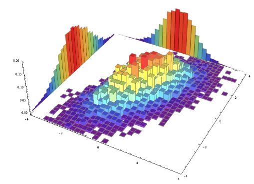
Kodiak Economic Simulations

SmolQuants

November 2023



Contents

1 Introduction

Kodiak is a concentrated liquidity AMM with vaults to be launched on Berachain.

SmolQuants was engaged by Kodiak for their V1 to:

- Suggest a positive expected value strategy (+EV) for their V1 concentrated liquidity vaults, with no external hedging.
- Backtest the strategy using historical data from Uniswap V3.

2 Summary

- The proposed strategy consists of providing liquidity over a fixed time period τ at a fixed half tick width Δ, and rebalancing around the current pool tick once this time period elapses.
- Yields an optimization problem where the LP must balance gains from a larger share of pro-rata swap fees with potential losses due to a higher probability of inactive liquidity and larger rebalancing costs. i.e. Maximize with respect to half tick width Δ the value of the LP position after the next rebalance swap:

$$\mathbb{E}_0[V_\tau(\Delta)]$$

- Provided script `optimize.py` uses `scipy` to determine the solution numerically, given the complexity of full expressions derived in the appendices of the supplementary note.
- Ignoring drift, +EV solutions appear to exist when the Uniswap V2 limit of infinite tick width is also +EV. To first order assuming GBM, this occurs when the fee volume per unit of pool virtual liquidity θ outside of the LP's own physical contribution l satisfies

$$\theta > \frac{\sigma^2}{8}(l+1)$$

- Suggested vault strategy is then relatively simple. LP over the full tick range when fee volume θ is less than the bound $(\sigma^2/8)(l+1)$ to minimize impermanent loss (IL). However, when θ exceeds the bound, concentrate liquidity around the critical half tick width Δ_c calculated by `optimize.py` that solves

$$0 = \partial_\Delta \mathbb{E}_0[V_\tau]|_{\Delta=\Delta_c}$$

- Using the Ethereum mainnet Uniswap V3 USDC/ETH 5 bps pool as an example, a tick width of about +/- 15% around the current price appears optimal under GBM assuming a 7 day rebalance period, LP providing \$10M of physical liquidity to the pool with existing virtual liquidity of about \$1.6B, and 24h fees of approximately \$500K. Expected yield would be close to 37 bps over the 7 days.
- The strategy was backtested assuming a 1 day rebalance period from block 16219692 (Dec-19-2022) to 17267692 (May-15-2023) on the mainnet Uniswap V3 USDC/ETH 5bps pool with ETH as the quote currency. Yields underperformed by 5.2% relative to simply LPing over the full tick range due to significant price shocks while concentrating the LP position's tick width, which caused more extreme realized IL at the end of the rebalance period.
- The most significant negative changes to strategy yield relative to full range LPing happened after each of these extreme price shocks. Coupled with the small probability of such shocks occurring under GBM gives credibility to the argument that the vault strategy is underperforming due to mismodeled price behavior (i.e. no tails modeled) when calculating the optimal tick width to LP over.
- Future work should consider a stochastic process with discontinuous jumps to more realistically model price behavior. The strategy is also hyper-sensitive to expected fee volumes over the next rebalance period, calibrated over the prior period (used as a constant) which isn't accurate. Modeling volumes would be very helpful for further improving the strategy.
- Alternatively, to ensure +EV LPing for most vaults, vault managers can supplement fee revenues with token incentives aiming for a reward yield greater than

$$R \gg \frac{\sigma^2}{8}$$

3 Background

Given a pair of tokens (X, Y) an LP is comfortable providing liquidity for on Uniswap V3 (or V4 in the future), what should the LP choose as their tick range to maximize expected yield?

This is an inherently probabilistic question relying on the nature of the stochastic price process p_t for the pool. The tradeoff the LP must consider is that a tighter tick range leads to a larger share of the currently active liquidity and thus a larger share of pro-rata swap fees. However, the downside to a tighter tick range is the higher probability of their liquidity quickly becoming inactive in addition to larger rebalancing costs (i.e. realized IL + swap fees, slippage) as price significantly deviates.

We'll start general by not assuming any particular model for the stochastic price process. Then focus on results for standard but unrealistic geometric Brownian motion (GBM).

The LP employs a strategy of deploying liquidity in a set tick range over fixed periods of time $[t_i, t_i + \tau]$, rebalancing around the current price once another period has elapsed. The problem reduces to the LP setting an optimal half tick width Δ given a chosen rebalance period τ, liquidity to deploy l, as well as existing pool conditions (i.e. liquidity, volume, and price history).

From a vault perspective, this is easier to implement with keepers v.s. existing work on this topic where LPs rebalance only when price moves out of a range. The latter strategy seems prone to LPs chasing price. Gas costs are less variable in the former as one roughly knows when the rebalance transactions will happen regardless of where the price has gone. It's also easier to analyze analytically.

4 Formulation

4.1 Setup

Assume the LP provides liquidity in a set tick range over a fixed period of time τ, and rebalances to a new optimal tick range every time this period has elapsed. To optimize over the next period $0 \leq t < \tau$, take the LP to enter a range position at the beginning of the period between prices $[p_a, p_b]$.

The LP aims to maximize their excess fee revenues relative to principal losses caused by rebalancing costs. The value of the LP's position will be the principal plus cumulative fee revenues gained over the period less swap fees and slippage incurred on the portion of principal that needs to be rebalanced at the end of the period due to swaps on the pool.

