Notes of Cryptography

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Preface

Course

密碼學設計與分析 Cryptography Design and Analysis (11320IIS500900) in NTHU

1 L1

1.1 Merkle 的故事

Merkle 在大學部修了一個課，然後要交一個 project。他在交這個作業的時候，提到了 Public Key Cryptography 的想法。當時的導師並不看好這個東西，所以 reject 了，最後他也退掉了這門課。之後他找到另一個很欣賞他的老師，覺得應該要「Publish it, win fame and fortune」，所以他將這篇文章那個投到了 CACM（Communications of the ACM）。第一次投期刊就因為「這個想法不是當今的主流想法」而被拒絕。在 Merkle 的某些堅持之下，過了快三年終於讓 CACM 接受了這篇文章。

這邊的故事及當時的論文，可以在 https://ralphmerkle.com/1974/ 找到。

另外影片中的 link 有誤，應該改成 https://ralphmerkle.com，不然你只會找到一間搞 CRM 和賣資料的公司。

1.2 Conventions

- 離散且有限的時間 (discrete and finite world)
  ⇒ 因為我們正在討論 computer science
- Data v.s. Information
- Machine (function/algorithm) 需要在 polynomial time 下執行
  ⇒ 因為我們需要能在一定時間內看到結果，不想要等到天荒地老
  ⇒ 不一定**強制**要求 polynomial time，但這堂課大部分會是這樣
- Alice and Bob：就是 sender 和 receiver，通常是 Alice 要傳訊息給 Bob
  ⇒ 還有其他角色，可以參見 Wikipedia：
  https://en.wikipedia.org/wiki/Alice_and_Bob
- 計算（computation）: 任何遵循 well-defined model（例如 algorithm、protocol）的 calculation。
- Efficiency
  Input size: $|x| = n$ bits
  其他的就是拿 complexity 概念來作為 efficiency 的概念
- Crypto 像是信仰 (Faith)？
  密碼學不一定總是對的，但我們需要相信某些東西才能繼續在密碼學上前進
  這些東西包含：
  ⇒ 某些數學問題很難被解決
  ⇒ 某些假設無法被打破（通常指在 poly-time 底下）
  ⇒ 某些底層的密碼工具 (underlying crypto primitives) 是安全的
  ⇒ P $\neq$ NP
  ⇒ 亂數/隨機 (randomness)，因為我們不知道真的亂數長什麼樣，所以無法驗證

1.3 Overview

§ 什麼是密碼學？

如果我們不在意安全，那麼我們不需要密碼學。
(If do not care security, we won't need crypto.)

安全 (security) 可以由以下兩點來定義：
- 目的 (purposes)：我們需要達到什麼效果
- 需求 (requirements)：為了達到目的，我們需要達成哪些目標

一些密碼學相關的內容：

- 加密 (entryption)
- 數位簽章 (signature)
- 零知識 (zero knowledge)
- 安全計算 (secure computation)

1.4 Notations

Private key encryption (or “secret key encryption”)
就是對稱式加密，加密和解密皆使用同一個 key

Public key encryption
公鑰系統。一個公鑰會對應一個私鑰。公鑰會公開，私鑰不公開。
若 Alice 要傳訊息給 Bob，則 Alice 會使用自己的公鑰加密，並且讓 Bob 使用「與 Alice 的公鑰相對應的」私鑰進行解密。

Zero knowledge
A 想向 B 證明某件事情，但不想透漏任何其他的額外資訊。
Ex1：我想向你證明我有 100 萬，但不想真的放 100 萬現金在你眼前（以免被你搶走），所以我可以要求銀行開立證明來達到這個目的。Ex2: 我想向你證明我真的知道「威利在哪裡」。我可以用一張比原圖更大張的紙，並且在上面挖一個威利形狀的洞，以此來達到目的。

1.5 Story of solving impossibility

（這邊的例子經過一點點調整）
你的上司要求你解決一個問題 Q，並且告知你如果無法解決問題就會被炒魷魚，並被另一個比你聰明的傢伙取代。你雖然不知道怎麼解決 Q，但你知道另一個**相關的**知名問題 $\widetilde{Q}$ (Q tilde) 在現今根本就沒人會解。最後你告訴你的上司，由於「現在根本沒人知道如何解 $\widetilde{Q}$」，所以「也沒人會解 Q」，因此這問題解不了，而另一個自稱聰明的傢伙其實是騙子。

重點就是
If there's a good algorithm for Q, then there exists a good one for another well-known problem $\widetilde{Q}$.
這句話的逆否命題就是
If there's no algorithm for $\widetilde{Q}$, then there's no algorithm for Q either.

這背後的概念就是 reduction（就演算法的那個 reduction）。

1.6 Principle of modern crypto

Kerckhoff's principle

「加密方法不能被要求是保密的，就算它落入敵人手中也不應該造成麻煩」
意即，整套加密方法的安全性只仰賴金鑰的保密。

（原文：It should not require secrecy, and it should not be a problem if it falls into enemy hands.）

Principle of modern crypto

1. Formal definition
    - System framework (model)：系統長什麼樣子
    - Security definition：如何定義安全

2. Precise assumption Π'
   通常會是已知難題
   從上一節的重點可以知道，我們通常會將加密法與某個已經被研究過的難題 (well-studied hardness) 做連結。若難題不是 well-studied，一來無法說服別人這個加密法安全，二來代表可能有人知道這個問題如何解決。

3. Construction Π
   加密法的步驟是什麼

4. Security proof
   基本上就是上一節的 reduction
   如果假象的攻擊者可以在 definition（即第一個要素）底下破解 Π，那麼我可以構造另一個攻擊者，使其破解已知難題 Π'。
   上面逆否命題的推論可以寫成：如果 Π' 是安全的（意即不被破解），那麼 Π 就是安全的。

加密系統 = 產生 key (key generation) + 加密 (encryption) + 解密 (decryption)

1.7 History of cryptography

§ Shift cipher

使用 private key encryption。
Key 是每個字母需要做 shift 的次數。

Key generation：選擇一個 $key \in \{0, 1, \ldots, 25\}$
Encryption：將每個字母對應的數字 shift key 位
Decryption：將每個字母對應的數字**反方向** shift key 位

破解：最多嘗試 26 次就可以找到答案

§ Substitution cipher

使用 private key encryption。

Key generation：將每個字母逐一對應到另一個字母，以此這個 mapping 作為 key

Encryption：將明文中的字母按照 key 逐一對應過去
Decryption：將密文中的字母按照 key 逐一對應回來

破解：字典攻擊（常用詞）+ 頻率分析（「E」在英文中出現的次數比較多）

加強：明文中不使用頻率較高的字母

§ Stronger cipher?

Vigenère cipher：設定偏移量為字母在明文中所在的位置。

DES (first published in 1975, and standardized in 1977)

AES

§ History about PKC

1974: Merkle proposed the notion
1976: Diffie-Hellman proposed the key exchange solution (Turing Awad 2015)
1977: Rivest-Shamir-Adleman proposed the first PKE (Turing Award 2002)
UK claimed their Government Communications Headquarters proposed such PKC idea before them.

Other impovements: ID-based encryption from Weil Pairing
使用了不同的 assumption，所以概念上較簡單，執行起來也較有效率（關於 ID-based 的概念，之後如果有時間，可能會提到）

2 L2: Perfect Secrecy

2.1 Encryption definition

三個 space：

- $\mathcal{M}$: message space
- $\mathcal{C}$: ciphertext space
- $\mathcal{K}$: key space

三種動作：

- Gen (key generation): probabilistic algorithm。
  $\mathrm{Gen}(1^\lambda) \to k \in \mathcal{K}$, where λ is security parameter, or a symbol length (usually related to enc/dec execution time).

- Enc (encryption): probabilistic algorithm。
  For $m \in \mathcal{M}$, $\mathrm{Enc}_k(m) \to c \in \mathcal{C}$

- Dec (decryption): deterministic algorithm。
  For $c \in \mathcal{C}$, $\mathrm{Dec}_k(c) \coloneqq m \in \mathcal{M}$

注意上述使用 $\to$ 表示 probabilistic algorithm；使用 $\coloneqq$ 表示 deterministic algorithm。Probabilistic algorithm 就是每次執行都有可能產生不同結果，而 deterministic algorthm 則代表每次執行必定產生出相同結果。

正確性 (Correctness) 定義：

$$\Pr[\mathrm{Dec}_k(c) \coloneqq m : c \leftarrow \mathrm{Enc}_k(m), k \leftarrow \mathrm{Gen}(1^\lambda)] = 1$$

即由正確的金鑰一定可以成功進行解密。
對於某些系統，我們不一定會要求其機率是 1，可能會是接近 1 （即 ≈ 1）

2.2 Notations

Distribution over $\mathcal{K}$：denoted as $\mathrm{dist}(\mathcal{K})$, which is defined by running Gen, and taking the output key。
一個好的 key generation algorithm 應該要均勻地 (uniformly) 選擇 key（即選擇 key space 中的每個 key 的機率都是相等的）。因為如果我們有意地提高某些 key 的選擇機率，那麼攻擊者便可以藉由頻率分析知道我們的偏好，進而增加破解的機率。

K：a random variable, denoting the value of key generated by Gen.

$\Pr[K = k]$：for all $k \in \mathcal{K}$, it denotes the probability that the key generated by Gen is equal to k.

上面三項皆可以套用至明文（$\mathrm{dist}(\mathcal{M})$、$M$、$\Pr[M = m]$）和密文（$\mathrm{dist}(\mathcal{C})$、$C$、$\Pr[C = c]$）。

當我們固定一個 encryption scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ 且 dist over $\mathcal{M}$，這就可以根據所給定的 $k \in \mathcal{K}$ 和 $m \in \mathcal{M}$，確定 $\mathrm{dist}(\mathcal{C})$。

2.3 Examples of notations

§ Example 1

一個 adversary A 知道訊息是「attack today」的機率是 70%、「not attack」的機率是 30%，所以

$$\Pr[M = \text{A.T.}] = 0.7, \quad \Pr[M = \text{N.A.}] = 0.3$$

Random variables K 和 M 會假設沒有關係 (independent)。因為 $\text{dist}(\mathcal{K})$ 由 Gen 決定，而 $\text{dist}(M)$ 由我們想要加密的 context 決定。

§ Example 2 - Shift cipher

$K = \{0, 1, 2, \ldots, 25\}$ with $\Pr[K = k] = \dfrac{1}{26}$ (aka uniformly distributed).

Let distribution of $\mathcal{M}$

$$\text{dist}(\mathcal{M}) = \begin{cases} \Pr[M = \text{'a'}] = 0.7 \\ \Pr[M = \text{'z'}] = 0.3 \end{cases}$$

Then

$$\begin{aligned} \Pr[C = \text{'b'}] &= \Pr[M = \text{'a'} \wedge K = 1] + \Pr[M = \text{'z'} \wedge K = 2] \\ &= \Pr[M = \text{'a'}] \cdot \Pr[K = 1] + \Pr[M = \text{'z'}] \cdot \Pr[K = 2] \quad \text{(By independence)} \\ &= 0.7 \cdot \frac{1}{26} + 0.3 \cdot \frac{1}{26} \\ &= \frac{1}{26} \end{aligned}$$

Condition probability

$$\begin{aligned} \Pr[M = \text{'a'} \mid C = \text{'b'}] &= \frac{\Pr[C = \text{'b'} \mid M = \text{'a'}] \cdot \Pr[M = \text{'a'}]}{\Pr[C = \text{'b'}]} \\ &= \frac{\frac{1}{26} \cdot 0.7}{\frac{1}{26}} \\ &= 0.7 \end{aligned}$$

where $\Pr[C = \text{'b'} \mid M = \text{'a'}]$ iff. $K = 1$, and $\Pr[K = 1] = \dfrac{1}{26}$

[Bayes' theorem]

$$\Pr[A \mid B] = \frac{\Pr[B \mid A] \cdot \Pr[A]}{\Pr[B]} \qquad \text{if} \quad \Pr[B] \neq 0$$

2.4 Intuition for security

Adversary 通常在收發兩端的中間進行竊聽 (eavesdrop)。
Adversary 知道 $\text{dist}(\mathcal{M})$ 和 encryption scheme $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$，而不知道 key。

A scheme Π meets **perfect secrecy** means observation (usually from adversary) on ciphertext c should give no additional infomation.
意即密文 c 不能給攻擊者有更多的資訊可以更準確地進行猜測，也可以說 c 不會洩漏更多的資訊。

2.5 Perfect secrecy

Formal definition of perfect secrecy (Definition 1)

An encrytion scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ with message space $\mathcal{M}$ is perfect secrecy if for every probability distribution over $\mathcal{M}$, every message $m \in \mathcal{M}$ and every chiphertext $c \in \mathcal{C}$ for $\Pr[C = c] > 0$

$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

簡單來說，就是在觀察 c 之後，所得知的 $\mathrm{dist}(\mathcal{M})$ 與在觀察 c 之前相等。
若 c 洩漏了某些資訊，則上式中的等號 (=) 應該改成大於符號 (>)。

Example: shift cipher

這邊用和前面一樣的例子：

$$\begin{aligned}
\Pr[C = \text{'b'}] &= \Pr[M = \text{'a'} \wedge K = 1] + \Pr[M = \text{'z'} \wedge K = 2] \\
&= \Pr[M = \text{'a'}] \cdot \Pr[K = 1] + \Pr[M = \text{'z'}] \cdot \Pr[K = 2] \quad \text{(By independence)} \\
&= 0.7 \cdot \frac{1}{26} + 0.3 \cdot \frac{1}{26} \\
&= \frac{1}{26}
\end{aligned}$$

$$\begin{aligned}
\Pr[M = \text{'a'} \mid C = \text{'b'}] &= \frac{\Pr[C = \text{'b'} \mid M = \text{'a'}] \cdot \Pr[M = \text{'a'}]}{\Pr[C = \text{'b'}]} \\
&= \frac{\frac{1}{26} \cdot 0.7}{\frac{1}{26}} \\
&= 0.7 \\
&= \Pr[M = \text{'a'}]
\end{aligned}$$

由此可知，shift cipher 是 prefect secrecy。

3 L3

3.1 Perfect secrecy II

Formal definition of perfect secrecy (Definition 2)

For every $m, m' \in \mathcal{M}$ and every $c \in \mathcal{C}$,

$$\Pr[\mathrm{Enc}_K(m) = c] = \Pr[\mathrm{Enc}_K(m') = c]$$

Example: shift cipher

$\Pr[M = \text{'a'}] = 0.7$
$\Pr[M = \text{'z'}] = 0.3$

Let $m = \text{'a'}$, and $m' = \text{'z'}$.
Then

$$\Pr[\mathrm{Enc}_K(\text{'a'}) = \text{'b'}] = \frac{1}{26} = \Pr[\mathrm{Enc}_K(\text{'z'}) = \text{'b'}]$$

(For further explanantion, if $\mathrm{Enc}_K(\text{'a'}) = \text{'b'}$, K must be 1, where probability is $\frac{1}{26}$; similarly, if $\mathrm{Enc}_K(\text{'z'}) = \text{'b'}$, K must be 2. That's why their probabilities are same.)

