Homework 6, MAT 258A

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Problem 1 (3)
For $C = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, let

$$f(x) = (x_1 - 1)^2 + x_2.$$

Illustrate the contours, and—using the Karush–Kuhn–Tucker (KKT) conditions—compute the set of minimisers

$$C_f \ := \ \arg\min_{x \in C} f(x).$$

Solution

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We minimize $f(x) = (x_1 - 1)^2 + x_2$ over the convex set

$$C = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0\}.$$

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = (x_1 - 1)^2 + x_2 - \lambda(x_1 + x_2 - 1) - \mu_1 x_1 - \mu_2 x_2,$$

with KKT conditions:

$$\begin{cases} 2(x_1 - 1) - \lambda - \mu_1 = 0, \\ 1 - \lambda - \mu_2 = 0, \\ x_1 + x_2 = 1, \quad x_1, x_2 \geq 0, \\ \mu_1 x_1 = 0, \quad \mu_2 x_2 = 0, \quad \mu_1, \mu_2 \geq 0. \end{cases}$$

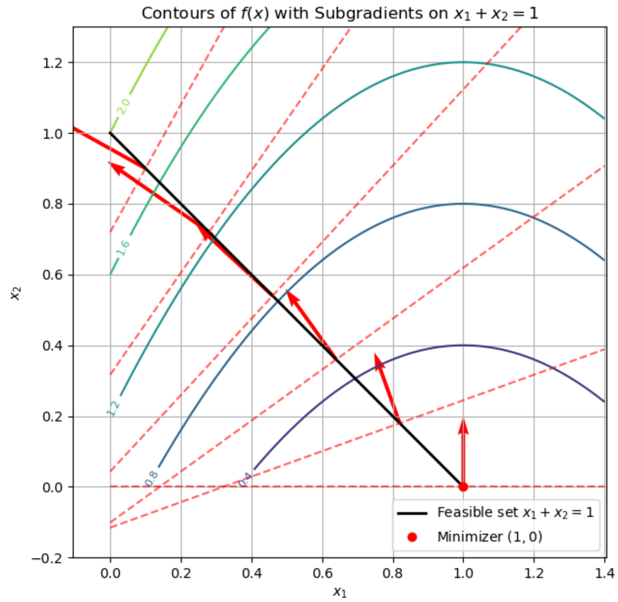
Case 1: $x_2 = 0 \Rightarrow x_1 = 1$. Then $\mu_2 \cdot 0 = 0$, and from the stationarity conditions:

$$2(1 - 1) - \lambda - \mu_1 = 0 \Rightarrow \lambda = -\mu_1, \quad 1 - \lambda - \mu_2 = 0.$$

Setting $\mu_1 = 0 \Rightarrow \lambda = 0,\ \mu_2 = 1$. All KKT conditions are satisfied.

Other cases (e.g., $x_1 = 0$) lead to higher objective values. Therefore, the unique minimizer is

$$C_f = \{(1, 0)\}.$$


Contours of $f(x)$ with Subgradients on $x_1 + x_2 = 1$
Feasible set $x_1 + x_2 = 1$
Minimizer $(1, 0)$
x_1
x_2
2.0
1.6
1.2
0.8
0.4