For each time period $t_i \leq t < t_i + \tau$, the same optimization procedure also takes place with the LP rebalancing to a new price range $[p_{a_i}, p_{b_i}]$. Assume the time-dependence of the stochastic process doesn't change.

We'll examine portfolio values assuming Y as the quote currency. We break down the value of the portfolio at the end of the period into two components:

- ψ_τ: value of accumulated fee revenues over the period $[0, \tau]$.
- ρ_τ: value of the principal tokens used for LPing *after* performing the rebalancing swap at τ.

From this portfolio value expression,

$$V_\tau = \psi_\tau + \rho_\tau$$

We calculate the yield of the LP's portfolio value over the period. We then maximize the expected value of the yield to determine an appropriate choice of tick width.

Take the LP to be providing active liquidity in an on-chain vault that does not hedge on other venues. Information the vault has access to is constained to the individual pool the LP is providing liquidity on.

4.2 Fee Revenues

The pool sees Y token volume in of v_t^Y with fees at fee tier f taken on that volume of

$$v_t^Y f dt$$

Cumulative fees between rebalances taken by the pool are

$$\int_0^\tau dt\ v_t^Y f$$

But the LP only sees a fraction of these fees based on whether the current price is in range $p_a \leq p_t \leq p_b$ at the time the swaps occur. If not in range, the LP receives no fees. If in range, the liquidity is active and the LP receives a pro-rata share based on the existing virtual liquidity at the current tick range.

Assume before the LP provides liquidity, the virtual liquidity on the pool is L. Virtual liquidity on the pool is taken to remain constant throughout the rebalance period, outside of the LP's actions. The LP's share of virtual liquidity after they add δL to the pool will be

$$\frac{\delta L}{L + \delta L}$$

Further, assume token volumes in are roughly constant throughout the rebalance period, with

$$v_0 = v_0^Y \approx p_0 v_0^X$$

Taking the external active liquidity and volumes to each be roughly constant between rebalances avoids estimating liquidity and volume distributions.

The LP receives fees on volume at time t of

$$d\psi_t^Y = v_0 f dt \cdot \frac{\delta L}{L + \delta L} \cdot \mathbb{1}_{p_a \leq p_t \leq p_b}$$

where $\mathbb{1}_A$ is the indicator function. Assume the LP provides liquidity symmetrically around the current tick range when rebalancing, such that

$$\frac{p_b}{p_0} = \frac{p_0}{p_a} = e^{\Delta}$$

for a full tick range width of 2Δ in natural log terms. One finds their contribution to Uniswap-style virtual liquidity will be

$$\delta L = \frac{\sqrt{\delta x_0 \cdot \delta y_0}}{1 - e^{-\Delta/2}}$$

for $(\delta x_0, \delta y_0)$ initial token amounts sent to the pool.

Y cumulative fees received by the LP at the end of the rebalance period are then

$$\psi_\tau^Y = \delta y_0 \frac{\theta}{1 - e^{-\Delta/2} + l} \int_0^\tau dt\ \mathbb{1}_{p_a \leq p_t \leq p_b}$$

where we've introduced shorthand

$$\begin{aligned}\theta &= \frac{v_0 f}{L\sqrt{p_0}}\\ l &= \frac{\delta y_0}{L\sqrt{p_0}}\end{aligned}$$

for fee volume per unit of external virtual liquidity (excluding the LP) and the amount of physical tokens the LP contributes to the pool per unit of external virtual liquidity, respectively.

Note that the payoff of the future LP fee revenues looks like a portfolio of double digital options with expiries at each successive time t between rebalances. This is useful context when thinking through replication strategies for any derivative protocols aiming to tokenize these fee streams and sell them off to interested buyers.

Assume the LP holds the accumulated fees until the end of the rebalance period, at which point they reinvest the fees into principal for the next period. Accumulated X token fees are then valued at the price at the end of the rebalance period.

Total value of the fees accumulated in quote terms at the end of the period will be

$$\begin{aligned}\psi_\tau &= \psi^Y_\tau + p_\tau \psi^X_\tau\\ &\approx \psi^Y_\tau \cdot \left[1 + \frac{p_\tau}{p_0}\right]\end{aligned}$$

given prior assumptions on volume.

4.3 Principal Losses

As a Uniswap V3 liquidity provider, token balances attributed to the LP change as price moves. At any time t before the rebalance swap happens, the LP has principal token balances in the pool of

$$\begin{aligned}\delta x_t &= \delta L\left[\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right]\mathbb{1}_{p_t<p_a} + \delta L\left[\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}}\right]\mathbb{1}_{p_a\le p_t\le p_b}\\ \delta y_t &= \delta L\left[\sqrt{p_b} - \sqrt{p_a}\right]\mathbb{1}_{p_t>p_b} + \delta L\left[\sqrt{p_t} - \sqrt{p_a}\right]\mathbb{1}_{p_a\le p_t\le p_b}\end{aligned}$$

At rebalance time τ, the LP removes all of their liquidity and swaps a portion of the returned principal through the same pool. The LP then adds the rebalanced liquidity to the pool. Assume arbitraguers immediately return the price back to its pre-swap value of p_τ, after the LP rebalances but prior to the LP adding liquidity again to simplify things.