Lemma

An encryption scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ with message space is perfectly secret (which means Π satisfies Def. 1), the above equation (which is Def. 2) holds for every $m, m' \in \mathcal{M}$ and every $c \in \mathcal{C}$.

意即 Def. 1 等價 (equivalent) 於 Def. 2.

Proof (Proof from Def. 2 to Def. 1)

Fix a $\mathrm{dist}(\mathcal{M})$, a message m and a ciphertext c for which $\Pr[C = c] > 0$.
If $\Pr[M = m] = 0$, then $\Pr[M = m \mid C = c] = \Pr[M = m]$. It always holds.
If $\Pr[M = m] > 0$:

(i) $\Pr[C = c \mid M = m] = \Pr[\mathrm{Enc}_K(M) = c \mid M = m] = \Pr[\mathrm{Enc}_K(m) = c] = \alpha$

(ii) For every $m' \in \mathcal{M}$,

$$\Pr[C = c \mid M = m'] = \Pr[\mathrm{Enc}_K(M) = c \mid M = m'] = \Pr[\mathrm{Enc}_K(m') = c] = \alpha$$

(iii) By Bayes' Theorem,

$$\begin{aligned}
\Pr[M = m \mid C = c] &= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{\Pr[C = c]} \\
&= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{\sum_{m' \in \mathcal{M}} \Pr[C = c \mid M = m'] \cdot \Pr[M = m']} && \text{(by (i) and (ii))} \\
&= \frac{\alpha \cdot \Pr[M = m]}{\sum_{m' \in \mathcal{M}} \alpha \cdot \Pr[M = m']} \\
&= \frac{\alpha \cdot \Pr[M = m]}{\alpha \cdot \sum_{m' \in \mathcal{M}} \Pr[M = m']} \\
&= \frac{\cancel{\alpha} \cdot \Pr[M = m]}{\cancel{\alpha} \cdot \sum_{m' \in \mathcal{M}} \Pr[M = m']} \\
&= \Pr[M = m]
\end{aligned}$$

□

Proof (Proof from Def. 1 to Def. 2 (Quiz))

Fix a $\mathrm{dist}(\mathcal{M})$, a message m and a ciphertext c for which $\Pr[C = c] > 0$.
If $\Pr[C = c] = 0$, then $\Pr[C = c \mid M = m] = \Pr[C = c \mid M = m'] = 0$. It always holds.
If $\Pr[C = c] > 0$:
(i) For $\Pr[\mathrm{Enc}_K(m) = c]$,

$$\begin{aligned}
\Pr[\mathrm{Enc}_K(m) = c] &= Pr[C = c \mid M = m] \\
&= \frac{\Pr[M = m \mid C = c] \cdot \Pr[C = c]}{\Pr[M = m]} \\
&= \frac{\Pr[M = m] \cdot \Pr[C = c]}{\Pr[M = m]} && \text{(by Def. 1)} \\
&= \frac{\cancel{\Pr[M = m]} \cdot \Pr[C = c]}{\cancel{\Pr[M = m]}} \\
&= \Pr[C = c]
\end{aligned}$$

(ii) For $\Pr[\mathrm{Enc}_K(m') = c]$,

$$\begin{aligned}
\Pr[\mathrm{Enc}_K(m) = c] &= Pr[C = c \mid M = m'] \\
&= \frac{\Pr[M = m' \mid C = c] \cdot \Pr[C = c]}{\Pr[M = m']} \\
&= \frac{\Pr[M = m'] \cdot \Pr[C = c]}{\Pr[M = m']} && \text{(by Def. 1)} \\
&= \frac{\cancel{\Pr[M = m']} \cdot \Pr[C = c]}{\cancel{\Pr[M = m']}} \\
&= \Pr[C = c]
\end{aligned}$$

From (i) and (ii), we know that

$$\Pr[\mathrm{Enc}_K(m) = c] = \Pr[\mathrm{Enc}_K(m') = c]$$

□

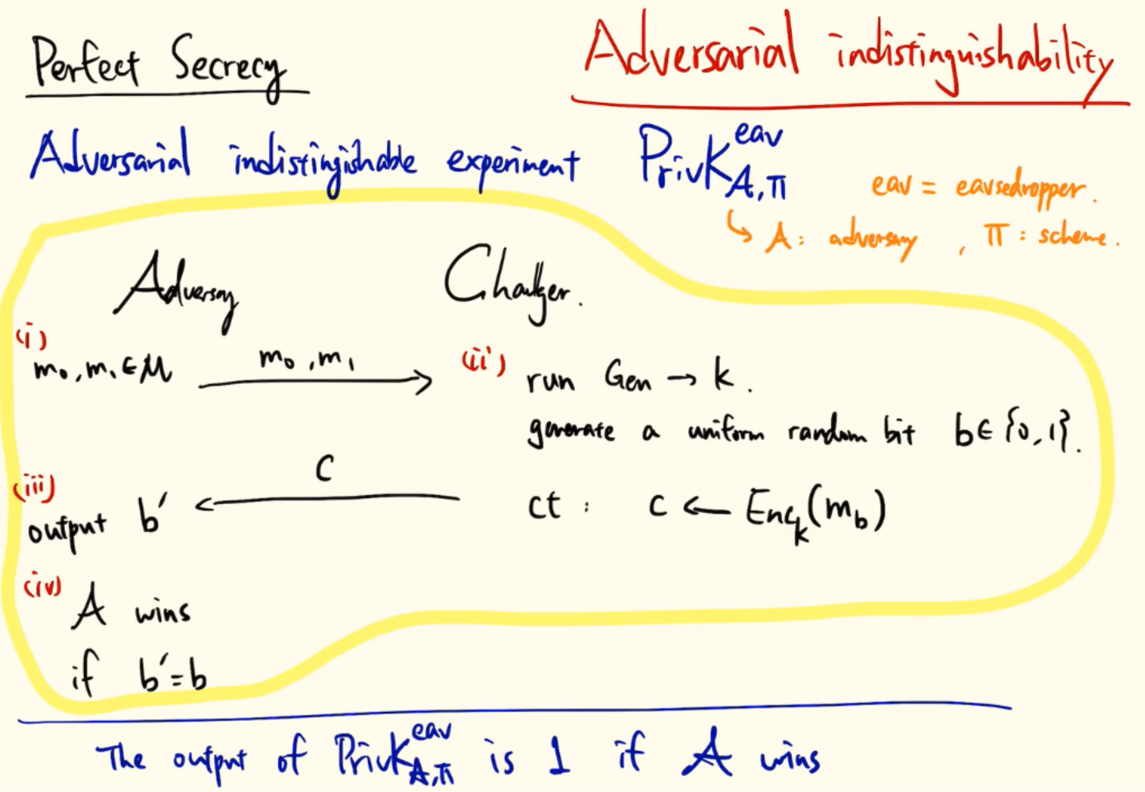
3.2 Perfect secrecy III

Adversarial indistinguishability

Adversarial indistinguishable experiment

$$PrivK^{eav}_{A,\Pi}$$

其中 A 代表 adversary, Π 代表 scheme, and eav 代表 eavesdropper.



這個 experiment 有兩個人：adversary 和 Challenger。

Step 1： Adversary 會從 message space 中選出兩份訊息 m_0 和 m_1，並這兩份訊息發送給 Challenger。

Step 2： Challenger 會執行 key generation algorithm Gen 來產生 key k，並 generate 一個 uniform random bit $b \in \{0, 1\}$。最後產生出 ciphertext $c \leftarrow \mathrm{Enc}_k(m_b)$，再將 c 回傳給 adversary。

Step 3： Adversary 會 output 一個 b' 來代表它猜測 b 的結果。

Step 4： 若 $b' = b$，則 adversary 成功猜對了。

這個 experiment $PrivK^{eav}_{A,\Pi}$ 的 output 就是 adversary 是否猜對；也可以說，當 $PrivK^{eav}_{A,\Pi} = 1$，則 $b' = b$。

Formal definition of perfect secrecy (Definition 3, defined by perfect indistinguishability)

$\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ with message space $\mathcal{M}$ is perfectly indistinguishable if for every adversary A, it holds

$$\Pr[PrivK^{eav}_{A,\Pi} = 1] = \frac{1}{2}$$

意思：猜中的機率為 $\frac{1}{2}$，和沒有 c 的前提下，隨便亂猜的機率（即 $\Pr[(\text{randomly output } b') \wedge (b' = b)] = \frac{1}{2}$）是一樣的。代表 c 並沒有洩漏任何額外資訊。

這個命題和 $\Pr[PrivK^{eav}_{A,\Pi} = 0] = \frac{1}{2}$ 是等價的。

注意：若 $\Pr[PrivK^{eav}_{A,\Pi} = 1] < \frac{1}{2}$ 並不代表攻擊者更不會猜。因為 $\Pr[PrivK^{eav}_{A,\Pi} = 1] + \Pr[PrivK^{eav}_{A,\Pi} = 0] = 1$，所以 $\Pr[PrivK^{eav}_{A,\Pi} = 0] > \frac{1}{2}$。因此猜另一種情況的正確機率會更高。

Lemma

Π is perfectly secret if and only if it is perfectly indistinguishable.

Proof (Proof from Def. 2 to Def.)

由 Def. 2 可知

$$\Pr[\mathrm{Enc}_K(m_0) = c] = \Pr[\mathrm{Enc}_K(m_1) = c]$$

又因為 $c \leftarrow \mathrm{Enc}_k(m_b)$，所以

$$\begin{aligned}\Pr[\mathrm{Enc}_K(m_0) = c] = \Pr[b = 0]\\ \Pr[\mathrm{Enc}_K(m_1) = c] = \Pr[b = 1]\end{aligned}$$

因此 $\Pr[b = 0] = \Pr[b = 1] = \frac{1}{2}$（因為在本例中 $\Pr[b = 0] + \Pr[b = 1] = 1$）。

$$\begin{aligned}
\Pr[PrivK^{eav}_{A,\Pi}] &= \Pr[b' = b] \\
&= \Pr[b' = b \wedge b = 0] + \Pr[b' = b \wedge b = 1] && \text{(rewrite)} \\
&= \Pr[b' = b \mid b = 0] \times \Pr[b = 0] + \Pr[b' = b \mid b = 1] \times \Pr[b = 1] && \text{(rewrite)} \\
&= \Pr[b' = 0] \times \Pr[b = 0] + \Pr[b' = 1] \times \Pr[b = 1] && \text{(rewrite)} \\
&= \Pr[b' = 0] \times \frac{1}{2} + \Pr[b' = 1] \times \frac{1}{2} && \text{(by Def. 2 denoted above)} \\
&= \frac{1}{2}(\Pr[b' = 0] + \Pr[b' = 1]) \\
&= \frac{1}{2} && (\because \Pr[b' = 0] + \Pr[b' = 1] = 1)
\end{aligned}$$

□

Proof of Def. 3 to Def. 2 (Bonus)

欲證 Def. 3（$\Pr[PrivK^{eav}_{A,\Pi} = 1] = \frac{1}{2}$）$\Rightarrow$ Def. 2（$\Pr[\mathrm{Enc}_K(m) = c] = \Pr[\mathrm{Enc}_K(m') = c]$）

Proof (Prove by contraposition)

□

3.3 One-Time Pad (OTP)

Construction of OTP

Fix an integer $l > 0$, and let $|\mathcal{M}| = |\mathcal{C}| = |\mathcal{K}| = l$.
(whch means all are binary strings of length l, i.e., $\{0,1\}^l$)

Key generation algorithm Gen: uniformly randomly chooses a key $k \in \mathcal{K}$, k is l-bit key.

Encryption algorithm Enc: given $k \in \{0,1\}^l$ and a message $m \in \{0,1\}^l$, Enc outputs a ciphertext $c = m \oplus k$.

Decryption algorithm Dec: given k, c, Dec outputs message $m = c \oplus k$.

Prove that OTP is perfectly secret

Proof (Proved by Def. 1)

(i) For an arbitrary $c \in \mathcal{C}$ and $m \in \mathcal{M}$

$$\Pr[C = c \mid M = m] = \Pr[\mathrm{Enc}_K(m) = c] = \Pr[m \oplus K = c] = \Pr[K = m \oplus c] = \frac{1}{2^l}$$

(ii) Fix any $\mathrm{dist}(\mathcal{M})$, for any $c \in \mathcal{C}$

$$\begin{aligned}\Pr[C = c] &= \sum_{m' \in \mathcal{M}} \Pr[C = c \mid M = m'] \cdot \Pr[M = m'] \\ &= \sum_{m' \in \mathcal{M}} \frac{1}{2^l} \cdot \Pr[M = m'] \\ &= 2^{-l}(\sum_{m' \in \mathcal{M}} \Pr[M = m']) \\ &= 2^{-l}\end{aligned}$$

(iii)

$$\begin{aligned}\Pr[M = m \mid C = c] &= \frac{\Pr[C = c \mid M = m] \cdot \Pr[M = m]}{Pr[C = c]} \\ &= \frac{2^{-l} \cdot \Pr[M = m]}{2^{-l}} \\ &= \Pr[M = m]\end{aligned}$$

□

4 L4

4.1 Limitation of Perfect Secrecy

Theorem 1 (Limitation of perfect secrecy)
If $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ is a perfectly secret encryption scheme with message space $\mathcal{M}$ and key space $\mathcal{K}$, then

$$|\mathcal{M}| \leqslant |\mathcal{K}|$$

Proof

Suppose $|\mathcal{K}| < |\mathcal{M}|$, Π cannot be perfectly secret.

Consider the uniform $\mathrm{dist}(\mathcal{M})$ and fix $c \in \mathcal{C}$, $\Pr[C = c] = 0$.

Let $\mathcal{M}(c)$ be the set of possible message which contains all possible messages decrypted by c. That is,

$$\mathcal{M}(c) \stackrel{\text{def}}{=} \{m \mid m = \mathrm{Dec}_K(c) \text{ for some } k \in \mathcal{K}\}$$

Dec is deterministic function, so $|\mathcal{M}(c)| \leqslant |\mathcal{K}|$.
(We know $\mathrm{Dec}_k(c) \coloneqq m$, and different values of k may map to the same m. If all m are distinct for different k, then equation holds; otherwise, $|\mathcal{M}(c)| < |\mathcal{K}|$.)

If $|\mathcal{K}| < |\mathcal{M}|$ and $\mathcal{M}(c) \leqslant |\mathcal{K}|$, there exist some $m' \in \mathcal{M}$ but $m' \notin \mathcal{M}(c)$.
$\Rightarrow \Pr[M = m' \mid C = c] = 0 \neq \Pr[M = m']$, which is not perfect secrecy. □

Quiz

We know that it's impossible to achieve pefect secrecy with shorter key size. So, what can we do or modify some factors to achieve shorter key? Any tradeoff (factor)?

