I. Varieties

The basic object of study in algebraic geometry is an arbitrary prescheme. However, among all preschemes, the classical ones known as varieties are by far the most accessible to intuition. Moreover, in dealing with varieties one can carry over without any great difficulty the elementary methods and results of the other geometric categories, i.e., of topological spaces, differentiable manifolds or of analytic spaces. Finally, in any study of general preschemes, the varieties are bound, for many reasons which I will not discuss here, to play a unique and central role. Therefore it is useful and helpful to have a basic idea of what a variety is before plunging into the general theory of preschemes. We will fix throughout an algebraically closed ground field k which will never vary. We shall restrict ourselves to the purely geometric operations on varieties in keeping with the aim of establishing an intuitive and geometric background: thus we will not discuss specialization, nor will we use generic points. This set-up is the one pioneered by Serre in his famous paper "Faisceaux algébriques cohérents". There is no doubt that it is completely adequate for the discussion of nearly all purely geometric questions in algebraic geometry.

§1. Some algebra

We want to study the locus V of roots of a finite set of polynomials $f_i(X_1, \ldots, X_n)$ in k^n (k being an algebraically closed field). However, the basic tool in this study is the ring of functions from V to k obtained by restricting polynomials from k^n to V. And we cannot get very far without knowing something about the algebra of such a ring. The purpose of this section is to prove 2 basic theorems from commutative algebra that are key tools in analyzing these rings, and hence also the loci such as V. We include these results because of their geometric meaning, which will emerge gradually in this chapter (cf. §7). On the other hand, we assume known the following topics in algebra:

1) The essentials of field theory (Galois theory, separability, transcendence degree).
2) Localization of a ring, the behaviour of ideals in localization, the concept of a local ring.
3) Noetherian rings, and the decomposition theorem of ideals in these rings.
4) The concept of integral dependence, (cf., for example, Zariski-Samuel, vol. 1).

D. Mumford: LNM 1358, pp. 1–63, 1999.

The first theorem is:

Noether's Normalization Lemma. *Let R be an integral domain, finitely generated over a field k. If R has transcendence degree n over k, then there exist elements $x_1, \ldots, x_n \in R$, algebraically independent over k, such that R is integrally dependent on the subring $k[x_1, \ldots, x_n]$ generated by the x's.*

Proof (Nagata). Since R is finitely generated over k, we can write R as a quotient:

$$R = k[Y_1, \ldots, Y_m]/P ,$$

for some prime ideal P. If $m = n$, then the images $y_1, \ldots, y_m$ of the Y's in R must be algebraically independent themselves. Then $P = (0)$, and if we let $x_i = y_i$, the lemma follows. If $m > n$, we prove the theorem by induction on m. It will suffice to find a subring S in R generated by $m - 1$ elements and such that R is integrally dependent on S. For, by induction, we know that S has a subring $k[x_1, \ldots, x_n]$ generated by n independent elements over which it is integrally dependent; by the transitivity of integral dependence, R is also integrally dependent on $k[x_1, \ldots, x_n]$ and the lemma is true for R.

Now the m generators $y_1, \ldots, y_m$ of R cannot be algebraically independent over k since $m > n$. Let

$$f(y_1, \ldots, y_m) = 0$$

by some non-zero algebraic relation among them (i.e., $f(y_1, \ldots, y_m)$ is a non-zero polynomial in P). Let $r_1, \ldots, r_m$ be positive integers, and let

$$z_2 = y_2 - y_1^{r_2}, \qquad z_3 = y_3 - y_1^{r_3}, \ldots, z_m = y_m - y_1^{r_m} .$$

Then

$$f(y_1, z_2 + y_1^{r_2}, \ldots, z_m + y_1^{r_m}) = 0 ,$$

i.e., $y_1, z_2, \ldots, z_m$ are roots of the polynomial $f(Y_1, Z_2 + Y_1^{r_2}, \ldots, Z_m + Y_1^{r_m})$.

Each term $a \cdot \prod_{i=1}^{m} y_i^{b_i}$ in f gives rise to various terms in this new polynomial, including one monomial term

$$a \cdot y_1^{b_1 + r_2 b_2 + \ldots + r_m b_m} .$$

A moment's reflection will convince the reader that if we pick the r_i's to be large enough, and increasing rapidly enough:

$$0 \ll r_2 \ll r_3 \ll \ldots \ll r_m,$$

then these new terms $a \cdot Y_1^{b_1 + \ldots + r_m b_m}$ will all have distinct degrees, and one of them will emerge as the term of highest order in this new polynomial. Therefore,

$$f(Y_1, Z_2 + Y_1^{r_2}, \ldots, Z_m + Y_1^{r_m}) = b \cdot Y_1^N + \quad [\text{ terms of degree } < N] ,$$

$(b \neq 0)$. This implies that the equation $f(y_1, z_2 + y_1^{r_2}, \ldots, z_m + y_1^{r_m}) = 0$ is an equation of integral dependence for y_1 over the ring $k[z_2, \ldots, z_m]$. Thus y_1 is integrally dependent on $k[z_2, \ldots, z_m]$, so $y_2, \ldots, y_m$ are too since $y_i = z_i + y_1^{r_i}$ $(i = 2, \ldots, m)$. Therefore the whole ring R is integrally dependent on the subring $S = k[z_2, \ldots, z_m]$. By induction, this proves the lemma. $\square$

The second important theorem is:

Going-up theorem of Cohen-Seidenberg. *Let R be a ring (commutative as always) and $S \subset R$ a subring such that R is integrally dependent on S. For all prime ideals $P \subset S$, there exist prime ideals $P' \subset R$ such that $P' \cap S = P$.*

Proof. Let M be the multiplicative system $S - P$. Then we may as well replace R and S by their localizations R_M and S_M with respect to M. For S_M is still a subring of R_M, and R_M is still integrally dependent on S_M. In fact, we get a diagram:

$$\begin{tikzcd} R \arrow[r, "j"] & R_M \\ \cup & \cup \\ S \arrow[r, "i"'] & S_M . \end{tikzcd}$$

Moreover S_M is a local ring, with maximal ideal $P_M = i(P) \cdot S_M$ and $P = i^{-1}(P_M)$. If $P^* \subset R_M$ is a prime ideal of R_M such that $P^* \cap S_M = P_M$, then $j^{-1}(P^*)$ is a prime ideal in R such that

$$j^{-1}(P^*) \cap S = i^{-1}(P^* \cap S_M) = i^{-1}(P_M) = P.$$

Therefore, it suffices to prove the theorem for R_M and S_M.

Therefore we may assume that S is a local ring and P is its unique maximal ideal. In this case, for all ideals $A \subset R$, $A \cap S \subseteq P$. I claim that for *all* maximal ideals $P' \subset S$, $P' \cap S$ equals P. Since maximal ideals are prime, this will prove the theorem. Take some maximal ideal P'. Then consider the pair of quotient rings:

$$\begin{tikzcd} R \arrow[r] & R/P' \\ \cup & \cup \\ S \arrow[r] & S/S \cap P' . \end{tikzcd}$$

Since P' is maximal, R/P' is a field. If we can show that the subring $S/S \cap P'$ is a field too, then $S \cap P'$ must be a maximal ideal in S, so $S \cap P'$ must equal P and the theorem follows. Therefore, we have reduced the question to:

Lemma. *Let R be a field, and $S \subset R$ a subring such that R is integrally dependent on S. Then S is a field.*

Note that this is a special case of the theorem: For if S were not a field, it would have non-zero maximal ideals and these could not be of the form $P' \cap S$ since R has no non-zero ideals at all.

Proof of lemma. Let $a \in S$, $a \neq 0$. Since R is a field, $1/a \in R$. By assumption, $1/a$ is integral over S, so it satisfies an equation

$$X^n + b_1 X^{n-1} + \ldots + b_n = 0,$$

$b_i \in S$. But this means that

$$\frac{1}{a^n} + \frac{b_1}{a^{n-1}} + \ldots + b_n = 0.$$

Multiply this equation by a^{n-1} and we find

$$\frac{1}{a} = -b_1 - ab_2 - \ldots - a^{n-1}b_n \in S.$$

Therefore S is a field. □

Using both of these results, we can now prove:

Weak Nullstellensatz. *Let k be an algebraically closed field. Then the maximal ideals in the ring $k[X_1, \ldots, X_n]$ are the ideals*

$$(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n),$$

where $a_1, \ldots, a_n \in k$.

Proof. Since the ideal $(X_1 - a_1, \ldots, X_n - a_n)$ is the kernel of the surjection:

$$\begin{array}{ccc} k[X_1, \ldots, X_n] & \longrightarrow & k \\ f[X_1, \ldots, X_n] & \longmapsto & f(a_1, \ldots, a_n), \end{array}$$

it follows that $k[X_1, \ldots, X_n] / (X_1 - a_1, \ldots, X_n - a_n) = k$, hence the ideal $(X_1 - a_1, \ldots, X_n - a_n)$ is maximal. Conversely, let $M \subset k[X_1, \ldots, X_n]$ be a maximal ideal. Let $R = k[X_1, \ldots, X_n]/M$. R is a field since M is maximal, and R is also finitely generated over k as a ring. Let r be the transcendence degree of R over k.

The crux of the proof consists in showing that $r = 0$: By the normalization lemma, find a subring $S \subset R$ of the form $k[y_1, \ldots, y_r]$ such that R is integral over S. Since the y_i's are algebraically independent, S is a polynomial ring in r variables. By the going-up theorem – in fact, by the special case given in the lemma – S must be a field too. But a polynomial ring in r variables is a field only when $r = 0$.

Therefore R is an algebraic extension field of k. Since k is algebraically closed, R must equal k. In other words, the subset k of $k[X_1, \ldots, X_n]$ goes *onto* $k[X_1, \ldots, X_n]/M$. Therefore

$$k + M = k[X_1, \ldots, X_n].$$

In particular, each variable X_i is of the form $a_i + m_i$, with $a_i \in k$ and $m_i \in M$. Therefore, $X_i - a_i \in M$ and M contains the ideal $(X_1 - a_1, \ldots, X_n - a_n)$. But the latter is maximal already, so $M = (X_1 - a_1, \ldots, X_n - a_n)$. □

The great importance of this result is that it gives us a way to translate affine space k^n into pure algebra. We have a bijection between k^n, on the one hand, and the set of maximal ideals in $k[X_1, \dots, X_n]$ on the other hand. This is the origin of the connection between algebra and geometry that gives rise to our whole subject.

§2. Irreducible algebraic sets

For the rest of this chapter k will denote a fixed algebraically closed field, known as the *ground field.*

Definition 1. A *closed algebraic subset* of k^n is a set consisting of all roots of a finite collection of polynomial equations: i.e.,

$$\{(x_1, \dots, x_n) \mid f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0\} .$$

It is clear that the above set depends only on the ideal $A = (f_1, \dots, f_m)$ generated by the f_i's in $[X_1, \dots, X_n]$ and not on the actual polynomials f_i. Therefore, if A is any ideal in $k[X_1, \dots, X_n]$, we define

$$V(A) = \{x \in k^n \mid f(x) = 0 \quad \text{for all } f \in A\} .$$

Since $k[X_1, \dots, X_m]$ is a noetherian ring, the subsets of k^n of the form $V(A)$ are exactly the closed algebraic sets. On the other hand, if Σ is a closed algebraic set, we define

$$I(\Sigma) = \{f \in k[X_1, \dots, X_n] \mid f(x) = 0 \quad \text{for all } x \in \Sigma\} .$$

Clearly $I(\Sigma)$ is an ideal such that $\Sigma = V(I(\Sigma))$. The key result is:

Theorem 1 (Hilbert's Nullstellensatz).

$$I(V(A)) = \sqrt{A} .$$

Proof. It is clear that $\sqrt{A} \subset I(V(A))$. The problem is to show the other inclusion. Put concretely this means the following:

> Let $A = (f_1, \dots, f_m)$. If $g \in k[X_1, \dots, X_n]$ satisfies:
>
> $$\{f_1(a_1, \dots, a_n) = \dots = f_m(a_1, \dots, a_n) = 0\} \Longrightarrow g(a_1, \dots, a_n) = 0$$
>
> then there is an integer ℓ and polynomials $h_1, \dots, h_m$ such that
>
> $$g^\ell(X) = \textstyle\sum_{i=1}^m h_i(X) \cdot f_i(X).$$

To prove this, introduce the ideal

$$B = A \cdot k[X_1, \dots, X_n, X_{n+1}] + (1 - g \cdot X_{n+1})$$

in $k[X_1, \ldots, X_{n+1}]$. There are 2 possibilities: either B is a proper ideal, or $B = k[X_1, \ldots, X_{n+1}]$. In the first case, let M be a maximal ideal in $k[X_1, \ldots, X_{n+1}]$ containing B. By the weak Nullstellensatz of §1,

$$M = (X_1 - a_1, \ldots, X_n - a_n, X_{n+1} - a_{n+1})$$

for some elements $a_i \in k$. Since M is the kernel of the homomorphism:

$$\begin{array}{rcl} k[X_1, \ldots, X_n, X_{n+1}] & \longrightarrow & k \\ f & \longmapsto & f(a_1, \ldots, a_{n+1}), \end{array}$$

$B \subset M$ means that:

$$i) \qquad f_1(a_1, \ldots, a_n) = \ldots = f_m(a_1, \ldots, a_n) = 0$$

and

$$ii) \qquad 1 = g(a_1, \ldots, a_n) \cdot a_{n+1}.$$

But by our assumptionon g, (i) implies that $g(a_1, \ldots, a_n) = 0$, and this contradicts (ii). We can only conclude that the ideal B would not have been a proper ideal.

But then $1 \in B$. This means that there are polynomials $h_1, \ldots, h_m, h_{m+1} \in k[X_1, \ldots, X_{n+1}]$ such that:

$$\begin{aligned} 1 \;=\; & \sum_{i=1}^{m} h_i(X_1, \ldots, X_{m+1}) \cdot f_i(X_1, \ldots, X_n) \\ & + (1 - g(X_1, \ldots, X_n) \cdot X_{n+1}) \cdot h_{m+1}(X_1, \ldots, X_{n+1}). \end{aligned}$$

Substituting g^{-1} for X_{n+1} in this formula, we get:

$$1 = \sum_{i=1}^{m} h_i(X_1, \ldots, X_n, 1/g) \cdot f_i(X_1, \ldots, X_n).$$

Clearing denominators, this gives:

$$g^\ell(X_1, \ldots, X_n) = \sum_{i=1}^{m} h_i^*(X_1, \ldots, X_n) \cdot f_i(X_1, \ldots, X_n)$$

for some new polynomials $h_i^* \in k[X_1, \ldots, X_n]$, i.e., $g \in \sqrt{A}$. □

Corollary. *V and I set up a bijection between the set of closed algebraic subsets of k^n and the set of ideals $A \subset k[X_1, \ldots, X_n]$ such that $A = \sqrt{A}$.*

This correspondence between algebraic sets and ideals is compatible with the lattice structures:

i) $A \subset B \Longrightarrow V(A) \supset V(B)$
ii) $\Sigma_1 \subset \Sigma_2 \Longrightarrow I(\Sigma_1) \supset I(\Sigma_2)$
iii) $V\left(\sum_\alpha A_\alpha\right) = \bigcap_\alpha V(A_\alpha)$

iv) $V(A \cap B) = V(A) \cup V(B)$

where A, B, A_α are ideals, Σ_1, Σ_2 closed algebraic sets.

Proof. All are obvious except possibly (iv). But by (i), $V(A \cap B) \supset V(A) \cup V(B)$. Conversely, if $c \notin V(A) \cup V(B)$, then there exist polynomials $f \in A$ and $g \in B$ such that $f(x) \neq 0$, $g(x) \neq 0$. But then $f \cdot g \in A \cap B$ and $(f \cdot g)(x) \neq 0$, hence $x \notin V(A \cap B)$. □

Definition 2. A closed algebraic set is *irreducible* if it is not the union of two strictly smaller closed algebraic sets. (We shall omit "closed" in referring to these sets).

Recall that by the noetherian decomposition theorem, if $A \subset k[X_1, \ldots, X_n]$ is an ideal such that $A = \sqrt{A}$, then A can be written in exactly one way as an intersection of a finite set of *prime* ideals, none of which contains any other. And a prime ideal is not the intersection of any two strictly bigger ideals. Therefore:

Proposition 2. *In the bijection of the Corollary to Theorem 1, the irreducible algebraic sets correspond exactly to the prime ideals of $k[X_1, \ldots, X_n]$. Moreover, every closed algebraic set Σ can be written in exactly one way as:*

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$$

where the Σ_i are irreducible sets and $\Sigma_i \not\subseteq \Sigma_j$ if $i \neq j$.

Definition 3. The Σ_i of Proposition 2 will be called the *components* of Σ.

In the early 19th century it was realized that for many reasons it was inadequate and misleading to consider only the above "affine" algebraic sets. Among others, Poncelet realized that an immense simplification could be introduced in many questions by considering "projective" algebraic sets (cf. Felix Klein, *Die Entwicklung der Mathematik*, part I, pp. 80–82). Even to this day, there is no doubt that projective algebraic sets play a central role in algebro-geometric questions; therefore we shall define them as soon as possible.

Recall that, by definition, $\mathbb{P}_n(k)$ is the set of $(n+1)$-tuples $(x_0, \ldots, x_n) \in k^{n+1}$ such that some $x_i \neq 0$, modulo the equivalence relation

$$(x_0, \ldots, x_n) \sim (\alpha x_0, \ldots, \alpha x_n), \qquad \alpha \in k^*,$$

(where k^* is the multiplicative group of non-zero elements of k). Then an $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$ is called a set of *homogeneous coordinates* for the point associated to it. $\mathbb{P}_n(k)$ can be covered by $n+1$ subsets $U_0, U_1, \ldots, U_n$, where

$$U_i = \left\{ \begin{array}{l} \text{points represented by homogeneous} \\ \text{coordinates } (x_0, x_1, \ldots, x_n) \text{ with } x_i \neq 0 \end{array} \right\}.$$

Each U_i is naturally isomorphic to k^n under the map

$$\begin{array}{rcl} U_i & \longrightarrow & k^n \\ (x_0, x_1, \ldots, x_n) & \longmapsto & \left(\dfrac{x_0}{x_i}, \dfrac{x_1}{x_i}, \ldots, \dfrac{x_n}{x_i}\right), \qquad \left(\dfrac{x_i}{x_i} \text{ omitted}\right). \end{array}$$

The original motivation for introducing $\mathbb{P}_n(k)$ was to add to the affine space $k^n \cong U_0$ the extra "points at infinity" $\mathbb{P}_n(k) - U_0$ so as to bring out into the open the mysterious things that went on at infinity.

Recall that to all subvectorspaces $W \subset k^{n+1}$ one associates the set of points $P \in \mathbb{P}_n(k)$ with homogeneous coordinates in W: the sets so obtained are called the linear subspaces L of $\mathbb{P}_n(k)$. If W has codimension 1, then we get a hyperplane. In particular, the points "at infinity" with respect to the affine piece U_i form the hyperplane associated to the subvectorspace $x_i = 0$. Moreover, by introducing a basis into W, the linear subspace L associated to W is naturally isomorphic to $\mathbb{P}_r(k)$, ($r = \dim W - 1$). The linear subspaces are the simplest examples of projective algebraic sets:

Definition 4. A *closed algebraic set* in $\mathbb{P}_n(k)$ is a set consisting of all roots of a finite collection of homogeneous polynomials $f_i \in k[X_0, \ldots, X_n]$, $1 \le i \le m$. This makes sense because if f is homogeneous, and $(x_0, \ldots, x_n)$, $(\alpha x_0, \ldots, \alpha x_n)$ are 2 sets of homogeneous coordinates of the same point, then

$$f(x_0, \ldots, x_n) = 0 \Longleftrightarrow f(\alpha x_0, \ldots, \alpha x_n) = 0.$$

We can now give a projective analog of the V, I correspondence used in the affine case. We shall, of course, now use only homogeneous ideals $A \subset k[X_0, \ldots, X_n]$: i.e., ideals which, when they contain a polynomial f, also contain the homogeneous components of f. Equivalently, homogeneous ideals are the ideals generated by a finite set of homogeneous polynomials. If A is a homogeneous ideal, define

$$V(A) = \left\{ P \in \mathbb{P}_n(k) \;\middle|\; \begin{array}{l} \text{If } (x_0, \ldots, x_n) \text{ are homogeneous coordinates} \\ \text{of } P, \text{ then } f(x) = 0, \text{ all } f \in A \end{array} \right\}.$$

If $\Sigma \subset \mathbb{P}_n(k)$ is a closed algebraic set, then define

$$I(\Sigma) = \left\{ \begin{array}{l} \text{ideal generated by all homogeneous polynomials} \\ \text{that vanish identically on } \Sigma \end{array} \right\}.$$

Theorem 3. *V and I set up a bijection between the set of closed algebraic subsets of $\mathbb{P}_n(k)$, and the set of all homogeneous ideals $A \subset k[X_0, \ldots, X_n]$, such that $A = \sqrt{A}$ except for the one ideal $A = (X_0, \ldots, X_n)$.*

Proof. It is clear that if Σ is a closed algebraic set, then $V(I(\Sigma)) = \Sigma$. Therefore, in any case, V and I set up a bijection between closed algebraic subsets of $\mathbb{P}_n(k)$ and those homogeneous ideals A such that:

$$(*) \qquad A = I(V(A)).$$

These ideals certainly equal their own radical. Moreover, the empty set is $V((X_0,\ldots,X_n))$, hence $1 \in I(V((X_0,\ldots,X_n)))$; so $(X_0,\ldots,X_n)$ does not satisfy $(*)$ and must be excluded. Finally, let A be any other homogeneous ideal which equals its own radical. Let $V^*(A)$ be the closed algebraic set corresponding to A in the *affine* space k^{n+1} with coordinates $X_0,\ldots,X_n$. Then $V^*(A)$ is invariant under the substitutions

$$(X_0,\ldots,X_n) \longrightarrow (\alpha X_0,\ldots,\alpha X_n) \ ,$$

all $\alpha \in k$. Therefore, either

1) $V^*(A)$ is empty,
2) $V^*(A)$ equals the origin only, or
3) $V^*(A)$ is a union of lines through the origin: i.e., it is the cone over the subset $V(A)$ in $\mathbb{P}_n(k)$.

