Trajectory Optimization

CISC 7404 - Decision Making

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University of Macau

Exam

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- After you are done, give me your exam and go relax outside, we resume class at 8:30

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- Good luck!

Exam

- 在所有学生收起电脑/笔记/手机后,我会分发试卷。
- 如果在此之后仍有电脑/笔记/手机未收,将视为作弊。
- 试卷会背面朝下发下,在我宣布开始前请勿翻面。
- 试卷翻面后,我会简要说明每道题的注意事项。
- 说明结束后,你们有 75 分钟完成考试。
- 交卷后请到教室外休息,8:30 恢复上课。
- 试卷可能存在不同版本,细节略有差异。
- 若你的试卷上出现其他版本的答案,将被判定为作弊。
- 试卷说明为中英双语,若内容冲突以英文为准。
- 祝各位考试顺利!

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If you thought the exam was easy, come talk to me after class

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Our lab is always looking for smart students to work on RL problems

Review

MDP Coding

Decision Making with a Model

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These methods are expensive in terms of compute

We usually only use these methods for simple problems

Decision Making with a Model

"Simple" problems have low dimensional and actions spaces

$$|S|, |A| = \text{small}$$

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One example is position and velocity control

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$$S \in \mathbb{R}^6, A \in \mathbb{R}^3$$

https://www.youtube.com/watch?v=6qj3EfRTtkE

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With modern GPUs, researchers are revisiting these methods

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They are applying them to more difficult tasks with high dimensional $|S|, |A|$

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https://youtu.be/_e3BKzK6xD0?si=Kr-KOccTDypgRjgJ&t=194

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Critical part of Alpha-* methods (AlphaGo, AlphaStar, AlphaZero)

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We want to find the trajectory $\tau = \begin{bmatrix} s_0 & a_0 \\ s_1 & a_1 \\ \vdots & \vdots \end{bmatrix}$ that provides the greatest discounted return

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To understand what is hiding, let us examine the reward function

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Cannot know s_{t+1} with certainty, only know the distribution!

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We should write it as $\mathcal{R} : S \mapsto \mathbb{R}$

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Random variable conditioned on s_t, a_t

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In English:

1. Compute the probability for each outcome $s \in S$, for each $a \in A$
2. Compute the reward for each possible outcome $s \in S$
3. The expected reward for $s \in S$ is probability times reward
4. Take the action $a_t \in A$ that produces the largest the expected reward

Question: Have we seen this before?

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$$\arg\max_{a \in \{1...k\}} \mathbb{E}[\mathcal{X}_a]$$

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$$\pi : S \times \Theta \mapsto \Delta A$$

$$\pi(a_t \mid s_t; \theta) = \begin{cases} 1 \text{ if } a_t = \arg\max_{a_t \in A} \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_t, a_t, \theta] \\ 0 \text{ otherwise} \end{cases}$$

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We have a name for a function that picks actions

We call this the **policy**, which usually has parameters $\theta \in \Theta$

$$\pi : S \times \Theta \mapsto \Delta A$$

$$\pi(a_t \mid s_t; \theta) = \begin{cases} 1 \text{ if } a_t = \arg\max_{a_t \in A} \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_t, a_t, \theta] \\ 0 \text{ otherwise} \end{cases}$$

The policy is the "brain" of the agent, it controls the agent

The Mysterious Reward

We figured out the mystery the reward function was hiding

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We have one last thing to do

Trajectory Optimization

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Trajectory Optimization

What we have:

Expected value of the reward, as a function of state and action

Trajectory Optimization

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$$\mathbb{E}[G(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] = ?$$

Question: Why depend on future actions?

Trajectory Optimization

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Question: Why depend on future actions?

Answer: To pick the actions that maximize G

Trajectory Optimization

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We want to find the best actions, so they must be in the expectation

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Trajectory Optimization

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Question: Any problems?

Trajectory Optimization

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Answer: $\mathcal{R}$ needs s_2, but we only have s_0!

