Algorithms

CISC 7404 - Decision Making

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University of Macau

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Quiz results on moodle

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If you have no score, come see me

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Do not forget individual participation grade!

Review

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Diffusion models

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https://arxiv.org/pdf/2006.11239

Algorithms

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The policy is the "brain" of the agent

It makes decisions for the agent

Algorithms

Policies can be good, bad, or even human!

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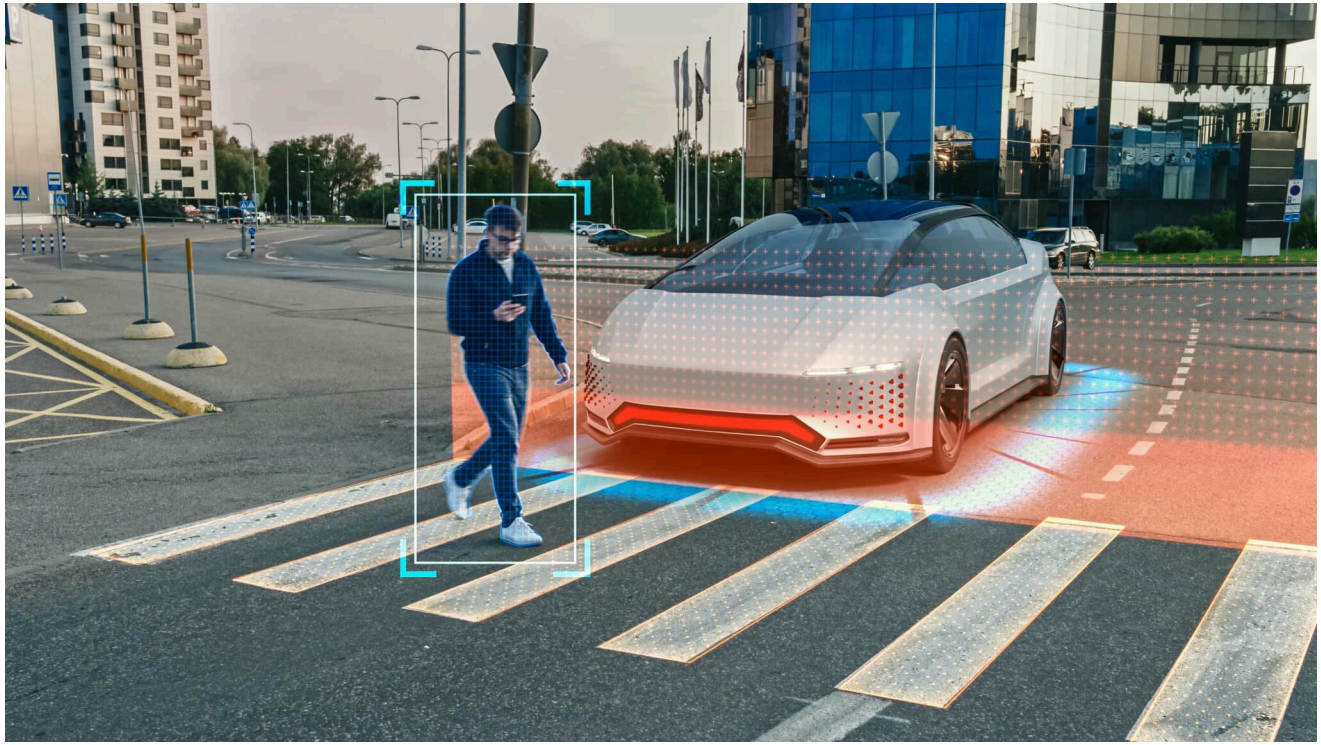

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The policy is guaranteed to maximize the discounted return

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https://www.youtube.com/watch?v=6qj3EfRTtkE

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Critical part of Alpha-* methods (AlphaGo, AlphaStar, AlphaZero)

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We want to find τ that provides the greatest discounted return

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To understand what is hiding, let us examine the reward function

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Cannot know s_{t+1} with certainty, only know the distribution!

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Question: Ok, now what is the definition of R?

