# Regression

## CISC 7026: Introduction to Deep Learning

University of Macau

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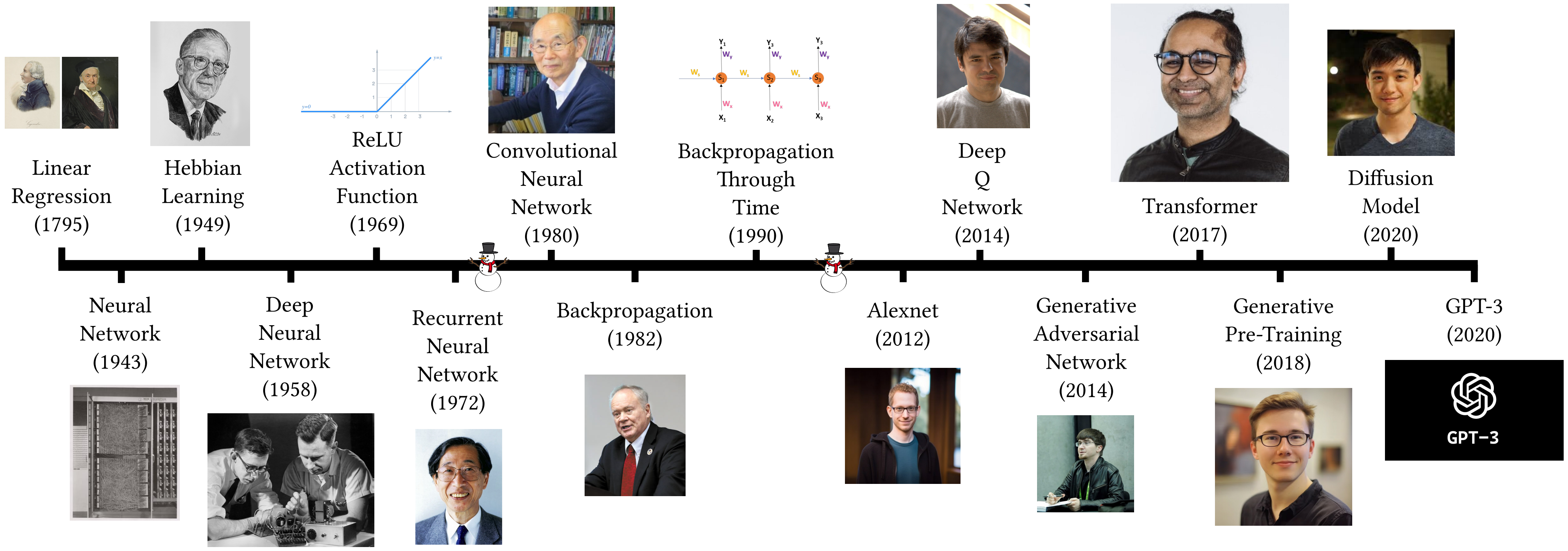

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Let us start with regression

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Available for free at https://www.who.int/data/gho/data/themes/mortality-and-global-health-estimates/ghe-life-expectancy-and-healthy-life-expectancy

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We can use this data to make future predictions

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# Linear Regression

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There are studies showing a causal effect on education on health

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- By staying in school, you are likely to live longer

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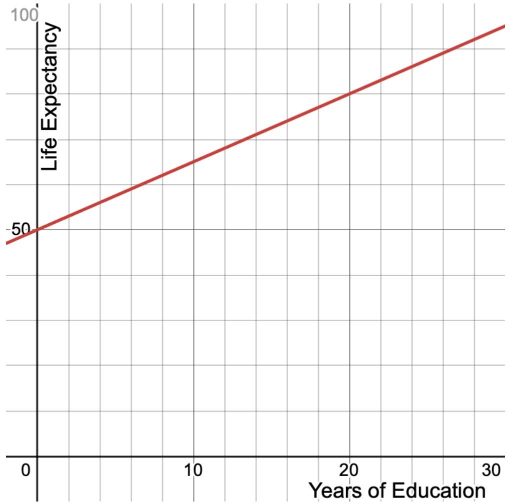

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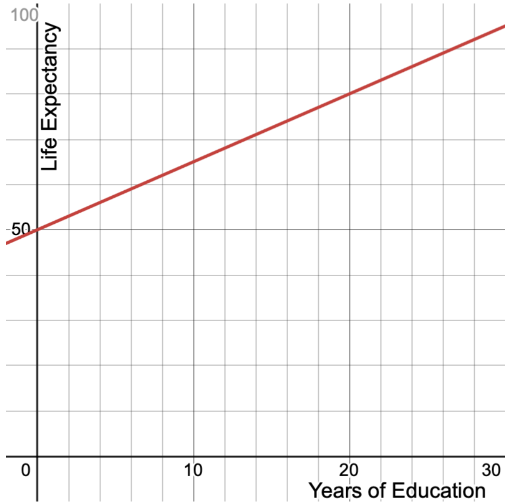


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**Question:** How do we find $\boldsymbol{\theta}$? (Hint: We want $f(x, \boldsymbol{\theta}) = y$)

**Answer:** We will minimize the **loss** (error) between $f(x, \boldsymbol{\theta})$ and $y$, for all

$$x \in X, y \in Y$$

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$$\text{error}(y, \hat{y}) = (y - \hat{y})^2$$

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Move y to LHS

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$$\text{error}(f(x, \boldsymbol{\theta}), y) = (f(x, \boldsymbol{\theta}) - y)^2$$

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**Answer:** We only consider a single datapoint! We want to learn $\boldsymbol{\theta}$ for the entire dataset

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For the entire dataset:

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top, \boldsymbol{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}^\top$$

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Minimizing this loss function will give us the optimal parameters!

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**Answer:** For now, magic! We need more knowledge before we can derive this.

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$$\boldsymbol{X}_D = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$$

# Linear Regression

We add the column of ones so that we can multiply $\boldsymbol{X}_D^\top$ with $\boldsymbol{\theta}$ to get a linear function $\theta_1 x + \theta_0$ evaluated at each data point

$$\boldsymbol{X}_D \boldsymbol{\theta} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} \theta_1 x_1 + \theta_0 \\ \theta_1 x_2 + \theta_0 \\ \vdots \\ \theta_1 x_n + \theta_0 \end{bmatrix}$$

# Linear Regression

With our design matrix $\boldsymbol{X}_D$ and desired output $\boldsymbol{y}$,

$$\boldsymbol{X}_D = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and our parameters $\boldsymbol{\theta}$,

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix},$$

$$\boldsymbol{\theta} = \left(\boldsymbol{X}_D^\top \boldsymbol{X}_D\right)^{-1} \boldsymbol{X}_D^\top \boldsymbol{y}$$

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You will be doing this in your first assignment!