Moreover, by the affine Nullstellensatz, we know that

$$(**) \qquad A = I(V^*(A)).$$

In case 1), $(**)$ implies that $A = k[X_0,\ldots,X_n]$, hence $I(V(A))$ – which always contains A – must equal A since there is no bigger ideal. In case 2), $(**)$ implies that $A = (X_0,\ldots,X_n)$ which we have excluded. In case 3), if f is a homogeneous polynomial, then f vanishes on $V(A)$ if and only if f vanishes on $V^*(A)$. Therefore by $(**)$, if f vanishes on $V(A)$, then $f \in A$, i.e.,

$$A \supset I(V(A)).$$

Since the other inclusion is obvious, Theorem 3 is proven. □

The same lattice-theoretic identities hold as in the affine case. Moreover, we define *irreducible* algebraic sets exactly as in the affine case. And we obtain the analog of Proposition 2:

Proposition 4. *In the bijection of Theorem 3, the irreducible algebraic sets correspond exactly to the homogeneous prime ideals ($(X_0,\ldots,X_n)$ being accepted). Moreover, every closed algebraic set Σ in $\mathbb{P}_n(k)$ can be written in excatly one way as:*

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k \ ,$$

where the Σ_i are irreducible algebraic sets and $\Sigma_i \not\subseteq \Sigma_j$ if $i \neq j$.

Problem. Let $\Sigma \subset \mathbb{P}_n(k)$ be a closed algebraic set, and let H be the hyperplane $X_0 = 0$. Identify $\mathbb{P}_n - H$ with k^n in the usual way. Prove that $\Sigma \cap (\mathbb{P}_n - H)$ is a closed algebraic subset of k^n and show that the ideal of $\Sigma \cap (\mathbb{P}_n - H)$ is derived from the ideal of Σ in a very natural way. For details on the relationship between the affine and projective set-up and on everything discussed so far, read Zariski-Samuel, vol. 2, Ch. 7, §§3,4,5 and 6.

Example A. Hypersurfaces. Let $f(X_0, \ldots, X_n)$ be an irreducible homogeneous polynomial. Then the principal ideal (f) is prime, so $f = 0$ defines an irreducible algebraic set in $\mathbb{P}_n(k)$ called a hypersurface (e.g. plane curve, surface in 3-space, etc.).

Example B. The twisted cubic in $\mathbb{P}_3(k)$. This example is given to show the existence of nontrivial examples: Start with the ideal:

$$A_0 = \left(xz - y^2, yw - z^2\right) \subset k[x, y, z, w].$$

$V(A_0)$ is just the intersection in $\mathbb{P}_3(k)$ of the 2 quadrics $xz = y^2$ and $yw = z^2$. Look in the affine space with coordinates

$$X = x/w, \qquad Y = y/w, \qquad Z = z/w$$

(the complement of $w = 0$). In here, $V(A_0)$ is the intersection of the ordinary cone $XZ = Y^2$, and of the cylinder over the parabola $Y = Z^2$. This intersection falls into 2 pieces: the line $Y = Z = 0$, and the twisted cubic itself. Correspondingly, the ideal A_0 is an intersection of the ideal of the line and of the twisted cubic:

$$A_0 = (y, z) \cap \underbrace{\left(xz - y^2,\ yw - z^2,\ xw - yz\right)}_{A}\ .$$

The twisted cubic is, by definition, $V(A)$. [To check that A is prime, the simplest method is to verify that A is the kernel of the homomorphism ϕ:

$$k[x, y, z, w] \xrightarrow{\phi} k[s, t]$$

$$\begin{aligned} \phi(x) &= s^3 \\ \phi(y) &= s^2 t \\ \phi(z) &= s t^2 \\ \phi(w) &= t^3 \ ]\,. \end{aligned}$$

In practice, it may be difficult to tell whether a given ideal is prime or whether a given algebraic set is irreducible. It is relatively easy for principal ideals, i.e., for hypersurfaces, but harder for algebraic sets of higher codimension. A good deal of effort used to be devoted to compiling lists of *all* types of irreducible algebraic sets of given dimension and "degree" when these were small numbers. In Semple and Roth, *Algebraic Geometry,* one can find the equivalent of such lists. A study of these will give one a fair feeling for the menagerie of algebraic sets that live in $\mathbb{P}_3$, $\mathbb{P}_4$ or $\mathbb{P}_5$ for example. As for the general theory, it is far from definitive however.

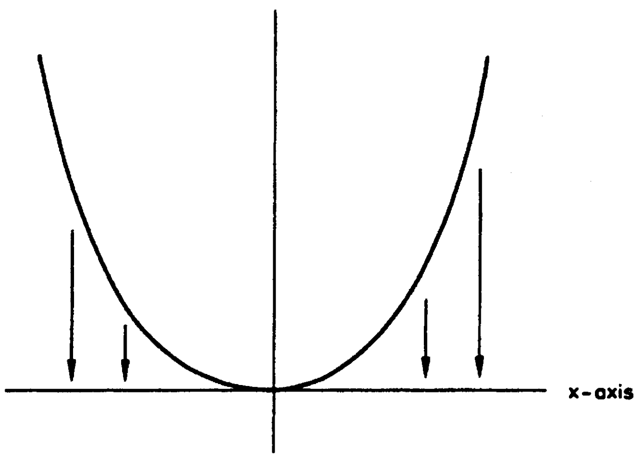
§3. Definition of a morphism

We will certainly want to know when 2 algebraic sets are to be considered isomorphic. More generally, we will need to define not just the *set* of all algebraic sets, but the *category* of algebraic sets (for simplicity, in Chapter 1, we will stick to the irreducible ones).

Example C. Look at

a) k, the affine line,
b) $y = x^2$ in k^2, the parabola.

Projecting the parabola onto the x-axis should surely be an isomorphism between these algebraic sets:



More generally, if $V \subset k^n$ is an irreducible algebraic set, and if $f \in k[X_1, \ldots, X_n]$, then the set of points:

$$V^* = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \mid (x_1, \ldots, x_n) \in V\} \subset k^{n+1}$$

is an irreducible algebraic set. And the projection

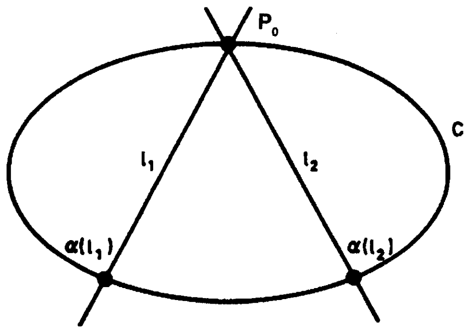
$$(x_1, x_2, \ldots, x_{n+1}) \longmapsto (x_1, x_2, \ldots, x_n)$$

should define an isomorphism from V^* to V.

Example D. An irreducible conic $C \subset \mathbb{P}_2(k)$ will turn out to be *isomorphic* to the projective line $\mathbb{P}_1(k)$ under the following map: fix a point $P_0 \in C$. Identify $\mathbb{P}_1(k)$ with the set of all lines through P_0 in the classical way. Then define a map

$$\mathbb{P}_1(k) \xrightarrow{\alpha} C$$

by letting $\alpha(\ell)$ for all lines ℓ through P_0 be the second point in which ℓ meets C, besides P_0. Also, if ℓ is the tangent line to C at P_0, define $\alpha(\ell)$ to be P_0 itself (since P_0 is a "double" intersection of C and this tangent line).



Example E. $\mathbb{P}_1(k)$ and the twisted cubic $\mathbb{P}_3(k)$ will be isomorphic. For $\mathbb{P}_1(k)$ consists of the pairs (s,t) modulo $(s,t) \sim (\alpha s, \alpha t)$, i.e., of the set of ratios $\beta = s/t$ including $\beta = \infty$. Define a map

$$\mathbb{P}_1(k) \xrightarrow{\alpha} \mathbb{P}_3(k)$$

by $\alpha(\beta)$ = the point with homogeneous coordinates $(1, \beta, \beta^2, \beta^3)$ and $\alpha(\infty) = (0,0,0,1)$.

The image points clearly satisfy $xz = y^2$, $yw = z^2$ and $zw = yz$, so they are on the twisted cubic. The reader can readily check that α maps $\mathbb{P}_1(k)$ *onto* the twisted cubic.

Example F. Let C be a cubic curve in $\mathbb{P}_2(k)$ and let $P_0 \in C$. For any point $P \in C$, let ℓ be the line through P and P_0 and let $\alpha(P)$ be the third point in which ℓ meets C. Although this may not seem as obvious as the previous examples, α will be an automorphism of C of order 2.

We shall use this example later to work out our definition in a nontrivial case.

Now turn to the problem of actually defining morphisms, and hence isomorphisms, of irreducible algebraic sets. First consider the case of 2 irreducible *affine* algebraic sets.

Definition 1. Let $\Sigma_1 \subset k^{n_1}$ and $\Sigma_2 \subset k^{n_2}$ be two irreducible algebraic sets. A map

$$\alpha : \Sigma_1 \longrightarrow \Sigma_2$$

will be called a *morphism* if there exist n_2 polynomials $f_1, \ldots, f_{n_2}$ in the variables $X_1, \ldots, X_{n_1}$ such that

$$(*) \qquad \alpha(x) = (f_1(x_1, \ldots, x_{n_1}), \ldots, f_{n_2}(x_1, \ldots, x_{n_1}))$$

for all points $x = (x_1, \ldots, x_{n_1}) \in \Sigma_1$.

Note one feature of this definition: it implies that every morphism α from Σ_1 to Σ_2 is the restriction of a morphism α' from k^{n_1} to k^{n_2}. This may look odd at first, but it turns out to be reasonable – cf. §4. Note also that with this definition the map in Example C above is an isomorphism, i.e., both it and its inverse are morphisms.

To analyze the definition further, suppose

$$\begin{array}{ccc} P_1 & \subset & k[X_1,\ldots,X_{n_1}] \\ P_2 & \subset & k[X_1,\ldots,X_{n_2}] \end{array}$$

are the prime ideals $I(\Sigma_1)$ and $I(\Sigma_2)$ respectively. Set

$$R_i = k[X_1,\ldots,X_{n_i}]/P_i, \qquad i = 1,2 \ .$$

Then R_1 (resp. R_2) is just the ring of k-valued functions on Σ_1 (resp. Σ_2) obtained by restricting the ring of polynomial functions on the ambient affine space. Suppose $g \in R_2$. Regarding g as a function on Σ_2, the definition of morphism implies that the function $g \cdot \alpha$ on Σ_1 is in R_1 – in fact

$$(g \cdot \alpha)(X_1,\ldots,X_{n_1}) = g(f_1(X_1,\ldots,X_{n_1}),\ldots,f_{n_2}(X_1,\ldots,X_{n_1})).$$

Therefore α induces a k-homomorphism:

$$\alpha^* : R_2 \longrightarrow R_1.$$

Moreover, note that α is determined by α^*. This is so because the polynomials $f_1,\ldots,f_{n_2}$ can be recovered – up to an element of P_1 – as $\alpha^*(X_1),\ldots,\alpha^*(X_{n_2})$; and the point $\alpha(x)$, for $x \in \Sigma_1$, is determined via $f_1,\ldots,f_{n_2}$ *modulo* P_1 by equation $(*)$. Even more is true. Suppose you start with an arbitrary k-homomorphism

$$\lambda : R_2 \longrightarrow R_1.$$

Let f_1 be a polynomial in $k[X_1,\ldots,X_{n_1}]$ whose image modulo P_1 equals $\lambda(X_i)$, for all $1 \le i \le n_2$. Then define a map

$$\alpha' : k^{n_1} \longrightarrow k^{n_2}$$

by

$$\alpha'(x_1,\ldots,x_{n_1}) = (f_1(x_1,\ldots,x_{n_1}),\ldots,f_{n_2}(x_1,\ldots,x_{n_1})).$$

If $x = (x_1,\ldots,x_{n_1}) \in \Sigma_1$, then actually $\alpha'(x)$ will be in Σ_2: for if $g \in P_2$, then

$$g(\alpha'(x)) = g(f_1(x),\ldots,f_{n_2}(x)).$$

But $g(f_1,\ldots,f_{n_2}) \equiv g(\lambda(X_1),\ldots,\lambda(X_{n_2}))$ modulo P_1

$$\begin{aligned} &\equiv \lambda(g) \text{ modulo } P_1 \\ &\equiv 0 \qquad \text{modulo } P_1 \ . \end{aligned}$$

Therefore, $g(\alpha'(x)) = 0$ and $\alpha'(x) \in \Sigma_2$.

We can summarize this discussion in the following:

Definition 2. Let $\Sigma \subset k^n$ be an irreducible algebraic set. Then the *affine ring* $\Gamma(\Sigma)$ is the ring of k-valued functions on Σ given by polynomials in the coordinates, i.e.,

$$k\left[X_1, \ldots, X_n\right] / I(\Sigma).$$

Proposition 1. *If Σ_1, Σ_2 are two irreducible algebraic sets, then the set of morphisms from Σ_1 to Σ_2 and the set of k-homomorphisms from $\Gamma(\Sigma_2)$ to $\Gamma(\Sigma_1)$ are canonically isomorphic:*

$$\hom\left(\Sigma_1, \Sigma_2\right) \cong \hom_k\left(\Gamma(\Sigma_2), \Gamma(\Sigma_1)\right).$$

Corollary. *If Σ is an irreducible algebraic set, then $\Gamma(\Sigma)$ is canonically isomorphic to the set of morphisms from Σ to k.*

Proof. Note that $\Gamma(k)$ is just $k[X]$.

Using $\Gamma(\Sigma)$, we can define a subset $V(A)$ of Σ for any ideal $A \subset \Gamma(\Sigma)$ as

$$\{x \in \Sigma \mid f(x) = 0 \text{ for all } f \in A\}.$$

Moreover, the Nullstellensatz for k^n implies immediately the Nullstellensatz for Σ:

$$\{f \in \Gamma(\Sigma) \mid f(x) = 0 \text{ for all } x \in V(A)\} = \sqrt{A}\ .$$

Even more than Proposition 1 is true:

Proposition 2. *The assignment*

$$\Sigma \longmapsto \Gamma(\Sigma)$$

extends to a contravariant functor Γ:

$$\left\{\begin{array}{l}\textit{Category of irreducible} \\ \textit{algebraic sets + morphisms}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{Category of finitely generated integral} \\ \textit{domains over } k + k\textit{-homomorphisms}\end{array}\right\}$$

which is an equivalence of categories.

Proof. Prop. 1 asserts that Γ is a fully faithful functor. The other fact to check is that every finitely generated integral domain R over k occurs as $\Gamma(\Sigma)$. But every such domain can be represented as:

$$R \cong k\left[X_1, \ldots, X_n\right] / \left(f_1, \ldots, f_m\right),$$

hence as $\Gamma(\Sigma)$ where Σ is the locus of zeroes of $f_1, \ldots, f_m$ in k^n. □

Because of the usefulness of continuity in topology and other parts of geometry, another natural question is whether there is a natural topology on irreducible algebraic sets in which all morphisms are continuous. We will certainly want points of k to be closed, so their inverse images by morphisms must be closed. If we take the weakest topology satisfying this condition, we get the following:

Definition 3. A *closed set* in k^n is to be a closed algebraic set $V(A)$. By the results of §2, these define a topology in k^n, called the *Zariski topology*. An irreducible algebraic set $\Sigma \subset k^n$ is given the induced topology, again called the Zariski topology. It is clear that the closed sets of Σ are exactly the sets $V(A)$, where A is an ideal in $\Gamma(\Sigma)$.

It is easy to check that all morphisms are continuous in the Zariski topology. A basis for the open sets in the Zariski topology on Σ is given by the open sets:

$$\Sigma_f = \{x \in \Sigma \mid f(x) \neq 0\}$$

for elements $f \in \Gamma(\Sigma)$. In fact, $\Sigma_f = \Sigma - V((f))$, hence Σ_f is open. And if $U = \Sigma - V(A)$ is an arbitrary open set, then

$$U = \bigcup_{f \in A} \Sigma_f.$$

One should notice that the Zariski topology is very weak. On k itself, for instance, it is just the topology of finite sets, the weakest T_1 topology (since any ideal A in $k[X]$ is principal – $A = (f)$ – therefore $V(A)$ is just the finite set of roots of f). It follows that any bijection $\alpha : k \to k$ is continuous, so not all continuous maps are morphisms. In any case this is a very unclassical type of topological space.

Definition 4. A topological space X is *noetherian* if its closed sets satisfy the descending chain condition (d.c.c.). It is equivalent to require that all open sets be quasi-compact (= having the Heine-Borel covering property, but not necessarily T2).

Now since ideals in $k[X_1, \ldots, X_2]$ satisfy the a.c.c., it follows that closed sets satisfy the d.c.c. – so the Zariski topology is noetherian.

Our simple definition of morphisms for affine algebraic sets does not work for projective algebraic sets. The trouble is that it automatically implied that the morphism will extend to a morphism of the ambient affine space. There is no analogous fact in the projective case. Look at the case of Example D. Let Σ_1 be the conic with homogeneous equation

$$(*) \qquad xz = y^2$$

in $\mathbb{P}_2(k)$. Let $\Sigma_2 = \mathbb{P}_1(k)$. Let $P_0 \in \Sigma_1$ be the point $(0,0,1)$. To every point $Q \in \mathbb{P}_2(k) - \{P_0\}$, we can associate the line P_0Q, and by identifying the pencil of lines through P_0 with $\mathbb{P}_1(k)$ we get a point of $\mathbb{P}_1(k)$. In terms of coordinates, this can be expressed by the map:

$$(a,b,c) \longmapsto (a,b)$$

as long as (a,b,c) are not homogeneous coordinates for P_0, i.e., a or b is not zero. Let (s,t) be homogeneous coordinates in $\mathbb{P}_1(k)$. Then the map from Σ_1 to Σ_2 should be defined by:

$$(A) = \begin{cases} s & = & x \\ t & = & y \end{cases} .$$

Unfortunately, this is undefined at P_0 itself. But consider the second map defined by:

$$(B) = \begin{cases} s & = & y \\ t & = & z \end{cases} .$$

This is defined except at the point $P_1 \in \Sigma_1$ with coordinates $(1,0,0)$; moreover, at points on Σ_1 not equal to P_1 or P_2, the ratios $(x:y)$ and $(y:z)$ are equal in view of the equation $(*)$. Therefore A and B together define everywhere a map from Σ_1 to Σ_2. On the other hand, it will turn out that there are no surjective morphisms at all from $\mathbb{P}_2(k)$ to $\mathbb{P}_1(k)$ (cf. §7).

Thus defining morphisms between projective sets is more subtle. We find that we must define morphisms locally and patch them together. But the problem arises: on which local places. We could use the affine algebraic sets

$$\Sigma - \Sigma \cap H$$

where $H \subset \mathbb{P}_n(k)$ is a hyperplane. But in general these will not be small enough. We shall need arbitrarily small open sets in the Zariski-topology:

Definition 5. A *closed set* in $\mathbb{P}_n(k)$ is to be a closed algebraic set $V(A)$. By the results of §2, these define a topology in $\mathbb{P}_n(k)$, the *Zariski-topology.* Irreducible algebraic sets themselves are again given the induced topology. As in the affine case, a basis for the open sets is given by:

$$[\mathbb{P}_n(k)]_f = \{x \in \mathbb{P}_n(k) \mid f(x) \neq 0\}$$

where f is a *homogeneous* polynomial. Moreover, it is clear that the Zariski topology on $\mathbb{P}_n(k)$ is noetherian.

Problem. Check that $\mathbb{P}_n(k)_{X_0}$ is *homeomorphic* to k^n under the usual map

$$(x_0, \ldots, x_n) \longmapsto (x_1/x_0, x_2/x_0, \ldots, x_n/x_0) .$$

Finally, to define morphisms locally, we will need to attach affine coordinate rings to a lot of the Zariski-open sets U and give a definition of affine morphism in terms of local properties. Clearly, we should begin by constructing the apparatus used for defining things locally.

§4. Sheaves and affine varieties

Definition 1. Let X be a topological space. A *presheaf* F on X consists of

i) for all open $U \subset X$, a set $F(U)$
ii) for all pairs of open sets $U_1 \subset U_2$, a map ("restriction")

$$\mathrm{res}_{U_2 U_1} : F(U_2) \longrightarrow F(U_1)$$

such that the following axioms are satisfied:

a) $\mathrm{res}_{U,U} = \mathrm{id}_{F(U)}$ for all U.
b) If $U_1 \subset U_2 \subset U_3$, then

$$\begin{tikzcd}
F(U_3) \arrow[dd, "\mathrm{res}_{U_3 U_1}"'] \arrow[dr, "\mathrm{res}_{U_3 U_2}"] & \\
& F(U_2) \arrow[dl, "\mathrm{res}_{U_2 U_1}"] \\
F(U_1) &
\end{tikzcd}$$

commutes.

