Appendix

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This is a formalization of the new results of Sections A.1–A.3 of my paper *Mixed ℓ-adic complexes for schemes over number fields* (`https://arxiv.org/pdf/1806.03096.pdf`), excluding the example of filtered derived categories and admitting all the results that can be found in the literature.

Contents

1 Section A.1: filtered triangulated categories

In Appendix A of [1], Beilinson introduced filtered triangulated categories over triangulated categories, that have all the abstract properties of filtered derived categories, and generalized the properties of filtered derived categories to this more general setting. We review his definition and results. Note that Section 6 of Schnürer's paper [4] gives more detailed proofs of many of the results of Appendix A of [1].

This section corresponds to the Lean file `Filtered_no_proof.lean`.

1.1 Definition A.1.1

When formalizing the definition of filtered triangulated categories, we run into our first issue immediately. Beilinson defines an filtered triangulated category as a triangulated category DF with a second shift functor s, which should be a triangulated self-equivalence (plus some extra structure). In particular, the functor s needs to commute with the shifts coming from the triangulated category structure. The easiest way to encode all the necessary compatibilities is actually to put a shift by $\mathbb{Z} \times \mathbb{Z}$ on our category DF, where the shift by the first factor will be part of the triangulated structure and the shift by the second factor will give the functor s. The beginning of the project is devoted to setting this shift structure. In particular, we chose to make the default shift by $\mathbb{Z}$ on the category DF to be the shift by the first factor, and to introduce a type synonym 'FilteredShift C' which carries a shift by $\mathbb{Z}$ encoding the functor s.

The following is part of Definition A.1 of [1]. (Almost: we actually define filtered pretriangulated categories in the Lean code, and a filtered triangulated catgeory is a filtered pretriangulated category that is also triangulated.)

Definition 1.1.1. A *filtered triangulated category* is the data of:

- a triangulated category DF;
- two full triangulated subcategories $\mathrm{DF}(\leq 0)$, $\mathrm{DF}(\geq 0)$ of DF that are stable by isomorphisms;
- a triangulated self-equivalence $s\colon \mathrm{DF} \longrightarrow \mathrm{DF}$ (called *shift of filtration*);
- a morphism of functors $\alpha\colon \mathrm{id}_{\mathrm{DF}} \longrightarrow s$;

satisfying the following conditions, where, for every $n \in \mathbb{Z}$, we set

$$\mathrm{DF}(\leq n) = s^n \,\mathrm{DF}(\leq 0) \text{ and } \mathrm{DF}(\geq n) = s^n \,\mathrm{DF}(\geq 0):$$

(i) We have $\mathrm{DF}(\geq 1) \subset \mathrm{DF}(\geq 0)$, $\mathrm{DF}(\leq 1) \supset \mathrm{DF}(\leq 0)$ and

$$\mathrm{DF} = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\leq n) = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\geq n).$$

(ii) For any $X \in \operatorname{Ob}\mathrm{DF}$, we have $\alpha_X = s(\alpha_{s^{-1}(X)})$.

(iii) For any $X \in \operatorname{Ob}\mathrm{DF}(\geq 1)$ and $Y \in \operatorname{Ob}\mathrm{DF}(\leq 0)$, we have $\operatorname{Hom}(X, Y) = 0$, and the maps $\operatorname{Hom}(s(Y), X) \longrightarrow \operatorname{Hom}(Y, X) \longrightarrow \operatorname{Hom}(Y, s^{-1}(X))$ induced by α_Y and $\alpha_{s^{-1}(X)}$ are bijective.

(iv) For every $X \in \operatorname{Ob}\mathrm{DF}$, there exists a distinguished triangle $A \longrightarrow X \longrightarrow B \xrightarrow{+1}$ with A in $\mathrm{DF}(\geq 1)$ and B in $\mathrm{DF}(\leq 0)$.

We need to setup extra definitions (for example, classes like 'IsLE' and 'IsGE'), then prove a lot of "easy" lemmas in order to make this definition usable. For example, there is a lemma `CategoryTheory.FilteredTriangulated.exists_triangle` saying that, for every X and DF and every $n \in \mathbb{Z}$, there exists a distinguished triangle $A \longrightarrow X \longrightarrow B \longrightarrow A[1]$ with A in $\mathrm{DF}(\geq n+1)$ and B in $\mathrm{DF}(\leq n)$.

Then we define filtered triangulated functors. In Lean, we first need to ask for a 'CommShift' structure by $\mathbb{Z} \times \mathbb{Z}$, and then we require three properties: preservation of objects that are ≤ 0, preservation of objects that are ≥ 0, and commutation with the natural transformation α.

Definition 1.1.2. If DF and DF' are filtered triangulated categories, a *filtered triangulated functor* from DF to DF' is the data of a triangulated functor $T\colon \mathrm{DF} \longrightarrow \mathrm{DF}'$ and a natural isomorphism $s' \circ T \xrightarrow{\sim} T \circ s$ such that $T(\mathrm{DF}(\leq 0)) \subset \mathrm{DF}'(\leq 0)$, $T(\mathrm{DF}(\geq 0)) \subset \mathrm{DF}(\geq 0)$ and that, for every $X \in \operatorname{Ob}\mathrm{DF}$, the following triangle commutes:

$$\begin{tikzcd} T(X) \arrow[r, "\alpha'_{T(X)}"] \arrow[dr, "T(\alpha(X))"'] & s'(T(X)) \arrow[d, "\wr"] \\ & T(s(X)) \end{tikzcd}$$

Definition 1.1.3. Let $\mathcal{D}$ be a triangulated category. A *filtered triangulated category over* $\mathcal{D}$ is a filtered triangulated category DF together with a fully faithful functor $i\colon \mathcal{D} \longrightarrow \mathrm{DF}$ whose essential image is $\mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0)$.

Definition 1.1.4. Let $\mathcal{D}$ be triangulated categories, DF be a filtered triangulated category over $\mathcal{D}$. We write *equiv* for the equivalence of categories $\mathcal{D} \longrightarrow \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0)$ induced by the fully faithful functor $i\colon \mathcal{D} \longrightarrow \mathrm{DF}$.

