Homework 4, MAT 258A

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May 5, 2025

We rewrite the task as optimisation problem

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) := (x-1)^2 + (y-2)^2 \quad \text{s.t.} \quad g(x,y) := y - \tfrac{1}{5}\,(x-1)^2 = 0.$$

(a) KKT points, LICQ and optimality. The Lagrangian is

$$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda\, g(x,y) = (x-1)^2 + (y-2)^2 + \lambda\Big(y - \tfrac{1}{5}(x-1)^2\Big).$$

Stationarity gives

$$\partial_x\mathcal{L}:\ 2(x-1) - \tfrac{2}{5}\lambda(x-1) = 0, \qquad \partial_y\mathcal{L}:\ 2(y-2) + \lambda = 0.$$

Hence

$$(x-1)\big(2 - \tfrac{2}{5}\lambda\big) = 0, \qquad y = 2 - \tfrac{\lambda}{2}.$$

Case 1: $x = 1$. Then $y = 0$ from the constraint, and $\lambda = 4$. *Case 2:* $\lambda = 5$. Then $y = -\frac{1}{2}$, but the constraint would require $y = -\frac{1}{2} = (1/5)(x-1)^2$, i.e. $(x-1)^2 = -\frac{5}{2}$, impossible.

Thus the only KKT point is $(x^\star, y^\star) = (1,0)$ with $\lambda^\star = 4$.

Because $\nabla g(1,0) = \big(0,1\big) \neq 0$, the gradient of the active constraint is linearly independent, so the LICQ holds. Since this is the *only* feasible stationary point, it must be the unique candidate for optimality.

Second–order test (for completeness). The Hessian of the Lagrangian is $\nabla^2_{xx}\mathcal{L} = 2I + \lambda\begin{pmatrix} -\frac{2}{5} & 0 \\ 0 & 0 \end{pmatrix}$. At $(1,0,\lambda^\star)$ this is $\operatorname{diag}\big(\frac{2}{5}, 2\big)$. The tangent space of the constraint is $T = \{v \in \mathbb{R}^2 : \nabla g^\top v = 0\} = \{(v_x, 0)\}$. For every non-zero $v \in T$, $v^\top \nabla^2_{xx}\mathcal{L}\, v = \frac{2}{5}\, v_x^2 > 0$. Hence the *second–order sufficient* condition holds and $(1,0)$ is a strict local (therefore global) minimiser.

(b) Lagrange dual problem and SOSC. Define the dual function $q(\lambda) = \inf_{x,y}\mathcal{L}(x,y,\lambda)$. From the stationarity equations we saw that for every $\lambda \neq 5$ the minimiser is $x = 1,\ y = 2 - \lambda/2$, yielding

$$q(\lambda) = 2\lambda - \tfrac{1}{4}\lambda^2, \qquad \lambda \in \mathbb{R}.$$

(The formula also gives $q(5) = \frac{15}{4}$.) Thus the dual problem is

$$\max_{\lambda\in\mathbb{R}} 2\lambda - \tfrac{1}{4}\lambda^2,$$

a concave quadratic attaining its maximum at $\lambda^\star = 4$ with value $q(4) = 4$, which equals the primal optimum $f(1,0) = 4$; strong duality holds.

(c) Eliminating the constraint. Setting $y = \frac{1}{5}(x-1)^2$ and substituting gives the single-variable problem

$$\min_{x \in \mathbb{R}} h(x) := (x-1)^2 + \left(\tfrac{1}{5}(x-1)^2 - 2\right)^2.$$

Writing $s := (x-1)^2 \geq 0$ we obtain $h(s) = \frac{1}{25}s^2 + \frac{1}{5}s + 4$, whose minimum on $[0, \infty)$ occurs at $s^\star = 0$. Hence $x^\star = 1$, so the unconstrained reformulation recovers the same solution.

(d) KKT without LICQ. Consider

$$\min_{x \in \mathbb{R}} x \quad \text{s.t.} \quad \underbrace{x}_{g_1(x)} = 0, \; \underbrace{2x}_{g_2(x)} = 0.$$

Both constraints are active at $x^\star = 0$ and g_1, g_2 are *linear*, so KKT holds with multipliers $\lambda_1^\star = 1$, $\lambda_2^\star = -1$. However, $\nabla g_1(0) = (1)$ and $\nabla g_2(0) = (2)$ are linearly *dependent*; the LICQ fails even though all KKT conditions are met.