Post rebalance swap, the LP holds token balances of

$$\begin{aligned}\delta x_{\tau'} &= \delta x_\tau + \epsilon^X_\tau \cdot \mathbb{1}_{p_\tau>0} - \epsilon^X_\tau(1+f)\cdot \mathbb{1}_{p_\tau>0}\\ \delta y_{\tau'} &= \delta y_\tau - \epsilon^Y_\tau(1+f)\cdot \mathbb{1}_{p_\tau>0} + \epsilon^Y_\tau \cdot \mathbb{1}_{p_\tau<0}\end{aligned}$$

Principal balances after the swap must satisfy

$$p_\tau \delta x_{\tau'} = \delta y_{\tau'}$$

for the LP to rebalance around the current price at the end of the period. The LP removes their full liquidity contribution then swaps through the remaining external virtual liquidity L. The pool pre- and post-swap obeys the Uniswap invariant

$$
\begin{aligned}
L^2 &= \tilde{x}_\tau \tilde{y}_\tau \\
&= (\tilde{x}_\tau \mp \epsilon_\tau^X)(\tilde{y}_\tau \pm \epsilon_\tau^Y)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{x}_\tau &= \frac{L}{\sqrt{p_\tau}} \\
\tilde{y}_\tau &= L\sqrt{p_\tau}
\end{aligned}
$$

are virtual token balances ignoring possible changes in external liquidity distribution between ticks (roughly ok for smaller LP sizes). $\pm, \mp$ signs are dictated by whether $p_\tau > p_0$ or $p_\tau < p_0$. Expanding terms to second order in $f, (\epsilon/\tilde{x}), (\delta x/x)$, we find

$$
\frac{\delta y_{\tau'}}{\tilde{y}_\tau} \approx \frac{1}{2}\left[\frac{\delta y_\tau}{\tilde{y}_\tau} + \frac{\delta x_\tau}{\tilde{x}_\tau}\right] - \frac{1}{2}\left|\frac{\delta y_\tau}{\tilde{y}_\tau} - \frac{\delta x_\tau}{\tilde{x}_\tau}\right| \left[\frac{f}{2} + \frac{1}{4}\left|\frac{\delta y_\tau}{\tilde{y}_\tau} - \frac{\delta x_\tau}{\tilde{x}_\tau}\right|\right]
$$

and

$$
\begin{aligned}
\frac{\delta y_\tau}{\tilde{y}_\tau} + \frac{\delta x_\tau}{\tilde{x}_\tau} = l\left[e^{\Delta/2} + 1\right] &\left[\mathbb{1}_{p_\tau < p_a}\sqrt{\frac{p_\tau}{p_0}} + \mathbb{1}_{p_\tau > p_b}\sqrt{\frac{p_0}{p_\tau}}\right] \\
+\mathbb{1}_{p_a \le p_\tau \le p_b} \frac{l}{1 - e^{-\Delta/2}} &\left[2 - e^{-\Delta/2}\left(\sqrt{\frac{p_0}{p_\tau}} + \sqrt{\frac{p_\tau}{p_0}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta y_\tau}{\tilde{y}_\tau} - \frac{\delta x_\tau}{\tilde{x}_\tau} = l\left[e^{\Delta/2} + 1\right] &\left[\mathbb{1}_{p_\tau > p_b}\sqrt{\frac{p_0}{p_\tau}} - \mathbb{1}_{p_\tau < p_a}\sqrt{\frac{p_\tau}{p_0}}\right] \\
+\mathbb{1}_{p_a \le p_\tau \le p_b} \frac{l}{e^{\Delta/2} - 1} &\left[\sqrt{\frac{p_\tau}{p_0}} - \sqrt{\frac{p_0}{p_\tau}}\right]
\end{aligned}
$$

This leads to a principal value at the end of the period post-swap of

$$
\rho_\tau \approx \tilde{y}_\tau \left\{\frac{\delta y_\tau}{\tilde{y}_\tau} + \frac{\delta x_\tau}{\tilde{x}_\tau} - \left|\frac{\delta y_\tau}{\tilde{y}_\tau} - \frac{\delta x_\tau}{\tilde{x}_\tau}\right| \left[\frac{f}{2} + \frac{1}{4}\left|\frac{\delta y_\tau}{\tilde{y}_\tau} - \frac{\delta x_\tau}{\tilde{x}_\tau}\right|\right]\right\}
$$

Each term represents

- $\delta y_\tau + p_\tau \delta x_\tau$: principal before rebalancing
- $-(f/2) \cdot |\delta y_\tau - p_\tau \delta x_\tau|$: swap fees paid on rebalance
- $-|\delta y_\tau - p_\tau \delta x_\tau|^2/(4\tilde{y}_\tau)$: slippage paid on rebalance

4.4 EV for LPs

The LP optimizes their expected gains at the end of the period from information known at the beginning of the period $\mathcal{F}_0$, i.e.

$$\mathbb{E}_0[V_\tau]$$

Assume the price process abides by GBM

$$\begin{aligned} dp_t &= \mu p_t dt + \sigma p_t dW_t \\ p_t &= p_0 e^{\mu' t + \sigma W_t} \\ \mu' &= \mu - \sigma^2/2 \end{aligned}$$

where W_t is a Wiener process. We'll make use of the identity that the expectation of the indicator function is the probability of the associated event occurring, as e.g.

$$\begin{aligned} \mathbb{E}_0[\mathbb{1}_{p_a \le p_\tau \le p_b}] &= \mathbb{P}_0[-\Delta \le \ln(p_\tau/p_0) \le \Delta] \\ &= \Phi\left(\frac{\Delta - \mu'\tau}{\sigma\sqrt{\tau}}\right) - \Phi\left(\frac{-\Delta - \mu'\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

$\Phi(z)$ is the standard normal CDF.

