Algorithms

CISC 7404 - Decision Making

Steven Morad

University of Macau

Quiz results on moodle

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If you have no score, come see me

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Do not forget individual participation grade!

Review

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Diffusion models

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https://arxiv.org/pdf/2006.11239

Algorithms

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The policy is the "brain" of the agent

It makes decisions for the agent

Algorithms

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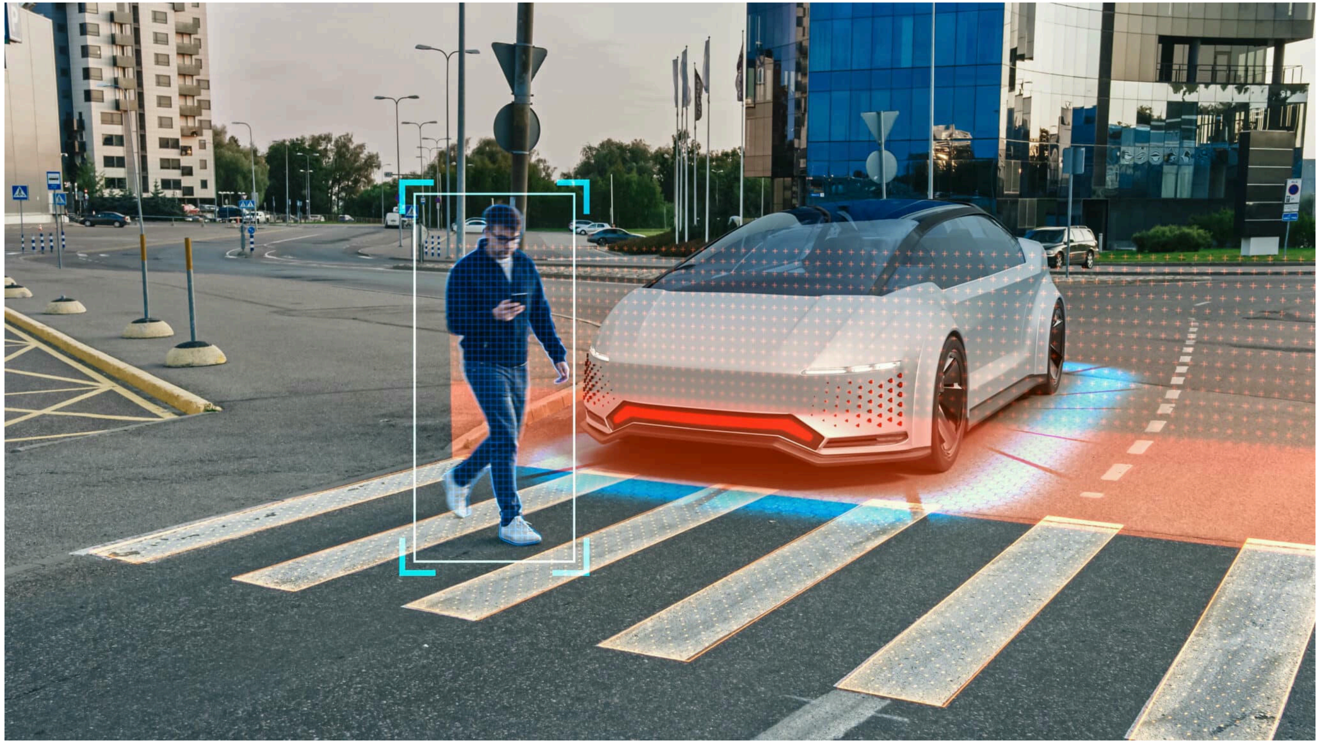

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The policy is guaranteed to maximize the discounted return

Algorithms

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These ideas appear in classical robotics and control theory

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https://www.youtube.com/watch?v=6qj3EfRTtkE

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Critical part of Alpha-* methods (AlphaGo, AlphaStar, AlphaZero)

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We want to find τ that provides the greatest discounted return

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To understand what is hiding, let us examine the reward function

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Cannot know s_{t+1} with certainty, only know the distribution!

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Question: Ok, now what is the definition of R?

Answer:

$$R : S \mapsto \mathbb{R}$$

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We should write it as $\mathcal{R} : S \mapsto \mathbb{R}$

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Question: Why do we like to take the expectation of random variables?

Answer: It maps random processes to something we can maximize

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Random variable conditioned on s_t, a_t

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Note: Cannot directly maximize $\mathcal{R}$ because s_{t+1} is random

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$$\arg\max_{a \in \{1...k\}} \mathbb{E}[\mathcal{X}_a]$$

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Let us turn this equation into a policy

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It will always act to maximize the expected reward

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Example: Online advertising, show users ads so they buy products

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This will create the best ad creator possible!