Definition 2. If F_1, F_2 are presheaves on X, a *map* $\varphi : F_1 \to F_2$ is a collection of maps $\varphi(U) : F_1(U) \to F_2(U)$ for each open U such that if $U \subset V$,

$$\begin{tikzcd}
F_1(V) \arrow[r, "\varphi(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & F_2(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
F_1(U) \arrow[r, "\varphi(U)"'] & F_2(U)
\end{tikzcd}$$

commutes.

Definition 3. A presheaf F is a *sheaf* if for every collection $\{U_i\}$ of open sets in X with $U = \cup U_i$, the diagram

$$F(U) \longrightarrow \prod F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact, i.e., the map

$$\prod \mathrm{res}_{U,U_i} : F(U) \longrightarrow \prod F(U_i)$$

is injective, and its image is the set on which

$$\prod \mathrm{res}_{U_i, U_i \cap U_j} : \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \cap U_j)$$

and

$$\prod \mathrm{res}_{U_j, U_i \cap U_j} : \prod_j F(U_j) \longrightarrow \prod_{i,j} F(U_i \cap U_j)$$

agree.

When we pull this high-flown terminology down to earth, it says this.

1) If $x_1, x_2 \in F(U)$ and for all i, $\mathrm{res}_{U,U_i} x_1 = \mathrm{res}_{U,U_i} x_2$, then $x_1 = x_2$. (That is, elements are uniquely determined by local data.)
2) If we have a collection of elements $x_i \in F(U_i)$ such that $\mathrm{res}_{U_i, U_i \cap U_j} x_j = \mathrm{res}_{U_j, U_i \cap U_j} x_j$ for all i and j then there is an $x \in F(U)$ such that $\mathrm{res}_{U,U_i} x = x_i$ for all i. (That is, if we have local data which are compatible, they actually "patch together" to form something in $F(U)$.)

Example G. Let X and Y be topological spaces. For all open sets $U \subset X$, let $F(U)$ be the set of continuous maps $U \to Y$. This is a presheaf with the restriction maps given by simply restricting maps to smaller sets; it is a sheaf because a function is continuous on $\cup U_i$ if and only if its restrictions to each U_i are continuous.

Example H. X and Y differentiable manifolds. $F(U) =$ differentiable maps $U \to Y$. This again is a sheaf because differentiability is a local condition.

Example I. X, Y topological spaces, $G(U) =$ continuous functions $U \to Y$ which have relatively compact image. This is a subpresheaf of the first example, but clearly need not be a sheaf.

Example J. X a topological space, $F(U) =$ the vector space of locally constant real-valued functions on U, modulo the constant functions on U. This is clearly a presheaf. But every $s \in F(U)$ goes to zero in $\prod F(U_i)$ for some covering $\{U_i\}$, while if U is not connected, $F(U) \neq (0)$. Therefore it is not a sheaf.

Sheaves are almost standard nowadays, and we will not develop their properties in detail. Recall two important ideas:

(1) *Stalks.* Let F be a sheaf on X, $x \in X$. The collection of $F(U)$, U open containing x, is an inverse system and we can form

$$F_x = \varinjlim_{x \in U} F(U) \ ,$$

called the *stalk of F at x*.

Example. Let $F(U) =$ continuous functions $U \to \mathbb{R}$. Then F_x is the set of *germs of continuous functions at x*. It is $\cup_{x \in U} F(U)$ modulo an equivalence relation: $f_1 \sim f_2$ if f_1 and f_2 agree in a neighbourhood of x.

(2) *Sheafification of a presheaf.* Let F_0 be a presheaf on X. Then there is a sheaf F and a map $f : F_0 \to F$ such that if $g : F_0 \to F'$ is any map with F' a sheaf, there is a unique map $h : F \to F'$ such that

$$\begin{tikzcd}
 & F \arrow[dr, "h"] & \\
F_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & F'
\end{tikzcd}$$

commutes.

(F is "the best possible sheaf you can get from F_0". It is easy to imagine how to get it: first identify things which have the same restrictions, and then add in all the things which can be patched together.) Thus in Example I above, if X is locally compact, the sheafification of this presheaf is the sheaf of all continuous functions in functions on X; and in example J, the sheafification of this presheaf is (0).

Notation. We may write $\Gamma(U, F)$ for $F(U)$, and call it the set of *sections of F over U*. $\Gamma(X, F)$ is the set of *global sections* of F. In other contexts we may denote $F(X)$ by $H^0(X, F)$ and call it the *zeroth cohomology group.* (In those contexts it will be a group, and there will be higher cohomology groups.)

Suppose that for all U, $F(U)$ is a group [ring, etc.] and that all the restriction maps are group [ring, etc.] homomorphisms. Then F is called a *sheaf of groups [rings, etc.]*. In this case F_x is a group [ring, etc.], and so on.

Example K. For any topological space X, let $F_{\text{cont},X}(U) =$ continuous functions $U \to \mathbb{R}$. Then $F_{\text{cont},X}(U)$ is a sheaf of rings.

Note that if $g : X \to Y$ is a continuous function, the operation $f \longmapsto f \cdot g$ gives us the following maps: for every open $U \subset Y$ a map $F_{\text{cont},Y}(U) \to F_{\text{cont},X}\left(g^{-1}U\right)$ such that

$$\begin{tikzcd}
F_Y(U) \arrow[r] \arrow[d, "\text{res}"'] & F_X(g^{-1}U) \arrow[d, "\text{res}"] \\
F_Y(V) \arrow[r] & F_X(g^{-1}V)
\end{tikzcd}$$

commutes for all open sets $V \subset U$. This set-up is called a morphism of the pair (X, F_X) to the pair (Y, F_Y).

Example L. Suppose that X and Y are differentiable manifolds, and that $F_{\text{diff},X}$ and $F_{\text{diff},Y}$ are the subsheaves of $F_{\text{cont},X}$ and $F_{\text{cont},Y}$ of differentiable functions. Let $g : X \to Y$ be a continuous map. Then g is differentiable if and only if for all open sets $U \subset Y$, $f \in F_{\text{diff},Y}(U) \Longrightarrow f \cdot g \in F_{\text{diff},X}\left(g^{-1}U\right)$.

Example M. Similarly, say X, Y are complex analytic manifolds. Let $F_{\mathrm{an},X}$ and $F_{\mathrm{an},Y}$ be the sheaves of holomorphic functions. Then a continuous map $g : X \to Y$ is holomorphic if and only if for all open sets $U \subset Y$, $f \in F_{\mathrm{an},Y}(U) \Longrightarrow f \cdot g \in F_{\mathrm{an},X}\left(g^{-1}U\right)$.

Thus the idea of using a "structure sheaf" to describe an object is useful in many contexts, and it will solve our problems too.

Definition 4. Let $X \subset k^n$ be an irreducible algebraic set, R its affine coordinate ring. Since X is irreducible, $I(X)$ is prime and R is an integral domain. Let K be its field of fractions. Recall that R has been identified with a ring of functions on X. For $x \in X$, let $m_x = \{f \in R \mid f(x) = 0\}$. This is a maximal ideal, the kernel of the homomorphism $R \to k$ given by $f \mapsto f(x)$. Let $\underline{o}_x = R_{m_x}$. We have then $\underline{o}_x = \{f/g \mid f, g \in R, g(x) \neq 0\} \subset K$. Now for U open in X, let

$$\underline{o}_X(U) = \bigcap_{X \in U} \underline{o}_x \quad .$$

All the $\underline{o}_X(U)$ are subrings of K. If $V \subset U$, then $\underline{o}_X(U) \subset \underline{o}_X(V)$; if we take the inclusion as the restriction map, this defines a sheaf $\underline{o}_X$.

The elements of $\underline{o}_X(U)$ can be viewed as functions on U. Say $F \in \underline{o}_X(U)$, and $x \in U$. Then $F \in \underline{o}_x$, so we can write $F = f/g$ with $g(x) \neq 0$. We then define $F(x) = f(x)/g(x)$. Clearly $F(x) = 0$ for all $x \in U$ implies $F = 0$, so we can identify $\underline{o}_X(U)$ with the associated ring of functions on U.

Proposition 1. *Let X be an irreducible algebraic set and let $R = \Gamma(\Sigma)$. Let $f \in R$, and $X_f = \{x \in X \mid f(x) \neq 0\}$. Then $\underline{o}_X(X_f) = R_f$.*

Proof. If $g/f^n \in R_f$, then $g/f^n \in \underline{o}_x$ for all $x \in X_f$, since by definition $f(x) \neq 0$. Thus $R_f \subset \underline{o}_X(X_f)$.

Now suppose $f \in \underline{o}_X(X_f) \subset K$. Let $B = \{g \in R \mid g \cdot F \in R\}$. If we can prove $f^n \in B$, for some n, that will imply $f \in R_f$, and we will be through. By assumption, if $x \in X_f$, then $F \in \underline{o}_x$, so there exist functions $g, h \in R$ such that $F = h/g$, $g(x) \neq 0$. Then $gF = h \in R$, so $g \in B$, and B contains an element not vanishing at x. That is, $V(B) \subset \{x \mid f(x) = 0\}$. By the Nullstellensatz, then $f \in \sqrt{B}$. □

In particular,

Corollary. $\Gamma(X, \underline{o}_X) = R$.

Remarks. I) Assume that $f \in \underline{o}_X(U)$ and that f vanishes nowhere on U. Then $1/f \in \underline{o}_X(U)$.

Proof. Obvious, since $f(x) \neq 0 \Longrightarrow 1/f \in \underline{o}_x$.

II) The stalk of $\underline{o}_X$ at x is $\underline{o}_x$.

Proof. Since the sets X_f are a basis of the Zariski topology of X, we have

$$\varinjlim_{x \in U} \underline{o}_X(U) = \varinjlim_{x \in X_f} \underline{o}_X(X_f) = \varinjlim_{f(x)\neq 0} R_f .$$

Since all restriction maps in our sheaf are injective, this is just $\bigcup_{f(x)\neq 0} R_f$, which is clearly $\underline{o}_x$. □

III) The field K can also be recovered from the sheaf $\underline{o}_X$. Recall that X is irreducible, i.e., not the union of two proper closed subsets. Equivalently, the intersection of any two nonempty open sets is nonempty. But this means that we actually have an inverse system of *all* open sets, just like our previous inverse systems of open sets containing a given point x; in this way we can define a *generic stalk* of any sheaf F on X. In particular, it is evident that K is the generic stalk of the structure sheaf $\underline{o}_X$.

IV) If $h \in \underline{o}_X(U)$ for some open $U \subset X$, then it need not be true that $h = f/g$ with $f, g \in R$ and g vanishing nowhere on U. For example, let $X \subset k^4$ be $V(xw - yz)$, and let $U = X_y \cup X_w$. The following function $h \in \underline{o}_X(U)$ is not equal to f/g, $g \neq 0$ in U: $h = x/y$ on X_y, and $h = z/w$ on X_w^*[2]. The proposition shows that this is true however if U has the form X_g.

Proposition 2. *Let $X \subset k^n$, $Y \subset k^m$ be irreducible algebraic sets, and let $f : X \to Y$ be a continuous map. The following conditions are equivalent:*

i) *f is a morphism*
ii) *for all $g \in \Gamma(Y, \underline{o}_Y)$, $g \cdot f \in \Gamma(X, \underline{o}_X)$*
iii) *for all open $U \subset Y$, and $g \in \Gamma(U, \underline{o}_Y) \Longrightarrow g \cdot f \in \Gamma(f^{-1}U, \underline{o}_X)$*
iv) *for all $x \in X$, and $g \in \underline{o}_{f(x)} \Longrightarrow g \cdot f \in \underline{o}_x$.*

Proof. Trivially iii) $\Longrightarrow$ ii), and iv) $\Longrightarrow$ iii) by the definition of $\underline{o}_X$. i) $\Longleftrightarrow$ ii) is essentially proved in Proposition 1, §3. We assume ii), then, and prove iv). Let $g \in \underline{o}_{f(x)}$. We write $g = a/b$, $a, b \in \Gamma(Y, \underline{o}_Y)$, $b(f(x)) \neq 0$. By ii), $a \cdot f, b \cdot f \in \Gamma(X, \underline{o}_X)$; hence $g \cdot f = a \cdot f / b \cdot f \in \underline{o}_x$, since we have $b \cdot f(x) \neq 0$. □

This shows, among other things, that our sheaf gives us all the information we need for defining morphisms. We are ready, then, to cut loose from the ambient spaces and define:

[2] The proof that this h is not of the form f/g, $g \neq 0$ in U, requires a later result but it goes like this: Assume $h = f/g$. Let $Z = V(y, w)$: then Z is a plane in X and $U = X - Z$. By assumption $V(g) \cap X \subset Z$. Since all components of $V(g) \cap X$ have dimension 2 (cf. §7), and since Z is irreducible, either $V(g) \cap X = z$ or $V(g) \cap X$ is empty. If $V(g) \cap X = \emptyset$, then $h = f/g \in \underline{o}_X(X)$ which is absurd (since $x = y \cdot h$ and at some points of $X, x = 1$ and $y = 0$). Now let $Z' = V(x, z)$. Then

$$\{(0,0,0,0)\} = Z \cap Z' = V(g) \cap Z' \ .$$

In other words, g would be a polynomial function on the plane Z' that vanishes only at the origin. This is impossible too.

Definition 5. An *affine variety* is a topological space X plus a sheaf of k-valued functions $\underline{o}_X$ on X which is isomorphic to an irreducible algebraic subset of some k^n plus the sheaf just defined.

Definition 6. The affine variety $(k^n, \underline{o}_{k^n})$ is $\mathbb{A}^n$, *affine n-space.*

Definition 5 and Proposition 2 set up the category of affine varieties in precise analogy with the category of topological spaces, differentiable manifolds, and analytic spaces. There are, however, some very categorical differences between these examples. Consider the following statement:

Bijective morphisms are isomorphisms.

This is correct, for example, in the category of compact topological spaces, of Banach spaces, and of complex analytic *manifolds*. On the other hand, it is false for differentiable manifolds – consider the map:

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

where $f(x) = x^3$.

The statement is also false in the category of affine varieties: a bijection $f : X_1 \to X_2$ of varieties may well correspond to an isomorphism of the ring of X_2 with a proper subring of the ring of X_1. Here are 3 key examples to bear in mind.

Example N. Let $\mathrm{char}(k) = p \neq 0$. Define the morphism

$$\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1$$

by $f(t) = t^p$. This is bijective. On the ring level, this corresponds to the inclusion map in the pair of rings:

$$k[X] \hookleftarrow k[X^p].$$

f is not an isomorphism since these rings are not equal.

Example O. Let k be any algebraically closed field. Define the morphism

$$\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^2$$

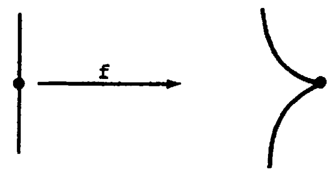
by $f(t) = \left(t^2, t^3\right)$. The image of this morphism is the irreducible closed curve

$$C : X^3 = Y^2.$$

The morphism f from $\mathbb{A}^1$ to C is a bijection which corresponds to the inclusion map in the pair of rings:

$$k[T] \hookleftarrow k\left[T^2, T^3\right].$$

These rings are not equal, so f is not an isomorphism.



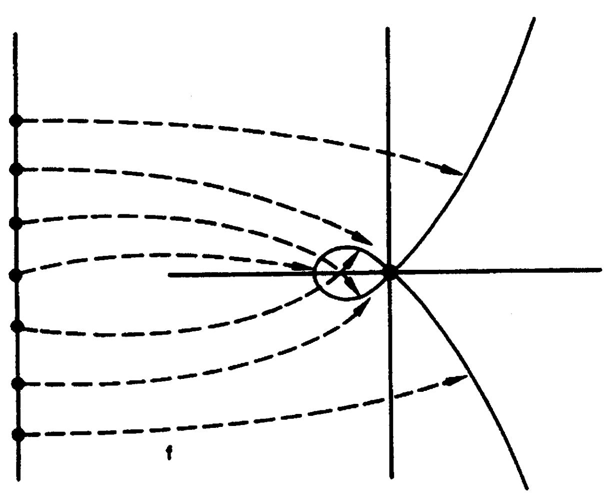
Example P. Define $\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^2$ by $X = t^2 - 1$, $Y = t\left(t^2 - 1\right)$. It is not hard to check that the image of this morphism is the curve D:

$$Y^2 = X^2(X+1)\ . \tag{1}$$

(Simply note that one can solve for the coordinate t of the point in $\mathbb{A}^1$ by the equation $t = Y/X$. Then substitute this into $X = t^2 - 1$.) Also, f is bijective between $\mathbb{A}^1$ and D except that both the points $t = -1$ and $t = 1$ are mapped to the origin. Let $X_1 = \mathbb{A}^1 - \{1\}$, an affine variety with coordinate ring $k\left[T, (T-1)^{-1}\right]$ (cf. Proposition 4 below). Then f restricts to a bijection f' from X_1 to D. This morphism corresponds to the inclusion in the pair of rings:

$$k\left[T, (T-1)^{-1}\right] \hookleftarrow k\left[T^2 - 1, T(T^2 - 1)\right].$$

Since these rings are unequal, f' is not an isomorphism.



The last topic we will take up in this section is the induced variety structure on open and closed subsets of affine varieties.

Let Y be an irreducible closed subset of an affine variety $(X, \underline{o}_X)$. [Irreducible, now, in the sense given by the topology on X.] Define an induced sheaf $\underline{o}_X$ of functions on Y as follows:

If V is open in Y,

$$\underline{o}_Y(V) = \left\{ \begin{array}{l|l} k\text{-valued functions} & \forall x \in V, \;\; \exists \text{ a neighbourhood } U \text{ of } x \text{ in } X \\ f \text{ on } V & \text{and a function } \; F \in \underline{o}_X(U) \;\text{ such that} \\ & f = \;\; \text{restriction to } U \cap V \text{ of } F. \end{array} \right\}.$$

Proposition 3. *$(Y, \underline{o}_Y)$ is an affine variety.*

Proof. Say X is isomorphic to $(\Sigma, \underline{o}_\Sigma)$ in k^n. Let $R = k[X_1, \ldots, X_n]$, $A = I(\Sigma) \subset R$. Let Y correspond to $\Sigma' \subset \Sigma$. Then Σ' is an irreducible algebraic set in k^n, so we have an affine variety $(\Sigma', \underline{o}_{\Sigma'})$. We claim $(Y, \underline{o}_Y)$ is isomorphic to $(\Sigma', \underline{o}_{\Sigma'})$.

It suffices to show that the sheaves are equal. Since the inclusion of Σ' in Σ is a morphism, the restrictions of functions in $\underline{o}_\Sigma$ to Σ' are functions in $\underline{o}_{\Sigma'}$. This shows that all the functions in $\underline{o}_Y$ correspond to functions in $\underline{o}_{\Sigma'}$. Conversely, every function $f' \in \underline{o}_{\Sigma'}(\Sigma'_g)$, $g \in R$, is a restriction of a function $f \in \underline{o}_\Sigma(\Sigma_g)$ since both of these rings are quotients of R_g. Therefore all functions in $\underline{o}_\Sigma$ correspond to functions in $\underline{o}_Y$ too. □

Proposition 4. *Let $(X, \underline{o}_X)$ be an affine variety, and let $f \in \Gamma(X, \underline{o}_X)$. Then $\left(X_f, \underline{o}_{X|X_f}\right)$ is an affine variety. [The restriction of the sheaf $\underline{o}_X$ to the open set X_f is defined in the obvious way.]*

Proof. Say we identify X with $\Sigma \subset k^n$. Let $A = I(\Sigma) \subset k[X_1, \ldots, X_n]$. Let $f_1 \in k[X_1, \ldots, X_n]$ be some element giving f as a function on Σ. Let B be the ideal in $k[X_1, \ldots, X_n, X_{n+1}]$ generated by A and by $1 - f_1 \cdot X_{n+1}$. We claim B is prime and, if $\Sigma^* = V(B) \subset k^{n+1}$, then $(\Sigma^*, \underline{o}_{\Sigma^*}) \cong \left(X_f, \underline{o}_{X|X_f}\right)$.

From the definition of B we see that

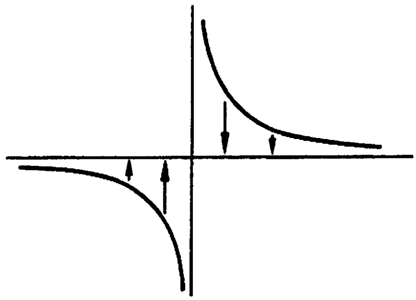
$$k[X_1, \ldots, X_{n+1}]/B = (k[X_1, \ldots, X_n]/A)_{f_1} \cong \Gamma(X, \underline{o}_X)_f,$$

which is an integral domain, so B is prime.

Define a morphism $\alpha : \Sigma^* \to \Sigma$ by $(x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, x_n)$. It's an injection with image Σ_{f_1}, since $(x_1, \ldots, x_n, x_{n+1}) \in \Sigma^*$ if and only if $(x_1, \ldots, x_n) \in \Sigma$ and $1 = f_1(x_1, \ldots, x_n)\, x_{n+1}$. We leave to the reader the verification that it is a homeomorphism onto Σ_{f_1}.

We saw above that $\Gamma(\Sigma^*, \underline{o}_{\Sigma^*}) \simeq \Gamma(X, \underline{o}_X)_f$. Therefore, by Proposition 2, α^{-1} is a morphism from Σ_{f_1} to Σ^*, i.e. $(\Sigma^*, \underline{o}_{\Sigma^*})$ and $\left(X_f, \underline{o}_{X|X_f}\right)$ are isomorphic. □

What we have done to get X_f is to push the zeroes of f out to infinity. For example, suppose $X = \mathbb{A}^1$ and f is the coordinator X_1. Then $B = (1 - X_1 X_2)$, giving a hyperbola:



Projection of the hyperbola down to the axis is an isomorphism with X_f.

