Autoencoders and Generative Models

CISC 7026: Introduction to Deep Learning

University of Macau

Agenda

1. Review
2. Compression
3. Autoencoders
4. Applications
5. Variational Models
6. Coding

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Answer: Shrek lives in a swamp, Lord Farquaad, dragons, etc

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Let us examine a more principled form of video compression

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$$X \in \mathbb{Z}_{255}^{3\times 3840\times 2160}, X^{90\times 60\times 24}$$

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H.264 MPEG-AVC **encoder** transforms videos into a more compact representation

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We achieve a compression ratio of $3000 \text{ GB} \ / \ 60 \text{ GB} = 50$

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You CPU has a H.264 decoder built in to make this fast

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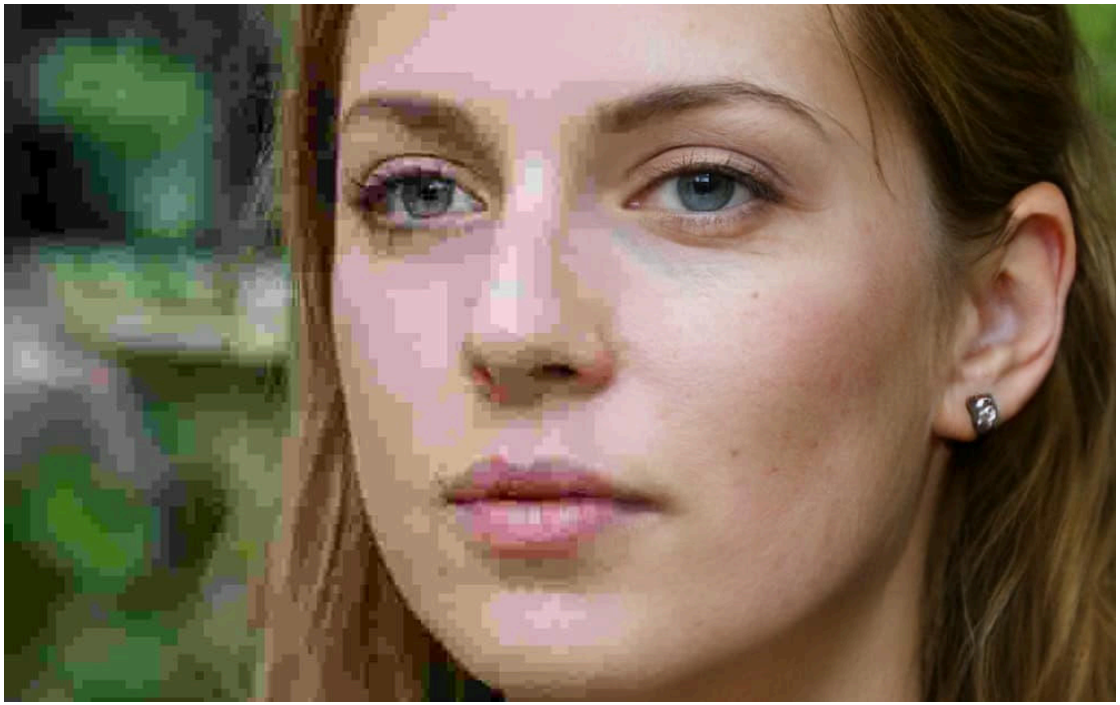

Question: Which is H.264?

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Notice there is no Y this time

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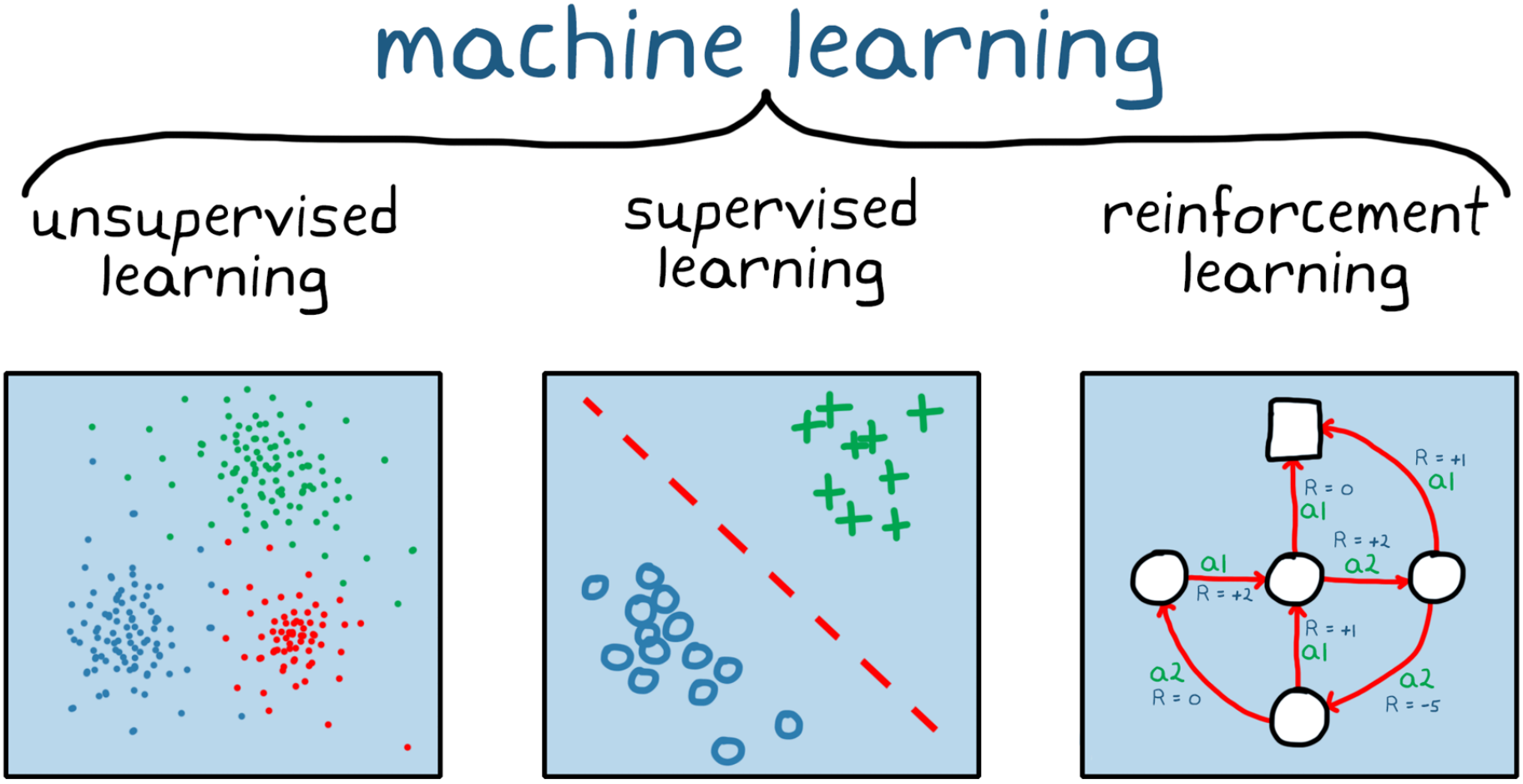
Training autoencoders is different than what we have seen before

Autoencoders



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The training algorithm may generate labels

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Task: Compress images for your clothing website to save on costs

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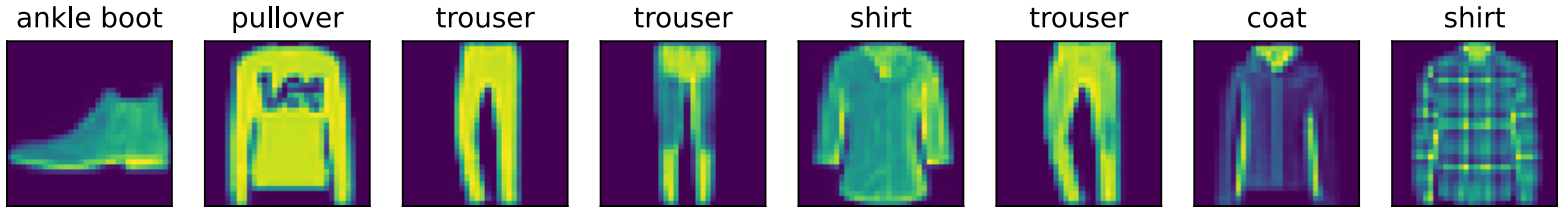

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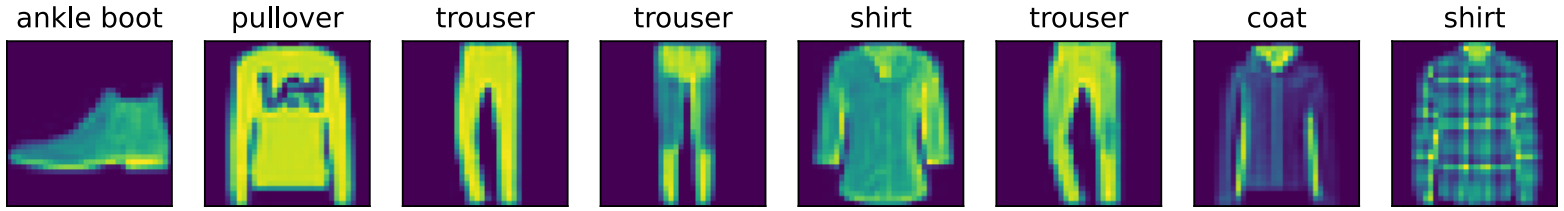

