Notes of Cryptography

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Preface

Course

密碼學設計與分析 Cryptography Design and Analysis (11320IIS500900) in NTHU

1 L1

1.1 Merkle 的故事

Merkle 在大學部修了一個課，然後要交一個 project。他在交這個作業的時候，提到了 Public Key Cryptography 的想法。當時的導師並不看好這個東西，所以 reject 了，最後他也退掉了這門課。之後他找到另一個很欣賞他的老師，覺得應該要「Publish it, win fame and fortune」，所以他將這篇文章那個投到了 CACM（Communications of the ACM）。第一次投期刊就因為「這個想法不是當今的主流想法」而被拒絕。在 Merkle 的某些堅持之下，過了快三年終於讓 CACM 接受了這篇文章。

這邊的故事及當時的論文，可以在 https://ralphmerkle.com/1974/ 找到。

另外影片中的 link 有誤，應該改成 https://ralphmerkle.com，不然你只會找到一間搞 CRM 和賣資料的公司。

1.2 Conventions

- 離散且有限的時間 (discrete and finite world)
  ⇒ 因為我們正在討論 computer science
- Data v.s. Information
- Machine (function/algorithm) 需要在 polynomial time 下執行
  ⇒ 因為我們需要能在一定時間內看到結果，不想要等到天荒地老
  ⇒ 不一定**強制**要求 polynomial time，但這堂課大部分會是這樣
- Alice and Bob：就是 sender 和 receiver，通常是 Alice 要傳訊息給 Bob
  ⇒ 還有其他角色，可以參見 Wikipedia：
  https://en.wikipedia.org/wiki/Alice_and_Bob
- 計算（computation）：任何遵循 well-defined model（例如 algorithm、protocol）的 calculation。
- Efficiency
  Input size: $|x| = n$ bits
  其他的就是拿 complexity 概念來作為 efficiency 的概念
- Crypto 像是信仰 (Faith)？
  密碼學不一定總是對的，但我們需要相信某些東西才能繼續在密碼學上前進
  這些東西包含：
  ⇒ 某些數學問題很難被解決
  ⇒ 某些假設無法被打破（通常指在 poly-time 底下）
  ⇒ 某些底層的密碼工具 (underlying crypto primitives) 是安全的
  ⇒ P $\neq$ NP
  ⇒ 亂數/隨機 (randomness)，因為我們不知道真的亂數長什麼樣，所以無法驗證

1.3 Overview

§ 什麼是密碼學？

如果我們不在意安全，那麼我們不需要密碼學。
(If do not care security, we won't need crypto.)

安全 (security) 可以由以下兩點來定義：

- 目的 (purposes)：我們需要達到什麼效果
- 需求 (requirements)：為了達到目的，我們需要達成哪些目標

一些密碼學相關的內容：

- 加密 (entryption)
- 數位簽章 (signature)
- 零知識 (zero knowledge)
- 安全計算 (secure computation)

1.4 Notations

Private key encryption (or “secret key encryption”)
就是對稱式加密，加密和解密皆使用同一個 key

Public key encryption
公鑰系統。一個公鑰會對應一個私鑰。公鑰會公開，私鑰不公開。
若 Alice 要傳訊息給 Bob，則 Alice 會使用自己的公鑰加密，並且讓 Bob 使用「與 Alice 的公鑰相對應的」私鑰進行解密。

Zero knowledge
A 想向 B 證明某件事情，但不想透漏任何其他的額外資訊。
Ex1：我想向你證明我有 100 萬，但不想真的放 100 萬現金在你眼前（以免被你搶走），所以我可以要求銀行開立證明來達到這個目的。Ex2: 我想向你證明我真的知道「威利在哪裡」。我可以用一張比原圖更大張的紙，並且在上面挖一個威利形狀的洞，以此來達到目的。

1.5 Story of solving impossibility

（這邊的例子經過一點點調整）
你的上司要求你解決一個問題 Q，並且告知你如果無法解決問題就會被炒魷魚，並被另一個比你聰明的傢伙取代。你雖然不知道怎麼解決 Q，但你知道另一個**相關的**知名問題 $\widetilde{Q}$ (Q tilde) 在現今根本就沒人會解。最後你告訴你的上司，由於「現在根本沒人知道如何解 $\widetilde{Q}$」，所以「也沒人會解 Q」，因此這問題解不了，而另一個自稱聰明的傢伙其實是騙子。

重點就是

If there's a good algorithm for Q, then there exists a good one for another well-known problem $\widetilde{Q}$.
這句話的逆否命題就是
If there's no algorithm for $\widetilde{Q}$, then there's no algorithm for Q either.

