An Introduction to Positional Number Systems

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Decimal system: Numbers such as 8, 10, 52, 64, and 2739 are said be in "decimal" form, or in "Base 10" form, because they are expressed as a combination of 10 distinct digits: A digit is simply some unique shape or symbol. We can choose any set of symbols, but we traditionally use the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Multi-digit numbers: Values 0 through 9 are each represented using a single digit (or symbol). Ten, the first value after 9 is expressed by combining the digits 1 and 0 and written "10". The next value, eleven, is expressed by combining digits 1 and 1, and so on until the value 19. After this point, the leading digit 1 is changed to 2, and combined again with digits 0 through 9 to express the next 10 values (20 through 29). This pattern repeats until the value 99, at which point, a third digit is added to the beginning to get the next higher value 100. Thus, arbitrarily large numbers can be expressed using only 10 distinct digits.

Choosing our own digits: Because digits can be any shape or symbol, we could use letters A, B, and so on until J as digits, instead of using 0 through 9. With this choice, the value of A will be 0, the value of B will be 1, and the value of J will be 9. Then, the number ten will be expressed as BA (letter B for 1 and A for 0). Likewise, the number twenty-five will be expressed as CF (C stands for 2 and F stands for 5).

Problem 1: Express the following decimal values using the letters A through J as digits:

1.1. Forty-two:

1.2. Thirty:

1.3. Ninety-seven:

1.4. One hundred:

1.5. One thousand two hundred and seven:

Nonary system: We do not have to represent numbers in only the decimal (base 10) form. For example, we can choose 9 as the base and write numbers in the "nonary system" or "Base 9", which has only nine digits: 0, 1, 2, 3, 4, 5, 6, 7, and 8. In this system, the number after 8 is written "10" but that number's decimal value is actually nine. The next number is written "11" but its decimal value is actually ten. Numbers after this are formed by combining digits just as in the decimal system.

Problem 2: Fill in nonary equivalents to the decimal numbers shown. The preceding paragraph already gives the nonary numbers for decimal values 9 and 10. Remember, the last digit in the nonary system is 8.

Decimal	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**27**	**30**
Nonary															

Problem 3: Repeat Problem 2 using the following symbols for nonary digits: α, β, γ, δ, ε, η, κ, λ, μ, where α stands for the digit 0, β stands for 1, and so on.

Decimal	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**27**	**30**
Nonary															

Distinguishing nonary numbers from decimal numbers: The nonary number "10" should be read "nonary one zero" or "one zero base 9". It should **not** be read "ten". Likewise, that number should be written 10_9 where the subscript 9 denotes the "base" of the number system. Writing 10_9 is shorthand for the longer phrase "the nonary number 10", but it should still be read "nonary one zero" or "one zero base 9".

If a number is written without a base, it is assumed to be a decimal number, but it is acceptable to explicitly show the base. For example, the decimal number 23 can also be written as 23_{10}.

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Undecimal system: We really can choose to represent numbers in an arbitrary base B as long as we also choose B distinct symbols as digits. Regardless of the symbols chosen, the first digit's decimal value will be 0, the second digit's value will be 1, and so on. If the base is larger than 10, it is common to use digits 0 through 9 as the first nine digits, and use upper case letters A, B, and so on for additional digits.

For example, we can represent numbers in the **undecimal system** or "Base 11" which has 11 distinct digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and A. Note that the symbol A is a single digit whose decimal value is actually ten. The number after A is written "10" but its decimal value is actually eleven. The number after that is written "11" but its decimal value is twelve.

Undecimal "10" should be written 10_{11} and read "undecimal one zero" or "one zero base eleven".

Problem 4: Fill in the undecimal numbers equivalent to the decimal numbers shown.

Decimal	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**31**	**32**
Undecimal																

Common non-decimal number systems: In computing, Binary (Base 2), Octal (Base 8), and Hexadecimal (Base 16) systems are common. (Computers store and manipulate numbers in binary; not in decimal.) The binary system has just two digits: 0 and 1; the octal system has eight digits: 0 through 7; and the hexadecimal system has 16 digits: 0 through 9 followed by the letters A through F.

Because the binary system has only two digits 0 and 1, the decimal number 2 is written "10" and decimal 3 is written "11". The next number is formed by adding a 1 to the beginning, just the way decimal 100 is formed after decimal 99. Thus, binary numbers generally use more digits than equivalent decimal numbers. For example, decimal 16 is 10000 in binary (why?).

Problem 5: Fill in the non-decimal numbers that are equivalent to the decimal numbers shown.