# Tips for assignment 1

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Everything you need is in the lecture notes

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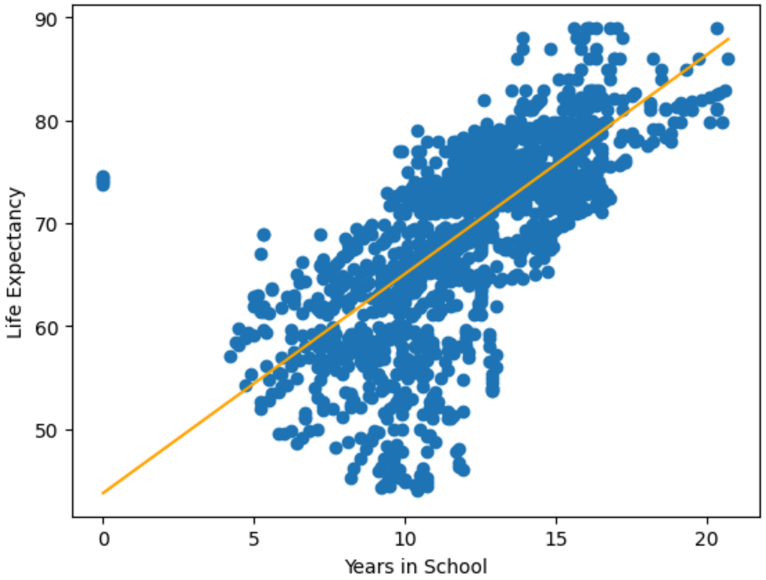

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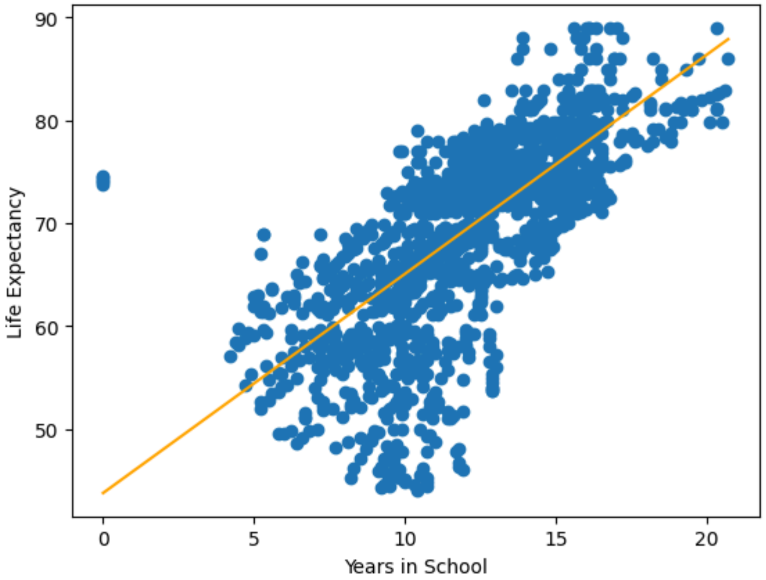

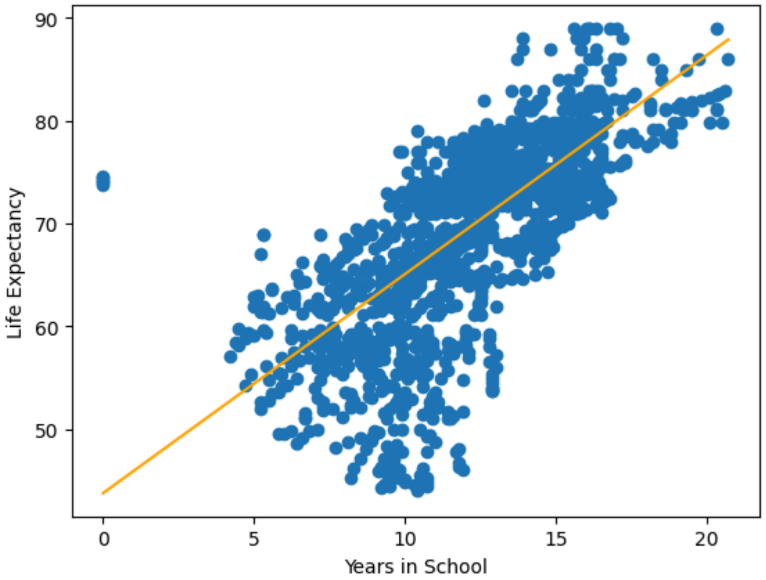


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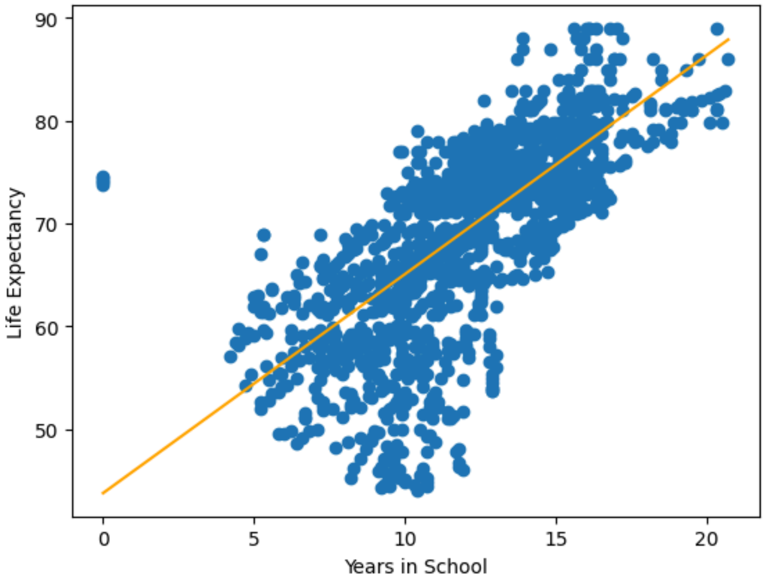


We figured out linear regression!

But can we do better?

1. Beyond linear functions

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2. Overfitting

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Does the data look linear?

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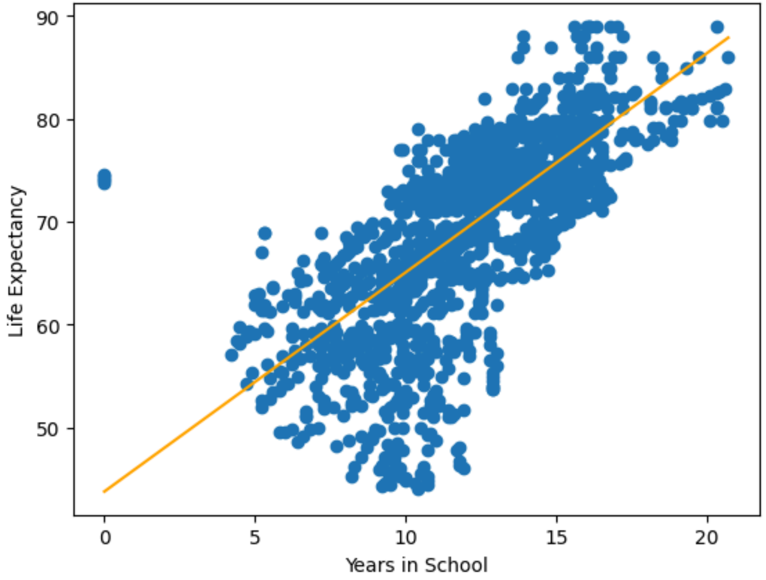

