This is an explanation of what happens in the Lean files leading up to FiniteWeightedComplex.lean. The goal is to define the ”weighted complex” (a subcomplex of the Coxeter complex of the symmetric group), to prove that it is shellable and to calculate its Euler-Poincaré characteristic. The sections correspond to separate Lean files, the subsectino to the internal organizations of these files.

1 General preorder stuff

1.1 General stuff

This contains general lemmas about preorder on a type α. If s is a preorder on α, we write $\simeq_s$ for the antisymmetrization relation of s (i.e. $a \simeq_s b$ if and only if $a \leq_s b$ and $b \leq_s a$). This is an equivalence relation, and the quotient of α by this equivalence relation is called the antisymmetrization; it carries a partial order induced by s.

Lemma 1.1.1 (TotalPreorder_trichotomy) *If s is a total preorder on α, then, for all $a, b \in \alpha$, we have $a <_s b$, $b <_s a$ or $a \simeq_s b$.*

Lemma 1.1.2 (LinearPreorder_trichotomy) *If s is a linear preorder on α (i.e. a preorder that happens to be a linear order), then, for all $a, b \in \alpha$, we have $a <_s b$, $b <_s a$ or $a = b$.*

Lemma 1.1.3 (TotalPreorder_lt_iff_not_le) *If s is a total preorder on α and $a, b \in \alpha$, then $\neg(a \leq_s b)$ if and onlyb if $b <_s a$.*

1.2 SuccOrders and antisymmetrization

A SuccOrder structure on s is the data of a "reasonable" successor function.

Definition 1.2.1 (SuccOrdertoAntisymmetrization) If hsucc is a SuccOrder on α for the preorder s, then we get a SuccOrder on the antisymmetrization of α (relative to s) by taking as successor function the function sending x to the image of the successor of a lift of x.

Definition 1.2.2 (SuccOrderofAntisymmetrization) If s is a preorder on α and hsucc is a SuccOrder on the antisymmetrization of α (for the canonical partial order), then we get a SuccOrder on α for s by sending a to any lift of the successor of the image of a in the antisymmetrization.

1.3 Essentially locally finite preorders

Definition 1.3.1 (EssentiallyLocallyFinitePreorder) We say that a preorder is *essentially locally finite* if its antisymmetrization partial order is locally finite (i.e. closed intervals are finite).

Definition 1.3.2 (EssentiallyLocallyFinite_ofLocallyFinite) Any structure of locally finite preorder on s defines a structure of essentially locally finite preorder.

Definition 1.3.3 (TotalELFP_SuccOrder) Any structure of essentially locally finite preorder on a total preorder s defines a SuccOrder structure.

Lemma 1.3.4 (ELFP_is_locally_WellFounded) *If s is an essentially locally finite preorder, then it is well-founded on any closed interval.*

1.4 Partial order on preorders

Definition 1.4.1 (instPreorder.le & Preorder.PartialOrder) We define a partial order on preorders by saying that $s \leq t$ if only, for all $a, b \in \alpha$, $a \leq_s b$ implies $a \leq_t b$.

1.5 The trivial preorder

Definition 1.5.1 (trivialPreorder) The trivial preorder on α is the preorder with graph $\alpha \times \alpha$.

Lemma 1.5.2 (trivialPreorder_is_greatest) *Every preorder is smaller than or equal to the trivial preorder.*

Lemma 1.5.3 (nontrivial_preorder_iff_exists_not_le) *A preorder s is different from the trivial preorder if and only if there exist $a, b \in \alpha$ such that $\neg(a \leq_s b)$.*

1.6 Partial order on preorders and antisymmetrization

Lemma 1.6.1 (AntisymmRel_monotone) *If s, t are preorders on α such that $s \leq t$, then the graph of $\simeq_s$ is contained in the graph of $\simeq_t$.*

Definition 1.6.2 (AntisymmetrizationtoAntisymmetrization) Let s, t be preorders such that $s \leq t$. We have a map from the antisymmetrization of s to that of t sending x to the image in the antisymmetrization of t of any lift of x.

Lemma 1.6.3 (AntisymmetrizationtoAntisymmetrization_lift) *If s, t are preorders on α such that $s \leq t$ and $a \in \alpha$, then the class of a in the antisymmetrization of t is the image of its class in the antisymmetrization of s.*

Lemma 1.6.4 (AntisymmetrizationtoAntisymmetrization_monotone) *If s, t are preorders on α such that $s \leq t$, then the map from the antisymmetrization of s to the antisymmetrization of t is monotone.*

Lemma 1.6.5 (AntisymmetrizationtoAntisymmetrization_image_interval) *If r, s are preorders on α such that $r \leq s$ and r is total, and if $a, b \in \alpha$ are such that $a \leq_r b$, the the image of the interval $[\overline{a}, \overline{b}]$ in the antisymmetrization of r is the corresponding interval in the antisymmetrization of s.*

1.7 Upper sets for the partial order on preorders

Lemma 1.7.1 (Total_IsUpperSet) *Total preorders form on upper set.*

Definition 1.7.2 (TotalELPF_IsUpperSet) If r is a total essentially locally finite preorder, then it defines a structure of essentially locally finite preorder on any s such that $r \leq s$.

