Convolution

CISC 7026: Introduction to Deep Learning

University of Macau

Agenda

1. Review
2. Signal Processing
3. Convolution
4. 2D Convolution
5. Coding

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1. **Review**
2. Signal Processing
3. Convolution
4. 2D Convolution
5. Coding

Review

Agenda

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4. 2D Convolution
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Signal Processing

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However, there is structure inherent in the real world

By representing this structure within neural networks, we can make neural networks that are more efficient and generalize better

To do so, we must think of the world as a collection of signals

Signal Processing

A **signal** represents information as a function of time, space or some other variable

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$$x(t) = ...$$

$$x(u, v) = ...$$

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Signal processing is a field of research that focuses on analyzing the meaning of signals

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Knowing the meaning of signals is very useful

Signal Processing

Signal Processing

$$x(t) = \text{stock price}$$

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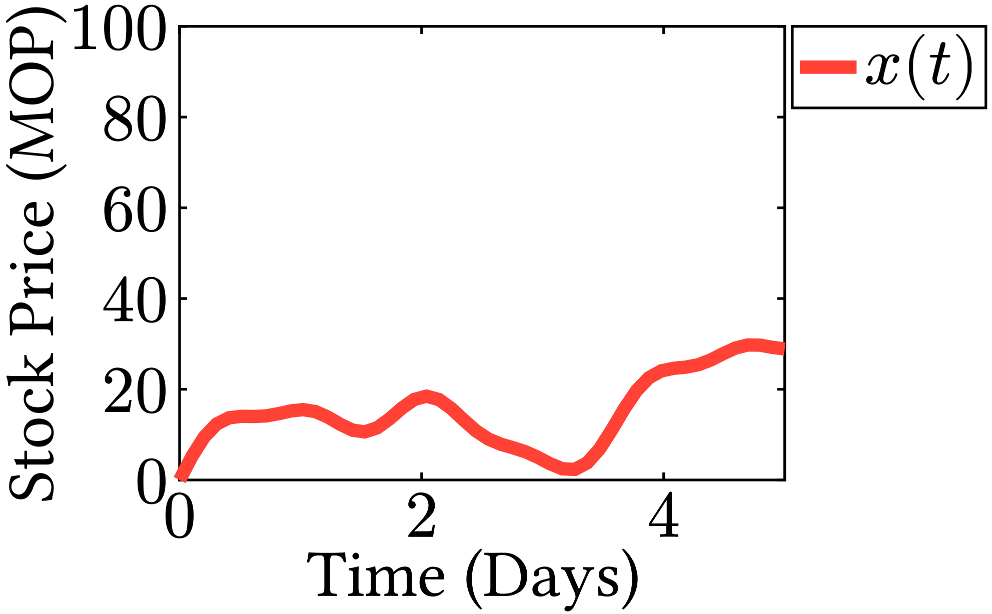
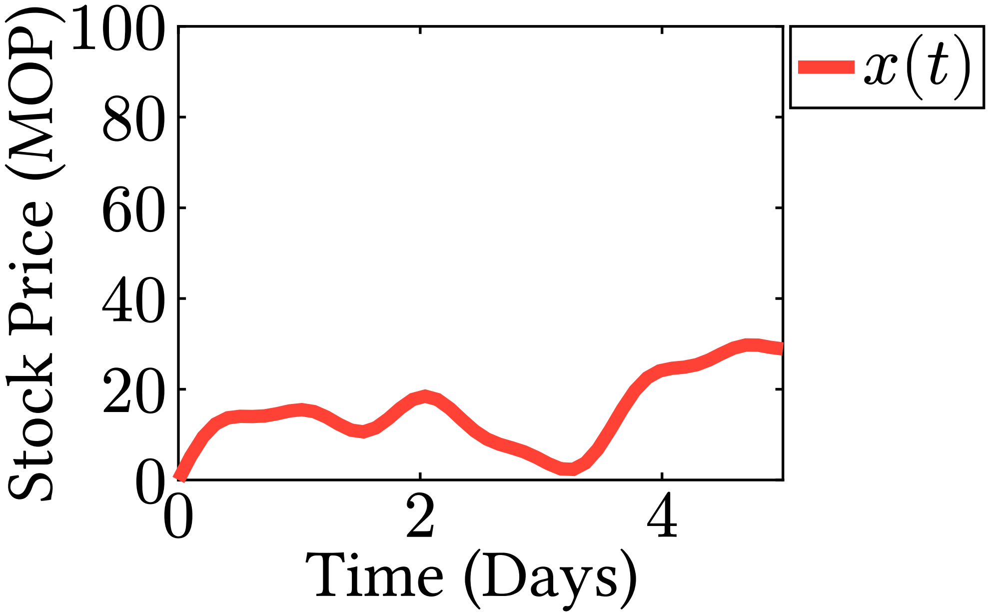