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We have one more thing to do

Trajectory Optimization

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Answer: It is more tricky

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For $\mathcal{R}(s_{n+1})$ we need an expression for $\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots)$

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$$\Pr(s_{n+1} \mid s_0, a_0, a_1, ..., a_{n-1}) = \sum_{s_1, s_2, ... s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

Question: How do we find $\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots)$?

Answer: In lecture 3 we found the probability of a future state in a Markov process

$$\Pr(s_{n+1} \mid s_0) = \sum_{s_1, s_2, \ldots s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t)$$

To extend to MDP, just need to include the actions!

$$\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots, a_{n-1}) = \sum_{s_1, s_2, \ldots s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

This predicts the future states of an MDP

Trajectory Optimization

Combine s_{n+1} distribution with $\mathcal{R}$ to predict future rewards

Trajectory Optimization

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Trajectory Optimization

$$\mathbb{E}[\mathcal{R}(s_{n+1}) \mid s_0, a_0, a_1, ..., a_n] = \sum_{s_{n+1} \in S} R(s_{n+1}) \sum_{s_1, ..., s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

Trajectory Optimization

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What does each piece mean?

Trajectory Optimization

s_{n+1} Distribution

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Trajectory Optimization

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What does each piece mean?

This is only for a single reward, must plug back into discounted return

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0, a_0, ..., a_t]$$

Trajectory Optimization

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] =$$

Trajectory Optimization

$$\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, ...] = \qquad \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0]$$

Trajectory Optimization

$$\begin{aligned}\mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \ldots] = & \quad \mathbb{E}[\mathcal{R}(s_1) \mid s_0, a_0] \\ + \gamma & \quad \mathbb{E}[\mathcal{R}(s_2) \mid s_0, a_0, a_1]\end{aligned}$$

Trajectory Optimization

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Trajectory Optimization

To maximize the return, we take the arg max over possible actions

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And turn it into a policy

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And turn it into a policy

$$\pi(a_t \mid s_t; \theta_\pi) = \begin{cases} 1 \text{ if } a_t = \arg\max_{a_t \in A} \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \dots] \\ 0 \text{ otherwise} \end{cases}$$

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We have a name for this policy in control theory

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Question: Anyone know what we call it?

Trajectory Optimization

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Question: Anyone know what we call it?

Answer: Model Predictive Control (MPC) or Receding Horizon Control

Trajectory Optimization

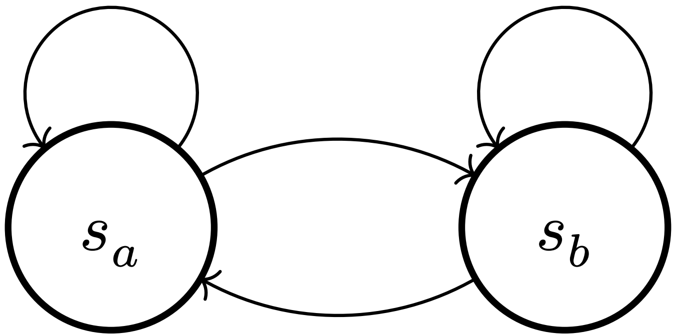
There is a lot of math behind trajectory optimization/MPC

Trajectory Optimization

There is a lot of math behind trajectory optimization/MPC

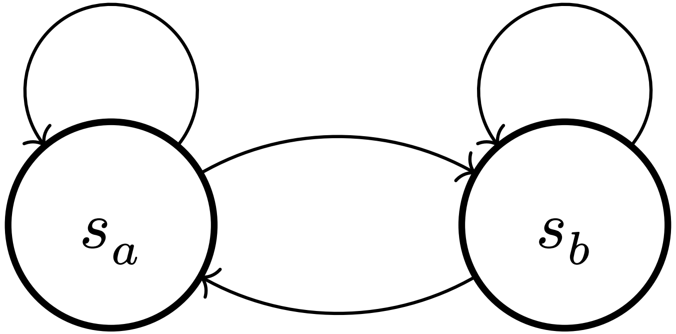
Let us do a visual example to help you understand