Answer:

$$R : S \mapsto \mathbb{R}$$

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We should write it as $\mathcal{R} : S \mapsto \mathbb{R}$

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Question: Why do we like to take the expectation of random variables?

Answer: It maps random processes to something we can maximize

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Random variable conditioned on s_t, a_t

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Note: Cannot directly maximize $\mathcal{R}$ because s_{t+1} is random

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Question: Have we seen something similar before?

Answer: Bandits!

$$\arg\max_{a \in \{1 \ldots k\}} \mathbb{E}[\mathcal{X}_a]$$

Reward Optimization

$$\arg\max_{a_t \in A} \mathbb{E}\left[\mathcal{R}(s_{t+1}) \mid s_t, a_t\right] = \arg\max_{a_t \in A} \sum_{s_{t+1} \in S} \mathcal{R}(s_{t+1}) \cdot \mathrm{Tr}(s_{t+1} \mid s_t, a_t)$$

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Let us turn this equation into a policy

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It will always act to maximize the expected reward

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Example: Online advertising, show users ads so they buy products

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This will create the best ad creator possible!

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We have one more thing to do

Trajectory Optimization

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Question: Do we know the second term?

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Answer: It is more tricky

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For $\mathcal{R}(s_{n+1})$ we need an expression for $\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots)$

Trajectory Optimization

Question: How do we find $\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots)$?

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To extend to MDP, just need to include the actions!

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Trajectory Optimization

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This predicts the future states of an MDP

Trajectory Optimization

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Trajectory Optimization

$$\mathbb{E}[\mathcal{R}(s_{n+1}) \mid s_0, a_0, a_1, \ldots, a_n] = \sum_{s_{n+1} \in S} R(s_{n+1}) \sum_{s_1, \ldots, s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

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What does each piece mean?

Trajectory Optimization

s_{n+1} Distribution

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Trajectory Optimization

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Answer: Model Predictive Control (MPC) or Receding Horizon Control

Trajectory Optimization

There is a lot of math behind trajectory optimization/MPC

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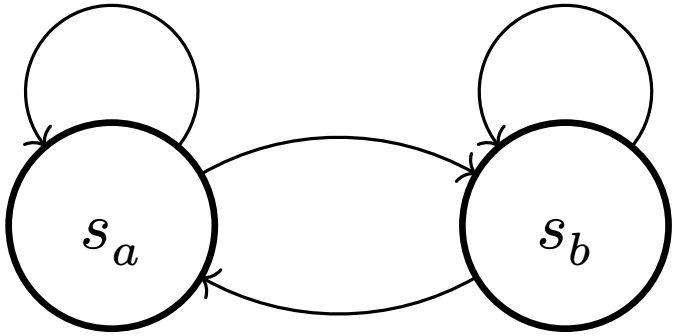
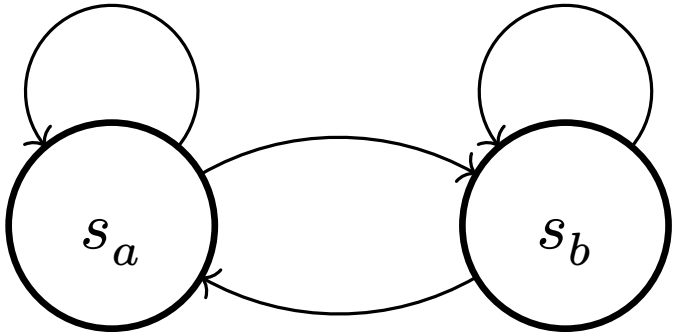
Let us do a visual example to help you understand

Trajectory Optimization



$\mathcal{R}(s_a) = 0$ $\mathcal{R}(s_b) = 1$

Trajectory Optimization



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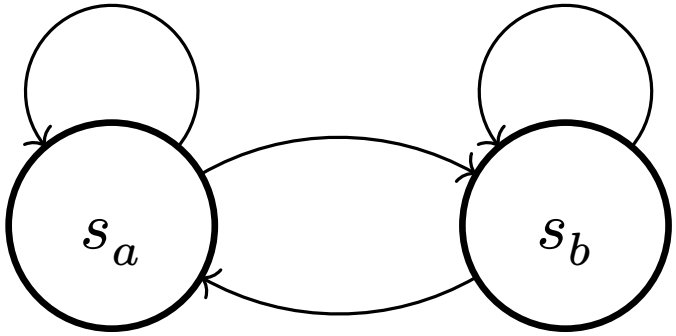
Trajectory Optimization



