# Algorithms

## CISC 7404 - Decision Making

Steven Morad

University of Macau

# Quiz results on moodle

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If you have no score, come see me

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Do not forget individual participation grade!

# Review

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Diffusion models

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https://arxiv.org/pdf/2006.11239

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The policy is the “brain” of the agent

It makes decisions for the agent

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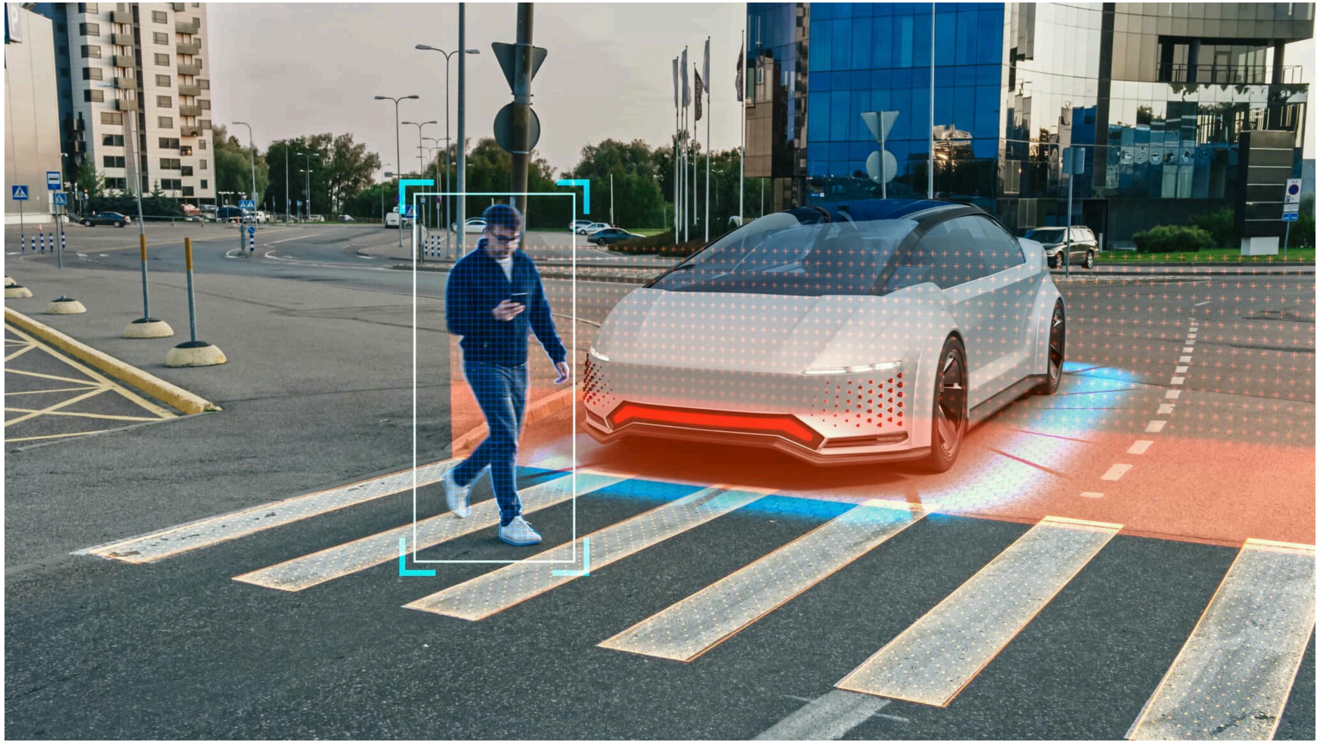

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The policy is guaranteed to maximize the discounted return

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https://www.youtube.com/watch?v=6qj3EfRTtkE

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Critical part of Alpha-* methods (AlphaGo, AlphaStar, AlphaZero)

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We want to find $\tau$ that provides the greatest discounted return

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To understand what is hiding, let us examine the reward function

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Cannot know $s_{t+1}$ with certainty, only know the distribution!

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**Question:** Ok, now what is the definition of $R$?

**Answer:**

$$R : S \mapsto \mathbb{R}$$

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We should write it as $\mathcal{R} : S \mapsto \mathbb{R}$

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**Question:** Why do we like to take the expectation of random variables?

