Neural Networks

CISC 7026: Introduction to Deep Learning

University of Macau

Notation Change

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Moving forward, I will differentiate between **data** indices $x_{[i]}$ and other indices x_i

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$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}$$

1. Review
2. Multivariate linear regression
3. Limitations of linear regression
4. History of neural networks
5. Biological neurons
6. Artificial neurons
7. Wide neural networks
8. Deep neural networks
9. Practical considerations

1. **Review**
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Review

Since you are very educated, we focused on how education affects life expectancy

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Studies show a causal effect of education on health

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Studies show a causal effect of education on health

- *The causal effects of education on health outcomes in the UK Biobank.* Davies et al. *Nature Human Behaviour.*
- By staying in school, you are likely to live longer
- Being rich also helps, but education alone has a **causal** relationship with life expectancy

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Task: Given your education, predict your life expectancy

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$X \in \mathbb{R}_+$: Years in school

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$$f : X \times \Theta \mapsto Y$$

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$\Theta \in \mathbb{R}^2$: Parameters

$$f : X \times \Theta \mapsto Y$$

Approach: Learn the parameters θ such that

$$f(x, \theta) = y; \quad x \in X, y \in Y$$

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Started with a linear function f

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$$f(x, \boldsymbol{\theta}) = f\left(x, \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix}\right) = \theta_1 x + \theta_0$$

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Then, we derived the square error function

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$$\text{error}(f(x, \boldsymbol{\theta}), y) = (f(x, \boldsymbol{\theta}) - y)^2$$

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We wrote the loss function for a single datapoint $x_{[i]}, y_{[i]}$ using the square error

$$\mathcal{L}\left(x_{[i]}, y_{[i]}, \boldsymbol{\theta}\right) = \text{error}\left(f\left(x_{[i]}, \boldsymbol{\theta}\right), y_{[i]}\right) = \left(f\left(x_{[i]}, \boldsymbol{\theta}\right) - y_{[i]}\right)^2$$

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But we wanted to learn a model over **all** the data, not a single datapoint

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$$\boldsymbol{x} = \begin{bmatrix} x_{[1]} & x_{[2]} & \cdots & x_{[n]} \end{bmatrix}^\top, \boldsymbol{y} = \begin{bmatrix} y_{[1]} & y_{[2]} & \cdots & y_{[n]} \end{bmatrix}^\top$$

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$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \text{error}\left(f\left(x_{[i]}, \boldsymbol{\theta}\right), y_{[i]}\right) = \sum_{i=1}^{n} \left(f\left(x_{[i]}, \boldsymbol{\theta}\right) - y_{[i]}\right)^2$$

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$$\arg\min_x f(x) = -1$$

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$$\begin{aligned}\arg\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\theta}) &= \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \text{error}\Big(f\Big(x_{[i]}, \boldsymbol{\theta}\Big), y_{[i]}\Big) \\ &= \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \Big(f\Big(x_{[i]}, \boldsymbol{\theta}\Big) - y_{[i]}\Big)^2\end{aligned}$$

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We use the design matrix to find an **analytical** solution to the optimization objective

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We use the design matrix to find an **analytical** solution to the optimization objective

$$\boldsymbol{\theta} = \left(\boldsymbol{X}_D^\top \boldsymbol{X}_D\right)^{-1} \boldsymbol{X}_D^\top \boldsymbol{y}$$

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With this analytical solution, we were able to learn a linear model

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Then, we used a trick to extend linear regression to nonlinear models

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$$\boldsymbol{X}_D = \begin{bmatrix} x_{[1]} & 1 \\ x_{[2]} & 1 \\ \vdots & \vdots \\ x_{[n]} & 1 \end{bmatrix} \Rightarrow \boldsymbol{X}_D = \begin{bmatrix} \log\left(1 + x_{[1]}\right) & 1 \\ \log\left(1 + x_{[2]}\right) & 1 \\ \vdots & \vdots \\ \log\left(1 + x_{[n]}\right) & 1 \end{bmatrix}$$

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We extended to polynomials, which are **universal function approximators**

