Extended Proofs for "Relay Pursuit for Multirobot Target Tracking on Tile Graphs"

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The document provides a detailed proof of properties of tile graphs, tile cover and minimal tile cover used in the submitted version of ICRA 2023 paper titled "Relay Pursuit for Multirobot Target Tracking on Tile Graphs".

A. Minimal Tile Cover Computation

Given an environment, a Minimal Tile Cover (MTC) is a tile cover of minimum cardinality.

MTCP. *An Apriori Structure is defined as a set of sets* $W = \{T_1, ..., T_u\}$ *where* $T_i \subseteq Q$ *where* $Q = \{c_1, \ldots, c_n\}$. *Given a Apriori Structure* W *and corners* Q, *what is the minimum size of a disjoint cover* F *such that* $x \in F \wedge x \subseteq T_p \wedge T_p \in W$ *and* $Q = \bigcup_{y \in F} y$?

Theorem 1. *MTCP is NP-Hard.*

Set Cover. *Given a universe of elements* $U = \{e_1, \ldots, e_m\}$ *and a set of sets* $\mathcal{S} = \{S_1, \ldots, S_t\}$ *where* $S_i \subseteq U$, *what is the minimum size of a set* $\mathcal{C}$ *such that* $\mathcal{C} \subseteq \mathcal{S}$ *and* $U = \bigcup_{j \in \mathcal{C}} j$?

Proof. Minimal Tile Cover Problem (MTCP) can be shown to be NP-Hard by reducing an instance of Set Cover (a known NP-Hard problem [1]) to it. An instance of Set Cover can be represented as $\langle U, \mathcal{S} \rangle$ and an instance of MTCP can be represented as $\langle Q, W \rangle$. To map an instance of Set Cover to an instance of MTCP, $U \mapsto Q$ since both describe the universe of elements. The Apriori Structure W is also a set of sets similar to $\mathcal{S}$. So $\mathcal{S} \mapsto W$. This can be done in polynomial time. For the reverse mapping, when we get a MTCP G (set of disjoint sets), the sets in G can be traced back to its superset in W. These sets are a part of $\mathcal{S}$ and this operation doesn't change the size of the cover and is also in polynomial time. Since, G covers all elements in Q and each of these elements can be mapped to a set in $\mathcal{S}$, we can guarantee that the resulting set cover is the minimum set cover. Hence, $\langle U, \mathcal{S} \rangle \mapsto \langle Q, W \rangle$. Since Set Cover is a known NP-Hard problem, MTCP has to be NP-Hard. ∎

Algorithm 3 uses a greedy strategy to generate a tile cover on an Apriori structure. Since higher levels of the Apriori structure combine more corners compared to lower levels, the algorithm greedily iterates from the top to the bottom of the Apriori structure. In Algorithm 3, U denotes the set of tiles that form the tile cover. At each level of the apriori structure, we add a set of disjoint tiles to U that maximizes its cardinality. This is done as follows. F denotes the set of tiles at a given level that does not have any corners already present in U (Line 5). Next, we construct a graph $G = (V, E)$ such that V is a set of tiles in F. An edge exists between two vertices if the corresponding tiles have a non-empty intersection (Line 9). Initially, all vertices are marked as *valid*. Next, vertices of G are scanned in the increasing order of their degree. At any stage, a vertex marked valid is added to U, and its neighbours are marked *invalid* before removing the vertex from G. This process continues till either G is empty or all vertices of G are marked invalid.

Algorithm 3 Algorithm for Tile Cover

Input: k Height of structure, L Set of items set till k
Output: U Tile Cover

1: **function** MINIMIZE TILES(L, k)
2: $U \leftarrow \phi$
3: **while** $k \geq 0$ **do**
4: **for** $i \in L_k$ **do**
5: **if** $\forall x \in i, \nexists y \in U \wedge x \notin y$ **then**
6: Add i to F
7: **end if**
8: **end for**
9: Create a graph $G = (V, E)$ s.t. $V = \{v | \forall v \in F\}$ and $E = \{(u, v) | u, v \in F \wedge u \cap v \neq \phi\}$.
10: $j \leftarrow 0$
11: **while** *valid* nodes remain in G **do**
12: Add nodes with degree j to U and set linked nodes to *invalid*
13: $j \leftarrow j + 1$
14: **end while**
15: $k \leftarrow k - 1$
16: **end while**
17: **return** U
18: **end function**

B. Tight bound on the number of guards and their speed

Refer to Figure 1. The polygon has a tile cover containing m tiles. If m guards are placed in each tile, the intruder cannot escape the line-of-sight of the team of guards since the entire polygon is covered. Next, we show that $m - 1$ guards need a minimum speed equal to that of the intruder to track it.