Not all open subsets of affine varieties are affine varieties. For instance, you cannot push the origin in $\mathbb{A}^2$ out to infinity, and $\mathbb{A}^2 - (0,0)$ is not an affine variety. In fact, no rational function is well-defined on $\mathbb{A}^2 - (0,0)$ but not at $(0,0)$; i.e., the intersection of the local rings $\underline{o}_x$, for all $x \in \mathbb{A}^2$, $x \neq (0,0)$, is contained in $\underline{o}_{(0,0)}$. Hence if we had any embedding $\mathbb{A}^2 - (0,0) \to k^n$, the coordinate functions giving this embedding would have to extend to all of $\mathbb{A}^2$, and so the image of $\mathbb{A}^2 - (0,0)$ would not be closed. There is an analogous statement about complex functions: a holomorphic function on $\mathbb{C} \times \mathbb{C} - (0,0)$ is necessarily holomorphic at $(0,0)$.

§5. Definition of prevarieties and morphisms

Definition 1. A topological space X plus a sheaf $\underline{o}_X$ of k-valued functions on X is a *prevariety* if

1) X is connected, and
2) there is a finite open covering $\{U_i\}$ of X such that for all i, $\left(U_i, \underline{o}_{X|U_i}\right)$ is an affine variety.

Definition 2. An open subset U of X is called an *open affine set* if $\left(U, \underline{o}_{X|U}\right)$ is an affine variety.

Note that the open affine sets are a basis of the topology. In fact, we know by Proposition 4, §4, that this is true within each of the open affines U_i, and they cover X.

Definition 3. A topological space is *irreducible* if it is not the union of two proper closed subsets (equivalently, the intersection of any two nonempty sets is nonempty).

Proposition 1. *Every prevariety X is an irreducible topological space.*

Proof. Let V be open and nonempty in X. Let U_1 be the union of all open affine sets meeting V, U_2 the union of all those disjoint from V; then $U_1 \cup U_2 = X$. Suppose $y \in U_1 \cap U_2$; then there are affine open sets W_1, W_2 containing y, such that $W_1 \cap V \neq \emptyset$, $W_2 \cap V = \emptyset$. But then $W_1 \cap V$ is a nonempty open set in the affine W_1, so it is dense in it; $W_2 \cap W_1$ is also a nonempty open set in W, so it meets $W_1 \cap V$. This shows that $W_2 \cap V$ cannot be empty. This is a contradiction, hence no such y exists. Since X is connected and U_1 is nonempty, $U_1 = X$.

Now let U be any other open set, and say $x \in U$. By the above, there is an affine open set W containing x and meeting V. Then both $V \cap W$ and $U \cap W$ are nonempty open sets in the affine W, so $V \cap W \cap U \neq \emptyset$, and *a fortiori* $U \cap V \neq \emptyset$. □

In particular, every open set is dense. Thus prevarieties are not like differentiable manifolds, which can have disjoint coordinate patches; to get a prevariety, we just put things around the edges of one affine piece.

Proposition 2. *If X is a prevariety, then the closed sets of X satisfy the descending chain condition, i.e., X is a noetherian space.*

Proof. Let $\{Z_i\}$ be a sequence of closed sets, such that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$. Since X is covered by finitely many affines, it suffices to show that $\{U \cap Z_i\}$ stationary for each affine open U in X. The result in the affine case follows immediately from the fact that $\Gamma(U, \underline{o}_X)$ is noetherian as we noted in §3. □

In particular, every variety is quasi-compact.

Proposition 3. *Let X be a noetherian topological space. Then every closed set Z in X can be written uniquely as an irredundant union of finitely many irreducible closed sets (called the components of Z).*

Proof. Suppose Z is a minimal closed set for which the Proposition is false; this exists since X is noetherian. Then Z is not itself irreducible, so $Z = Z_1 \cup Z_2$, where Z_1, Z_2 are smaller and hence are unions of the required type. Then so is Z. □

Let X be a prevariety. Since X is an irreducible topological space, any 2 nonempty open subsets have a nonempty intersection. Therefore, all sheaves have "generic" stalks:

Definition 4. The *function field* $k(X)$ is the generic stalk of $\underline{o}_X$, i.e.

$$k(X) = \varinjlim_{\substack{\text{all non-empty} \\ \text{open } U}} \underline{o}_X(U) \ .$$

In fact, $k(X)$ equals the function field of each open affine set U in X, since the open subsets of U are cofinal. In particular, this shows that $k(X)$ is really a field. The elements of $k(X)$ are called *rational functions on* X, although they are, strictly speaking, only functions on open dense subsets of X.

Another type of $\varinjlim$ over $\underline{o}_X(U)$'s is sometimes very useful. This is intermediate between the $\varinjlim$ that leads to $\underline{o}_x$ and that which leads to $k(X)$. Let $Y \subset X$ be an irreducible closed subset of X. Then let:

$$\underline{o}_{Y,X} = \varinjlim_{\substack{\text{open sets } U \\ \text{such that} \\ U \cap Y \neq \emptyset}} \underline{o}_X(U) \ .$$

To express more simply the ring which you get in this way, fix one open affine $U \subset X$ which meets Y. Let R be the coordinate ring of U and $P = I(Y \cap U)$ the ideal in R determined by Y. Then:

$$\underline{o}_{Y,X} = \varinjlim_{\substack{\text{open sets } U_f \\ f \in R,\ f \notin P}} \underline{o}_X(U_f) = R_P \ .$$

In particular, $\underline{o}_{Y,X}$ is a local ring with quotient field $k(X)$ and residue field $k(Y)$.

Proposition 4. *An open subset of a prevariety is a prevariety.*

Proof. Let $U \subset X$ be open. Since X is irreducible, U is connected. U is of course a union of affine open subsets. But since X is noetherian, U is quasi-compact and hence it is covered by finitely many affines. □

Now let Y be a closed irreducible subset of a prevariety X. The sheaf $\underline{o}_X$ induces a sheaf $\underline{o}_Y$ on Y as follows:

If V is open in Y,

$$\underline{o}_Y(V) = \left\{ \begin{array}{l|l} k\text{-valued functions} & \forall\, x \in V,\ \exists \text{ a neighbourhood } U \text{ of } x \text{ in } X \\ f \text{ on } V & \text{and a function } F \in \underline{o}_X(U) \text{ such that} \\ & f = \text{ restriction to } U \cap V \text{ of } F \end{array} \right\} .$$

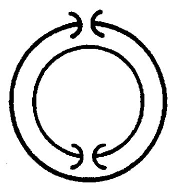
Proposition 5. *The pair $(Y, \underline{o}_Y)$ is a prevariety.*

Proof. This follows immediately from the definition and Proposition 3, §4. □

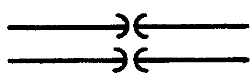
Combining Propositions 4 and 5, we can even give a prevariety structure to every *locally closed* subset of a prevariety X. The set of all prevarieties so obtained are called the *sub-prevarieties* of X.

Example Q. $\mathbb{P}_1$:
Take two copies U and V of $\mathbb{A}^1$. Let u, v be the coordinates on these 2 affine lines. Let $U_0 \subset U$ (resp. $V_0 \subset V$) be defined by $u \neq 0$ (resp. $v \neq 0$). Then $\Gamma(U, \underline{o}_U) = k[u]$, so $\Gamma(U_0, \underline{o}_U) = k\left[u, u^{-1}\right]$. Similarly $\Gamma(V_0, \underline{o}_V) = k\left[v, v^{-1}\right]$. Define a map $\varphi : U_0 \to V_0$ taking the point with coordinate $u = a$ to the point with coordinate $v = \frac{1}{a}$; this gives a map $\varphi^* : k\left[v, v^{-1}\right] \to k\left[u, u^{-1}\right]$ taking v to u^{-1}, v^{-1} to u. φ^* is an isomorphism of rings, so φ is an isomorphism of varieties

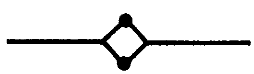
(φ has an inverse since φ^* does). Now we patch together U and V *via* φ, i.e., we form $U \cup V$ with U_0 and V_0 identified *via* φ. This has a sheaf on it, in the obvious way, and is a prevariety. The space is homeomorphic to $\mathbb{P}_1$, and we call it the variety $\mathbb{P}_1$. Our patching can be pictured as follows:



We could have patched U and V differently: $v \to u$, $v^{-1} \to u^{-1}$ also gives an isomorphism of U_0 onto V_0. But this is a silly way to patch; we are leaving out the same point each time:



and the result is $\mathbb{A}^1$ with a point doubled:



This is a prevariety, of course, but, in fact, not a variety (cf. §6).

We could define all projective varieties by this kind of scissors and glue method, but there is a more intrinsic definition.

Definition of Projective Varieties. Let $P \subset k[X_0, \ldots, X_n]$ be a homogeneous prime ideal, $X = V(P) \subset \mathbb{P}_n(k)$. We want to make X (with the Zariski topology) into a prevariety. We do it by defining a function field, getting local rings, and intersecting them, just as for affine varieties.

The elements of $k[X_0, \ldots, X_n]$, even the homogeneous ones, do not give functions on X; but the ratio of any two having the same degree is a function. Since P is homogeneous, $R = k[X_0, \ldots, X_n]/P$ is in a natural way a graded ring $\bigoplus_{n=0}^{\infty} R_n$ and an integral domain. We let $k(X)$ be the zeroth graded piece of the localization of R with respect to homogeneous elements, i.e., $\{f/g \mid f, g \in R_n \text{ for the same } n\}$.

If $x \in X$, and $g \in R_n$ it makes sense to say $g(x) \neq 0$, even though g is not a function on X; for g changes by a nonzero factor as we change the homogeneous coordinates of x. Hence we can define a ring $\underline{o}_x$ in $k(X)$ as $\{f/g \in k(X) \mid g(x) \neq 0\}$. The set

$$m_x = \left\{ f/g \in k(X) \,\middle|\, \begin{matrix} f(x) = 0 \\ g(x) \neq 0 \end{matrix} \right\}$$

is clearly an ideal in the ring $\underline{o}_x$, and any element not in m_x is invertible in $\underline{o}_x$. Thus $\underline{o}_x$ is a local ring.

We now define a sheaf $\underline{o}_X$ on X by

$$\underline{o}_X(U) = \bigcap_{x \in U} \underline{o}_x, \qquad \text{for all } U \subset X \text{ open}.$$

We can identify $\underline{o}_X$ with a sheaf of k-valued functions. For suppose $x \in U$, and $a \in \Gamma(U, \underline{o}_X)$. Then a can be written f/g with $f, g \in R_n$, $g(x) \neq 0$, and we define $a(x) = f(x)/g(x)$. [That is, we lift f, g to F, G homogeneous polynomials of degree n in $k[X_0, \dots, X_n]$, let $\tilde{x}$ be a set of homogeneous coordinates of x, and take $F(\tilde{x})(/G(\tilde{x})$. It is clear that this value is unchanged if we take a different set of homogeneous coordinates, or if we change F and G by members of P_n, so we have a well-defined function.] We still should check that if $a \in \Gamma(U, \underline{o}_x)$ and $a(x) = 0$ for all $x \in U$, then $a = 0$. But this also comes out of the next step, which consists in checking that $(X, \underline{o}_X)$ is locally isomorphic to an affine variety. In fact, we claim that for all i, $0 \leq i \leq n$,

$$(X \cap \mathbb{P}_n(k)_{X_i}, \quad \text{restriction of } \underline{o}_X)$$

is an affine variety. We will check this only for X_0, since the general case goes just the same.

For every homogeneous polynomial $F \in P$, $F/X_0^{\deg F}$ can be written as a polynomial F' in the variables $\frac{X_1}{X_0}, \dots \frac{X_n}{X_0}$. Let $P' \subset k[Y_1, \dots, Y_n]$ be the ideal generated by all these F'. We can map $k[Y_1, \dots, Y_n]$ into a subring of $k(X)$ by taking Y_i to the function given by X_i/X_0; the kernel of this map is exactly P', so P' is prime. It is easy to see that we get an isomorphism $\varphi : X \cap \mathbb{P}_n(k)_{X_n} \to X' = V(P') \subset k^n$ by taking x to $\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$; φ is actually a homeomorphism (cf. Problem at end of §3).

Now for $x \in X \cap \mathbb{P}_n(k)_{X_0}$, the local ring $\underline{o}_x$ is the set of all elements of $k(X)$ having the form f/g for f, g in some R_m, $g(x) \neq 0$. X' has the affine coordinate ring $R' = k[Y_1, \dots, Y_n]/P$. If $k(X')$ is its quotient field, $\underline{o}_{\varphi(x)}$ is the set of all elements in $k(X')$ having the form F/G for $F, G \in R'$, $G(\varphi(x)) \neq 0$. The map defined above taking R' into $k(X)$ extends to an isomorphism $k(X') \xrightarrow{\sim} k(X)$. I claim that this map takes $\underline{o}_{\varphi x}$ precisely onto $\underline{o}_x$. First of all it clearly maps $\underline{o}_{\varphi x}$ into $\underline{o}_x$; and if $f, g \in R_m$, $g(x) \neq 0$, then $f/g = f/X_0^m/g/X_0^m$ in $k(X)$ and f/X_0^m, g/X_0^m come from F, G in R' with $G(\varphi(x)) \neq 0$. Thus the local rings correspond; since the sheaves were defined by intersecting local rings, they also correspond, and $X \cap \mathbb{P}_n(k)_{X_0}$ is indeed an affine variety.

Definition 5. Let X and Y be prevarieties. A map $f : X \to Y$ is a *morphism* if f is continuous and, for all open sets V in Y,

$$g \in \Gamma(V, \underline{o}_Y) \Longrightarrow g \cdot f \in \Gamma(f^{-1}V, \underline{o}_X).$$

Proposition 6. *Let $f : X \to Y$ be any map. Let (V_i) be a collection of open affine subsets covering Y. Suppose that $\{U_i\}$ is an open covering of X such that 1) $f(U_i) \subset V_i$ and 2) f^* maps $\Gamma(V_i, \underline{o}_Y)$ into $\Gamma(U_i, \underline{o}_X)$. Then f is a morphism.*

Proof. We may assume the U_i are affine; for if $U \subset U_i$ is affine, f^* certainly maps $\Gamma(V_i, \underline{o}_Y)$ into $\Gamma(U, \underline{o}_X)$, and we can replace U_i by a set of affines that cover U_i. First of all, the restriction f_i of f to a map from U_i to V_i is a morphism. In fact, the homomorphism

$$f_i^* : \Gamma(V_i, \underline{o}_Y) \longrightarrow \Gamma(U_i, \underline{o}_X)$$

is also induced by some morphism $g_i : U_i \to V_i$ (Proposition 1, §3). And since the functions in $\Gamma(V_i, \underline{o}_Y)$ separate points, a map from U_i to V_i is determined by the contravariant map from $\Gamma(U_i, \underline{o}_X)$ to $\Gamma(V_i, \underline{o}_Y)$. Therefore $f_i = g_i$ and f_i is a morphism. In particular, f_i is continuous and this implies immediately that f itself is continuous. It remains to check that f^* always takes sections $\underline{o}_Y$ to sections $\underline{o}_X$. But if $V \subset Y$ is open, and $g \in \Gamma(V, \underline{o}_Y)$, then $g \cdot f$ is at least a section of $\underline{o}_X$ on the sets $f^{-1}(V \cap V_i)$, hence on the sets $f^{-1}(V) \cap U_i$. Since $\underline{o}_X$ is a sheaf, $g \cdot f$ is actually a section of $\underline{o}_X$ in $f^{-1}(V)$. □

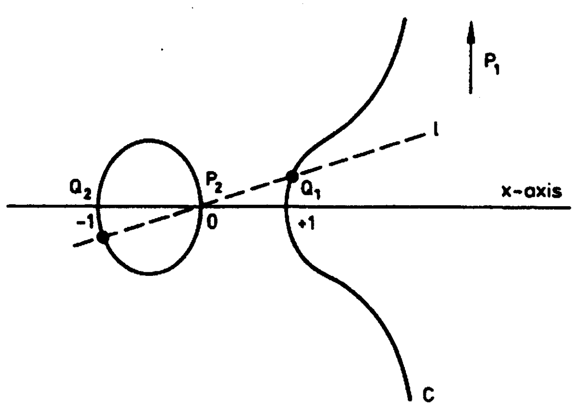
To illustrate the meaning of our definitions, it seems worthwhile to work out in detail a non-trivial example. We shall reconsider Example F, §3. Let C be the plane cubic curve defined in homogeneous coordinates by:

$$zy^2 = x(x^2 - z^2) .$$

Look first at $C \cap \mathbb{P}_2(k)_z$, with affine coordinates $X = x/z, Y = y/z$. The equation of C becomes:

$$Y^2 = X \cdot (X^2 - 1) .$$

For all lines ℓ through the origin, we want to interchange the 2 points in $\ell \cap C$ (other than the origin). Start with a point $(a, b) \in C$. This is joined to the origin by the line



$$\begin{aligned} X &= at \\ Y &= bt. \end{aligned}$$

Intersecting this with the cubic, we get the equation

$$b^2t^2 = at\left(a^2t^2 - 1\right)$$

or

$$0 = at(t-1)\left(a^2t+1\right) .$$

Thus the 2nd point of intersection is given by $t = -1/a^2$. In other words, the morphism on C is to be given by:

$$(a,b) \longmapsto \left(-1/a, -b/a^2\right).$$

These are not polynomials, so at any rate they do not define a map from this affine piece of C into itself; this is as it should be, since as we can see from the drawing, we want the origin itself to go to the one point at infinity on the cubic.

To describe the subsets on which we *will* get a morphism, we must throw out the various "bad" points one at a time. We need names for them:

$P_1 = (0,1,0)$ (the only point at ∞ with respect to the affine piece X, Y)

$P_2 = (0,0,1)$ (the origin)

$$\left.\begin{aligned} Q_1 &= (1,0,1) \\ Q_2 &= (-1,0,1) \end{aligned}\right\} \quad \text{other points on the } X\text{-axis} .$$

The morphism – call it f – should interchange P_1 and P_2, Q_1 and Q_2. Define

$$\begin{aligned} U_1 &= C - \{P_1, P_2\} \\ U_2 &= C - \{P_1, Q_1, Q_2\} \\ U_3 &= C - \{P_2, Q_1, Q_2\} \\ V_1 &= C - \{P_1\} = C \cap \mathbb{P}_2(k)_z \\ V_2 &= C - \{P_2, Q_1, Q_2\} = C \cap \mathbb{P}_2(k)_y . \end{aligned}$$

Then 1) U_1, U_2, U_3 is an open covering of C, 2) V_1, V_2 is an *affine* open covering of C, and 3) if f is defined set-theoretically as above, then $f(U_1) \subset V_1$, $f(U_2) \subset V_2$ and $f(U_3) \subset V_1$. Therefore, by Proposition 6, it suffices to check that

$$f^*\left[\Gamma\left(V_1, \underline{o}_C\right)\right] \subset \Gamma\left(U_1, \underline{o}_C\right) \cap \Gamma\left(U_3, \underline{o}_C\right)$$

and

$$f^*\left[\Gamma\left(V_2, \underline{o}_C\right)\right] \subset \Gamma\left(U_2, \underline{o}_C\right),$$

and then it follows that f is a morphism. In more down to earth terms: note that X, Y are affine coordinates in V_1 and

$$\begin{aligned} S &= x/y \\ T &= z/y \end{aligned}$$

are affine coordinates in V_2. Then what we have to check is that f, described via one of these 2 sets of coordinates, is given by polynomials in U_1, U_2, and U_3.

In U_1: X, Y and $1/X$ are functions in $\Gamma(U_1, \underline{o}_C)$. Thus describing the image point also by coordinates X, Y, f is given by:

$$(a,b) \longmapsto \left(-1/a, -b/a^2\right)$$

and these are polynomials in $a, b, 1/a$.

In U_2: X, Y and $\frac{1}{x^2-1}$ are functions in $\Gamma(U_2, \underline{o}_C)$. Describe the image of the point (a,b) by coordinates S, T now:

$$\begin{aligned} S[f(a,b)] &= a/b \\ T[f(a,b)] &= -a^2/b\,. \end{aligned}$$

This does not look very promising until we use the relation $b^2 = a\cdot(a^2-1)$ to rewrite these as:

$$\begin{aligned} S &= b/a^2-1 \\ T &= -ab/a^2-1 \end{aligned}$$

which are polynomials in a, b, $1/a^2-1$.

In U_3: S and T are functions in $\Gamma(U_3, \underline{o}_C)$. Describe the image of the point $S = c$, $T = d$ by coordinates X, Y:

$$\begin{aligned} X[f(c,d)] &= -d/c \\ Y[f(c,d)] &= -d/c^2\,. \end{aligned}$$

However c and d are related by $d = c\cdot\left(c^2-d^2\right)$, so we get:

$$\begin{aligned} X &= d^2-c^2 \\ Y &= d\left(c^2-d^2\right)-c\,. \end{aligned}$$

□

Problem. Generalize the result by which we covered $\mathbb{P}_n$ by open affine sets as follows: for all homogeneous polynomials $H \in k[X_0, \ldots, X_n]$ of positive degree, show that $\mathbb{P}_n(k)_H$ is an affine variety.