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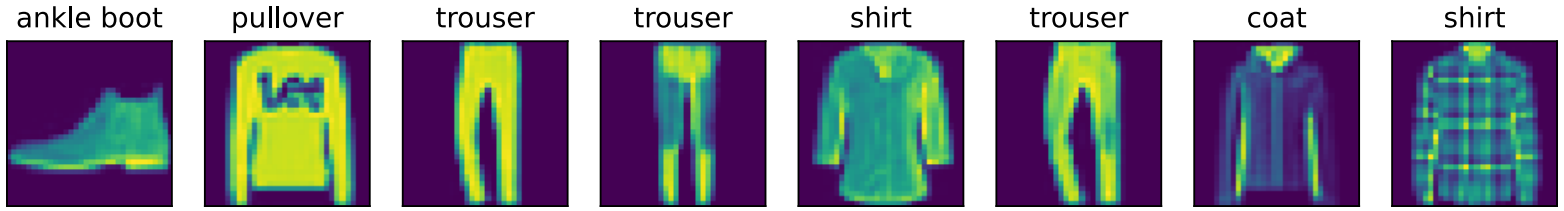

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Autoencoders

Let us find f, then find the inverse

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$$\boldsymbol{z} = \sigma(\boldsymbol{\theta}^\top \overline{\boldsymbol{x}})$$

Solve for $\boldsymbol{x}$ to find the inverse

$$\sigma^{-1}(\boldsymbol{z}) = \sigma^{-1}(\sigma(\boldsymbol{\theta}^\top \overline{\boldsymbol{x}}))$$

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Look for another solution

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Let us try another way

$$\boldsymbol{z} = f(\boldsymbol{x}, \boldsymbol{\theta}_e) = \sigma(\boldsymbol{\theta}_e^\top \overline{\boldsymbol{x}})$$

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What if we plug $\boldsymbol{z}$ into the second equation?

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$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_{j=1}^{d_x} \left(x_j - f^{-1}(f(\boldsymbol{x}, \boldsymbol{\theta}_e), \boldsymbol{\theta}_d)_j\right)^2$$

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We call this the **reconstruction loss**

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It is an unsupervised loss because we only provide $\boldsymbol{X}$ and not $\boldsymbol{Y}$!

Autoencoders

First coding exercise

https://colab.research.google.com/drive/1UyR_W6NDIujaJXYlHZh6O3NfaCAMscpH#scrollTo=nmyQ8aE2pSbb

https://www.youtube.com/watch?v=UZDiGooFs54

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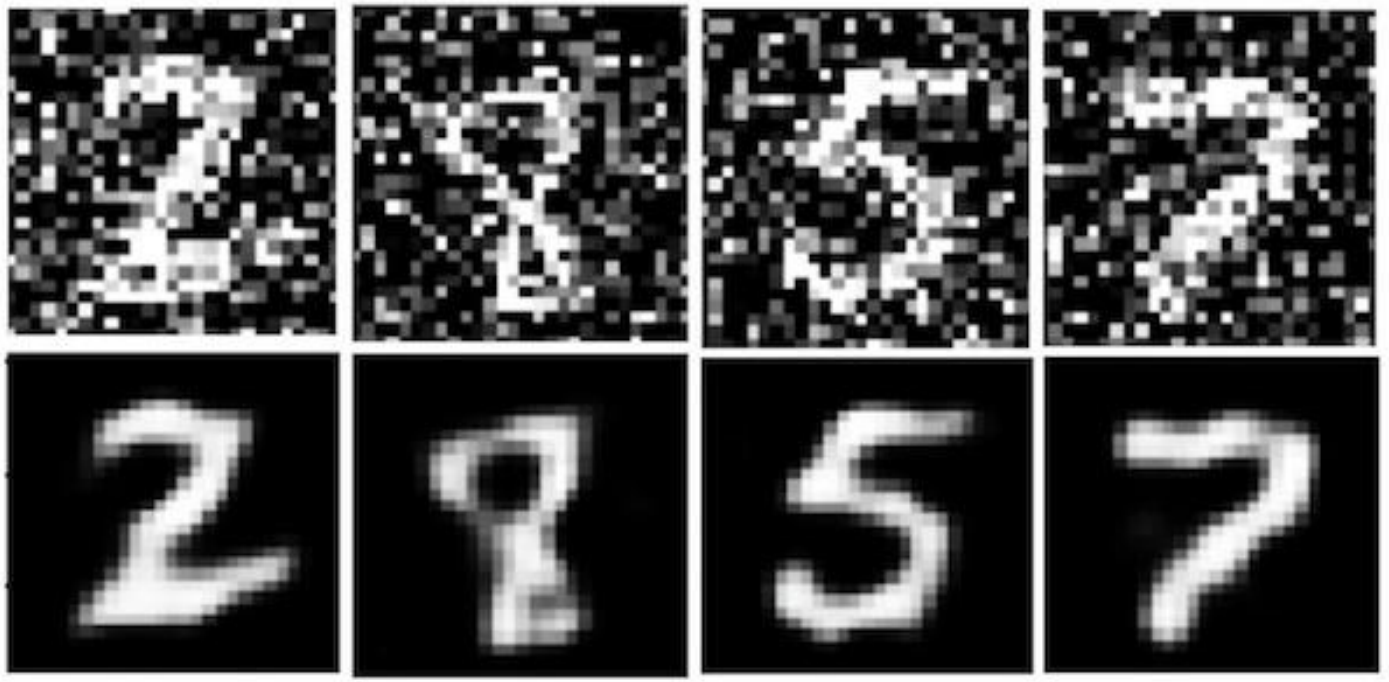

Applications

Generate some noise $\boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Add noise to the image $\boldsymbol{x} + \boldsymbol{\varepsilon}$

Original loss $$\mathcal{L}(\boldsymbol{X}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{d_x} \left(x_{[i],j} - f^{-1}\left(f\left(\boldsymbol{x}_{[i]}, \boldsymbol{\theta}_e \right), \boldsymbol{\theta}_d \right)_j \right)^2$$

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Denoising loss $$\mathcal{L}(\boldsymbol{X}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{d_x} \left(x_{[i],j} - f^{-1}\left(f\left(\boldsymbol{x}_{[i]} + \boldsymbol{\varepsilon}, \boldsymbol{\theta}_e \right), \boldsymbol{\theta}_d \right)_j \right)^2$$

Applications

We can add camera blur too

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$$\text{blur}(\boldsymbol{x} + \boldsymbol{\varepsilon})$$

Applications

$$\text{blur}(\boldsymbol{x} + \boldsymbol{\varepsilon})$$

Denoising deblur loss

$$\mathcal{L}(\boldsymbol{X}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{d_x} \left(x_{[i],j} - f^{-1}\Big(f\Big(\text{blur}\Big(\boldsymbol{x}_{[i]} + \boldsymbol{\varepsilon}\Big), \boldsymbol{\theta}_e\Big), \boldsymbol{\theta}_d\Big)_j \right)^2$$

Applications

Now we can “enhance” images like in crime tv shows

Applications

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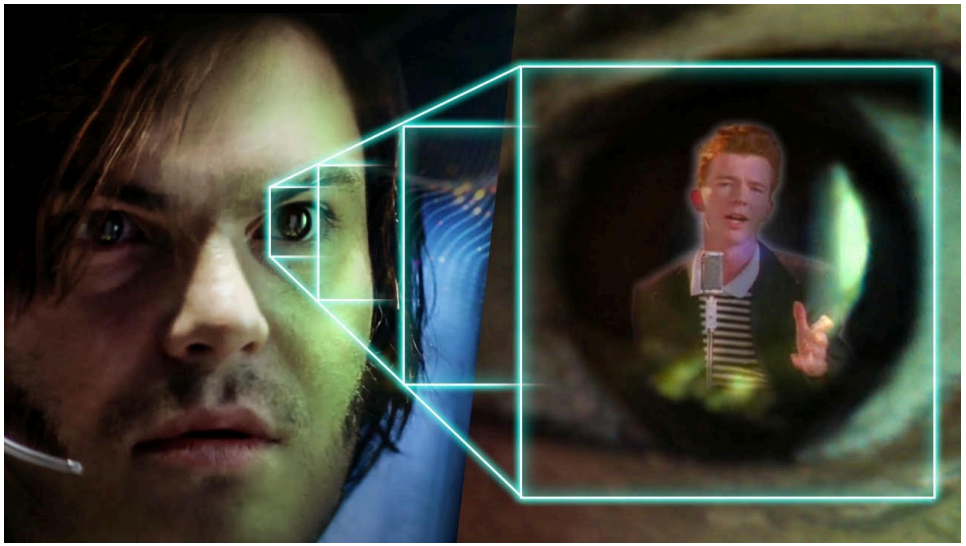
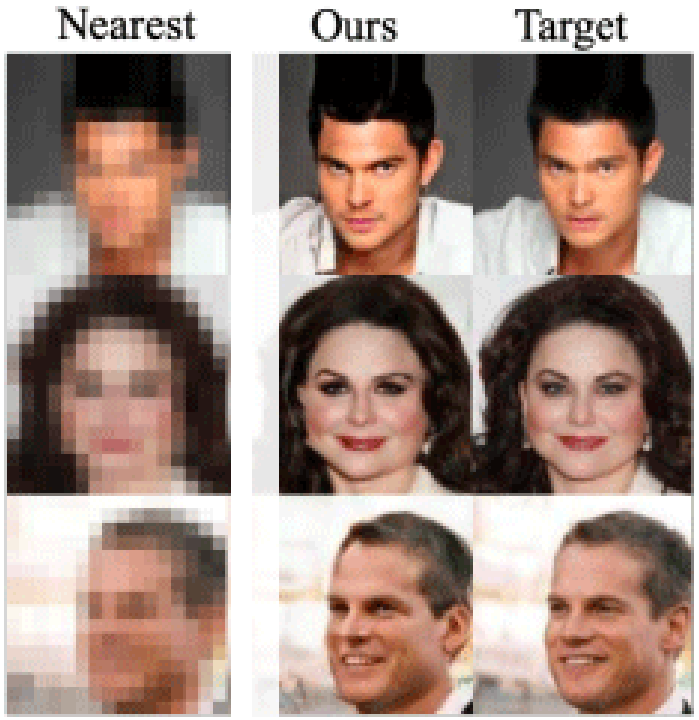