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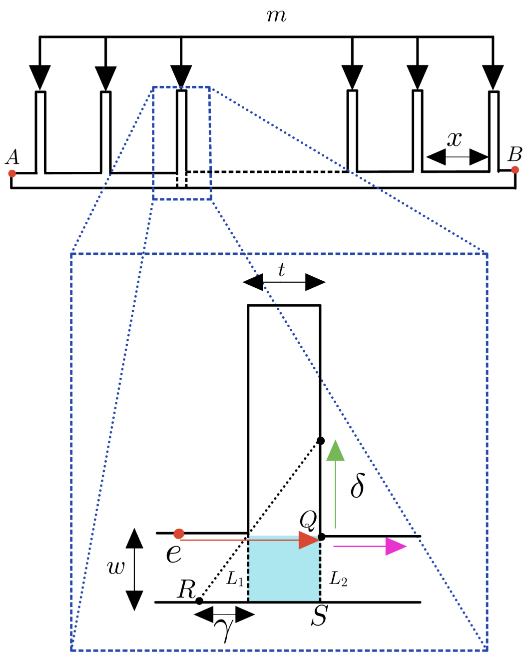

Fig. 1: A "comb" environment

Refer to Figure 1. It shows an environment wherein m rectangular rooms lie on one side of a long horizontal corridor. Let x denote the distance between two neighboring rooms. Let t and w denote the width of each room and corridor, respectively. Consider a straight line path from A to B that grazes the walls of the corridor along which the rooms lie. As the intruder moves along the path at its maximum speed, it passes through the corners at the entrance of the rooms which provide an opportunity to break the line-of-sight with the guards. Figure 1 shows such an instance. For sufficiently small $\frac{w}{x}$ and wt (exact threshold depend on the value of r), the intruder can ensure that no guard crosses l from the left as the intruder crosses D if it takes a sharp left turn at c_2. Therefore, for very small values of w, an additional guard has to be initially located either to the left of L_2 or very close to L_2 if it lies on the right, to prevent the intruder from breaking the line-of-sight around c_2.

Let $r = \frac{v_e}{v_p} > 1$, where v_p and v_e are the maximum speed of the guards and intruder, respectively. and tries to escape via L_2 at Q of every tooth (green arrow) if an escape is possible. $\gamma = \frac{tw}{\delta}$ is the maximum distance from the line from L_1 at which the guard can still see the intruder when it is located at **name of the point**. Since $\gamma \propto tw$ if we reduce tw we can reduce the field of visibility of the evader from the hallway. If the intruder tries to escape via L_2, no guard can see the intruder right of the line L_2 and has to be within γ distance left of L_1 to see the intruder. If $tw \lll \delta$, R is very close to L_1. At each tooth, the intruder decides at point Q whether it can escape via L_2, so a guard has to be between R and S to prevent the escape. If the guard is at L_2 or behind it, the intruder can leave the guard and move towards the next tooth (pink arrow) since $v_e > v_p$. So the same guard cannot be responsible for tracking the evader in the next tooth. If a guard from a previous tooth wanted to reach R then it would need to travel from L_2 of the previous tooth to R of the current tooth when evader travels form Q of the previous tooth to the Q of the current tooth. The minimum value of x to prevent this can be solved by

$$
\begin{aligned}
r &= \frac{x+t}{x-\gamma} \\
\Longrightarrow x &= \frac{t+r\gamma}{r-1}
\end{aligned}
\tag{1}
$$

Then if $x > \frac{t+r\gamma}{r-1}$, the previous guard cannot keep up with the evader. So we can show that at each tooth, any guard behind it cannot interfere with its escape around the current tooth if $tw \lll \delta$ and $x > \frac{t+r\gamma}{r-1}$. After each tooth, a guard is left behind by the evader so, after j teeth, j guards are behind the evader i.e. $k-j$ guards are remaining to guard the rest of the teeth. Then after k teeth, there are no more guards left since $k < m$ hence the evader can escape. Therefore, we can say the evader cannot be tracked for any guard speed less than v_e no matter what the policy.

References

[1] R. M. Karp, *Reducibility among Combinatorial Problems*. Boston, MA: Springer US, 1972, pp. 85–103. [Online]. Available: https://doi.org/10.1007/978-1-4684-2001-2$_9$