To simplify expressions, let

$$\begin{aligned} d_t^+ &= \frac{\Delta - \mu' t}{\sigma\sqrt{t}} \\ d_t^- &= \frac{-\Delta - \mu' t}{\sigma\sqrt{t}} \end{aligned}$$

We find for the fee revenue term (see Appendix A of the supplementary note)

$$\begin{aligned} \mathbb{E}_0[\psi_\tau] = \delta y_0 \frac{\theta}{1 - e^{-\Delta/2} + l} \int_0^\tau dt \, \Big\{ \Phi(d_t^+) - \Phi(d_t^-) \\ + e^{\mu\tau}\Big[\Phi(d_t^+ - \sigma\sqrt{t}) - \Phi(d_t^- - \sigma\sqrt{t})\Big]\Big\} \end{aligned}$$

and below for the principal value terms (see Appendix B of the supplementary note).

Principal before rebalance swap:

$$\begin{aligned} &\mathbb{E}_0[\delta y_\tau + p_\tau \delta x_\tau] \\ &= \delta y_0 \Big\{ \Big[e^{\Delta/2} + 1\Big]\Big[e^{\mu\tau}\Phi(d_\tau^- - \sigma\sqrt{\tau}) + 1 - \Phi(d_\tau^+)\Big] \\ &+ \frac{1}{1 - e^{-\Delta/2}}\Big[2e^{\frac{1}{2}(\mu - \frac{1}{4}\sigma^2)\tau}\Big(\Phi(d_\tau^+ - \sigma\sqrt{\tau}/2) - \Phi(d_\tau^- - \sigma\sqrt{\tau}/2)\Big) \\ &- e^{-\Delta/2}\Big(\Phi(d_\tau^+) - \Phi(d_\tau^-) + e^{\mu\tau}[\Phi(d_\tau^+ - \sigma\sqrt{\tau}) - \Phi(d_\tau^- - \sigma\sqrt{\tau})]\Big)\Big]\Big\} \end{aligned}$$

Swap fees paid on rebalance:

$$
\begin{aligned}
&\mathbb{E}_0[-(f/2)\cdot|\delta y_\tau - p_\tau \delta x_\tau|] \\
&= -\delta y_0 \cdot (f/2)\Bigg\{\Big[e^{\Delta/2}+1\Big]\Big[e^{\mu\tau}\Phi(d_\tau^- - \sigma\sqrt{\tau}) + 1 - \Phi(d_\tau^+)\Big] \\
&\quad + \frac{1}{e^{\Delta/2}-1}\Bigg[e^{\mu\tau}\Bigg(\Phi(d_\tau^+ - \sigma\sqrt{\tau}) + \Phi(d_\tau^- - \sigma\sqrt{\tau}) \\
&\quad -2\Phi(-\mu'\tau/(\sigma\sqrt{\tau}) - \sigma\sqrt{\tau})\Bigg) \\
&\quad +2\Phi(-\mu'\tau/(\sigma\sqrt{\tau})) - \Phi(d_\tau^+) - \Phi(d_\tau^-)\Bigg]\Bigg\}
\end{aligned}
$$

Slippage paid on rebalance:

$$
\begin{aligned}
&\mathbb{E}_0[-|\delta y_\tau - p_\tau\delta x_\tau|^2/(4\tilde{y}_\tau)] \\
&= -\delta y_0 \cdot (l/4)e^{\frac{1}{2}(\mu+\frac{3}{4}\sigma^2)\tau}\Bigg\{\Big[e^{\Delta/2}+1\Big]^2\Bigg[e^{-\mu\tau}\Bigg(1-\Phi(d_\tau^+ + \sigma\sqrt{\tau}/2)\Bigg) \\
&\quad +e^{\mu\tau}\Phi(d_\tau^- - 3\sigma\sqrt{\tau}/2)\Bigg] \\
&\quad \frac{1}{[e^{\Delta/2}-1]^2}\Bigg[e^{\mu\tau}\Bigg(\Phi(d_\tau^+ - 3\sigma\sqrt{\tau}/2) - \Phi(d_\tau^- - 3\sigma\sqrt{\tau}/2)\Bigg) \\
&\quad +e^{-\mu\tau}\Bigg(\Phi(d_\tau^+ + \sigma\sqrt{\tau}/2) - \Phi(d_\tau^- + \sigma\sqrt{\tau}/2)\Bigg) \\
&\quad -2e^{-\frac{1}{2}\sigma^2\tau}\Bigg(\Phi(d_\tau^+ - \sigma\sqrt{\tau}/2) - \Phi(d_\tau^- - \sigma\sqrt{\tau}/2)\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

Hard to have clarity about these expressions from just looking at them. Desmos plots presented in Figure 1 make life easier.

The solid black line is the expected LP value $\mathbb{E}_0[V_\tau]$ at the end of the period as a function of half tick width Δ, given strategist set parameters for liquidity provided l and rebalance period length τ as well as assumed pool fee volume θ. Solid blue line is the associated expected yield relative to initial principal $y_\tau = \mathbb{E}_0[V_\tau]/V_0 - 1$. This is also given as a function of half tick width Δ, but scaled by a multiplier of 100 to express in percentage points.

Optimization for half tick width choice reduces to finding the critical point Δ_c

$$0 = \partial_\Delta \mathbb{E}_0[V_\tau]|_{\Delta=\Delta_c}$$

at which the yield peaks. Script `optimize.py` in the Kodiak simulations repo uses the `scipy.optimize` package to find this value.