這背後的概念就是 reduction（就演算法的那個 reduction）。

1.6 Principle of modern crypto

Kerckhoff's principle

「加密方法不能被要求是保密的，就算它落入敵人手中也不應該造成麻煩」
意即，整套加密方法的安全性只仰賴金鑰的保密。

（原文：It should not require secrecy, and it should not be a problem if it falls into enemy hands.）

Principle of modern crypto

1. Formal definition
   - System framework (model)：系統長什麼樣子
   - Security definition：如何定義安全

2. Precise assumption Π'
   通常會是已知難題
   從上一節的重點可以知道，我們通常會將加密法與某個已經被研究過的難題 (well-studied hardness) 做連結。若難題不是 well-studied，一來無法說服別人這個加密法安全，二來代表可能有人知道這個問題如何解決。

3. Construction Π
   加密法的步驟是什麼

4. Security proof
   基本上就是上一節的 reduction
   如果假象的攻擊者可以在 definition（即第一個要素）底下破解 Π，那麼我可以構造另一個攻擊者，使其破解已知難題 Π'。
   上面逆否命題的推論可以寫成：如果 Π' 是安全的（意即不被破解），那麼 Π 就是安全的。

加密系統 = 產生 key (key generation) + 加密 (encryption) + 解密 (decryption)

1.7 History of cryptography

§ Shift cipher

使用 private key encryption。

Key 是每個字母需要做 shift 的次數。

Key generation：選擇一個 $key \in \{0, 1, ..., 25\}$
Encryption：將每個字母對應的數字 shift key 位
Decryption：將每個字母對應的數字**反方向** shift key 位

破解：最多嘗試 26 次就可以找到答案

§ Substitution cipher

使用 private key encryption。

Key generation：將每個字母逐一對應到另一個字母，以此這個 mapping 作為 key
Encryption：將明文中的字母按照 key 逐一對應過去
Decryption：將密文中的字母按照 key 逐一對應回來

破解：字典攻擊（常用詞）+ 頻率分析（「E」在英文中出現的次數比較多）

加強：明文中不使用頻率較高的字母

§ Stronger cipher?

Vigenère cipher：設定偏移量為字母在明文中所在的位置。

DES (first published in 1975, and standardized in 1977)

AES

§ History about PKC

1974: Merkle proposed the notion
1976: Diffie-Hellman proposed the key exchange solution (Turing Awad 2015)
1977: Rivest-Shamir-Adleman proposed the first PKE (Turing Award 2002)
UK claimed their Government Communications Headquarters proposed such PKC idea before them.

Other impovements: ID-based encryption from Weil Pairing
使用了不同的 assumption，所以概念上較簡單，執行起來也較有效率（關於 ID-based 的概念，之後如果有時間，可能會提到）

2 L2: Perfect Secrecy

2.1 Encryption definition

三個 space：
- $\mathcal{M}$: message space
- $\mathcal{C}$: ciphertext space
- $\mathcal{K}$: key space

三種動作：
- Gen (key generation): probabilistic algorithm。
  $\mathrm{Gen}(1^\lambda) \to k \in \mathcal{K}$, where λ is security parameter, or a symbol length (usually related to enc/dec execution time).

- Enc (encryption): probabilistic algorithm。
  For $m \in \mathcal{M}$, $\mathrm{Enc}_k(m) \to c \in \mathcal{C}$

- Dec (decryption): deterministic algorithm。
  For $c \in \mathcal{C}$, $\mathrm{Dec}_k(c) \coloneqq m \in \mathcal{M}$

注意上述使用 $\to$ 表示 probabilistic algorithm；使用 $\coloneqq$ 表示 deterministic algorithm。Probabilistic algorithm 就是每次執行都有可能產生不同結果，而 deterministic algorthm 則代表每次執行必定產生出相同結果。

正確性 (Correctness) 定義：

$$\Pr[\mathrm{Dec}_k(c) \coloneqq m : c \leftarrow \mathrm{Enc}_k(m), k \leftarrow \mathrm{Gen}(1^\lambda)] = 1$$