Trajectory Optimization



$\mathcal{R}(s_a) = 0$ $\qquad$ $\mathcal{R}(s_b) = 1$

Trajectory Optimization



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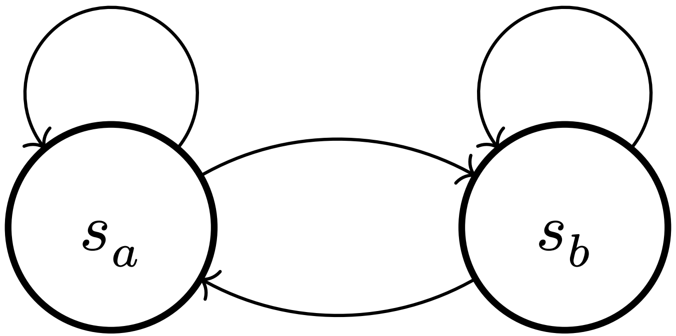
$$S = \{s_a, s_b\} \quad A = \{a_a, a_b\}$$

Trajectory Optimization



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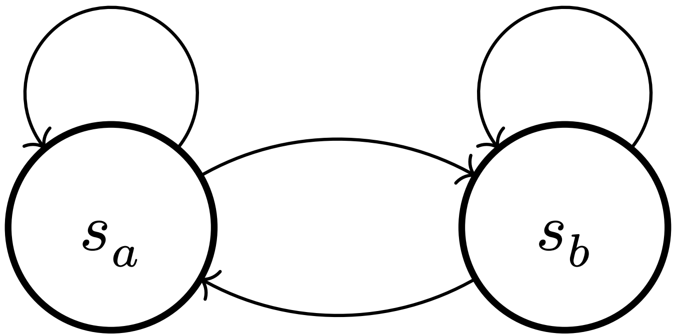
Trajectory Optimization



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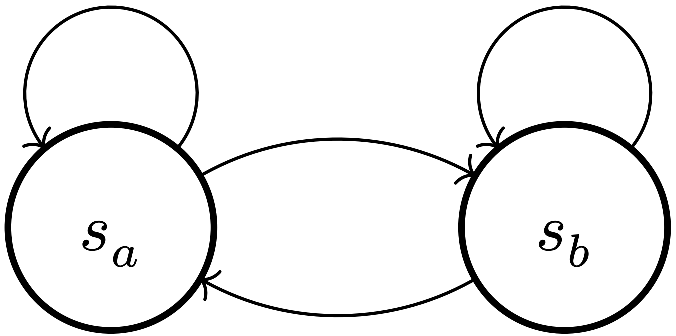
Trajectory Optimization



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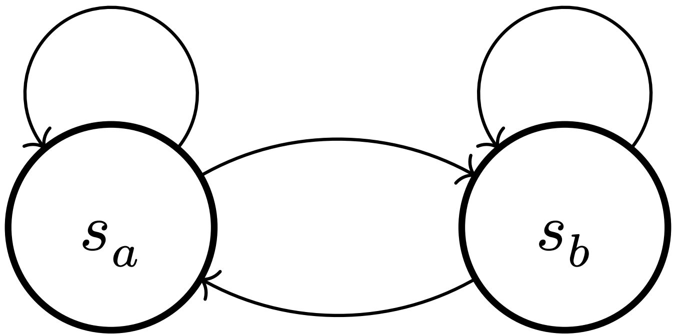
Trajectory Optimization



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$$\Pr(s_a \mid s_b, a_b) = 0.1; \;\; \Pr(s_b \mid s_a, a_b) = 0.9$$

Trajectory Optimization

We can build this into a decision tree using trajectory optimization

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The root node of the tree corresponds to s_0