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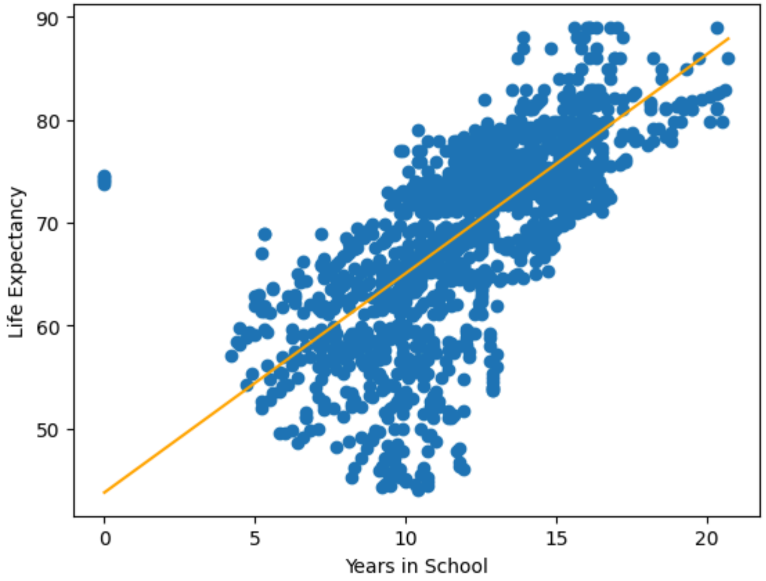
Does the data look linear?




Or maybe more logarithmic?

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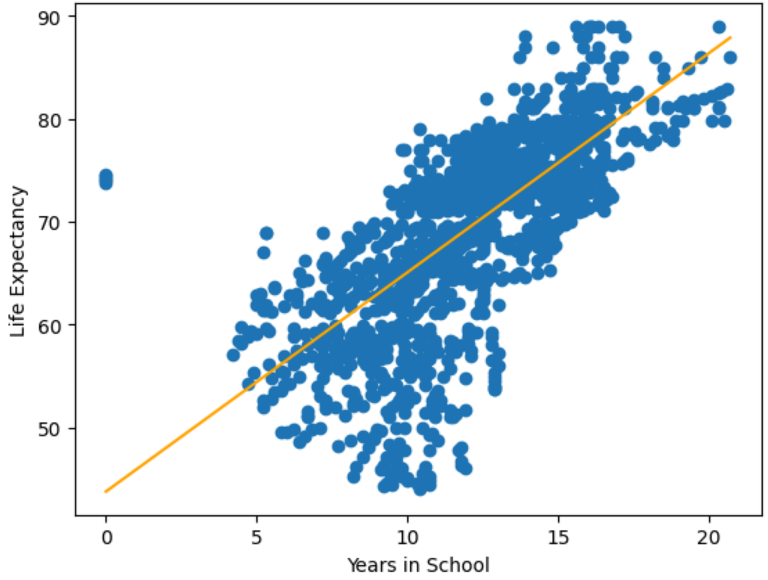
Does the data look linear?

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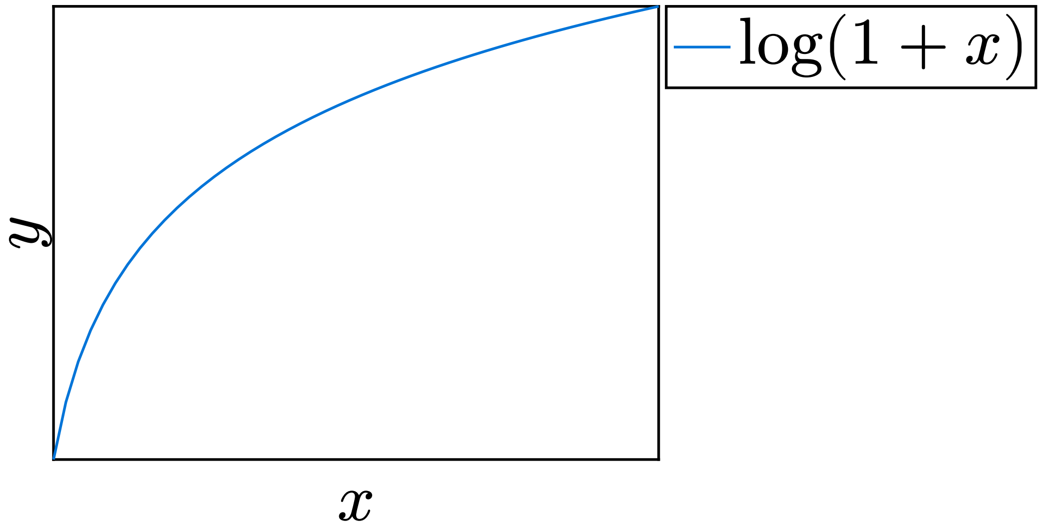
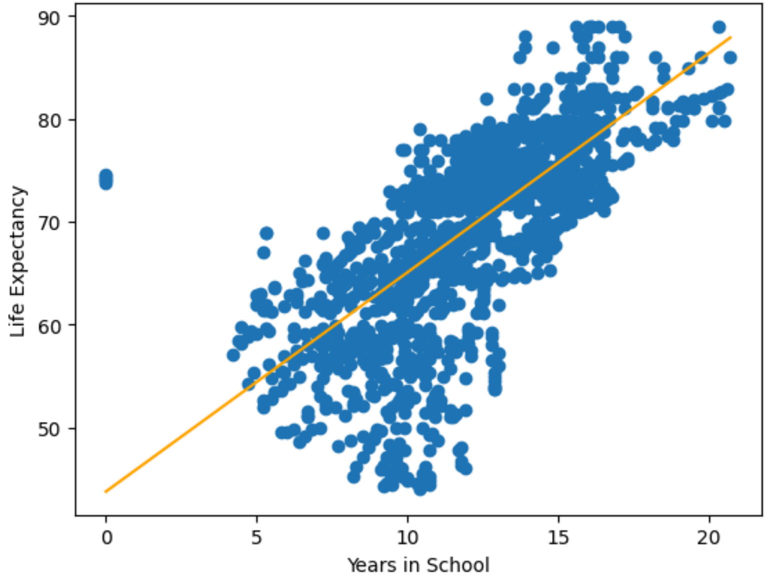
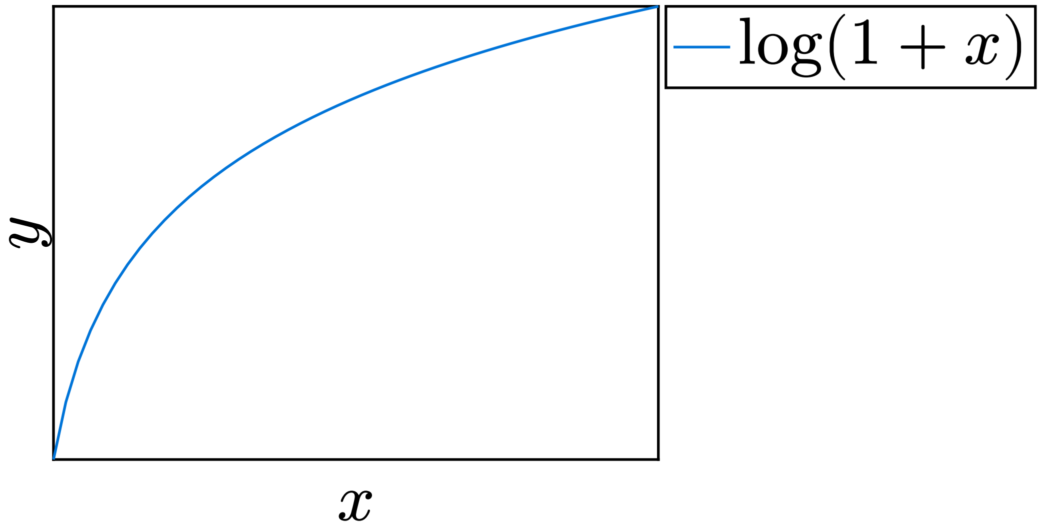