When ignoring drift $\mu = 0$, expected yield appears to peak with the same sign as expected yield when $\Delta \to \infty$. In the Uniswap V2 limit with zero drift, expected yield is positive when fee volume exceeds

$$\frac{l+1}{\tau}\left[1 - e^{-\frac{1}{8}\sigma^2\tau}\right]$$

or to first order

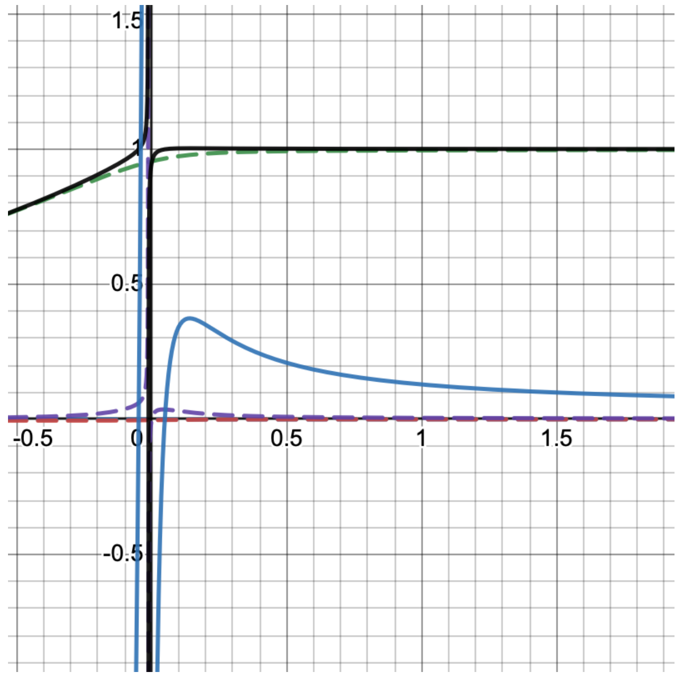


Figure 1: Expected LP yield × 100 (blue) v.s. half tick width Δ.

$$\theta > \frac{\sigma^2}{8}(l+1)$$

We used the Ethereum mainnet Uniswap V3 USDC/ETH 5 bps pool history to fit and set parameters in the Desmos plot above, ignoring drift param results. For a 7 day rebalance period providing about \$10M of physical liquidity to a pool with virtual liquidity of about \$1.6B and assuming 24h fees of about \$500K, a tick width of approximately `0.14 * 2 // log(1.0001) = 2800` (i.e. ∼ +/- 15% around current price) appears optimal under GBM with an expected yield at the end of the period after rebalancing of 37 bps.

Alternatively, to ensure +EV LPing for most vaults, vault managers can supplement fee revenues with token incentives. Aim for a reward yield greater than

$$R \gg \frac{\sigma^2}{8}$$

which would result in $\theta + R > \frac{\sigma^2}{8}(l+1)$ for most LPs. This also likely subsidizes a flywheel effect as

$$\begin{aligned}\theta + R > \frac{\sigma^2}{8}(l+1) \rightarrow \Delta_c \text{ exists} \rightarrow \text{LPs concentrate liquidity at +EV}\\ \rightarrow \text{Less slippage for traders} \rightarrow \text{More volume and fees for LPs}\\ \rightarrow \Delta_c \downarrow \text{ due to increase in } \theta \uparrow \text{ from higher volumes}\\ \rightarrow \text{Even less slippage for traders} \ldots\end{aligned}$$

5 Backtesting

Backtests were performed using the SmolQuants `backtest-ape` package, which is built on the ApeWorX `ape` framework. To backtest, `backtest-ape` begins by forking the chain at a user-specified historical start block.

It then deploys duplicate mock contracts, which are updated at each block of the simulation to what the state of a set of reference contracts (e.g. Uniswap V3 pool) had been at that historical block. A user-implemented runner instance submits transactions to a separate `Backtest.sol` contract deployed on the forked chain to implement their strategy on these mock contracts. Over each block of the simulation, relevant user-defined values (e.g. principal, fee values of an LP position) are queried from the `Backtest.sol::values` function to generate a timeseries of historical data. At the end of each block iteration, the runner can update its strategy based off of the current state of the fork by sending additional transactions through `Backtest.sol`.

For Kodiak economic simulations, we used historical data on Uniswap V3 pools and implemented two types of runners.

The first is a `UniswapV3LPSimpleRunner` runner class that creates a hypothetical LP position on the mock of the reference pool at a pre-determined tick width. Liquidity is provided symmetrically around the current tick. The tick width used at each rebalance is fixed over the entire simulation (i.e. no optimization calculations are performed). The user specifies:

- `refs["pool"]`: Uniswap V3 pool to reference.
- `tick_width`: Tick width 2Δ to LP over.
- `blocks_between_rebalance`: The rebalance period τ in blocks.
- `compound_fees_at_rebalance`: Whether to fold in accumulated fees into the rebalance swap.
- `amount1`: The physical amount of quote tokens δy_0 to provide.

The second is a `UniswapV3LPOptimizedRunner` runner class that inherits from `UniswapV3LPSimpleRunner`, but optimizes the tick width after each rebalance based on average fee revenues per unit of external liquidity θ over the prior period.

The strategy is as follows:

- If $\theta \leq \frac{\sigma^2}{8}(l+1)$, the runner provides liquidity over the full tick range for the next period to minimize IL.
- If $\theta > \frac{\sigma^2}{8}(l+1)$, the runner opportunistically concentrates liquidity around the current tick using the optimal tick width $2\Delta_c$, which is computed via the optimization procedure in `optimize.py`.