§ Shannon's Theorem

Theorem 2 (Shannon's theorem)
Let $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ be an encryption scheme with message space M for which $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$.
The scheme is perfectly secret if and only if:

1. Every key $k \in \mathcal{K}$ is chosen with probability $\dfrac{1}{|\mathcal{K}|}$ by Gen
2. For every $m \in \mathcal{M}$ and every $c \in \mathcal{C}$, there exists a unique key $k \in \mathcal{K}$ such that $\mathrm{Enc}_k(m)$.

Quiz

Design a tricky scheme Π that $k \in \mathcal{K}$ is **NOT** uniformly chosen. Show Π is **NOT** perfectly secret by using Definition 1, 2 or 3.
(Hint: modify shift cipher or one-time pad)

4.2 Private Key Encryption

§ Computational Security

Perfect secrecy 的缺點 (weakness)：

- 只能用一次 (one-time use)
- key 的長度一定要大於訊息的長度 ($|\mathcal{K}| \geqslant |\mathcal{M}|$)

Computational security 是從計算上保證安全的一種安全性。它不像 pefect secrecy 那樣地完美，但可以更靈活地建立 scheme (如減少 key 的長度)。

從 adversary 的觀點來看：

Adversary's power	time/space	success probability
Perfect secrecy	unbounded	= random guess
Computational security	polynomial time	= random guess + small probability

目的：減少安全性，來換取更好的效率 (by weakening the security, to achieve better efficiency)。

§ Concrete Definition

Definition 1 (Concrete definition)
A scheme Π is (t, ϵ)-secure if any adversary A running for time at most t, succeeds in breaking Π with probability at most ϵ.

Ex: $t = 2^{10}$, $\epsilon = \dfrac{1}{2^{100}}$

§ Asymptotic Definition

在這裡的 adversary A 的能力 (power) 是以漸進式術語來定義的 (asymptotic setting)：

- Efficient adversary：這種 adversary 會執行可以在 polynomial time 內跑完的演算法。這種演算法的執行時間是 $p(n)$，其中 p 為多項式集合，而 n 為安全參數 (security parameter)。
- Small probability of success: 成功機率小於任何 polynomial 的倒數。也就是

$$\Pr[\text{success}] < \frac{1}{p(n)}, \text{ where } p \text{ is arbitary polynomial}$$

PPT = Probabilistic Polynomial Time

Definition 2 (Asymptotic definition)
A scheme is secure if for any PPT adversary succeeds in breaking the scheme with at most **negligible** probability.

§ Negligible Probability

Negligible function 是漸進小於 (asymptotic smaller) 任何 polynomial function 的函數。

Definition 3
A function f is negligible if
for every positive polynomial p, there exists a number N such that $f(n) < \dfrac{1}{p(n)}$ where $n > N$.

Example:
Let $g(x) = \dfrac{1}{2^x}$.
There exists N such that $g(n) < \dfrac{1}{p(n)}$.

$$\begin{aligned} & g(n) < \frac{1}{p(n)} \\ \Rightarrow \quad & \frac{1}{2^n} < \frac{1}{n^k} \\ \Rightarrow \quad & 2^n > n^k \\ \Rightarrow \quad & n > k \cdot log_2(n) \\ \Rightarrow \quad & \frac{n}{log_2(n)} > k \end{aligned} \qquad \text{(k is positive constant)}$$

If $n > k^2$, this inequality holds.

Quiz
Let $\mathrm{negl}(x), \mathrm{negl}'(x)$ be negligible functions.

1. A function f_1, defined by $f_1(x) = \mathrm{negl}(x) + \mathrm{negl}'(x)$
2. A function f_2, defined by $f_2(x) = p(x) \cdot \mathrm{negl}(x)$, where $p(x)$ is positive polynomial.

Are f_1 and f_2 are still negligible functions? **Yes**

Summary
任何關於 computational security 的 security definition 都由下列組成：
1. 破解 scheme 的定義（也就是怎麼樣才叫 scheme 被破解了）
2. 關於 adversary 的能力

我們通常將 adversary 塑造 (model) 成有效率（有計算能力）的演算法，且只考慮 adversary 可以在 polynomial time 之內執行的 probabilistic stratigies。

Definition 4
A scheme is secure if for every PPT adversary A carrying out an attack of some formally specified attack type, and the probability that A succeeds is negligible.

§ Private Key Encryption

Definition 5 (Private key encryption)
A private key encryption is a tuple of PPT algorithm $(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$

- Key generation: $\mathrm{Gen}(1^k) \to k$. 這裡 n 的意義是 $|\mathcal{K}| \geqslant n$ 或 $|\mathcal{K}| = \mathrm{poly}(n)$。
- Encryption: $\mathrm{Enc}_k(m) \to c$, where key k and $m \in \{0,1\}*$ are inputs. 若 $m \in \{0,1\}^{l(n)}$，我們會稱這個等式為 fixed-length private key encryption with message length $l(n)$。
- Decryption: $\mathrm{Dec}_k(c) \coloneqq m$. If c cannot be decrypted, then outuput $\perp$ (error).

Basic definition of security

Eavesdropping（竊聽）: adversary 的策略或能力

這裡和之前的 $PrivK^{eav}_{A,\Pi}$ 大致一樣，參見 3.2 Perfect secrecy III。

差異：

- Perfect secrecy：沒有 security parameter，因為不在意 adversary 有多少的能力

$$\Pr[PrivK^{eav}_{A,\Pi} = 1] = \frac{1}{2}$$

- Computational security：有 security parameter n

$$\Pr[PrivK^{eav}_{A,\Pi} = 1] \leqslant \frac{1}{2} + \mathrm{negl}(n)$$

5 L5

5.1 Basics

§ Scenario

Sender S 和 receiver R 彼此有有一把相同的 key k，且 S 想要發送訊息給 R。
在發送訊息前，S 會先使用 k 將明文 m 加密為密文 c（$c \leftarrow \mathrm{Enc}_k(m)$），之後 S 將 c 傳送給 R。
R 在收到 c 後，使用同一把 key k 將 c 解密（$m \coloneqq \mathrm{Dec}_k(c)$）來得到 m。

關於這個 scenario 的正式的定義可以參見 Definition 5 Private key encryption。

§ 安全性定義

使用前面提到的 $PrivK^{eav}_{A,\Pi}$，參見 3.2 Perfect secrecy III。

5.2 EAV-security

EAV = eavesdropping

Definition 6 (EAV-secruity of private key encryption)
A private key encryption scheme Π is **EAV-secure** if for all PPT adversary A, there is a negligible function negl such that for all n,

$$\Pr[PrivK^{eav}_{A,\Pi}(n) = 1] \leqslant \frac{1}{2} + \mathrm{negl}(n)$$

(The probability is taken over randomness used by adversary and used in experiment.)

§ Equivalent Formulation of EAV-security

前一節 EAV-security 的定義等價於下面這句話：
「無論 PPT adversary A 看到由 m_0 或 m_1 加密過後的密文，其表現都相同。」
（Every PPT adversary behaves the same whether it sees ciphertext of m_0 or m_1.）

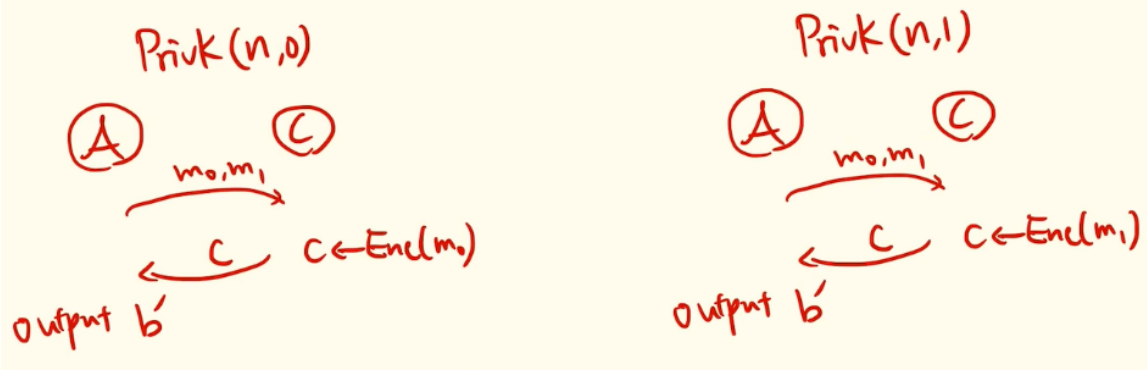
更精確的定義是：

- 修改之前的定義為 $PrivK^{eav}_{A,\Pi}(n, b)$，其定義都和之前一樣，除了 b 是固定的，而不是隨機選擇的。
- 定義 $out_A(PrivK^{eav}_{A,\Pi}(n, b)) = b'$，其中 b' 是 A 的 output。
- 沒有 PPT adversary A 可以知道現在是 experiment $PrivK^{eav}_{A,\Pi}(n, 0)$ 或 $PrivK^{eav}_{A,\Pi}(n, 1)$。

正式定義如下：

Definition 7 (Equivalent formulation of EAV-security)
Π is EAV-secure if for all PPT adversary A, there is a negligible function negl such that

$$|\Pr[out_A(PrivK^{eav}_{A,\Pi}(n, 0)) = 1] - \Pr[out_A(PrivK^{eav}_{A,\Pi}(n, 1)) = 1]| \leqslant \mathrm{negl}(n)$$



Quiz

In $PrivK$, we define A to choose two messages with the same length.
Please write your thought for the impossibility to support arbitrary-length messages.

5.3 Private Key Encryption

§ Pseudorandom Generator

Definition 8 (pseudorandom generator, PRG)
Let l be a polynomial and G is a deterministic polynomial-time algorithm. For any n and input $s \in \{0,1\}^n$, the output of $G(s)$ is $l(n)$-length.
We say G is a PRG if:

- Expansion: for every n, it holds $l(n) > n$. l is a so-called expansion factor of G.
- Pseudorandomness: for any PPT algorithm D (aka distinguisher), there is a negligible function negl such that
$$|\Pr[D(G(s)) = 1] - \Pr[D(r) = 1]| \leqslant \mathrm{negl}(n)$$
where $s \in \{0,1\}^n$ and $r \in \{0,1\}^{l(n)}$ is a turly random variable.

§ PRG-based Construction of Fixed-length Private Key Encryption

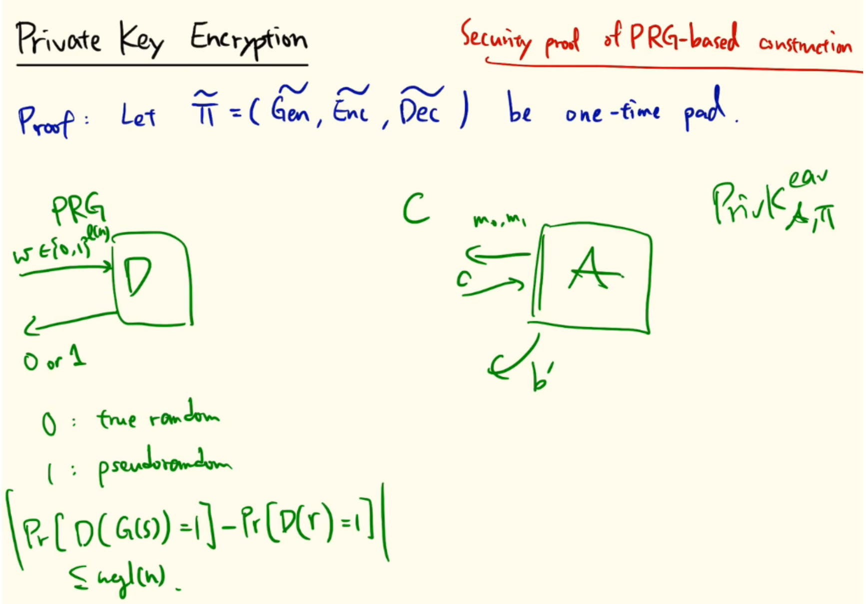
Let G be a PRG with expansion factor l.
Scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$.

- $\mathrm{Gen}(1^n)$: on input 1^n, choose uniform $k \in \{0,1\}^n$.
- $\mathrm{Enc}(k,m)$: with input of a message $m \in \{0,1\}^{l(n)}$ and outputs a ciphertext $c = G(k) \oplus m$
- $\mathrm{Dec}(k,c)$: with input of a ciphertext $c \in \{0,1\}^{l(n)}$ and outputs a message $m = G(k) \oplus c$

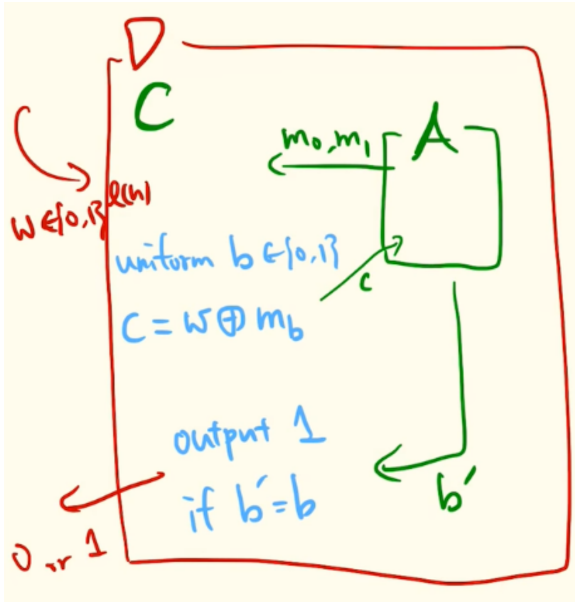
這種構造法和 OTP（見 3.3 One-Time Pad (OTP)）很像。那時候的 OTP 會遇到 perfect secrecy 的限制，也就是 key 的長度至少要和 message 一樣長（$|\mathcal{K}| \geqslant |\mathcal{M}|$）。在這裡，我們通過 PRG 來將原本的 key 長度 n 擴展成 $l(n)$，藉此來降低 key 的長度。而其代價就是，這種使用 PRG 的方法一定不是 perfect secrecy。

P.S. 由於 private key encryption 要求雙方要事先使用安全通道交換同一把 key。若在這種情景下使用和 message 一樣長的 key，那我們就可以直接使用這個安全通道交換訊息本身了，而無需進行加密。

§ PRG-based construction is EAV-secure



(a) Distinguisher D and adversary A



(b) Reduction in proof

Theorem 3
If G is a pseudorandom generator, then the construction Π is a EAV-secure.

其逆否命題為「如果 Π 不是 EAV-secure，則 G 也不是 PRG」。

證明思路
由 D 扮演 challenger。
在 reduction 時是 D 包在 A 的外面。

Let $\tilde{\Pi} = (\widetilde{\text{Gen}}, \widetilde{\text{Enc}}, \widetilde{\text{Dec}})$ be one-time pad.