Trajectory Optimization

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For $t = 1$, the reward relies on the distribution $\mathrm{Tr}(s_1 \mid s_0, a_0)$

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For $t = 1$, the reward relies on the distribution $\mathrm{Tr}(s_1 \mid s_0, a_0)$

For $t = 2$, the reward relies on $\mathrm{Tr}(s_2 \mid s_1, a_1)$ and $\mathrm{Tr}(s_1 \mid s_0, a_0)$

Trajectory Optimization

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For $\mathcal{R}(s_{n+1})$ we need an expression for $\Pr(s_{n+1} \mid s_0, a_0, a_1, ...)$

Trajectory Optimization

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We are predicting the future state of an MDP

Trajectory Optimization

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TODO write out expectation so we can plug in $R(s_t) \Pr(s_t \mid s_0, a_0, \ldots)$

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \ldots] = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0, a_0, a_1, \ldots]$$

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Trajectory Optimization

Goal: Given an initial state and some actions, predict the expected discounted return

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$$\mathbb{E}[R(s_1) \mid s_0, a_0] = \sum_{s_1 \in S} R(s_1) \Pr(s_1 \mid s_0, a_0)$$

$$\mathbb{E}[R(s_2) \mid s_0, a_0, a_1] = \sum_{s_2 \in S} R(s_2) \sum_{s_1 \in S} \Pr(s_2 \mid s_1, a_1) \Pr(s_1 \mid s_0, a_0)$$

$$\mathbb{E}[R(s_{n+1}) \mid s_0, a_0, a_1, \ldots a_n] = \sum_{s_{n+1} \in S} R(s_{n+1}) \sum_{s_1, \ldots, s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

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Trajectory Optimization

$$\mathbb{E}[R(s_{n+1}) \mid s_0, a_0, a_1, ...a_n] = \underbrace{\sum_{s_{n+1} \in S} R(s_{n+1})}_{\text{Mean reward over possible } s_{n+1}} \sum_{s_1, ..., s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

s_{n+1} Distribution

$$\mathbb{E}[R(s_{n+1}) \mid s_0, a_0, a_1, ... a_n] = \sum_{s_{n+1} \in S} R(s_{n+1}) \sum_{s_1, ..., s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Mean reward over possible s_{n+1}

Trajectory Optimization

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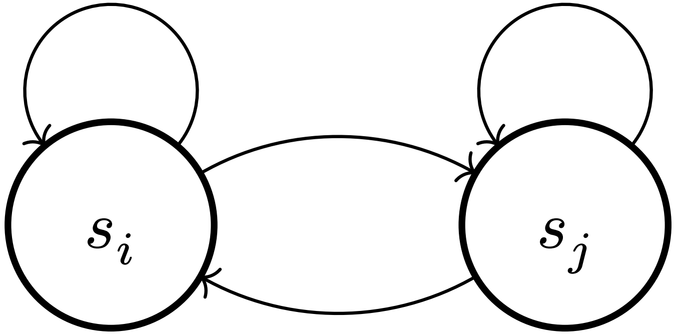
Trajectory Optimization

$$
\begin{aligned}
\mathbb{E}[G \mid s_0, a_0, a_1, ...] = & \quad \mathbb{E}[R(s_1) \mid s_0, a_0] \\
& + \gamma \quad \mathbb{E}[R(s_2) \mid s_0, a_0, a_1] \\
& + \gamma^2 \quad \mathbb{E}[R(s_3) \mid s_0, a_0, a_1, a_2] \\
& + \quad ... \\
= & \quad \sum_{s_1 \in S} R(s_1) \Pr(s_1 \mid s_0, a_0) \\
& + \gamma \quad \sum_{s_2 \in S} R(s_2) \sum_{s_1 \in S} \Pr(s_2 \mid s_1, a_1) \Pr(s_1 \mid s_0, a_0) \\
& + \gamma^2 \quad \sum_{s_3 \in S} R(s_3) \sum_{s_2 \in S} \Pr(s_3 \mid s_2, a_2) \sum_{s_1 \in S} \Pr(s_2 \mid s_1, a_1) ... \\
& + ...
\end{aligned}
$$

