1 Preorders

Let α be a set. We consider the maps LS and PLS from the $\mathrm{Pre}(s)$ of preorders on α to the powerset of the powerset of α, sending a preorder s to the set of its nonempty lower sets (resp nonempty proper lower sets), and the map SL from the powerset of the powerset of α to $\mathrm{Pre}(\alpha)$ sending a set of subsets E of α to the preorder s defined by: "$a \leq_s b$ if anf only if, for every $\beta \in E$, we have $b \in \beta \Longrightarrow a \in \beta$".

We have, for $s \in \mathrm{Pre}(\alpha)$ and E a set of subsets of α:

1. $s = SL(LS(s))$;
2. $E \subset LS(SL(E))$;

Also, if s is total and essentially locally finite, then every ideal of s is principal, and we have $E \cup \{\top\} = LS(s)$ for every E such that $s = SL(E)$.

We are mostly interested in the case where s is total. Then $LS(s)$ (resp. $PLS(s)$) is the set of ideals (resp. proper ideals) of s. Also, both these sets are totally ordered by inclusion. Indeed, let $\beta, \gamma \in PLS(s)$, and suppose that $\beta \not\subset \gamma$. Choose $a \in \beta$ such that $a \notin \gamma$. If $b \in \gamma$, we cannot have $a \leq b$ (otherwise a would be in γ, as γ is a lower set for s), so $b \leq a$, which implies that $b \in \beta$; so $\gamma \subset \beta$.

Conversely, if E is totally ordered by inclusion, then the preorder $s = SL(E)$ is total. Indeed, let $a, b \in \alpha$, and suppose that $a \not\leq b$. Then there exists $\beta \in E$ such that $b \in \beta$ and $a \notin \beta$. If $\gamma \in E$ and $a \in \gamma$, we cannot have $\gamma \subset \beta$ because $a \notin \beta$, so $\beta \subset \gamma$, so $b \in \gamma$; this implies that $b \leq a$.

We want to investigate the link because E being totally ordered by inclusion and locally finite, and s being total and essentially locally finite. What I think holds:

1. If s is total and essentially locally finite, then $LS(s)$ is total and locally finite, and isomorphic as a partial order to the quotient partial order of s or that quotient partial with a biggest element added, depending on whether α has no maximal element or has one.
2. If E is totally ordered by inclusion and locally finite, and if, for every $a \in \alpha$, there exists an element of E containing a, then s is total and essentially locally finite. The extra condition is necessary: if $\alpha = \mathbb{N} \cup \infty$ and s is the usual order on it (i.e. it makes ∞ biggest than everything else), and if $E = \{[0, n],\ n \in \mathbb{N}\}$, then $s = SL(E)$, but E is locally finite and s is not.

Can we similary characterize the (total?) preorders that have a finite number of finite blocks and at most one infinite block, which is maximal? This would mean that s is essentially locally finite and every proper ideal is finite. So on E, it means that E is totally ordered by inclusion and every proper element of E is finite and E itself is finite; if we have these conditions, then it implies that $s = SL(E)$ is total essentially locally finite, then that proper initial intervals of s are finite, so the blocks of s are finite except maybe for the last one; also, there is a finite number of blocks because E is finite.

2 Ideals in the face complex of an abstract simplicial complex (and shellability)

Let K be an abstract simplicial complex on a set V. We say that K is *bounded* if it admits no strictly increasing infinite sequence of faces; this holds for example if the dimension of the faces of K is bounded. We denote by $V_K \subset V$ the set of vertices of K.

Facts:

1. Every filter of K is principal (because it has minimal elements, as initial intervals are finite).
2. Ideals of K correspond to nonempty subsets S of V_K with the property that every finite nonempty subset of S is a face of K.

3. If K is bounded, every ideal of K is principal, and the maximal ideals are exactly the ones generated by facets.

4. In fact, the condition that K is bounded is equivalent to the condition that every ideal is principal.

5. Prime ideals: these are ideals I such that, for all a, b not in I, there exists c not in I such that $c \leq a$ and $c \leq b$. For a simplicial complex, this means that, for all a, b not in I, the intersection $a \cap b$ is a face (or equivalently nonempty) and not in I. Suppose that I corresponds to a set of vertices S, vertices of K except exactly one. Note that does not define an ideal in general, since it requires every finite subset of S to be a face. then I being prime implies that S contains all but at most one of the vertices of K; since prime ideals are proper, it is equivalent to the fact that I contains all the

We are interested in the following case: $\alpha = \mathbb{N}$ with its usual order, we consider the set of total preorders on s that have a finite number of finite blocks and at most one infinite block, which is maximal. We take to be the image of this set by PLS (minus the empty set, which corresponds to the trivial preorder), then K is an abtract simplicial set on the powerset of $\mathbb{N}$. Its vertices correspond to finite subsets of $\mathbb{N}$, it has no facets. Actually we don't need α to be $\mathbb{N}$, it could be any set. It is empty if α is empty or a singleton, and if α is finite of cardinality at least 2 then K is pure of dimension $\operatorname{card}(\alpha) - 2$; if α is infinite then K has no facets.

What are the ideals/maximal ideals of K? We suppose that α is infinite, otehrwise any ideal is principal. Then we have the principal ideals, but none of them is maximal. In general, the set of vertices corresponding to an ideal will be a set of proper finite subsets of E that is totally ordered by inclusion. It will be maximal if it contains an element of cardinality n for every positive integer n. Then the corresponding total preorder will be ω followed by one block containing an equivalence class of biggest element; this last block can be empty, though only if α is countable of course.