Applications

We can deblur faces from security cameras



Applications

We can even hide parts of the image

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A **masked autoencoder** will reconstruct the missing data

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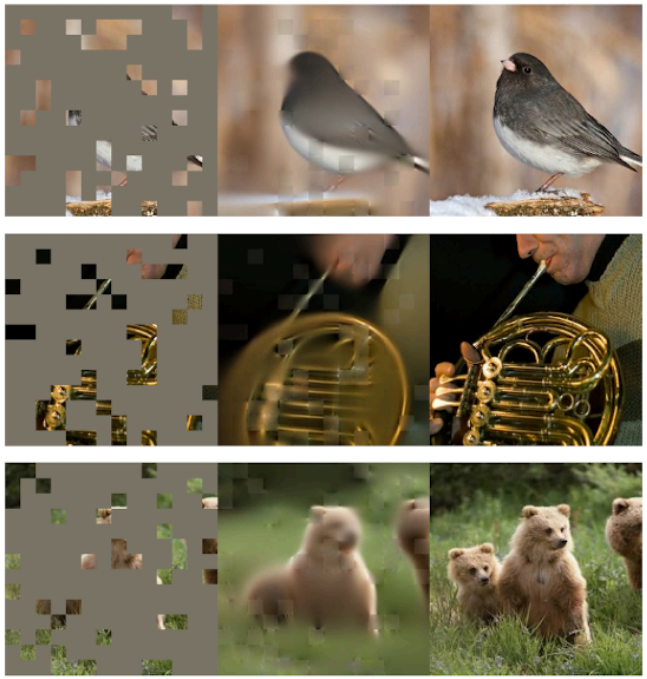

Applications

What is happening here? How can these models do this?

Applications

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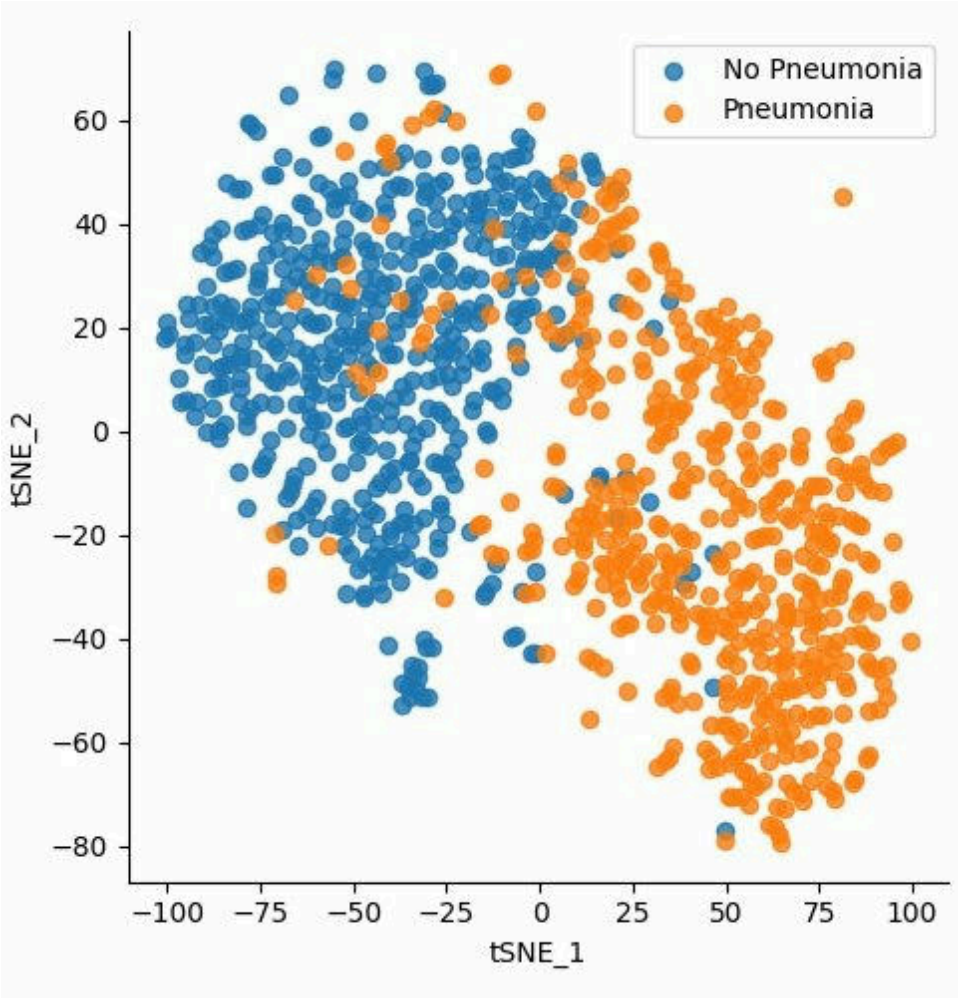
Autoencoders learn the structure of the dataset

Applications



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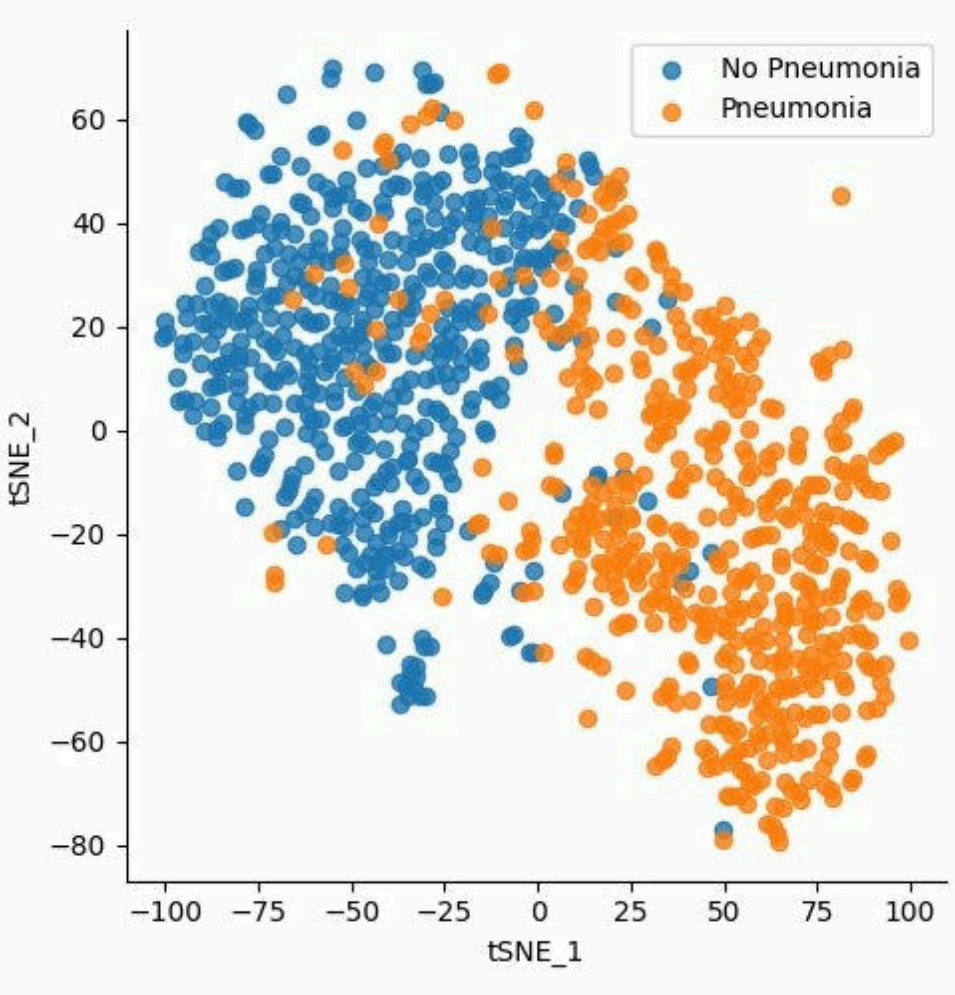
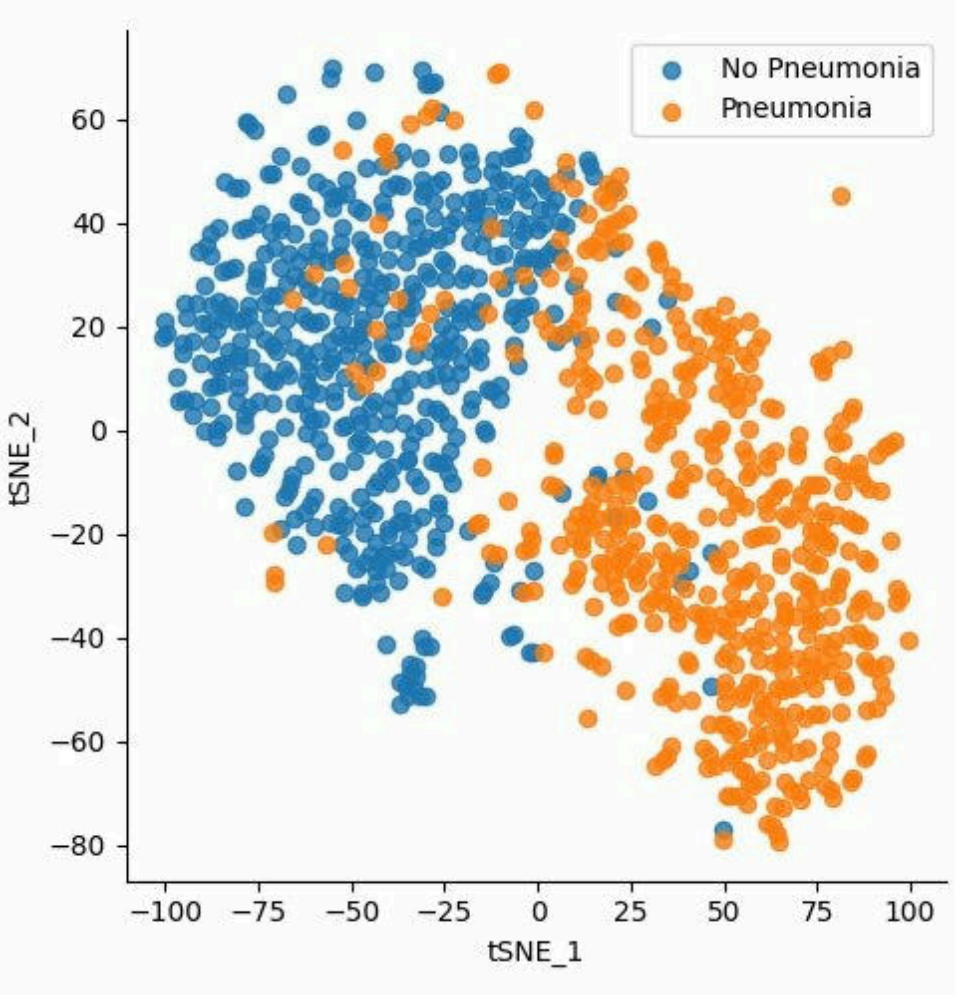