In addition to the parameters for the simple runner, the user also specifies:

- `mu`: Drift μ calculated from fits to historical log-price data on the pool.
- `sigma`: Volatility σ calculated from fits to historical log-price data on the pool.
- `max_tick_width`: Max optimal tick width, above which default to full tick range.
- `rewards`: Rewards per unit of external virtual liquidity R for active LP incentives

Backtests were simulated from block 16219692 (Dec-19-2022) to 17267692 (May-15-2023) on the Ethereum mainnet Uniswap V3 USDC/ETH 5bps pool with ETH as the quote currency. In each backtest simulation, the runner rebalances every $\tau = 7200$ blocks or about once per day, and compounds fees at each rebalance swap. Runner deploys 1000 ETH of liquidity and the corresponding amount of USDC at the start tick.

We run simulations for the following fixed tick widths with the simple runner: `[1400, 2800, 5600, 8400, 11200, 14000, 1774540]`. The last element in the tick width list is full tick range for the pool.

We also run a simulation with the optimized runner strategy. Price was taken to be driftless $\mu = 0$ over the 1 day rebalance period, and per-day volatility taken from fits to be $\sigma = 0.03858$. Max tick width set to $\Delta_c \leq 14000$ above which the runner provides liquidity over the full tick range. No token incentives $R = 0$ were assumed.

Backtest results of strategy yields relative to passively holding initial capital $p_t \delta x_0 + \delta y_0$ are produced in Figure 2. We also plot accumulated fee returns per strategy in Figure 3 and relative price changes in Figure 4 to compare with anticipated principal losses due to realized IL. Notebook to generate the plots is provided in the repo as `backtest.ipynb`.

For the full range LP, which does not need to rebalance given its chosen infinite tick width, anticipated losses relative to passively holding using the end of simulation price deviation figure of `-0.35` would be $2\sqrt{p_t/p_0}/(p_t/p_0+1) - 1 =$ `-0.02276`.

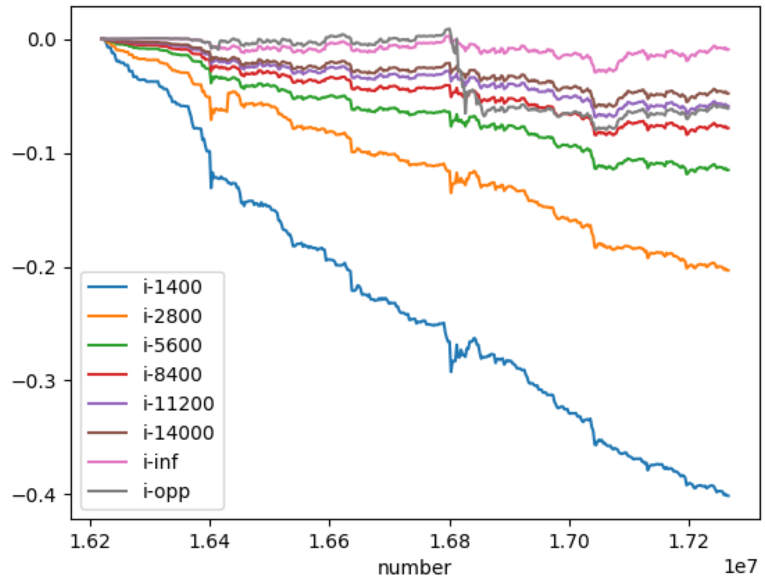


Figure 2: Backtested strategy yields benchmarked relative to passively holding initial capital, with ETH as quote. Simple runner simulations labeled in the legend as `i-[tick_width]`, where `tick_width` is the fixed tick width used over the entire simulation. `i-inf` simulates full range LPing by the simple runner and `i-opp` the optimized runner strategy.

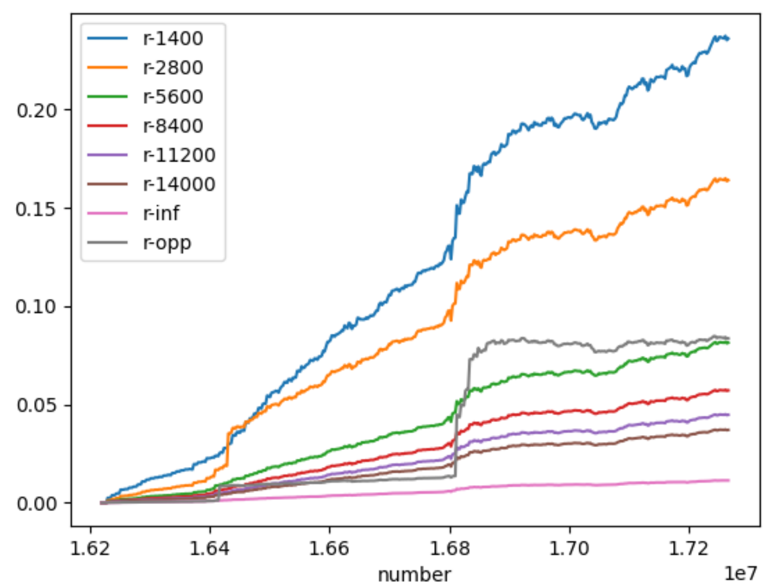


Figure 3: Backtested strategy fee returns benchmarked relative to initial value of LP principal. Simple runner simulations labeled in the legend as `r-[tick_width]`, where `tick_width` is the fixed tick width used over the entire simulation. `r-inf` simulates full range LPing by the simple runner and `r-opp` the optimized runner strategy.