**Answer:** It maps complex random processes to a single value, which is much easier to work with

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Random variable conditioned on $s_t, a_t$

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$$\underset{a \in \{1 \ldots k\}}{\arg\max}\, \mathbb{E}[\mathcal{X}_a]$$

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This policy will always act to maximize the expected reward

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We have one more thing to do

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**Answer:** Agent picks actions, optimize over actions to maximize $G$

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**Answer:** It is more tricky

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For $\mathcal{R}(s_{n+1})$ we need an expression for $\Pr(s_{n+1} \mid s_0, a_0, a_1, \ldots)$

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This predicts the future states of an MDP

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# Trajectory Optimization

$$\mathbb{E}[\mathcal{R}(s_{n+1}) \mid s_0, a_0, a_1, ..., a_n] = \sum_{s_{n+1} \in S} R(s_{n+1}) \sum_{s_1, ..., s_n \in S} \prod_{t=0}^{n} \Pr(s_{t+1} \mid s_t, a_t)$$

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And turn it into a policy

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$$\pi(a_t \mid s_t; \theta) = \begin{cases} 1 \text{ if } a_t = \arg\max_{a_t \in A} \mathbb{E}[\mathcal{G}(\boldsymbol{\tau}) \mid s_0, a_0, a_1, \dots] \\ 0 \text{ otherwise} \end{cases}$$

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We have a name for this policy in control theory

**Question:** Anyone know what we call it?

**Answer:** Model Predictive Control (MPC) or Receding Horizon Control

# Trajectory Optimization

MPC is arguably the best practical method for control

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Most robots and autonomous vehicles today use some form of MPC

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Example application of trajectory optimization/MPC:

https://www.youtube.com/watch?v=bjlT-6KVQ7U

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There is a lot of math behind trajectory optimization/MPC

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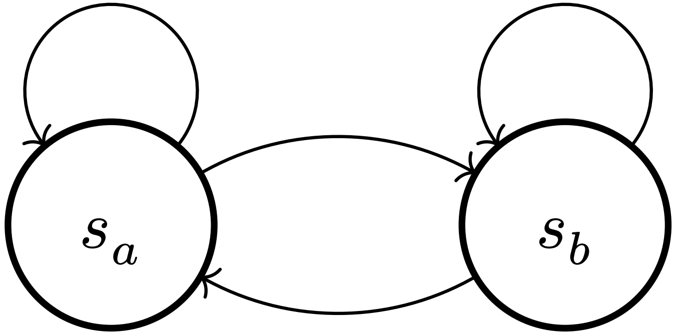
Let us do a visual example to help you understand

# Trajectory Optimization




$R(s_a) = 0$ $\qquad$ $R(s_b) = 1$

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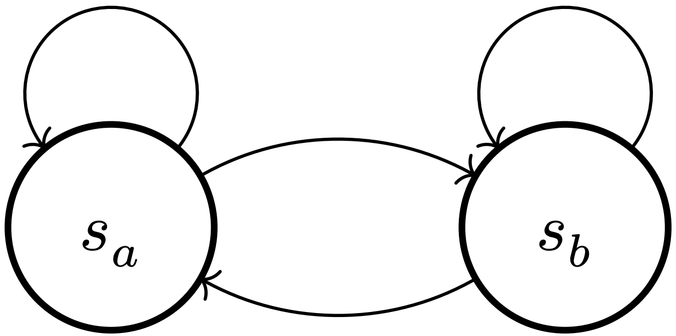


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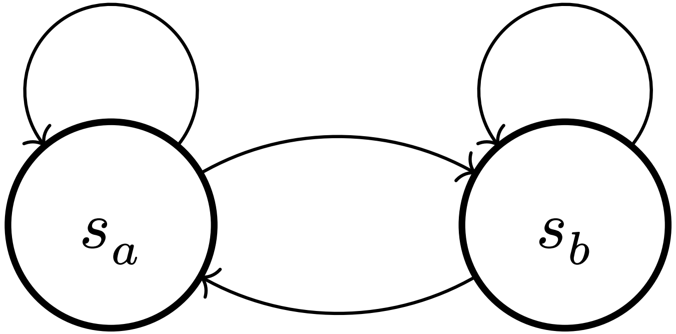


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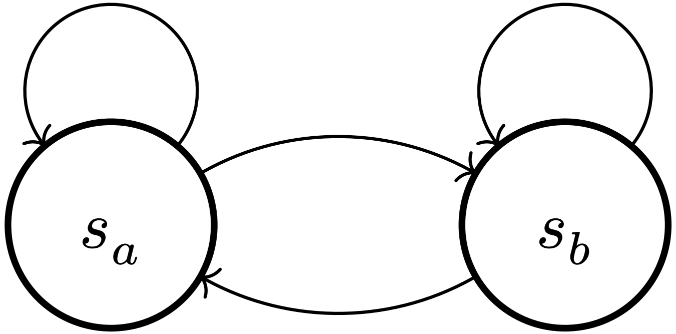