## Question:

Does the data look linear?

Or maybe more logarithmic?




However, linear regression must be linear!

**Question:** What does it mean when we say linear regression is linear?

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**Trick:** Change of variables to make $f$ nonlinear: $x_{\text{new}} = \log(1 + x_{\text{data}})$

$$\boldsymbol{X}_D = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \Rightarrow \boldsymbol{X}_D = \begin{bmatrix} \log(1 + x_1) & 1 \\ \log(1 + x_2) & 1 \\ \vdots & \vdots \\ \log(1 + x_n) & 1 \end{bmatrix}$$

Now, $f$ is a linear function of $\log(1 + x)$ – a nonlinear function of $x$!

New design matrix...

$$\boldsymbol{X}_D = \begin{bmatrix} \log(1 + x_1) & 1 \\ \log(1 + x_2) & 1 \\ \vdots & \vdots \\ \log(1 + x_n) & 1 \end{bmatrix}$$

New function...

$$f\left(x, \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix}\right) = \theta_1 \log(1 + x) + \theta_0$$

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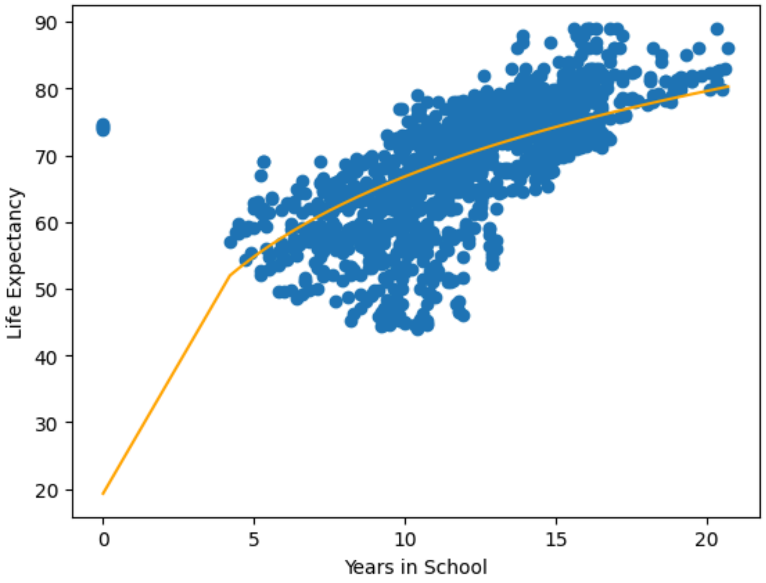
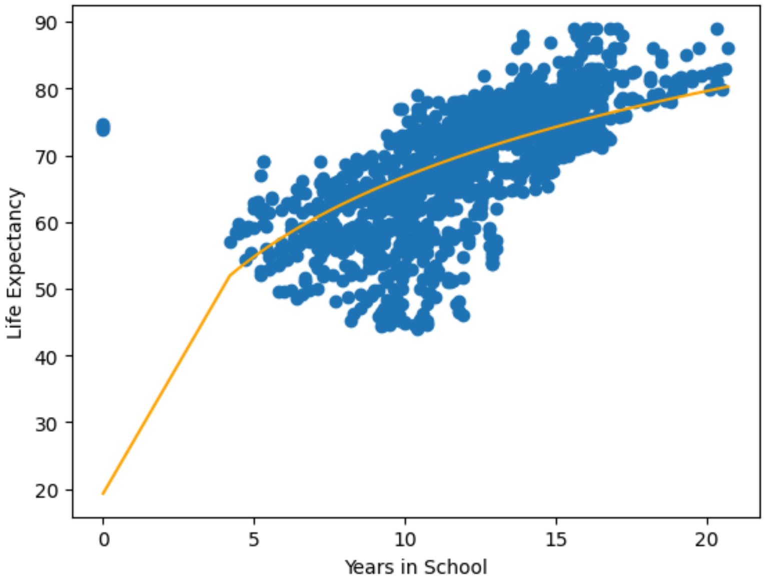
Same solution...

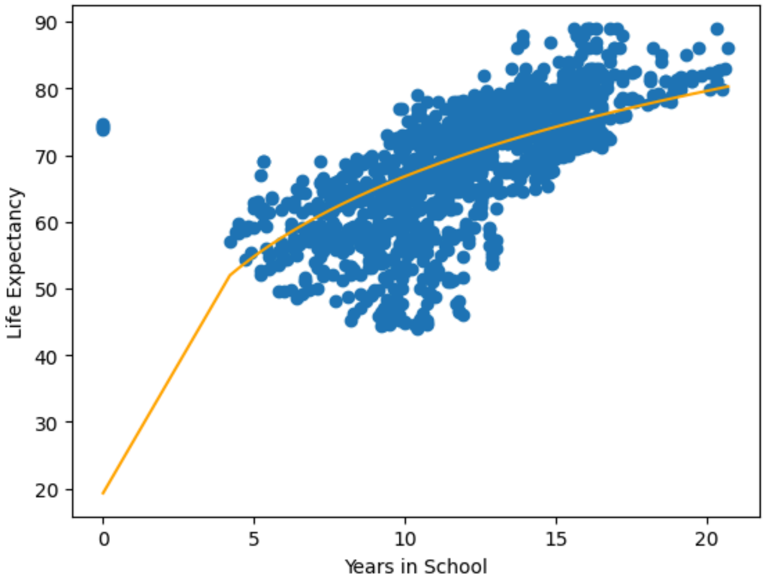
$$\boldsymbol{\theta} = \left(\boldsymbol{X}_D^\top \boldsymbol{X}_D\right)^{-1} \boldsymbol{X}_D^\top \boldsymbol{y}$$



Life Expectancy
Years in School
90
80
70
60
50
40
30
20
0
5
10
15
20




Better, but still not perfect




Better, but still not perfect

Can we do even better?

