Regression

CISC 7026: Introduction to Deep Learning

University of Macau

ML

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Let us start with regression

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Approach: Learn the parameters θ such that

$$f(x, \theta) = y$$

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Goal: Given someone's education, you can predict how long they will live

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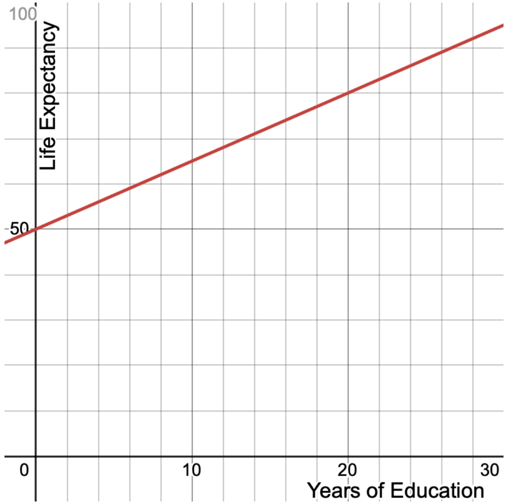
$$f(x, \boldsymbol{\theta}) = f\left(x, \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}\right) = \theta_1 x + \theta_0$$



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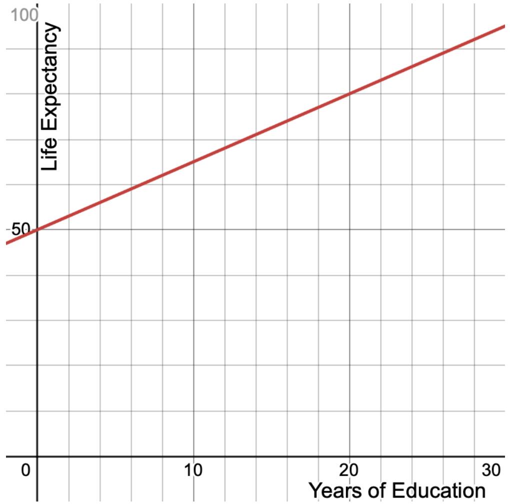

Now, we need to find the parameters $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix}$ that makes $f(x, \boldsymbol{\theta}) = y$

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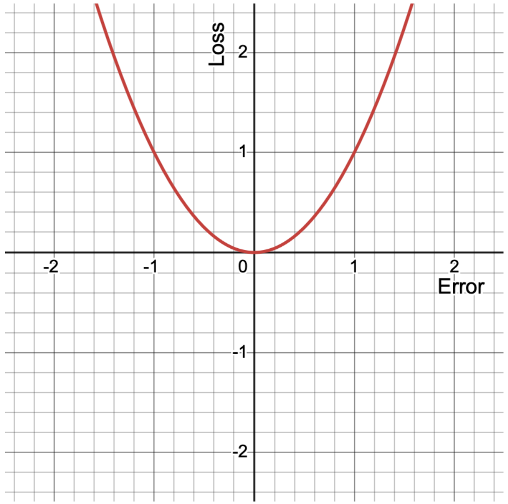
Question: How do we find $\boldsymbol{\theta}$? (Hint: We want $f(x, \boldsymbol{\theta}) = y$)

Answer: We will minimize the **loss** (error) between $f(x, \boldsymbol{\theta})$ and y

E.g.,

$$\min_{\boldsymbol{\theta}} (f(x, \boldsymbol{\theta}) - y)^2 = 0$$



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$$\text{error}(y, \hat{y}) = (y - \hat{y})^2$$

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We can write the loss function for a single datapoint x_i, y_i as

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Question: Any issues with $\mathcal{L}$?

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Answer: We only consider a single datapoint! We want to learn $\boldsymbol{\theta}$ for the entire dataset

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For a single x_i, y_i:

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Minimizing this loss function will give us the optimal parameters!

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Answer: For now, magic! We need more knowledge before we can derive this.

