# Homework 3, MAT 258A

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**1. KKT conditions at $(4,2)$ show non-optimality.** Write the constraints in the form

$$g_1(x) = x_1 - x_2 - 2 \le 0, \qquad g_2(x) = -x_1 \le 0, \qquad g_3(x) = -x_2 \le 0.$$

For a maximisation problem the Lagrangian is $L(x,\lambda) = f(x) - \sum_{i=1}^{3} \lambda_i g_i(x)$ with $\lambda_i \ge 0$. The objective is $f(x) = x_1/(x_2+1)$ and its gradient $\nabla f(x) = \big((x_2+1)^{-1},\, -x_1\,(x_2+1)^{-2}\big)$. At the candidate point $(4,2)$ the active set is $g_1 = 0$; $g_2, g_3 < 0$. Stationarity therefore requires

$$\nabla f(4,2) - \lambda_1 \nabla g_1(4,2) = 0 \Longrightarrow (1/3,\, -4/9) - \lambda_1(1,-1) = 0,$$

which yields $\lambda_1 = 1/3$ from the first component and $\lambda_1 = -4/9$ from the second—a contradiction. Hence $(4,2)$ violates KKT and cannot be optimal.

**2. A point satisfying the KKT conditions.** Because the denominator $x_2 + 1$ should be as small as possible while the numerator $x_1$ should be as large as possible, optimality must occur on the *boundary* $x_1 - x_2 = 2$ with $x_2$ minimal, i.e. $x_2 = 0,\ x_1 = 2$.

At $(2,0)$ we have $g_1 = g_3 = 0$, $g_2 < 0$. With multipliers $\lambda_1 = \lambda_3 = 1,\ \lambda_2 = 0$,

$$\nabla f(2,0) - \lambda_1 \nabla g_1 - \lambda_3 \nabla g_3 = (1,-2) - (1,-1) - (0,-1) = (0,0),$$

and complementary slackness holds. Thus $(2,0)$ satisfies all KKT conditions.

**3. The problem is *not* convex.** A maximisation problem is convex only when the objective is *concave* on a convex feasible set. The feasible region here is a simplex and is therefore convex; the issue is the objective $f(x) = x_1/(x_2+1)$.

Take $A = (0,2)$ and $B = (2,0)$, both feasible. Their midpoint is $C = (1,1)$. Now

$$f(A) = \frac{0}{3} = 0, \quad f(B) = \frac{2}{1} = 2, \quad f(C) = \frac{1}{2} = 0.5.$$

Concavity would demand $f(C) \ \ge \ \frac{1}{2} f(A) + \frac{1}{2} f(B) = 1$. But $f(C) = 0.5 < 1$, violating the concavity inequality. Hence $f$ is *not* concave on the feasible region, so the optimisation problem is *non-convex.*

**4. Optimal solution.** On the active face $x_1 - x_2 = 2$ we have $f = (x_2+2)/(x_2+1) = 1 + 1/(x_2+1)$, which is strictly decreasing in $x_2 \ge 0$. Hence the maximum is attained at the minimal admissible $x_2$, namely $x_2^\star = 0$, with $x_1^\star = 2$. Therefore

$$\boxed{x^\star = (2,0),\ f^\star = 2}.$$

(Joint work with Ian Gallagher)