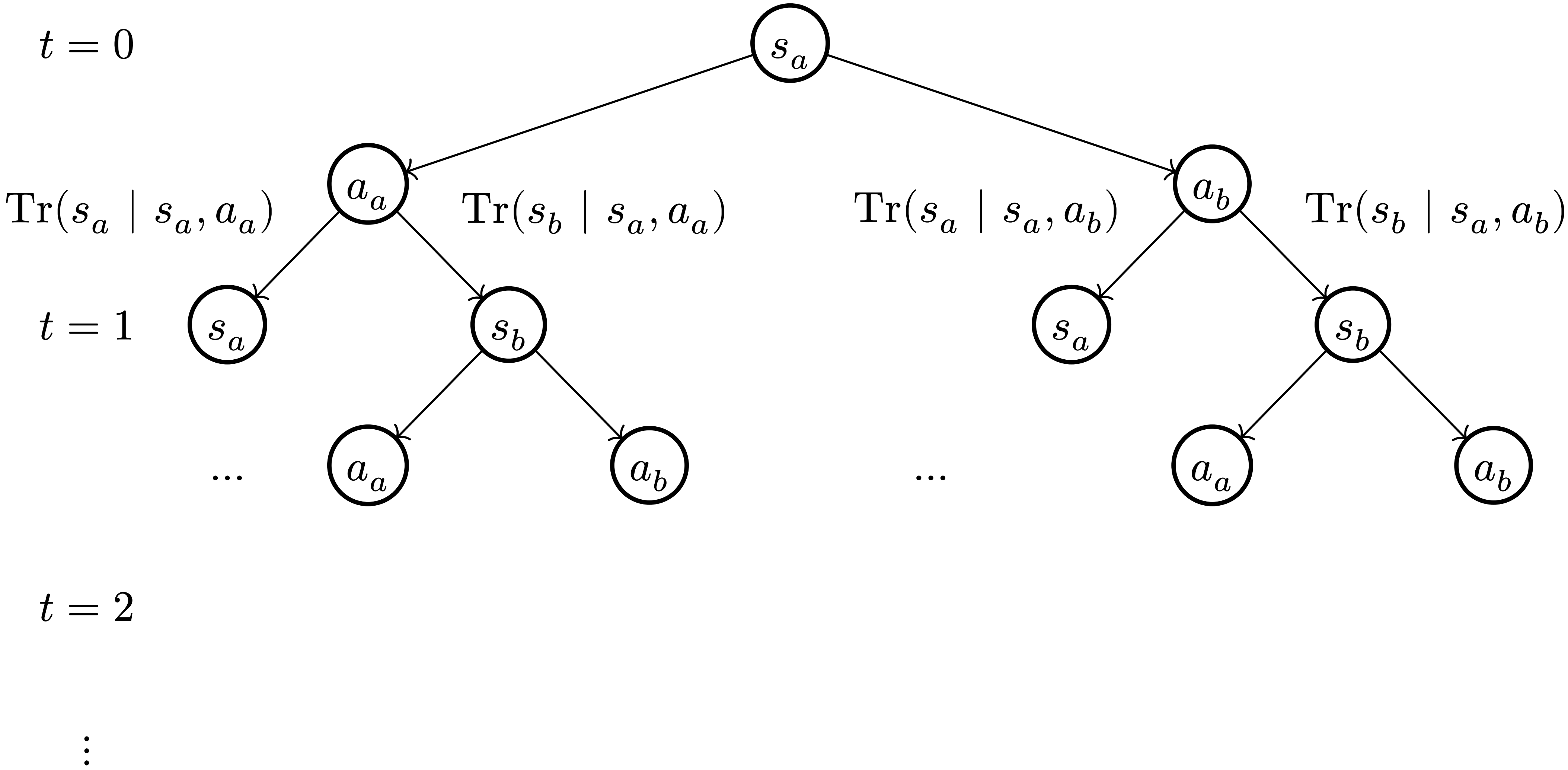
Trajectory Optimization

We can build this into a decision tree using trajectory optimization

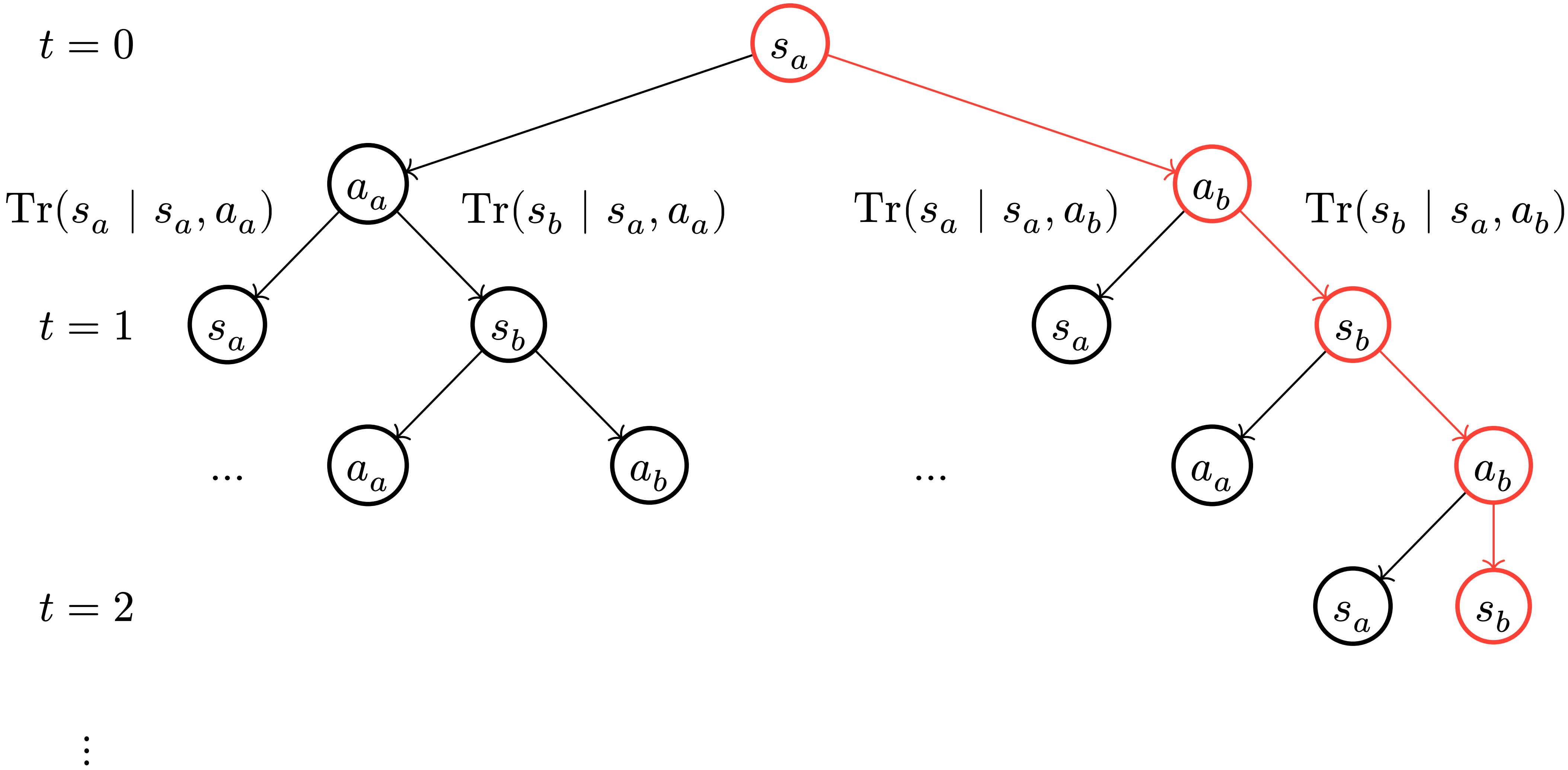
The root node of the tree corresponds to s_0

Each level of the tree enumerates possible outcomes

Trajectory Optimization



Trajectory Optimization



Trajectory Optimization

Question: How many nodes does our tree have?

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Answer: $O(|S| \cdot |A|)^\infty$

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Answer: Do not have the memory/compute to evaluate all possibilities

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Answer: $O(|S| \cdot |A|)^{\infty}$

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We have some tricks to make this tractable

Trajectory Optimization

$$\underset{a_0, a_1, \ldots \in A}{\arg\max}\ \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \ldots] = \underset{a_0, a_1, \ldots}{\arg\max} \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[\mathcal{R}(s_{t+1}) \mid s_0, a_0, \ldots, a_t]$$

Trick 1: Introduce a **horizon** n

Trajectory Optimization

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Now we can limit computation to $O(|S| \cdot |A|)^n$

Trajectory Optimization

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Question: Drawback?

Trajectory Optimization

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Now we can limit computation to $O(|S| \cdot |A|)^n$