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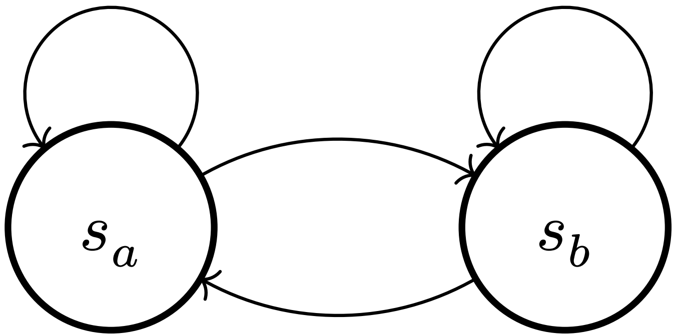


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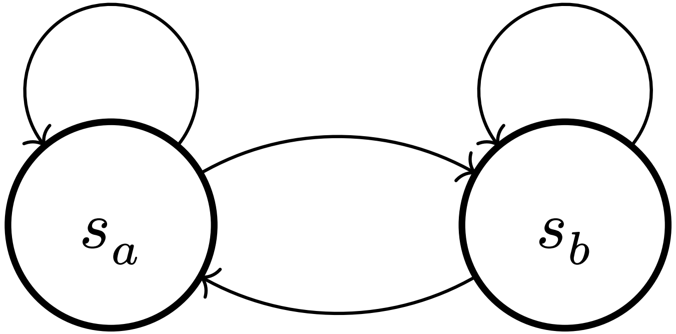


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We can build this into a decision tree

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Each level of the tree enumerates possible outcomes

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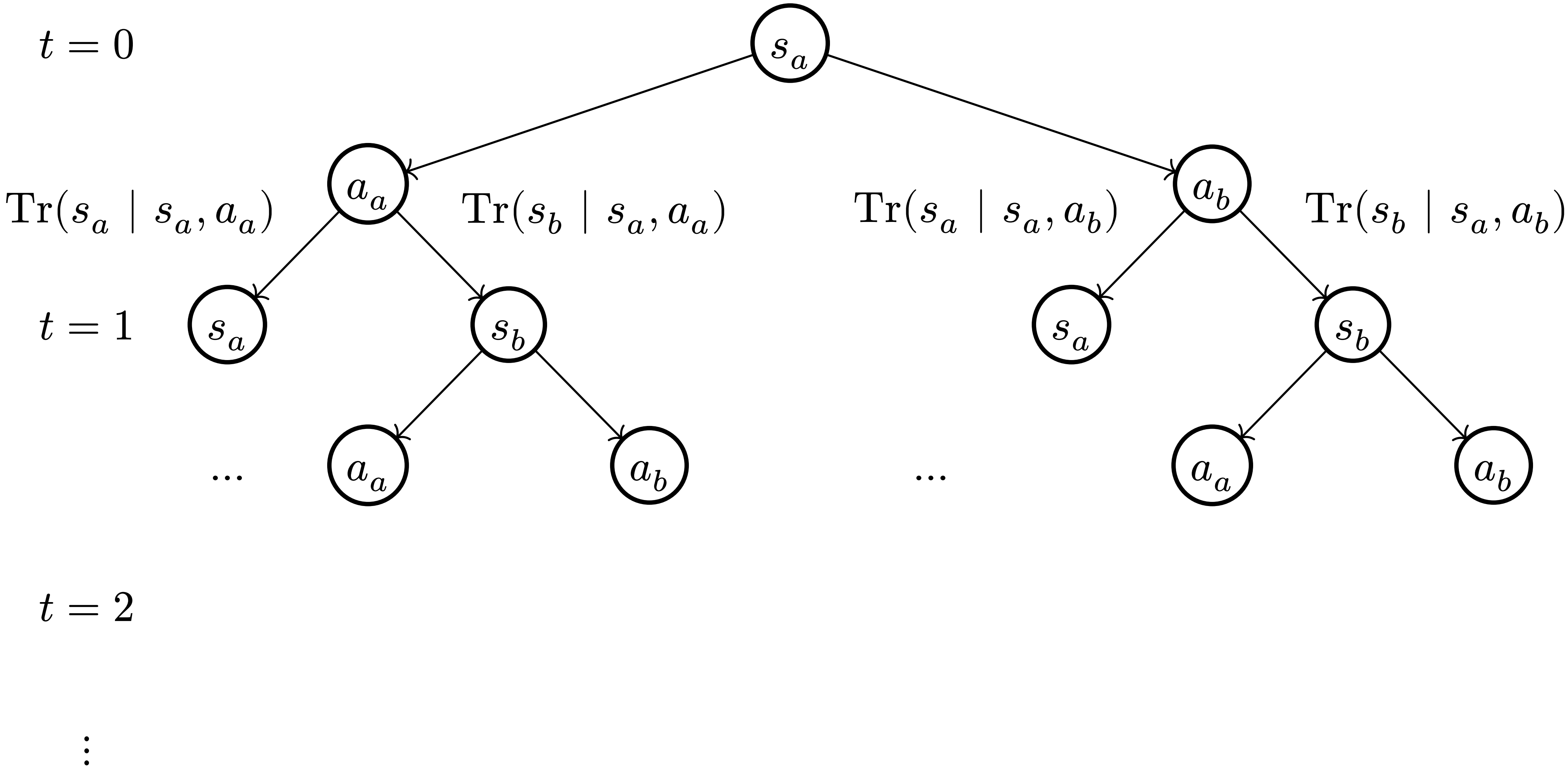