§6. Products and the Hausdorff Axiom

We want to define the product $X \times Y$ of any two prevarieties X, Y. Now we will certainly want to have $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$. But the product of the Zariski topologies in $\mathbb{A}^n$ and $\mathbb{A}^m$ does not give the Zariski topology in $\mathbb{A}^{n+m}$; in $\mathbb{A}^1 \times \mathbb{A}^1$, for instance, the only closed sets in the product topology are finite unions of horizontal and vertical lines. The only reliable way to find the correct definition is to use the general category-theoretic definition of product.

Definition 1. Let $\mathcal{C}$ be a category, X, Y objects in $\mathcal{C}$. An object Z plus two morphisms

$$\begin{tikzcd} & X \\ Z \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & Y \end{tikzcd}$$

is a *product* if it has the following universal mapping property: for all objects W and morphisms

$$\begin{tikzcd} & X \\ W \arrow[ur, "r"] \arrow[dr, "s"'] & \\ & Y \end{tikzcd}$$

there is a unique morphism $t : W \to Z$ such that $r = p \cdot t$, $s = q \cdot t$, i.e., such that

$$\begin{tikzcd} & X \\ W \arrow[ur, "r"] \arrow[r, "t"] \arrow[dr, "s"'] & Z \arrow[u, "p"'] \arrow[d, "q"] \\ & Y \end{tikzcd}$$

commutes. The induced morphism t in this situation will always be denoted (r, s). We call p and q the *projections* of the product onto its factors. Clearly a product, if it exists, is unique up to a unique isomorphism commuting with the projections.

The requirement of the definition can be rephrased to say $\hom(W,Z) \xrightarrow{\sim} \hom(W,X) \times \hom(W,Y)$ (under the obvious map induced by p and q).

We shall prove that products exist in the category of prevarieties over k. Note that we have no choice for the underlying set; for if $X \times Y$ is a product of the prevarieties X and Y, $X \times Y$ as a point set must be the usual product of the point sets X and Y. To see this, let W be a simple point; this is a prevariety ($\mathbb{A}^0$, in fact). The maps of W to any prevariety S clearly correspond to the points of S, and by definition $\hom(W, X \times Y) \simeq \hom(W,X) \times \hom(W,Y)$.

Proposition 1. *Let X and Y be affine varieties, with coordinate rings R and S. Then*

1) there is a product prevariety $X \times Y$.
2) $X \times Y$ is affine with coordinate ring $R \otimes_k S$.
3) a basis of the topology is given by the open sets

$$\sum f_i(x) g_i(y) \neq 0, \qquad f_i \in R, g_i \in S .$$

4) $\underline{o}_{(x,y)}$ is the localization of $\underline{o}_x \otimes_k \underline{o}_y$ at the maximal ideal $m_x \cdot \underline{o}_y + \underline{o}_x \cdot m_y$.

Proof. We recall the following result from commutative algebra: let R and S be integral domains over the algebraically closed field k. Then $R \otimes_k S$ is an integral domain. [Cf. Zariski-Samuel, vol. 1, Ch. 3, §15.]

Represent

$$\begin{array}{lll} X \subset k^{n_1} & \text{as} & V(f_1, \ldots, f_{m_1}) \ . \\ Y \subset k^{n_2} & \text{as} & V(g_1, \ldots, g_{m_2}) \ . \end{array}$$

Then the set $X \times Y \subset k^{n_1+n_2}$ is the locus of zeroes of $f_j(X_i), g_j(Y_i)$ in $k[X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}]$. Moreover

$$\begin{aligned} k[X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}] / (f_j, g_j) &\simeq k[X_i] / (f_j) \otimes_k k[Y_i] / (g_j) \\ &= R \otimes_k S. \end{aligned}$$

But $R \otimes_k S$ is an integral domain; hence (f_j, g_j) is prime, $X \times Y$ is irreducible, and $R \otimes_k S$ is its coordinate ring.

This gives us an affine variety $X \times Y$. The next step is to prove that it is a categorical product. We have natural projections

$$p, q : X \times Y \longrightarrow X, Y \quad [\text{e.g., } p(x_1, \ldots, y_{n_2}) = (x_1, \ldots, x_{n_1})]$$

which are clearly morphisms. Suppose we are given morphisms $f : Z \to X$, $s : Z \to Y$. There is just one map of point sets $t : Z \to X \times Y$ such that $r = p \cdot t$, $s = g \cdot t$ (since $X \times Y$ as a point set is the product of X and Y), and to verify the universal mapping property we need only check that t is always a morphism.

But this is simple. Since $X \times Y$ is affine, it suffices to check that $g \in \Gamma(X \times Y, \underline{o}_{X \times Y}) \Longrightarrow g \cdot t \in \Gamma(Z, \underline{o}_Z)$. Now $\Gamma(X \times Y, \underline{o}_{X \times Y})$ is generated by the images of $\Gamma(X, \underline{o}_X) = R$ and $\Gamma(Y, \underline{o}_Y) = S$. Both of these by composition with t

go into $\Gamma(Z, \underline{o}_Z)$ since r and s are morphisms; therefore all of $\Gamma(X \times Y, \underline{o}_{X \times Y})$ goes into $\Gamma(Z, \underline{o}_Z)$.

We have proved 1) and 2), and 3) follows from 2). Now $\underline{o}_{(x,y)}$ is the localization of $R \otimes_k S$ at the ideal of all functions vanishing at (x, y). Clearly $R \otimes_k S \subset \underline{o}_x \otimes_k \underline{o}_y \subset \underline{o}_{(x,y)}$, and therefore we can get $\underline{o}_{(x,y)}$ by localizing $\underline{o}_x \otimes_k \underline{o}_y$ at the ideal of all functions in it vanishing at (x, y). We claim that ideal is precisely $m_x \cdot \underline{o}_y + \underline{o}_x \cdot m_y$. Evidently all these functions do vanish there. Conversely, if we take any $h = \sum f_1 \otimes g_i \in \underline{o}_x \otimes_k \underline{o}_y$, with, say, $f_i(x) = \alpha_i$, $g_i(y) = \beta_i$, then we claim $h - \sum \alpha_i \beta_i \in m_x \cdot \underline{o}_y + \underline{o}_x \cdot m_y$. Indeed it equals

$$\sum f_i \otimes g_i - \sum \alpha_i \otimes \beta_i = \sum (f_i - \alpha_i) \otimes g_i + \sum \alpha_i \otimes (g_i - \beta_i) \in m_x \cdot \underline{o}_y + \underline{o}_x \cdot m_y \, .$$

□

We can now "glue together" these affine products to obtain:

Theorem 2. *Let X and Y be prevarieties over k. Then they have a product.*

Proof. We start, of course, with the product *set*. For all open affine $U \subset X$, $V \subset Y$ and all finite sets of elements $f_i \in \Gamma(U, \underline{o}_X)$, $g_i \in \Gamma(V, \underline{o}_Y)$ we form $(U \times V)_{\Sigma f_i g_i}$; these we take as a basis of the open sets. (They do form a basis, since $(U \times V)_{\Sigma f_i g_i} \cap (U' \times V')_{\Sigma f'_j g'_j}$ contains $(U'' \times V'')_{\Sigma f_i g_i \cdot \Sigma f'_j g'_j}$, where U'' [resp. V''] is an affine contained in $U \cap U'$ [resp. $V \cap V'$].) Note that on $U \times V$ this induces the topology of their true product.

Let K be the quotient field of $k(X) \otimes_k k(Y)$ [which, as before, is an integral domain]. For $x \in X$, $y \in Y$ we let $\underline{o}_{(x,y)} \subset K$ be the localization of $\underline{o}_x \otimes_k \underline{o}_y$ at the ideal $m_x \cdot \underline{o}_y + \underline{o}_x \cdot m_y$, and we set

$$\Gamma(U, \underline{o}_{X \times Y}) = \bigcap_{(x,y) \in U} \underline{o}_{(x,y)} .$$

This gives us a sheaf of functions. Furthermore, it coincides on each $U \times V$ (U, V affine) with the product of the affine varieties. Clearly $X \times Y$ is connected and covered by finitely many affines, so it is a prevariety.

Now suppose $Z \xrightarrow{r} X$, $Z \xrightarrow{s} Y$ are morphisms. Set-theoretically there is a unique map $(r, s) : Z \to X \times Y$ composing properly with the projections; we want to check that it is a morphism. For each U in X, V in Y affine, look at $Z_{U,V} = r^{-1}(U) \cap s^{-1}(V)$. These are open sets covering Z, and since being a morphism is a local property it is enough to prove $(r, s) \mid Z_{U.V}$ is a morphism. That is, we may assume $r(Z) \subset U$, $s(Z) \subset V$. But in the last proposition we saw that then the product map $Z \to U \times V$ is a morphism; and $U \times V$ is an open subprevariety of $X \times Y$, so the composite map $Z \to U \times V \to X \times Y$ is a morphism.

Remarks. I. If U is any open subprevariety of X, then $U \times Y$ is an open subprevariety of $X \times Y$.

II. If Z is a closed subprevariety of X, then $Z \times Y$ is a closed subprevariety of $X \times Y$. [It is enough to prove $(Z \cap U) \times V$ a closed subprevariety of $U \times V$ for U, V affine, and this is easily checked.]

Theorem 3. *The product of two projective varieties is a projective variety.*

Proof. Since a closed subvariety of a projective variety is a projective variety, it is enough to show $\mathbb{P}_n \times \mathbb{P}_m$ is a projective variety. In fact, we can embed it as a closed subvariety of $\mathbb{P}_{nm+n+m}$.

Take homogeneous coordinates $X_0, \ldots, X_n$ in $\mathbb{P}_n$, $Y_0, \ldots, Y_m$ in $\mathbb{P}_m$, and U_{ij} $(i = 0, \ldots, n,\ j = 0, \ldots, m)$ in $\mathbb{P}_{(n+1)(m+1)-1}$. Define a map:

$$I : \mathbb{P}_n \times \mathbb{P}_m \longrightarrow \mathbb{P}_{nm+n+m}$$

by

$$(x_0, \ldots, x_n) \times (y_0, \ldots, y_m) \longrightarrow \begin{array}{l}\text{a point with homogeneous coordinates} \\ U_{ij} = x_i y_j\end{array} .$$

[This makes sense; multiplying all x_i for all y_j by λ multiplies all U_{ij} by λ; and some U_{ij} is nonzero.] Clearly

$$I^{-1}\left((\mathbb{P}_{nm+n+m})_{U_{ij}}\right) = (\mathbb{P}_n)_{X_i} \times (\mathbb{P}_m)_{Y_i} .$$

We claim that first I is injective. Assume that for some $\lambda \neq 0$, $x_i y_j = \lambda x_i' y_j'$ for all i and j; then we want to prove that for some $\mu, \nu \neq 0$, $x_i = \mu x_i'$, $y_j = \nu y_j'$. We may by symmetry assume $x_0 \neq 0$, $y_0 \neq 0$. Then we have $0 \neq x_0 y_0 = x_0' y_0'$, so $x_0' \neq 0$, $y_0' \neq 0$. We have $x_i/x_0 = x_i y_0 / x_0 y_0 = x_i' y_0' / x_0' y_0' = x_i'/x_0'$ and similarly for y_j/y_0. This proves what we want.

Now we claim I is an isomorphism of $(\mathbb{P}_n)_{X_i} \times (\mathbb{P}_m)_{Y_j}$ onto a closed subvariety of $(\mathbb{P}_{n+m+nm})_{U_{ij}}$. We may assume $i = 0, j = 0$ for simplicity. On $(\mathbb{P}_n)_{X_0}$ we take affine coordinates $S_i = X_i/X_0$, $i = 1, \ldots, n$. On $(\mathbb{P}_m)_{Y_0}$ we take affine coordinates $T_j = Y_j/Y_0$, $j = 1, \ldots, m$. On $(\mathbb{P}_{n+m+nm})_{U_{00}}$ we take affine coordinates $R_{ij} = U_{ij}/U_{00}$, $i \geq 1$ or $j \geq 1$. In these coordinates, I takes (s, t) to the point

$$\begin{array}{ll} R_{ij} = s_i t_j & \text{if } i, j \geq 1 \\ R_{i0} = s_i, & R_{0j} = t_j \ . \end{array}$$

Hence the image is the locus of points satisfying $R_{ij} = R_{i0} R_{0j}$ for all $i, j \geq 1$. This is certainly closed. Its affine coordinate ring is $k[R_{ij}] / (R_{ij} - R_{i0} R_{0j})$, which is clearly isomorphic to the polynomial ring $k[R_{i0}, R_{0j}]$. Under I^*, this is mapped isomorphically to $k[S_i, T_j]$ which is the affine coordinate ring of $(\mathbb{P}_n)_{X_0} \times (\mathbb{P}_m)_{Y_0}$. Hence we do have an isomorphism.

Let $Z = I(\mathbb{P}_n \times \mathbb{P}_m)$. Since $Z \cap (\mathbb{P}_{n+m+nm})_{U_{ij}}$ is closed for all i, j, Z is closed. I is a homeomorphism on each of these affines, so it is a homeomorphism globally, and in particular Z is irreducible. Thus Z is a projective variety. Since I is an isomorphism on each affine piece, it is an isomorphism globally and hence Z is isomorphic to $\mathbb{P}_n \times \mathbb{P}_m$. □

The classical example is the embedding $\mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_3$. Take coordinates u, v on $\mathbb{P}_1$, s, t on $\mathbb{P}_1$, x, y, z, w on $\mathbb{P}_3$. I is defined by $x = us$, $y = vt$, $z = ut$, $w = vs$. It is easy to see that the image of I is the quadric $xy = zw$. [Thus, over an algebraically closed field all nondegenerate quadrics (those corresponding to nondegenerate quadric forms) are isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$.]

One can show in general that the homogeneous ideal of the image is generated by the elements $U_{ij}U_{i'j'} - U_{ij'}U_{i'j}$, so the image is an intersection of quadrics.

We now take up the main topic of this section – the Hausdorff axiom:

Definition 2. Let X be a prevariety. X is a *variety* if for all prevarieties Y and for all morphisms

$$Y \underset{g}{\overset{f}{\rightrightarrows}} X.$$

$\{y \in Y \mid f(y) = g(y)\}$ is a closed subset of Y.

One case of this criterion is particularly simple: let $Y = X \times X$, $f = p_1$, $g = p_2$. Also let

$$\Delta : X \longrightarrow X \times X$$

be a morphism $(\mathrm{id}_X,\ \mathrm{id}_X)$: Δ is called the diagonal morphism. Then

$$\Delta(X) = \{z \in X \times X \mid p_1(z) = p_2(z)\}.$$

Therefore the Hausdorff axiom implies $\Delta(X)$ is closed. But the converse is also true:

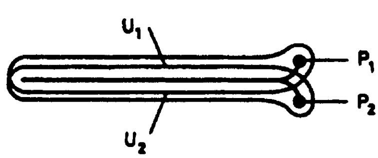
Proposition 4. *Let X be a prevariety. Then X is a variety if and only if $\Delta(X)$ is closed in $X \times X$.*

Proof. Suppose $f, g : Y \to X$ are given. Then induce a morphism $(f, g) : Y \to X \times X$. Since

$$\{y \in Y \mid f(y) = g(y)\} = (f, g)^{-1}[\Delta(X)],$$

$\Delta(X)$ being closed implies the Hausdorff axiom. $\square$

Example R. Let X be:



i.e., 2 copies U_1, U_2 of $\mathbb{A}^1$, say with coordinates x_1 and x_2, patched by the map $x_1 = x_2$ on the open sets $x_1 \neq 0$ and $x_2 \neq 0$. Consider the isomorphisms of $\mathbb{A}^1$ with each of the 2 copies:

$$\begin{tikzcd} & U_1 \arrow[dr, phantom, "\subset" sloped] & \\ \mathbb{A}^1 \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & X \\ & U_2 \arrow[ur, phantom, "\subset" sloped] & \end{tikzcd}$$

Call these $i_1, i_2 : \mathbb{A}^1 \to X$. Then

$$\{y \in \mathbb{A}^1 \mid i_1(y) = i_2(y)\} = \mathbb{A}^1 - \{0\}$$

is not closed in $\mathbb{A}^1$, hence X is not a variety.

Remarks. I. A subprevariety of a variety is a variety. A product of 2 varieties is a variety.

II. An affine variety is a variety: in fact, if X is affine, Y is an arbitrary prevariety, and $f, g : Y \to X$ are 2 morphisms, then

$$\{y \in Y \mid f(y) = g(y)\} = \left\{ \begin{array}{l} \text{locus of zeroes of the functions} \\ s.f. - s.g., \quad \text{all } s \in \Gamma(X, \underline{o}_X) \end{array} \right\}$$

and this set is closed.

III. From II., we can check that if $f, g : Y \to X$ are any morphisms of *any* prevarieties, then $\{y \in Y \mid f(y) = g(y)\}$ is always *locally closed.* In fact, call this set Z. If $z \in Z$, let V be an affine open neighbourhood of $f(z)$ $(= g(z))$. Then

$$Z \cap [f^{-1}(V) \cap g^{-1}(V)] = \left\{ z \in f^{-1}(V) \cap g^{-1}(V) \;\middle|\; \begin{array}{l} f(z) = g(z) \\ \text{in the affine} \\ \text{variety} \end{array} \right\}$$

and this set is closed since V is a true variety.

IV. Another useful way to use the Hausdorff property is this: if $f : X \to Y$ is a morphism of prevarieties and Y is a variety, then the image of the morphism

$$(\mathrm{id}, f) : X \longrightarrow X \times Y$$

which is the *graph* of f, is closed. Moreover, if we let Γ_g be this image, then Γ_f is even a closed subprevariety of $X \times Y$ isomorphic to X under the mutually inverse morphisms:

$$\begin{tikzcd} \Gamma_f \arrow[d, bend left, "p_1|_{\Gamma_f}"] \\ X \arrow[u, bend left, "(\mathrm{id}, f)"] \end{tikzcd}$$

(here $p_1 : X \times Y \to X$ is the projection).

Proposition 5. *Let X be a prevariety. Assume that for all $x, y \in X$ there is an open affine U containing both x and y. Then X is a variety.*

Proof. Suppose $f, g : Y \to X$ are 2 morphisms such that $Z = \{y \in Y \mid f(y) = g(y)\}$ is not closed. Let $z \in \overline{Z}$, $x = f(z)$, $y = g(z)$. By assumption, there is an open affine V containing x and y. Let $U = f^{-1}(V) \cap g^{-1}(V)$: U is an open neighbourhood of Z. If f', g' are the restrictions of f, g to morphisms from U to V, then

$$Z \cap U = \{y \in U \mid f'(y) = g'(y)\}$$

is closed in U, since V, being affine, is a variety. Therefore $z \in Z \cap U$, and Z is closed. □

Corollary. *Every projective variety X is a variety.*

Proof. For all $x, y \in \mathbb{P}_n(k)$, there is a hyperplane not containing x or y, i.e., an element $H = \Sigma \alpha_i x_i \in k[X_0, \ldots, X_n]$ such that $x, y \in \mathbb{P}_n(k)_H$. But $\mathbb{P}_n(k)_H$ is affine, hence $X \cap \mathbb{P}_n(k)_H$ is open and affine in X. □

Proposition 6. *Let X be a variety, and let U, V be affine open subsets with coordinate rings R, S. Then $U \cap V$ is an affine open subset with coordinate ring $R \cdot S$ (the compositum being formed in $k(X)$).*

Proof. $U \times V$ is an affine open subset of $X \times X$ with coordinate ring $R \otimes_k S$. Let $Z = \Delta(X)$. Then $Z \cap (U \times V)$ is a closed subset of $U \times V$ isomorphic via Δ to $U \cap V$. Since Δ is an isomorphism, $Z \cap (U \times V)$ is irreducible, hence it is a closed subvariety of $U \times V$. Therefore it is affine, and its coordinate ring T is a quotient of the coordinate ring of $U \times V$. But it is also open in Z, hence $Z \subset k(Z)$, i.e., T is the image of $R \otimes_k S$ in $k(Z)$. Again since Δ is an isomorphism, this proves that

$$U \cap V = \Delta^{-1}\{Z \cap (U \times V)\}$$

is an open affine subvariety of X, and that its coordinate ring is $\Delta^*(R \otimes S)$ in $k(X)$. But $\Delta^*(f \otimes g) = f \cdot g$, hence $\Delta^*(R \otimes S) = R \cdot S$. □

Problem. Prove the following converse: let X be a prevariety, $\{U_i\}$ an affine open covering of X. Let R_i be the coordinate ring of U_i. Then if $U_i \cap U_j$ is an affine subset of X with coordinate ring $R_i \cdot R_j$, X is a variety.

Zariski and Chevalley have used a quite different definition of a variety which is useful for many questions, especially "birational" ones, i.e., questions dealing with various different prevarieties with a common function field.

Definition 3. Let $\mathcal{O} \subset \mathcal{O}'$ be local rings. We say $\mathcal{O}'$ *dominates* $\mathcal{O}$ or $\mathcal{O}' > \mathcal{O}$, if $m' \supset m$ (equivalently $m' \cap \mathcal{O} = m$) where m, m' are the maximal ideals of $\mathcal{O}$, $\mathcal{O}'$ respectively.

Local Criterion: Let X be a prevariety. Then X is a variety if and only if for all $x, y \in X$ such that $x \neq y$, there is *no* local ring $\mathcal{O} \subset k(X)$ such that $\mathcal{O} > \underline{o}_x$ and $\mathcal{O} > \underline{o}_y$.

The fact that this holds for varieties will be proven in Ch. II, §6. We omit the converse which we will not use. If one starts with this criterion, however, one can take the elegant Chevalley-Nagata definition of a variety: we identify a variety X with the set of local rings $\underline{o}_x$. For this definition, a variety is a finitely generated field extension K of k plus a collection of local subrings of K satisfying various conditions. [Notice: the topology can be recovered as follows – for each $f \in K$, let U_f be the set of those local rings in X containing f.]