Definition 1.1.5. Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories, DF (resp. DF$'$) be a filtered triangulated category over $\mathcal{D}$ (resp. $\mathcal{D}'$) and $T\colon \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor. A *filtered lifting* of T is a filtered triangulated functor $FT\colon \mathrm{DF} \longrightarrow \mathrm{DF}'$ and a natural isomorphism *compat* : $i' \circ T \simeq TF \circ i$.

I am guessing that the isomorphism *compat* should satisfy some more compatibilities, notably with the "commutation with shifts" isomorphisms. (This will probably come up when I actually formalize properties of filtered triangulated functors.)

1.2 Proposition A.1.3

We fix a filtered triangulated catgeory DF.

Proposition 1.2.1. *For every $n \in \mathbb{Z}$, the full subcategory* $\mathrm{DF}(\leq n)$ *is reflective.*

Proposition 1.2.2.
For every $n \in \mathbb{Z}$, the full subcategory $\mathrm{DF}(\leq n)$ *is coreflective.*

Definition 1.2.3.
For every $n \in \mathbb{Z}$, we denote by $\sigma_{\leq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ the composition of a left adjoint of the inclusion functor $\iota : \mathrm{DF}(\leq n) \to \mathrm{DF}$ and of ι.

Definition 1.2.4.
For every $n \in \mathbb{Z}$, we denote by $truncLE\pi n : \mathbf{1}\,\mathrm{DF} \longrightarrow \sigma_{\leq n}$ the unit of the adjunction.

Definition 1.2.5.
For every $n \in \mathbb{Z}$, we denote by $\sigma_{\geq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ the composition of a right adjoint of the inclusion functor $\iota : \mathrm{DF}(\geq n) \to \mathrm{DF}$ and of ι.

Definition 1.2.6.
For every $n \in \mathbb{Z}$, we denote by $truncGE\iota n : \sigma_{\geq n} \longrightarrow \mathbf{1}\,\mathrm{DF}$ the counit of the adjunction.

We have some lemmas about these, among others concerning the essential image of $\sigma_{\leq n}$ and $\sigma_{\geq n}$, as well as lemmas expressing the universal property of the adjunctions.

Definition 1.2.7.
For every $n \in \mathbb{Z}$, the functor $\sigma_{\leq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ commutes with the triangulated shift.

Proposition 1.2.8.
For every $n \in \mathbb{Z}$, the functor $\sigma_{\leq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ is triangulated.

Definition 1.2.9.
For every $n \in \mathbb{Z}$, the functor $\sigma_{\geq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ commutes with the triangulated shift.

Proposition 1.2.10.
For every $n \in \mathbb{Z}$, the functor $\sigma_{\geq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ is triangulated.

Proposition 1.2.11.
For all $n, m \in \mathbb{Z}$, the functor $\sigma_{\leq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ sends $\mathrm{DF}(\leq m)$ *to itself.*

Proposition 1.2.12.
For all $n, m \in \mathbb{Z}$, the functor $\sigma_{\leq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ sends $\mathrm{DF}(\geq m)$ *to itself.*

Proposition 1.2.13.
For all $n, m \in \mathbb{Z}$, the functor $\sigma_{\geq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ sends $\mathrm{DF}(\leq m)$ *to itself.*

Proposition 1.2.14.
For all $n, m \in \mathbb{Z}$, the functor $\sigma_{\geq n} : \mathrm{DF} \longrightarrow \mathrm{DF}$ sends $\mathrm{DF}(\geq m)$ to itself.

We need to switch the order of statements compared to the paper, because the proof of Proposition A.1.3 (ii) uses Proposition A.1.3 (iii), but with general indices.

The way we state the existence of the triangle cheating in a way, because the connecting morphism in the triangle is not arbitrary, it's given by the axioms. (The statements are still okay thanks to the uniqueness.)

Definition 1.2.15.
For $n \in \mathbb{Z}$, this is the natural transformation $\delta \colon \sigma_{\leq n} \to (\mathit{shiftFunctor}\ 1) \circ (\sigma_{\geq n+1})$.

Definition 1.2.16.
For $n \in \mathbb{Z}$, this is the functor *triangleGELEn* from DF to the category of triangles of DF sending X to the triangle

$$\sigma_{\geq n+1} X \xrightarrow{truncGE\iota} X \xrightarrow{truncLE\pi} \sigma_{\leq n} \xrightarrow{\delta} (\sigma_{\geq n+1} X)[1].$$

Proposition 1.2.17.
For X in DF and $n \in \mathbb{Z}$, the triangle triangleGELEnX is distinguished.

The second part of Proposition A.1.3(iii) is a uniqueness statement for the triangle. In the paper, this says that any distinguished triangle $A \longrightarrow X \longrightarrow B \longrightarrow A[1]$ with A in $\mathrm{DF}(\leq n)$ and B in $\mathrm{DF}(\geq n+1)$ is isomorphic to *triangleGELEnX* in a unique way. Actually, this is not quite correct, because we only have uniqueness if we require the morphism of triangles to be the identity of X on the second objects. Also, the other morphisms are already explicit and uniquely determined, they are given by $descTruncLE$ and $liftTruncGE$, so the real content is that these morphisms are isomorphisms.

Proposition 1.2.18.
Let $n \in \mathbb{Z}$ and T be a distinguished triangle in DF. Assume that the first object $T.X_1$ of T is in $\mathrm{DF}(\geq n+1)$ and that the third object $T.X_3$ of T is in $\mathrm{DF}(\leq n)$. Then the morphism $\sigma_{\leq n} T.X_2 \longrightarrow T.X_3$ induced by the second morphism of T and the universal property of truncLE is an isomorphism.

Proposition 1.2.19.
Let $n \in \mathbb{Z}$ and T be a distinguished triangle in DF. Assume that the first object $T.X_1$ of T is in $\mathrm{DF}(\geq n+1)$ and that the third object $T.X_3$ of T is in $\mathrm{DF}(\leq n)$. Then the morphism $T.X_1 \longrightarrow \sigma_{\geq n+1} T.X_2$ induced by the first morphism of T and the universal property of truncGE is an isomorphism.