Linear Regression

First, construct a **design matrix** $\boldsymbol{X}$ containing input data x and a constant 1 for the bias. Also construct a $\boldsymbol{y}$ vector!

$$\boldsymbol{X} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

And remember the parameters $\boldsymbol{\theta}$

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix},$$

$$\boldsymbol{\theta} = \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{y}$$

The solution to linear regression

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Back to the example...

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Task: Given your education, predict your life expectancy

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You will be doing this in your first assignment!

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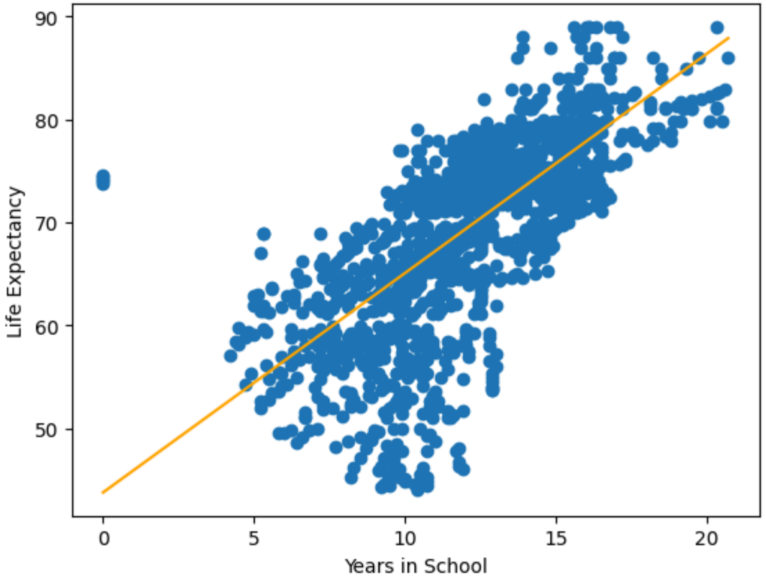

Tips for assignment 1

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def f(theta, design):
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  return theta @ design
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Not all matrices can be inverted! Ensure the matrices are square and the condition number is low

```python
A.shape
cond = jax.numpy.linalg.cond(A)
```

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Relax

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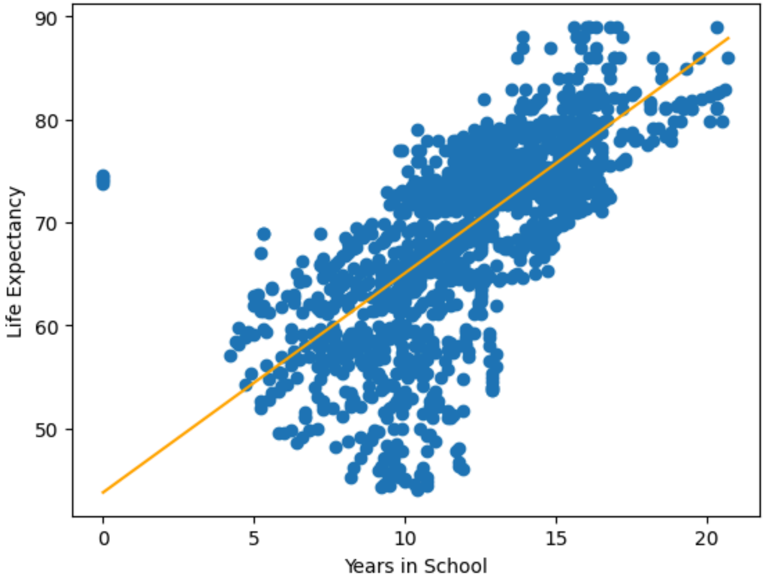