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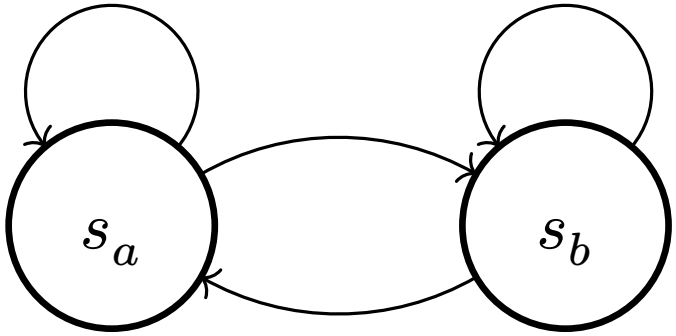
Trajectory Optimization



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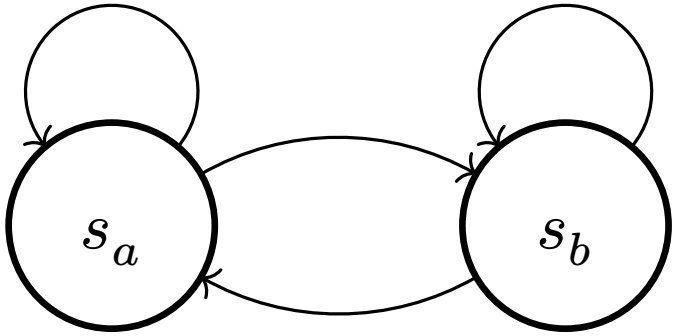
Trajectory Optimization



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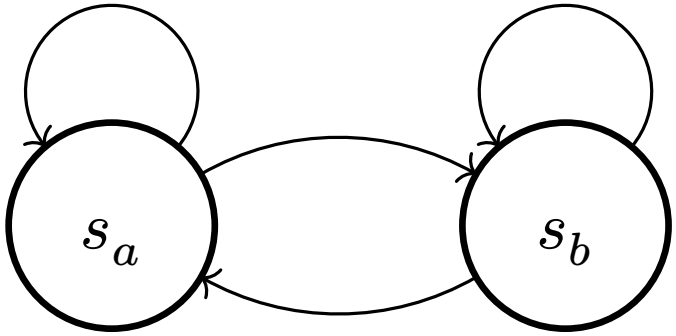
Trajectory Optimization



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Trajectory Optimization

We can build this into a decision tree using trajectory optimization

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The root node of the tree corresponds to s_0