即由正確的金鑰一定可以成功進行解密。
對於某些系統，我們不一定會要求其機率是 1，可能會是接近 1（即 ≈ 1）

2.2 Notations

Distribution over $\mathcal{K}$：denoted as $\mathrm{dist}(\mathcal{K})$, which is defined by running Gen, and taking the output key。
一個好的 key generation algorithm 應該要均勻地 (uniformly) 選擇 key（即選擇 key space 中的每個 key 的機率都是相等的）。因為如果我們有意地提高某些 key 的選擇機率，那麼攻擊者便可以藉由頻率分析知道我們的偏好，進而增加破解的機率。

K：a random variable, denoting the value of key generated by Gen.

$\Pr[K = k]$：for all $k \in \mathcal{K}$, it denotes the probability that the key generated by Gen is equal to k.

上面三項皆可以套用至明文（$\mathrm{dist}(\mathcal{M})$、$M$、$\Pr[M = m]$）和密文（$\mathrm{dist}(\mathcal{C})$、$C$、$\Pr[C = c]$）。

當我們固定一個 encryption scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ 且 dist over $\mathcal{M}$，這就可以根據所

給定的 $k \in \mathcal{K}$ 和 $m \in \mathcal{M}$，確定 $\operatorname{dist}(\mathcal{C})$。

2.3 Examples of notations

§ Example 1

一個 adversary A 知道訊息是「attack today」的機率是 70%、「not attack」的機率是 30%，所以

$$\Pr[M = \text{A.T.}] = 0.7, \quad \Pr[M = \text{N.A.}] = 0.3$$

Random variables K 和 M 會假設沒有關係 (independent)。因為 $\operatorname{dist}(\mathcal{K})$ 由 Gen 決定，而 $\operatorname{dist}(M)$ 由我們想要加密的 context 決定。

§ Example 2 - Shift cipher

$K = \{0, 1, 2, \ldots, 25\}$ with $\Pr[K = k] = \dfrac{1}{26}$ (aka uniformly distributed).

Let distribution of $\mathcal{M}$

$$\operatorname{dist}(\mathcal{M}) = \begin{cases} \Pr[M = \text{'a'}] = 0.7 \\ \Pr[M = \text{'z'}] = 0.3 \end{cases}$$

Then

$$\begin{aligned} \Pr[C = \text{'b'}] &= \Pr[M = \text{'a'} \wedge K = 1] + \Pr[M = \text{'z'} \wedge K = 2] \\ &= \Pr[M = \text{'a'}] \cdot \Pr[K = 1] + \Pr[M = \text{'z'}] \cdot \Pr[K = 2] \quad \text{(By independence)} \\ &= 0.7 \cdot \frac{1}{26} + 0.3 \cdot \frac{1}{26} \\ &= \frac{1}{26} \end{aligned}$$

Condition probability

$$\begin{aligned} \Pr[M = \text{'a'} \mid C = \text{'b'}] &= \frac{\Pr[C = \text{'b'} \mid M = \text{'a'}] \cdot \Pr[M = \text{'a'}]}{\Pr[C = \text{'b'}]} \\ &= \frac{\frac{1}{26} \cdot 0.7}{\frac{1}{26}} \\ &= 0.7 \end{aligned}$$

where $\Pr[C = \text{'b'} \mid M = \text{'a'}]$ iff. $K = 1$, and $\Pr[K = 1] = \dfrac{1}{26}$

[Bayes' theorem]

$$\Pr[A \mid B] = \frac{\Pr[B \mid A] \cdot \Pr[A]}{\Pr[B]} \qquad \text{if} \quad \Pr[B] \neq 0$$

2.4 Intuition for security

Adversary 通常在收發兩端的中間進行竊聽 (eavesdrop)。
Adversary 知道 $\text{dist}(\mathcal{M})$ 和 encryption scheme $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$，而不知道 key。

A scheme Π meets **perfect secrecy** means observation (usually from adversary) on ciphertext c should give no additional infomation.
意即密文 c 不能給攻擊者有更多的資訊可以更準確地進行猜測，也可以說 c 不會洩漏更多的資訊。

2.5 Perfect secrecy

Formal definition of perfect secrecy (Definition 1)

An encrytion scheme $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$ with message space $\mathcal{M}$ is perfect secrecy if for every probability distribution over $\mathcal{M}$, every message $m \in \mathcal{M}$ and every chiphertext $c \in \mathcal{C}$ for $\Pr[C = c] > 0$