Signal Processing

$$x(t) = \text{stock price}$$



Structure: Tomorrow's stock price will be close to today's stock price

Signal Processing

$$x(t) = \text{audio}$$

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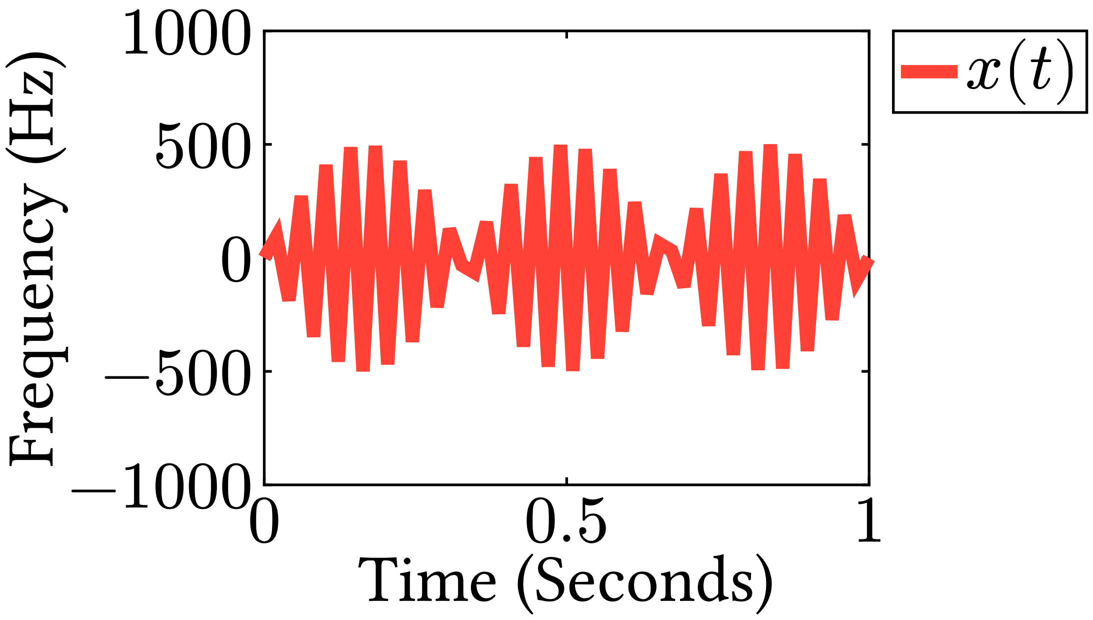
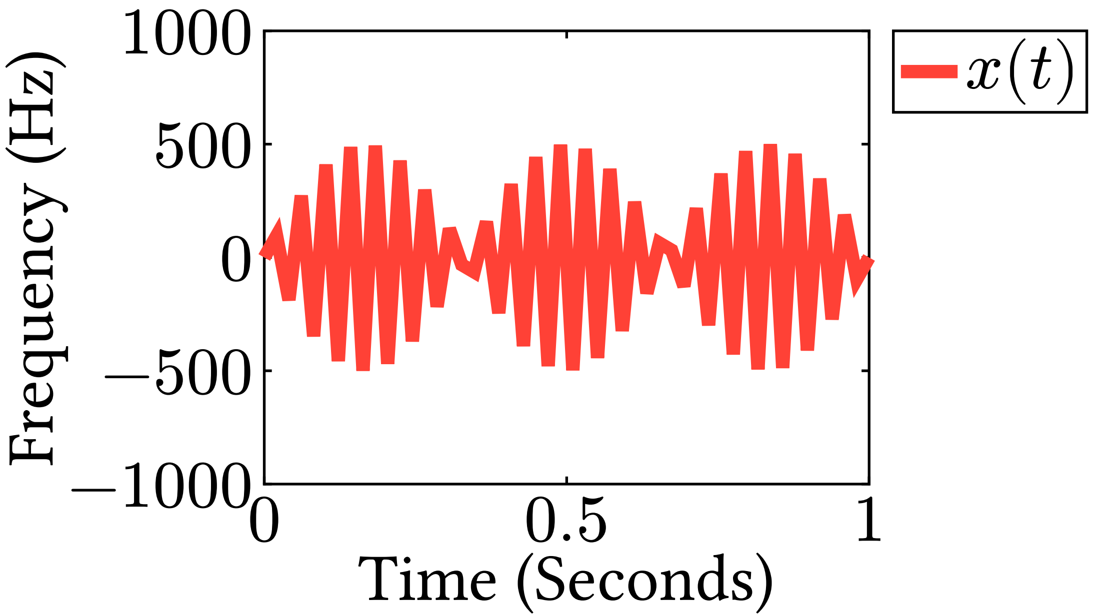


Signal Processing

$$x(t) = \text{audio}$$



Structure: Nearby waves form syllables

Signal Processing

$$x(u, v) = \text{image}$$

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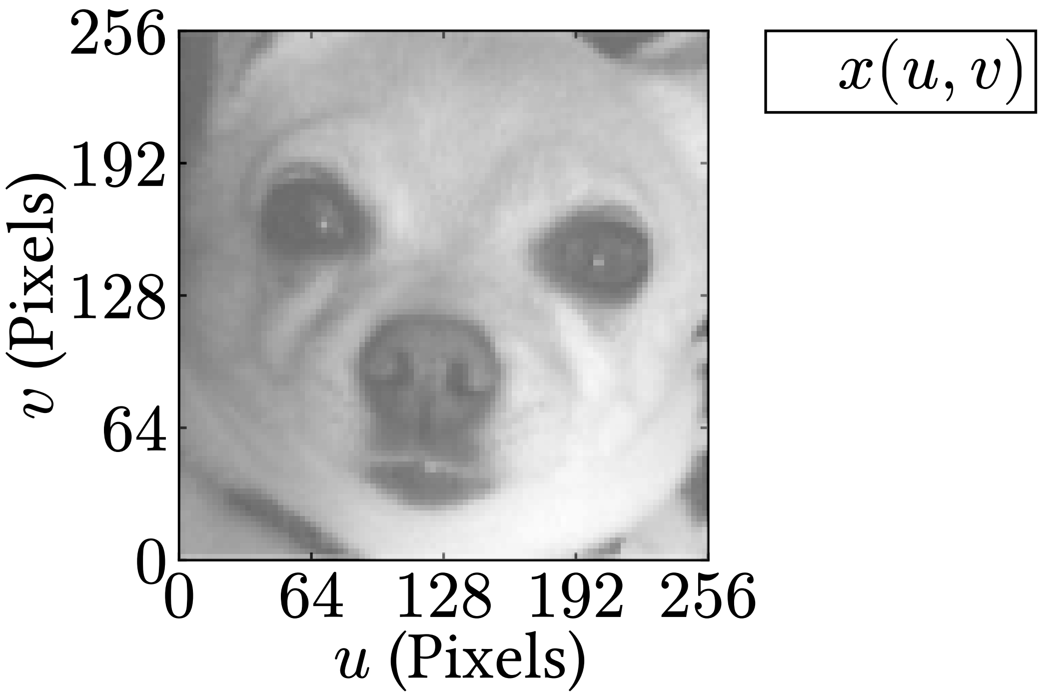

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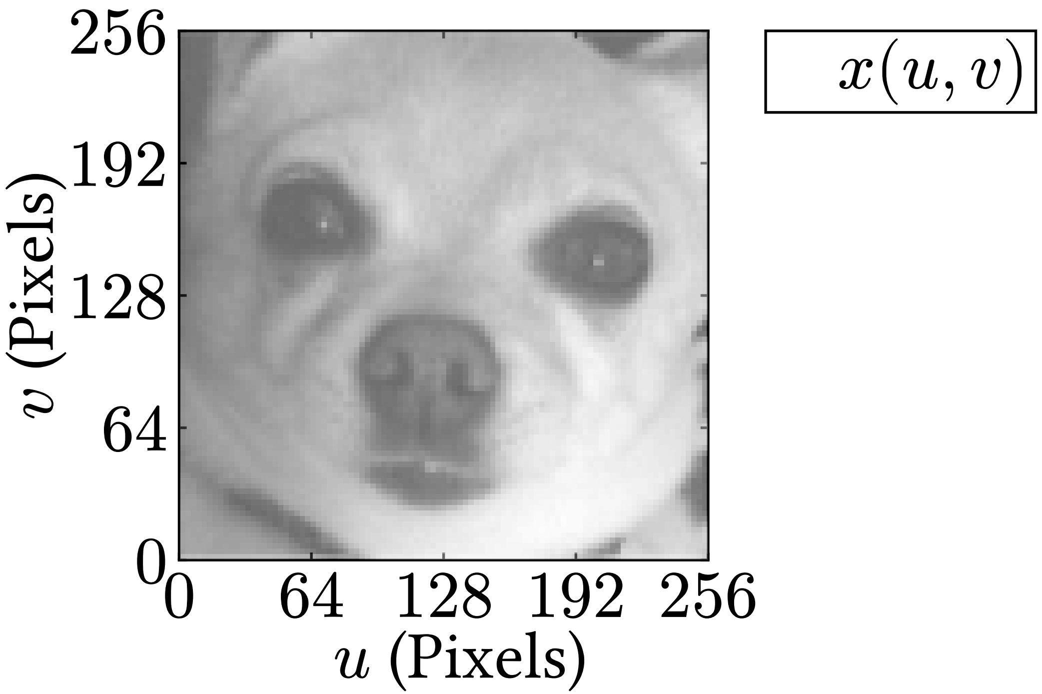