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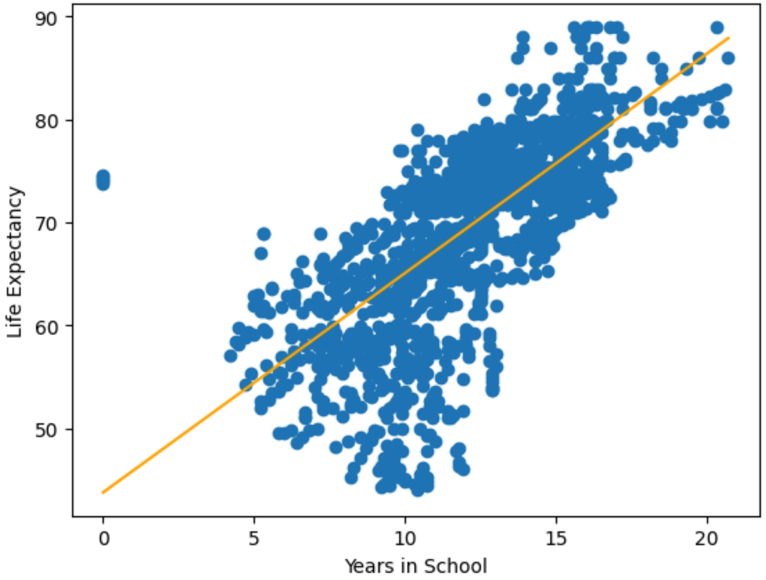

Could we do better than a linear function f?

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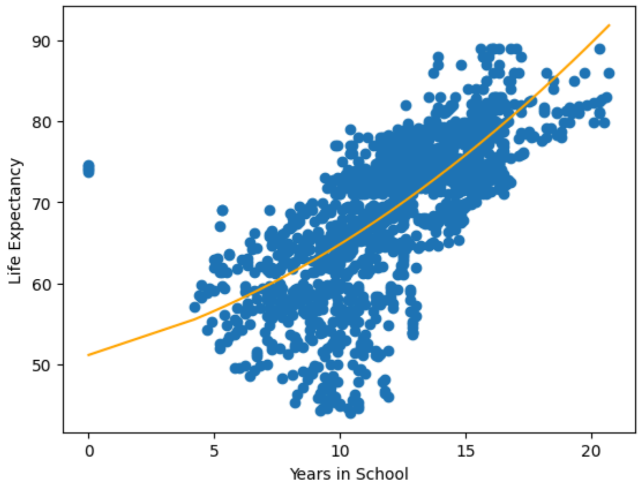

What if we used a polynomial instead?

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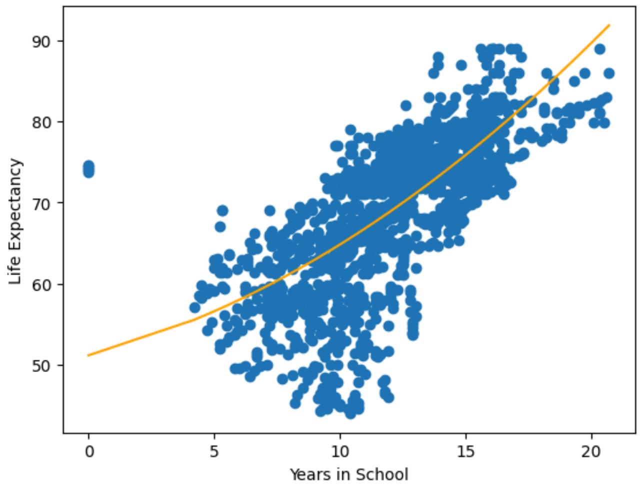

What if we used a polynomial instead?

$$f(x) = \theta_n x^n + \theta_{n-1} x^{n-1}, ..., \theta_1 + x^1 + \theta_0$$

Example

$$f(x, \boldsymbol{\theta}) = f\left(x, \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \right) = \theta_n x^n + \theta_{n-1} x^{n-1}, ..., \theta_1 + x^1 + \theta_0$$

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$$f(x, \boldsymbol{\theta}) = \begin{bmatrix} \theta_n & \theta_{n-1} & \dots & \theta_1 & b \end{bmatrix} \begin{bmatrix} x^n \\ x^{n-1} \\ \vdots \\ x^1 \\ 1 \end{bmatrix}$$