Problem. Let $X \subset \mathbb{P}^n$ be defined by a homogeneous ideal $P \subset k[X_0, \ldots, X_n]$. Let P define the *affine* variety $X^* \subset \mathbb{A}^{n+1}$. Show that for all i,

$$\left\{ \begin{array}{l} \text{Subset of } X^* \\ \text{where } X_i \neq 0 \end{array} \right\} \cong \mathbb{A}^1 \times \left\{ \begin{array}{l} \text{Subset of } X \\ \text{where } X_i \neq 0 \end{array} \right\} .$$

§7. Dimension

Definition 1. Let X be a variety. $\dim X = \text{tr.d.}_k k(X)$.

If U is an open and nonempty set in X, $\dim U = \dim X$, since $k(U) = k(X)$. Since k is algebraically closed, the following are equivalent:

i) $\dim X = 0$
ii) $k(X) = k$
iii) X is a point.

Proposition 1. *Let Y be a proper closed subvariety of X. Then* $\dim Y < \dim X$.

Proof .

Lemma. *Let R be an integral domain over k, $P \subset R$ a prime. Then* $\text{tr.d.}_k R \geq \text{tr.d.}_k R/P$, *with equality only if $P = \{0\}$ or both sides are ∞. [By convention,* tr.d.R *is the* tr.d. *of the quotient field of R.]*

Proof. Say $P \neq 0$, $\text{tr.d.}_k R = n < \infty$. If the statement is false, there are n elements $x_1, \ldots, x_n$ in R such that their images $\bar{x}_i$ in R/P are algebraically independent. Let $0 \neq p \in P$. Then $p, x_1, \ldots, x_n$ cannot be algebraically independent over k, so there is a polynomial $P(Y, X_1, \ldots, X_n)$ over k such that $P(p, x_1, \ldots, x_n) = 0$. Since R is an integral domain, we may assume P is irreducible. The polynomial P cannot equal αY, $\alpha \in k$, since $p \neq 0$. Therefore P is not even a multiple of Y. But then $P(0, \bar{x}_1, \ldots, \bar{x}_n) = 0$ in R/P is a nontrivial relation on the $\bar{x}_1, \ldots, \bar{x}_n$.

Now choose $U \subset X$ an open affine set with $U \cap Y \neq \emptyset$. Let R be the coordinate ring of U, P the prime ideal corresponding to the closed set $U \cap Y$. Then $P \neq 0$, since $U \cap Y \neq U$. $k(X)$ is the quotient field of R, and $k(Y)$ is the quotient field of R/P. Therefore the Proposition follows immediately from the lemma. □

In the situation of this Proposition, $\dim X - \dim Y$ will be called the *codimension* of X in Y. This is half of what we want so that our definition gives a good dimension function. The other half is that it does not go down too much.

Theorem 2. *Let X be a variety, $U \subset X$ open, $g \in \Gamma(U, \underline{o}_X)$, Z an irreducible component of $\{x \in U \mid g(x) = 0\}$. Then if $g \neq 0$, $\dim Z = \dim X - 1$.*

Proof. Take $U_0 \subset U$ an open affine set with $U_0 \cap Z \neq \emptyset$. Let $R = \Gamma(U_0, \underline{o}_X)$, $f = \mathrm{res}_{U,U_0} g \in R$. Then $Z \cap U_0$ corresponds to a prime $P \subset R$. Z is by hypothesis a maximal irreducible subset of the locus $g = 0$, so $Z \cap U_0$ is a maximal irreducible subset of the locus $f = 0$, i.e., P is a maximal prime containing f. Thus we have translated Theorem 2 into:

Algebraic Version (Krull's Principal Ideal Theorem). *Let R be a finitely generated integral domain over k, $f \in R$, P an isolated prime ideal of (f) (i.e., minimal among the prime ideals containing it). Then if $f \neq 0$, $\mathrm{tr.d.}_k R/P = \mathrm{tr.d.}_k R - 1$.*

This is a standard result in commutative algebra (cf. for example, Zariski-Samuel, vol. 2, Ch. 7, §7). However there is a fairly straightforward and geometric proof, using only the Noether normalization lemma and based on the Norm, so I think it is worthwhile proving the algebraic version too. This proof is due to J. Tate.

At this point, it is convenient to translate the results of §1 into the geometric framework that we have built up:

Definition 2. Let $f : X \to Y$ be a morphism of affine varieties. Let R and S be the coordinate rings of X and Y, and let $f^* : S \to R$ be the induced homeomorphism. Then f is *finite* if R is integrally dependent on the subring $f^*(S)$.

Note that the restriction of a finite morphism from X to Y to a closed subvariety Z of X is also finite. Examples N and O in §3 are finite morphisms, but example P is not. The main properties of finite morphisms are the following:

Proposition 2. *Let $f : X \to Z$ be a finite morphism of affine varieties.*

(1.) Then f is a closed map, i.e., maps closed sets onto closed sets.
(2.) For all $y \in Y$, $f^{-1}(y)$ is a finite set.
(3.) f is surjective if and only if the corresponding map f^* of coordinate rings is injective.

Proof. Let f be dual to the map $f^* : S \to R$ of coordinate rings. As usual, the maps V, I set up bijections between the points of X (resp. Y) and the maximal ideals m of R (resp. S). If $V(A)$ is a closed set in X, then $f(V(A))$ corresponds to the set of maximal ideals $f^{*^{-1}}(m)$, $m \subset R$ a maximal ideal containing A. But by the going-up theorem, whenever $R \supset S$ and R is integral over S every *prime* ideal $P \subset S$ is of the form $P' \cap S$ for some prime ideal $P' \subset R$ (cf. §1). If $B = f^{*^{-1}}(A)$, apply the going-up theorem to $S/B \subset R/A$; therefore $f(V(A))$

corresponds to the set of maximal ideals m such that $m \supset B$, i.e., to $V(B)$. Therefore, f is closed. Moreover, $f(X) = Y$ if and only if $\mathrm{Ker}(f^*) = (0)$. (2.) is equivalent to saying that for all maximal ideals $m \subset S$, there are only a *finite* number of maximal ideals $m' \subset R$ such that $m = {f^*}^{-1}(m')$. Since such m' all contain $f^*(m) \cdot R$, it is enough to check that $R/f^*(m) \cdot R$ contains only a finite number of maximal ideals. But $R/f^*(m) \cdot R$ is integral over S/m, i.e., it is a finite-dimensional algebra over k, so this is clear. □

Via this concept, we can state:

Geometric form of Noether's normalization lemma. *Let X be an affine variety of dimension n. Then there exists a finite surjective morphism π:*

$$X \xrightarrow{\pi} \mathbb{A}^n \quad .$$

Proof of Theorem 2. We first reduce the proof to the case $P = \sqrt{(f)}$, i.e., geometrically, to the case $Z = V((f))$. To do this, look back at the way in which we related the geometric and algebraic versions of the Theorem. Note that the algebraic version is essentially identical to Theorem 2 in the case $X = U$ is affine with coordinate ring R. Suppose we have the decomposition

$$\sqrt{(f)} = P \cap P_1' \cap \ldots \cap P_t'$$

in R. Geometrically, if $Z_i' = V(P_i')$, this means that $Z, Z_1', \ldots, Z_t'$ are the components of $V((f))$. Now we pick an affine open $U_0 \subset X$ such that:

$$\begin{aligned} &U_0 \cap Z \neq \emptyset \\ &U_0 \cap Z_i' = \emptyset, \qquad i = 1, \ldots, t \ . \end{aligned}$$

For example, let $U_0 = X_g$, where:

$$g \in P_1' \cap \ldots \cap P_t', \qquad g \notin P \ .$$

Then replace X by U_0, R by $R_{(g)}$, and in the new set-up,

$$\begin{aligned} V_{U_0}((f)) &= V_X((f)) \cap U_0 \\ &= Z \cap U_0 \end{aligned}$$

is irreducible; hence in $R_{(g)}$, $\sqrt{(f)} = P \cdot R_{(g)}$ is prime.

Now use the normalization lemma to find a morphism as follows:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & R \\ \mathbb{A}^n & k[X_1, \ldots, X_n] = S. \arrow[u, hook, "\pi^*"] \end{tikzcd}$$

Let K (resp. L) be the quotient field of R (resp. S). Then K/L is a finite algebraic extension. Let

$$f_0 = N_{K/L}(f) \ .$$

Then I claim $f_0 \in S$ and

$$(*) \qquad P \cap S = \sqrt{(f_0)} \ .$$

If we prove $(*)$, the theorem follows. For R/P is an integral extension of $S/S \cap P$, so

$$\text{tr.d.}_k R/P = \text{tr.d.}_k S/S \cap P \ .$$

But S is a UFD, so the primary decomposition of a principal ideal in S is just the product of decomposition of the generator into irreducible elements. Therefore $(*)$ implies that f_0 is a unit times f_{00}^{ℓ} for some integer ℓ and some irreducible f_{00}, and that $P \cap S = (f_{00})$. Hence

$$\text{tr.d.}_k S/S \cap P = \text{tr.d.}_k k\left[X_1, \ldots, X_n\right]/(f_{00}) = n^{-1}.$$

We check first that $f_0 \in P \cap S$. Let

$$Y^n + a_1 Y^{n-1} + \ldots + a_n = 0$$

be the irreducible equation satisfied by f over the field L. Then f_0 is a power $(a_n)^m$ of a_n. Moreover, all the a_i's are symmetric functions in the conjugates of f (in some normal extension of L): therefore the a_i are elements of L integrally dependent on S. Therefore $a_i \in S$. In particular, $f_0 = a_n^m \in S$, and since

$$\begin{aligned} 0 &= a_n^{m-1} \cdot \left(f^n + a_1 f^{n-1} + \ldots + a_n\right) \\ &= f \cdot \left(a_n^{m-1} f^{n-1} + a_n^{m-1} a_1 f^{n-2} + \ldots + a_n^{m-1} a_{n-1}\right) + f_0 \ , \end{aligned}$$

$f_0 \in P$ too.

Finally suppose $g \in P \cap S$. Then $g \in P = \sqrt{(f)}$, hence

$$g^n = f \cdot h, \qquad \begin{array}{l} \text{some integer } n \ , \\ \text{some } h \in R \ . \end{array}$$

Taking norms, we find that

$$\begin{aligned} g^{n \cdot [K:L]} &= N_{K/L}(g^n) \\ &= N_{K/L}(f) \cdot N_{K/L}(h) \in (f_0) \end{aligned}$$

since $N_{K/L} h$ is an element of S, by the reasoning used before. Therefore $g \in \sqrt{(f_0)}$, and $(*)$ is proven. □

Definition 3. Let X be a variety, and let $Z \subset X$ be a closed subset. Then Z has *pure dimension* r if each of its components has dimension r (similarly for *pure codimension* r).

The conclusion of Theorem 2 may be stated as: $V((g))$ has pure codimension 1, for any non-zero $g \in \Gamma(X, \underline{o}_X)$. The Theorem has an obvious converse: Suppose Z is an irreducible closed subset of a variety X of codimension 1. Then for all open sets U such that $Z \cap U \neq \emptyset$ and for all non-zero functions f on U vanishing on Z, $Z \cap U$ is a component of $f = 0$. In fact, if W were a component of $f = 0$ containing $Z \cap U$, then we have

$$\dim X > \dim W \geq \dim Z \cap U = \dim X - 1$$

hence $W = Z \cap U$ by Proposition 1.

Corollary 1. *Let X be a variety and Z a maximal closed irreducible subset, smaller than X itself. Then* $\dim Z = \dim X - 1$.

Corollary 2 (Topological characterization of dimension). *Suppose*

$$\emptyset \neq Z_1 \not\subseteq Z_2 \not\subseteq \ldots Z_1 \not\subseteq Z$$

is any maximal chain of closed irreducible subsets of X. Then $\dim X = r$.

Proof. Induction on $\dim X$.

Corollary 3. *Let X be a variety and let Z be a component of $V((f_1, \ldots, f_r))$, where $f_1, \ldots, f_r \in \Gamma(X, \underline{o}_X)$. Then* codim $Z \leq r$.

Proof. Induction on r. Z is an irreducible subset of $V((f_1, \ldots, f_{r-1}))$, so it is contained in some component Z' of $V((f_1, \ldots, f_{r-1}))$. Then Z is a component of $Z' \cap V((f_r))$, since $Z' \cap V((f_r)) \subset V((f_1, \ldots, f_r))$. By induction codim $Z' \leq r-1$. If $f_r \equiv 0$ on Z', $Z = Z'$. If f_r does not vanish identically on Z', then by the theorem, $\dim Z = \dim Z' - 1$, so codim $Z \leq r$. □

Of course, equality need not hold in the above result: e.g., take $f_1 = \ldots = f_1$, $r > 1$.

Corollary 4. *Let U be an affine variety, Z a closed irreducible subset. Let $r =$* codim Z. *Then there exist $f_1, \ldots, f_r$ in $R = \Gamma(U, \underline{o}_U)$ such that Z is a component of $V((f_1, \ldots, f_r))$.*

Proof. In fact, we prove the following. Let $Z_1 \supset Z_2 \supset \ldots \supset Z_r = Z$ be a chain of irreducibles with codim $Z_i = i$ (by Cor. 2). Then there are $f_1, \ldots, f_r$ in R such that Z_s is a component of $V((f_1, \ldots, f_s))$ and all components of $V((f_1, \ldots, f_s))$ have codim s.

We prove this by induction on s. For $s = 1$, take $f_1 \in I(Z_1)$, $f_1 \neq 0$, and we have just the converse of the theorem. Now say $f_1, \ldots, f_{s-1}$ have been chosen. Let $Z_{s-1} = Y_1, \ldots, Y_\ell$ be the components of $V((f_1, \ldots, f_{s-1}))$. For all $i, Z_s \not\supset Y_i$ (because of their dimensions), so $I(Y_i) \not\supset I(Z_s)$. Since the $I(Y_i)$ are prime, $\bigcup_{i=1}^{\ell} I(Y_i) \not\supset I(Z_s)$ [Zariski-Samuel, vol. 1, p. 215; Bourbaki, Ch. 2, p. 70]. Hence we can choose an element $f_s \in I(Z_s)$, $f_s \notin \bigcup_{i=1}^{\ell} I(Y_i)$.

If Y is any component of $V((f_1,\dots,f_s))$, then (as in the proof of Cor. 3) Y is a component of $Y_i \cap V((f_s))$ for some i. Since f_s is not identically zero on Y_i, $\dim Y = \dim Y_i - 1$, so codim $Y = s$.

By the choice of f_s, $Z_s \subset V((f_1,\dots,f_s))$. Being irreducible, Z_s is contained in some component of $V((f_1,\dots,f_s))$, and it must equal this component since it has the same dimension. □

In the theory of local rings, it is shown that one can attach to every noetherian local ring $\mathcal{O}$ an integer called its Krull dimension. This number is defined as either

a) the length r of the longest chain of prime ideals:

$$P_0 \subsetneqq P_1 \subsetneqq \dots \subsetneqq P_n = M$$

$$(M \text{ the maximal ideal of } \mathcal{O})$$

or b) the least integer n such that there exist elements $f_1,\dots,f_n \in M$ for which

$$M = \sqrt{(f_1,\dots,f_n)}\ .$$

(Cf. Zariski-Samuel, vol. 1, pp. 240–242; vol. 2, p. 288. Also, cf. Serre, *Algèbre Locale*, Ch. 3B). Recall that in §5 we attached a local ring $\underline{o}_{Z,X}$ to every irreducible closed subset Z of every variety X. We now have:

Corollary 5. *The Krull dimension* $\underline{o}_{Z,X}$ *is* $\dim X - \dim Z$.

Proof. By Corollary 2, for all maximal chains of irreducible closed subvarieties:

$$Z = Z_n \subsetneqq Z_{n-1} \subsetneqq \dots \subsetneqq Z_0 = Z\ ,$$

$n = \dim X - \dim Z$. But it is not hard to check that there is an order reversing isomorphism between the set of irreducible closed subvarieties Y and X containing Z, and the set of prime ideals $P \subset \underline{o}_{Z,X}$. Or else, use the second definition of Krull dimension: first note that if $f_1,\dots,f_n \in \underline{o}_{Z,X}$, then

$$M_{Z,X} = \sqrt{(f_1,\dots,f_n)}$$

if and only if there is an open set $U \subset X$ such that

a) $U \cap Z \neq \emptyset$,
b) $f_1,\dots,f_n \in \Gamma(U,\underline{o}_X)$,
c) $U \cap Z = V((f_1,\dots,f_n))$.

Then using Corollaries 3 and 4, it follows that the smallest n for which such f_i's exist is $\dim X - \dim Z$. □

Suppose $Z \subset X$ is irreducible and of codimension 1. A natural question to ask is whether, for all $y \in Z$, there is some neighbourhood U of y in X and some function $f \in \Gamma(U,\underline{o}_X)$ such that $Z \cap U$ is not just a component of $f = 0$, but actually equal to the locus $f = 0$. More generally, if $Z \subset X$ is a closed subset of pure codimension r, one may ask whether, for all $y \in Z$, there is a neighbourhood U of y and functions $f_1,\dots,f_r \in \Gamma(U,\underline{o}_X)$ such that

$$Z \cap U = V((f_1, \ldots, f_r)) \ .$$

This is unfortunately not always true even in the special case where Z is irreducible of codimension 1. A closed set Z with this property is often referred to as a *local set-theoretic complete intersection* and it has many other special properties. There is one case where we can say something however:

Proposition 4. *Let X be an affine variety with coordinate ring R. Assume R is a UFD. Then every closed subset $Z \subset X$ of pure codimension 1 equals $V((f))$ for some $f \in R$.*

Proof. Since R is a UFD, every minimal prime ideal of R is principal [i.e., say $P \subset R$ is minimal and prime. Let $f \in P$. Since P is prime, P contains one of the prime factors f' of f. By the UFD property, (f') is also a prime ideal and since $(f') \subset P$, we must have $(f') = P$]. Let $Z_1, \ldots, Z_t$ be the components of Z. Then $I((Z_1), \ldots, I(Z_t))$ are minimal prime ideals. If $I(Z_i) = (f_i)$, then

$$Z = V((f_1, \ldots, f_t)) \ .$$

□

Proposition 5. $\dim X \times Y = \dim X + \dim Y$.

The proof is easy.

The results and methods of this section all have projective formulations which give some global as well as some local information:

Let $X \subset \mathbb{P}_n(k)$ be a projective variety, and let $I(X) \subset k[X_0, \ldots, X_n]$ be its ideal.

Theorem 2*. *If $f \in k[X_0, \ldots, X_n]$ is homogeneous and not a constant and $f \notin I$, then $X \cap V((f))$ is non-empty and of pure codimension 1 in X, unless* $\dim X = 0$.

Proof. All this follows from Theorem 2, except for the fact that $X \cap V((f))$ is not empty if $\dim X \geq 1$. But let $X^* \subset \mathbb{A}^{n+1}$ be the cone over X, i.e., the affine variety defined by the ideal $I(X)$ in $k[X_0, \ldots, X_n]$. By the problem in §6, we know that $\dim X^* = \dim X + 1$, hence $\dim X \geq 2$. Let $V^*((f))$ be the locus $f = 0$ in $\mathbb{A}^{n+1}$. Since

$$(0, 0, \ldots, 0) \in X^* \cap V^*((f)) \ ,$$

therefore $X^* \cap V^*((f)) \neq \emptyset$ and by Theorem 2, $X^* \cap V^*((f))$ has a component of dimension at least 1. Therefore $X^* \cap V^*((f))$ contains points other than $(0, 0, \ldots, 0)$; but the affine coordinates of such points are homogeneous coordinates of points in $X \cap V((f))$. □

Corollary 3*. *If $f_1, \ldots, f_r \in k[X_0, \ldots, X_n]$ are homogeneous and not constants, then all components of $X \cap V((f_1, \ldots, f_r))$ have codimension at most r in X. And if* $\dim X \geq r$, *then $X \cap V((f_1, \ldots, f_r))$ is non-empty.*

Corollary 4*. *If $Y \subset X$ is a closed subvariety, (resp. Y is the empty subset), then there exist homogeneous non-constant elements $f_1, \ldots, f_r \in k[X_0, \ldots, X_n]$ where $r =$ codimension of Y (resp. $r = \dim X + 1$) such that Y is a component of $X \cap V((f_1, \ldots, f_r))$ (resp. $X \cap V((f_1, \ldots, f_r))) = \emptyset$.*

Proof. One can follow exactly the inductive proof of Corollary 4 above, using $k[X_0, \ldots, X_n]$ instead of the affine coordinate ring. In case $Y = \emptyset$, the last step is slightly different. By induction, we have $f_1, \ldots, f_r$ such that

$$X \cap V((f_1, \ldots, f_r))$$

is a finite set of points. Then let f_{r+1} be the equation of any hypersurface not containing any of these points. □

This implies, for instance, that every *space curve*, i.e., one-dimensional subvariety of $\mathbb{P}_3(k)$, is a component of an intersection $H_1 \cap H_2$ of 2 surfaces.