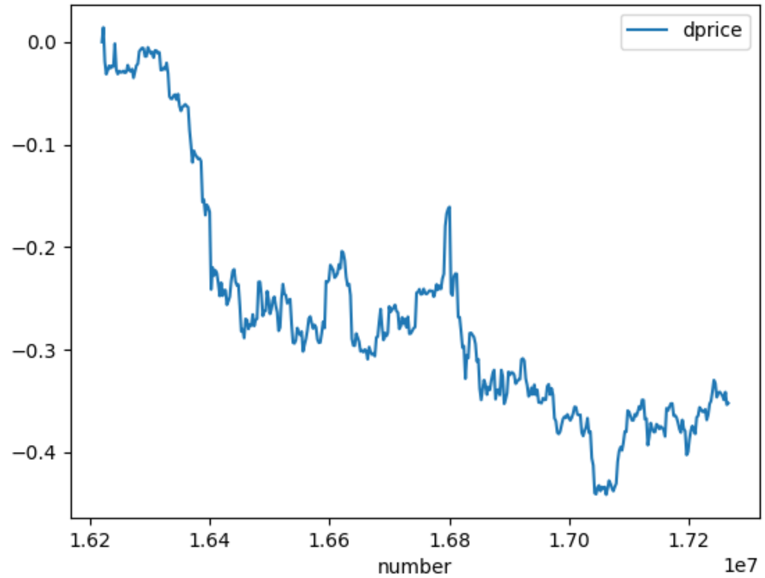


Figure 4: Relative change in price v.s. block number on the Uniswap V3 USDC/ETH 5 bps pool. ETH as the quote token.

Backtest results in Figure 2 show the more concentrated the tick width, the more significant the losses as IL over the simulation becomes realized at each rebalance. The best performing strategy for the USDC/ETH pair was the simple full range LP, which minimizes IL losses and does not require a rebalance swap of LP principal. This is consistent with 0xfbifemboy's analysis of simple vault strategies.

Why though does the opportunistic LP strategy implemented by the optimized runner also appear to underperform when it concentrates down from full range only if calculated to be +EV to do so?

Figure 5 plots strategy yields relative to passively holding only for the opportunistic strategy and the full range LP. In the same panel, we also plot log-price changes (8 hour candles), the optimized tick widths used by the opportunistic strategy, and fee revenues per unit of external virtual liquidity θ calculated over the prior rebalance period by the optimized runner.

There are only a few instances where the optimized runner decides to concentrate liquidity over the simulation, given the reduction in volume during a bear market. The opportunistic strategy begins by outperforming the full range LP given an uptick in fee revenues earned by an initial stab at concentrated liquidity from block 16409293 to 16414093. There was a significant log-price delta in the period prior ($\sim 4\sigma$ candle) accompanied by elevated fees which triggered the runner.

Over the course of the actual rebalance period (1 day) over which the optimized runner was initially concentrating liquidity at a tick width of `400`, there were no large price deviations which caused the strategy to outperform full range as minimal IL was incurred v.s. the gain in fees.

The next stab at concentrated liquidity with the opportunistic strategy did not fare so well. From block 16802893 (Mar-11-2023) to 16846093 (Mar-17-2023) the optimized runner concentrated liquidity down to a tick width range between `250` and `750`. Figure 6 plots the same panel over the shorter timeframe but with strategy yields benchmarked relative to the full range LP performance. Associated timeseries values provided in Table 1. After the initial $\sim 4.5\sigma$ price shock, two subsequent $2\sigma+$ drops in price cause significant losses to strategy yield in the form of realized IL. Under log-normal assumptions, the probability of each of these subsequent events occurring over their 8 hour candles are `0.005891` and `0.018825`, respectively. Or about once every 56 and 18 days. Taken together with the two $\sim 4\sigma+$ events over the entire 6 month backtest

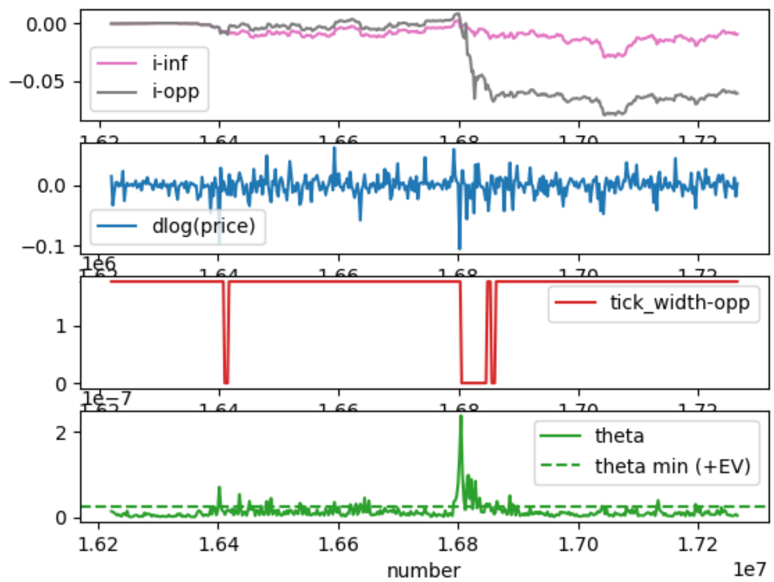


Figure 5: Backtested strategy yields benchmarked relative to passively holding initial capital for only the full range LP (pink) and optimized runner strategy (gray). Plots in the panel also display log-price differences per 8 hour candle (blue), the tick width 2Δ used by the optimized runner (red), and the fee revenues per unit of external liquidity θ (green).