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How do we choose n? Let us try different n

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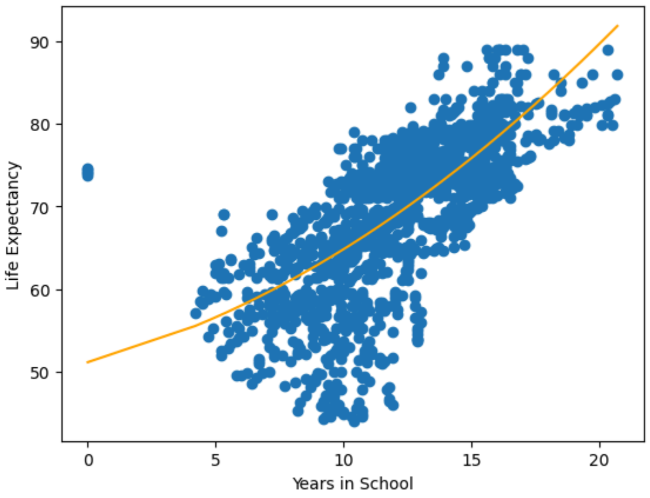
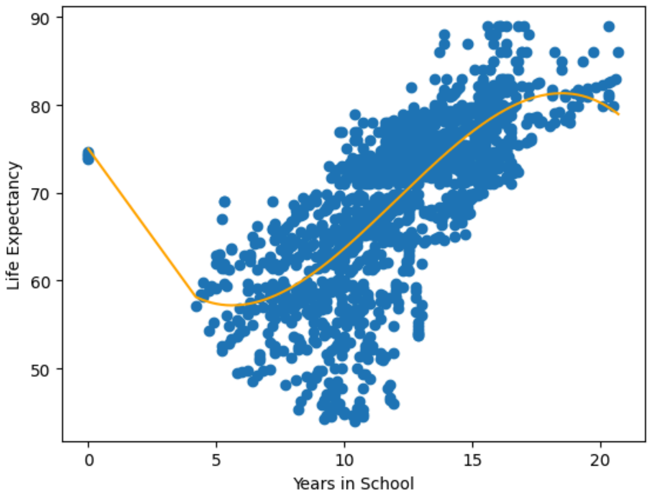


$n = 2$



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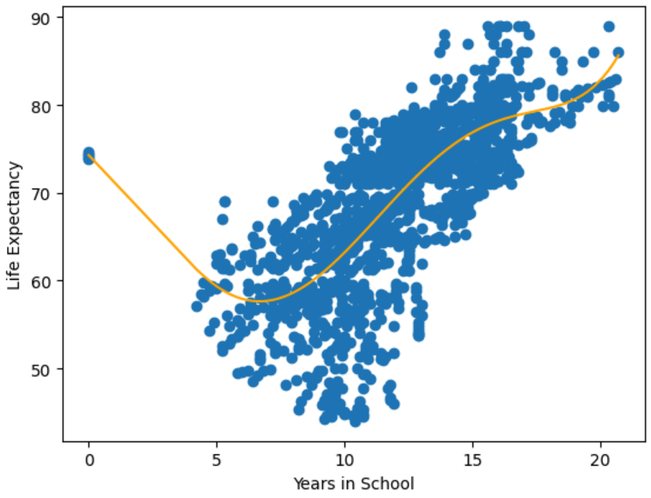