Proposition 4*. *Every closed subset of $\mathbb{P}_n(k)$ of pure codimension 1 is a hypersurface, i.e., equals $V((f))$ for some homogeneous element $f \in k[X_0, \ldots, X_n]$.*

Proof. Exactly the same as Proposition 4, using the fact that $k[X_0, \ldots, X_n]$ is a UFD. □

An interesting corollary of these results is the following global theorem:

Proposition 6. *Suppose*

$$\mathbb{P}_n \xrightarrow{f} \mathbb{P}_m$$

is a morphism. Assume $W = f(\mathbb{P}_m)$ is closed (actually this is always true as we will see in §9). Then either W is a single point, or

$$\dim W = n.$$

Proof. Let $r = \dim W$ and assume $1 \leq r \leq n-1$. By Cor. 4* applied to the empty subvariety Y of W, there are homogeneous non-constant elements

$$f_1, f_2, \ldots, f_{r+1} \in k[X_0, \ldots, X_m]$$

such that

$$W \cap V((f_1, \ldots, f_{r+1})) = \emptyset.$$

Also, by Cor. 2*, $W \cap V((f_i)) \neq \emptyset$ for all i. Let $Z_i = f^{-1}V((f_i))$. Then

$$Z_1 \cap \ldots \cap Z_{r+1} = \emptyset, \quad \text{and } Z_i \neq \emptyset, \qquad 1 \leq i \leq r+1.$$

Note that the hypersurface $V((f_i))$ in $\mathbb{P}_m$ is defined locally by the vanishing of a single function. Therefore the closed subset Z_i in $\mathbb{P}_n$ is also defined locally by the vanishing of a single function. Therefore $Z_i = \mathbb{P}_n$ or else Z_i is of pure codimension 1, hence by Cor. 4* a hypersurface. Since $r + 1 \leq n$, the intersection of $r+1$ hypersurfaces in $\mathbb{P}_n$ cannot be empty because of Cor. 3*. This is a contradiction, so in fact $r = 0$ or $r = n$. □

§8. The fibres of a morphism

Let $f : X \to Y$ be a morphism of varieties. The purpose of this section is to study the family of closed subsets of X consisting of the sets $f^{-1}(y)$, $y \in Y$.

Definition 1. A morphism $f : X \to Y$ is *dominating* if its image is dense in Y, i.e., $Y = \overline{f(X)}$.

Proposition 1. *If $f : X \to Y$ is any morphism, let $Z = \overline{f(X)}$. Then Z is irreducible, the restricted morphism $f' : X \to Z$ is dominating and f'^* induces an injection*

$$k(Z) \overset{f'^*}{\hookrightarrow} k(X) .$$

Proof. Suppose $Z = W_1 \cup W_2$, where W_1 and W_2 are closed subsets. Then $X = f^{-1}(W_1) \cup f^{-1}(W_2)$. Since X is irreducible, $X = f^{-1}(W_1)$ or $f^{-1}(W_2)$, i.e., $f(X) \subset W_1$ or W_2. Therefore $Z = \overline{f(X)}$ is equal to W_1 or W_2: hence Z is irreducible. f^* is clearly dominating and since $f'(X)$ is dense in Z, for *all* nonempty open sets $U \subset Z$, $f^{-1}(U)$ is nonempty and open in X; therefore, we obtain a map:

$$k(Z) = \varinjlim_{\substack{U \subset Z \\ \text{open} \\ \text{nonempty}}} \Gamma(U, \underline{o}_Z) \xrightarrow{f'^*} \varinjlim_{\substack{V \subset X \\ \text{open} \\ \text{nonempty}}} \Gamma(V, \underline{o}_X) = k(X) .$$

□

This reduces the study of the fibres of an arbitrary morphism to the case of dominating morphisms. Note also that a finite morphism is dominating if and only if it is surjective.

Theorem 2. *Let $f : X \to Y$ be a dominating morphism of varieties and let $r = \dim X - \dim Y$. Let $W \subset Y$ be a closed irreducible subset and let Z be a component of $f^{-1}(W)$ that dominates W. Then*

$$\dim Z \geq \dim W + r$$

or

$$\operatorname{codim}\ (Z \text{ in } X) \leq \operatorname{codim}\ (W \text{ in } Y).$$

Proof. If U is an affine open subset of Y such that $U \cap W \neq \emptyset$, then to prove the theorem we may as well replace Y by U, X by $f^{-1}(U)$, W by $W \cap U$, and Z by $Z \cap f^{-1}(U)$. Therefore we assume that Y is affine. By Cor. 4 to Th. 2, §7, if $s = \operatorname{codim}\ (W \text{ in } Y)$, there are functions $f_1, \ldots, f_s \in \Gamma(Y, \underline{o}_Y)$ such that W is a component of $V((f_1, \ldots, f_s))$. Let $g_i \in \Gamma(X, \underline{o}_X)$ denote the function $f^*(f_i)$. Then $Z \subset V((g_1, \ldots, g_s))$ and I claim Z is a component of $V((g_1, \ldots, g_s))$. Suppose

$$Z \subset Z' \subset V((g_1, \ldots, g_s)) ,$$

where Z' is a component of $V((g_1,\ldots,g_s))$. Then

$$W = \overline{f(Z)} \subset \overline{f(Z')} \subset V((f_1,\ldots,f_s)).$$

Since W is a component of $V((f_1,\ldots,f_s))$ and $\overline{f(Z')}$ is irreducible, it follows that $W = \overline{f(Z')}$. Therefore, $Z' \subset f^{-1}(W)$. But Z is a component of $f^{-1}(W)$. Therefore $Z = Z'$, and Z is also a component of $V((g_1,\ldots,g_s))$. By Cor. 3 to Th. 2, §7, this proves that codim $(Z \text{ in } X) \leq s$. □

Corollary. *If Z is a component of $f^{-1}(y)$, for some $y \in Y$, then* $\dim Z \geq r$.

Theorem 3. *Let $f : X \to Y$ be a dominating morphism of varieties and let $r = \dim X - \dim Y$. Then there exists a nonempty open set $U \subset Y$ such that:*

i) $U \subset f(X)$
ii) for all irreducible closed subsets $W \subset Y$ such that $W \cap U \neq \emptyset$, and for all components Z of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$,

$$\dim Z = \dim W + r$$

or

$$\text{codim } (Z \text{ in } X) = \text{codim } (W \text{ in } Y).$$

Proof. As in Theorem 2, we may as well replace Y by a nonempty open affine subset; therefore, assume that Y is affine. Moreover, we can also reduce the proof easily to the case where X is affine. In fact, cover X by affine open sets $\{X_i\}$ and let $f_i : X_i \to Y$ by the restriction of f. Let $U_i \subset Y$ satisfy i) and ii) of the theorem for f_i. Let $U = \cap U_i$. Then with this U, i) and ii) are correct for f itself.

Now assume X and Y are affine, and let R and S be their coordinate rings. f defines a homomorphism

$$f^* : S \longrightarrow R$$

which is an injection by Proposition 1. Let $K = k(Y)$, the quotient field of S. Apply the normalization lemma to the K-algebra $R \otimes_S K$. Note that $R \otimes_S K$ is just the localization of R with respect to the multiplicative system S^*, hence it is an integral domain with the same quotient field as R, i.e., $k(X)$. In particular,

$$\begin{aligned} \text{tr.d.}_K (R \otimes_S K) &= \text{tr.d.}_{k(Y)} k(X) \\ &= \text{tr.d.}_k k(X) - \text{tr.d.}_k k(Y) = r. \end{aligned}$$

Therefore, there exists a subring:

$$K[Y_1,\ldots,Y_r] \subset R \otimes_S K$$

such that $R \otimes_S K$ is integrally dependent on $K[Y_1,\ldots,Y_r]$. We can assume that the elements Y_i are actually in the subring R: for any element of $R \otimes_S K$ is the product of an element of R and a suitable "constant" in K. Now consider the 2 rings:

$$S[Y_1,\ldots,Y_r] \subset R.$$

R is not necessarily integral over $S[Y_1,\ldots,Y_r]$; however, if $\alpha \in R$, then α satisfies an equation:

$$X^n + P_1(Y_1,\ldots,Y_r)X^{n-1} + \ldots + P_n(Y_1,\ldots,Y_r) = 0 ,$$

where the P_i are polynomials with coefficients in K. If g is a common denominator of all the coefficients, α is integral over $S_{(g)}[Y_1,\ldots,Y_r]$. Applying this reasoning to a finite set of generators of R as an S-algebra, we can find some $g \in S$ such that $R_{(g)}$ is integral over $S_{(g)}[Y_1,\ldots,Y_r]$. Define $U \subset Y$ as Y_g, i.e., $\{y \in Y \mid g(y) \neq 0\}$. The subring $S_{(g)}[Y_1,\ldots,Y_r]$ in $R_{(g)}$ defines a factorization of f restricted to $f^{-1}(U)$:

$$\begin{tikzcd} X \arrow[dd, "f"'] & \supset & f^{-1}(u) \arrow[d, "\pi"'] \\ & & U \times \mathbb{A}^r \arrow[d, "p_1"'] \\ X & \supset & U \end{tikzcd}$$

where π is finite and surjective. This shows first of all that $U \subset f(X)$ which is (i). To show (ii), let $W \subset Y$ be an irreducible closed subset that meets U, and let $Z \subset X$ be a component of $f^{-1}(W)$ that meets $f^{-1}(U)$. It suffices to show that

$$\dim Z \leq \dim W + r$$

since the other inequality has been shown in Theorem 2. Let $Z_0 = Z \cap f^{-1}(U)$ and let $W_0 = W \cap U$. Then

$$\overline{\pi(Z_0)} \subset W_0 \times \mathbb{A}^r .$$

Therefore

$$\dim \overline{\pi(Z_0)} \leq \dim(W_0 \times \mathbb{A}^r) = \dim W + r .$$

The restriction π' of π to a map from Z_0 to $\overline{\pi(Z_0)}$ is still dominating and finite. Therefore it induces an inclusion of $k\left(\overline{\pi(Z_0)}\right)$ in $k(Z_0)$ such that $k(Z_0)$ is algebraic over $k\left(\overline{\pi(Z_0)}\right)$. Therefore

$$\dim Z \leq \dim \overline{\pi(Z_0)} \leq \dim W + r .$$

□

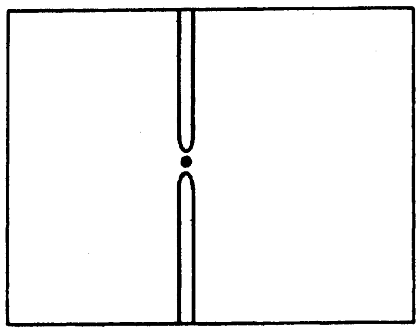
Corollary 1. *f as above. Then there is a nonempty open set $U \subset Y$ such that, for all $y \in Y$, $f^{-1}(y)$ is a nonempty "pure" r-dimensional set, i.e., all its components have dimension r.*

Theorems 2 and 3 together give a good qualitative picture of the structure of a morphism. We can work this out a bit by some simple inductions.

Definition 2. Let X be a variety. A subset A of X is *constructible* if it is a finite union of locally-closed subsets of X.

The constructible sets are easily seen to form a Boolean algebra of subsets of X; in fact, they are the smallest Boolean algebra containing all open sets. A typical constructible set which is not locally closed is

$$\{\mathbb{A}^2 - V((X))\} \cup \{(0,0\}$$



Corollary 2 (Chevalley). *Let $f : X \to Y$ be any morphism. Then the image of f is a constructible set in Y. More generally, f maps constructible sets in X to constructible sets in Y.*

Proof. The second statement follows immediately from the first. To prove the first, use induction on $\dim Y$.

a. If f is not dominating, let $Z = \overline{f(Y)}$. Then $f(Y) \subset Z$ and $\dim Z < \dim Y$, so the result follows by induction.
b. If f is dominating, let $U \subset Y$ be a nonempty open set as in Theorem 3. Let $Z_1, \ldots, Z_t$ be the components of $Y - U$ and let $W_{il}, \ldots, W_{is_i}$ be the components of $f^{-1}(Z_i)$. Let $g_{ij} : W_{ij} \to Z_i$ be the restriction of f. Since $\dim Z_i < \dim Y$, $g_{ij}(W_{ij})$ is constructible. Since

$$f(X) = U \cup \bigcup_{i,j} g_{ij}(W_{ij}) \ ,$$

$f(X)$ is constructible too.

□

Corollary 3 (Upper semi-continuity of dimension). *Let $f : X \to Y$ be any morphism. For all $x \in X$, define*

$$e(x) = \left\{ \max(\dim Z) \,\middle|\, \begin{array}{l} Z \ \text{ a component of} \\ f^{-1}(f(x)) \ \text{ containing } x \end{array} \right\} \ .$$

Then e is upper semi-continuous, i.e., for all integers n

$$S_n(f) = \{x \in X \mid e(x) \geq n\}$$

is closed.

Proof. Again do an induction on $\dim Y$. Again, we may well assume that f is dominating. Let $U \subset Y$ be a set as in Theorem 3. Let $r = \dim X - \dim Y$. First of all, if $n \leq r$, then $S_n(f) = X$ by Theorem 2, so $S_n(f)$ is closed. Secondly, if $n > r$, then $S_n(f) \subset X - f^{-1}(U)$ by Theorem 3. Let $Z_1, \ldots, Z_t$ be the components of $Y - U$, $W_{il}, \ldots, W_{is_i}$ the components of $f^{-1}(Z_i)$ and g_{ij} the restriction of f to a morphism from W_{ij} to Z_i. If $S_n(g_{ij})$ is the subset of W_{ij} defined for the morphism g_{ij}, just as $S_n(f)$ is for f, then $S_n(g_{ij})$ is closed by the induction hypothesis. But if $n > r$, then it is easy to check that

$$S_n(f) = \bigcup_{i,j} S_n(g_{ij}) \ ,$$

so $S_n(f)$ is closed too. □

Definition 3. A morphism $f : X \to Y$ is *birational* if it is dominating and the induced map

$$f^* : k(Y) \longrightarrow k(X)$$

is an isomorphism.

Theorem 4. *If $f : X \to Y$ is a birational morphism, then there is a nonempty open set $U \subset Y$ such that f restricts to an isomorphism from $f^{-1}(U)$ to U.*

Proof. We may as well assume that Y is affine with coordinate ring S. Let $U \subset X$ be any nonempty open affine set, with coordinate ring R. Let $W = \overline{f(X - U)}$. Since all components of $X - U$ are of lower dimension than X, also all components of W are of lower dimension than Y. Therefore W is a proper closed subset of Y. Pick $g \in S$ such that $g = 0$ on W, but $g \neq 0$. Then it follows that

$$f^{-1}(Y_g) \subset U \ .$$

If $g' = f * g$ is the induced element in R, then in fact

$$f^{-1}(Y_g) = U_{g'} \ ,$$

so by replacing Y by Y_g and X by $U_{g'}$, we have reduced the proof of the theorem to the case where Y and X are affine.

Now assume that R and S are the coordinate rings of X and Y. Then f defines the homomorphism

$$\begin{tikzcd} R \arrow[r, phantom, "\subset"] & k(X) \arrow[d, dash, "\cong"'] \\ S \arrow[u, "f^*"] \arrow[r, phantom, "\subset"] & k(Y). \end{tikzcd}$$

Let $x_1, \ldots, x_n$ be a set of generators of R, and write $x_i = y_i/g$, $y_1, \ldots, y_n$, $g \in S$. Then f^* localizes to an isomorphism from $S_{(g)}$ to $R_{(g)}$. Therefore Y_g satisfies the requirements of the theorem. □

The theory developed in this section cries out for examples. Theorem 3 and its corollaries are illustrated in the following:

Example S. $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2$ defined by:

$$f(x,y) = (xy, y) .$$

i) The image of f is the union of $(0,0)$ and

$$\left(\mathbb{A}^2\right)_y = \mathbb{A}^2 - \{\text{points where } y = 0\} .$$

This set is *not* locally closed.

ii) f is birational, and if $U = \mathbb{A}^2_y$, then $f^{-1}(U) = \mathbb{A}^2_y$ and the restriction of f to a map from $f^{-1}(U)$ to U is an isomorphism.

iii) On the other hand, $f^{-1}((0,0))$ is the whole line of points $(x, 0)$.

iv) $S_0(f) = \mathbb{A}^2$, $S_1(f) = \{(x,0)\}$, $S_2(f) = \emptyset$ (notation as in Corollary 3, Theorem 3).

To illustrate Theorem 4, look again at:

Examples O, P bis. In example O, §4, we defined a finite birational morphism

$$f : \mathbb{A}^1 \longrightarrow C ,$$

where C is the affine plane curve $X^3 = Y^2$. If $U = C - \{(0,0)\}$, then $f^{-1}(U) = \mathbb{A}^1 - \{(0)\}$, and $f^{-1}(U) \xrightarrow{\sim} U$. On the ring level:

$$k[T] \supsetneq k\left[T^2, T^3\right] ,$$

but

$$k\left[T, T^{-1}\right] = k\left[T^2, T^3, T^{-2}\right] .$$

In example P, we defined a finite birational morphism

$$f : \mathbb{A}^1 \longrightarrow D$$

and then considered its restriction

$$f' : \mathbb{A}^1 - \{1\} \longrightarrow D$$

to get a bijection. If $U = D - \{(0,0)\}$, then $f'^{-1}(U) = \mathbb{A}^1 - \{(1), (-1)\}$ and $f^{-1}(U) \xrightarrow{\sim} U$.

§9. Complete varieties

An affine variety can be embedded in a projective variety, by a birational inclusion. Can a projective variety be embedded birationally in anything even bigger? The answer is no; there is a type of variety, called complete, which in our algebraic theory plays the same role as compact spaces do in the theory of topological spaces. These are "maximal" and projective varieties turn out to be complete.

Recall the main result of classical elimination theory (which we will reprove later):

Given r polynomials, with coefficients in k:

$$f_1(x_0, \ldots, x_n;\ y_1, \ldots, y_m)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_r(x_0, \ldots, x_n;\ y_1, \ldots, y_m) \quad ,$$

all of which are homogeneous in the variables $x_0, \ldots, x_n$, there is a second set of polynomials (with coefficients in k):

$$g_1(y_1, \ldots, y_m)$$
$$\cdots\cdots$$
$$g_\nu(y_1, \ldots, y_m)$$

such that for all m-tuples $(a_1, \ldots, a_m)$ in k, $g_i(a_1, \ldots, a_m) = 0$, all i if and only if there is a non-zero $(n+1)$-tuple $(b_0, \ldots, b_n)$ in k such that $f_i(b_0, \ldots, b_m;\ a_1, \ldots, a_m) = 0$, all i. (Cf. van der Waerden, §80). In our language, the equations $f_1 = \ldots = f_r = 0$ define a closed subset

$$X \subset \mathbb{P}_n \times \mathbb{A}^m \quad .$$

Let p_2 be the projection of $\mathbb{P}_n \times \mathbb{A}^m$ onto $\mathbb{A}^m$. The conclusion asserted is that $p_2(X)$ is a closed subset of $\mathbb{A}^m$; in fact that

$$p_2(X) = V((g_1, \ldots, g_\nu)) \quad .$$

In other words, the theorem is:

$$p_2 : \mathbb{P}_n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m \text{ is a } \textit{closed} \text{ map,}$$
i.e., it maps closed sets onto closed sets,

(modulo the fact that every closed subset of $\mathbb{P}_n \times \mathbb{A}^m$ is described by a set of equations $f_1, \ldots, f_r$ as above). This property easily implies the apparently stronger property – $p_2 : \mathbb{P}_n \times X \to X$ is closed, for all varieties X. This motivates:

Definition 1. A variety X is *complete* if for all varieties Y, the projection morphism

$$p_2 : X \times Y \longrightarrow Y$$

is a closed map.

The analogous property in the category of topological spaces characterizes compact spaces X, at least as long as X is a reasonable space – say completely regular or with a countable basis of open sets. This definition is very nice from a category-theoretic point of view. It gives the elementary properties of completeness very easily:

i) Let $f : X \to Y$ be a morphism, with X complete, then $f(X)$ is closed in Y and is complete.
ii) If X and Y are complete, then $X \times Y$ is complete.
iii) If X is complete and $Y \subset X$ is a closed subvariety, then Y is complete.
iv) An affine variety X is complete only if $\dim X = 0$, i.e., X consists in a single point.

[In fact, (iv) follows from (i) by embedding the affine variety X in its closure $\overline{X}$ in a suitable projective space; and noting that $\overline{X} - Y = \overline{X} \cap$ (hyperplane at ∞) is non-empty by Theorem 2*, §7].

It is harder to prove the main theorem of elimination theory:

Theorem 1. *$\mathbb{P}_n$ is complete.*

Proof (Grothendieck). We must show that for all varieties Y, $p_2 : \mathbb{P}_n \times Y \to Y$ is closed. The problem is clearly local on Y, so we can assume that Y is affine. Let $R = \Gamma(Y, \underline{o}_Y)$.

Note that $\mathbb{P}_n \times Y$ is covered by affine open sets $U_i = (\mathbb{P}_n)_{X_i} \times Y$, whose coordinate rings are $R\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right]$. Now suppose Z is a closed subset of $\mathbb{P}_n \times Y$. The first problem is to describe Z by a homogeneous ideal in the graded ring $S = R[X_0, \ldots, X_n]$ over R. Let S_m be the graded piece of degree m. Let $A_m \subset S_m$ be the vector space of homogeneous polynomials $f(X_0, \ldots, X_m)$, of degree m, coefficients in R, such that for all i,

$$f\left(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right) \in I(Z \cap U_i) \ .$$

Then $A = \Sigma A_m$ is a homogeneous ideal in S.