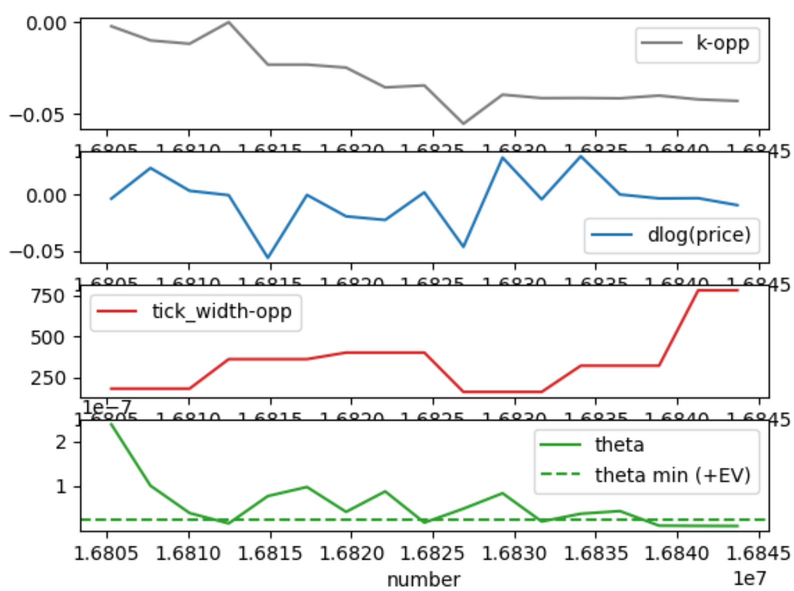
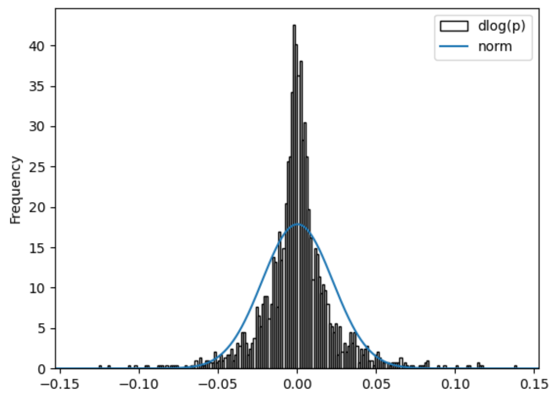
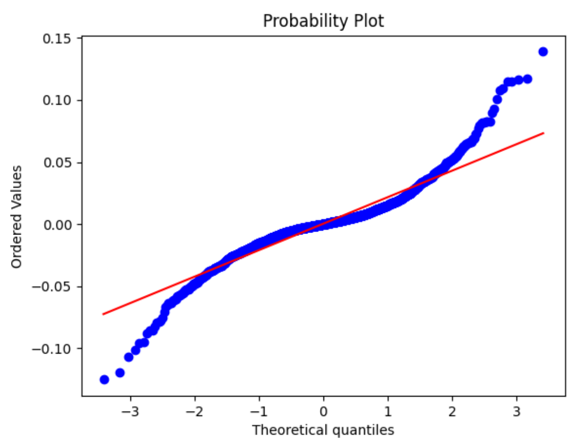


Figure 6: Backtested strategy yields for the optimized runner benchmarked relative to the full range LP, labeled `k-opp`. Yields examined from block 16802893 (Mar-11-2023) to 16846093 (Mar-17-2023) after the large spike in price and fee volume, when the runner is most active.



(a) Histogram



(b) Probability plot

Figure 7: Histogram and probability plots of log-price 8 hour candles taken from the Uniswap V3 USDC/ETH 5 bps pool. Data queried from block 13143698 to 18399698.

Block number	$\log(P_{t_{i+1}}/P_{t_i})$	$\mathbb{P}[\lvert\log(P_{t_{i+1}}/P_{t_i})\rvert > z]$	$2\Delta_{opp}$	`k-opp`
16800493	0.002300	0.458879	1774540	0.006149
16802893	-0.105195	0.000001	1774540	0.006149
16805293	-0.003300	0.441110	180	-0.002084
16807693	0.023999	0.140631	180	-0.009820
16810093	0.003700	0.434034	180	-0.011643
16812493	-0.000200	0.496418	360	0.000143
16814893	-0.056097	0.005891	360	-0.023039
16817293	0.000000	0.500000	360	-0.023040
16819693	-0.019099	0.195584	400	-0.024670
16822093	-0.022299	0.158373	400	-0.035484
16824493	0.002300	0.458879	400	-0.034398
16826893	-0.046298	0.018825	160	-0.055332
16829293	0.033298	0.067455	160	-0.039435
16831693	-0.003900	0.430504	160	-0.041322
16834093	0.034498	0.060704	320	-0.041244
16836493	0.000400	0.492836	320	-0.041431
16838893	-0.003100	0.444656	320	-0.039956
16841293	-0.002900	0.448206	780	-0.042025
16843693	-0.009100	0.341436	780	-0.042862
16846093	-0.024599	0.134704	780	-0.050166
16848493	0.003300	0.441110	1774540	-0.046250

Table 1: Performance `k-opp` of the opportunistic strategy relative to the full range LP over time. Candle probabilities are calculated assuming GBM with zero drift and per day volatility of $\sigma = 0.03858$.

period, which individually should only occur every 90 years under log-normal assumptions, suggests we're mismodelling the price process. Underestimating extreme events through mismodelling the price process would lead to an optimized tick width that we presume to be +EV, but may not really be in actuality. Meaning, we should likely expect more extreme IL than what we're currently modelling.

Histogram and probability plots of USDC/ETH log-price changes in Figure 7 support this conjecture. Excess kurtosis exhibited by the price process is calculated to be `5.45` for the 8 hour candles fitted. Generalized central limit theorem arguments would suggest using a simple log-stable model to address these issues in future work.

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