Proposition A.1.3(ii) asserts that the functors $\sigma_{\leq n}$ and $\sigma_{\geq m}$ commute. Before proving it, we establish a criterion for triangulated endofunctors of DF to commute with the truncation functors (up to an isomorphism which will arise naturally). It is better to make this more general, as it will be used again.

We consider another filtered triangulated category DF'.

Definition 1.2.20.
Let $F : \mathrm{DF} \longrightarrow \mathrm{DF}'$ be a filtered triangulated functor and $n \in \mathbb{Z}$. If F preserves the subcategory $\mathrm{DF}(\leq n)$, we get a natural transformation $\sigma_{\leq n} \circ F \longrightarrow F \circ \sigma_{\leq n}$.

Definition 1.2.21.
Let $F : \mathrm{DF} \longrightarrow \mathrm{DF}'$ be a filtered triangulated functor and $n \in \mathbb{Z}$. If F preserves the subcategory $\mathrm{DF}(\geq n)$, we get a natural transformation $F \circ \sigma_{\geq n} \longrightarrow \sigma_{\geq n} \circ F$.

Remark 1.2.22. It would be natural to use mates in this construction, but the properties were easier to prove with the direct definition we adopted.

Proposition 1.2.23.
Let $F : \mathrm{DF} \longrightarrow \mathrm{DF}'$ be a filtered triangulated functor and $n \in \mathbb{Z}$. Suppose that F preserves the subcategories $\mathrm{DF}(\leq n)$. and $\mathrm{DF}(\geq n+1)$.
Then the natural transformation $\sigma_{\leq n} \circ F \longrightarrow F \circ \sigma_{\leq n}$ is an isomorphism.

Proposition 1.2.24.
Let $F : \mathrm{DF} \longrightarrow \mathrm{DF}'$ be a filtered triangulated functor and $n \in \mathbb{Z}$. Suppose that preserves the subcategories $\mathrm{DF}(\leq n)$. and $\mathrm{DF}(\geq n+1)$.
Then the natural transformation $F \circ \sigma_{\geq n} \longrightarrow \sigma_{\geq n} \circ F$ is an isomorphism.

Now we write the existence statement of Proposition A.1.3(ii).

Definition 1.2.25.
For $a, b \in \mathbb{Z}$, we define $\sigma'_{[a,b]} : \mathrm{DF} \longrightarrow \mathrm{DF}$ as $\sigma_{\leq b} \circ \sigma_{\geq a}$.

Definition 1.2.26.
For $a, b \in \mathbb{Z}$, we define $\sigma_{[a,b]} : \mathrm{DF} \longrightarrow \mathrm{DF}$ as $\sigma_{\geq a} \circ \sigma_{\leq b}$.

Definition 1.2.27.
For $a, b \in \mathbb{Z}$, we have a natural transformation $\sigma'_{[a,b]} \longrightarrow \sigma_{[a,b]}$ given by `commute_truncLE`.

Definition 1.2.28.
For $a, b \in \mathbb{Z}$, the natural transformation $\sigma'_{[a,b]} \longrightarrow \sigma_{[a,b]}$ of Definition 1.2.27 is an isomorphism.

Remark 1.2.29. Because $\sigma_{[a,b]}$ and $\sigma'_{[a,b]}$ are isomorphic (by a "canonical" isomorphism), they only get one notation in the paper, but of course you need two notations in Lean.

The uniqueness statement of Proposition A.1.3(ii):

Proposition 1.2.30.
Let $a, b \in \mathbb{Z}$, let X be an object of DF, and let $f : \sigma'_{[a,b]}X \longrightarrow \sigma_{[a,b]}X$. Suppose that f makes the following diagram commute:

$$\begin{tikzcd}
\sigma_{\geq b} \arrow[rr] \arrow[dr] & & \mathrm{id}_{\mathrm{DF}} \arrow[rr] & & \sigma_{\leq a} \\
& \sigma_{\leq a}\sigma_{\geq b} \arrow[rr, "f"'] & & \sigma_{\geq b}\sigma_{\leq a} \arrow[ur] &
\end{tikzcd}$$

Then f is equal to the morphism of Definition 1.2.27.

We can now state a more general version of Proposition A.1.3(iii).

Definition 1.2.31.
Let $a, b, c \in \mathbb{Z}$ with $b \leq c$. Then we have a natural transformation $\sigma_{[a,c]} \longrightarrow \sigma_{[a,b]}$.

Definition 1.2.32.
Let $a, b, c \in \mathbb{Z}$ with $a \leq b$. Then we have a natural transformation $\sigma_{[b,c]} \longrightarrow \sigma_{[a,c]}$.

Definition 1.2.33.
For $a, b, c \in \mathbb{Z}$, we have a natural transformation $\delta \colon \sigma_{[a,b]} \to (\mathit{shiftFunctor}\ 1) \circ (\sigma_{[b+1,c]})$.

Definition 1.2.34.
For $a, b, c \in \mathbb{Z}$ such that $a \leq b \leq c$, this is the functor *triangleGELEabc* from DF to the category of triangles of DF sending X to the triangle

$$\sigma_{[b+1,c]}X \longrightarrow \sigma_{[a,c]}X \longrightarrow \sigma_{[a,b]} \xrightarrow{\delta} (\sigma_{[b+1,c]}X)[1].$$

Proposition 1.2.35. *For X in* DF *and $a, b, c \in \mathbb{Z}$ such that $a \leq b \leq c$, the triangle of Definition 1.2.34 is distinguished.*

We did not write the uniqueness statement for the triangle, though there is a statement similar to Propositions 1.2.18 and 1.2.19.

Proposition A.1.3 (iv) uses the adjective "canonical" and an equality sign, which is bad. We need to explain what compatibilities hide under it, and to make the equality sign by an isomorphism (that will be given by the universal property of the adjoint).