Trajectory Optimization

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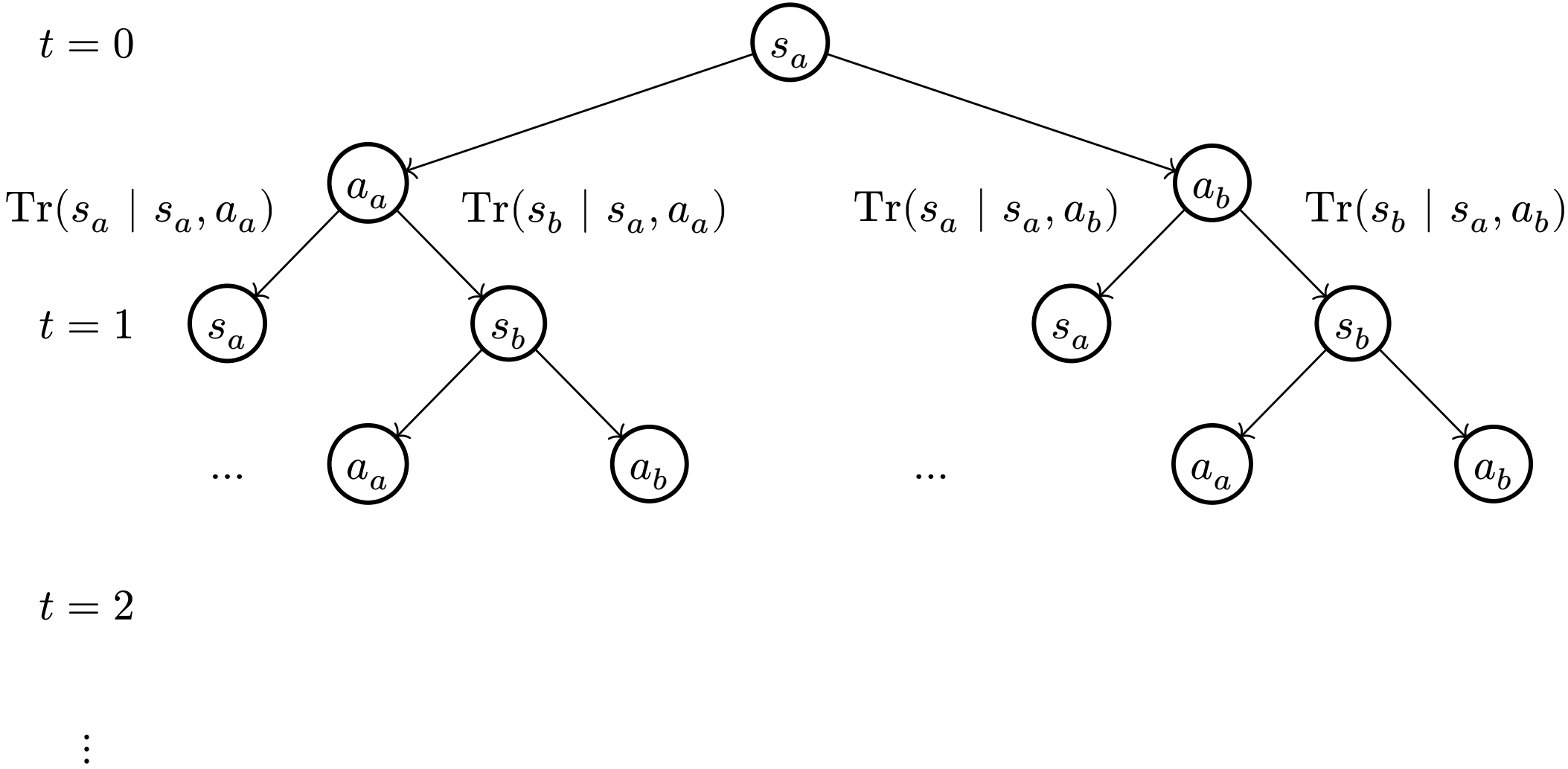
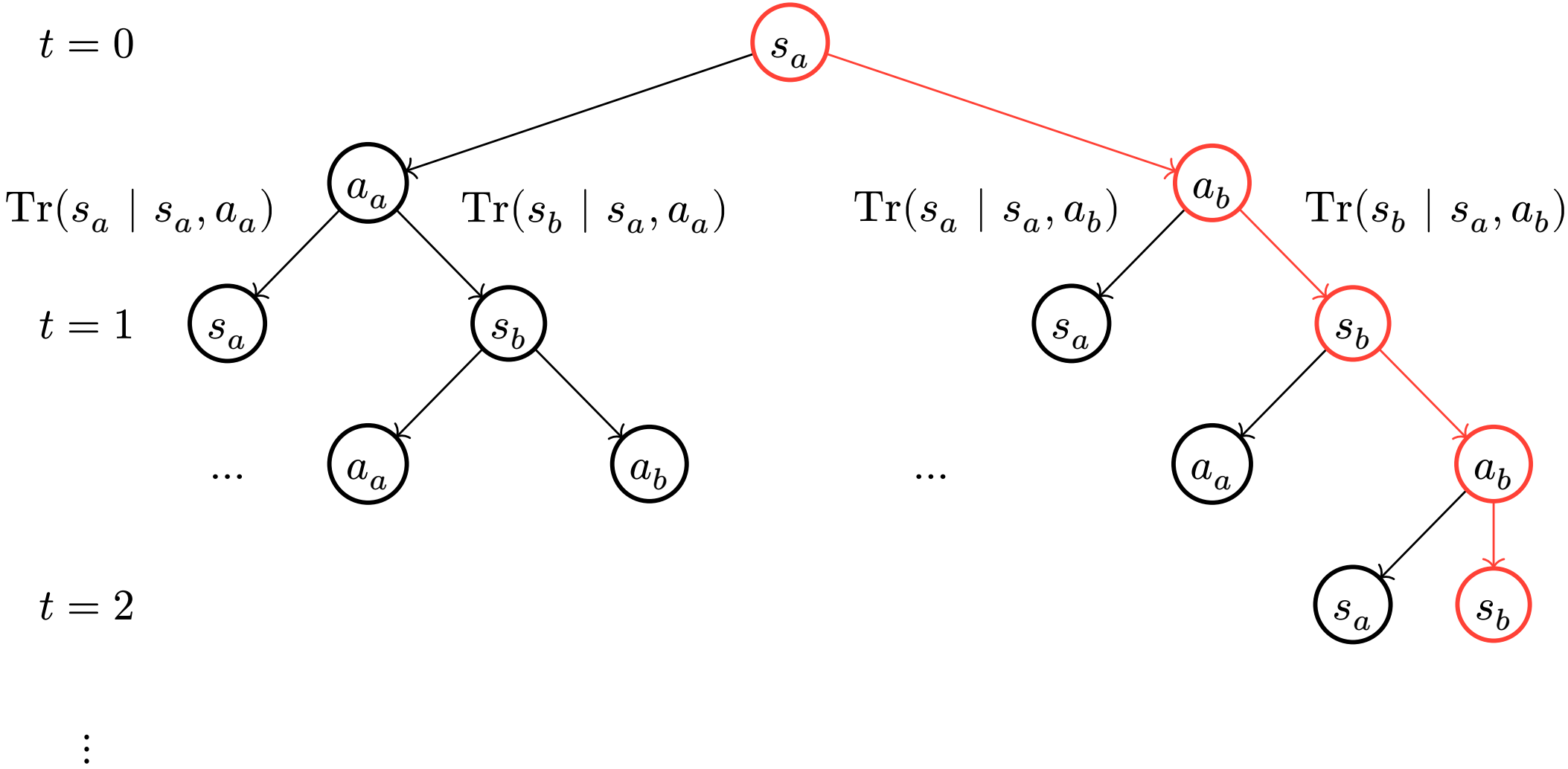
The root node of the tree corresponds to s_0