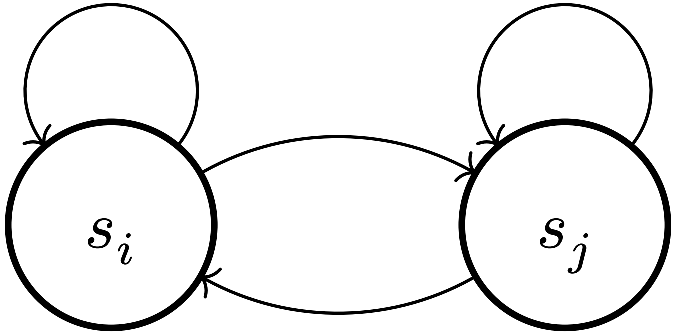
Trajectory Optimization



$R(s_i) = 0$ $\quad$ $R(s_j) = 1$

$$S = \{s_i, s_j\} \quad A = \{a_i, a_j\}$$

Trajectory Optimization



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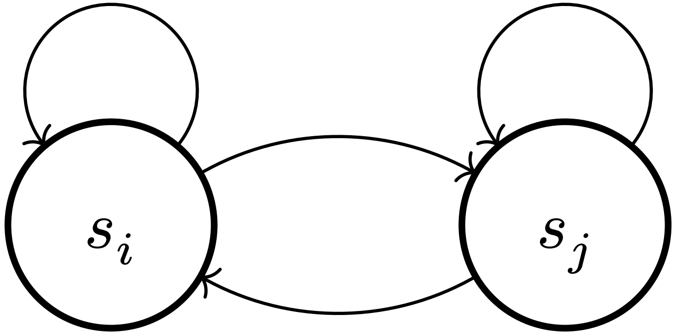
Trajectory Optimization



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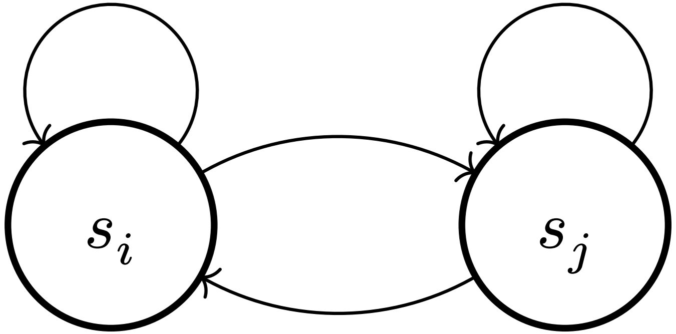
Trajectory Optimization



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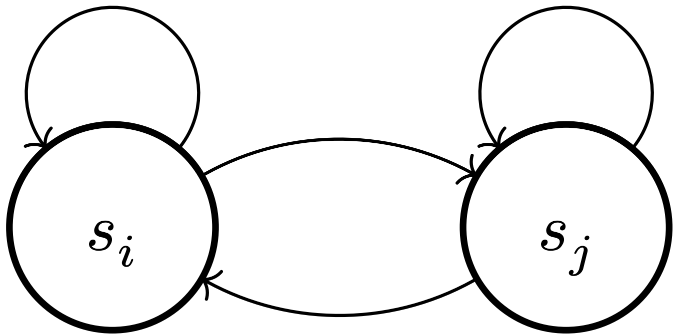
Trajectory Optimization