$n = 5$

Example

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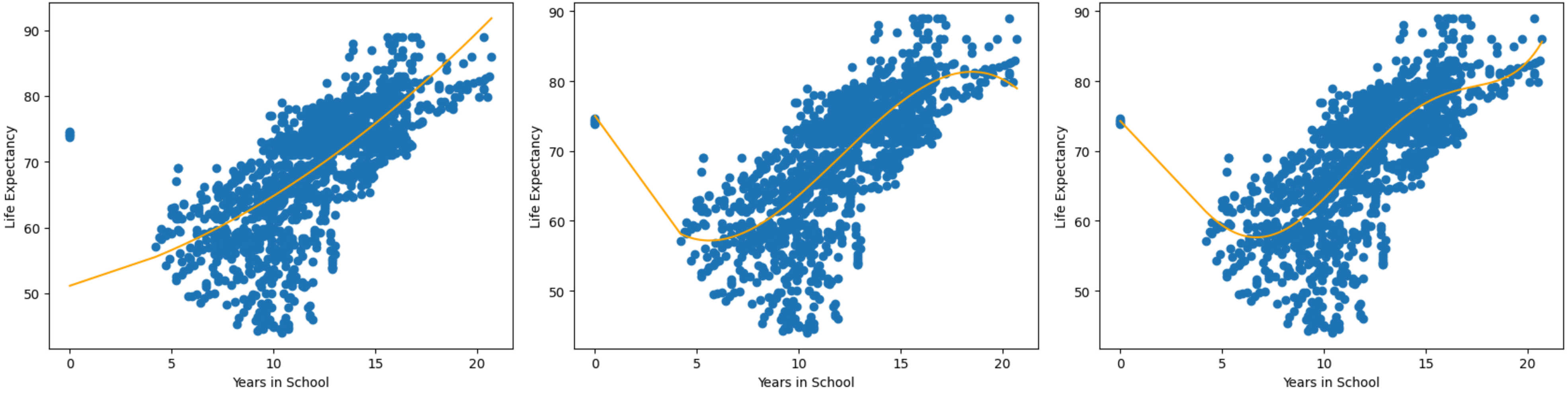

$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 5$

Question: Which n should we pick? Why?

Example

Data is inherently noisy

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The world is governed by random processes

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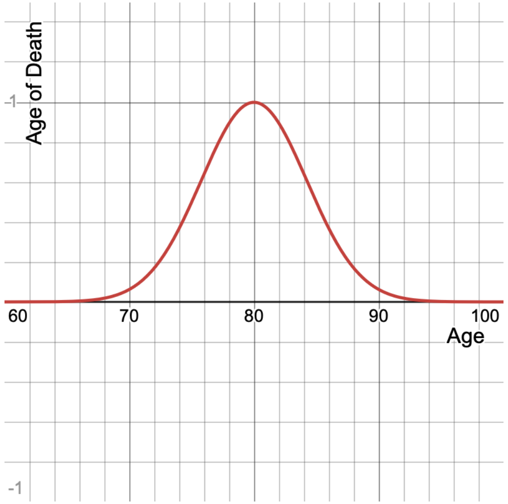

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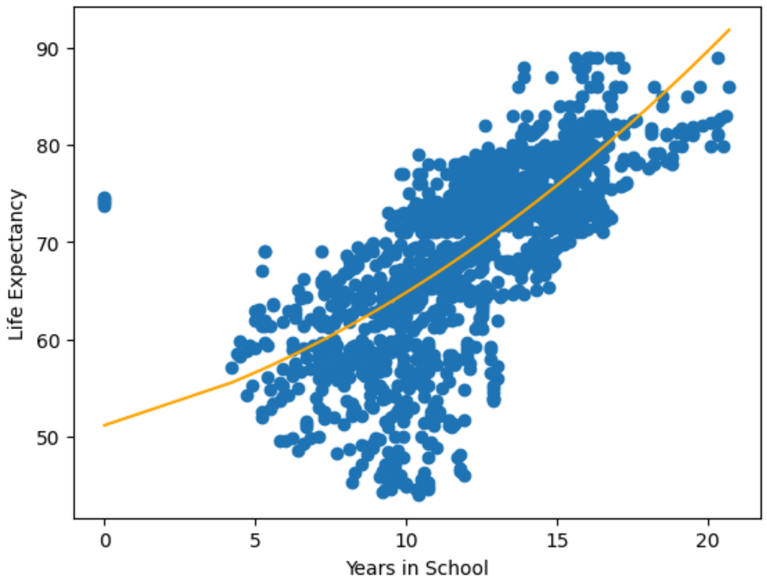
This is just an estimate