$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

簡單來說，就是在觀察 c 之後，所得知的 $\text{dist}(\mathcal{M})$ 與在觀察 c 之前相等。
若 c 洩漏了某些資訊，則上式中的等號 (=) 應該改成大於符號 (>)。

Example: shift cipher

這邊用和前面一樣的例子：

$$\begin{aligned}
\Pr[C = \text{'b'}] &= \Pr[M = \text{'a'} \wedge K = 1] + \Pr[M = \text{'z'} \wedge K = 2] \\
&= \Pr[M = \text{'a'}] \cdot \Pr[K = 1] + \Pr[M = \text{'z'}] \cdot \Pr[K = 2] \quad \text{(By independence)} \\
&= 0.7 \cdot \frac{1}{26} + 0.3 \cdot \frac{1}{26} \\
&= \frac{1}{26}
\end{aligned}$$

$$\begin{aligned}
\Pr[M = \text{'a'} \mid C = \text{'b'}] &= \frac{\Pr[C = \text{'b'} \mid M = \text{'a'}] \cdot \Pr[M = \text{'a'}]}{\Pr[C = \text{'b'}]} \\
&= \frac{\frac{1}{26} \cdot 0.7}{\frac{1}{26}} \\
&= 0.7 \\
&= \Pr[M = \text{'a'}]
\end{aligned}$$

由此可知，shift cipher 是 prefect secrecy。

3 L3

3.1 Perfect secrecy II

Formal definition of perfect secrecy (Definition 2)

For every $m, m' \in \mathcal{M}$ and every $c \in \mathcal{C}$,

$$\Pr[\mathrm{Enc}_K(m) = c] = \Pr[\mathrm{Enc}_K(m') = c]$$

Example: shift cipher

$\Pr[M = \text{'a'}] = 0.7$
$\Pr[M = \text{'z'}] = 0.3$

Let $m = \text{'a'}$, and $m' = \text{'z'}$.
Then

$$\Pr[\mathrm{Enc}_K(\text{'a'}) = \text{'b'}] = \frac{1}{26} = \Pr[\mathrm{Enc}_K(\text{'z'}) = \text{'b'}]$$

(For further explananntion, if $\mathrm{Enc}_K(\text{'a'}) = \text{'b'}$, K must be 1, where probability is $\frac{1}{26}$; similarly, if $\mathrm{Enc}_K(\text{'z'}) = \text{'b'}$, K must be 2. That's why their probabilities are same.)

Lemma

An encryption scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ with message space is perfectly secret (which means Π satisfies Def. 1), the above equation (which is Def. 2) holds for every $m, m' \in \mathcal{M}$ and every $c \in \mathcal{C}$.

意即 Def. 1 等價 (equivalent) 於 Def. 2.

Proof of Def. 2 to Def. 1

Fix a $\mathrm{dist}(\mathcal{M})$, a message m and a ciphertext c for which $\Pr[C = c] > 0$.
If $\Pr[M = m] = 0$, then $\Pr[M = m \mid C = c] = \Pr[M = m]$. It always holds.
If $\Pr[M = m] > 0$:

(i) $\Pr[C = c \mid M = m] = \Pr[\mathrm{Enc}_K(M) = c \mid M = m] = \Pr[\mathrm{Enc}_K(m) = c] = \alpha$

(ii) For every $m' \in \mathcal{M}$,

$$\Pr[C = c \mid M = m'] = \Pr[\mathrm{Enc}_K(M) = c \mid M = m'] = \Pr[\mathrm{Enc}_K(m') = c] = \alpha$$

(iii) By Bayes' Theorem,

$$\begin{aligned}
\Pr[M = m \mid C = c] &= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{\Pr[C = c]} \\
&= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{\sum_{m' \in \mathcal{M}} \Pr[C = c \mid M = m'] \cdot \Pr[M = m']} && \text{(by (i) and (ii))} \\
&= \frac{\alpha \cdot \Pr[M = m]}{\sum_{m' \in \mathcal{M}} \alpha \cdot \Pr[M = m']} \\
&= \frac{\alpha \cdot \Pr[M = m]}{\alpha \cdot \sum_{m' \in \mathcal{M}} \Pr[M = m']} \\
&= \frac{\cancel{\alpha} \cdot \Pr[M = m]}{\cancel{\alpha} \cdot \sum_{m' \in \mathcal{M}} \Pr[M = m']} \\
&= \Pr[M = m]
\end{aligned}$$