Lemma. *For all i and all $g \in I(Z \cap U_i)$, there is a polynomial $f \in A_m$ for some m such that*

$$g = f\left(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right) \ .$$

Proof. If m is large enough, $X_i^m \cdot g$ is a homogeneous polynomial $f' \in S_m$. To check whether $f' \in A_m$, look at the functions

$$g_j = \frac{f'}{X_j^m} \in R\left[\frac{X_0}{X_j}, \ldots, \frac{X_n}{X_j}\right] \ .$$

g_j is clearly zero on $Z \cap U_i \cap U_j$. And even if it is not zero on $Z \cap U_j$, $\frac{X_i}{X_j} \cdot g_j$ is zero there. Therefore $f = X_i \cdot f' \in A_{m+1}$ and this f does the trick. □

Now suppose $y \in Y - p_2(Z)$. Let $M = I(y)$ be the corresponding maximal ideal. Then $Z \cap U_i$ and $(\mathbb{P}_n)_{X_i} \times \{y\}$ are disjoint closed subsets of U_i. Therefore

$$I(Z \cap U_i) + M \cdot R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] = R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] .$$

In particular

$$1 = a_i + \sum_j m_{ij} \cdot g_{ij} ,$$

where $a_i \in I(Z \cap U_i)$, $m_{ij} \in M$, and $g_{ij} \in R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$. If we multiply this equation through by a high power of X_i and use the lemma, it follows that

$$X_i^N = a_i' + \sum_j m_{ij} \cdot g_{ij}' ,$$

where $a_i' \in A_N$, $g_{ij} \in S_N$. (N may as well be chosen large enough to work for all i). In other words, $X_i^N \in A_N + M \cdot S_N$, for all i. Taking N even larger, it follows that all monomials in the X_i of degree N are in $A_N + M \cdot S_N$, i.e.,

$$(*) \qquad S_N = A_N + M \cdot S_N .$$

Now by the lemma of Nakayama applied to S_N/A_N there is an element $f \in R - M$ such that $f \cdot S_N \subset A_N$. In fact:

Nakayama's Lemma. *Let M be a finitely generated R-module, and let $A \subset R$ be an ideal such that*

$$M = A \cdot M .$$

Then there is an element $f \in 1 + A$ which annihilates M.

Proof. Let $m_1, \dots, m_n$ be generators of M as an R-module. By assumption,

$$m_i = \sum_{j=1}^{n} a_{ij} \cdot m_j$$

for suitable elements $a_{ij} \in A$. But then:

$$\sum_{j=1}^{n} (\delta_{ij} - a_{ij}) \cdot m_j = 0.$$

Solving these linear equations directly, it follows that

$$\det(\delta_{ij} - a_{ij}) \cdot m_k = 0, \qquad \text{all } k .$$

But if $f = \det(\delta_{ij} - a_{ij})$, then $1 - f \in A$. □

Now if $f \cdot S_n \subset A_N$, then $f \cdot X_i^N \in A_N$, hence $f \in I(Z \cap U_i)$. This shows that $f = 0$ at all points of $p_2(Z)$, i.e., $p_2(Z) \cap Y_f = \emptyset$.

This proves that the complement of $p_2(Z)$ contains a neighbourhood of every point in it, hence it is open. □

Putting the theorem and remark iii) together, it follows that every projective variety is complete. For some years people were not sure whether or not all complete varieties might not actually be projective varieties. In the next section, we will see that even if a complete variety is not projective, it can still be dominated by a projective variety with the same function field. Thus the problem is a "birational" one, i.e., concerned with the comparison of the collection of all varieties with a common function field. An example of a non-projective complete variety was first found by Nagata.

Theorem 1 can be proven also by valuation-theoretic methods, invented by Chevalley. This method is based on the:

Valuative Criterion. *A variety X is complete if and only if for all valuation rings $R \subset k(X)$ containing k and with quotient field $f(X)$, $R \supset \underline{o}_x$ for some $x \in X$.*

§10. Complex varieties

Suppose that our algebraically closed ground field k is given a topology making it into a topological field. The most interesting case of this is when $k = \mathbb{C}$, the complex numbers. However, we can make at least the first definition in complete generality. Namely, I claim that when k is a topological field, then there is a unique way to endow all varieties X over k with a new topology, which we will call the *strong topology*, such that the following properties hold:

i) the strong topology is stronger than the Zariski-topology, i.e., a closed (resp. open) subset $Z \subset X$ is always strongly closed, (resp. strongly open).
ii) all morphisms are strongly continuous,
iii) if $Z \subset X$ is a locally closed subvariety, then the strong topology on Z is the one induced by the strong topology on X,
iv) the strong topology on $X \times Y$ is the product of the strong topologies on X and Y;
v) the strong topology on $\mathbb{A}^1$ is exactly the given topology on k.

(These are by no means independent requirements.)

We leave it to the reader to check that such a set of strong topologies exists; it is obvious that there is at most one such set. Note that all varieties X are Hausdorff spaces in their strong topology. In fact, if $\Delta : X \to X \times X$ is the diagonal map, then $\Delta(X)$ is strongly closed by (i). Since $X \times X$ has the product strong topology by (iv), this means exactly that X is a Hausdorff space.

From now on, suppose $k = \mathbb{C}$ with its usual topology. Then varieties not only have the strong topologies: they even have "strong structure sheaves", or in more conventional language, they are complex analytic spaces[3]. This means

[3] The standard definition of a complex analytic space is completely analogous to our definition of a variety: i.e., it is a Hausdorff topological space X, plus a sheaf of $\mathbb{C}$-valued continuous functions Ω_X on X, which is locally isomorphic to one of

that there is a unique collection of sheaves (in the strong topology) of strongly continuous $\mathbb{C}$-valued functions $\{\Omega_X\}$, one for each variety X, such that:

i) for each Zariski-open set $U \subset X$,

$$\underline{o}_X(U) \subset \Omega_X(U),$$

ii) all morphisms $f : X \to Y$ are "holomorphic", i.e., f^* takes sections of Ω_Y to sections of Ω_X,
iii) if $Z \subset X$ is a locally closed subvariety, then Ω_Z is the sheaf of $\mathbb{C}$-valued functions on Z induced by the sheaf Ω_X on X,
iv) if $X = \mathbb{A}^n \underset{\text{as a set}}{=} \mathbb{C}^n$, then Ω_X is the usual sheaf of holomorphic functions on $\mathbb{C}^n$.

(Again these properties are not independent.) We leave it to the reader to check that this set of sheaves exists: the uniqueness is obvious. Moreover, it follows immediately that (X, Ω_X) is a complex analytic space for all varieties X.

The first non-trivial comparison theorem relating the 2 topologies states that the strong topology is not "too strong":

Theorem 1. *Let X be a variety, and U a nonempty open subvariety. Then U is strongly dense in X.*

Proof (Based on suggestions of G. Stolzenberg). Since the theorem is a local statement (in the Zariski topology), we can suppose that X is affine. By Noether's normalization lemma (geometric form), there exists a finite surjective morphism

$$\pi : X \longrightarrow \mathbb{A}^n .$$

Let $Z = X - U$. Then $\pi(Z)$ is a Zariski closed subset of $\mathbb{A}^n$. Since all components of Z have dimension $< n$, so do all components of $\pi(Z)$, hence $\pi(Z)$ is even a proper closed subset of $\mathbb{A}^n$. In particular, there is a non-zero polynomial $g(X_1, \ldots, X_n)$ such that

$$\pi(Z) \subset \{(x_1, \ldots, x_n) \mid g(x_1, \ldots, x_n) = 0\} .$$

Now choose a point $x \in X - U$. Let's first represent $\pi(x)$ as a limit of points $y^{(i)} = \left(x_1^{(i)}, \ldots, x_n^{(i)}\right) \in \mathbb{A}^n$ such that $g\left(y^{(i)}\right) \neq 0$. To do this, choose any point $y^{(1)} \in \mathbb{A}^n$ such that $g\left(y^{(1)}\right) \neq 0$, and let

$$h(t) = g\left((1-t) \cdot \pi(x) + t \cdot y^{(1)}\right), \qquad t \in \mathbb{C}$$

(i.e., $\pi(x)$ and $y^{(1)}$ are regarded as *vectors*). Then $h \not\equiv 0$ since $h(1) \neq 0$. Therefore $h(t)$ has only a finite number of zeroes, and we can choose a sequence of numbers $t_i \in \mathbb{C}$ such that $t_i \to 0$, as $i \to \infty$, and $h(t_i) \neq 0$. Then let

the standard objects: i.e., the locus of zeroes in a polycylinder of a finite set of holomorphic functions, plus the sheaf of functions induced on it by the sheaf of holomorphic functions on the polycylinder. For details, see Gunning-Rossi, Ch. 5.

$$y^{(i)} = (1-t_i)\cdot\pi(x) + t_i\cdot y^{(1)}\ .$$

Then $y^{(i)} \to \pi(x)$ strongly, as $i\to\infty$, and $g\left(y^{(i)}\right)\neq 0$.

The problem now is to lift each $y^{(i)}$ to a point $z^{(i)}\in X$ such that $z^{(i)}\to x$. Since $y^{(i)}\notin\pi(Z)$, all the points $z^{(i)}$ must be in U, hence it will follow that x is the strong closure of U. We will do this in 2 steps. First, let $\pi^{-1}(\pi(x)) = \{x, x_2, \ldots, x_n\}$. Choose a function $g\in\Gamma(X,\underline{o}_X)$ such that $g(x)=0$, but $g(x_i)\neq 0$, $2\le i\le n$. Let $F(X_1,\ldots,X_n,g)=0$ be the irreducible equation satisfied by $X_1,\ldots,X_n$ and g in $\Gamma(X,\underline{o}_X)$. We shall work with the 3 rings and 3 affine varieties:

$$\begin{array}{cccc}
 & \Gamma(X,\underline{o}_X) & & X \\
 & \cup & & \downarrow{\scriptstyle\pi_1} \\
k[X_1,\ldots,X_n,Y]/(F) \cong & k[X_1,\ldots,X_n,g] & & V(F)\subset\mathbb{A}^{n+1} \\
 & \cup & & \downarrow{\scriptstyle\pi_2} \\
 & k[X_1,\ldots,X_n] & , & \mathbb{A}^n \quad , \quad \pi=\pi_2\cdot\pi_1 .
\end{array}$$

Since g is integrally dependent on $k[X_1,\ldots,X_n]$, F has the form:

$$F(X_1,\ldots,X_n,Y) = Y^d + A_1(X_1,\ldots,X_n)\cdot Y^{d-1}+\ldots+A_d(X_1,\ldots,X_n)\,.$$

Writing $(X_1,\ldots,X_n)$ as a vector, we abbreviate $F(X_1,\ldots,X_n,Y)$ to $F(X,Y)$. Now since $g(x)=0$,

$$0 = F\big(\pi(x),g(x)\big) = A_d\big(\pi(x)\big)\,.$$

Therefore, $A_d\left(y^{(i)}\right)\to 0$ as $i\to\infty$. On the other hand,

$$A_d\left(y^{(i)}\right) = \left\{\begin{array}{l}\text{Product of the roots of the equation in } t\\ F\left(y^{(i)},t\right)=0\end{array}\right\}\,.$$

Therefore we can find roots $t^{(i)}$ of $F\left(y^{(i)},t\right)\equiv 0$ such that $t^{(i)}\to 0$. Then $\left(y^{(i)},t^{(i)}\right)$ is a sequence of points of $V(F)$ converging strongly to $\pi_1(x)$. This is the 1st step.

Now choose generators $h_1,\ldots,h_N$ of the ring $\Gamma(X,\underline{o}_X)$. Via the h_i's, we can embed X in $\mathbb{A}^N$, so that its strong topology is induced by the strong topology in $\mathbb{A}^N$. Each h_i satisfies an equation of integral dependence:

$$h_i^m + a_{i1}\cdot h_i^{m-1}+\ldots+a_{im}=0$$

with $a_{ij}\in k[X_1,\ldots,X_n,g]$. Therefore, if $\Sigma\in V(F)$ is a relatively compact subset, all the polynomials a_{ij} are bounded on Σ, so each of the functions h_i is bounded on $\pi_1^{-1}(\Sigma)$. Since $|h_i|\le C$, all i, is a compact subset of $\mathbb{A}^N$, $\pi_1^{-1}(\Sigma)$ is a relatively compact subset of X. On the other hand, π_1 is a surjective map since π_1

is a finite morphism. So choose points $z^{(i)} \in X$ such that $\pi_1\left(z^{(i)}\right) = \left(y^{(i)}, t^{(i)}\right)$. Since the points $\left(y^{(i)}, t^{(i)}\right)$ converge in $V(F)$, they are a relatively compact set. Therefore $\left\{z^{(i)}\right\}$ is a relatively compact subset of X. Suppose they did not converge to x: then some subsequence $z^{(i_k)}$ would converge to some $x' \neq x$. Since

$$y^{(i_k)} = \pi\left(z^{(i_k)}\right) \longrightarrow \pi(x'), \qquad (\text{as } k \to \infty)$$

$\pi(x') = x$, so $x' = x_i$, for some $2 \leq i \leq n$. But then

$$t^{(i_k)} = g\left(z^{(i_k)}\right) \longrightarrow g(x') \neq 0 \qquad (\text{as } k \to \infty).$$

This is a contradiction, so $z^{(i)} \to x$. □

Corollary 1. *If $Z \subset X$ is a constructible subset of a variety, then the Zariski closure and the strong closure of Z are the same.*

The main result of this section is:

Theorem 2. *Let X be a variety over $\mathbb{C}$. Then X is complete if and only if X is compact in its strong topology.*

Proof. Suppose first that X is strongly compact. Let Y be another variety, let $p_2 : X \times Y \to Y$ be the projection, and let $Z \subset X \times Y$ be a closed subvariety. Since X is compact, p_2 is a proper map in the strong topology. Therefore p_2 takes strongly closed sets to strongly closed sets (cf. Bourbaki, *Topologie Générale*, Ch. I, §10). Therefore $p_2(Z)$ is strongly closed. Since it is also constructible (§8, Th. 3, Cor. 3), it is Zariski closed by the Cor. to Theorem 1.

Conversely, we must show that complete varieties are strongly compact. First of all, it is clear that $\mathbb{P}_n(\mathbb{C})$ is strongly compact. For example, it is a continuous image of the sphere in the space of homogeneous coordinates:

$$\begin{array}{l} \Sigma = \left\{(z_0, \ldots, z_n) \;\middle|\; \Sigma_i |z_i|^2 = 1\right\} \\ \quad \Big\downarrow \text{surjective} \\ \mathbb{P}_n(\mathbb{C}) \quad . \end{array}$$

Therefore all closed subvarieties of $\mathbb{P}_n(\mathbb{C})$ are strongly compact. The general case follows from:

Chow's Lemma. *Let X be a complete variety (over any algebraically closed field k). Then there exists a closed subvariety Y of $\mathbb{P}_n(k)$ for some n and a surjective birational morphism:*

$$\pi : Y \longrightarrow X \ .$$

Proof. Cover X by open affine subsets U_i with coordinate rings A_i for $1 \leq i \leq m$, and let $U^* = U_1 \cap \ldots \cap U_m$. Embed all the U_i's as closed subvarieties of $\mathbb{A}^n$ (for some n). With respect to the composite inclusion:

$$U_i \subset \mathbb{A}^n \subset \mathbb{P}_n(k)$$

U_i is a locally closed subvariety of $\mathbb{P}_n(k)$; let $\overline{U_i}$ be its closure in $\mathbb{P}_n(k)$. Note that $\overline{U}_1 \times \ldots \times \overline{U}_m$ is isomorphic to a closed subvariety of $\mathbb{P}_N(k)$ for some N by Theorem 3, §6.

Consider the composite morphism:

$$U^* \longrightarrow U^* \times \ldots \times U^* \subset \overline{U}_1 \times \ldots \times \overline{U}_m \ .$$

The first morphism is an isomorphism of U^* with a closed subvariety of $(U^*)^m$ – the "multidiagonal"; the second morphism is the product of all the inclusions $U^* \subset U_i \subset \overline{U}_i$, i.e., it is an isomorphism of $(U^*)^m$ with an open subvariety of $\overline{U}_1 \times \ldots \times \overline{U}_m$. Therefore the image is a locally closed subvariety of $\overline{U}_1 \times \ldots \times \overline{U}_m$ isomorphic to U^*. Let Y be the closure of the image. Y is certainly a projective variety and we will construct a morphism $\pi : Y \to X$.

To construct π, consider the morphism

$$U^* \xrightarrow{\Delta} U^* \times U^* \subset X \times Y$$

induced by a) the inclusion of U^* in Y, b) the inclusion of U^* in X. Let $\widetilde{Y}$ be the closure of the image. Since X and Y are complete, therefore $X \times Y$ and $\widetilde{Y}$ are complete. The projections of $X \times Y$ onto X and Y give the diagram:

$$\begin{tikzcd}
 & U^* \arrow[dl, "\cong"'] \arrow[drr, "\cong"] & & \subset & & \widetilde{Y} \arrow[dlll, "p"'] \arrow[d, "q"] \\
U^* & \subset & X & U^* & \subset & Y
\end{tikzcd}$$

This shows that the projections p and q are both isomorphisms on U^*, hence they are birational morphisms. Moreover, since $\widetilde{Y}$ is complete, $p(\widetilde{Y})$ and $q(\widetilde{Y})$ are closed in X and Y respectively; since $p(\widetilde{Y}) \supset U^*$, $q(\widetilde{Y}) \supset U^*$, this implies that p and q are surjective. I claim

(∗) $\qquad\qquad$ q is an isomorphism.

When (∗) is proven, we can set $\pi = p \cdot g^{-1}$ and everything is proven. $\widetilde{Y}$ is a closed subvariety of the product $X \times \overline{U}_1 \times \ldots \times \overline{U}_m$. We want to analyze its projection on the product $X \times \overline{U}_i$ of only 2 factors. Look at the diagram:

$$\begin{tikzcd}
 & U^* \times \ldots \times U^* \arrow[d] & \subset & X \times \bar{U}_1 \times \ldots \times \bar{U}_m \arrow[d, "r_i"', "\text{projection onto 1st + (i+1)st factors}"] \\
U^* \arrow[ur] \arrow[r, hook] & U^* \times U^* & \subset & X \times \bar{U}_i
\end{tikzcd}$$

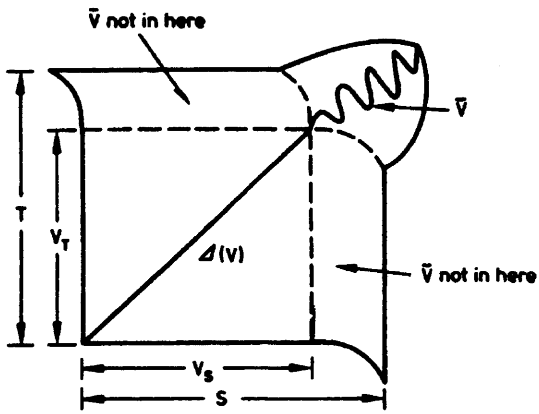
Since the projection $r_i(\widetilde{Y})$ is closed (since $\widetilde{Y}$ is complete) and contains the image of U^* as a dense subset, it follows that $r_i(\widetilde{Y})$ is just the closure of U^* in $X \times \overline{U}_i$ via the bottom arrows.

Sublemma. *Let S and T be varieties, with isomorphic open subsets $V_S \subset S$, $V_T \subset T$. For simplicity identify V_S with V_T and look at the morphism*

$$V \xrightarrow{\Delta} V \times V \subset S \times T\ .$$

If $\overline{V}$ is the closure of the image, then

$$\overline{V} \subset (S \times V) = \overline{V} \cap (V \times T) = \Delta(V)\ .$$



Proof. It suffices to show that $\Delta(V)$ is already closed in $V \times T$ and in $S \times V$. But $\Delta(V) \cap (V \times T)$, say, is just the graph of the inclusion morphism $V \to T$. Hence it is closed (cf. Remark II, following Def. 2, §6). □

Therefore

$$\begin{aligned} r_i(\widetilde{Y}) \cap (X \times U_i) &= r_i(\widetilde{Y}) \cap (U_i \times \overline{U}_i) \\ &= \{(x,x) \mid x \in U_i\}\ . \end{aligned}$$

Therefore

$$\widetilde{Y} \cap \left(X \times \overline{U}_1 \times \ldots \times U_i \times \ldots \times \overline{U}_m\right) = \widetilde{Y} \cap \left(U_i \times \overline{U}_1 \times \ldots \times \overline{U}_m\right)\ .$$

Call this set $\widetilde{Y}_i$. From the second form of the intersection it follows that $\left\{\widetilde{Y}_i\right\}$ is an open covering of $\widetilde{Y}$. From the first form of the intersection, it follows that

$$\widetilde{Y}_i = q^{-1}(Y_i)$$

if

$$Y_i = Y \cap \left(\overline{U}_1 \times \ldots \times U_i \times \ldots \times \overline{U}_M\right) .$$

Since q is surjective, this implies that $\{Y_i\}$ must be an open covering of Y. But now define:

$$\begin{aligned} &\sigma_i : Y_i \longrightarrow \widetilde{Y}_i \\ &\sigma_i\,(u_1, \ldots, u_m) = (u_i, u_1, \ldots, u_m) \ , \end{aligned}$$

(which makes sense exactly because the i^{th} component u_i is in U_i, hence is a point of X too). Then σ_i is an inverse of q restricted to $\widetilde{Y}_i$:

a) $q\,(\sigma_i\,(u_1, \ldots, u_m)) = q\,(u_i, u_1, \ldots, u_m) = (u_1, \ldots, u_m)$
b) by the sublemma, all points $(v, u_1, \ldots, u_m)$ of $\widetilde{Y}_i$ satisfy $v = u_i$, hence

$$\sigma_i\,(q\,(v, u_1, \ldots, u_m)) = (u_i, u_1, \ldots, u_m) = (v, u_1, \ldots, u_m) .$$

Therefore q is an isomorphism and π can be constructed.

QED for Chow's lemma and Th. 2.