Also, we actually want the isomorphisms for "second" shifts by any integer, compatible with the zero and the addition, like in 'Functor.CommShift'. We introduce a new structure for this, similar to `Functor.CommShift` and called `familyCommShift`. It expresses the fact that a family of endofunctors $F : \mathbb{Z} \longrightarrow \mathrm{DF}$ has a family of isomorphisms $F(n+m) \circ shiftFunctor_2 m \xrightarrow{\sim} shiftFunctor_2 m \circ Fn$, where $shiftFunctor_2 a$ is the second shift by $a \in \mathbb{Z}$ (i.e., s^a), and that these isomorphisms are equal to the obvious ones when $m = 0$ and compatible with addition.

Definition 1.2.36.
The family of functors $n \mapsto truncLEn$ has a `familyCommShift` structure.

Definition 1.2.37.
The family of functors $n \mapsto truncGEn$ has a `familyCommShift` structure.

1.3 The "forget the filtration" functor (Proposition A.1.6)

The next thing in the paper is the definition, when we have a filtered triangulated category DF over a triangulated category $\mathcal{D}$, of the "graded pieces" functors $\mathrm{Gr}^n : \mathrm{DF} \to \mathcal{D}$, which use an arbitrary quasi-inverse of the fully faithful functor $i : \mathcal{D} \to \mathrm{DF}$ on the essential image of i.

Rather than using an arbitrary quasi-inverse, it makes things much simpler to use the one given by the "forget the filtration" functor $\omega : \mathrm{DF} \to \mathcal{D}$, which has the additional pleasant property that it is defined on all of DF and so avoids an `ObjectProperty.lift`. In fact, this is even better, as ω intertwines the second shift and the identity of $\mathcal{D}$, so we can directy define Gr^n as $\omega \circ \sigma_{[n,n]}$ (noting that Gr^n was originally defined as $shiftFunctor_2(-n) \circ \sigma_{[n,n]}$ followed by a quasi-inverse of the equivalence $\mathcal{D} \xrightarrow{\sim} \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0)$).

For this, we need to change the order of statements and do Proposition A.1.6 first (this is possible as that proposition makes no use of the functors Gr^n).

Definition 1.3.1.
The functor $\omega : \mathrm{DF} \to \mathcal{D}$.

Definition 1.3.2.
The functor $\omega : \mathrm{DF} \to \mathcal{D}$ restricted to the full subcategory $\mathrm{DF}(\leq 0)$ is left adjoint to the functor $\mathcal{D} \to \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0) \subset \mathrm{DF}(\leq 0)$.

Definition 1.3.3.
The functor $\omega : \mathrm{DF} \to \mathcal{D}$ restricted to the full subcategory $\mathrm{DF}(\geq 0)$ is right adjoint to the functor $\mathcal{D} \to \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0) \subset \mathrm{DF}(\geq 0)$.

Proposition 1.3.4.
For every X in DF*, the image by ω of the map $\alpha(X) : X \to s(X)$ is an isomorphism.*

This implies that ω intertwines the second shift s and the identity of $\mathcal{D}$. Right now this is expressed via a custom structure called a `leftCommShift`, but I don't know if that's optimal.

Proposition 1.3.5.
Let X, Y be objects of DF*. If X is in* $\mathrm{DF}(\leq 0)$ *and Y is in* $\mathrm{DF}(\geq 0)$*, then the map* $\mathrm{Hom}_{\mathrm{DF}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\omega(X), \omega(Y))$ *is bijective.*

The functor ω should also be triangulated. (This actually follows from the other conditions, but is not stated in the paper. Note that the first statement contains data, so I am actually cheating here, because the data is determined by the other properties of ω.)

Definition 1.3.6.
The functor $\omega : \mathrm{DF} \to \mathcal{D}$ commutes with the triangulated shifts.

Proposition 1.3.7.
The functor $\omega : \mathrm{DF} \to \mathcal{D}$ is triangulated.

We don't write the uniqueness statements here, as they are painful (which probably means that I haven't yet found the correct way to talk about ω).

Property (a) implies that we have an isomorphism $\omega \circ i \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}}$ given by the counit of the adjunction of Definition 1.3.2. Property (b) gives an isomorphism in the other direction, given by the unit of the adjunction of Definition 1.3.3. We give the definition of the first isomorphism, and the result after that says that these isomorphisms are inverses of each other. This compatibility is not stated in the paper.

Definition 1.3.8.
The isomorphism $\omega \circ i \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}}$ given by the counit of the adjunction of Definition 1.3.2.

Proposition 1.3.9.
The isomorphism $\omega \circ i \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}}$ given by the counit of the adjunction of Definition 1.3.2 is the inverse of the isomorphism $\mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} \omega \circ i$ given by the unit of the adjunction of Definition 1.3.3

The composition in the other direction, i.e. $i \circ \omega$, is isomorphic to $\sigma_{[0,0]}$. (Obviously, this is not an arbitrary isomorphism.)

Definition 1.3.10.
An isomorphism $i \circ \omega \xrightarrow{\sim} \sigma_{[0,0]}$.

1.4 Definition A.1.4 and Proposition A.1.5

Definition 1.4.1.
For every $n \in \mathbb{Z}$, we set $\mathrm{Gr}^n = \omega \circ \sigma_{n,n}$.

The functors Gr^n are triangulated:

Definition 1.4.2.
For every $n \in \mathbb{Z}$, the functor Gr^n commutes with the triangulated shifts.

Proposition 1.4.3.
For every $n \in \mathbb{Z}$, the functor Gr^n is triangulated.

Proposition 1.4.4. *For every X in* DF*, there exists $n \in \mathbb{Z}$ such that* $\mathrm{Gr}^m X$ *is zero if $m > n$.*

Proposition 1.4.5. *For every X in* DF*, there exists $n \in \mathbb{Z}$ such that* $\mathrm{Gr}^m X$ *is zero if $m < n$.*

We now state Proposition A.1.5.

Definition 1.4.6.
For every $n \in \mathbb{Z}$, the functor Gr^n intertwines the second shift of DF and the identity functor of $\mathcal{D}$.