Proof of Def. 1 to Def. 2 (Quiz)

Fix a $\mathrm{dist}(\mathcal{M})$, a message m and a ciphertext c for which $\Pr[C=c]>0$.
If $\Pr[C=c]=0$, then $\Pr[C=c \mid M=m]=\Pr[C=c \mid M=m']=0$. It always holds.
If $\Pr[C=c]>0$:
(i) For $\Pr[\mathrm{Enc}_K(m)=c]$,

$$\begin{aligned}
\Pr[\mathrm{Enc}_K(m)=c] &= Pr[C=c \mid M=m] \\
&= \frac{\Pr[M=m \mid C=c]\cdot\Pr[C=c]}{\Pr[M=m]} \\
&= \frac{\Pr[M=m]\cdot\Pr[C=c]}{\Pr[M=m]} && \text{(by Def. 1)} \\
&= \frac{\cancel{\Pr[M=m]}\cdot\Pr[C=c]}{\cancel{\Pr[M=m]}} \\
&= \Pr[C=c]
\end{aligned}$$

(ii) For $\Pr[\mathrm{Enc}_K(m')=c]$,

$$\begin{aligned}
\Pr[\mathrm{Enc}_K(m)=c] &= Pr[C=c \mid M=m'] \\
&= \frac{\Pr[M=m' \mid C=c]\cdot\Pr[C=c]}{\Pr[M=m']} \\
&= \frac{\Pr[M=m']\cdot\Pr[C=c]}{\Pr[M=m']} && \text{(by Def. 1)} \\
&= \frac{\cancel{\Pr[M=m']}\cdot\Pr[C=c]}{\cancel{\Pr[M=m']}} \\
&= \Pr[C=c]
\end{aligned}$$

From (i) and (ii), we know that

$$\Pr[\mathrm{Enc}_K(m)=c]=\Pr[\mathrm{Enc}_K(m')=c]$$

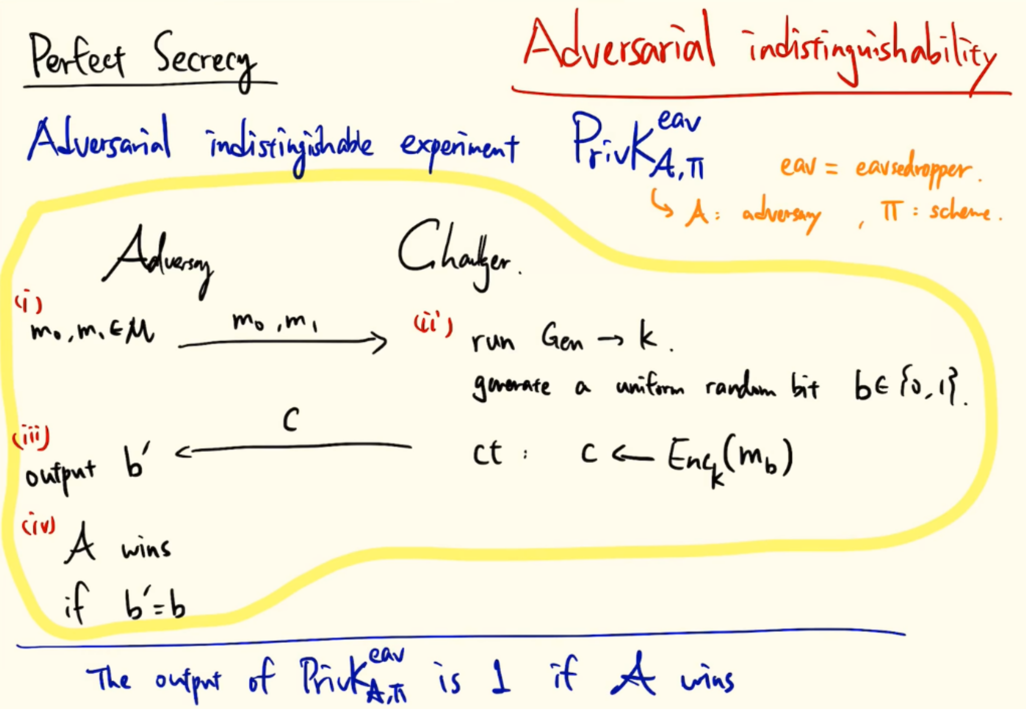
3.2 Perfect secrecy III

Adversarial indistinguishability

Adversarial indistinguishable experiment

$$PrivK^{eav}_{A,\Pi}$$

其中 A 代表 adversary, Π 代表 scheme, and eav 代表 eavesdropper.