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$$\boldsymbol{X}_D = \begin{bmatrix} x_{[1]} & 1 \\ x_{[2]} & 1 \\ \vdots & \vdots \\ x_{[n]} & 1 \end{bmatrix} \Rightarrow \boldsymbol{X}_D = \begin{bmatrix} x_{[1]}^m & x_{[1]}^{m-1} & \dots & x_{[1]} & 1 \\ x_{[2]}^m & x_{[2]}^{m-1} & \dots & x_{[2]} & 1 \\ \vdots & \vdots & \ddots & & \\ x_{[n]}^m & x_{[n]}^{m-1} & \dots & x_{[n]} & 1 \end{bmatrix}$$

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$$f : X \times \Theta \mapsto \mathbb{R}$$

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$$\Theta \in \mathbb{R}^2 \Rightarrow \Theta \in \mathbb{R}^{m+1}$$

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$$f(x, \boldsymbol{\theta}) = \theta_m x^m + \theta_{m-1} x^{m-1}, ..., \theta_1 x^1 + \theta_0$$

$m = 2$

$m = 3$

$m = 5$

Review

We care about **generalization** in machine learning

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So we should always split our dataset into a training dataset and a testing dataset

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Last time, we assumed a single-input system

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We can solve these problems using linear regression too

For multivariate problems, we will define the input dimension as d_x

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We will write the vectors as

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$x_{[i],1}$ refers to the first dimension of training data i

The design matrix for a **multivariate** linear system is

$$\boldsymbol{X}_D = \begin{bmatrix} x_{[1],d_x} & x_{[1],d_x-1} & \cdots & x_{[1],1} & 1 \\ x_{[2],d_x} & x_{[2],d_x-1} & \cdots & x_{[2],1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ x_{[n],d_x} & x_{[n],d_x-1} & \cdots & x_{[n],1} & 1 \end{bmatrix}$$

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Remember $x_{[n],d_x}$ refers to dimension d_x of training data n

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Remember $x_{[n],d_x}$ refers to dimension d_x of training data n

The solution is the same as before

$$\boldsymbol{\theta} = \left(\boldsymbol{X}_D^\top \boldsymbol{X}_D\right)^{-1} \boldsymbol{X}_D^\top \boldsymbol{y}$$

Agenda

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Linear models are useful for certain problems

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One-dimensional polynomial functions

$$\boldsymbol{X}_D = \begin{bmatrix} x_{[1]}^m & x_{[1]}^{m-1} & \dots & x_{[1]} & 1 \\ x_{[2]}^m & x_{[2]}^{m-1} & \dots & x_{[2]} & 1 \\ \vdots & \vdots & \ddots & & \\ x_{[n]}^m & x_{[n]}^{m-1} & \dots & x_{[n]} & 1 \end{bmatrix}$$

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Combine them to create multi-dimensional polynomial functions

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Let us do an example

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Task: predict how many ♥ a photo gets on social media

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$$X \in \mathbb{Z}_+^{256 \times 256} = \mathbb{Z}_+^{65536}; \quad Y \in \mathbb{Z}_+$$

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Highly nonlinear task, use a polynomial with order $m = 20$

$$\boldsymbol{X}_D = \begin{bmatrix} \boldsymbol{x}_{D,[1]} & \cdots & \boldsymbol{x}_{D,[n]} \end{bmatrix}^\top$$

$$\boldsymbol{X}_D = \left[\boldsymbol{x}_{D,[1]} \;\; \cdots \;\; \boldsymbol{x}_{D,[n]}\right]^\top$$

$$\boldsymbol{x}_{D,[i]} =$$

$$\left[\underbrace{x^m_{[i],d_x} x^m_{[i],d_x-1} \cdots x^m_{[i],1}}_{(d_x \Rightarrow 1, x^m)} \;\; \underbrace{x^m_{[i],d_x} x^m_{[i],d_x-1} \cdots x^m_{[i],2}}_{(d_x \Rightarrow 2, x^m)} \;\; \cdots \;\; \underbrace{x^{m-1}_{[i],d_x} x^{m-1}_{[i],d_x-1} \cdots x^m_{[i],1}}_{(d_x \Rightarrow 1, x^{m-1})} \;\; \cdots\right]$$

Question: How many columns in this matrix?