X : Pictures of human lungs

Applications



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$Z \in \mathbb{R}^2$

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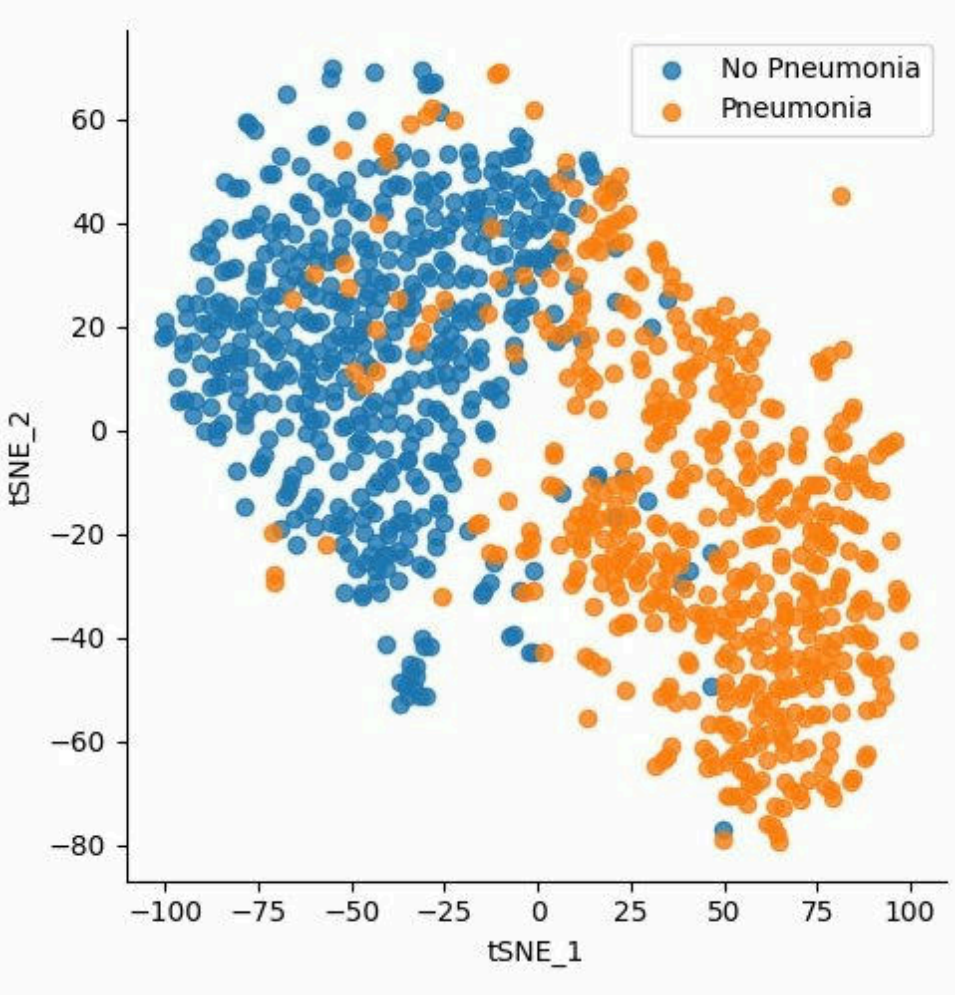

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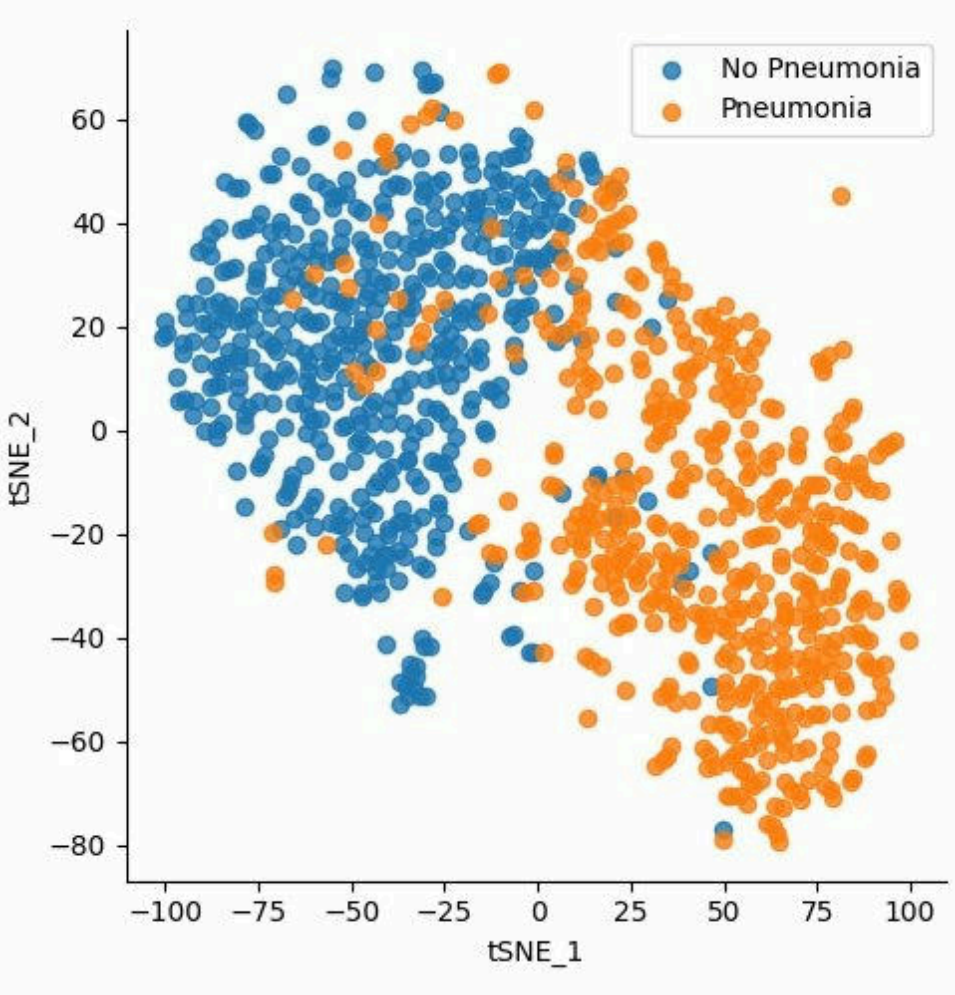

X : Pictures of human lungs

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Learns the structure of lungs from images

Differentiates sick and healthy lungs without being told

Applications

If the dataset is pictures from our world, then the autoencoders learn the structure of the world

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Nobody tells them what a dog or cat is

