Value

CISC 7404 - Decision Making

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Review

Policy-Conditioned Returns

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Trajectory optimization is model-based algorithm

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Guaranteed optimal policy, given infinite compute

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Today, we will look at new algorithms based on the notion of **value**

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Expensive to train, but very cheap to use

Policy-Conditioned Returns

Recall the return from trajectory optimization

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$$[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \ldots] = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0, a_0, a_1, \ldots]$$

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This is an **action-conditioned** discounted return

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- Random
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- Maximize $\mathcal{G}$

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$$\pi : S \times \Theta \mapsto \Delta A$$

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$$\pi(a_t \mid s_t; \theta_\pi) = \begin{cases} 1 \text{ if } a_t = \arg\max_{a_t \in A} \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] \\ 0 \text{ otherwise} \end{cases}$$

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Conditioning the return on actions is annoying

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Must compute infinitely many actions and outcomes for the return

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What if we condition on a policy, instead of specific actions?

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$$a_0 \sim \pi(\cdot \mid s_0; \theta_\pi), \quad a_1 \sim \pi(\cdot \mid s_1; \theta_\pi), \quad a_2 \sim \pi(\cdot \mid s_2; \theta_\pi), \quad ...$$

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Condition on a function parameterized by θ_π instead of many actions

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The function outputs a distribution over the action space $\pi(a \mid s; \theta_\pi)$

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But remember, $\mathcal{R}(s_{t+1})$ hides much of the magic

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How does $\mathbb{E}[\mathcal{R}(s_{t+1})]$ change when we condition on θ_π?

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$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] = \sum_{t=0}^{\infty} \gamma^t \mathcal{R}(s_{t+1}) \sum_{s_{t+1} \in S} \Pr(s_{t+1} \mid s_0, a_0, ..., a_t)$$

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Answer: State transition function

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Combine the policy distribution with next state distribution

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Write out the first few timesteps

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$$\Pr(s_2 \mid s_0; \theta_\pi) = \sum_{s_1 \in S} \sum_{a_1 \in A} \mathrm{Tr}(s_2 \mid s_1, a_1) \cdot \pi(a_1 \mid s_1; \theta_\pi) \\ \cdot \sum_{a_0 \in A} \mathrm{Tr}(s_1 \mid s_0, a_0) \cdot \pi(a_0 \mid s_0; \theta_\pi)$$

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Derive a general form for $\Pr(s_{n+1} \mid s_0; \theta_\pi)$

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$$\Pr(s_{n+1} \mid s_0; \theta_\pi) = \sum_{s_1, \dots, s_n \in S} \prod_{t=0}^{n} \left(\sum_{a_t \in A} \mathrm{Tr}(s_{t+1} \mid s_t, a_t) \cdot \pi(a_t \mid s_t; \theta) \right)$$

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Plug back into our expected reward

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Plug back into our expected reward

$$\mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0; \theta_\pi] = \sum_{s_{t+1} \in S} \mathcal{R}(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_0; \theta_\pi)$$

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Discounted return is discounted sum of rewards

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Goal: find the θ_π (policy parameters) to maximize the expected return

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It is a critical part of decision making

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We can compute

$$V(s_a, \theta_\pi), V(s_b, \theta_\pi), V(s_c, \theta_\pi)$$

To find the value of any state

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Question: Why does Prof. Steven keep showing stupid equations? How is the value function useful?

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With value, we can be sure we make the right decision

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For each location $s_0 \in \{s_a, s_b\}$:

- Estimate probability of reaching each goal $\Pr(s_g \mid s_0)$; $s_g \in \{s_x, s_y, s_z\}$
- Estimate time to accomplish each goal $t = \ldots$

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We call the following equation the **Monte Carlo** value function

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- Or build the infinite decision tree

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Let us try to delete the infinite sum

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$$V(s_0, \theta_\pi) = \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} \mathcal{R}(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_0; \theta_\pi)$$

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Factor out initial timestep $t = 0$ out of the outer sum

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Rewrite sum starting from $t = 0$

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Factor out γ

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Split Pr using Markov property

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Move sum and Pr outside

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Question: What is this term?

