our strategy for the evader always starts from left of each fringe as it travels to the right of the fringe along the upper edge of the hallway. As the evader reaches L_2, a guard is necessary to be present at L_2 or to the left of it since the evader can decide to escape by following L_2 inside the fringe. Since the guard has to be at L_2 or behind it, the evader can travel to the next fringe and this guard cannot keep up with the evader since $v_p < v_e$. So we see that after each fringe a guard is left behind by the evader so after j fringes, j guards are behind the evader i.e. $k - j$ guards are remaining to guard the rest of the fringes. Then after k fringes, there are no more guards left since $k < m$ hence the evader can escape. Therefore, we can say the evader cannot be tracked for any guard speed less than v_e no matter what the policy.

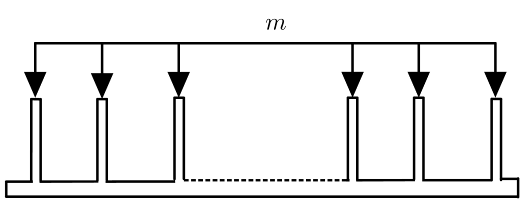

Fig. 1: A "comb" environment

References

[1] R. M. Karp, *Reducibility among Combinatorial Problems*. Boston, MA: Springer US, 1972, pp. 85–103. [Online]. Available: https://doi.org/10.1007/978-1-4684-2001-2$_9$