Structure: Repeated components (eyes, nostrils, etc)

Signal Processing

In signal processing, we often consider:

Signal Processing

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- Locality

Signal Processing

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- Locality
- Translation invariance

Signal Processing

Locality: Information concentrated over small regions of space/time

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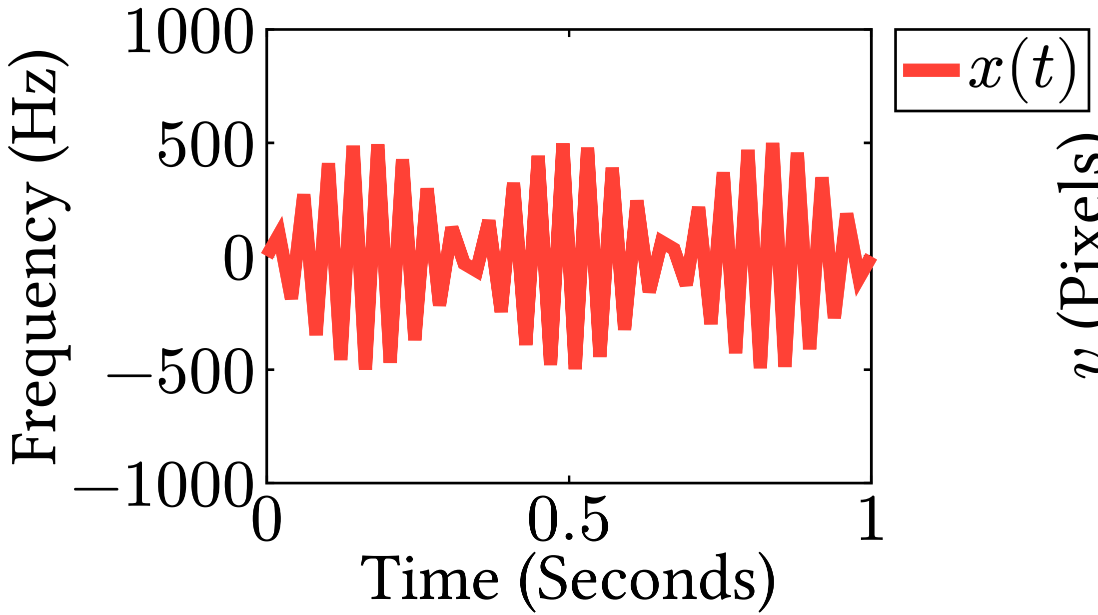

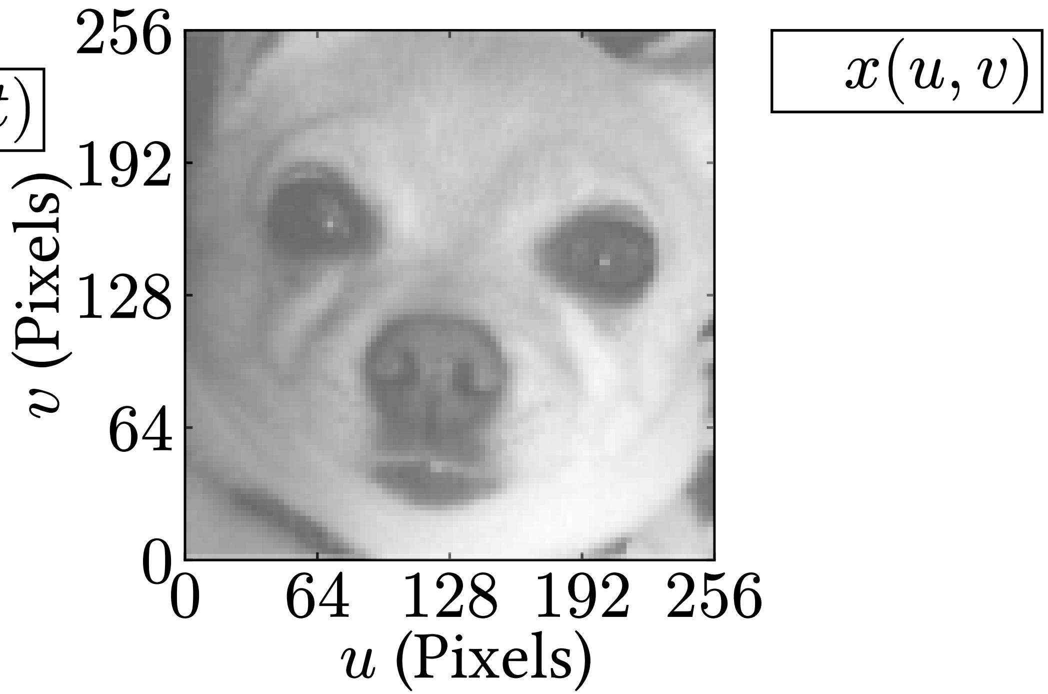

Signal Processing

Translation Invariance: Signal does not change when shifted in space/ time

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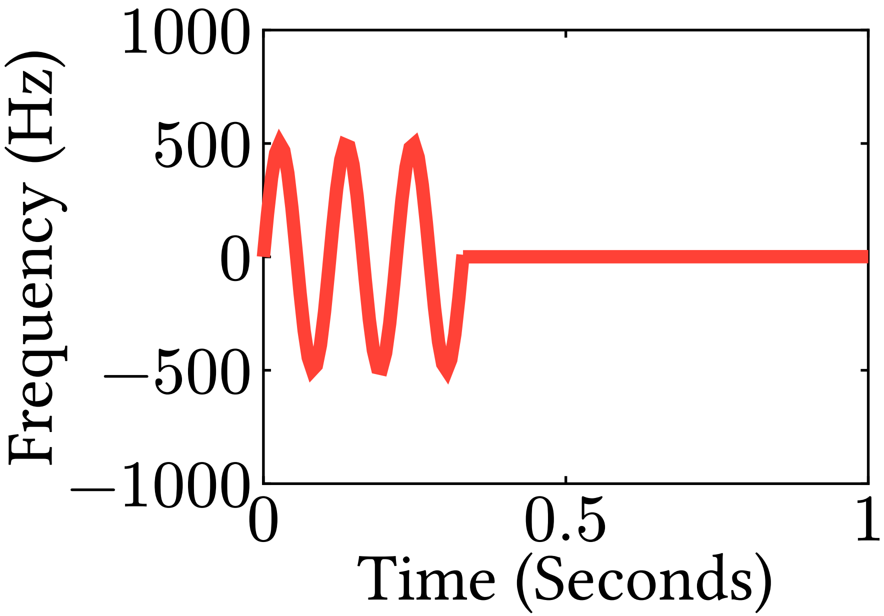

