Math for Sling Shot Simulation

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Notation:

$$
\begin{aligned}
G &\equiv \text{Gravitational constant} \\
M &\equiv \text{Sun's mass} \\
m &\equiv \text{Planet's mass} \\
R &\equiv \text{Planet's orbital radius} \\
\omega &\equiv \text{Planet's orbital angular velocity} \\
\phi &\equiv \text{Planet's phase} \\
v &\equiv \text{Speed of planet along its orbit} \\
v_e &\equiv \text{Escape velocity of planet}
\end{aligned}
$$

Basic facts:

$$
\begin{aligned}
\text{Gravitational attraction between sun and planet} &= \frac{GMm}{R^2}. \\
\text{Acceleration of the planet due to the sun} &= \frac{GM}{R^2}. \\
\text{Centripetal acceleration of the planet} &= \omega^2 R.
\end{aligned}
$$

The centripetal acceleration of the planet must be equal to the acceleration due to the sun, and so

$$\omega^2 R = \frac{GM}{R^2}$$

from which it follows that

$$\omega = \sqrt{\frac{GM}{R^3}}$$

and

$$
\begin{aligned}
v &= \omega R \\
&= \sqrt{\frac{GM}{R}}.
\end{aligned}
$$

The *phase* ϕ of a planet is its angular position at time 0.

$$
\begin{aligned}
\text{Position of planet at time } t \text{ in the } XY \text{ plane} &= (R\cos(\phi + \omega\, t),\ R\sin(\phi + \omega\, t)), \\
\text{Velocity of planet at time } t \text{ in the } XY \text{ plane} &= (-R\omega\,\sin(\phi + \omega\, t),\ R\omega\,\cos(\phi + \omega\, t)).
\end{aligned}
$$

The equation of an orbit with eccentricity e is

$$r = R\,\frac{1+e}{1+e\cos\theta}.$$

If $e = 0$, the orbit is a circle with radius R.

At perihelion (closest to sun), $\theta = 0$ and

$$\begin{aligned} r &= R\,\frac{1+e}{1+e\cos 0} \\ &= R. \end{aligned}$$

At aphelion (furthest from sun), $\theta = \pi$ and

$$\begin{aligned} r &= R\,\frac{1+e}{1+e\cos \pi} \\ &= R\,\frac{1+e}{1-e}. \end{aligned}$$

A satellite is at distance R from the sun. The velocity for a circular orbit is v, as given above. Suppose, instead, that it is launched with velocity $u \geq v$. It enters an orbit with eccentricity

$$e = \left(\frac{u}{v}\right)^2 - 1.$$

The orbit is $\begin{cases} \text{circular} & \text{if } e = 0, \\ \text{elliptic} & \text{if } 0 < e < 1, \\ \text{parabolic} & \text{if } e = 1, \\ \text{hyperbolic} & \text{if } e > 1. \end{cases}$

If $u = \sqrt{2}v$, then $e = 1$, and the satellite leaves the system. In general, at aphelion, the distance from the sun is

$$\begin{aligned} R_a &= R\,\frac{1+e}{1-e} \\ &= R\,\frac{(u/v)^2}{2-(u/v)^2} \\ &= \frac{Ru^2}{2v^2-u^2}. \end{aligned}$$

Conversely, if we wish to achieve a distance R_a from the sun, the launch velocity must be

$$u = v\sqrt{\frac{2R_a}{R_a+R}}.$$

For escape velocity:

$$\begin{aligned} \text{Kinetic energy of an object of mass } m_o \text{ moving with velocity } v_o &= \tfrac{1}{2}m_o v_o^2. \\ \text{Gravitational binding energy at surface of planet} &= \frac{Gmm_o}{r}. \end{aligned}$$

At escape velocity v_e, these two energies must balance:

$$\tfrac{1}{2}m_o v_e^2 = \frac{Gmm_o}{r}$$

and so

$$v_e = \sqrt{\frac{2Gm}{r}}.$$