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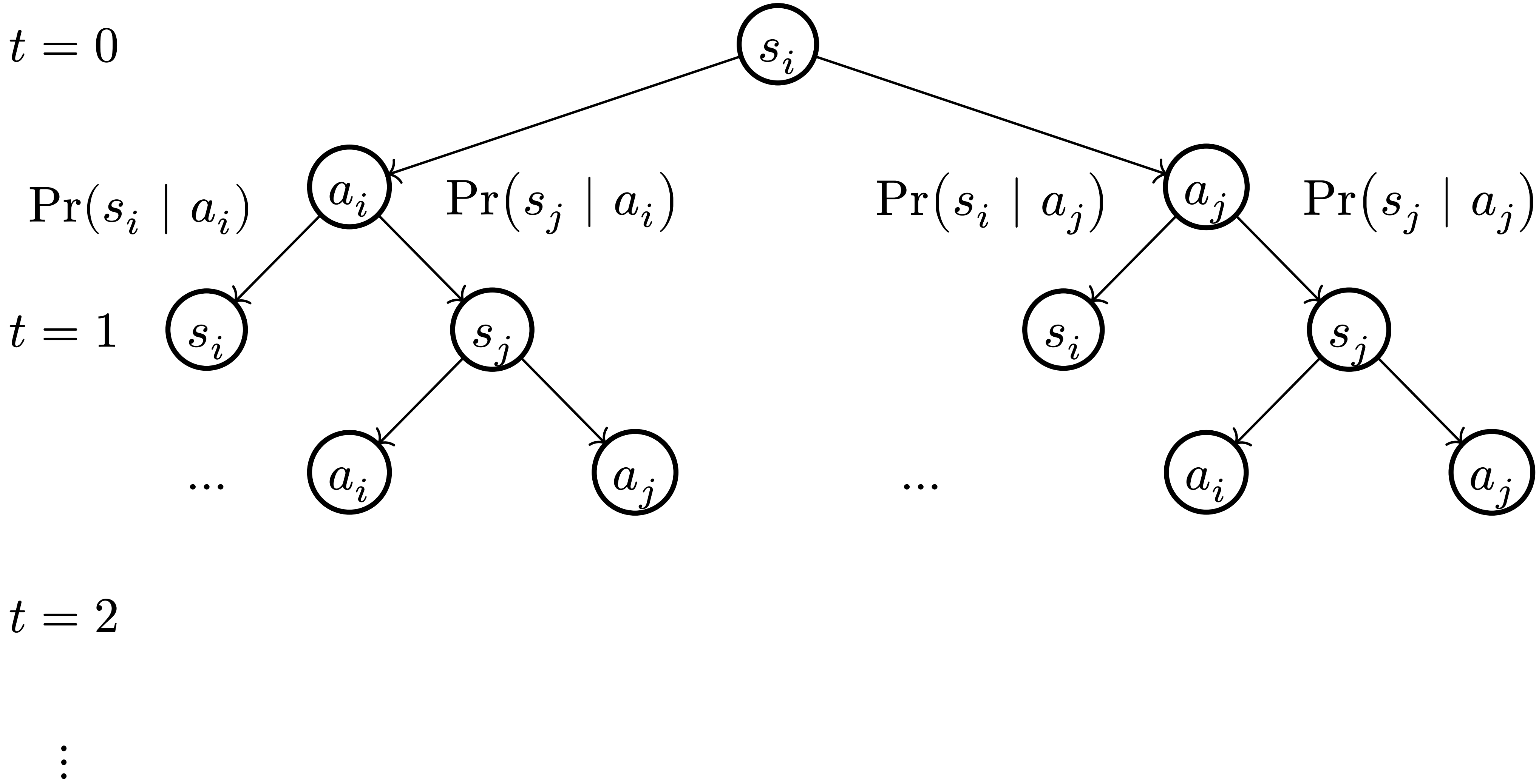