Question: Drawback?

Answer: We no longer consider the infinite future, our agent may get greedy and be trapped

Trajectory Optimization

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Trick 2: Only simulate j actions and k states

Trajectory Optimization

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Trick 2: Only simulate j actions and k states

$$\underset{a_0,\ldots,a_n \sim A^j}{\arg\max}\; \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}_n) \mid s_0, a_0, \ldots, a_n] = \underset{a_0,\ldots,a_n \sim A^j}{\arg\max} \sum_{t=0}^{n} \gamma^t \hat{\mathbb{E}}_k[\mathcal{R}(s_{t+1}) \mid s_0, a_0, \ldots, a_n]$$

Now, computation is $O(j \cdot k)^n$

Trajectory Optimization

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Trajectory Optimization

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Trick 2: Only simulate j actions and k states

$$\underset{a_0, \dots, a_n \sim A^j}{\arg\max}\ \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}_n) \mid s_0, a_0, \dots, a_n] = \underset{a_0, \dots, a_n \sim A^j}{\arg\max} \sum_{t=0}^{n} \gamma^t \hat{\mathbb{E}}_k[\mathcal{R}(s_{t+1}) \mid s_0, a_0, \dots, a_n]$$

Now, computation is $O(j \cdot k)^n$

Question: Drawbacks?

Answer: Optimal action may not be sampled, results in less-optimal trajectory

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I plan to release assignment 1 next lecture