Proposition 1.4.7.
For every $n \in \mathbb{Z}$ such that $n \neq 0$, for every X in $\mathcal{D}$, the object $\mathrm{Gr}^n(i(X))$ *is zero.*

Definition 1.4.8.
An isomorphism $\mathrm{Gr}^0 \circ i \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}}$.

Proposition A.1.5(iii) actually should say that some particular morphisms are isomorphisms, so we state it like that.

Proposition 1.4.9.
For all $r, n \in \mathbb{Z}$ such that $n < r$, the functor $\mathrm{Gr}^r \circ \sigma_{\leq n}$ *is zero.*

Proposition 1.4.10.
For all $r, n \in \mathbb{Z}$ such that $r \leq n$, the morphism $\mathrm{Gr}^r \to \mathrm{Gr}^r \circ \sigma_{\leq n}$ *given by truncLEπ is an isomorphism.*

Proposition 1.4.11.
For all $r, n \in \mathbb{Z}$ such that $r < n$, the functor $\mathrm{Gr}^r \circ \sigma_{\geq n}$ *is zero.*

Proposition 1.4.12.
For all $r, n \in \mathbb{Z}$ such that $n \leq r$, the morphism $\mathrm{Gr}^r \circ \sigma_{\geq n} \to \mathrm{Gr}^r$ *given by truncGEι is an isomorphism.*

1.5 Compatibility with filtered triangulated functors (Proposition A.1.8)

Let $\mathrm{DF}, \mathrm{DF}'$ be filtered triangulated categories, and $FT : \mathrm{DF} \to \mathrm{DF}'$ be a filtered triangulated functor.

Proposition A.1.8 is a mess as written. It is not precise, the natural isomorphisms are not arbitrary! Also, the square with $\sigma_{\geq n}$ is missing, and we need more squares $\sigma_{[a,b]}$, as well as compatibilities with the connecting morphisms in the triangles of $\sigma_{[a,b]}$.

By Propositions 1.2.23 and 1.2.24, the commutative squares for $\sigma_{\leq n}$ and $\sigma_{\geq n}$ follow from the facts that a filtered triangulated functor is triangulated and preserves all the categories $\mathrm{DF}(\leq m)$ and $\mathrm{DF}(\geq m)$.

The squares for the functors $\sigma_{[a,b]}$ and Gr^n then follow from the ones for the truncation functors.

The square for ω is harder to define, as the definition (and characterization) of ω is a little complicated. There should be some compatibilities there, but I am not yet sure what they are. We will see what is needed later.

The "commutative square" for $\sigma_{\leq n}$:
$$\begin{tikzcd} \mathrm{DF} \arrow[r, "FT"] \arrow[d, "\sigma_{\leq n}"'] & \mathrm{DF}' \arrow[d, "\sigma_{\leq n}"] \\ \mathrm{DF} \arrow[r, "FT"'] & \mathrm{DF}' \end{tikzcd}$$

Proposition 1.5.1.
Let $n \in \mathbb{Z}$. The morphism $\sigma_{\leq n} \circ FT \to FT \circ \sigma_{\leq n}$ of Definition 1.2.20 is an isomorphism.

The "commutative square" for $\sigma_{\geq n}$: $\begin{tikzcd} \mathrm{DF} \arrow[r, "FT"] \arrow[d, "\sigma_{\geq n}"'] & \mathrm{DF}' \arrow[d, "\sigma_{\geq n}"] \\ \mathrm{DF} \arrow[r, "FT"'] & \mathrm{DF}' \end{tikzcd}$

Proposition 1.5.2.
Let $n \in \mathbb{Z}$. The morphism $FT \circ \sigma_{\geq n} \to \sigma_{\geq n} \circ FT$ of Definition 1.2.21 is an isomorphism.

Definition 1.5.3.
For $n \in \mathbb{Z}$, an isomorphism $\mathrm{Gr}^n \circ FT \to FT \circ \mathrm{Gr}^n$.

$$\begin{tikzcd} \mathrm{DF} \arrow[r, "FT"] \arrow[d, "\mathrm{Gr}^n"'] & \mathrm{DF}' \arrow[d, "\mathrm{Gr}^n"] \\ \mathcal{D} \arrow[r, "T"'] & \mathcal{D}' \end{tikzcd}$$

Definition 1.5.4.
An isomorphism $\omega \circ FT \to FT \circ \omega'$.

$$\begin{tikzcd} \mathrm{DF} \arrow[r, "FT"] \arrow[d, "\omega"'] & \mathrm{DF}' \arrow[d, "\omega"] \\ \mathcal{D} \arrow[r, "T"'] & \mathcal{D}' \end{tikzcd}$$

2 Section A.2: compatible t-structures and the realization functor

We review the construction of the realization functor from [2] 3.1 and Appendix A of [1].
This section corresponds to the Lean file `TStructure_no_proof.lean`.

2.1 Compatible t-structures

We fix a filtered triangulated category DF over a triangulated category $\mathcal{D}$.

Definition 2.1.1. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ and a t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ on DF. We say that these t-structures are *compatible* if $i \colon \mathcal{D} \longrightarrow \mathrm{DF}$ is t-exact and if $s^n(\mathrm{DF}^{\leq a}) \subset \mathrm{DF}^{\leq b}$ for all $a, b, n \in \mathbb{Z}$ such that $a = b + n$.