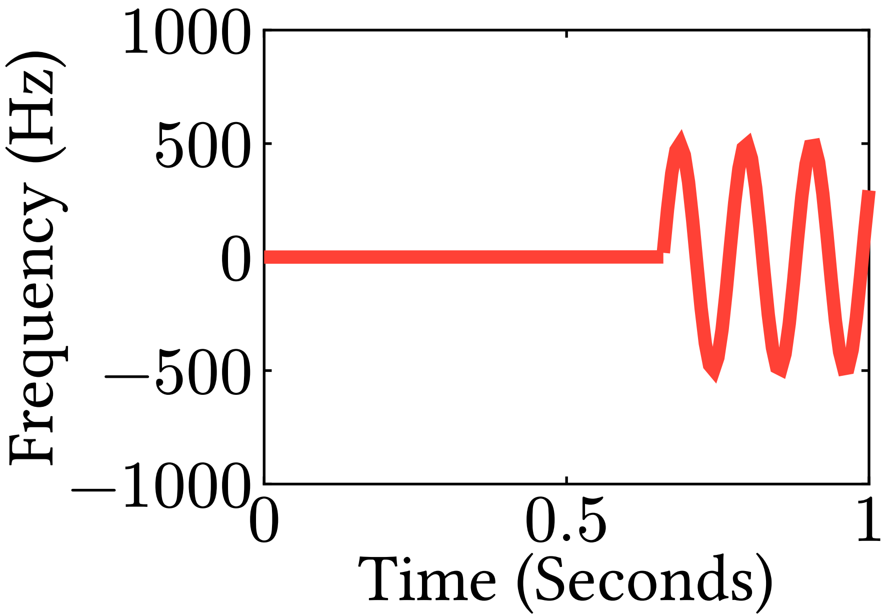

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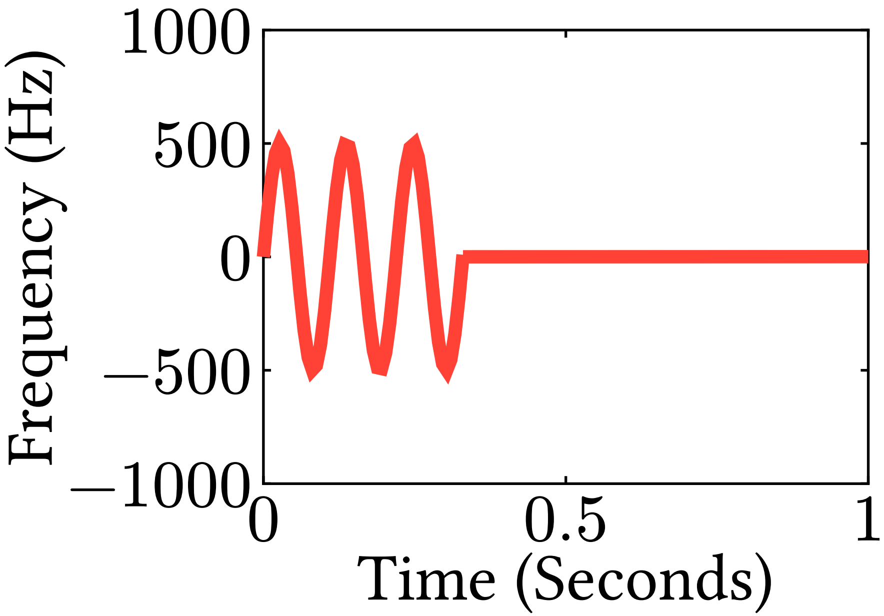

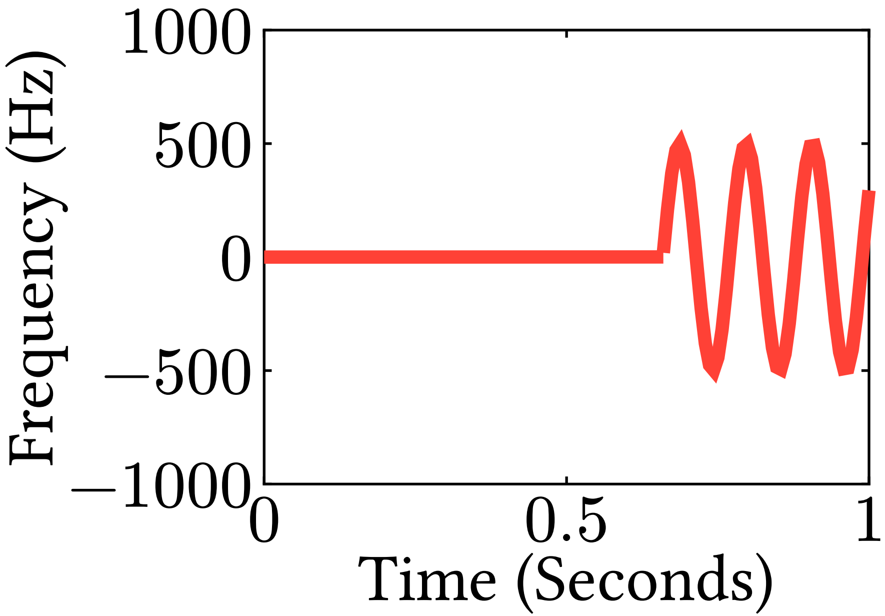

Both say "hello"

Signal Processing

Translation Invariance: Signal does not change when shifted

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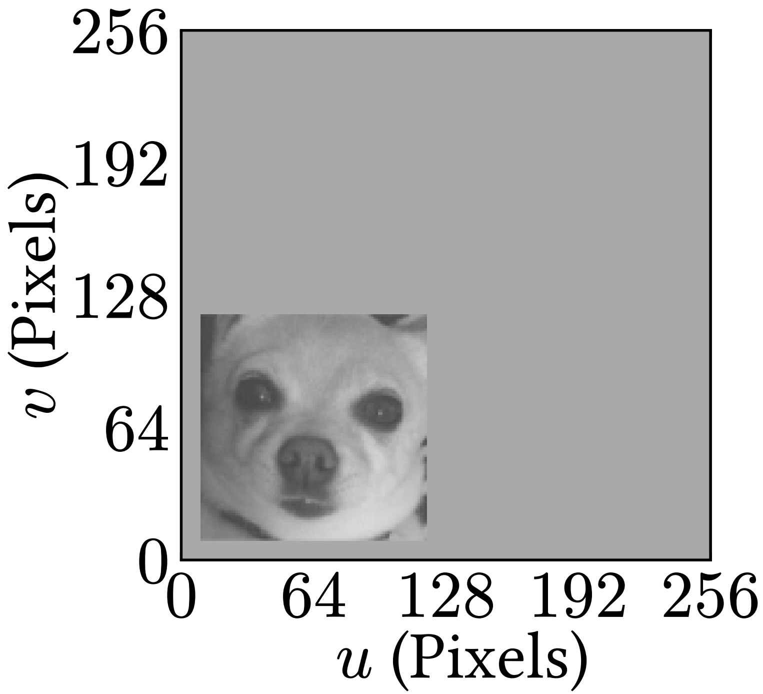