這個 experiment 有兩個人：adversary 和 Challenger。

Step 1： Adversary 會從 message space 中選出兩份訊息 m_0 和 m_1，並這兩份訊息發送給 Challenger。

Step 2： Challenger 會執行 key generation algorithm Gen 來產生 key k，並 generate 一個 uniform random bit $b \in \{0,1\}$。最後產生出 ciphertext $c \leftarrow \mathrm{Enc}_k(m_b)$，再將 c 回傳給 adversary。

Step 3： Adversary 會 output 一個 b' 來代表它猜測 b 的結果。

Step 4： 若 $b' = b$，則 adversary 成功猜對了。

這個 experiment $PrivK^{eav}_{A,\Pi}$ 的 output 就是 adversary 是否猜對；也可以說，當 $PrivK^{eav}_{A,\Pi} = 1$，則 $b' = b$。

Formal definition of perfect secrecy (Definition 3, defined by perfect indistinguishability)

$\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ with message space $\mathcal{M}$ is perfectly indistinguishable if for every adversary A, it holds

$$\Pr[PrivK^{eav}_{A,\Pi} = 1] = \frac{1}{2}$$

意思：猜中的機率為 $\frac{1}{2}$，和沒有 c 的前提下，隨便亂猜的機率（即 $\Pr[(\text{randomly output } b') \wedge (b' = b)] = \frac{1}{2}$）是一樣的。代表 c 並沒有洩漏任何額外資訊。

這個命題和 $\Pr[PrivK^{eav}_{A,\Pi} = 0] = \frac{1}{2}$ 是等價的。

注意：若 $\Pr[PrivK^{eav}_{A,\Pi} = 1] < \frac{1}{2}$ 並不代表攻擊者更不會猜。因為 $\Pr[PrivK^{eav}_{A,\Pi} = 1] + \Pr[PrivK^{eav}_{A,\Pi} = 0] = 1$，所以 $\Pr[PrivK^{eav}_{A,\Pi} = 0] > \frac{1}{2}$。因此猜另一種情況的正確機率會更高。

Lemma

Π is perfectly secret if and only if it is perfectly indistinguishable.

Proof of Def. 2 to Def. 3

由 Def. 2 可知

$$\Pr[\mathrm{Enc}_K(m_0) = c] = \Pr[\mathrm{Enc}_K(m_1) = c]$$

又因為 $c \leftarrow \mathrm{Enc}_k(m_b)$，所以

$$\begin{aligned}\Pr[\mathrm{Enc}_K(m_0) = c] = \Pr[b = 0] \\ \Pr[\mathrm{Enc}_K(m_1) = c] = \Pr[b = 1]\end{aligned}$$

因此 $\Pr[b = 0] = \Pr[b = 1] = \frac{1}{2}$（因為在本例中 $\Pr[b = 0] + \Pr[b = 1] = 1$）。

$$\begin{aligned}
\Pr[PrivK^{eav}_{\mathrm{A},\Pi}] &= \Pr[b' = b] \\
&= \Pr[b' = b \wedge b = 0] + \Pr[b' = b \wedge b = 1] && \text{(rewrite)} \\
&= \Pr[b' = b \mid b = 0] \times \Pr[b = 0] + \Pr[b' = b \mid b = 1] \times \Pr[b = 1] && \text{(rewrite)} \\
&= \Pr[b' = 0] \times \Pr[b = 0] + \Pr[b' = 1] \times \Pr[b = 1] && \text{(rewrite)} \\
&= \Pr[b' = 0] \times \frac{1}{2} + \Pr[b' = 1] \times \frac{1}{2} && \text{(by Def. 2 denoted above)} \\
&= \frac{1}{2}(\Pr[b' = 0] + \Pr[b' = 1]) \\
&= \frac{1}{2} && (\because \Pr[b' = 0] + \Pr[b' = 1] = 1)
\end{aligned}$$

Proof of Def. 3 to Def. 2 (Bonus)

3.3 One-Time Pad (OTP)

Construction of OTP

Fix an integer $l > 0$, and let $|\mathcal{M}| = |\mathcal{C}| = |\mathcal{K}| = l$.
(whch means all are binary strings of length l, i.e., $\{0,1\}^l$)

Key generation algorithm Gen: uniformly randomly chooses a key $k \in \mathcal{K}$, k is l-bit key.