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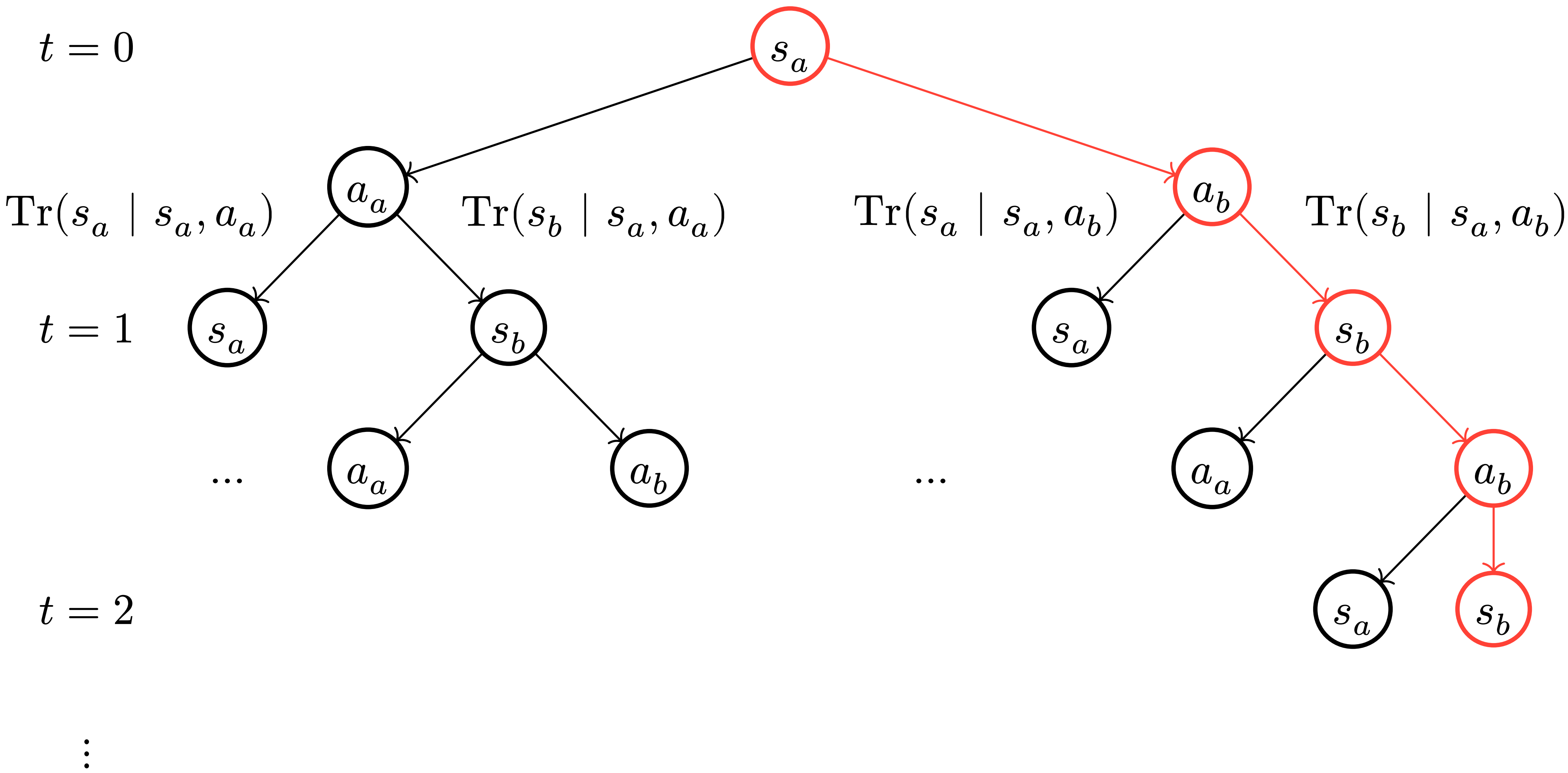

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**Answer:** We no longer consider the infinite future, our agent may get greedy and be trapped

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**Question:** Drawbacks?

**Answer:** Optimal action may not be sampled, results in less-optimal trajectory

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Often because it is limited by compute

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Next time, we will see what happens when we don't have a model