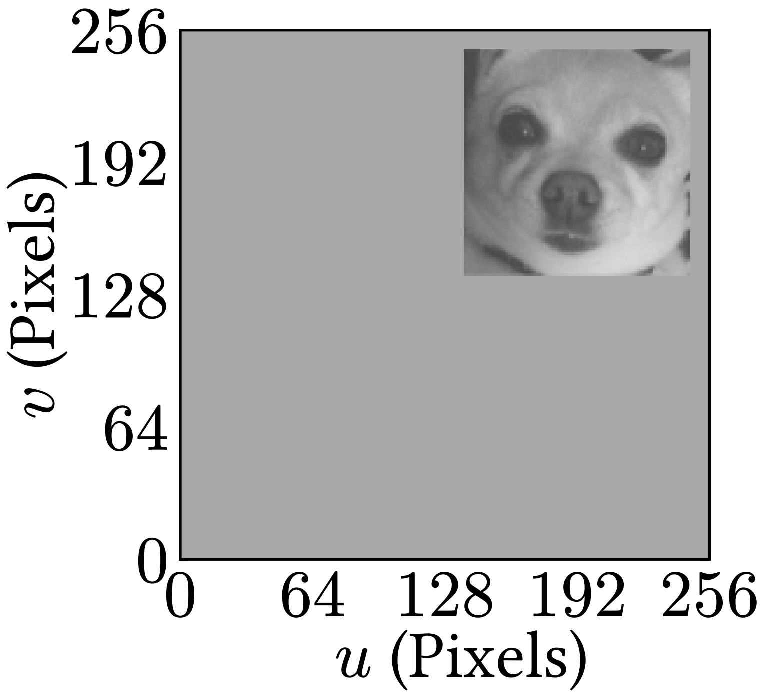

Both contain a dog

Signal Processing

Perceptrons are not local or translation invariant, each pixel is an independent neuron

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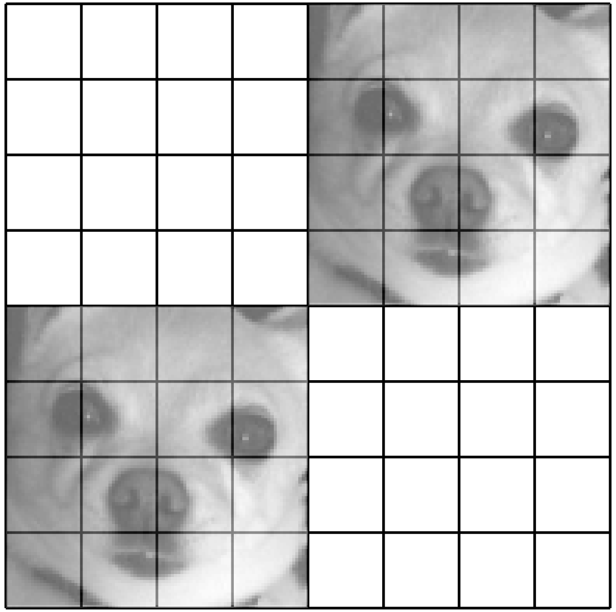

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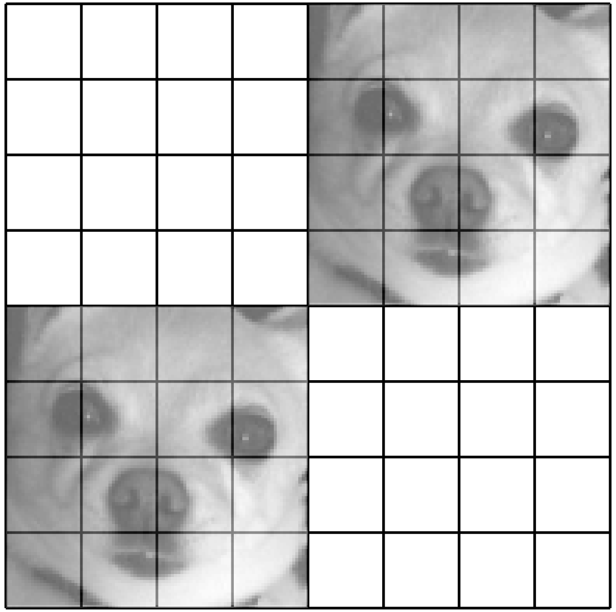

How can we get these properties in neural networks?

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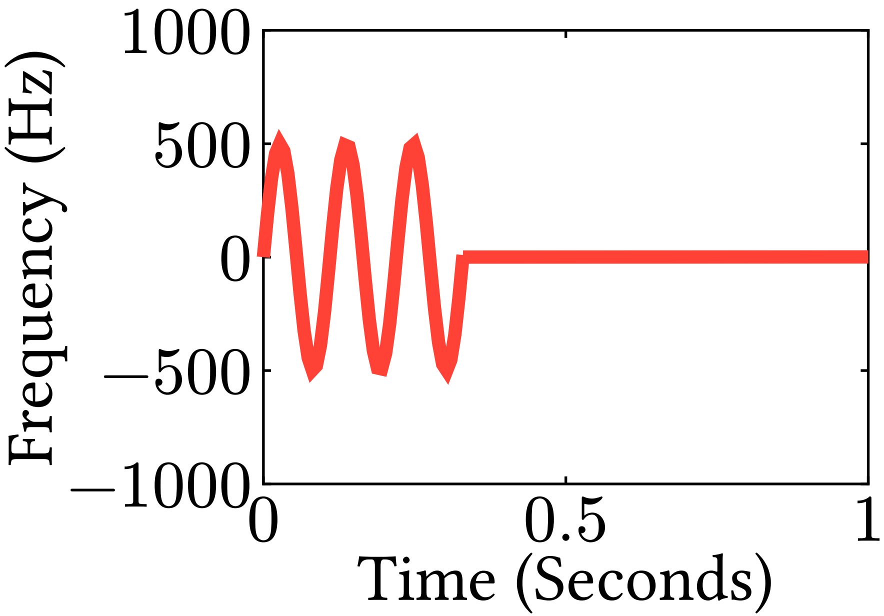