1.8 Noetherian preordered sets

Definition 1.8.1 (IsNoetherianPoset) A preorder s on α is called *Noetherian* if the relation $>_s$ is well-founded.

1.9 Maximal nonproper order ideals

We fix a preorder on α.

Definition 1.9.1 (Order.Ideal.IsMaximalNonProper) An order ideal is called *maximal nonproper* if it is maximal among all order ideals of α.

This definition is only interesting if α itself is not an order ideal, i.e. if the preorder on α is not directed.

Lemma 1.9.2 (OrderIdeals_inductive_set) *Order ideals of α form an inductive set (i.e. any nonempty chain has an upper bound).*

Lemma 1.9.3 (Order.Ideal.contained_in_maximal_nonproper) *Any order ideal of α is contained in a maximal nonproper order ideal.*

Lemma 1.9.4 (Order.Ideal.generated_by_maximal_element) *If I is an order ideal of α and if a is a maximal element of I, then I is generated by a.*

Lemma 1.9.5 (Order.PFilter.generated_by_minimal_element) *If F is an order filter of α and if a is a minimal element of F, then F is generated by a.*

Lemma 1.9.6 (Noetherian_iff_every_ideal_is_principal_aux & Noetherian_iff_every_ideal_is_principal) *The preorder on α is Noetherian if and only every order ideal is principal (= generated by one element).*

1.10 Map from α to its order ideals

We fix a partial order on α.

Definition 1.10.1 (Elements_to_Ideal) We have an order embedding from α to the set of its order ideals (ordered by inclusion) sending a to the ideal generated by a.

1.11 Locally finite partial order on finsets of α

Lemma 1.11.1 (FinsetIic_is_finite) *If s is a finset of α, then the half-infinite interval $]\leftarrow, s]$ is finite.*

Lemma 1.11.2 (FinsetIcc_is_finite) *If s and t are finsets of α, then the closed interval $[s, t]$ is finite.*

Definition 1.11.3 (FinsetLFB) A structure of locally finite order with smallest element on the set of finsets of α.

Definition 1.11.4 (FacePosetLF) A structure of locally finite order with smallest element on the set of finsets of α.

1.12 Two-step preorders

Definition 1.12.1 (twoStepPreorder) If a is an element of α, we define a preorder twoStepPreorder(a) that makes a strictly smaller than every other element and all other elements of α equal.

Lemma 1.12.2 (twoStepPreorder_smallest) *If a is an element of α, then it is the smallest element for twoStepPreorder(a).*

Lemma 1.12.3 (twoStepPreorder_greatest) *If a, b are elements of α such that $a \neq b$, then any element of α is smaller than or equal to b for twoStepPreorder(a).*

Lemma 1.12.4 (twoStepPreorder_IsTotal) *If a is an element of α, then twoStepPreorder(a) is a total preorder.*

Lemma 1.12.5 (twoStepPreorder_nontrivial) *If a, b are elements of α such that $a \neq b$, then twoStepPreorder(a) is nontrivial.*

Definition 1.12.6 (twoStepPreorder_singleton_toAntisymmetrization) For a an element of α, a map from $\{a\}$ to the antisymmetrization of α for twoStepPreorder(a).

Definition 1.12.7 (twoStepPreorder_nonsingleton_toAntisymmetrization) If a, b are elements of α, a map from $\{a\} \cup *$ to the antisymmetrization of α for twoStepPreorder(a).

Lemma 1.12.8 (twoStepPreorder_singleton_toAntisymmetrization_surjective) *If a is an element of α such that every element of α is equal to a, then the map from $\{a\}$ to the antisymmetrization of α for twoStepPreorder(a) is surjective.*

Lemma 1.12.9 (twoStepPreorder_nonsingleton_toAntisymmetrization_surjective) *If a, b are elements of α such that $a \neq b$, then the map from $\{a\} \cup *$ to the antisymmetrization of α for twoStepPreorder(a) is surjective.*

Lemma 1.12.10 (twoStepPreorder_Antisymmetrization_finite) *If a is an element of α, then the antisymmetrization of α for twoStepPreorder(a) is finite.*

Lemma 1.12.11 (twoStepPreorder_Antisymmetrization.card) *If a, b are elements of α such that $a \neq b$, then the antisymmetrization of α for twoStepPreorder(a) has cardinality* 2.

1.13 A linear order on any type

Definition 1.13.1 (ArbitraryLinearOrder) A definition of a linear order on α (by embedding α into the class of cardinals and lifting the linear order on cardinals).