University of Waterloo

Pmath 450 - Summer 2015
Assignment 3

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Problem 1

Assume $f : \mathbb{R} \to \mathbb{R}$ is integrable. Then since f^+ and f^- are non-negative and measurable, so we have:

$$\begin{aligned}
\int_{\mathbb{R}} f(x+y)dm(x) &= \int_{\mathbb{R}} f^+(x+y)dm(x) - \int_{\mathbb{R}} f^-(x+y)dm(x) \\
&= \int_{\mathbb{R}} f^+(x)dm(x) - \int_{\mathbb{R}} f^-(x)dm(x) \\
&= \int_{\mathbb{R}} f(x)dm(x)
\end{aligned}$$

Now if $f : \mathbb{R} \to \mathbb{C}$ is integrable, we have that $Re(f)$ and $Im(f)$ are integrable and therefore we have:

$$\begin{aligned}
\int_{\mathbb{R}} f(x+y)dm(x) &= \int_{\mathbb{R}} Re(f(x+y))dm(x) + i\int_{\mathbb{R}} Im(f(x+y))dm(x) \\
&= \int_{\mathbb{R}} Re(f(x))dm(x) - \int_{\mathbb{R}} Im(f(x))dm(x) \\
&= \int_{\mathbb{R}} f(x)dm(x)
\end{aligned}$$

Problem 2

Part a

We know that $\sup\{|f(x)+g(x)| : x \in A\} \le \sup\{|f(x)| : x \in A\} + \sup\{|g(x)| : x \in A\}$. This implies that

$$\inf\{\sup|f(x)+g(x)| : x \in A\} \le \inf\{\sup|f(X)| : x \in A\} + \inf\{\sup|g(x)| : x \in A\}$$

Hence $||f+g||_\infty \le ||f||_\infty + ||g||_\infty$.

Part b

We know that $m\{x : |h(x)| > ||h||_\infty\} = 0$. Thus

$$\inf\{\alpha \in \mathbb{R} : m\{x : |h(x)| > \alpha\} = 0\} \le ||h||_\infty$$

Now assume for a contradiction that there exist $\alpha < ||h||_\infty$ such that $m\{x : |h(x)| > \alpha\} = 0$. Since $||h||_\infty = \inf_{m(E\setminus A)=0}\{\sup|h(x)| : x \in A\}$ and $\alpha < ||h||_\infty$, for every set A with $m(E \setminus A) = 0$, we have that $\alpha < \{\sup|h(x)| : x \in A\}$. This means that there exist no set A with $m(E\setminus A) = 0$ such that $A = \{x : |h(x)| \le \alpha\}$. Thus $m\{x : |h(x)| > \alpha\} > 0$. Hence

$$\inf\{\alpha \in \mathbb{R} : m\{x : |h(x)| > \alpha\} = 0\} = ||h||_\infty$$

Problem 3

By lotus lemma we have:

$$\int \liminf f_n \le \liminf \int f_n$$

Since $f_n \to f$, we have $\int f \le \liminf \int f_n$. Thus f is integrable. Now dominated convergence theorem readily implies $\int f = \lim_n \int f_n$.

Problem 4

Part a

Let $f(x) = 0$ when $x < 1$ and $f(x) = \frac{1}{x}$ when $x \geq 1$. We have:

$$\int_{\mathbb{R}} |f|^2 = \int_1^\infty \frac{1}{x^2} = 1$$

Thus $f \in L^2(\mathbb{R})$. But $\int_{\mathbb{R}} |f| = \int_1^\infty \frac{1}{x}$ does not converge, thus $f \notin L^1(\mathbb{R})$.

Part b

Let $f^2 \in L^1[0,1]$. So $\int_0^1 |f^2| = \int_0^1 |f|^2 < \infty$, thus $f \in L^2[0,1]$. We have:

$$\begin{aligned} \int_0^1 |f| &= \int_0^1 |f|.1 \\ &\leq ||f||_2 ||1||_2 \ \ by \ \ holder's \ \ inequality \end{aligned}$$

Since $f \in L^2[0,1]$, $||f||_2 < \infty$, so $||f||_2||1||_2 < \infty$ which implies $\int_0^1 |f| < \infty$. Hence $f \in L^1[0,1]$.

Problem 5

Let $||f||_\infty > \epsilon > 0$. Let $A_\epsilon = \{x : |f(x)| \geq ||f||_\infty - \epsilon\}$. So by definition of maximum norm we get that $m(A_\epsilon) > 0$, so we have:

$$||f||_p \geq \left(\int_{A_\epsilon} (||f||_\infty - \epsilon)^p\right)^{\frac{1}{p}} = (||f||_\infty - \epsilon)(m(A_\epsilon))^{\frac{1}{p}} \to ||f||_\infty - \epsilon \;\; as \;\; p \to \infty$$

Thus, $\lim_{p\to\infty} \inf ||f||_p \geq ||f||_\infty$.
We also have:

$$\begin{aligned} ||f||_p &= \left(\int |f|^{p-1}|f|\right)^{\frac{1}{p}} \\ &\leq ||f||_\infty^{\frac{p-1}{p}} ||f||_1^{\frac{1}{p}} \quad by \quad holder's \quad inequality \\ &\to ||f||_\infty \quad as\ p \to \infty \end{aligned}$$

Thus, $\lim_{p\to\infty} \sup ||f||_p \leq ||f||_\infty$.
Hence $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Problem 6

Claim: S is dense in $C[0,1]$ with respect to L^2 norm.
Proof:
Let $\epsilon > 0$. Let $f \in C[0,1]$.
Let $a = \sup\{|f(x)| : x \in [0,\epsilon]\}$. Let $b = \sup\{|f(x)| : x \in [1-\epsilon, 1]\}$.
We define a new function $g : [0,1] \to \mathbb{C}$ as follows:
$g(x) = f(x)$ for all $x \in (\epsilon, 1-\epsilon)$.
On $[0,\epsilon]$, g is the line from 0 to $f(\epsilon)$ ($g(0) = 0$ and $g(\epsilon) = f(\epsilon)$).
On $[1-\epsilon, 1]$, g is the line from $f(1-\epsilon)$ to 0 ($g(1-\epsilon) = f(1-\epsilon)$ and $g(1) = 0$).
Note that $g \in S$. We have:

$$\begin{aligned}
||f-g||_2^2 &= \int_0^1 |f-g|^2 \\
&= \int_{[0,\epsilon]} |f-g|^2 + \int_{(\epsilon,1-\epsilon)} |f-g|^2 + \int_{[1-\epsilon,1]} |f-g|^2 \\
&\leq a\epsilon + 0 + b\epsilon \\
&= \epsilon(a+b)
\end{aligned}$$

Proving the claim .
In class we proved $C[a,b]$ is dense in $L^2[a,b]$ and now by the claim we have S is dense in $L^2[a,b]$.
Now we have a function $f \in L^2[0.1]$ such that $\int_0^1 fg = 0$ for all $g \in S$.
So for any $g \in S$, we have:

$$\begin{aligned}
0 &= \int_0^1 fg \\
&= \int_0^1 |fg| \\
&\leq ||f||_2 ||g||_2 \quad by \quad holder's \quad inequality
\end{aligned}$$