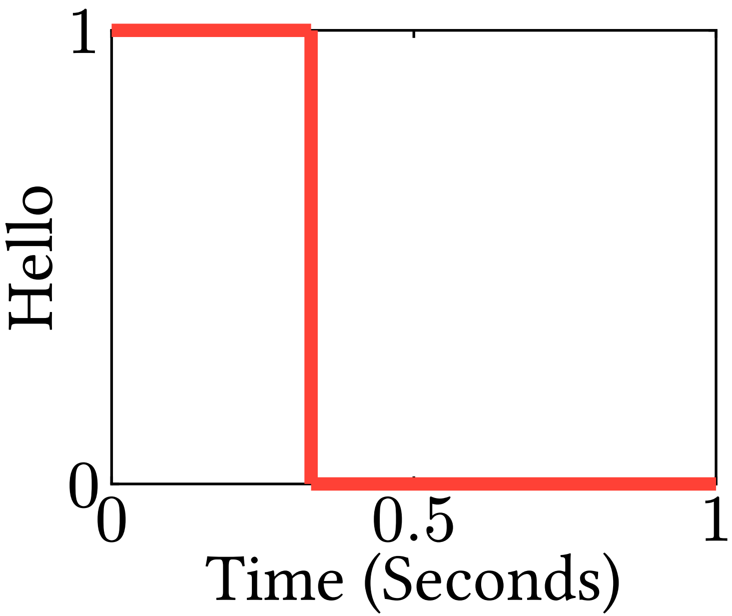

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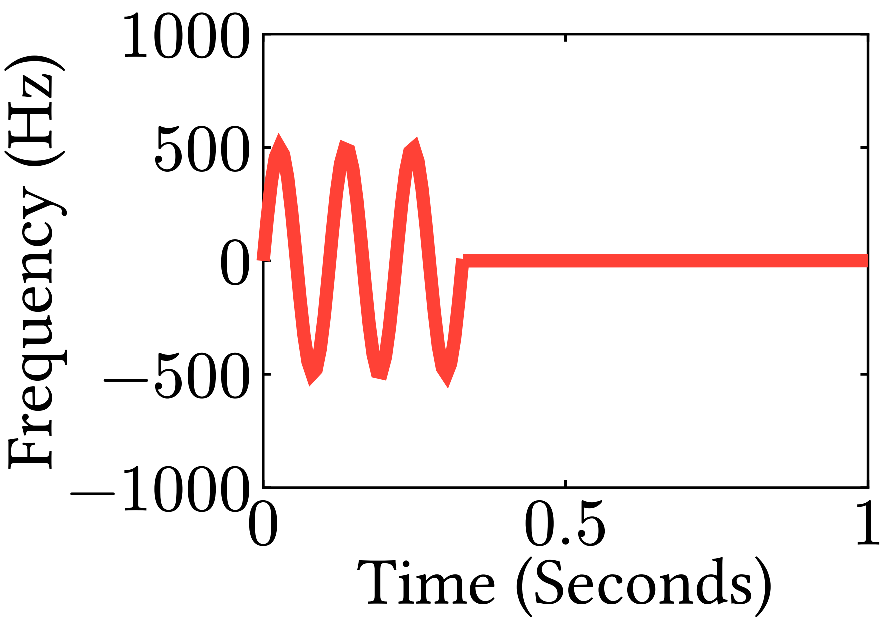

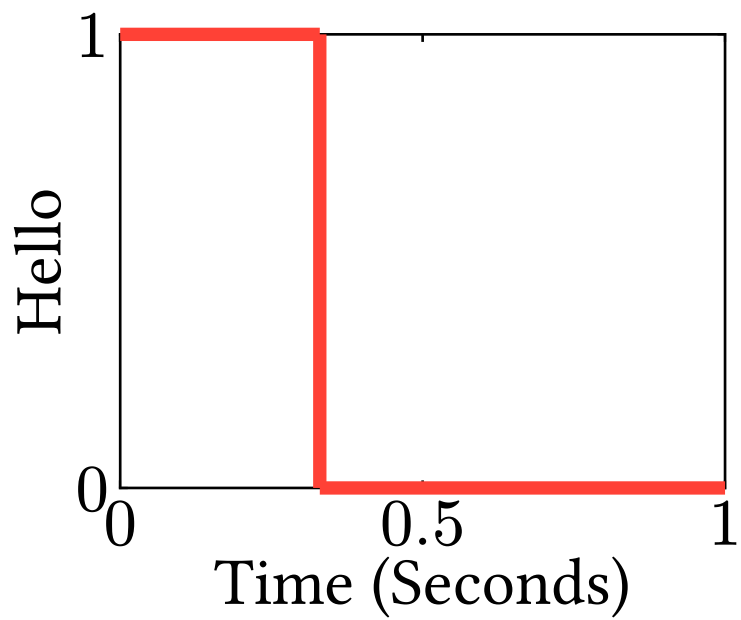

A standard way to transform signals is **convolution**

Convolution

Convolution is the sum of products of a signal $x(t)$ and a **filter** $g(t)$

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If time and space is continuous, we write convolution as

$$x(t) * g(t) = \int_{-\infty}^{\infty} x(t - \tau) g(\tau) d\tau$$

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We slide the filter across the signal, taking the product as we go

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We slide the filter across the signal, taking the product as we go

Convolution

Example: Let us examine a low-pass filter

Convolution

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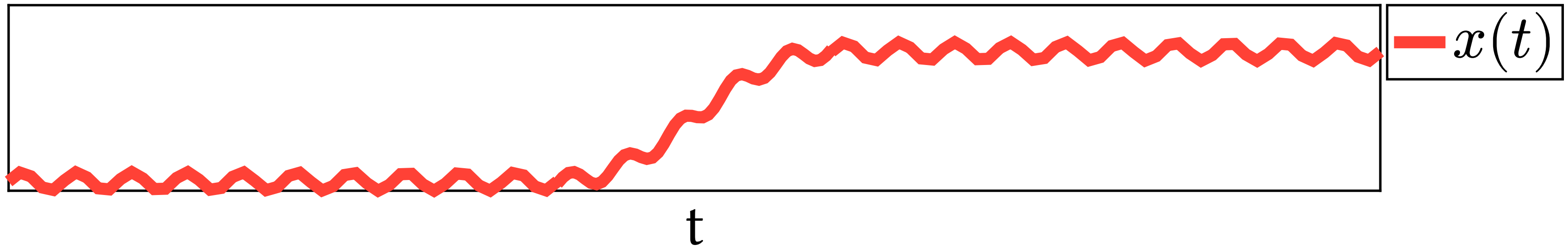
The filter will take a signal and remove noise, producing a cleaner signal

Convolution



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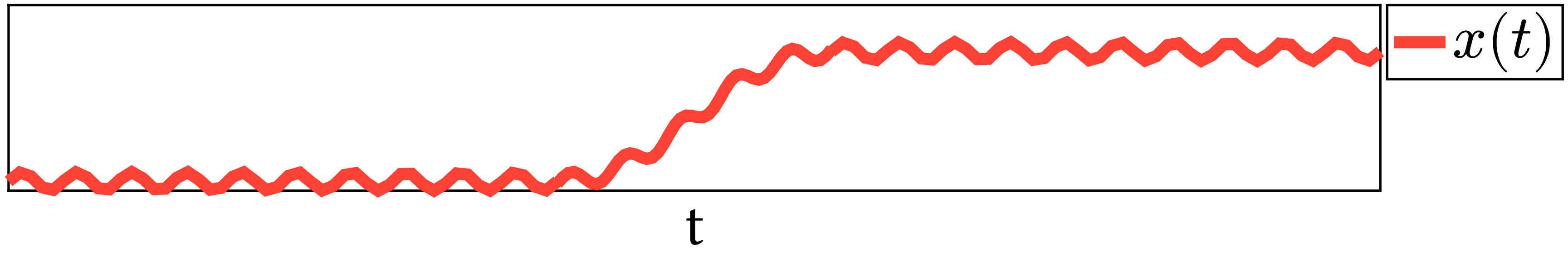