Definition 2.1.2. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$. Then there exists a t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ on DF such that

$$\mathrm{Ob}\,\mathrm{DF}^{\leq 0} = \{X \in \mathrm{Ob}\,\mathrm{DF} \mid \forall i \in \mathbb{Z},\ \mathrm{Gr}^i X[i] \in \mathrm{Ob}\,\mathcal{D}^{\leq 0}\}$$

$$\mathrm{Ob}\,\mathrm{DF}^{\geq 0} = \{X \in \mathrm{Ob}\,\mathrm{DF} \mid \forall i \in \mathbb{Z},\ \mathrm{Gr}^i X[i] \in \mathrm{Ob}\,\mathcal{D}^{\geq 0}\}.$$

Proposition 2.1.3. *Suppose that we are given a t-structure on $\mathcal{D}$. Then it is compatible with the t-structure on* DF *of Definition 2.1.2.*

Proposition 2.1.4. *Suppose that we are given compatible t-structures t on $\mathcal{D}$ and tF on* DF*. Then tF is equal to the t-structure of Definition 2.1.2.*

2.2 Realization

We fix a filtered triangulated category DF over a triangulated category $\mathcal{D}$, as well as compatible t-structure t on $\mathcal{D}$ and tF on DF, with hearts $\mathcal{D}^\heartsuit$ and $\mathrm{DF}^\heartsuit$ and cohomology functors H_t^* and H_{tF}^*. The realization functor is a triangulated functor $\mathrm{D^b}\,\mathcal{D}^\heartsuit \to \mathcal{D}$ extending the inclusion $\mathcal{D}^\heartsuit \subset \mathcal{D}$.

To construct it, we first need to construct a functor $H_F : \mathrm{DF} \to \mathrm{C^b}\,\mathcal{D}^\heartsuit$, which will induce an equivalence of categories when restricted to $\mathrm{DF}^\heartsuit$. In the paper, the nth degree of $H_F(X)$ is defined as $\mathrm{H}_t^n(\mathrm{Gr}^n(X))$. Given the new definition of Gr^n, this is equal to $\mathrm{H}_t^n(\omega(\sigma_{[n,n]}(X)))$; that last formula makes it easier to construct the differentials.

Definition 2.2.1. For $n \in \mathbb{Z}$, we define a functor $H_F^n : \mathrm{DF} \to \mathcal{D}^\heartsuit$ by $\mathrm{H}_F^n = \mathrm{H}_t^n \circ \mathrm{Gr}^n$.

Definition 2.2.2. For $n \in \mathbb{Z}$, we define a natural transformation $d^n : H_F^n \to H_F^{n+1}$ by setting $d^n(X)$ to be the image by ω of the degree 1 map in the triangle $\sigma_{[n+1,n+1]}(X) \to \sigma_{[n,n+1]}(X) \to \sigma_{n,n}(X)$ of Definition 1.2.34.

Proposition 2.2.3. *For every $n \in \mathbb{Z}$, we have $d^{n+1} \circ d^n = 0$.*

Definition 2.2.4. This is the complex of functors $\mathrm{DF} \to \mathcal{D}^\heartsuit$ whose degree n term is H_F^n, and whose differentials are given by the d^n.

Definition 2.2.5. This is the functor $H'_F : \mathrm{DF} \to \mathrm{C}(\mathcal{D}^\heartsuit)$ corresponding to the complex of Definition 2.2.4.

We check that H'_F lands in the subcategory of bounded complexes.

Proposition 2.2.6. *For every X in* DF*, the complex $HF'(X)$ is bounded.*

Definition 2.2.7. This is the functor $H_F : \mathrm{DF} \to \mathrm{C^b}(\mathcal{D}^\heartsuit)$ given by HF'.

Theorem 2.2.8. *The restriction to* $\mathrm{DF}^\heartsuit$ *of the functor* DF *is an equivalence* $\mathrm{DF}^\heartsuit \xrightarrow{\sim} \mathrm{C^b}(\mathcal{D}^\heartsuit)$

Proposition 2.2.9. *The composition of a quasi-inverse of the equivalence* $\mathrm{DF}^\heartsuit \xrightarrow{\sim} \mathrm{C^b}(\mathcal{D}^\heartsuit)$ *of Theorem 2.2.8, of the inclusion* $\mathrm{DF}^\heartsuit \subset \mathrm{DF}$ *and of* $\omega : \mathrm{DF} \to \mathcal{D}$ *sends quasi-isomorphisms to isomorphisms.*

Definition 2.2.10. The functor $\mathrm{real} : \mathrm{D^b}(\mathcal{D}^\heartsuit) \to \mathcal{D}^\heartsuit$ coming from the universal property of the localization $\mathrm{C^b}(\mathcal{D}^\heartsuit) \to \mathrm{D^b}(\mathcal{D}^\heartsuit)$ applied to the statement of Proposition 2.2.9.

3 Section A.3: homological algebra calculations and functor reconstruction

This section corresponds to the Lean files `Acyclic.lean` and `Derived.lean`.

3.1 T-acyclic objects

Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories and $T : \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor. Suppose that we are given a t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ (resp. $({\mathcal{D}'}^{\le 0}, {\mathcal{D}'}^{\ge 0})$) on $\mathcal{D}$ (resp. $\mathcal{D}'$), and denote the cohomology functors of this t-structure by H_t^i (resp. $\mathrm{H}_{t'}^i$) and its heart by $\mathcal{D}^\heartsuit$ (resp. ${\mathcal{D}'}^\heartsuit$).

Definition 3.1.1. We say that an object X of $\mathcal{D}^\heartsuit$ is *T-acyclic* if $T(X)$ is in ${\mathcal{D}'}^\heartsuit$.

Lemma 3.1.2. *If X is a T-acyclic object and $n \neq 0$ is an integer, then $\mathrm{H}^n_{t'}(F(X))$ is zero.*

Proof. ☐

Note that the converse is true if the t-structure t' is non-degenerate, but not in general.

Lemma A.3.1(i) says that the full subcategory of T-acyclic objects is stable by extensions. In Lean, we also need to prove the "obvious" facts that it is stable by isomorphisms, binary products and contains zero objects, hence is an additive category.

Lemma 3.1.3. *The full subcategory of T-acyclic objects is closed under isomorphisms.*

Proof. ☐

Lemma 3.1.4. *Zero objects are T-acyclic.*

Proof. ☐

Lemma 3.1.5. *The full subcategory of T-acyclic objects is closed under binary products.*

Proof. ☐

Lemma 3.1.6. *The full subcategory of T-acyclic objects is closed under extensions.*

Proof. ☐

Definition 3.1.7. The functor $T' : \mathcal{D}^\heartsuit \to \mathcal{D}'^\heartsuit$ given by the restriction of $\mathrm{H}^0_{t'} \circ T$.