1. If w is uniform chosen form $\{0,1\}^{l(n)}$,

$$\Pr[D(w) = 1] = \Pr[PrivK^{eav}_{A,\tilde{\Pi}}(n) = 1] = \frac{1}{2}$$

   這種情況是 one-time pad 的情況，也就是使用 true randomness。

2. If $w = G(k)$ by choosing uniform $k \in \{0,1\}^n$,

$$\Pr[D(G(k)) = 1] = \Pr[PrivK^{eav}_{A,\Pi}(n) = 1]$$

   這種情況是使用 pseudorandomness。
   這個機率是我們所要證明的，可以透過第三點來反推其機率為 $\leqslant \frac{1}{2} + \text{negl}(n)$

3. If G is PRG,

$$|\Pr[D(G(k)) = 1] - \Pr[D(w) = 1]| \ \leqslant \ \text{negl}(n)$$

Proof details

Let A be a PPT adversary. Our goal is to contract a distinguisher D (which is going to break PRG) that takes a string w as input.

Goal of D: determine whether
- (i) w was chosen uniformly (where $w \in \{0,1\}^{l(n)}$)
- (ii) w was generated by choosing uniform $k \in \{0,1\}^n$ and computing $w = G(k)$ (where $w \in \{0,1\}^{l(n)}$ and $l(n) > n$)

Output of D: outputs 1 if case (i) mentioned above; otherwise, outputs 0

Theorem used:

$$|\Pr[D(r) = 1] - \Pr[D(G(k)) = 1]| \leqslant \text{negl}(n)$$

where $r \leftarrow \{0,1\}^{l(n)}$, and $k \leftarrow \{0,1\}^n$.

Activites of D: (connect A and D)
Emulate the eav experiment $PrivK^{eav}_{A,\Pi}$ for A

— If A wins, D thinks $w \coloneqq G(k)$.
— If A fails, D thinks w is uniform chosen.

Proof

(Refer to figure Reduction in proof)
Distinguisher D get an input of a string $w \in \{0,1\}^{l(n)}$.

Step 1： Run A to obtain a pair of messages $m_0, m_1 \in \{0,1\}^{l(n)}$

Step 2： Choose a uniform bit $b \in 0, 1$. Set $c = w \oplus m_b$

Step 3： Send c to A

Step 4： Later, A returns b'

D outputs

— 1, if $b' = b$
— 0, if $b' \neq b$

Note that probability of output of D is related to $\Pr[PrivK^{eav}_{A,\Pi}]$.
If $\Pr[PrivK^{eav}_{A,\Pi}] > \dfrac{1}{2} + \text{negl}$,

$$\Pr[out_D = 1] > \frac{1}{2} + \text{negl}$$
$$\Pr[out_D = 0] \leqslant \frac{1}{2} - \text{negl}$$

□

5.4 Chosen Plaintext attack & CPA-security

CPA = Chosen Plaintext Attack

§ CPA security

在這個情景下的 adversary A 可以存取 encryption oracle。

Encryption oracle：是一個黑盒子，我們不知道其運作原理，但給它輸入和取得它的輸出。A 可以將明文 m 給 oracle，之後 oracle 會將明文加密為密文 $c \leftarrow \text{Enc}_k(m)$ 回傳給 A。

Experiment $PrivK^{cpa}_{A,\Pi}$

Step 1： A 可以選擇明文 m_i 給 C

Step 2： C 建立密鑰 $k \leftarrow \text{Gen}(1^n)$，並將明文加密為密文 $c_i \leftarrow \text{Enc}\ \ m_i)$ 回傳給 A。

Step 3： A 此時可以將這些收集到明文-密文對（plaintext-ciphertext pair）儲存起來。由於 A 是 PPT adversary，所以 A 可以收集的 pair 數為 poly-many。

Step 4： A 選擇 m_0 和 m_1 傳給 C 進行 chanllenge。之後的事情都和之前的 EAV-secure 的 experiment 一樣。

Step 5： 若 A 贏了，則 $PrivK^{cpa}_{A,\Pi}(1^n) = 1$。

P.S. 前三步稱為 encryption oracle query。而 challenge 之後一樣可以進行 eneryption oracle query，直到 A output b'。

Quiz

Show PRG-based construction Π is not CPA-secure.
(Hint: give A in $PrivK^{cpa}_{A,\Pi}$ to break Π)

6 L6

6.1 CPA-secure Encryption

§ Pseudorandom Function (PRF)

Let $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ be an efficient length-perserving keyed function.
F is a pseudorandom function (PRF) if all PPT distinguisher D, there is a negligible function such that

$$|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]| \leqslant \mathrm{negl}(n)$$

where $k \leftarrow \{0,1\}^n$, and $f \leftarrow \mathrm{Func}_n$ is a random function .

Note that Func_n is a set containing all posibilities of $\{0,1\}^n \to \{0,1\}^n$.

簡而言之，無法區分是否為 random function 的 function，即為 pseudorandom function。

Quiz

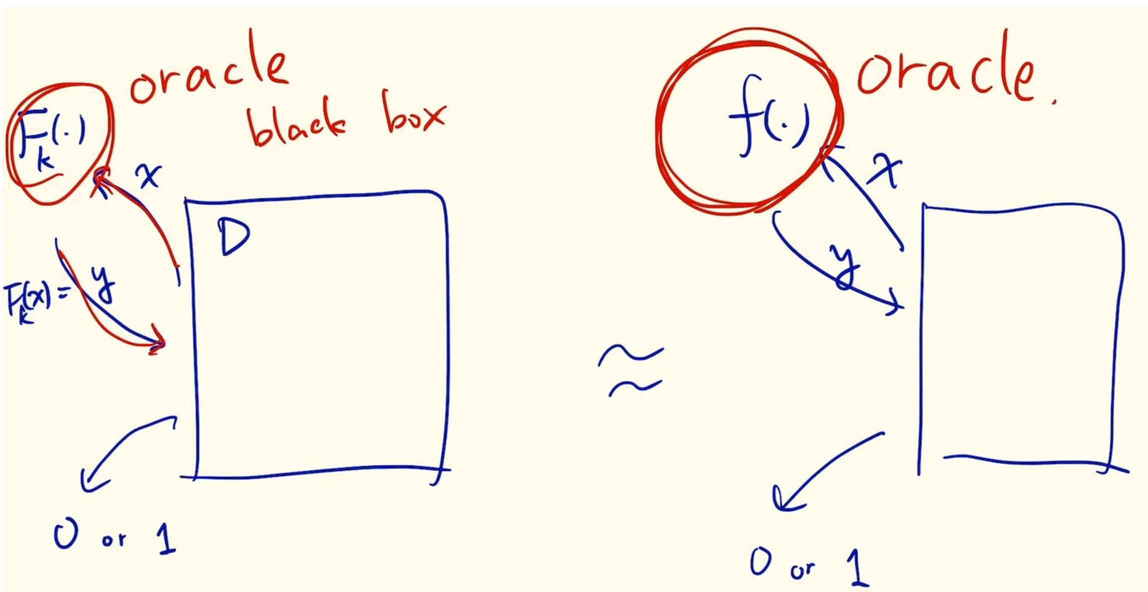
Show that the size of Func_n (aka $|\mathrm{Func}_n|$) equals to $2^{n \cdot 2^n}$.

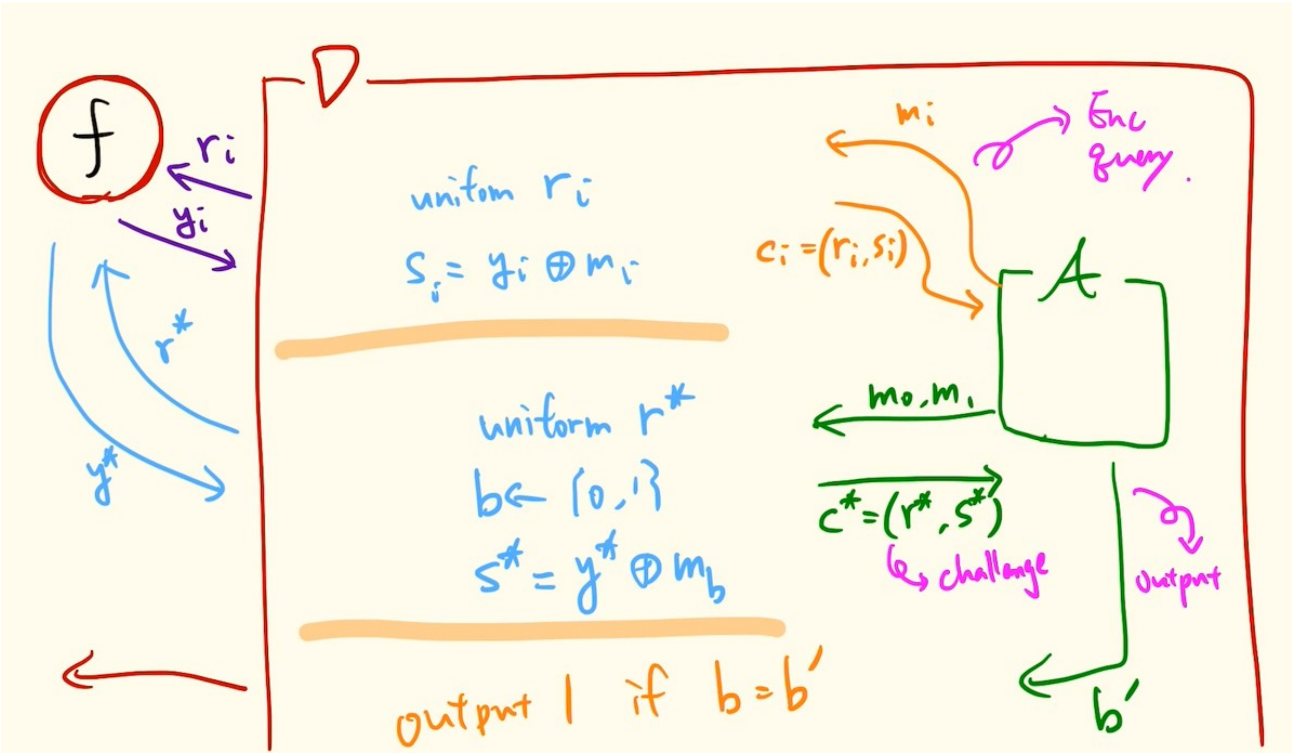
Ans:
The domain $\{0,1\}^n$ has 2^n elements, and the codomains $\{0,1\}^n$ also has 2^n elements.
For each of the 2^n inputs, a function can assign any of 2^n outputs.
So the number of such functions is

$$(2^n)^{2^n} = 2^{n \cdot 2^n}.$$

§ PRF-based Construction

這裡的 distinguisher D 有一個特別的能力，可以詢問 $F(\cdot)$ (可以將它視為是一種 oracle)，而 $F(\cdot)$ 可能是 PRF $F_k(\cdot)$ 或是 random function $f(\cdot)$，但 D 無法區分到底是哪一種。





Let F be a PRF and $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$:

- $\text{Gen}(1^n)$: uniformly choose $k \in \{0,1\}^n$ as the key.
- $\text{Enc}(k, m)$: $m \in \{0,1\}^n$, uniformly choose $r \in \{0,1\}^n$, and compute $s = F_k(r) \oplus m$ and $c = (r, s)$.
- $\text{Dec}(k, c)$: parse $c = (r, s)$, output $m = F_k(r) \oplus s$

Theorem 4 (PRF-based construction is CPA-secure)
If F is a PRF, the construction Π is CPA-secure.

證明思路

Contraposition: If Π is not CPA-secure, then F is not PRF.

Proof

Let $\tilde{\Pi} = (\widetilde{\text{Gen}}, \widetilde{\text{Enc}}, \widetilde{\text{Dec}})$ be one-time pad.

By modeling D and A:

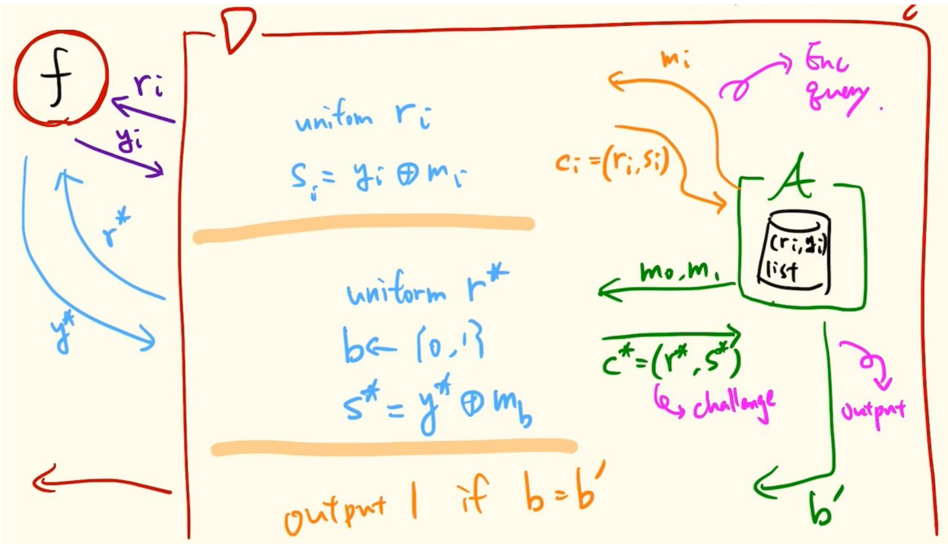
(i)
$$\Pr[D^{F_k(\cdot)}(1^n) = 1] = \Pr[PrivK^{cpa}_{A,\Pi}(n) = 1]$$

(ii)
$$\Pr[D^{f(\cdot)}(1^n) = 1] = \Pr[PrivK^{cpa}_{A,\tilde{\Pi}}(n) = 1]$$

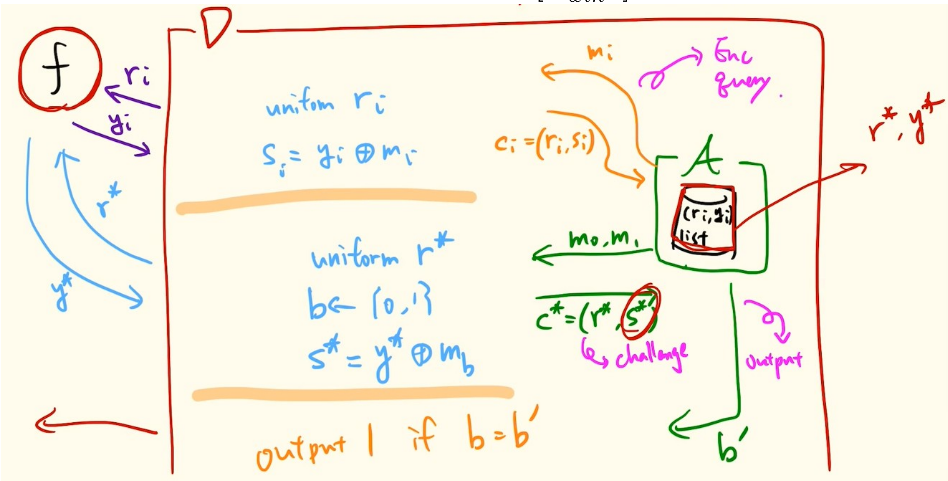
(iii) By assumption,
$$|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]| \leqslant \text{negl}(n)$$