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$$\boldsymbol{x}_{D,[i]} =$$

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Question: How many columns in this matrix?

Hint: $d_x = 2, m = 3$: $x^3 + y^3 + x^2y + y^2x + xy + x + y + 1$

$$\boldsymbol{X}_D = \left[\boldsymbol{x}_{D,[1]} \ \cdots \ \boldsymbol{x}_{D,[n]}\right]^\top$$

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Hint: $d_x = 2, m = 3$: $x^3 + y^3 + x^2y + y^2x + xy + x + y + 1$

Answer: $(d_x)^m = 65536^{20} + 1 \approx 10^{96}$

How big is 10^{96}?

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Question: How many atoms are there in the universe?

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Answer: 10^{82}

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Polynomial regression does not scale to large inputs

Issues arise with other problems

1. **Poor scalability**
2. Polynomials do not generalize well

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What happens to polynomials outside of the support (dataset)?

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Take the limit of polynomials to see their behavior

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Take the limit of polynomials to see their behavior

$$\lim_{x \to \infty} \theta_m x^m + \theta_{m-1} x^{m-1} + \ldots$$

Equation of a polynomial

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$$\lim_{x \to \infty} \theta_m x^m + \theta_{m-1} x^{m-1} + \ldots$$

Equation of a polynomial

$$\lim_{x \to \infty} x^m \left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots \right)$$

Factor out x^m

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$$\lim_{x \to \infty} x^m \left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots \right)$$

Factor out x^m

$$\lim_{x \to \infty} x^m \cdot \lim_{x \to \infty} \left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots \right)$$

Split the limit (limit of products)

$$\lim_{x \to \infty} x^m \cdot \lim_{x \to \infty} \left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots \right)$$

Split the limit (limit of products)

$$\left(\lim_{x \to \infty} x^m \right) \cdot (\theta_m + 0 + \ldots)$$

Evaluate right limit

$$\lim_{x\to\infty} x^m \cdot \lim_{x\to\infty}\left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots\right)$$

Split the limit (limit of products)

$$\left(\lim_{x\to\infty} x^m\right) \cdot (\theta_m + 0 + \ldots)$$

Evaluate right limit

$$\theta_m \lim_{x\to\infty} x^m$$

Rewrite

$$\lim_{x\to\infty} x^m \cdot \lim_{x\to\infty}\left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots\right)$$

Split the limit (limit of products)

$$\left(\lim_{x\to\infty} x^m\right) \cdot (\theta_m + 0 + \ldots)$$

Evaluate right limit

$$\theta_m \lim_{x\to\infty} x^m$$

Rewrite

$$\theta_m \lim_{x\to\infty} x^m = \infty$$

If $\theta_m > 0$

$$\lim_{x\to\infty} x^m \cdot \lim_{x\to\infty}\left(\theta_m + \frac{\theta_{m-1}}{x} + \ldots\right)$$ Split the limit (limit of products)

$$\left(\lim_{x\to\infty} x^m\right) \cdot (\theta_m + 0 + \ldots)$$ Evaluate right limit

$$\theta_m \lim_{x\to\infty} x^m$$ Rewrite

$$\theta_m \lim_{x\to\infty} x^m = \infty$$ If $\theta_m > 0$

$$\theta_m \lim_{x\to\infty} x^m = -\infty$$ If $\theta_m < 0$

Polynomials quickly tend towards $-\infty, \infty$ outside of the support

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$$f(x) = x^3 - 2x^2 - x + 2$$

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Polynomials quickly tend towards $-\infty, \infty$ outside of the support

$$f(x) = x^3 - 2x^2 - x + 2$$

Remember, to predict new data we want our functions to generalize

Linear regression has issues

Linear regression has issues

1. Poor scalability

Linear regression has issues

1. Poor scalability
2. Polynomials do not generalize well

We can use neural networks as an alternative to linear regression

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Neural network benefits:

We can use neural networks as an alternative to linear regression

Neural network benefits:

1. Scale to large inputs

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Neural network benefits:

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2. Slightly better generalization

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We can use neural networks as an alternative to linear regression

Neural network benefits:

1. Scale to large inputs
2. Slightly better generalization

Drawbacks:

1. No analytical solution
2. High data requirement

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In 1939-1945, there was a World War

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Militaries invested funding for research, and invented the computer

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Meanwhile, a neuroscientist and mathematician (McCullough and Pitts) were trying to understand the human brain

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They designed the theory for the first neural network

Rosenblatt implemented this neural network theory on a computer a few years later

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At the time, computers were very slow and expensive

Through advances in theory and hardware, neural networks became slightly better

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Through advances in theory and hardware, neural networks became slightly better

Around 2012, these improvements culminated in neural networks that perform like humans

So what is a neural network?

So what is a neural network?

It is a function, inspired by how the brain works

So what is a neural network?

It is a function, inspired by how the brain works

$$f : X \times \Theta \mapsto Y$$

Brains and neural networks rely on **neurons**

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Brain: Biological neurons $\rightarrow$ Biological neural network

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First, let us review biological neurons

Brains and neural networks rely on **neurons**

Brain: Biological neurons $\rightarrow$ Biological neural network

Computer: Artificial neurons $\rightarrow$ Artificial neural network

First, let us review biological neurons

Note: I am not a neuroscientist! I may make simplifications or errors with biology

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A simplified neuron consists of many parts

Neurons send messages based on messages received from other neurons

Incoming electrical signals travel along dendrites

Dendrites are not all equal! Different dendrites have different diameters and structures

Electrical charges collect in the Soma (cell body)

The axon outputs an electrical signal to other neurons

The axon terminals will connect to dendrites of other neurons through a synapse

The synapse converts electrical signal, to chemical signal, back to electrical signal

The synapse converts electrical signal, to chemical signal, back to electrical signal

Synaptic weight determines how well a signal crosses the gap

For our purposes, we can model the axon terminals, dendrites, and synapses to be one thing

The neuron takes many inputs, and produces a single output

The neuron will only output a signal down the axon ("fire") at certain times

How does a neuron decide to send an impulse (“fire”)?

How does a neuron decide to send an impulse ("fire")?

Incoming impulses (via dendrites) change the electric potential of the neuron

How does a neuron decide to send an impulse ("fire")?

Incoming impulses (via dendrites) change the electric potential of the neuron

In a parallel circuit, we can sum voltages together

How does a neuron decide to send an impulse ("fire")?

Incoming impulses (via dendrites) change the electric potential of the neuron

In a parallel circuit, we can sum voltages together

Many active dendrites will add together and trigger an impulse

Pain triggers initial nerve impulse, starts a chain reaction into the brain

When the signal reaches the brain,
we will think

After thinking, we will take action

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Direction message travels
Nucleus
Soma
(cell body)
Axon terminals
Dendrites
Myelin
Sheaths
Axon

Question: How could we write a neuron as a function? $f : __ \mapsto __$

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Answer:

$$f : \underbrace{\mathbb{R}^{d_x}}_{\text{Dendrite voltages}} \times \underbrace{\mathbb{R}^{d_x}}_{\text{Synaptic weight}} \mapsto \underbrace{\mathbb{R}}_{\text{Axon voltage}}$$

Let us implement an artifical neuron as a function

Let us implement an artifical neuron as a function

Neuron has a structure of dendrites with synaptic weights

Let us implement an artifical neuron as a function

Neuron has a structure of dendrites with synaptic weights

$$f\left(\begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_{d_x}\end{bmatrix}\right)$$

$$f(\boldsymbol{\theta})$$

Let us implement an artifical neuron as a function

Each incoming dendrite has some voltage potential

Let us implement an artifical neuron as a function

Each incoming dendrite has some voltage potential

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{d_x} \end{bmatrix}\right)$$

$$f(\boldsymbol{x}, \boldsymbol{\theta})$$

Let us implement an artifical neuron as a function

Voltage potentials sum together to give us the voltage in the cell body

Let us implement an artifical neuron as a function

Voltage potentials sum together to give us the voltage in the cell body

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{d_x} \end{bmatrix}\right) = \sum_{j=1}^{d_x} \theta_j x_j$$