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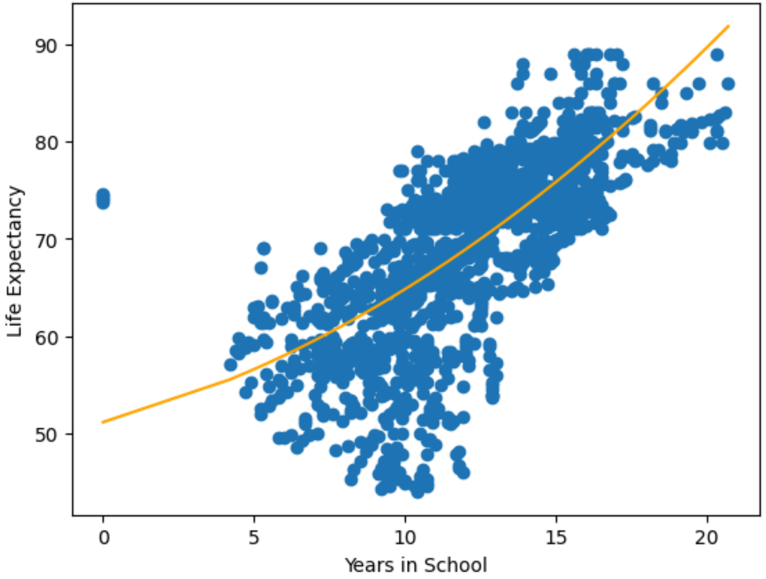
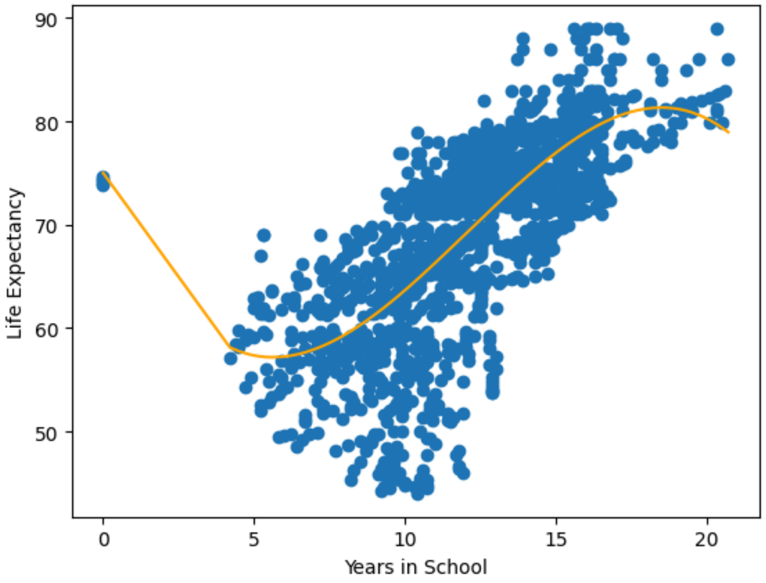


Going to school for 20 years will not save you from a hungry bear

Example



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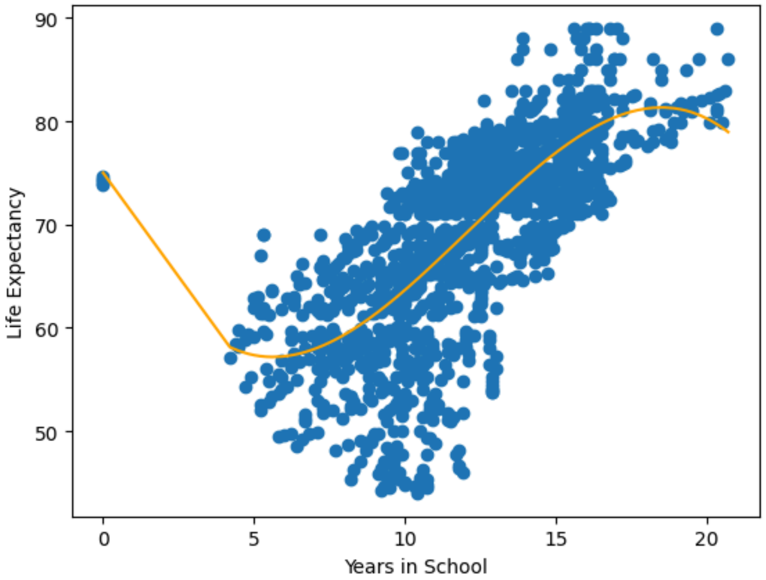