Decimal	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**15**
Binary												

Decimal	**7**	**8**	**9**	**10**	**14**	**15**	**16**	**17**	**20**
Octal									

Decimal	**9**	**10**	**11**	**15**	**16**	**17**	**18**	**30**	**31**	**32**	**33**
Hexadecimal											

Digit significance: In any number, in any positional number system, all digits except leading zeros are called significant digits. The left-most non-zero digit is called the *most-significant digit* (MSD). The rightmost digit (even if it is zero) is called the *least-significant digit* (LSD). For example:

- 324 has three significant digits; 3 is the MSD and 4 is the LSD
- 40_8 has two significant digits; 4 is the MSD, 0 is the LSD (the number's base does not matter)
- 8_9 has one significant digit, which is 8, and that digit is both the MSD and the LSD
- 0741 has three significant digits; 7 is the MSD (it is the left-most non-zero digit) and 1 is the LSD

To reiterate, leading zeros in a number must be discarded because they are not significant (they do not add any value). However, embedded and trailing zeros must not be discarded.

Problem 6: Fill in the number of significant digits, the MSD and LSD in each of the following numbers:

Number	#Significant digits	MSD	LSD
42_{10}			
100_2			
6_8			
$1AF_{16}$			
000462_9			
$2A50_{11}$			

Place value: The number systems described in this note is called the place-value system or positional system because in this approach every digit in a number has a value ("place value") based on its place in the number. The place value of a digit is the contribution that digit makes to the entire number. The LSD has the lowest place value (contributes the least value), whereas the MSD has the highest place value (contributes the most).

The place value of a digit is determined by first assigning a place (position) to each digit starting with the LSD and increasing the place number by 1 for each digit until the MSD is reached. The place number of the LSD is always zero. For example, in the number 324, the LSD 4 is in place 0, the digit 2 is in place 1, and the MSD is in place 2.

The place value of any digit in a number of any base *B* is given by the formula B^p*v, where *p* is the place number of the digit, B^p denotes *B* raised to *p* (that is, *B* to the *p*-th power), *v* is the decimal value of the digit, and * denotes multiplication. The following table shows some examples using 3-digit numbers. The expressions in parentheses show the calculations used to obtain the result in each cell:

Number	Place value of MSD	Place value of middle digit	Place value of LSD
324	300 (10^2*3)	20 (10^1*2)	4 (10^0*4)
324_9	243 (9^2*3)	18 (9^1*2)	4 (9^0*4)
324_8	192 (8^2*3)	16 (8^1*2)	4 (8^0*4)
324_{11}	363 (11^2*3)	22 (11^1*2)	4 (11^0*4)
$2AC_{16}$	512 (16^2*2)	160 (16^1*10) 10 is the decimal value of A	12 (16^0*12) 12 is the decimal value of C

Problem 7: Fill in place value of the MSD and LSD for each of the following numbers:

Number	Place value of MSD	Place value of LSD
42_{10}		
100_2		
6_8		
$1AF_{16}$		
000462_9		
$2A50_{11}$		

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Converting a non-decimal number to decimal: A number in any base can be converted to a decimal number by simply adding the place value of every significant digit in the number. For example:

- 324_9 is 265 in decimal (result of 9^2*3 + 9^1*2 + 9^0*4)
- 324_8 is 212 in decimal (result of 8^2*3 + 8^1*2 + 8^0*4)
- 1101_2 is 13 in decimal (result of 2^3*1 + 2^2*1 + 2^1*0 + 2^0*1)

In general, the decimal equivalent *D* of any number *M* in base *B*, and containing *n* significant digits, is given by the following formula, where B^i represents the *i*-th power of *B* and v_i represents the decimal value of the *i*-th digit, with v_0 being the decimal value of the LSD:

$$D = \sum_{i=0}^{n-1} B^i * v_i$$

Problem 8: Fill in the decimal equivalent of each number shown.

Number	Decimal value
100_2	
6_8	
$1AF_{16}$	
000462_9	
$2A50_{11}$	

Problem 9: Answer the following closing questions (the last two questions may need a bit of thinking):

9.1. What is the smallest base possible in positional number systems?

9.2. Recall that Base 9 and Base 2 are given the names nonary and binary systems respectively. Likewise, what are the names given to following number systems: Base 3, Base 5, Base 12?

9.3. What is the result of the expression $1AF_{16} + 6_8 – 100_2$ in decimal?

9.4. What are the digits of the Base 36 system?

9.5. What are the digits of the Base 37 system?

9.6. How many significant digits are in the number 0? What is the MSD? What is the LSD?