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \boldsymbol{x}$$

Let us implement an artifical neuron as a function

The axon fires only if the voltage is over a threshold

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$$\sigma(x) = H(x) =$$

Let us implement an artifical neuron as a function

The axon fires only if the voltage is over a threshold

$$\sigma(x) = H(x) =$$

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}\right) = \sigma\left(\sum_{j=1}^{d_x} \theta_j x_j\right)$$

Maybe we want to vary the activation threshold

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$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{d_x} \end{bmatrix}\right) = \sigma\left(\theta_0 + \sum_{j=1}^{d_x} \theta_j x_j\right)$$

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$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \theta_0 + \boldsymbol{\theta}_{1:d_x}^\top \boldsymbol{x}$$

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(\theta_0 + \boldsymbol{\theta}_{1:d_x}^\top \boldsymbol{x}\right)$$

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This is the artificial neuron!

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Let us write out the full equation for a neuron

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$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(\theta_{d_x} x_{d_x} + \theta_{d_x - 1} x_{d_x - 1} + \ldots + \theta_0\right)$$

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Question: Does this look familiar to anyone?

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(\theta_0 + \boldsymbol{\theta}_{1:d_x}^\top \boldsymbol{x}\right)$$

This is the artificial neuron!

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Question: Does this look familiar to anyone?

Answer: Inside σ is the multivariate linear model!

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \theta_{d_x} x_{d_x} + \theta_{d_x - 1} x_{d_x - 1} + \ldots + \theta_0$$

We model a neuron using a linear model and activation function

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Sometimes, we will write $\boldsymbol{\theta}$ as a bias and weight $b, \boldsymbol{w}$

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$$\boldsymbol{\theta} = \begin{bmatrix} b \\ \boldsymbol{w} \end{bmatrix}; \quad \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{d_x} \end{bmatrix} = \begin{bmatrix} b \\ w_1 \\ \vdots \\ w_{d_x} \end{bmatrix}$$

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$$f\left(\boldsymbol{x}, \begin{bmatrix} b \\ \boldsymbol{w} \end{bmatrix}\right) = b + \boldsymbol{w}^\top \boldsymbol{x}$$

Relax

1
x_1
x_2
x_{d_x}
θ_0
θ_1
θ_2
θ_{d_x}
$\sum$
σ
y

In machine learning, we represent functions

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What kinds of functions can our neuron represent?

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What kinds of functions can our neuron represent?

Let us consider some **boolean** functions

In machine learning, we represent functions

What kinds of functions can our neuron represent?

Let us consider some **boolean** functions

Let us start with a logical AND function

Review: Activation function (Heaviside step function)

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$$\sigma(x) = H(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$$

Implement AND using an artificial neuron

Implement AND using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

Implement AND using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

$$\boldsymbol{\theta} = [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top = [-1 \;\; 1 \;\; 1]^\top$$

Implement AND using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

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x_1	x_2	y	$f(x_1, x_2, \boldsymbol{\theta})$	$\hat{y}$
0	0	0	$\sigma(-1 + 1 \cdot 0 + 1 \cdot 0) = \sigma(-1)$	0
0	1	0	$\sigma(-1 + 1 \cdot 0 + 1 \cdot 1) = \sigma(0)$	0
1	0	0	$\sigma(-1 + 1 \cdot 1 + 1 \cdot 0) = \sigma(0)$	0
1	1	1	$\sigma(-1 + 1 \cdot 1 + 1 \cdot 1) = \sigma(1)$	1

