Extended Proofs for “Relay Pursuit for Multirobot Target Tracking on Tile Graphs”

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The document provides a detailed proof of properties of tile graphs, tile cover and minimal tile cover used in the submitted version of ICRA 2023 paper titled “Relay Pursuit for Multirobot Target Tracking on Tile Graphs”.

A. Minimal Tile Cover Computation

Given an environment, a Minimal Tile Cover (MTC) is a tile cover of minimum cardinality.

MTCP. *An Apriori Structure is defined as a set of sets $W = \{T_1, ..., T_u\}$ where $T_i \subseteq Q$ where $Q = \{c_1, \ldots, c_n\}$. Given a Apriori Structure W and corners Q, what is the minimum size of a disjoint cover F such that $x \in F \wedge x \subseteq T_p \wedge T_p \in W$ and $Q = \bigcup_{y \in F} y$?*

Theorem 1. *MTCP is NP-Hard.*

Set Cover. *Given a universe of elements $U = \{e_1, \ldots, e_m\}$ and a set of sets $\mathcal{S} = \{S_1, \ldots, S_t\}$ where $S_i \subseteq U$, what is the minimum size of a set $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{S}$ and $U = \bigcup_{j \in \mathcal{C}} j$?*

Proof. Minimal Tile Cover Problem (MTCP) can be shown to be NP-Hard by reducing an instance of Set Cover (a known NP-Hard problem [1]) to it. An instance of Set Cover can be represented as $\langle U, \mathcal{S} \rangle$ and an instance of MTCP can be represented as $\langle Q, W \rangle$. To map an instance of Set Cover to an instance of MTCP, $U \mapsto Q$ since both describe the universe of elements. The Apriori Structure W is also a set of sets similar to $\mathcal{S}$. So $\mathcal{S} \mapsto W$. This can be done in polynomial time. For the reverse mapping, when we get a MTCP G (set of disjoint sets), the sets in G can be traced back to its superset in W. These sets are a part of $\mathcal{S}$ and this operation doesn’t change the size of the cover and is also in polynomial time. Since, G covers all elements in Q and each of these elements can be mapped to a set in $\mathcal{S}$, we can guarantee that the resulting set cover is the minimum set cover. Hence, $\langle U, \mathcal{S} \rangle \mapsto \langle Q, W \rangle$. Since Set Cover is a known NP-Hard problem, MTCP has to be NP-Hard. ∎

Algorithm 3 uses a greedy strategy to generate a tile cover on an Apriori structure. Since higher levels of the Apriori structure combine more corners compared to lower levels, the algorithm greedily iterates from the top to the bottom of the Apriori structure. In Algorithm 3, U denotes the set of tiles that form the tile cover. At each level of the apriori structure, we add a set of disjoint tiles to U that maximizes its cardinality. This is done as follows. F denotes the set of tiles at a given level that does not have any corners already present in U (Line 5). Next, we construct a graph $G = (V, E)$ such that V is a set of tiles in F. An edge exists between two vertices if the corresponding tiles have a non-empty intersection (Line 9). Initially, all vertices are marked as *valid*. Next, vertices of G are scanned in the increasing order of their degree. At any stage, a vertex marked valid is added to U, and its neighbours are marked *invalid* before removing the vertex from G. This process continues till either G is empty or all vertices of G are marked invalid.

Algorithm 3 Algorithm for Tile Cover

Input: k Height of structure, L Set of items set till k
Output: U Tile Cover

1: **function** MINIMIZE TILES(L, k)
2: $U \leftarrow \phi$
3: **while** $k \geq 0$ **do**
4: **for** $i \in L_k$ **do**
5: **if** $\forall x \in i, \nexists y \in U \wedge x \notin y$ **then**
6: Add i to F
7: **end if**
8: **end for**
9: Create a graph $G = (V, E)$ s.t. $V = \{v | \forall v \in F\}$ and $E = \{(u, v) | u, v \in F \wedge u \cap v \neq \phi\}$.
10: $j \leftarrow 0$
11: **while** *valid* nodes remain in G **do**
12: Add nodes with degree j to U and set linked nodes to *invalid*
13: $j \leftarrow j + 1$
14: **end while**
15: $k \leftarrow k - 1$
16: **end while**
17: **return** U
18: **end function**

B. Tight bound on the number of guards and their speed

Refer to Figure 1. It shows an environment wherein m equi-spaced rectangular rooms lie on one side of a long horizontal corridor. The polygon has a tile cover containing m tiles. If m guards are placed in each tile, the intruder cannot escape the line-of-sight of the team of guards since the entire polygon is covered.

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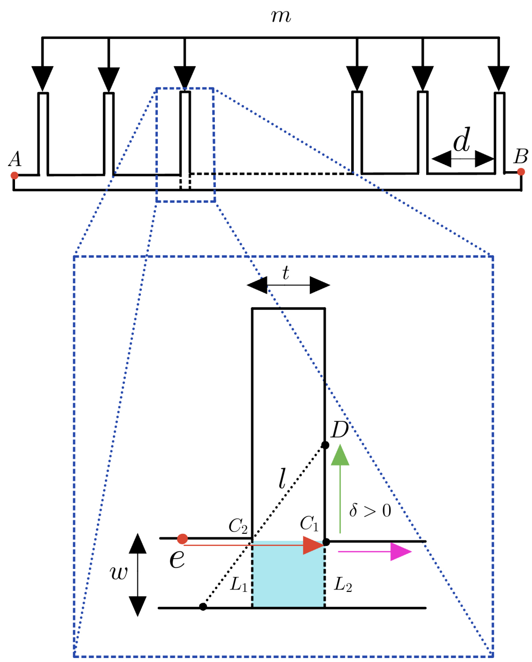

Fig. 1: A "comb" environment

Next, we show that $m-1$ guards need a minimum speed equal to that of the intruder to track it in this environment. Consider a straight line path from A to B that grazes the walls of the corridor along which the rooms lie. As the intruder moves along the path at its maximum speed (which is assumed to more than the maximum speed of the guards), it passes through the corners at the entrance of the rooms which provide an opportunity to break the line-of-sight with the guards. Figure 1 shows such an instance. For sufficiently small $\frac{w}{d}$ and $w*t$ (exact threshold depend on the value of r), the intruder can ensure that no guard crosses l from the left if the intruder crosses D after taking a sharp left turn at c_2 (since the maximum speed of the intruder is greater than the maximum speed of the guards). Therefore, for very small values of w, an additional guard has to be located either to the left of L_2 (in the shaded tile) or very close to L_2 on its right when the intruder is at c_2. As before, for sufficiently small $\frac{w}{d}$ and $w*t$, the intruder can ensure that the additional guard (and guards behind it) is unable to secure the corners associated with the next room along the corridor. Therefore, irrespective of the initial placement of the guards, we need to deploy one guard for each room in order to prevent the intruder from breaking the line-of-sight around the corners associated with the room when the maximum speed of the guards is less than the maximum speed of the intruder. Therefore, if $m-1$ guards are deployed to track the intruder, their maximum speed has to be at least equal to the maximum speed of the intruder for tracking.

REFERENCES

[1] R. M. Karp, *Reducibility among Combinatorial Problems*. Boston, MA: Springer US, 1972, pp. 85–103. [Online]. Available: https://doi.org/10.1007/978-1-4684-2001-2_9