# What about polynomials?

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$$f(x) = ax^n + bx^{n-1} + ... + cx + d$$

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Can we extend linear regression to polynomials?

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And add some new parameters...

$$\boldsymbol{\theta} = [\theta_1 \ \ \theta_0]^\top \Rightarrow \boldsymbol{\theta} = [\theta_n \ \ \theta_{n-1} \ \ \dots \ \ \theta_1 \ \ \theta_0]^\top$$

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**Summary:** By changing the input space, we can fit a polynomial to the data using a linear fit!

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How do we choose $n$ (polynomial order) that provides the best fit?

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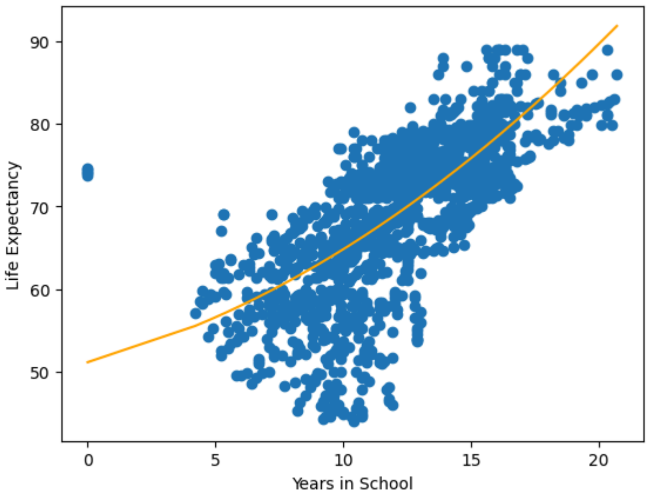
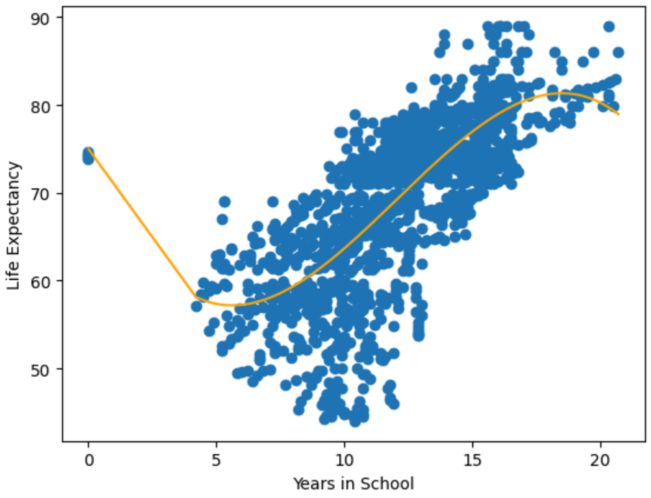



$n = 2$

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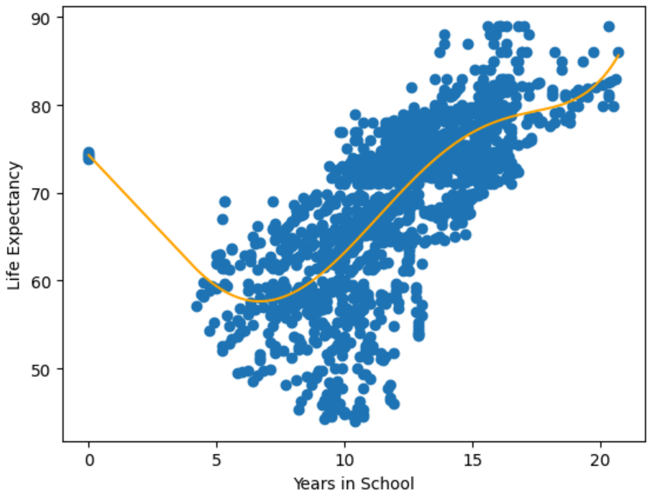


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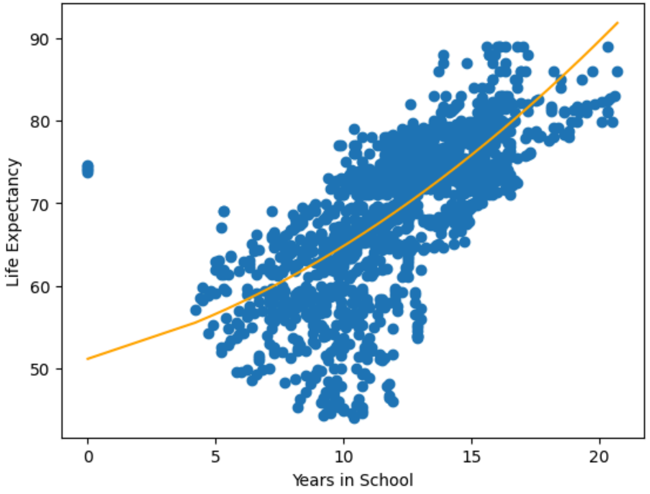


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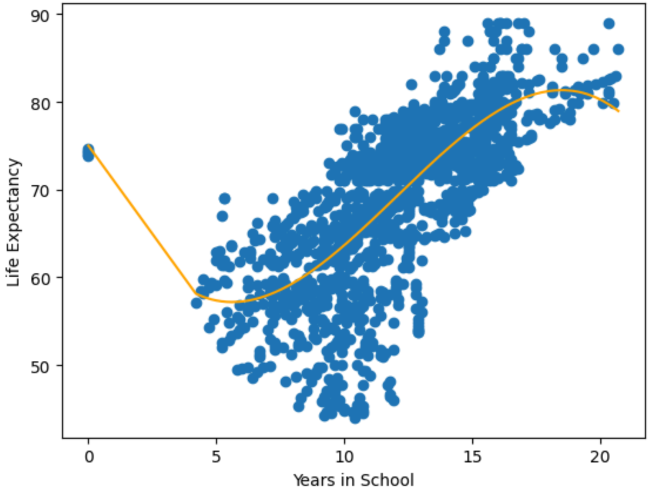