Implement OR using an artificial neuron

Implement OR using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

Implement OR using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

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Implement OR using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

$$\boldsymbol{\theta} = [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top = [0 \;\; 1 \;\; 1]^\top$$

x_1	x_2	y	$f(x_1, x_2, \boldsymbol{\theta})$	$\hat{y}$
0	0	0	$\sigma(0 + 1 \cdot 0 + 1 \cdot 0) = \sigma(0)$	0
0	1	0	$\sigma(0 + 1 \cdot 1 + 1 \cdot 0) = \sigma(1)$	1
1	0	1	$\sigma(0 + 1 \cdot 0 + 1 \cdot 1) = \sigma(1)$	1
1	1	1	$\sigma(1 + 1 \cdot 1 + 1 \cdot 1) = \sigma(2)$	1

Implement XOR using an artificial neuron

Implement XOR using an artificial neuron

$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

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$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

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$$f\left([x_1 \;\; x_2]^\top, [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top\right) = \sigma(\theta_0 + x_1\theta_1 + x_2\theta_2)$$

$$\boldsymbol{\theta} = [\theta_0 \;\; \theta_1 \;\; \theta_2]^\top = [? \;\; ? \;\; ?]^\top$$

x_1	x_2	y	$f(x_1, x_2, \boldsymbol{\theta})$	$\hat{y}$
0	0	0	This is IMPOSSIBLE!	
0	1	1		
1	0	1		
1	1	0		

Why can't we represent XOR using a neuron?

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We want to represent any function, not just linear functions

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We can only represent $\sigma(\text{linear function})$

XOR is not a linear combination of x_1, x_2!

We want to represent any function, not just linear functions

Let us think back to biology, maybe it has an answer

Brain: Biological neurons $\rightarrow$ Biological neural network

Brain: Biological neurons $\rightarrow$ Biological neural network

Computer: Artificial neurons $\rightarrow$ Artificial neural network

Connect artificial neurons into a network

Neuron

Neural Network

Adding neurons in **parallel** creates a **wide** neural network

Adding neurons in **parallel** creates a **wide** neural network

Adding neurons in **series** creates a **deep** neural network

Adding neurons in **parallel** creates a **wide** neural network

Adding neurons in **series** creates a **deep** neural network

Today's powerful neural networks are both **wide** and **deep**

Adding neurons in **parallel** creates a **wide** neural network

Adding neurons in **series** creates a **deep** neural network

Today's powerful neural networks are both **wide** and **deep**

Let us try to implement XOR using a wide and deep neural network

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How do we express a **wide** neural network mathematically?

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A single neuron:

$$f : \mathbb{R}^{d_x} \times \Theta \mapsto \mathbb{R}$$

$$\Theta \in \mathbb{R}^{d_x+1}$$

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A single neuron:

$$f : \mathbb{R}^{d_x} \times \Theta \mapsto \mathbb{R}$$

$$\Theta \in \mathbb{R}^{d_x+1}$$

d_y neurons (wide):

$$f : \mathbb{R}^{d_x} \times \Theta \mapsto \mathbb{R}^{d_y}$$

$$\Theta \in \mathbb{R}^{(d_x+1)\times d_y}$$

For a single neuron:

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_{d_x} \end{bmatrix}\right) = \sigma\left(\theta_0 + \sum_{i=1}^{d_x} \theta_i x_i\right)$$

For a single neuron:

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_{d_x} \end{bmatrix}\right) = \sigma\left(\theta_0 + \sum_{i=1}^{d_x} \theta_i x_i\right)$$

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma(\theta_0 + \boldsymbol{\theta}_{1:n}^\top \boldsymbol{x})$$

For a wide network:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_{1,0} & \theta_{2,0} & \cdots & \theta_{d_x,0} \\ \theta_{1,1} & \theta_{2,1} & \cdots & \theta_{d_x,1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1,d_y} & \theta_{2,d_y} & \cdots & \theta_{d_y,d_x} \end{bmatrix}\right) = \begin{bmatrix} \sigma\left(\theta_{1,0} + \sum_{i=1}^{d_x} x_i \theta_{1,i}\right) \\ \sigma\left(\theta_{2,0} + \sum_{i=1}^{d_x} x_i \theta_{2,i}\right) \\ \vdots \\ \sigma\left(\theta_{d_y,0} + \sum_{i=1}^{d_x} x_i \theta_{d_y,i}\right) \end{bmatrix}$$