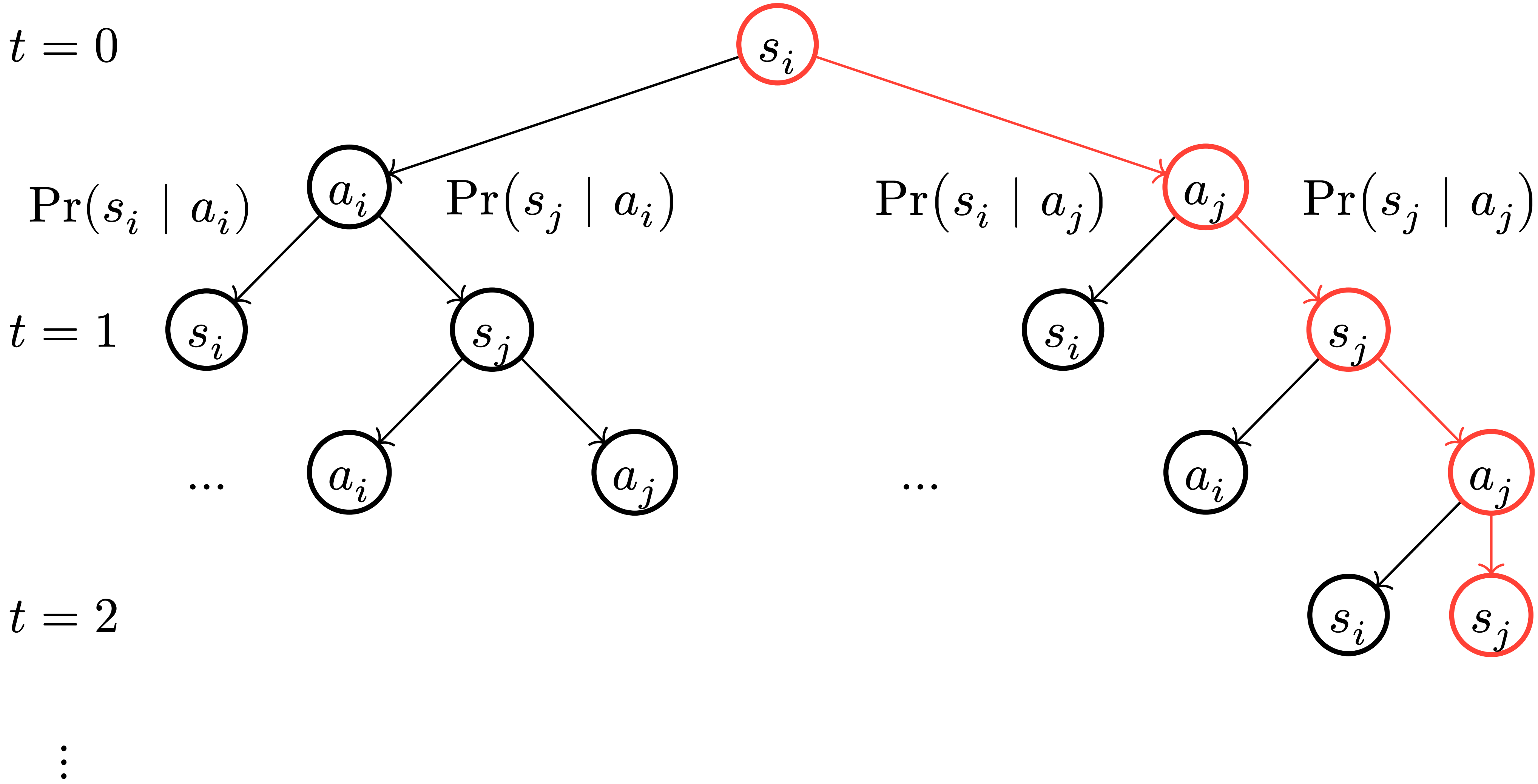
$$\Pr(s_i \mid s_j, a_j) = 0.1; \;\; \Pr(s_j \mid s_i, a_j) = 0.9$$