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This is a huge finding!

TD Value Functions

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Represent policy-conditioned discounted return without an infinite sum

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We call this the **Temporal Difference** (TD) value function

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Compute the return with a single transition $s_0 \rightarrow s_1$

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Represent policy-conditioned discounted return without an infinite sum

We call this the **Temporal Difference** (TD) value function

Compute the return with a single transition $s_0 \to s_1$

Evaluate infinite-depth decision tree with a single function call

TD Value Functions

To summarize, we can represent the value function in two ways:

TD Value Functions

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The Monte Carlo value function

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Q Functions

Q Functions

We saw two forms of the value function

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But it does not tell us the policy

Q Functions

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The value function relies on a policy

But it does not tell us the policy

How can we use the value function to find an optimal policy?

Q Functions

Consider the value function

$$V(s_0, \theta_\pi) = \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0; \theta_\pi] = \mathbb{E}[\mathcal{R}(s_1) \mid s_0; \theta_\pi] + \gamma V(s_1, \theta_\pi)$$

With trajectory optimization we conditioned on actions

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...]$$

We conditioned the value function on policy parameters

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0; \theta_\pi]$$

What if we wanted a mix of both?

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0; \theta_\pi]$$

Q Functions

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0; \theta_\pi]$$

We call this the **Q function**

$$Q(s, a, \theta_\pi) = \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0; \theta_\pi]$$

We can derive the Q function from the value function

$$V(s_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0; \theta_\pi] + \gamma V(s_1, \theta_\pi)$$

First, introduce the action a_0

$$V(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0; \theta_\pi] + \gamma V(s_1, \theta_\pi)$$

Q Functions

$$V(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0; \theta_\pi] + \gamma V(s_1, \theta_\pi)$$

Condition the initial reward on the action

$$V(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

Call it the Q function

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

Q Functions

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

The Q function tells us:

- The value of an action a_0
- In state s_0
- If we follow $\pi(a_t \mid s_t; \theta_\pi)$ afterwards

Question: How can we use the Q function for decision making?

Hint: We can evaluate Q for every possible action

$$\arg\max_{a_0 \in A} Q(s_0, a_0, \theta_\pi) = \arg\max_{a_0 \in A} \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