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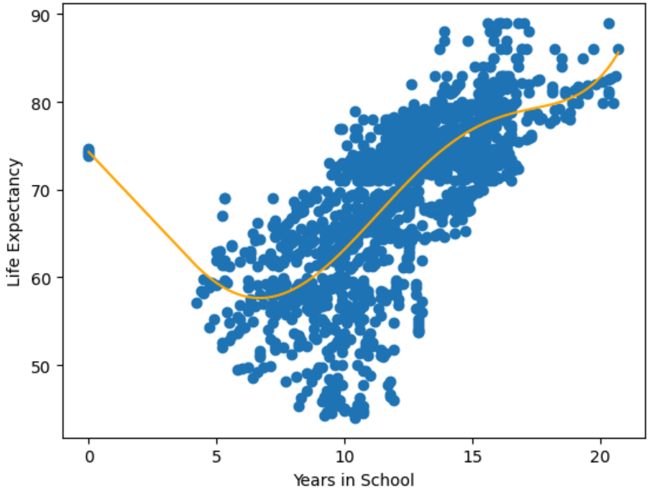
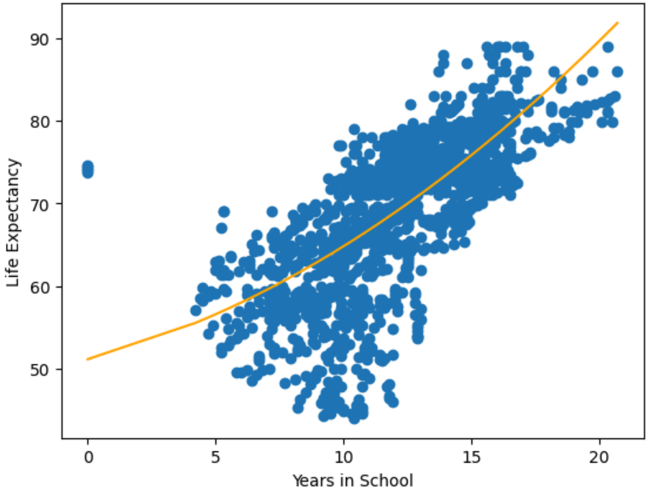



$n = 5$

Pick the $n$ with the smallest loss

$$\arg\min_{\boldsymbol{\theta}, n} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}, (\boldsymbol{\theta}, n))$$

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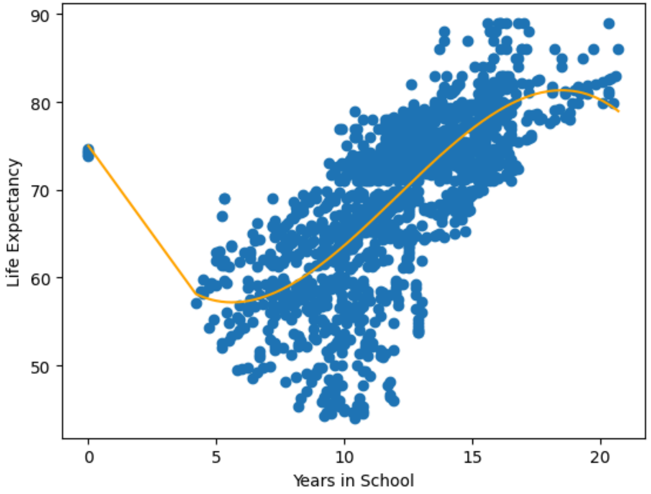


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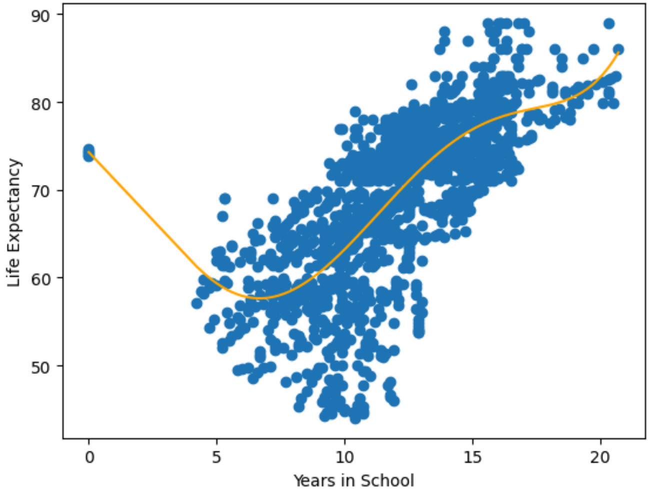


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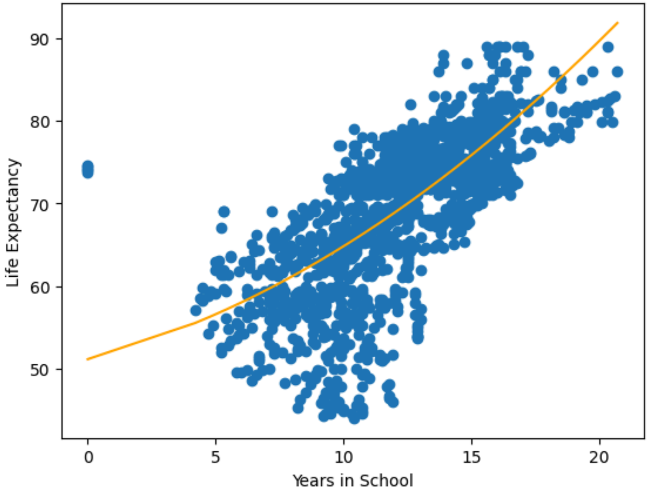
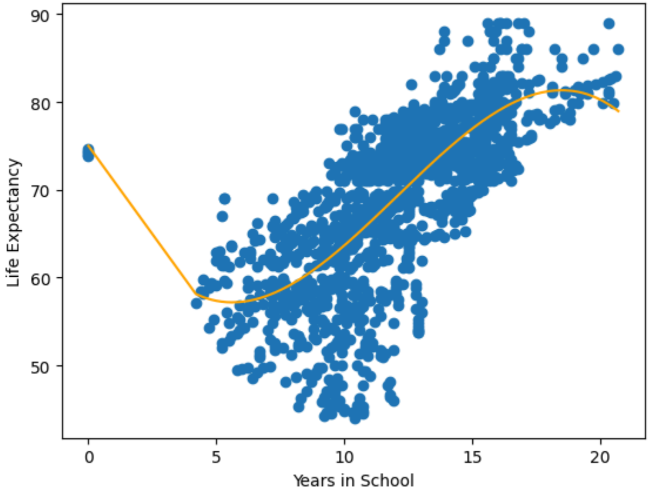
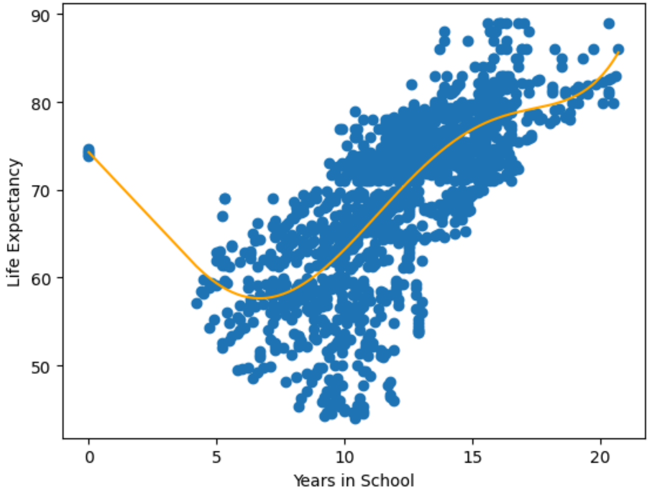
**Question:** Which $n$ do you think has the smallest loss?