$\Pr[PrivK^{cpa}_{A,\tilde{\Pi}}(n) = 1] =?$

- Case 1: If r^* is never used in Enc query, $\Pr[A^{case1}_{win}] = \frac{1}{2}$



- Case 2: If r^* is used in Enc query, $\Pr[A_{win}^{case2}] = 1$



Define an event: $Repeat$, if r^* is used.

$$
\begin{aligned}
\Pr[PrivK^{cpa}_{A,\widetilde{\Pi}}(n) = 1] &= \Pr[PrivK^{cpa}_{A,\tilde{\Pi}}(n) = 1 \wedge Repeat] + \Pr[PrivK^{cpa}_{A,\tilde{\Pi}}(n) = 1 \wedge \neg Repeat] \\
&\leqslant \Pr[Repeat] + \Pr[PrivK^{cpa}_{A,\tilde{\Pi}}(n) = 1 \wedge \neg Repeat] \\
&= \frac{q(n)}{2^n} + \frac{1}{2} \\
&= \mathrm{negl}(n) + \frac{1}{2}
\end{aligned}
$$

Use this result to the previous (ii), and then we can get the result of

$$\Pr[D^{F_k(\cdot)}(1^n) = 1] \leqslant \frac{1}{2} + \mathrm{negl}(n)$$

□

6.2 Encryption for Arbitrary Length Message

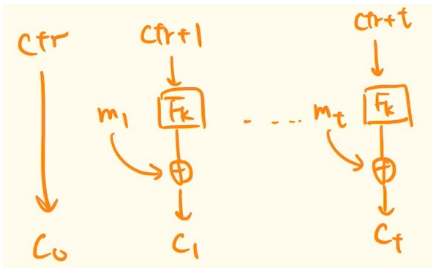
CPA-security ⇒ multiple encryption

當我們有任意長度 L 的訊息需要加密，我們可以對每 n bit 為一塊的訊息個別進行加密，如此便可以達到加密任意長度訊息的目的。
但前面提到的 CPA-secure 的方法會讓密文長度變成明文長度的兩倍，原本 n-bit block message 就會變成 $2n$-bit ciphertext，最終使得長度為 L 的訊息在加密後會變成長度的 $2L$ 的 ciphertext。

接下來會介紹數個解決這問題的方法，統稱為 mode of encryption。

§ Counter Mode (CTR Mode)

$\text{Enc}_k(m_1, \ldots, m_t)$, whose total length is $n \cdot t$
- Ramdomly choose $\text{ctr} \leftarrow \{0,1\}^n$, set $c_0 = \text{ctr}$, whose length is n
- For $i = 1$ to t, compute $c_i = m_i \oplus F_k(\text{ctr} + i)$, where F is PRF
- Output ciphertext $(c_0, c_1, \ldots, c_t)$, whose length is $n \cdot (t+1)$



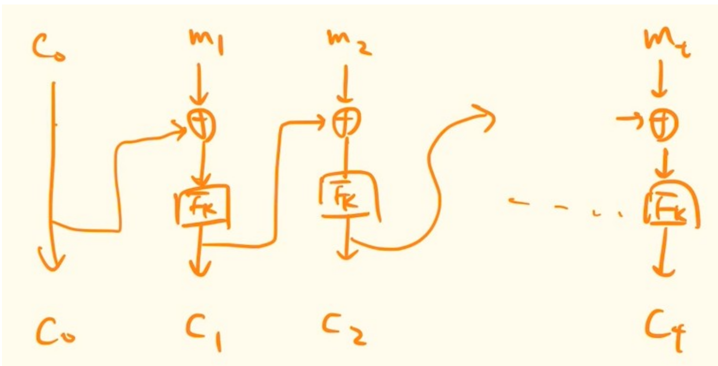
Theorem 5
If F is PRF, then CTR mode is CPA-secure.

§ Cipher Block Chaining (CBC mode)

CBC mode is more practical and used in our life.

$\text{Enc}_k(m_1, \ldots, m_t)$
- Randomly choose $c_0 \leftarrow \{0,1\}^n$
- For $i = 1$ to t, compute $c_i = F_k(m \oplus c_{i-1})$
- Output ciphertext $(c_0, c_1, \ldots, c_t)$

Note that decryption needs F_k^{-1}.



Theorem 6
F is PRF, CBC mode is CPA-secure.

Quiz

Show decryptiong of CBC. Draw a flowchart.

§ Electronic Codebook (ECB mode)

$\mathrm{Enc}_k(m_1, \ldots, m_t) \to F_k(m_1), \ldots, F_k(m_t)$

Decryption also needs F_k^{-1}.

ECB is not EAV-secure and CPA-secure ($\because$ ECB is deterministic).

Quiz

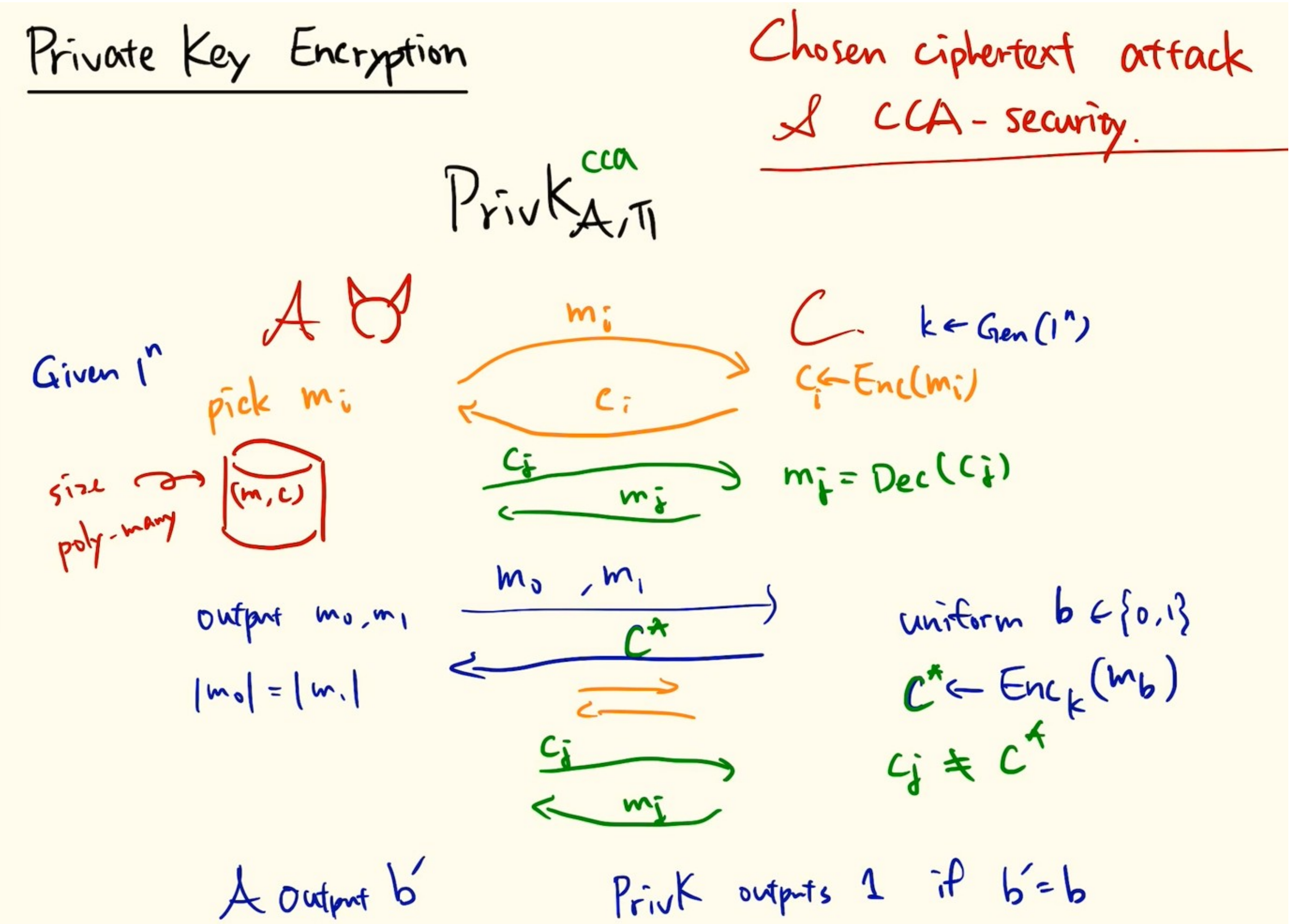
Prove ECB is not EAV-secure or CPA-secure.
寫出 EAV 的攻擊手段。

7 L7

7.1 CCA-security

CCA = Chosen Ciphertext Attack

允許攻擊者使用 decryption oracle，給予其密文，它會回傳明文。但限制攻擊者不可使用欲 challenge 的密文 c^*。



§ Remark on CCA-security

CCA => Lunch time attack

Is CCA realistic（現實可行）?
No, but still have weak decryption oracle which only leak 1-bit message from decrypted ciphertext, which is suffice to learn the entire message (plaintext).

§ Padding for Arbitrary Length

Assuming block size is L bytes.
If message length = $L(t-1)+2$, then we need padding which is $L-2$ bytes.

One of the pratical padding solution is PKCS #5:

- Block length: L byte
- b bytes to apppend the message to a multiple of L, where $1 \leqslant b \leqslant L$. Note that $b \neq 0$.
- Append b (encoded 1 byte), b time(s).
  i.e., b=3 $\Rightarrow$ 0x03 03 03, where underlines indicate 1 byte.

Quiz

當最後一個 block 本來就是滿的，應該如何進行 padding？

Ans：
額外補上一個 block，並在每個 byte 填入 block size 大小的數值。
E.g. 設 block size 為 8 bytes，則額外新增一個 block，並在八個 bytes 中填入 0x08。

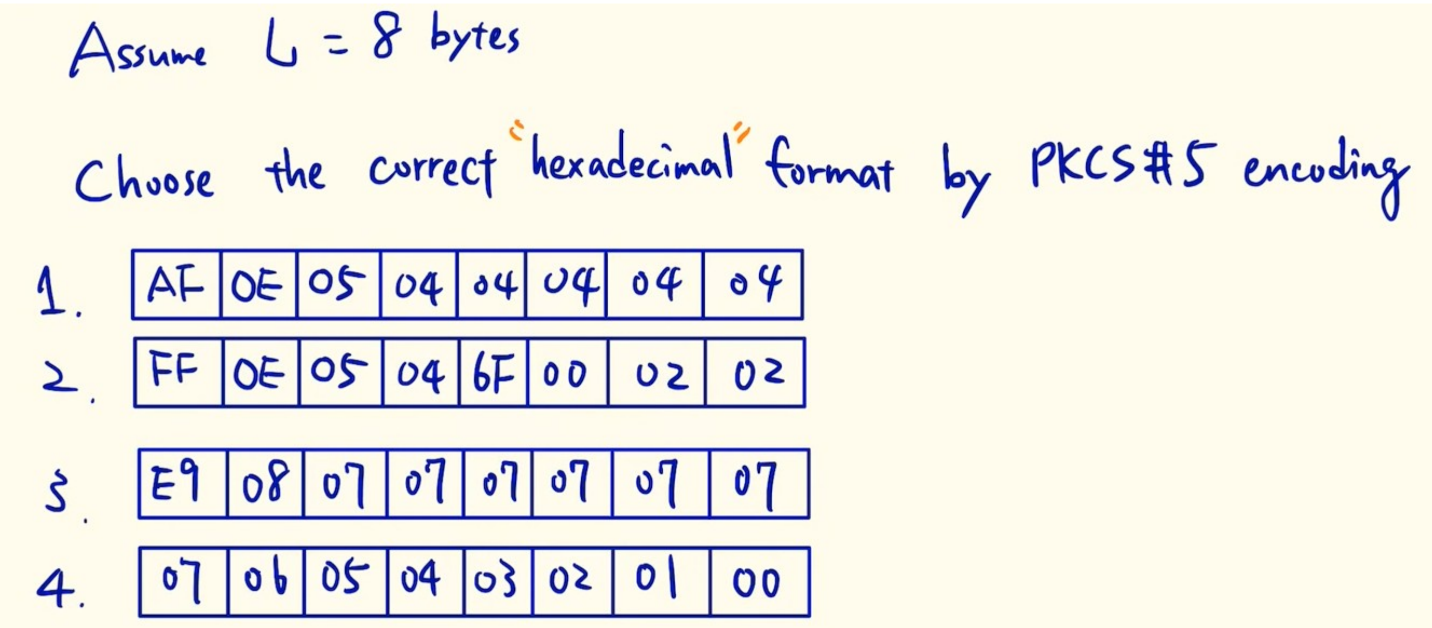
§ Decryption

使用 CBC mode 解密。

在 decryption 後檢查 encoded data，設最後一個 byte 為 b:

- 若 $b = 0$ 或 $b > L$，return error
- 若最後 b 個 bytes 並不全都等於 b，return error
- 否則，去除 padding 的部分，並 return message