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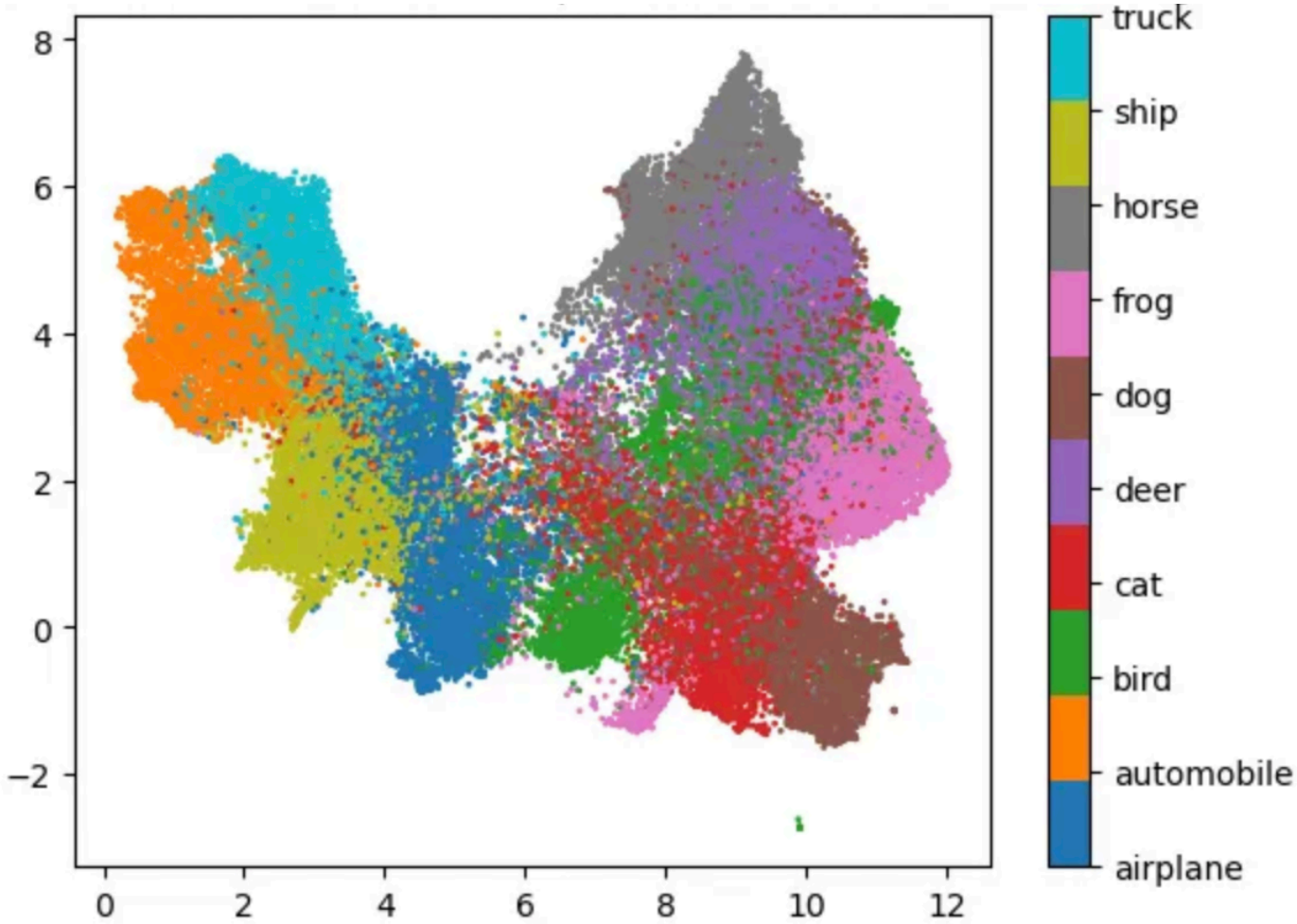
If the dataset is pictures from our world, then the autoencoders learn the structure of the world

Nobody tells them what a dog or cat is

They learn that on their own

Applications



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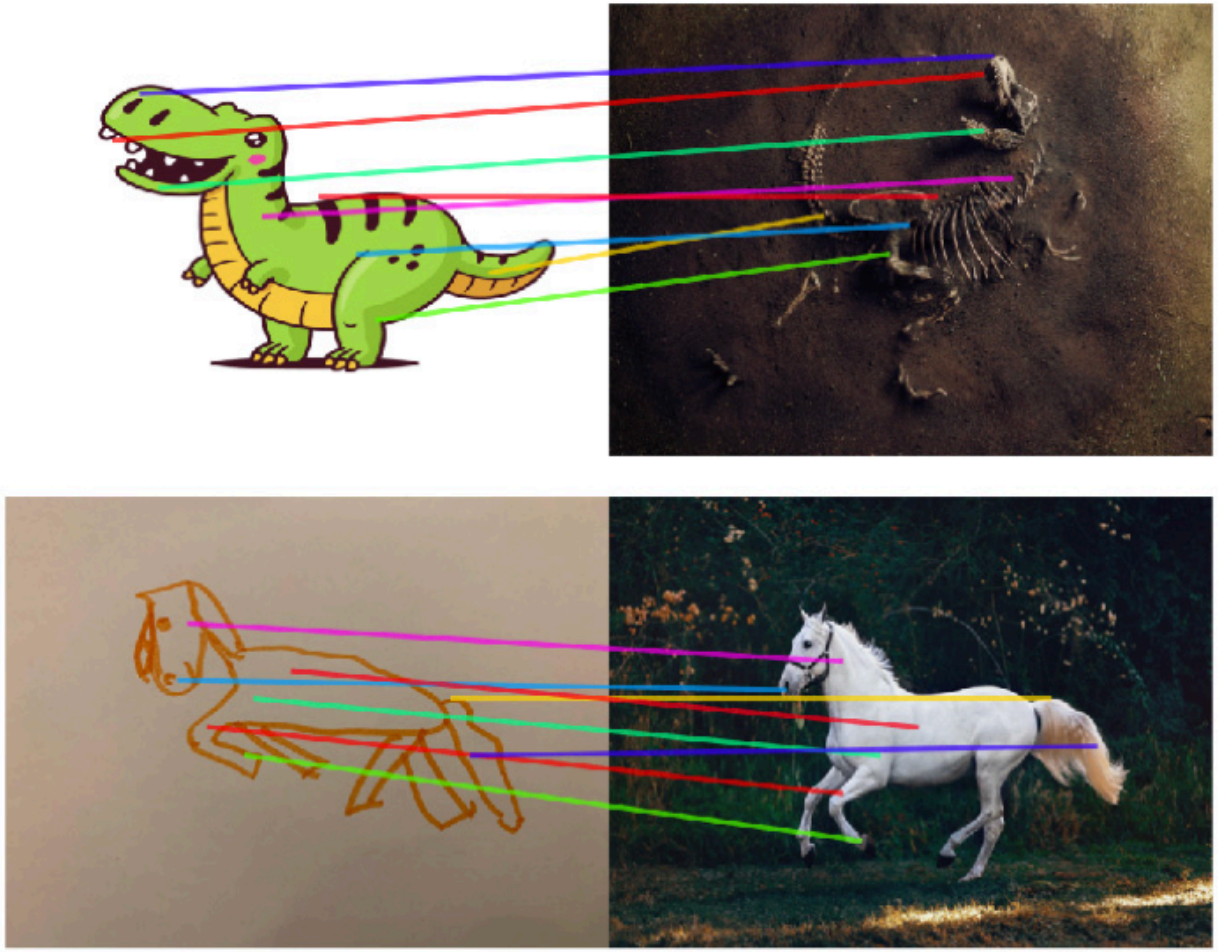

Applications

Some say “neural networks do not understand”, they just learn patterns

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Humans are also pattern recognition machines

Applications

Some say “neural networks do not understand”, they just learn patterns

Humans are also pattern recognition machines

Clearly, these networks can understand our world

Agenda

1. Review
2. Compression
3. Autoencoders
4. **Applications**
5. Variational Models
6. Coding

Agenda

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2. Compression
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4. Applications
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6. Coding

Variational Models

Variational Models

These pictures were created by an autoencoder

Variational Models

These pictures were created by an autoencoder

But these people do not exist!

Variational Models

Autoencoders are useful for compression and denoising

Variational Models

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But we can also use them as **generative models**

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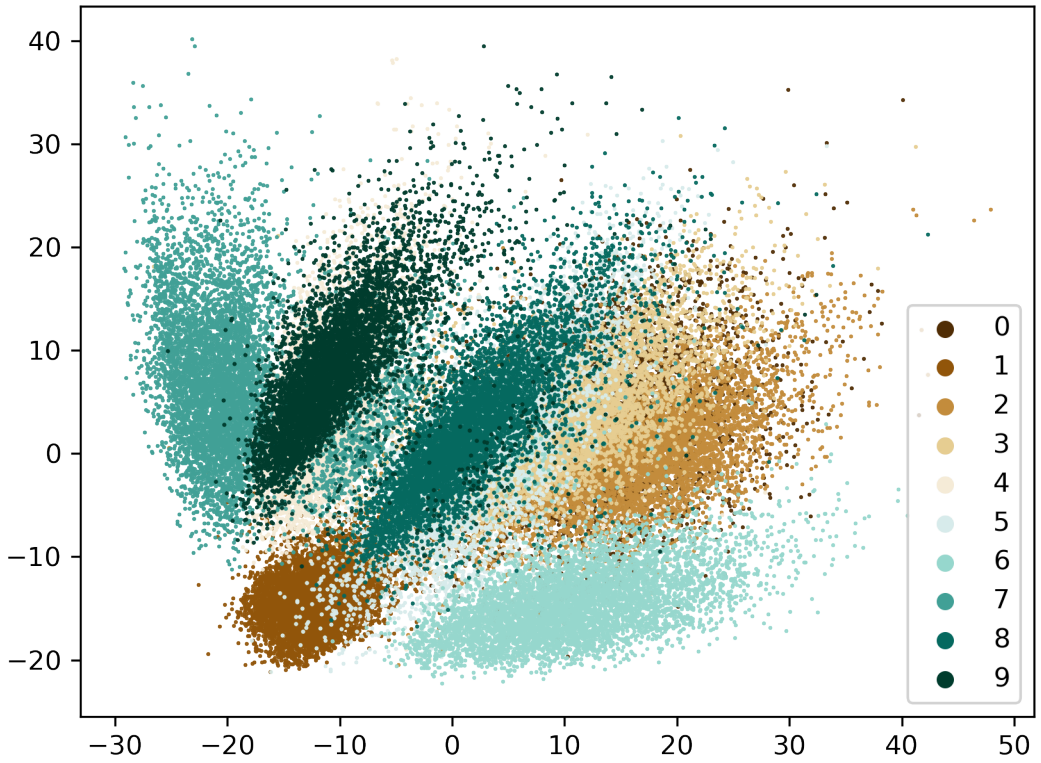
How does this work?