Q Functions

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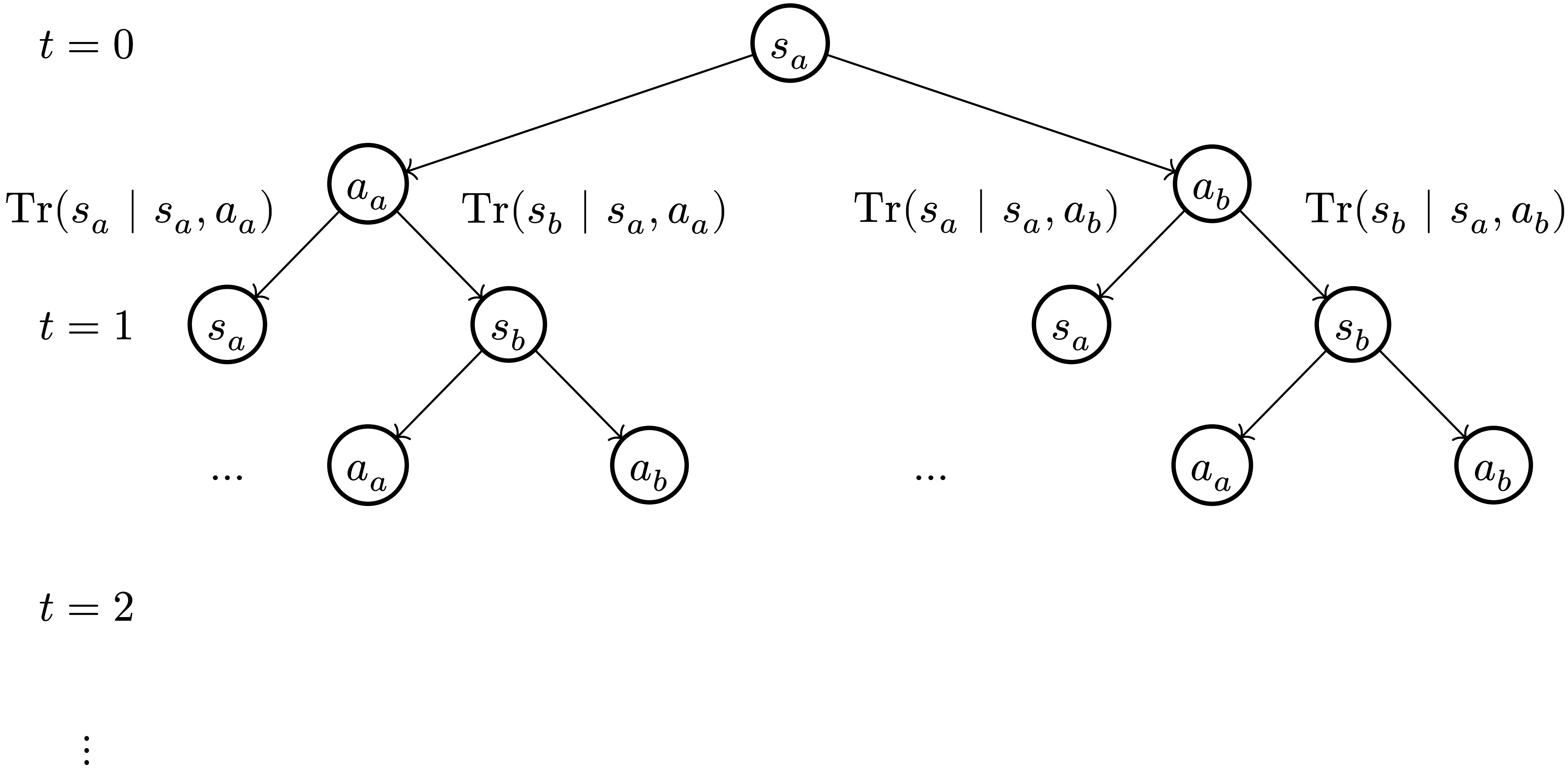
This is a very powerful equation

- Compute $Q(s_0, a_0)$ for all a_0
- Pick the a_0 that maximizes $Q(s_0, a_0)$

This a_0 is **guaranteed** to be the optimal action for the **infinite** future

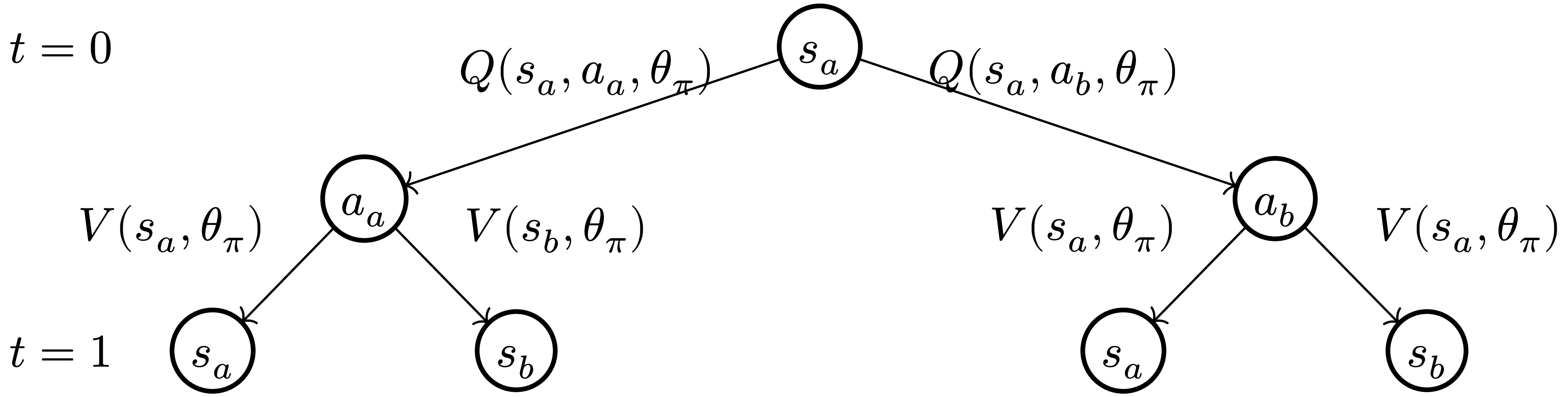
We collapsed the infinite decision tree into a single level