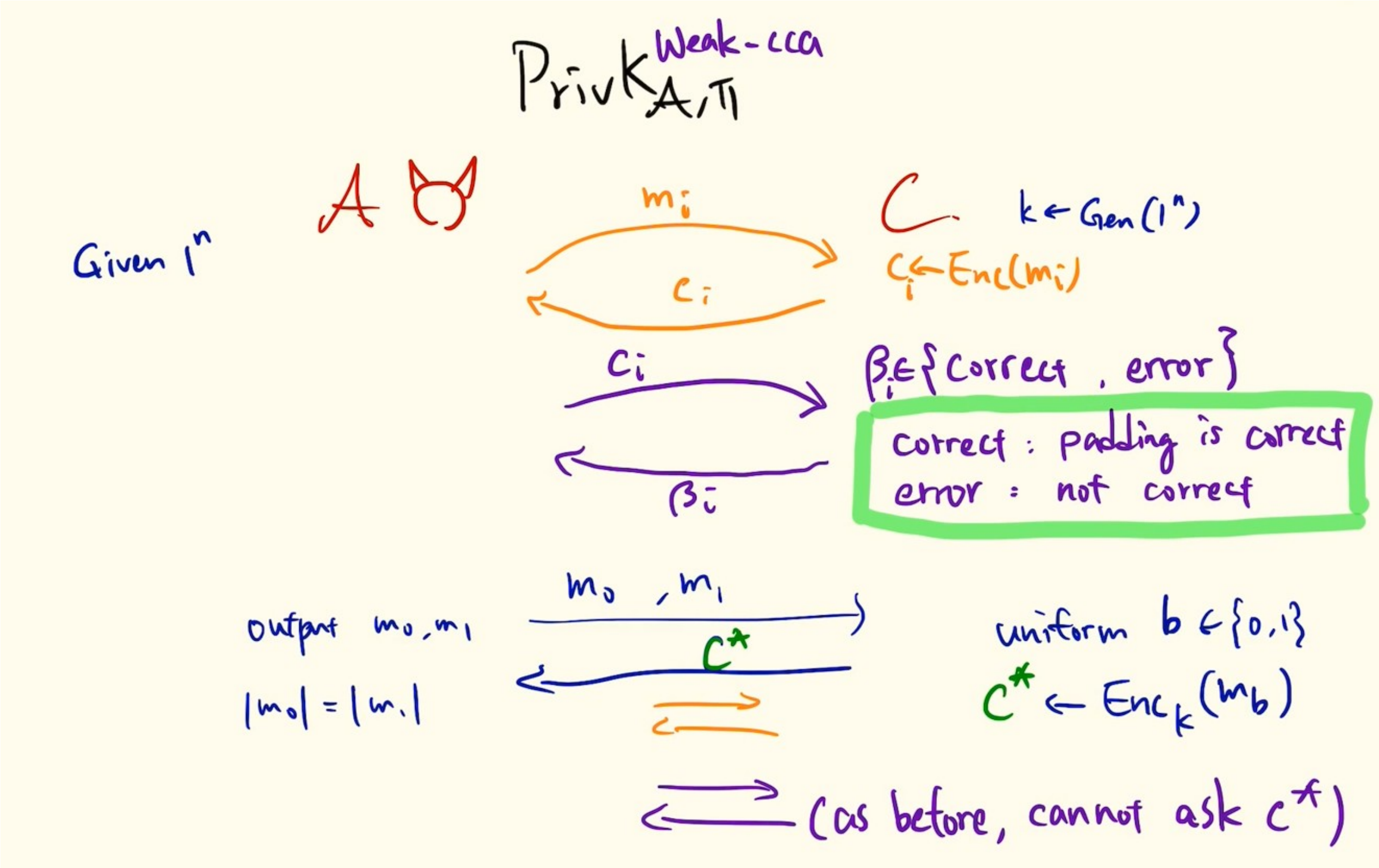
Quiz



Ans: (2)

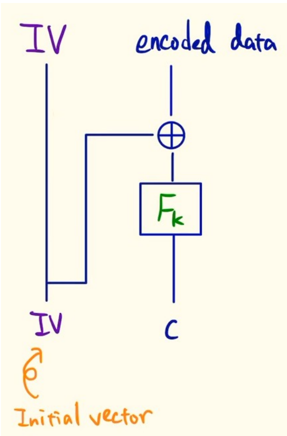
§ Weak CCA with Padding Oracle

這裡出現了一種新的 oracle。給定 ciphertext，它會 return padding 是正確或錯誤。



§ Padding Oracle Attack

基本原理

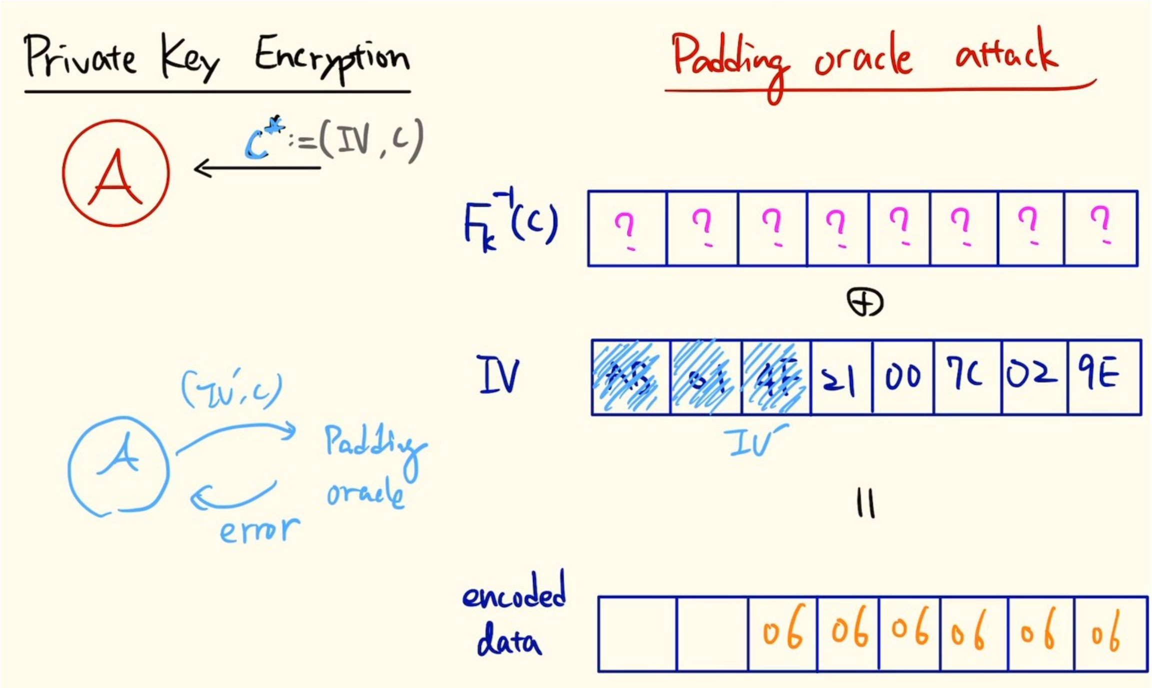


其中 encoded data = $F_k^{-1}(c) \oplus IV$

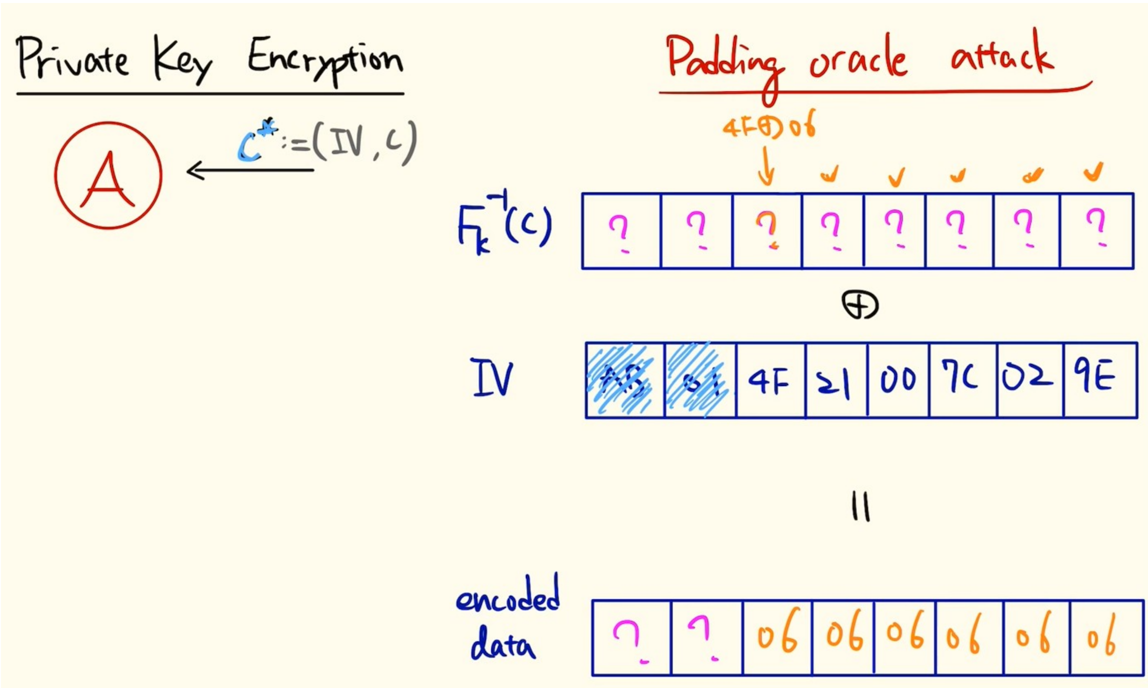
我們可以觀察到，若 attacker A 修改了 IV 的第 i 個 byte，這個動作只會影響到 encoded data 的第 i 個 byte。(∵ CBC 使用 XOR 運算)

攻擊過程

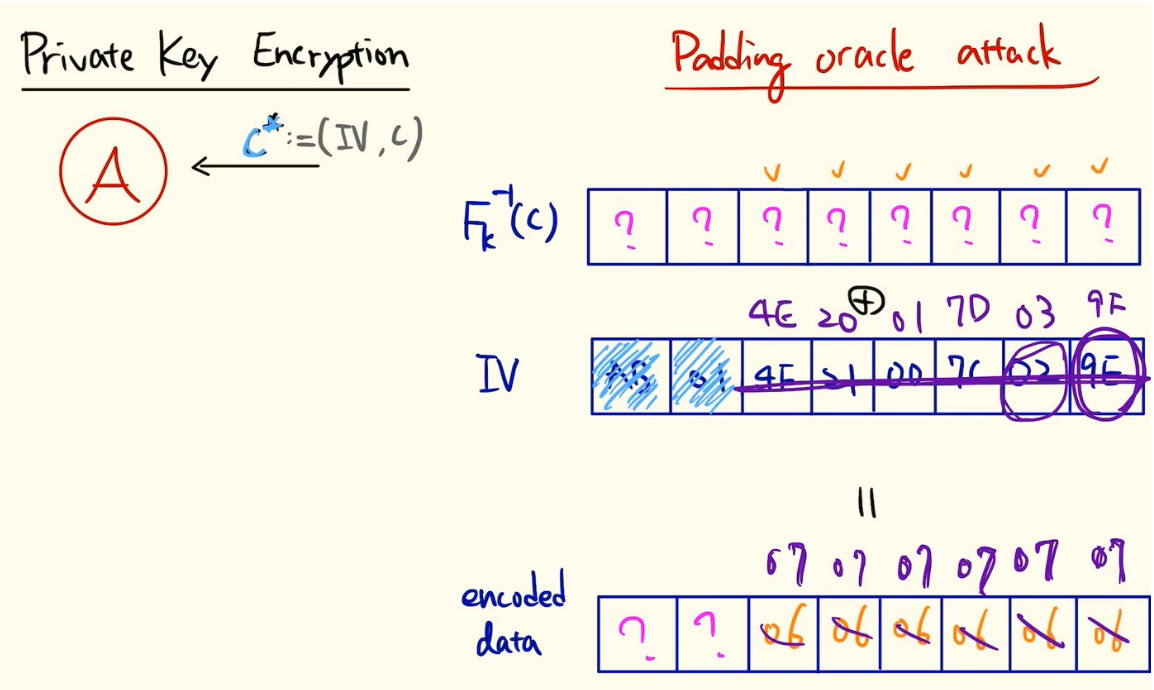
Attacker 會先由左至右逐個修改 byte，並在每次修改完之後都詢問 oracle。直到 oracle return error，該 byte 到最右邊即為 padding bytes：



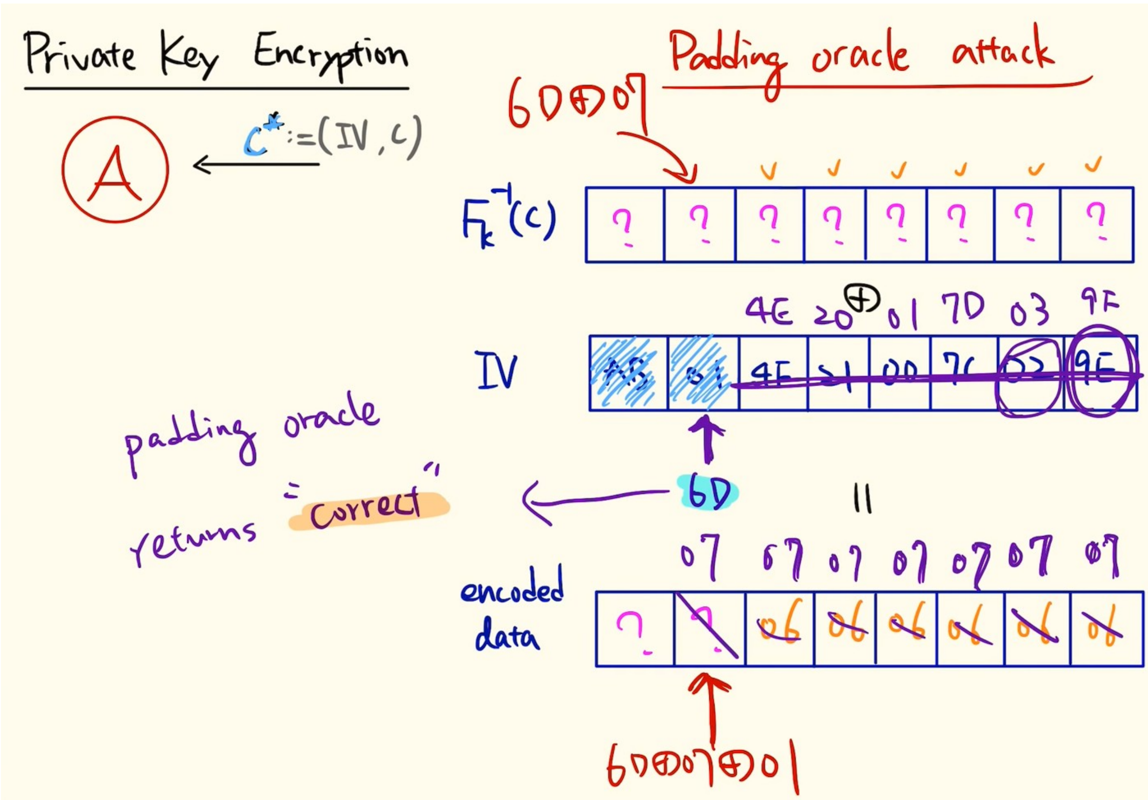
之後 attacker 便可藉由 IV 和 encoded data 反推 $F_k^{-1}(c)$ 的 padding 部分為何：



為了得到非 padding 部分的 message 為何，attacker 可以藉由修改 padding 部分為原本的值再加一，並重新計算新 IV 的 padding 部分：



再來是持續修改非 padding 部分的最後（最右）一個 byte，直到 padding oracle return correct，attacker 就可以知道此時的 encoded data 中的對應 byte 為原本的 padding 值再加一。最後依序計算 $F_k^{-1}(c)$ 得到密鑰，再將密鑰與 IV 做 XOR 得到原本的 encoded data。



持續進行這些步驟就可以得到完整的 encoded data 為何。

Remark on Padding Oracle Attack

- # of pading bytes: $< L$ padding oracle queries（確定 padding byte 的數量所需的次數）
- contain of one byte of the message: $\leqslant 2^8 = 256$ padding oracle queries（最多嘗試 256 次即可猜到 encoded data 中非 padding 部分的一個 byte）
- In $PrivK_{A,\Pi}^{weak-cca}$ with padding oracle, A choose m_0, m_1 such that $|m_0| = |m_1|$ and last significant byte of $m_0$is different from correspondence of m_1. And it only needs $\leqslant L + 2^8$ padding

oracle queries to finish it.