Variational Models

Latent space Z after training on the clothes dataset with $d_z = 2$



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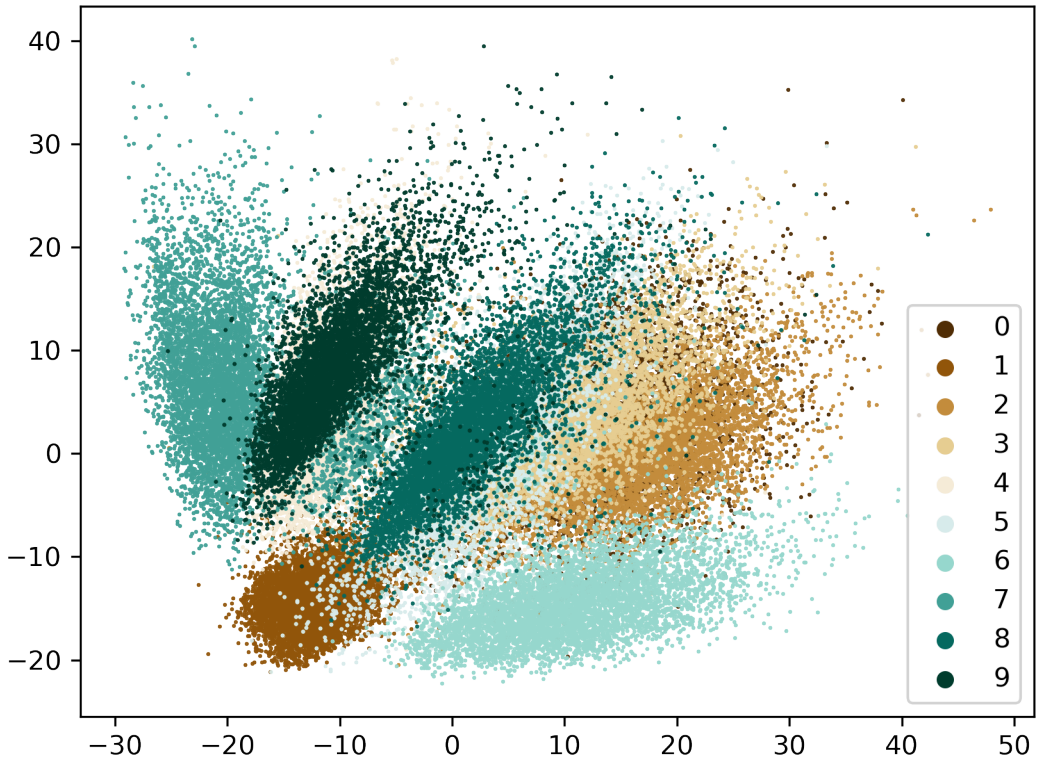
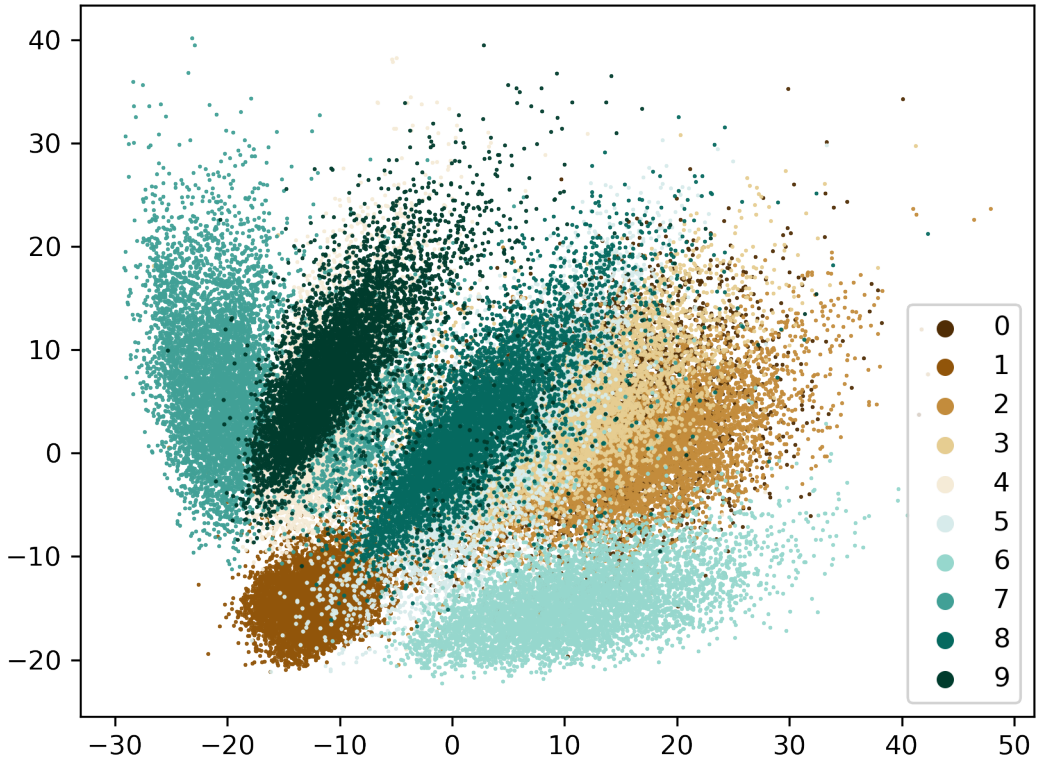
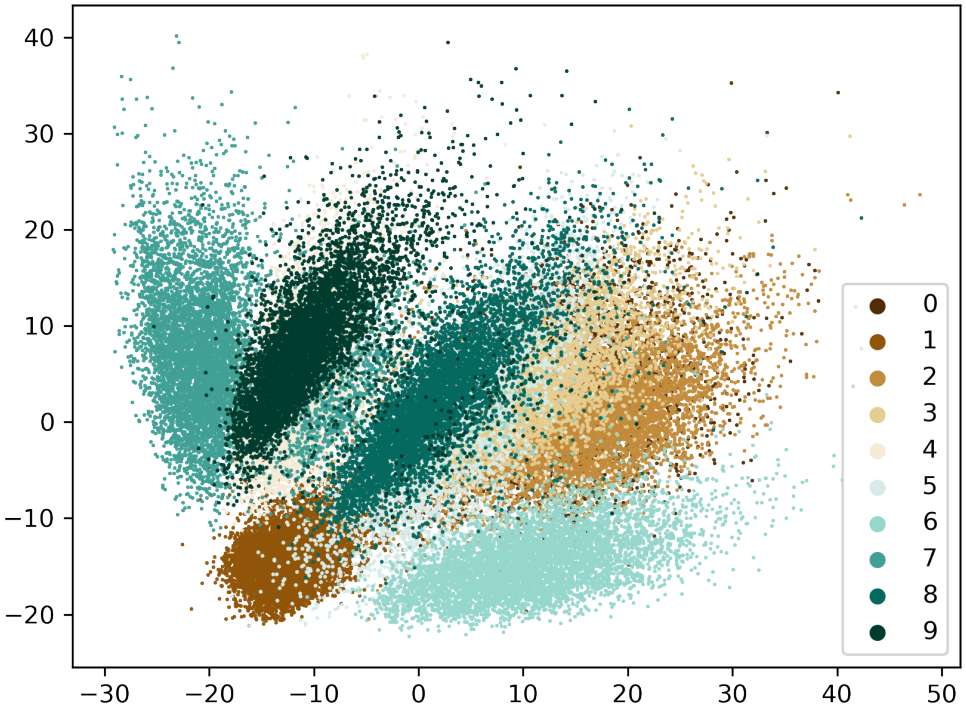


Variational Models

What happens if we decode a new point?

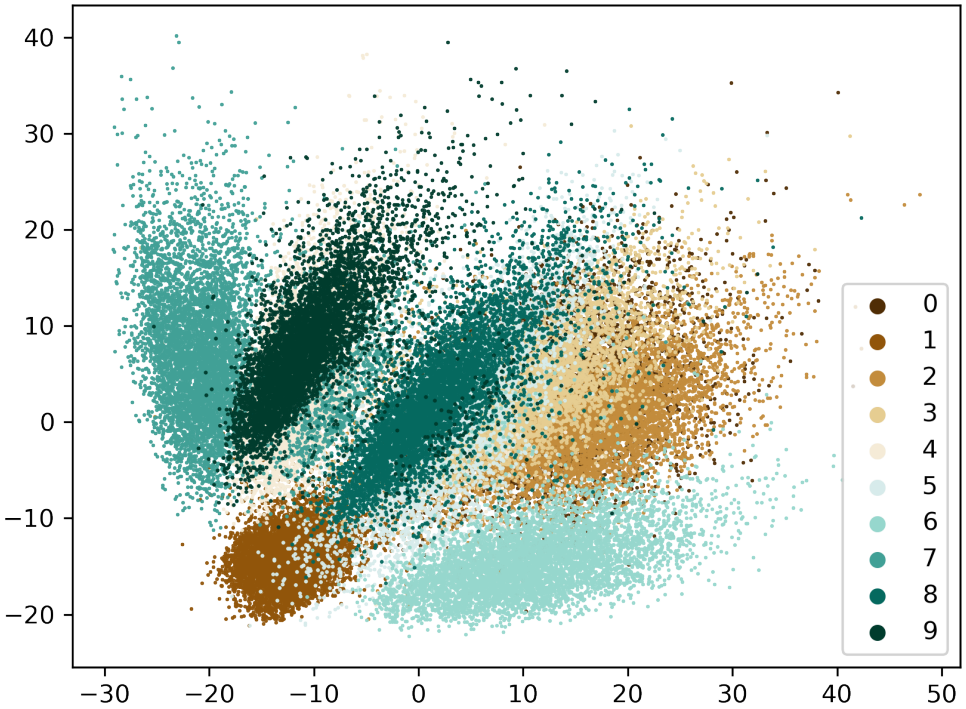


Variational Models



Autoencoder generative model:

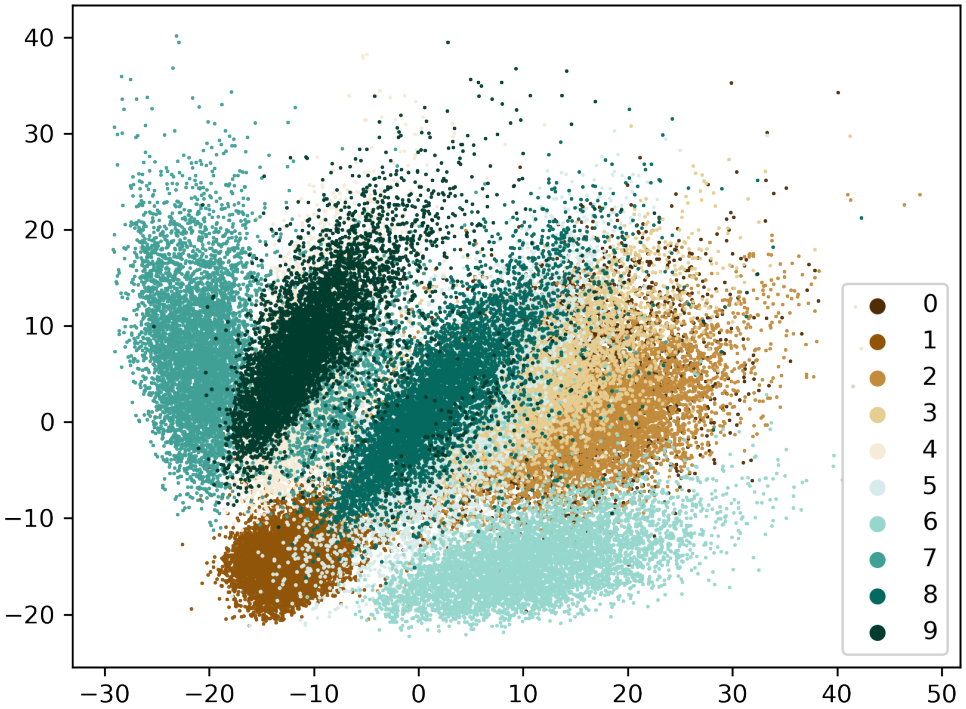
Variational Models



Autoencoder generative model:

$$\text{Encode} \begin{bmatrix} \boldsymbol{x}_{[1]} \\ \vdots \\ \boldsymbol{x}_{[n]} \end{bmatrix} \text{ into } \begin{bmatrix} \boldsymbol{z}_{[1]} \\ \vdots \\ \boldsymbol{z}_{[n]} \end{bmatrix}$$

Variational Models



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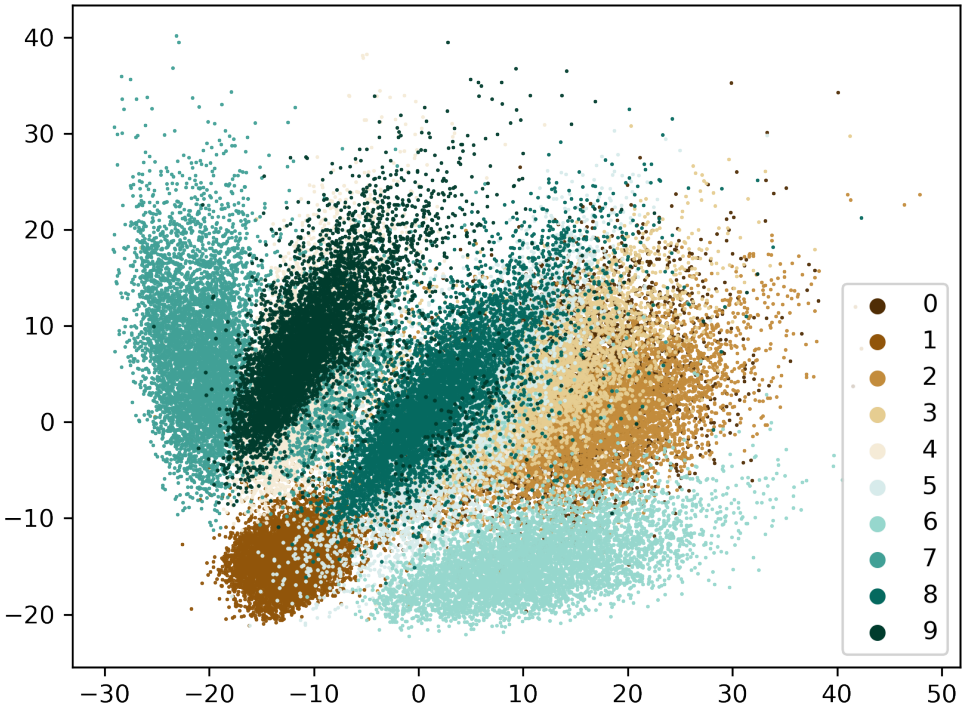
Variational Models



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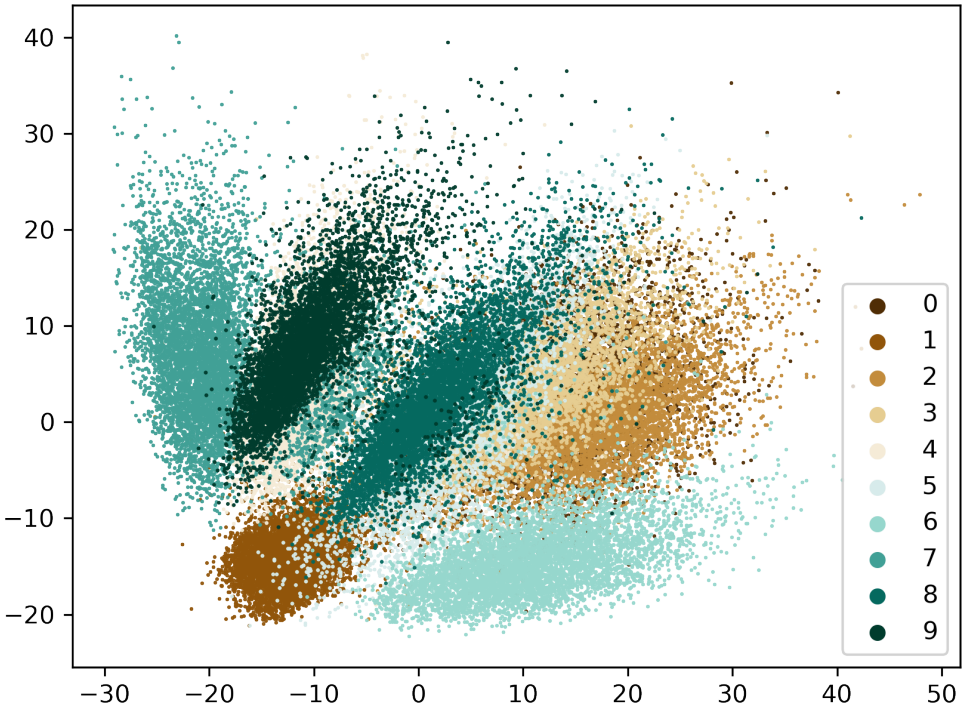
Variational Models



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Pick a point $\boldsymbol{z}_{[k]}$

Add some noise $\boldsymbol{z}_{\text{new}} = \boldsymbol{z}_{[k]} + \boldsymbol{\varepsilon}$

Decode $\boldsymbol{z}_{\text{new}}$ into a new $\boldsymbol{x}$

Variational Models

Variational Models

$$f^{-1}(\boldsymbol{z}_k + \boldsymbol{\varepsilon}, \boldsymbol{\theta}_d)$$

Variational Models

But there is a problem

Variational Models

But there is a problem

As d_z increases, it becomes difficult to find useful parts of latent space

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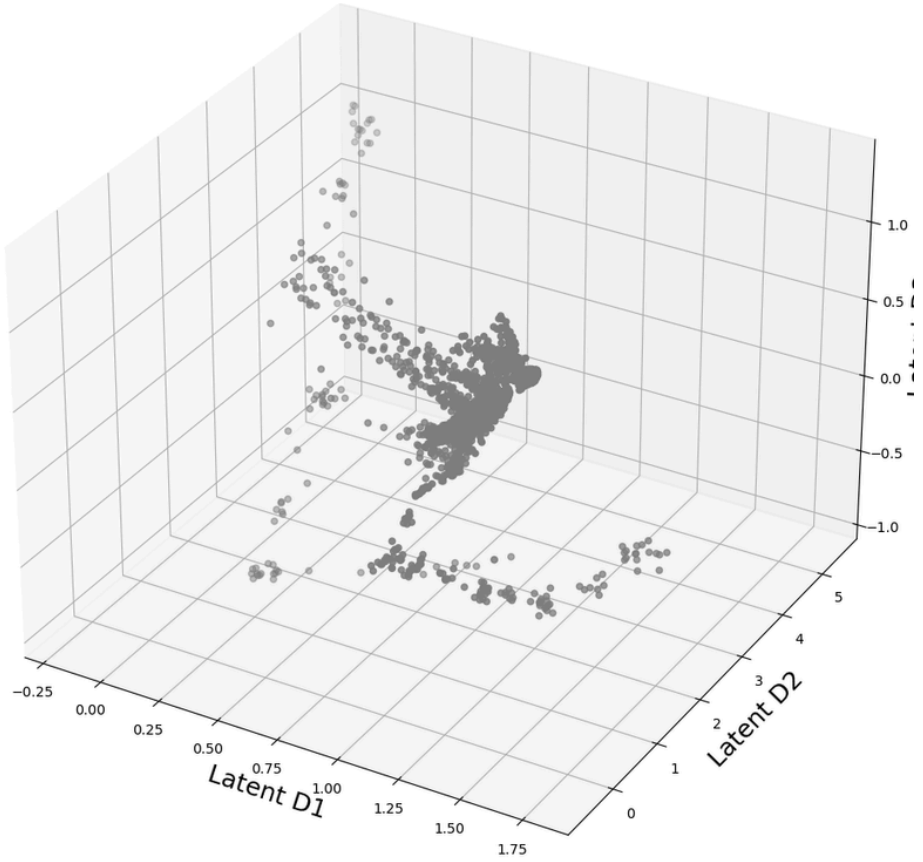
Points $\boldsymbol{z} + \boldsymbol{\varepsilon}$ decode into garbage



Variational Models

Question: What can we do?