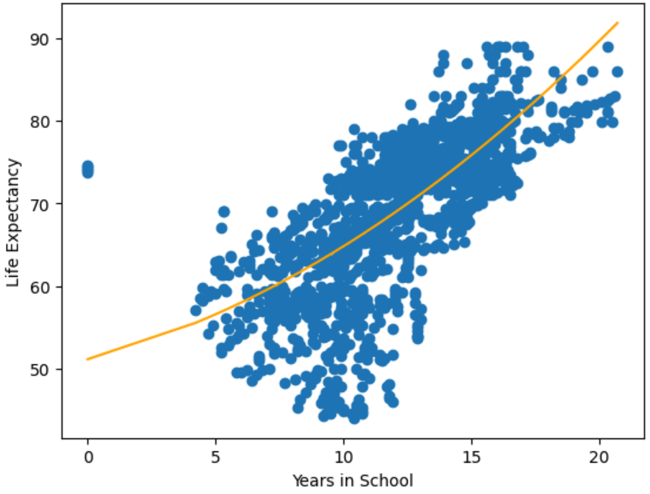




$n = 2$ $\quad$ $n = 3$ $\quad$ $n = 5$

**Question:** Which $n$ do you think has the smallest loss?

**Answer:** $n = 5$, but intuitively, $n = 5$ does not seem very good...

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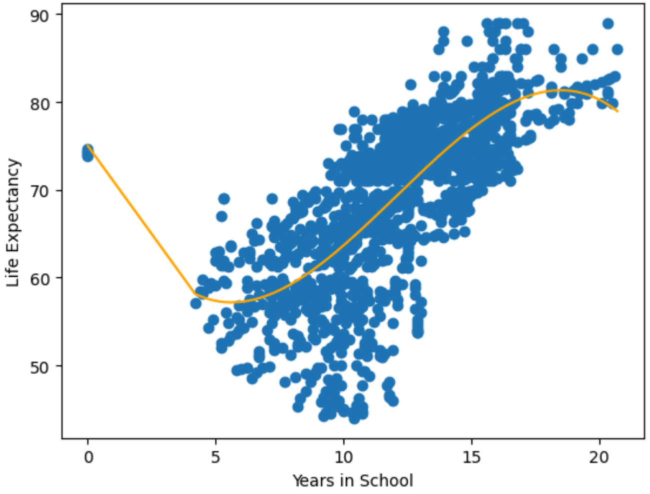


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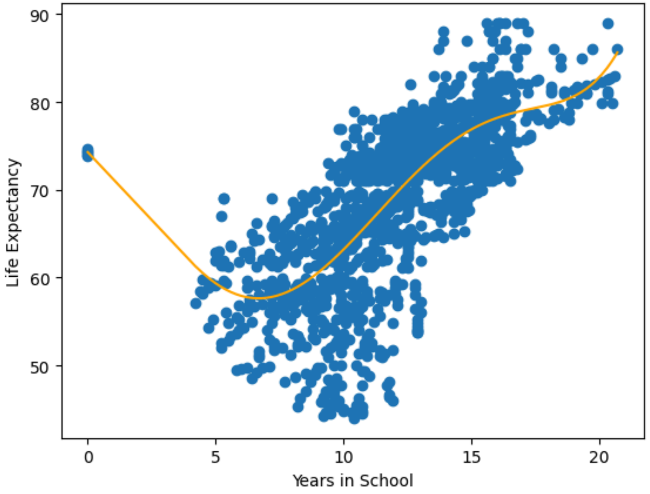
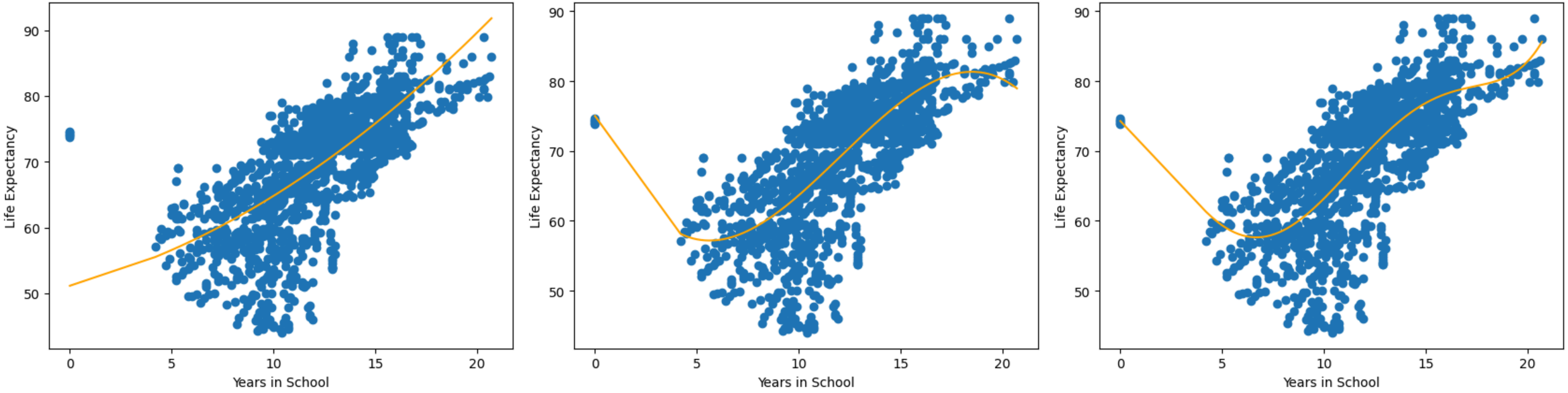



$n = 5$

More specifically, $n = 5$ will not generalize to new data

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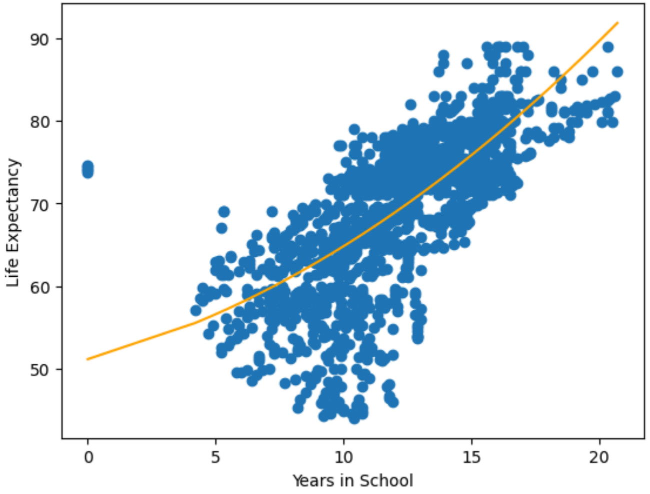
We will only use our model for new data (we already have the $y$ for a known $x$)!