7.2 Message Authentication Code (MAC)

Secrecy：由 Enc 提供、adversary 無法知道訊息內容、不能涵蓋所有的 concerns（例如：訊息篡改）
Integrity：確保訊息不被篡改 (tampering)、驗證訊息的正確性

MAC = Message Authentication Code

§ Syntax

Alice 傳 message m 及一個可以驗證 message 的 tag t 給 Bob，而 Bob 在收到訊息後，透過 t 來驗證 m。

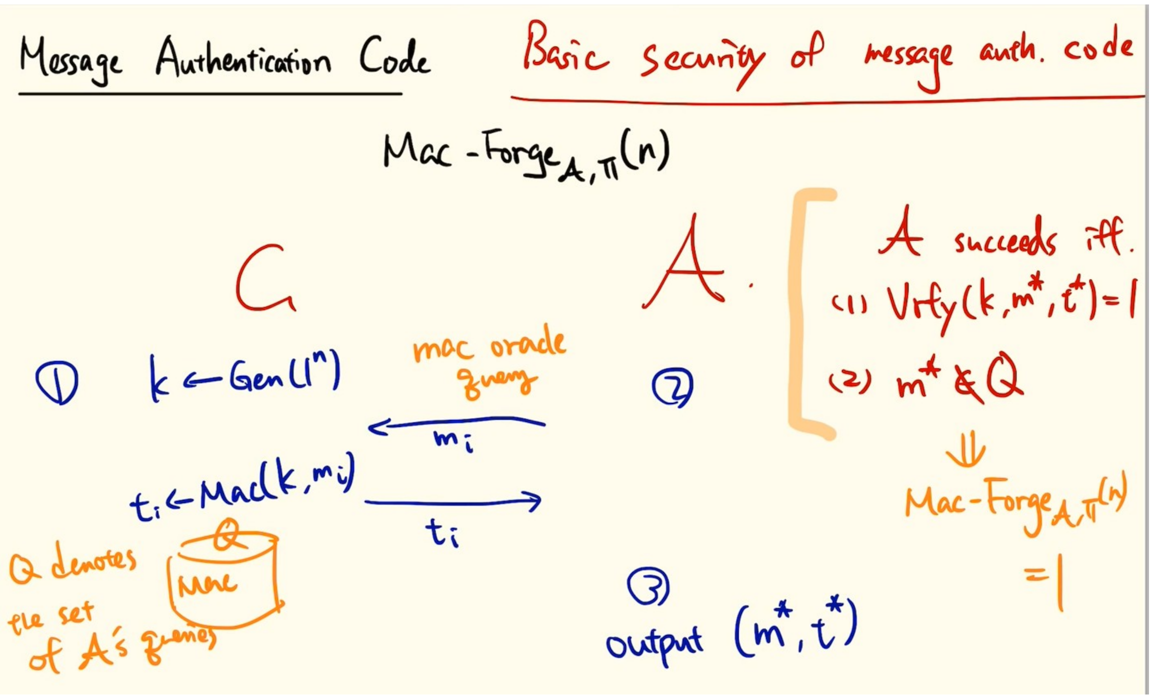
Π is a MAC construction. $\Pi = (\mathrm{Gen}, \mathrm{Mac}, \mathrm{Vrfy})$

- Key generation $\mathrm{Gen}(1^n) \to k$: a key
- Message authentication code $\mathrm{Mac}(k, m) \to t$: a tag
- Verification $\mathrm{Vrfy}(k, m, t) \coloneqq 0$ or 1, where 0 stands for rejection, 1 stands for acceptance.

Remark

- m is not hidden for Vrfy.
- 對於一個 deterministic MAC，Vrfy 在做的事情就是重新建立一次 t，並比較其是否與接收到的 t 相同。
- If $m \in \{0,1\}^{l(n)}$ where $l(n)$ is a polynomial, then MAC is fixed-length MAC.

§ Experiment $\mathrm{MAC\text{-}Forge}_{A,\Pi}(n)$



有 challenger C 和 adversary A：

Step 1： C 產生 key，即 $k \leftarrow \mathrm{Gen}(1^n)$

Step 2： A 選擇一個 message m_i，以此詢問 C，而 C 再 return 一個計算過後的 tag $t_i \leftarrow \mathrm{Mac}(k, m_i)$ 給 A。此外，C 可以使用一個 list Q 來收集所有來自 A 的 queries。

Step 3： A outputs 他的偽造 (m^*, t^*)

A 成功的條件（即 $\text{MAC-Forge}_{A,\Pi}(n) = 1$）有兩個（都要符合，if and only if）:

(1) $\mathrm{Vrfy}(k, m^*, t^*) = 1$

(2) $m^* \notin Q$

Remark

MAC-Forge 的 strong 版本是 MAC-sForge。

MAC-sForge：比 MAC-Forge 寬鬆一點，只要求 (m^*, t^*) 也不在 Q 中，即 $(m^*, t^*) \notin Q$。

If a deterministic MAC satisfies existential unforgeability in MAC-Forge, it also satisifeis strong unforgeability in MAC-sForge.

解釋：因為 deterministic MAC 會將一個 m 只對應到一個 t，所以如果 m^* 不在 Q 中，則與之對應的 t^* 也不會在 Q 中。因此 $(m^*, t^*) \notin Q$。

§ Strong Verison of Previous Experiment $\text{MAC-sForge}_{A,\Pi}(n)$

大致和之前相同，唯一不同處在於 A 成功的第二個條件改成 $(m^*, t^*) \notin Q$。這意味著 A 可以向 C query m^*，但只要可以偽造另一組不在 Q 中的 (m^*, t^*) 並且仍可以使用它通過 Vrfy，那麼就算 A 成功了。

只要在 MAC-sForge 是安全的，那它在 MAC-Forge 也是安全的。反之則不成立。

Definition 9 (Strong Security)

$$\Pr[\text{MAC-sForge}_{A,\Pi}(n) = 1] \leqslant \mathrm{negl}(n)$$

§ Security Definition of MAC

Definition 10

A message authentication code $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Vrfy})$ is existentially unforgeable under adaptive chosen message attacks, if for all PPT adversaries A there is a ngeligible function negl such that

$$\Pr[\text{MAC-Forge}_{A,\Pi}(n) = 1] \leqslant \mathrm{negl}(n)$$

§ Construction

Pseudorandom Function

Pseudorandom function (PRF) 的定義請參見 6.1 CPA-secure Encryption。

Construction of Fixed-length MAC

Let $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ be a PRF.

$\Pi = (\mathrm{Gen}, \mathrm{Mac}, \mathrm{Vrfy})$ is a fixed-length MAC for message fo length n.

- $\mathrm{Gen}(1^n)$: uniform $k \in \{0,1\}^n$
- $\mathrm{Mac}(k, m)$: on input k and message $m \in \{0,1\}^n$, outputs a tag $t = F_k(m)$ where t is t-bit tag.
- $\mathrm{Vrfy}(k, m, t)$: on input (k, m, t), outputs 1 if and only if $t = F_k(m)$; otherwise, outputs 0.

8 L8

接著 L7 的 MAC。

8.1 MAC

§ Security of Fixed-length MAC

Theorem 7
If F is PRF, then Π is existentially unforgeable under adaptive chosen message attack (in MAC-Forge experiment).

Proof (Security proof of PRF-based construction)

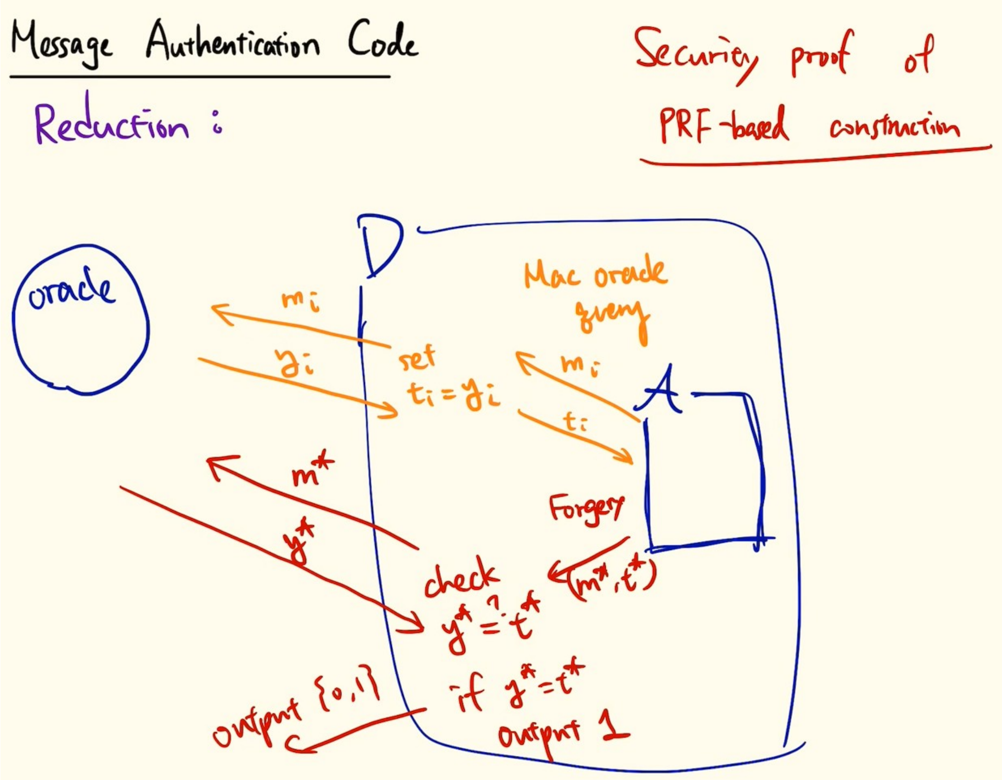
Consider $\widetilde{\Pi} = (\widetilde{\text{Gen}}, \widetilde{\text{Mac}}, \widetilde{\text{Vrfy}})$ constructed by the random function.
Then we know that $\Pr[\text{MAC-Forge}_{A,\widetilde{\Pi}}(n) = 1] \leqslant 2^{-n}$
because for any $m \notin Q$, the tag $t = f(m)$ is uniformly distributed in $\{0,1\}^n$ from A's view.

Our goal of proof:

$$\begin{aligned}
& |\Pr[\text{MAC-Forge}_{A,\Pi}(n) = 1] - \Pr[\text{MAC-Forge}_{A,\widetilde{\Pi}}(n) = 1]| \leqslant \text{negl}(n) \\
\Rightarrow \quad & \Pr[\text{MAC-Forge}_{A,\Pi}(n) = 1] \leqslant 2^{-n} + \text{negl}(n) \\
\Rightarrow \quad & \Pr[\text{MAC-Forge}_{A,\Pi}(n) = 1] \leqslant \text{negl}(n)
\end{aligned}$$

We then contruct a distringuisher D who want to break PRF security. D can do oracle access to some functions, and finally determines such function is pseudorandom (i.e. F_k for uniform $k \in \{0,1\}^n$) or truly random (i.e. f for uniform $f \in \text{Func}_n$).

Reduction:



(i) If D's oracle is PRF,

$$\Pr[D^{F_k(\cdot)}(1^n) = 1] = \Pr[\text{MAC-Forge}_{A,\Pi}(1^n) = 1]$$

(ii) If D's oracle is random function,

$$\Pr[D^{f(\cdot)}(1^n) = 1] = \Pr[\text{MAC-Forge}_{A,\tilde{\Pi}}(n) = 1] \leqslant 2^{-n}$$

(iii) If By assumption,

$$|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]| \leqslant \text{negl}(n)$$

綜合上面三點，

$$\Pr[\text{MAC-Forge}_{A,\Pi}(n) = 1] \quad \leqslant 2^{-n} + \text{negl}(n) \quad \leqslant \text{negl}(n)$$

□

Quiz

Say $\Pi = (\text{Gen}, \text{Mac}, \text{Vrfy})$ is a MAC with tag size $t(n)$.
Show that if $t(n) = O(\log(n))$, then Π is not secure MAC.
(Hint: PPT adversary runs **random guess**)

§ MAC for Arbitrary-length Messages (domain extension of MAC)

Let $\Pi' = (\text{Gen}', \text{Mac}', \text{Vrfy}')$ be a fixed-length MAC for message size n, and $\Pi = (\text{Gen}, \text{Mac}, \text{Vrfy})$ is a MAC for arbitrary-length messages.

- Gen: identical to $\text{Gen}'(1^n) \to k$, where $k \in \{0,1\}^n$
- Mac: on input $k \in \{0,1\}^n$ and a message $\{0,1\}^*$ of length $l < \frac{n}{4}$.
  (i) Parse m into d blocks $m_1, m_2, \ldots, m_d$ (each of length $\frac{n}{4}$).
  The final block is padded with 0s.
  (ii) choose a uniform $r \in \{0,1\}^{\frac{n}{4}}$

(iii) For $i = 1, 2, \ldots, d$, compute $t_i \leftarrow \mathrm{Mac}'_k(r\,||\,l\,||\,i\,||\,m_i)$ where all the length of r, l, i, m are $\frac{n}{4}$ bits respectively.
(iv) Output $t = (r, t_1, t_2, \ldots, t_d)$.

- Vrfy: On input k, t, m,
  (i) Parse m and t, such that $m = (m_1, m_2, \ldots, m_d)$ and $t = (r, t_1, t_2, \ldots, t_{d'})$.
  (ii) Output 1 if the below are both satisfied:
    - $d' = d$
    - For $1 \leqslant i \leqslant d$, $\mathrm{Vrfy}'(r\,||\,l\,||\,i\,||\,m_i) = 1$.

Remark of Mac

It seems r, l, i are important for security.

如果 Mac' 中沒有 l，則 $t_i \leftarrow \mathrm{Mac}'_k(r\,||\,i\,||\,m_i\,||0^{\frac{n}{4}})$。
此時 adversary 的可以這樣進行攻擊：

Step 1： 選取一個 m 來詢問 MAC oracle，並從 MAC oracle 收到 $t = (r, t_1, \ldots, t_d)$

Step 2： 偽造 $m^* = (m_1, m_2, \ldots, m_{d-1})$ 和 $t^* = (r, t_1, t_2, \ldots, t_{d-1})$。

由於 m^* 之前並沒有被拿來問 MAC oracle，所以 adversary 是成功的。

這說明 l 是重要的。

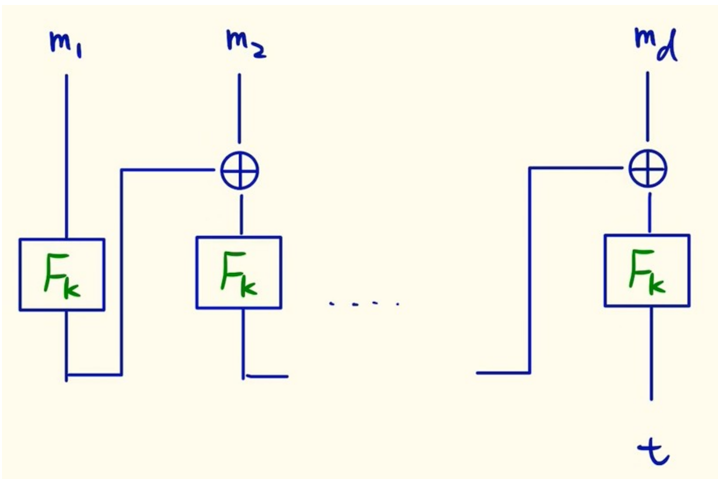
Quiz

In domain extension of MAC, it is in secure if there is no r or i.
(1) Show an attack if no r.
(2) Show an attack if no i.