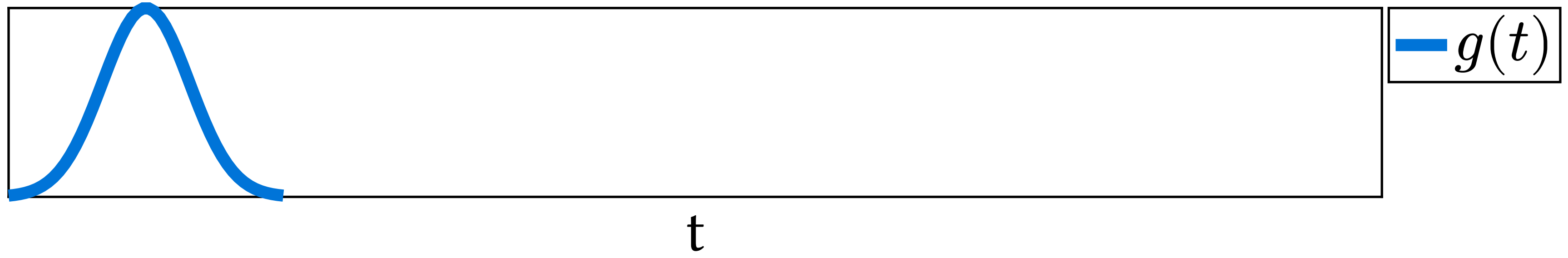

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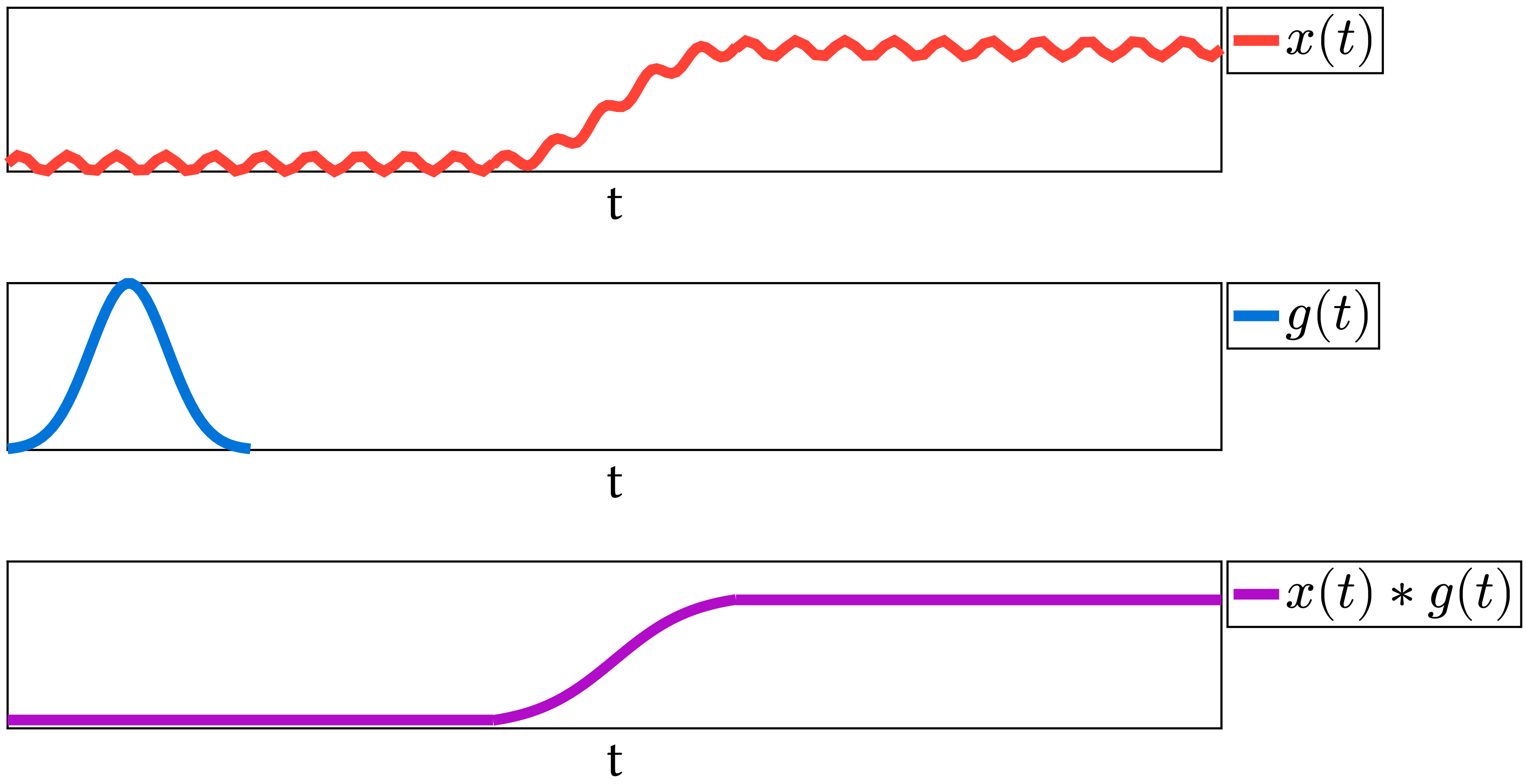