Encryption algorithm Enc: given $k \in \{0,1\}^l$ and a message $m \in \{0,1\}^l$, Enc outputs a ciphertext $c = m \oplus k$.

Decryption algorithm Dec: given k, c, Dec outputs message $m = c \oplus k$.

Prove that OTP is perfectly secret

Prove by Def. 1

(i) For an arbitrary $c \in \mathcal{C}$ and $m \in \mathcal{M}$

$$\Pr[C = c \mid M = m] = \Pr[\mathrm{Enc}_K(m) = c] = \Pr[m \oplus K = c] = \Pr[K = m \oplus c] = \frac{1}{2^l}$$

(ii) Fix any $\mathrm{dist}(\mathcal{M})$, for any $c \in \mathcal{C}$

$$\begin{aligned}
\Pr[C = c] &= \sum_{m' \in \mathcal{M}} \Pr[C = c \mid M = m'] \cdot \Pr[M = m'] \\
&= \sum_{m' \in \mathcal{M}} \frac{1}{2^l} \cdot \Pr[M = m'] \\
&= 2^{-l}(\sum_{m' \in \mathcal{M}} \Pr[M = m']) \\
&= 2^{-l}
\end{aligned}$$

(iii)

$$\begin{aligned}
\Pr[M = m \mid C = c] &= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{Pr[C = c]} \\
&= \frac{2^{-l} \cdot \Pr[M = m]}{2^{-l}} \\
&= \Pr[M = m]
\end{aligned}$$

4 L4

4.1 Limitation of Perfect Secrecy

Theorem 1 (Limitation of perfect secrecy)
If $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ is a perfectly secret encryption scheme with message space $\mathcal{M}$ and key space $\mathcal{K}$, then

$$|\mathcal{M}| \leq |\mathcal{K}|$$

Proof. Suppose $|\mathcal{K}| < |\mathcal{M}|$, Π cannot be perfectly secret.

Consider the uniform $\mathrm{dist}(\mathcal{M})$ and fix $c \in \mathcal{C}$, $\Pr[C = c] = 0$.

Let $\mathcal{M}(c)$ be the set of possible message which contains all possible messages decrypted by c.
That is,

$$\mathcal{M}(c) \stackrel{\mathrm{def}}{=} \{m \mid m = \mathrm{Dec}_K(c) \text{ for some } k \in \mathcal{K}\}$$

Dec is deterministic function, so $|\mathcal{M}(c)| \leq |\mathcal{K}|$.
(We know $\mathrm{Dec}_k(c) := m$, and different values of k may map to the same m. If all m are distinct for different k, then equation holds; otherwise, $|\mathcal{M}(c)| < |\mathcal{K}|$.)

If $|\mathcal{K}| < |\mathcal{M}|$ and $\mathcal{M}(c) \leq |\mathcal{K}|$, there exist some $m' \in \mathcal{M}$ but $m' \notin \mathcal{M}(c)$.
$\Rightarrow \Pr[M = m' \mid C = c] = 0 \neq \Pr[M = m']$, which is not perfect secrecy. $\square$

Quiz

We know that it's impossible to achieve pefect secrecy with shorter key size. So, what can we do or modify some factors to achieve shorter key? Any tradeoff (factor)?

§ Shannon's Theorem

Theorem 2 (Shannon's theorem)
Let $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ be an encryption scheme with message space M for which $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$.
The scheme is perfectly secret if and only if:

1. Every key $k \in \mathcal{K}$ is chosen with probability $\dfrac{1}{|\mathcal{K}|}$ by Gen
2. For every $m \in \mathcal{M}$ and every $c \in \mathcal{C}$, there exists a unique key $k \in \mathcal{K}$ such that $\mathrm{Enc}_k(m)$.