Theorem 8
If MAC for fixed length Π' is secure, then MAC for arbitrary length Π is secure.

§ CBC-MAC

每個 block 計算完後就跟下一個 block 做 xor 再繼續計算，最終只會得到一個 tag t。



Quiz

Analyze advantages and disadvantages of CBC-MAC and domain extension.
(Hint: compare in performance, security, adjustment of parameters, etc.)

9 L9: Hash

Hash 是一種 compression function。長的 input 經過 hash 之後會變成短的 ouput。
Hash 也被稱為 fingerprint / hash value / digest。

因為是壓縮，所以會存在一些碰撞 (collision)。
Collision: a pair of distinct items x, x' for which Hash(x) = Hash(x')。

9.1 Syntax

Definition 11
A hash function (with output length $l(n)$) is a pair of PPT algorithm $(\mathrm{Gen}, \mathrm{H})$

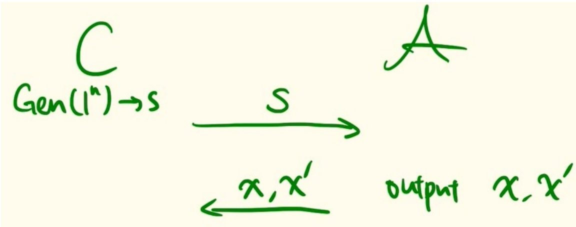
- Gen: takes a security parameter 1^n and outputs a key s.
- H: takes input as a key s and a string $x \in \{0,1\}^*$, and outputs a string $\mathrm{H}^s(x) \in \{0,1\}^{l(n)}$.

If $x \in \{0,1\}^{l'(n)}$ and $l'(n) > l(n)$, then we say it's a fixed-length input hash.

Definition 12 (Collision resistant)
A hash function $\Pi = (\mathrm{Gen}, \mathrm{H})$ is collision resistant if $\forall$ PPT adversaries A, there is a negligible function negl such that

$$\Pr[\text{Hash-coll}_{A,\Pi}(n) = 1] \leqslant \mathrm{negl}(n)$$

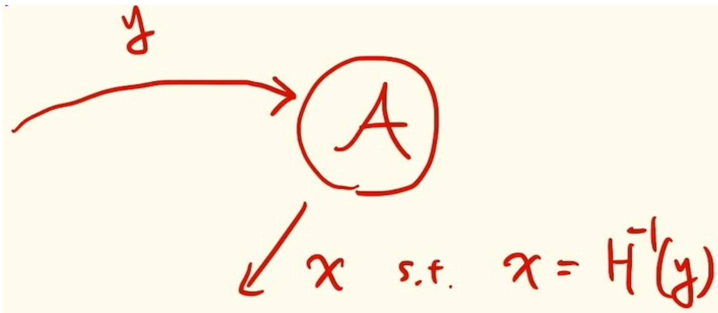
$\text{Hash-coll}_{A,\Pi}(n)$ 的 scenario 如下：



首先 challenger C 會先產生一個 key s 給 adversary A。A 可以通過自己的計算，試圖猜出一個 pair (x, x')，再傳給 C。若 $\mathrm{H}^s(x) = H^s(x')$ 且 $x \neq x'$，則 output 1，也就是 $\text{Hash-coll}_{A,\Pi}(n) = 1$。

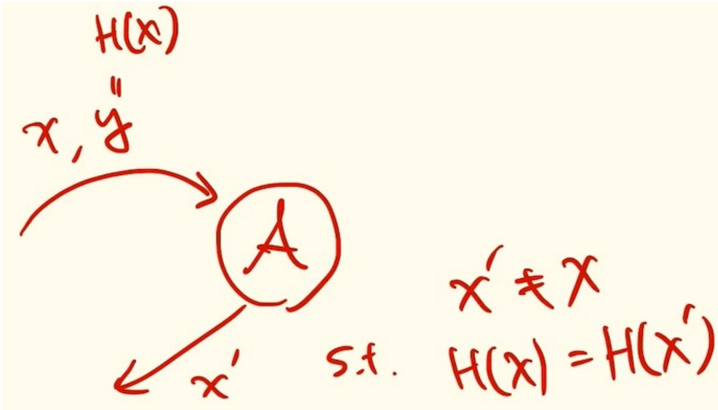
§ Preimage resistant (one-wayness)

Adversary 會拿到一個 key s（下圖省略）和一個 hash function H 的 output y，並且當 adversary 給出 $x = \mathrm{H}^{-1}(y)$ 時，視為破解成功。



§ Second-preimage resistant

Adversary 會拿到一個 key s (下圖省略) 和一對 value — hash function H 的 input x 和 output $y = H(x)$，並且當 adversary 給出 $x' \neq x$ 且 $\mathrm{H}(x') = \mathrm{H}(x)$ 時，視為破解成功。



若要以上述情況作為 security 的定義，則兩者成功的機率都是要 $\leqslant \mathrm{negl}$。

Quiz

Compare the security notions of hash function. 比較各種 hash function 的安全定義的難易程度 (易、難、無法比較)
(Ex:Second-preimage resistant is harder than collision resistant.)

9.2 SIS

§ Short Integer Solution (SIS) Problem

$\mathbb{Z}_q^n$: n-dimensional vectors modulo q (e.g. $q \approx n^3$)

Goal: find non-trivial small (ex: $\{0,1\}$) $z_1, z_2, \ldots, z_m \in \mathbb{Z}$ such that

$$z_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{13} \end{bmatrix} + z_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{23} \end{bmatrix} + z_m \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{m3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{Z}_q^n$$

Remark:

- $z_1, z_2, \ldots, z_m = 0 \quad \Rightarrow \quad$ “trival”
- $z_1, z_2, \ldots, z_m \notin \{0,1\} \quad \Rightarrow \quad$ “easy”

§ SIS-based Hash Function

Rewrite the forementioned definition of SIS problem.

$\mathbb{Z}_q^n$: n-dimensional vectors modulo q (e.g. $q \approx n^3$)

Goal: find non-trivial small (ex: $\{0,1\}$) $z_1, z_2, \ldots, z_m \in \mathbb{Z}$ such that

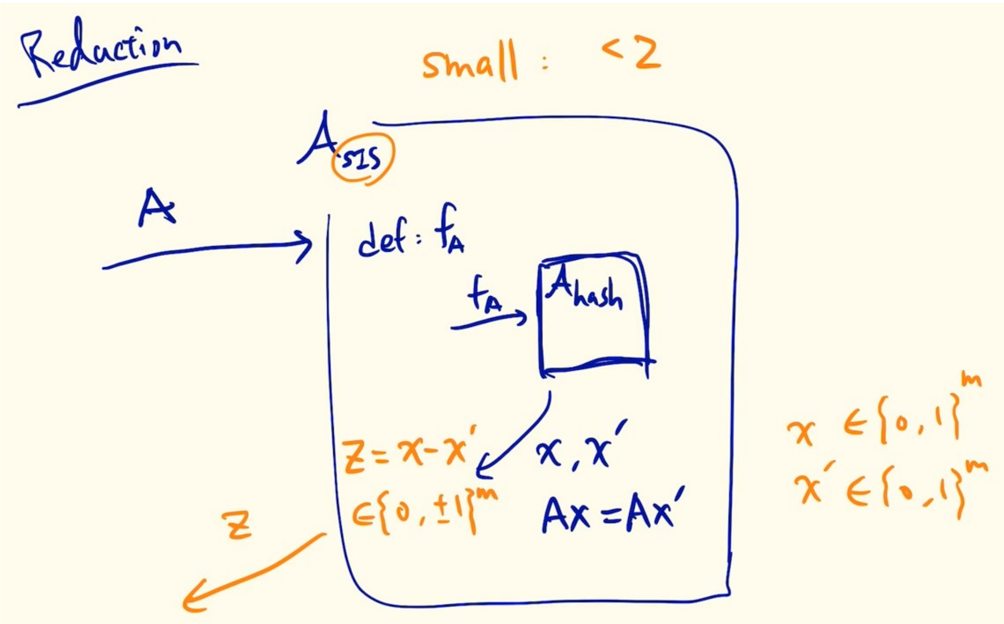
$$\mathbf{A}\mathbf{z} = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} = \mathbf{0} \in \mathbb{Z}_q^n$$

Construction of Hash

Set $m > n \log q$ (for compression).
Define $f_{\mathbf{A}} : \{0,1\}^m \to \mathbb{Z}_q^n$ as $f_{\mathbf{A}}(x) = \mathbf{A}\mathbf{x}$

Collision: $\mathbf{x}, \mathbf{x}' \in \{0,1\}^m$ where $\mathbf{x} \neq \mathbf{x}'$ and $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}'$

§ Collision Resistant SIS-based Hash



Quiz

We do not formally write down the security proof, and only provide proof intuition.

(i) Please show the assumption ($\Pr[success] \leqslant \text{negl}$), aka SIS, which is used in the proof.
(ii) Please complete the proof with probability analysis.

9.3 Arbitrary-length Hash Function

前面介紹的 SIS-based hash function 和現實中使用的 hash function 都是 fixed-length compression function。我們可以透過 Merkel-Damgard transformation 來做到 arbitrary-length hash function。

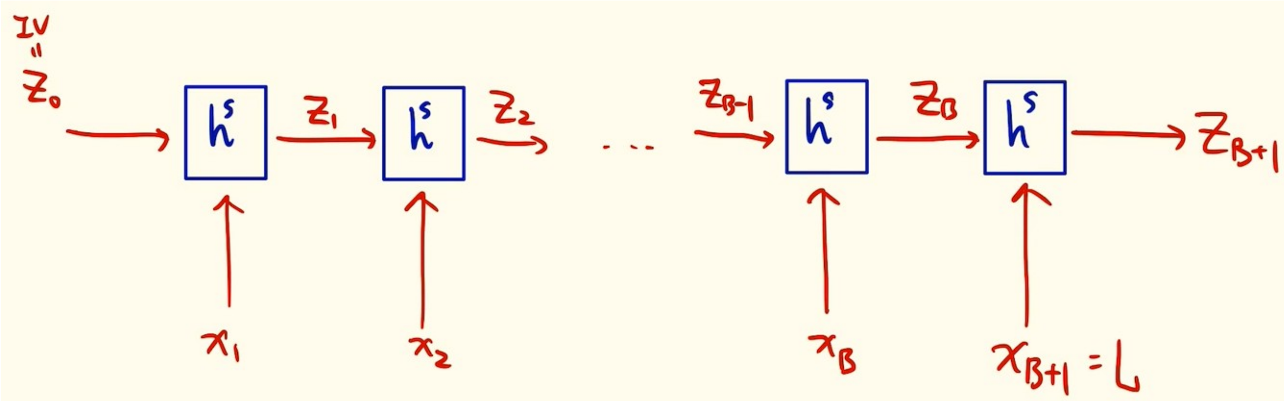
§ Merkle-Damgard Transformation

和 CBC-MAC 的概念相似。

Let $(\mathrm{Gen}', \mathrm{h})$ be a fixed input length hash. $\mathrm{h} : \{0,1\}^{2n} \to 0, 1^n$
Let $(\mathrm{Gen}, \mathrm{H})$ be a fixed input length hash. $\mathrm{H} : \{0,1\}^{*} \to 0, 1^n$

Use $(\mathrm{Gen}, \mathrm{h})$ to build $(\mathrm{Gen}, \mathrm{H})$

- Gen: run $\mathrm{Gen}'(1^n) \to s$ (key)
- H: on input s and a string $x \in \{0,1\}^*$ of length L where $L < 2^n$.
  (i) Set $B = \lceil \frac{L}{n} \rceil$ (B: number of blocks)
  Pad x with 0s, so length will be a multiple of n.
  Parse x to $x_1, x_2, \ldots, x_B$, and set $x_{B+1} = L$
  (ii) Set $z_0 = 0^n$ as IV
  (iii) For $i = 1, 2, \ldots, B+1$, compute $z_i = \mathrm{h}^s(z_{i-1} || x_i)$



  (iv) Output z_{B+1} as the hash value of x.

Quiz

We found some cute trick in Merkle-Damgard transformation:

(i) The purpose of L? (Hint: related to collision)
(ii) Suppose the fixed-length hash is $h : 0, 1^{n+1} \to 0, 1^n$
How to build an arbitrary length has from the above?

§ Security of Merkel-Damgard Transformation

Theorem 9

If $(\mathrm{Gen}', \mathrm{h})$ is collision resistant, then $(\mathrm{Gen}, \mathrm{h})$ is collision resistant.

Proof

For any s, a collision in H^s yields a collision h^s.
Assume two distinct strings (x, x') of length (L, L') such that $\mathrm{H}^s(x) = \mathrm{H}^s(x')$.

Let $x_1, x_2, \ldots, x_B$ are the blocks of padded x and $x_{B+1} = L$,
and $x'_1, x'_2, \ldots, x'_B$ are the blocks of padded x' and $x_{B'+1} = L'$.

Case 1: $L \neq L'$
In the last step of $\mathrm{H}^s(x)$ (resp. $\mathrm{H}^s(x')$),
$z_{B+1} = \mathrm{h}^s(z_B || L)$ (resp. $z'_{B'+1} = \mathrm{h}^s(z'_{B'} || L')$)
Assume $\mathrm{H}^s(x) = \mathrm{H}^s(x')$
$\Rightarrow \quad h^s(z_B || L) = h^s(z'_{B'} || L')$ which is a collision in h^s

Case 2: $L = L'$ (implies $B = B'$)
Let $I_i \stackrel{\text{def}}{=} z_{i-1} || x_i$. ($i$-th input of h^s) (I'_i, resp.)
Set $I_{B+2} \stackrel{\text{def}}{=} z_{B+1}$.

Assume $\mathrm{H}^s(x) = \mathrm{H}^s(x')$. Let N be the largest index for $I_N \neq I'_N$.
Since $|x| = |x'|$, but $x \neq x'$, there must exist an i with $x_i \neq x'_i$.
$I_{B+2} = z_{B+1} = \mathrm{H}^s(x) = \mathrm{H}^s(x') = z'_{B+1} = I'_{B+2}$
$\Rightarrow \quad N \leqslant B + 1 \Rightarrow \quad I_{N+1} = I'_{N+1}$
For this N, I_N, I'_N are collision in h^s.

$\square$