Q Functions



$t = 0$
s_a
a_a
a_b
$\mathrm{Tr}(s_a \mid s_a, a_a)$
$\mathrm{Tr}(s_b \mid s_a, a_a)$
$\mathrm{Tr}(s_a \mid s_a, a_b)$
$\mathrm{Tr}(s_b \mid s_a, a_b)$
$t = 1$
s_a
s_b
s_a
s_b
...
a_a
a_b
...
a_a
a_b
$t = 2$
$\vdots$

Q Functions



$t = 0$
$t = 1$
s_a
$Q(s_a, a_a, \theta_\pi)$
$Q(s_a, a_b, \theta_\pi)$
a_a
a_b
$V(s_a, \theta_\pi)$
$V(s_b, \theta_\pi)$
$V(s_a, \theta_\pi)$
$V(s_a, \theta_\pi)$
s_a
s_b
s_a
s_b

Q Functions

$$\arg\max_{a_0 \in A} Q(s_0, a_0, \theta_\pi) = \arg\max_{a_0 \in A} (\mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi))$$

Q Functions

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

We will use either Q or V in every other algorithm in the course!

It is the core of decision making

Q Learning

Q Learning

Q learning is a model-free algorithm first invented in the 1980s

It is still used heavily today

In fact, I am using it in our research right now

We now have all the information we need to implement Q learning

Q Learning

Our Q function relies on the value function for some θ_π

Right now, it is not clear what the policy is

So how can we use the Q function without knowing the policy?

Q Learning

Start with the Q function

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

We want to take the action that maximizes Q

$$\arg\max_{a_0 \in A} Q(s_0, a_0, \theta_\pi) = \arg\max_{a_0 \in A} (\mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi))$$

Recall the definition of value function

$$V(s_0, \theta_\pi) = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0; \theta_\pi]$$

Q Learning

$$V(s_0, \theta_\pi) = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0; \theta_\pi]$$

What should our policy be?

Well we know the following is optimal

$$\arg \max_{a_0 \in A} Q(s_0, a_0, \theta_\pi) = \arg \max_{a_0 \in A} (\mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi))$$

What if we say we follow a policy that maximizes Q?

Q Learning

$$\pi(a_0 \mid s_0; \theta_\pi) = \begin{cases} 1 \text{ if } a_0 = \arg\max_{a \in A} Q(s_0, a, \theta_\pi) \\ 0 \text{ otherwise} \end{cases}$$

What is the value function for this policy?

$$V(s_0, \theta_\pi) = \max_{a \in A} Q(s_0, a, \theta_\pi)$$

So we can rewrite the Q function without V

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma V(s_1, \theta_\pi)$$

$$Q(s_0, a_0, \theta_\pi) = \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] + \gamma \arg\max_{a \in A} Q(s_1, a, \theta_\pi)$$

Homework

Homework

Due in 2 weeks (Weds 12 March, 23:59)

Download and submit `.py` and `.ipynb` files

Uses turnitin for checking

https://colab.research.google.com/drive/1xtBxAaVc3ax6_j59RC3NLQQPFcIEoau-?usp=sharing