Proof. ☐

Definition 3.1.8. The functor T'' from the full subcategory of T-acyclic objects to $\mathcal{D}'^\heartsuit$ given by the restriction of T.

Lemma A.1.3(ii):

Lemma 3.1.9. *The image by the functor T' of Definition 3.1.7 of a short exact complex of T-acyclic objects is an exact complex in $\mathcal{D}'^\heartsuit$.*

Proof. ☐

Remark 3.1.10. Lemma A.3.1(ii) actually says that the image is short exact, which we do not prove yet. See the next statements.

Lemma 3.1.11. *The image by the functor T' of Definition 3.1.7 of a monomorphism with an acyclic cokernel is a monomorphism.*

Proof. ☐

Lemma 3.1.12. *The image by the functor T' of Definition 3.1.7 of an epimorphism with an acyclic kernel is an epimorphism.*

Proof. ☐

Lemma 3.1.13. *The image by the functor T' of Definition 3.1.7 of a short exact complex of T-acyclic objects is a short exact complex in $\mathcal{D}'^\heartsuit$.*

Proof. ☐

For the proof of Lemma A.1.3(iii), we need a number of facts about preservation of kernels/cokernels/etc. First we prove a statement about preservation of images.

Definition 3.1.14. If a map f in $\mathcal{D}^{\heartsuit}$ has an acyclic kernel and an acyclic cokernel, then the functor T' of Definition 3.1.7 preserves the image of 'f.

Remark 3.1.15. This isomorphism is not given by `CategoryTheory.PreservesImage.iso`, because we do not yet know at this point that the kernel and the cokernel of f are preserved. So we actually construct an image factorization of $T'(f)$ by hand in `imageFactorisation_map_of_acyclic_ker_coker` which has the desired image. Then we need to check that the isomorphism of Definition 3.1.14 is compatible with `Abelian.image.` and `Abelian.factorThruImage`, this is done in `CategoryTheory.Functor.isoImageOfAcyclic_comp_` and `CategoryTheory.Functor.factorThruImage_comp_isoImageOfAcyclic`.

We then look at the preservation of kernels and cokernels.

Lemma 3.1.16. *If f is a morphism in $\mathcal{D}$.heart with an acyclic image, then its kernel comparison morphism for the functor T' of Definition 3.1.7 is a mono.*

Proof. □

Lemma 3.1.17. *If f is a morphism in $\mathcal{D}$.heart with a T-acyclic image, a T-acyclic kernel and a T-acyclic cokernel, then its kernel comparison morphism for the functor T' of Definition 3.1.7 is an epi.*

Proof. □

Lemma 3.1.18. *If f is a morphism in $\mathcal{D}$.heart with a T-acyclic image, a T-acyclic kernel and a T-acyclic cokernel, then its kernel comparison morphism for the functor T' of Definition 3.1.7 is an isomorphism.*

Proof. □

Lemma 3.1.19. *If f is a morphism in $\mathcal{D}$.heart with an acyclic image, then its cokernel comparison morphism for the functor T' of Definition 3.1.7 is an epi.*

Proof. □

Lemma 3.1.20. *If f is a morphism in $\mathcal{D}$.heart with a T-acyclic image, a T-acyclic kernel and a T-acyclic cokernel, then its cokernel comparison morphism for the functor T' of Definition 3.1.7 is a mono.*

Proof. □

Lemma 3.1.21. *If f is a morphism in $\mathcal{D}$.heart with a T-acyclic image, a T-acyclic kernel and a T-acyclic cokernel, then its cokernel comparison morphism for the functor T' of Definition 3.1.7 is an isomorphism.*

Proof. □

Definition 3.1.22. Let S be a short exact sequence in $\mathcal{D}^{\heartsuit}$ whose second object $S.X_2$ is T-acyclic. Then, for every $n \in \mathbb{Z}$, the connecting morphism in the long exact sequence of homology of the distinguished triangle in $\mathcal{D}'$ image of S by T (i.e. given by `CategoryTheory.Functor.shortExactComplex_to_triangle`) is an isomorphism from nth homology of $T(S.X_3)$ to the $(n+1)$st homology of $T(S.X_1)$.

Lemma A.1.3(iii):

Definition 3.1.23. Let S be a cochain complex in $\mathcal{D}^\heartsuit$, and let $r \in \mathbb{Z}$. Suppose that S is exact in degree $r+1$ and that the rth entry of S is T-acyclic. Then, for every $n \in \mathbb{Z}$ different from 0 and -1, we have an isomorphism between the nth cohomology of $T(\operatorname{Ker}(S.d^{r+1}))$ and the $(n+1)$st homology of $T(\operatorname{Ker}(S.d^r))$.

We turn to the proof of Lemma A.1.3(iv).

Lemma 3.1.24. *Let S be a cochain complex in $\mathcal{D}^\heartsuit$, and let $k \in \mathbb{Z}$. Suppose that S is exact in degree $\leq k$, and that the entries of S are T-acyclic in degree $\leq k$ and zero in small enough degree.*

Then the homology of $T(\operatorname{Ker}(S.d^r))$ is zero in positive degree.

Proof. □

Lemma 3.1.25. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^\heartsuit$, and let $k \in \mathbb{Z}$. Suppose that S is exact in degree $\leq k$, and that the entries of S are T-acyclic in degree $\leq k$.*

Then the homology of $T(\operatorname{Ker}(S.d^r))$ is zero in positive degree.

Proof. □

Lemma 3.1.26. *Let S be a cochain complex in $\mathcal{D}^\heartsuit$, and let $k \in \mathbb{Z}$. Suppose that S is exact in degree $> k$, and that the entries of S are T-acyclic in degree $\geq k$ and zero in large enough degree.*

Then the homology of $T(\operatorname{Ker}(S.d^r))$ is zero in negative degree.

Proof. □

Lemma 3.1.27. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^\heartsuit$, and let $k \in \mathbb{Z}$. Suppose that S is exact in degree $> k$, and that the entries of S are T-acyclic in degree $\geq k$.*

Then the homology of $T(\operatorname{Ker}(S.d^r))$ is zero in negative degree.