Variational Models

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Answer: Force the points to be close together!

Variational Models

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We will use a **variational autoencoder** (VAE)

Variational Models

VAE discovered by Diederik Kingma (also adam optimizer)

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Variational Models

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How?

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How?

Make $\boldsymbol{z}_{[i]}, \ldots, \boldsymbol{z}_{[n]}$ normally distributed

$$\boldsymbol{z} \sim \mathcal{N}(\mu, \sigma), \quad \mu = 0, \sigma = 1$$

Variational Models

If $\boldsymbol{z}_{[i]}, \ldots, \boldsymbol{z}_{[n]}$ are distributed following $\mathcal{N}(0, 1)$:

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If $\boldsymbol{z}_{[i]}, \ldots, \boldsymbol{z}_{[n]}$ are distributed following $\mathcal{N}(0, 1)$:

1. 99.7% of $\boldsymbol{z}_{[1]}, \ldots, \boldsymbol{z}_{[n]}$ lie within $3\sigma = [-3, 3]$
2. Make it easy to generate new $\boldsymbol{z}$, just sample $\boldsymbol{z} \sim \mathcal{N}(0, 1)$

Variational Models

So how do we ensure that $\boldsymbol{z}_{[i]}, \ldots, \boldsymbol{z}_{[n]}$ are normally distributed?

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I have done my best to simplify it, but it is still complex

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There might be small mistakes!

Variational Models

Key idea:

Step 1: Encoder produces a distribution $P(\boldsymbol{z})$ from an input $\boldsymbol{x}$

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VAE is generative because we sample from a distribution

We are sampling new $\boldsymbol{z}$ that were not in the dataset!

Variational Models

Let us start with the encoder

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We want to turn an input x into a latent normal distribution over Z

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Start with the joint probability $P(\boldsymbol{z}, \boldsymbol{x})$

$$P(\boldsymbol{z}, \boldsymbol{x}) = \underbrace{P(\boldsymbol{z} \mid \boldsymbol{x})}_{\text{Likelihood}} \underbrace{P(\boldsymbol{x})}_{\text{Prior}}$$

$P(\boldsymbol{x})$ Probability of a specific input (image) existing in the world

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$P(\boldsymbol{x})$

Probability of a specific input (image) existing in the world

$P(\boldsymbol{z} \mid \boldsymbol{x})$

Latent distribution that summarizes $\boldsymbol{x}$

Variational Models

We can marginalize to get the latent distribution $P(\boldsymbol{z})$

$$P(\boldsymbol{z}) = \int_{\boldsymbol{x}} P(\boldsymbol{z} \mid \boldsymbol{x}) P(\boldsymbol{x}) \, \mathrm{d}x$$

We approximate this function using the encoder

$$f(\boldsymbol{x}, \boldsymbol{\theta}_e) = P(\boldsymbol{z})$$

Question: What distribution should $P(\boldsymbol{z})$ be?

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Answer: A normal distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma})$

Variational Models

We represent a normal distribution with a **mean** $\mu \in M$ and a **standard deviation** $\sigma \in \Sigma$

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$$f : X \times \Theta \mapsto M \times \Sigma$$

Variational Models

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$$f : X \times \Theta \mapsto M \times \Sigma$$

So our encoder should output two values

$$M \in \mathbb{R}^{d_z}, \Sigma \in \mathbb{R}_+^{d_z}$$

Variational Models

```
core = nn.Sequential(...)
mu_layer = nn.Linear(d_h, d_z)
# Neural networks output real numbers
# But sigma must be positive
# So we output log sigma, because e^(sigma) is always
positive
log_sigma_layer = nn.Linear(d_h, d_z)

tmp = core(x)
mu = mu_layer(tmp)
log_sigma = log_sigma_layer(tmp)
dist = (mu, exp(sigma))
```

Variational Models

Decoder maps the distribution $P(\boldsymbol{z})$ to $P(\boldsymbol{x})$

$$P(\boldsymbol{x}) = \int_z P(\boldsymbol{x} \mid \boldsymbol{z}) P(\boldsymbol{z})$$

$$\boldsymbol{x} \sim P(\boldsymbol{x})$$

$P(\boldsymbol{z})$ Normal dist. given by encoder

Variational Models

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$P(\boldsymbol{z})$	Normal dist. given by encoder
$P(\boldsymbol{x} \mid \boldsymbol{z})$	Output distribution
$\boldsymbol{x}$	Output (e.g., an image)

Variational Models

$$P(\boldsymbol{x}) = \int_z P(\boldsymbol{x} \mid \boldsymbol{z}) P(\boldsymbol{z})$$

$$\boldsymbol{x} \sim P(\boldsymbol{x})$$

Integral is intractable, so the decoder inputs and outputs a **sample**

$$f^{-1} : Z \times \Theta \mapsto X$$

Variational Models

```
decoder = nn.Sequential(
  nn.Linear(d_z, d_h),
  ...
  nn.Linear(d_h, d_x)
)
```

Decoder is the same as a normal autoencoder

Variational Models

So far, we have the encoder

$$f : X \times \Theta \mapsto M \times \Sigma$$

Variational Models

So far, we have the encoder

$$f : X \times \Theta \mapsto M \times \Sigma$$

And the decoder

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Variational Models

So far, we have the encoder

$$f : X \times \Theta \mapsto M \times \Sigma$$

And the decoder

$$f^{-1} : Z \times \Theta \mapsto X$$

Question: Any problems?

Answer: The encoder output is a distribution $M \times \Sigma$, but the decoder input is a sample Z

Variational Models

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We can draw samples but still use gradient descent

Variational Models

Put it all together

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Step 1: Encode the input to a latent distribution

$$\boldsymbol{\mu}, \boldsymbol{\sigma} = f(\boldsymbol{x}, \boldsymbol{\theta}_e)$$

Step 2: Generate a sample from distribution

$$\boldsymbol{z} = \boldsymbol{\mu} + \boldsymbol{\sigma} \odot \boldsymbol{\varepsilon}$$

Step 3: Decode the sample

$$\boldsymbol{x} = f^{-1}(\boldsymbol{z}, \boldsymbol{\theta}_d)$$

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Todo loss

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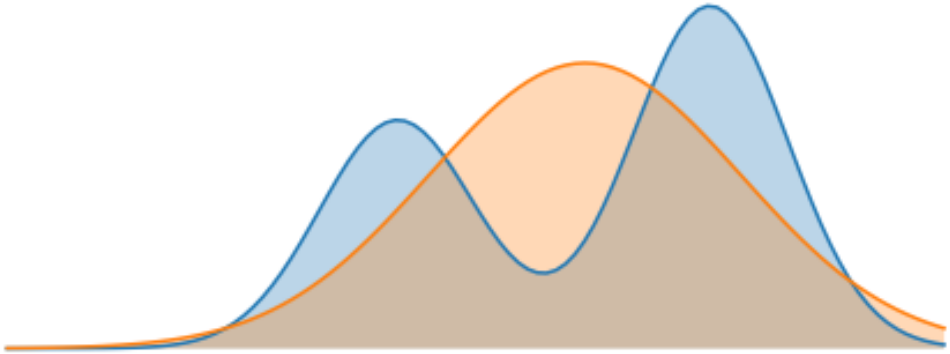
Question: How to measure distance between two distributions?

Variational Models

Answer: KL divergence



$$\mathrm{KL}(P, Q) = \sum_i P(i) \log \frac{P(i)}{Q(i)}$$

Variational Models

$$\mathcal{L}_{\text{KL}}(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_{j=1}^{d_z} \text{KL}\big(P\big(z_j \mid \boldsymbol{x}\big), \mathcal{N}(\mathbf{0}, \mathbf{1})\big)$$

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This ensures that the latent representations $\boldsymbol{z}$ are easy to sample

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Remember, our neural network outputs $P(\boldsymbol{z})$ as $\boldsymbol{\mu}, \boldsymbol{\sigma}$

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The KL divergence between two normal distributions simplifies to

$$\mathcal{L}_{\text{KL}}(\boldsymbol{x}, \boldsymbol{\theta}) = \left(\sum_{j=1}^{d_z} 1 + \sigma_j^2 + \mu_j^2 - \exp\big(\sigma_j^2\big) \right)$$

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Over the whole dataset

$$\mathcal{L}_{\text{KL}}(\boldsymbol{X}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \left(\sum_{j=1}^{d_z} 1 + \sigma_{[i],j}^2 + \mu_{[i],j}^2 - \exp\left(\sigma_{[i],j}^2\right) \right)$$

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We also want to make sure we reconstruct the input!

$$\mathcal{L}_{\text{recon}}(\boldsymbol{X}, \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{d_x} \left(x_{[i],j} - f^{-1}\left(f\left(\boldsymbol{x}_{[i]}, \boldsymbol{\theta}_e \right), \boldsymbol{\theta}_d \right)_j \right)^2$$

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cal(L)(bold(X), bold(theta))

Variational Models

The loss for a VAE contains the reconstruction error and the KL loss