Each level of the tree enumerates possible outcomes

Trajectory Optimization



Trajectory Optimization



Trajectory Optimization

Question: How many nodes does our tree have?

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We have some tricks to make this tractable

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$$\underset{a_0,a_1,\ldots\in A}{\arg\max}\,\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0,a_0,a_1,\ldots] = \underset{a_0,a_1,\ldots}{\arg\max}\sum_{t=0}^{\infty}\gamma^t\mathbb{E}\big[\mathcal{R}(s_{t+1}) \mid s_0,a_0,\ldots,a_t\big]$$

Trick 1: Introduce a **horizon** n

Trajectory Optimization

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Now we can limit computation to $O(|S| \cdot |A|)^n$

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Answer: We no longer consider the infinite future, our agent may get greedy and be trapped

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Trick 2: Only simulate j actions and k states

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Now, computation is $O(j \cdot k)^n$

Trajectory Optimization

$$\arg\max_{a_0,\ldots,a_n \in A} \mathbb{E}[\mathcal{G}(\tau_n) \mid s_0, a_0, \ldots, a_n] = \arg\max_{a_0,\ldots,a_n \in A} \sum_{t=0}^{n} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0, a_0, \ldots, a_n]$$

Trick 2: Only simulate j actions and k states

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Answer: Optimal action may not be sampled, results in less-optimal trajectory

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I plan to release assignment 1 next lecture