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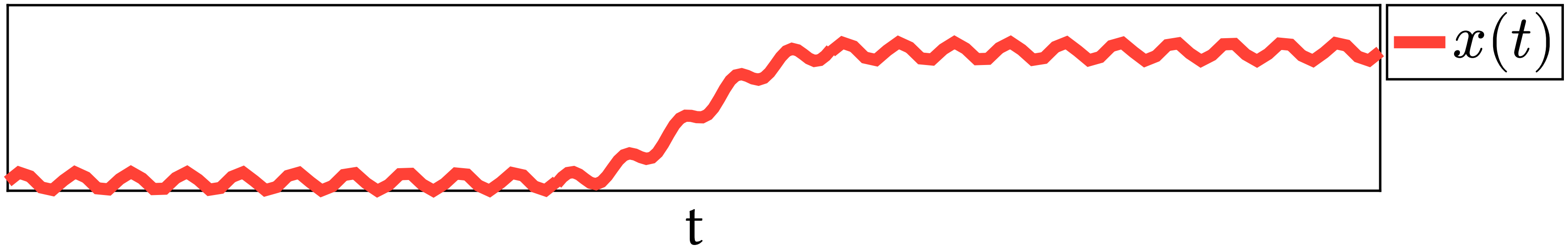
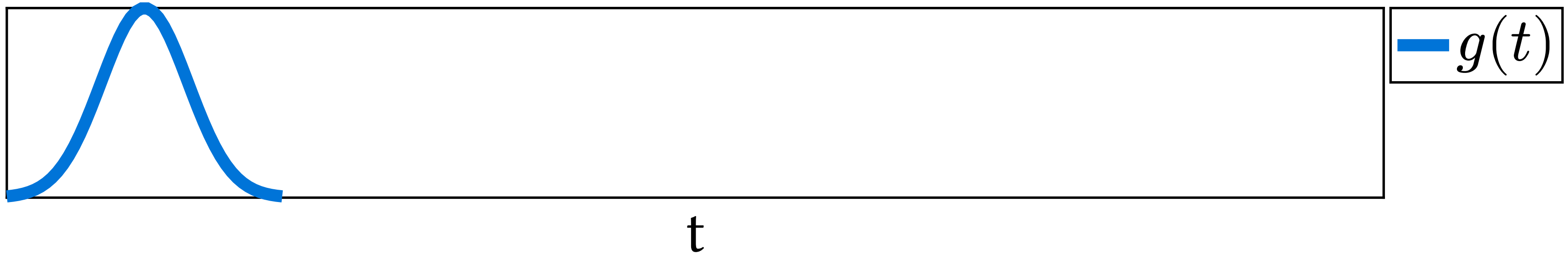
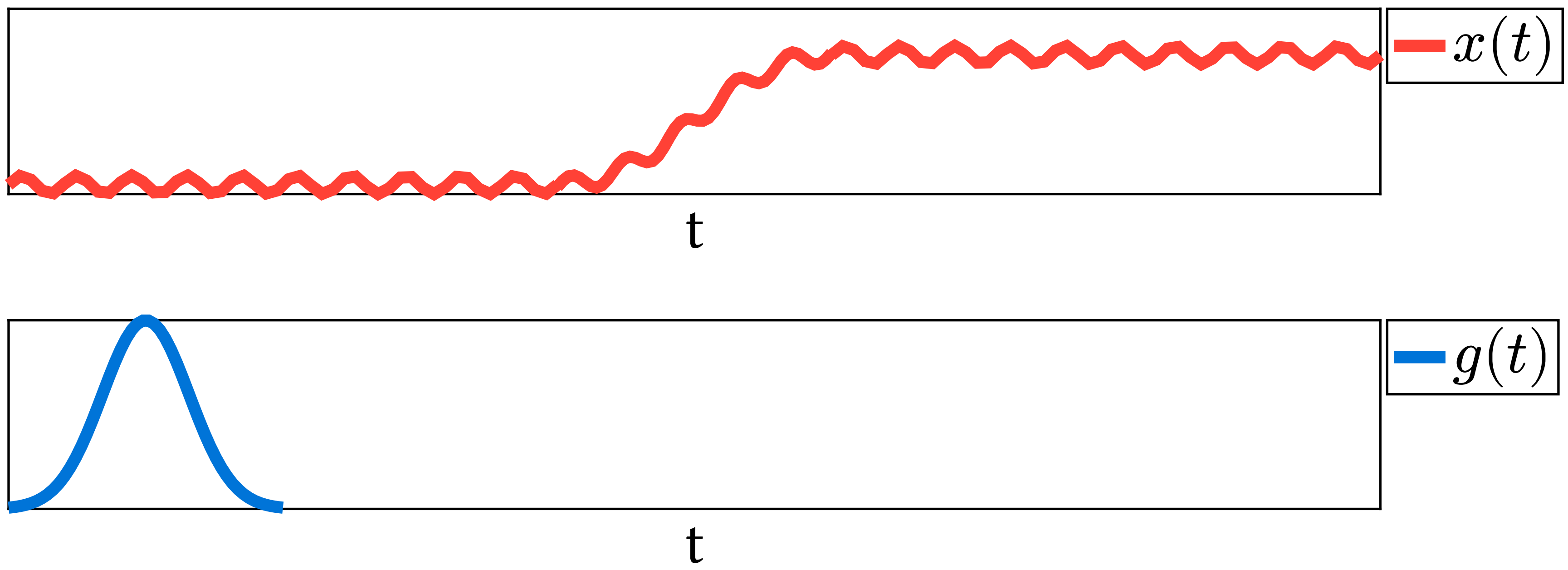




Convolution is **local** to the filter $g(t)$

Convolution



Convolution is **local** to the filter $g(t)$

Convolution is also **invariant** to time/space shifts

Convolution

Often, we use continuous time/space convolution for analog signals

Convolution

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For digital signals, we use discrete time/space

Convolution

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 3 & 4 & 5 \\ & & & & \end{bmatrix}$$

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Convolution

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} & \color{red}{2} & \color{red}{1} & & \\ 1 & \color{red}{2} & \color{red}{3} & 4 & 5 \\ 4 & \color{red}{5} & & & \end{bmatrix}$$

Convolution

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} & & 2 & 1 & \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 10 & & \end{bmatrix}$$

Convolution

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} & & & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 10 & 13 & \end{bmatrix}$$

Convolution

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$$

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Question: Does anybody see a connection to neural networks?

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Hint: What if I rewrite the filter?

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Hint: What if I rewrite the filter?

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta_2 & \theta_1 & & & \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 10 & 13 & \end{bmatrix}$$

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$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta_2 & \theta_1 & & & \\ 1 & 2 & 3 & 4 & 5 \\ \theta_2 + 2\theta_1 & 5 & 10 & 13 & \end{bmatrix}$$

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Convolution

$$\begin{bmatrix} g(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} & \color{red}{\theta_2} & \color{red}{\theta_1} & & \\ 1 & \color{red}{2} & \color{red}{3} & 4 & 5 \\ \theta_2 + 2\theta_1 & \color{red}{2\theta_2 + 3\theta_1} & 10 & 13 & \end{bmatrix}$$

Just like neural networks, convolution is a linear operation

Convolution

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It is a weighted sum of the inputs, just like a neuron

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Question: How does convolution differ from a neuron?

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Just like neural networks, convolution is a linear operation

It is a weighted sum of the inputs, just like a neuron

Question: How does convolution differ from a neuron?

Answer: In a neuron, each input x_i has a different parameter θ_i. In convolution, we reuse θ_i on $x_j, x_k, ...$

Convolution

Neuron: $$\boldsymbol{\theta}^\top \overline{\boldsymbol{x}} = \sum_{i=0}^{d_x} \theta_i \overline{x}_i$$

Convolution

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$$\boldsymbol{\theta}^\top \overline{\boldsymbol{x}} = \sum_{i=0}^{d_x} \theta_i \overline{x}_i$$

Convolution:

$$\boldsymbol{\theta}_1^\top \overline{\boldsymbol{x}}(t) + \boldsymbol{\theta}_2^\top \overline{\boldsymbol{x}}(t+1) = \left(\sum_{i=0}^{d_x} \theta_{1,i} \overline{x}_i(t) \right) + \left(\sum_{i=0}^{d_x} \theta_{2,i} \overline{x}_i(t+1) \right)$$

Convolution

Neuron:

$$\boldsymbol{\theta}^\top \overline{\boldsymbol{x}} = \sum_{i=0}^{d_x} \theta_i \overline{x}_i$$

Convolution:

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$$\begin{bmatrix} \boldsymbol{\theta_1} \\ \boldsymbol{\theta_2} \end{bmatrix} * \overline{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{\theta}_1^\top \overline{\boldsymbol{x}}(0) + \boldsymbol{\theta}_2^\top \overline{\boldsymbol{x}}(1) & \theta_1^\top \overline{\boldsymbol{x}}(1) + \boldsymbol{\theta}_2^\top \overline{\boldsymbol{x}}(2) & \ldots \end{bmatrix}$$

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We call this a **convolutional layer**

Convolution

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Question: Anything missing?

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Question: Anything missing?

Answer: Activation function!

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$$\begin{bmatrix} \sigma(\boldsymbol{\theta}_1^\top \overline{\boldsymbol{x}}(0) + \boldsymbol{\theta}_2^\top \overline{\boldsymbol{x}}(1)) & \sigma(\theta_1^\top \overline{\boldsymbol{x}}(1) + \boldsymbol{\theta}_2^\top \overline{\boldsymbol{x}}(2)) & \dots \end{bmatrix}$$

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Much better

Convolution

Convolution is **local**, in this example, we only consider two consecutive timesteps

Convolution

Convolution is **local**, in this example, we only consider two consecutive timesteps

Convolution is **shift invariant**, if $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ detect "hello", it does not matter whether "hello" occurs at $x(0), x(1)$ or $x(100), x(101)$

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