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(\boldsymbol{\theta}_{\cdot,0} + \boldsymbol{\theta}_{\cdot,1:d_x}\boldsymbol{x}\right); \quad \boldsymbol{\theta} \in \mathbb{R}^{(d_y+1)\times d_x}$$

For a wide network:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d_x} \end{bmatrix}, \begin{bmatrix} \theta_{1,0} & \theta_{2,0} & \cdots & \theta_{d_x,0} \\ \theta_{1,1} & \theta_{2,1} & \cdots & \theta_{d_x,1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1,d_y} & \theta_{2,d_y} & \cdots & \theta_{d_y,d_x} \end{bmatrix}\right) = \begin{bmatrix} \sigma\left(\theta_{1,0} + \sum_{i=1}^{d_x} x_i \theta_{1,i}\right) \\ \sigma\left(\theta_{2,0} + \sum_{i=1}^{d_x} x_i \theta_{2,i}\right) \\ \vdots \\ \sigma\left(\theta_{d_y,0} + \sum_{i=1}^{d_x} x_i \theta_{d_y,i}\right) \end{bmatrix}$$

$$f(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(\boldsymbol{\theta}_{\cdot,0} + \boldsymbol{\theta}_{\cdot,1:d_x}\boldsymbol{x}\right); \quad \boldsymbol{\theta} \in \mathbb{R}^{(d_y+1)\times d_x}$$

$$f\left(\boldsymbol{x}, \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{W} \end{bmatrix}\right) = \sigma(\boldsymbol{b} + \boldsymbol{W}\boldsymbol{x}); \quad \boldsymbol{b} \in \mathbb{R}^{d_y}, \boldsymbol{W} \in \mathbb{R}^{d_y \times d_x}$$

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$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \;\; \boldsymbol{\theta}_2 \;\; \ldots \;\; \boldsymbol{\theta}_\ell]^\top = [\boldsymbol{\varphi} \;\; \boldsymbol{\psi} \;\; \ldots \;\; \boldsymbol{\xi}]^\top$$

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$$f(\boldsymbol{x}, \boldsymbol{\theta}) = f_\ell(\ldots f_2(f_1(\boldsymbol{x}, \boldsymbol{\varphi}), \boldsymbol{\psi})\ldots\boldsymbol{\xi})$$

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A deep neural network is made of many layers

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$$
\begin{aligned}
f(x_1, x_2, \boldsymbol{\theta}) = \sigma\big(\theta_{3,0} & \\
+\theta_{3,1} \quad & \cdot \quad \sigma\big(\theta_{1,0} + x_1\theta_{1,1} + x_2\theta_{1,2}\big) \\
+\theta_{3,2} \quad & \cdot \quad \sigma\big(\theta_{2,0} + x_1\theta_{2,1} + x_2\theta_{2,2}\big)\big)
\end{aligned}
$$

Proof Sketch: Approximate a function $g(x)$ using a linear combination of Heaviside functions

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Very powerful finding! The basis of deep learning.

Task: predict how many ♥ a photo gets on social media

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I will explain them again very simply

A **layer** is a linear operation and an activation function

$$f\left(\boldsymbol{x}, \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{W} \end{bmatrix}\right) = \sigma(\boldsymbol{b} + \boldsymbol{W}\boldsymbol{x})$$

Many layers makes a deep neural network

$$\boldsymbol{z}_1 = f\left(\boldsymbol{x}, \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{W}_1 \end{bmatrix}\right)$$

$$\boldsymbol{z}_2 = f\left(\boldsymbol{z}_1, \begin{bmatrix} \boldsymbol{b}_2 \\ \boldsymbol{W}_2 \end{bmatrix}\right)$$

$$\boldsymbol{y} = f\left(\boldsymbol{z}_2, \begin{bmatrix} \boldsymbol{b}_2 \\ \boldsymbol{W}_2 \end{bmatrix}\right)$$

Let us create a wide neural network in colab! https://colab.research.google.com/drive/1bLtf3QY-yROIif_EoQSU1WS7svd0q8j7?usp=sharing

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