Trajectory Optimization



Trajectory Optimization



Trajectory Optimization

$$J(a_0, a_1, \ldots) = \mathbb{E}[G \mid s_0, a_0, a_1, \ldots] = \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

This expression gives us the **expected discounted return** J

Question: How can we maximize J?

$$\arg\max_{a_0, a_1, \ldots \in A} J(a_0, a_1, \ldots) = \arg\max_{a_0, a_1, \ldots \in A} \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

$$\underset{a_0, a_1, \ldots \in A}{\arg\max}\ J(a_0, a_1, \ldots) = \underset{a_0, a_1, \ldots \in A}{\arg\max} \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

In RL, we call this **trajectory optimization**

Question: What do we need to know about the problem to use trajectory optimization?

Answer:

- Must know the reward function R
- Must know the state transition function $T = \Pr(s_{t+1} \mid s_t, a_t)$

Trajectory Optimization

$$\underset{a_0, a_1, \ldots \in A}{\arg\max}\ J(a_0, a_1, \ldots) = \underset{a_0, a_1, \ldots \in A}{\arg\max} \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

Approach: Try all possible actions sequences and pick the one with the best return

Question: Any problem?

Answer: $a_0, a_1, \ldots$ is infinite, how can we try infinitely many actions?

We can't

Trajectory Optimization

$$\underset{a_0, a_1, \ldots \in A}{\arg\max}\, J(a_0, a_1, \ldots) = \underset{a_0, a_1, \ldots \in A}{\arg\max} \sum_{t=0}^{\infty} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

In trajectory optimization, we must introduce a **horizon** n

$$\underset{a_0, a_1, \ldots, a_n \in A}{\arg\max}\; J(a_0, a_1, \ldots, a_n) =$$

$$\underset{a_0, a_1, \ldots a_n \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

Now, we can perform a search/optimization

Trajectory Optimization

$$\underset{a_0, ..., a_n \in A}{\arg\max} \; J(a_0, ..., a_n) = \underset{a_0, ... a_n, \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

Question: What are the consequences of using a finite horizon n?

Answer:

- Our model can only consider rewards n steps into the future
- Actions will **not** be optimal

In certain cases, we do not care much about the distant future

Trajectory Optimization

$$\underset{a_0, \ldots, a_n \in A}{\arg\max}\ J(a_0, \ldots, a_n) = \underset{a_0, \ldots a_n, \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

For example, we often use trajectory optimization to avoid crashes

If we can avoid any crash in 10 actions, then $n = 10$ is enough for us

One application of trajectory optimization:

https://www.youtube.com/watch?v=6qj3EfRTtkE

Trajectory Optimization

$$\underset{a_0,...,a_n \in A}{\arg\max} \; J(a_0, ..., a_n) = \underset{a_0,...a_n, \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

How do we optimize J in practice?

- Try all possible sequences $a_0, ..., a_n$, pick the best one
- Randomly pick some sequences, pick the best one
- Use gradient descent to find $a_0, ..., a_n$
  - **Note:** The state transition function and reward function must be differentiable

Policies

Policies

With trajectory optimization, we plan all of our actions at once

$$\underset{a_0, a_1, \ldots \in A}{\arg\max} J(a_0, a_1, \ldots) = \underset{a_0, a_1, \ldots a_n \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

It is difficult to think about many actions and states at once

Policies

To simplify, we introduce the **policy** π with parameters $\theta \in \Theta$

$$\pi : S \times \Theta \mapsto \Delta A$$

$$\Pr(a \mid s; \theta)$$

It maps a current state to a distribution of actions

The policy determines the behavior of our agent, it is the “brain”

Policies

$$J(a_0, a_1, ...) = \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

We can rewrite the expected return using the policy π and parameters θ

$$J(\theta) = \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t) \cdot \pi(a_t \mid s_t; \theta)$$

Policies

$$\underset{a_0, a_1, \ldots \in A}{\arg\max}\ J(a_0, a_1, \ldots) = \underset{a_0, a_1, \ldots a_n \in A}{\arg\max} \sum_{t=0}^{n} \gamma^t \sum_{s_{t+1} \in S} R(s_{t+1}) \cdot \Pr(s_{t+1} \mid s_t, a_t)$$

In controls and robotics, we call this **model-predictive control** (MPC)

Where do we use trajectory optimization/MPC?

https://www.youtube.com/watch?v=Kf9WDqYKYQQ

Policies

Trajectory optimization is expensive

The optimization process requires us to simulate thousands/millions of possible trajectories

However, as GPUs get faster these methods become more interesting

TODO: Visualization

TODO: What is the state transition function

Value Functions