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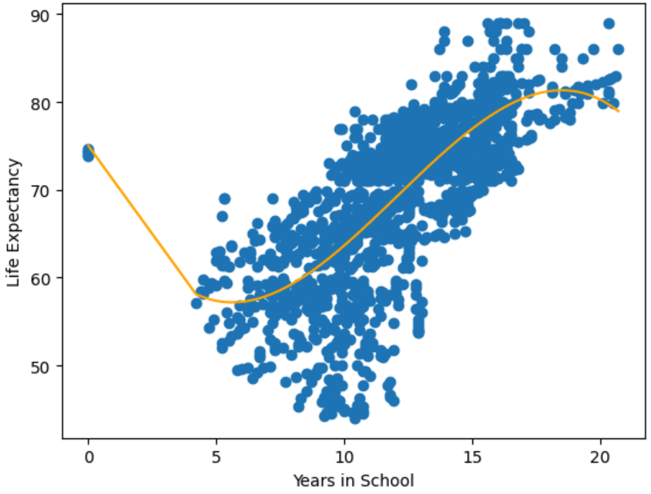


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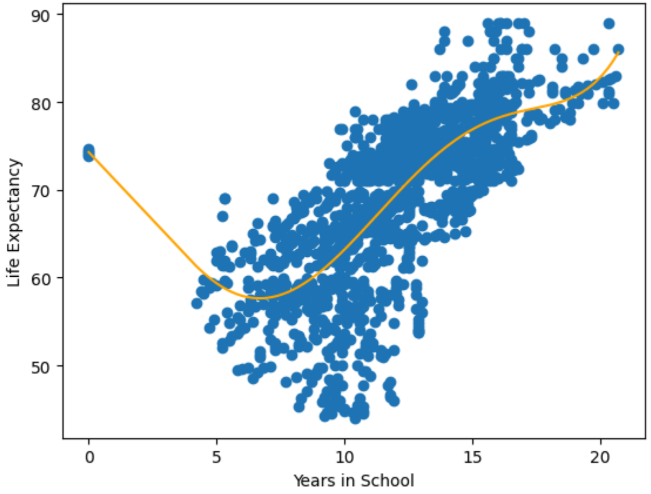


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**Question:** How do we choose $n$ such that our polynomial model works for unseen/new data?

**Answer:** Compute the loss on unseen data!

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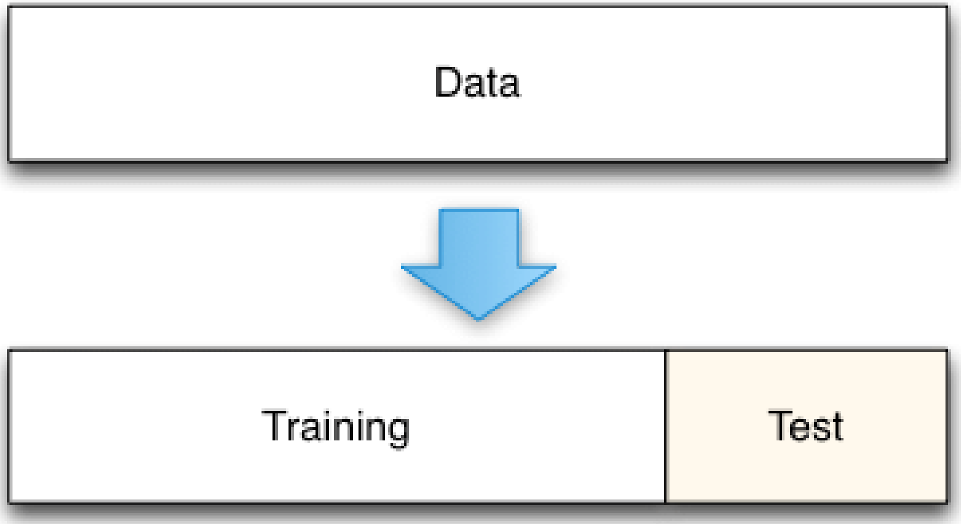

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**Note:** The model must never see the testing dataset during training. This is very important!

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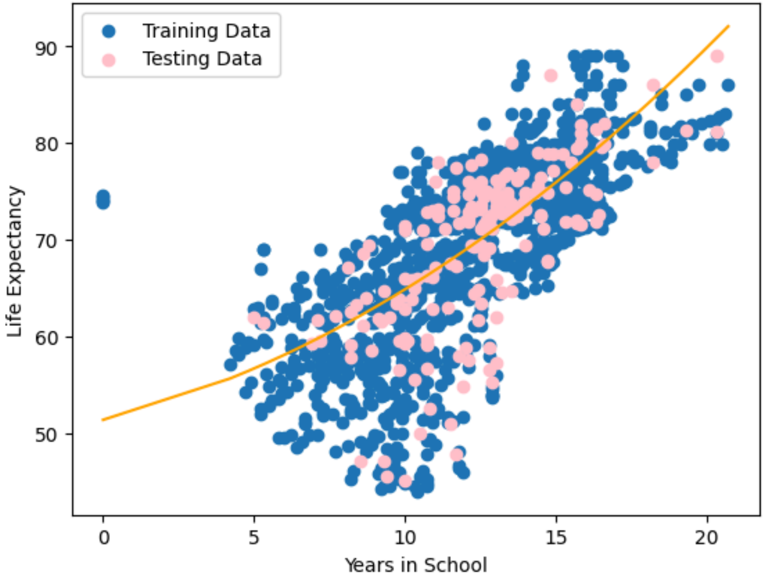


Learn parameters from the train dataset, evaluate on the test dataset

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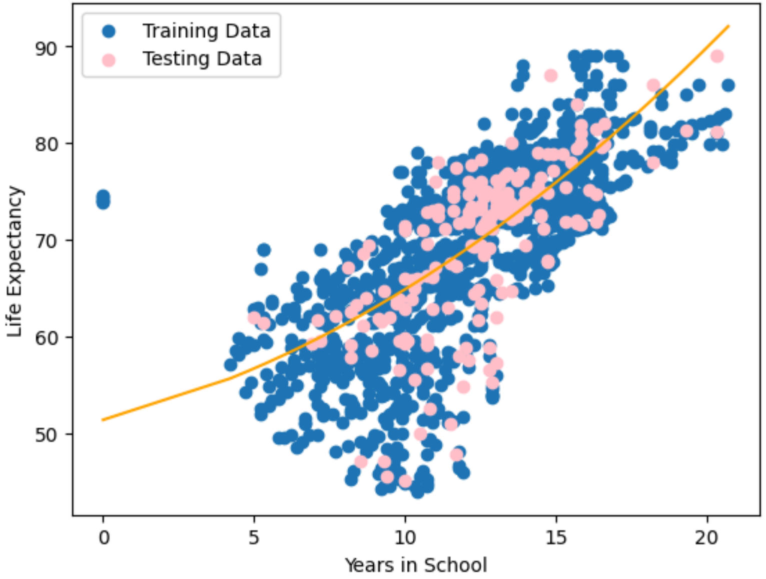


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$$\mathcal{L}(\boldsymbol{X}_{\text{train}}, \boldsymbol{y}_{\text{train}}, \boldsymbol{\theta}) \qquad \mathcal{L}(\boldsymbol{X}_{\text{test}}, \boldsymbol{y}_{\text{test}}, \boldsymbol{\theta})$$

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