Proof. □

From now on, we suppose that the t-structure t' on $\mathcal{D}'$ is non-degenerate.

Lemma 3.1.28. *Let S be a cochain complex in $\mathcal{D}^\heartsuit$. Suppose that S is exact, and that the entries of S are T-acyclic in every degree and zero outside of a bounded interval.*

Then $\operatorname{Ker}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma 3.1.29. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^\heartsuit$. Suppose that S is exact and that the entries of S are T-acyclic in every degree.*

Then $\operatorname{Ker}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma 3.1.30. *Let S be a cochain complex in $\mathcal{D}^\heartsuit$. Suppose that S is exact, and that the entries of S are T-acyclic in every degree and zero outside of a bounded interval.*

Then $\operatorname{Im}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma 3.1.31. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^{\heartsuit}$. Suppose that S is exact and that the entries of S are T-acyclic in every degree.*

Then $\mathrm{Im}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma 3.1.32. *Let S be a cochain complex in $\mathcal{D}^{\heartsuit}$. Suppose that S is exact, and that the entries of S are T-acyclic in every degree and zero outside of a bounded interval.*

Then $\mathrm{Coker}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma 3.1.33. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^{\heartsuit}$. Suppose that S is exact and that the entries of S are T-acyclic in every degree.*

Then $\mathrm{Coker}(S.d^r)$ is T-acyclic for every $k \in \mathbb{Z}$.

Proof. □

Lemma A.3.1(iv)(a) :

Lemma 3.1.34. *Let S be a cochain complex in $\mathcal{D}^{\heartsuit}$. Suppose that S is exact, and that the entries of S are T-acyclic in every degree and zero outside of a bounded interval.*

Then the image of S by the functor T' of Definition 3.1.7 is exact.

Proof. □

Lemma A.3.1(iv)(b):

Lemma 3.1.35. *Suppose that T has finite cohomological dimension (relative to the t-structures t and t'). Let S be a cochain complex in $\mathcal{D}^{\heartsuit}$. Suppose that S is exact and that the entries of S are T-acyclic in every degree.*

Then the image of S by the functor T' of Definition 3.1.7 is exact.

Proof. □

There is no explicit formalization of Lemma A.3.1(v).

3.2 Functor reconstruction

We keep the notation of the previous subsection, and we assume that the t-structure t' on $\mathcal{D}'$ is non-degenerate. We also assume that we have filtered triangulated categories DF over $\mathcal{D}$ and DF' over $\mathcal{D}'$, equipped with compatible t-structures tF and tF', as well as a filtered triangulated functor $FT : \mathrm{DF} \to \mathrm{DF}'$ lifing $T : \mathcal{D} \to \mathcal{D}'$; this is condition (a) of Proposition A.3.2.

Condition (b) of Proposition A.3.2 says that the "obvious" functor from the bounded homotopy category of complexes of T-acyclic objects in the heart $\mathcal{D}$ to the derived category of the heart of $\mathcal{D}$ is a localization functor for the class of morphisms with acyclic cone (i.e. quasi-isomorphisms).

First statement of Proposition A.3.2:

Proposition 3.2.1.

The functor on the bounded homotopy categories induced by the functor T'' of Definition 3.1.8 sends a (bounded) exact complex of T-acyclic objects to an exact complex.

Definition 3.2.2.

The functor $DT : \mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}'^{\heartsuit})$ given by the universal property of the localization functor of condition (b) above, and the property proved in Proposition 3.2.1.

Second statement of Proposition A.3.2:

Definition 3.2.3.

There is an isomorphism of functors $\mathrm{real}' \circ DT \xrightarrow{\sim} T \circ \mathrm{real}$.

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit}) \arrow[r, "DT"] \arrow[d, "\mathrm{real}"'] & \mathrm{D}^{\mathrm{b}}(\mathcal{D}'^{\heartsuit}) \arrow[d, "\mathrm{real}'"] \\
\mathcal{D} \arrow[r, "T"'] & \mathcal{D}'
\end{tikzcd}$$

Remark 3.2.4. Suppose that we are in the situation of Proposition A.3.2. If moreover $T\colon \mathcal{D} \longrightarrow \mathcal{D}'$ is left t-exact and if $\mathcal{J}$ is cogenerating in $\mathcal{C}$ (i.e. every object of $\mathcal{C}$ has a monomorphism into an object of $\mathcal{J}$), then the functor $\mathrm{H}^0(T)\colon \mathcal{C} \longrightarrow \mathcal{C}'$ admits a right derived functor $RT\colon \mathrm{D}^+(\mathcal{C}) \longrightarrow \mathrm{D}^+(\mathcal{C}')$ by Proposition 13.3.5 of [3], and the construction of RT in that proposition shows that RT sends that $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ to $\mathrm{D}^{\mathrm{b}}(\mathcal{C}')$ and that DT is the restriction of RT to $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$. We have a similar statement if T is right t-exact and $\mathcal{J}$ is generating in $\mathcal{C}$.

Remark 3.2.5. By Proposition 10.2.7 of [3], to check assumption (b) of Proposition A.3.2, it suffices to find triangulated subcategories $\mathcal{D}_0 = \mathrm{K}^{\mathrm{b}}(\mathcal{C}) \supset \mathcal{D}_1 \supset \ldots \supset \mathcal{D}_r = \mathrm{K}^{\mathrm{b}}(\mathcal{J})$ of $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$ such that, for every $i \in \{1, \ldots, r-1\}$, one of the following conditions holds:

- For every $X \in \mathrm{Ob}\,\mathcal{D}_i$, there exists a quasi-isomorphism $X \longrightarrow Y$ with $Y \in \mathrm{Ob}\,\mathcal{D}_{i+1}$.
- For every $X \in \mathrm{Ob}\,\mathcal{D}_i$, there exists a quasi-isomorphism $Y \longrightarrow X$ with $Y \in \mathrm{Ob}\,\mathcal{D}_{i+1}$.

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