Quiz

Design a tricky scheme Π that $k \in \mathcal{K}$ is **NOT** uniformly chosen. Show Π is **NOT** perfectly secret by using Definition 1, 2 or 3.
(Hint: modify shift cipher or one-time pad)

4.2 Private Key Encryption

§ Computational Security

Perfect secrecy 的缺點 (weakness)：
- 只能用一次 (one time use)
- key 的長度一定要大於訊息的長度 ($|\mathcal{K}| \geq |\mathcal{M}|$)

Computational security 是從計算上保證安全的一種安全性。它不像 pefect secrecy 那樣地完美，但可以更靈活地建立 scheme（如減少 key 的長度）。

從 adversary 的觀點來看：

Adversary's power	time/space	success probability
Perfect secrecy	unbounded	= random guess
Computational security	polynomial time	= random guess + small probability

目的：減少安全性，來換取更好的效率 (by weakening the security, to achieve better efficiency)。

§ Concrete Definition

Definition 1 (Concrete definition)
A scheme Π is (t, ϵ)-secure if any adversary A running for time at most t, succeeds in breaking Π with probability at most ϵ.

ex: $t = 2^{10}$, $\epsilon = \frac{1}{2^{100}}$

§ Asymptotic Definition

在這裡的 adversary A 的能力 (power) 是以漸進式術語來定義的 (asymptotic setting)：

- "Efficient adversary"：這種 adversary 會執行可以在 polynomial time 內跑完的演算法。這種演算法的執行時間是 $p(n)$，其中 p 為多項式集合，而 n 為安全參數 (security parameter)。
- "Small probability of success": 成功機率小於任何 polynomial 的倒數。也就是

$$\Pr[\text{success}] < \frac{1}{p(n)}, \text{ where } p \text{ is arbitary polynomial}$$

PPT = Probabilistic Polynomial Time

Definition 2 (Asymptotic definition)
A scheme is secure if for any PPT adversary succeeds in breaking the scheme with at most **negligible** probability.

§ Negligible Probability

Negligible function 是漸進小於 (asymptotic smaller) 任何 polynomial function 的函數。

Definition 3
A function f is negligible if
for every positive polynomial p, there exists a number N such that $f(n) < \dfrac{1}{p(n)}$ where $n > N$.

Example:
Let $g(x) = \dfrac{1}{2^x}$.
There exists N such that $g(n) < \dfrac{1}{p(n)}$.

$$\begin{aligned} & g(n) < \frac{1}{p(n)} \\ \Rightarrow \quad & \frac{1}{2^n} < \frac{1}{n^k} \\ \Rightarrow \quad & 2^n > n^k \\ \Rightarrow \quad & n > k \cdot log_2(n) \\ \Rightarrow \quad & \frac{n}{log_2(n)} > k \end{aligned}$$

(k is positive constant)

If $n > k^2$, this inequality holds.

Quiz
Let $\mathrm{negl}(x), \mathrm{negl}'(x)$ be negligible functions.

1. A function f_1, defined by $f_1(x) = \mathrm{negl}(x) + \mathrm{negl}'(x)$
2. A function f_2, defined by $f_2(x) = p(x) \cdot \mathrm{negl}(x)$, where $p(x)$ is positive polynomial.

Are f_1 and f_2 are still negligible functions? **Yes**

Summary
任何關於 computational security 的 security definition 都由下列組成：
1. 破解 scheme 的定義（也就是怎麼樣才叫 scheme 被破解了）
2. 關於 adversary 的能力

我們通常將 adversary 塑造 (model) 成有效率（有計算能力）的演算法，且只考慮 adversary 可以在 polynomial time 之內執行的 probabilistic stratigies。

Definition 4
A scheme is secure if for every PPT adversary A carrying out an attack of some formally specified attack type, and the probability that A succeeds is negligible.

§ Private Key Encryption

Definition 5 (Private key encryption)
A private key encryption is tuple of PPT algorithm $(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$

- Key generation: $\mathrm{Gen}(1^k) \to k$. 這裡 n 的意義是 $|\mathcal{K}| \geq n$ 或 $|\mathcal{K}| = \mathrm{poly}(n)$。
- Encryption: $\mathrm{Enc}_k(m) \to c$, where key k and $m \in \{0,1\}*$ are inputs. 若 $m \in \{0,1\}^{l(n)}$，我們會稱這個等式為 fixed-length private key encryption with message length $l(n)$。
- Decryption: $\mathrm{Dec}_k(c) \coloneqq m$. If c cannot be decrypted, then outupt $\perp$ (error).