When we fit to noise instead of the trend, we call it **overfitting**

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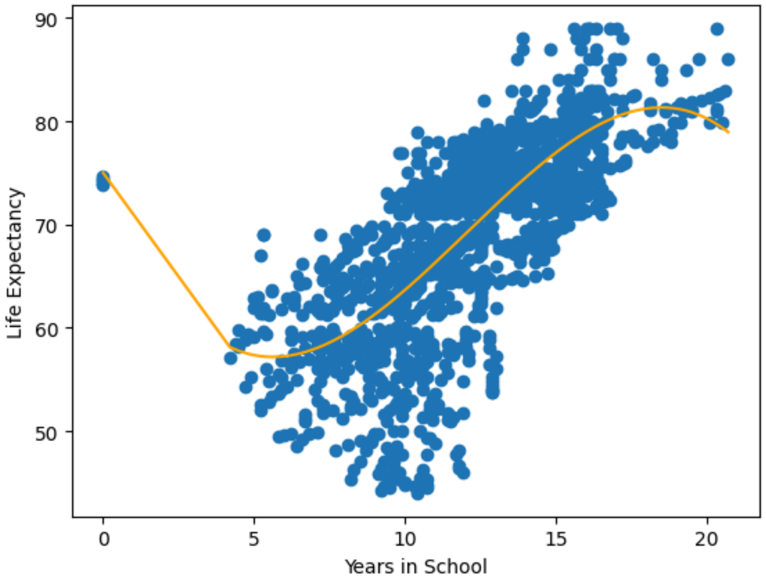

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Overfitting is bad because our predictions will be inaccurate

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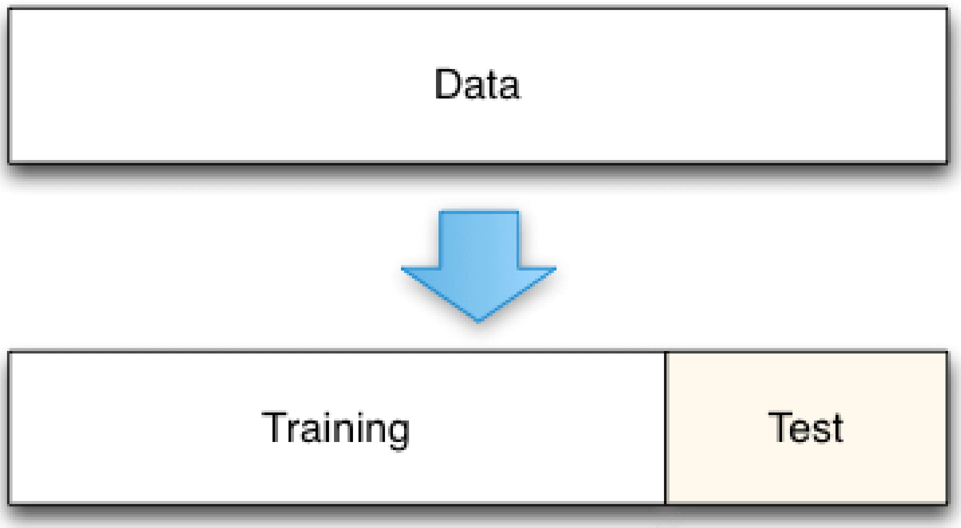

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ML relies on the **Independent and Identically Distributed (IID)** assumption

Example

- Overfitting
- Outliers
- Regularization
- Etc