# University of Waterloo
# Algebraic Geometry - Summer 2015
# Assignment 3

**Sina Motevalli 20455091**

## Problem 1

### Part a

We know from linear algebra that every element of $O(n,k)$ has determinant either 1 or $-1$. Thus $O(n,k) = (V(\det_n -1) \cap O(n,k)) \cup (V(\det_n +1) \cap O(n,k))$. So the set is reducible and therefore is not a variety.

### Part b

We just need to note that

$$SU(2,\mathbb{C}) = \{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \}$$

As I argued in Assignment 1, this intersects the line $z = 1$ infinitely many times and therefore is not an algebraic set.

### Part c

The polynomial map $\phi : \mathbb{A}^1 \to V(xz - y^2, yz - x^3, z^2 - x^y)$ that sends $t$ to $(t^3, t^4, t^5)$ is surjective. Therefore since $\mathbb{A}^1$ is irreducible, so is $X = \phi(\mathbb{A}^1)$.

## Problem 2

It is sufficient to show that $k(\mathbb{A}^1)$ is not isomorphic to $k(V(y^2 - x^3))$.
We have that $k(V(y^2 - x^3))$ is isomorphic to $k[t^2, t^3]$ and $k(\mathbb{A}^1)$ is isomorphic to $k[t]$.
Note that $k[t]$ is a UFD, but $k[t^2, t^3]$ is not because $t^8 = t^3 t^3 t^2 = (t^2)^3$ has two factorizations.
Hence $k(\mathbb{A}^1)$ is not isomorphic to $k(V(y^2 - x^3))$.

# Problem 3

## Part a

Note that $\phi^*$ sends $g + I(Y)$ to $g \circ \phi + I(X)$.
So $\phi^*$ is injective means $g \in I(Y) \iff g \circ \phi \in I(X)$.
Since $g \circ \phi \in I(X) \iff g \in I(\phi(X))$, we have that $\phi^*$ is injective if and only if

$$I(Y) = I(\phi(X))$$

which is equivalent to image of $\phi$ under $X$ being dense in $Y$.

## Part b

Assume $\phi$ has a polynomial left-inverse $\psi$.
Let $p + I(X)$ be an arbitrary element of the coordinate ring of $X$.
Note that $p \circ \psi \in k[y_1, ..., y_m]$.
We have $\phi^*(p \circ \psi + I(Y)) = p \circ \psi \circ \phi + I(x) = p(\psi \circ \phi) + I(X) = p + I(X)$.
Thus $\phi^*$ is surjective.
Conversely assume $\phi^*$ is surjective.
Since $\phi^*$ is surjective, there exist $g_i \in k[y_1, ..., y_m]$ such that $\phi(g_i + I(Y)) = x_i + I(X)$ for every $i \in \{1, 2, ..., n\}$.
So $(g_i \circ \phi) + I(X) = x_i + I(X) \to (g_i \circ \phi)(x) = x_i(x)\ \forall x \in X$.
Let $\psi = (g_1, g_2, ..., g_n)$, then we clearly have $\psi \circ \phi = id_X$.

# Problem 4

Since $k$ is a field, to prove bijectivity, it is sufficient to prove that $\phi$ is a ring homomorphism.
Let $a, b \in \mathbb{A}^1$. We have:

$$\begin{array}{rcl} \phi(a+b) & = & (a+b)^p \\ & = & a^p + \binom{p}{1} a^{p-1} b + .. + \binom{p}{i} a^{p-1i} b^i + ... + b^p \\ & = & a^p + b^p \quad since \quad char(k) = p \ \ p \ \ is \ \ prime \\ & = & \phi(a) + \phi(b) \end{array}$$

Also $\phi(ab) = (ab)^p = a^p b^p = \phi(a)\phi(b)$. Thus $\phi$ is a ring homomorphism.
Let $g \in k[x]$. Note that $\phi^*(g) = g \circ \phi$. So $\phi^* : k[x] \to k[x^p]$. Thus $\phi^*$ is not an isomorphism as $k[x]$ is not isomorphic to $k[x^p]$ ($\phi^*$ is not even surjective). Hence $\phi$ is not an isomorphism.

# Problem 5

$I(y^2 - x^2(x+1)) =< y^2 - x^2(x+1) >$ since $y^2 - x^2(x+1)$ is irreducible.
So we have $\bar{y}^2 = \bar{x}^2(\bar{x}+\bar{1})$ in $\Gamma(V(y^2 - x^2(x+1)))$. Thus

$$z = \frac{\bar{y}}{\bar{x}} = \frac{\bar{x}^2(\bar{x}+\bar{1})}{\bar{x}\bar{y}} = \frac{\bar{x}(\bar{x}+\bar{1})}{\bar{y}}$$

and

$$z^2 = \frac{\bar{y}^2}{\bar{x}^2} = \frac{\bar{x}^2(\bar{x}+\bar{1})}{\bar{x}^2} = \bar{x}+\bar{1}$$

So $z^2$ has no poles since it has a polynomial representation and also $z^2 \in \Gamma(X)$.
We can also see that $z$ has no pole at $(a,b)$ if $a \neq 0$ or $b \neq 0$. So the only possible pole for $z$ is $(0,0)$.
Assume for a contradiction that $z$ is defined on $(0,0)$. So there exist $g, h \in \Gamma(X)$ such that $z = \frac{g}{h}$ and $h(0,0) \neq 0$. Equivalently $h\bar{y} = g\bar{x}$.
Because of the relation $\bar{y}^2 = \bar{x}^2(\bar{x}+\bar{1})$ any element of $\Gamma(X)$ can be written uniquely in the form $a(\bar{x}) + b(\bar{x})\bar{y}$.
So we can write $h\bar{y} = g\bar{x}$ as:

$$(h_1(\bar{x}) + h_2(\bar{x})\bar{y})\bar{y} = (g_1(\bar{x}) + g_2(\bar{x})\bar{y})\bar{x}$$

where $h_1(0) \neq 0$. So

$$h_1(\bar{x})\bar{y} + h_2(\bar{x})\bar{x}^2(\bar{x}+\bar{1}) = g_1(\bar{x})\bar{x} + g_2(\bar{x})\bar{x}\bar{y}$$

Now by uniqueness we have $h_1(\bar{x}) = g_2(\bar{x})\bar{x}$. Thus $h_1(0) = 0$ which is a contradiction.
Hence $(0,0)$ is the only pole of $z$ and $z \notin \Gamma(X)$.

# Problem 6

## Part a

Let $F(x,y) = ax^2 + by^2 + cxy + dx + ey + f \in k[x,y]$ be irreducible. We break the problem down into a few cases and subcases:

Case 1: Either $a$ or $b$ is nonzero. WLOG assume $a \neq 0$.
Let $X_1 = \sqrt{a}(x + \frac{c}{2a}y)$. There exists $b_1$ such that $F = X_1^2 + b_1y^2 + dx + ey + f$.
There exist constants $d_1, e_1, f_1$ such that $F = X_1^2 + b_1y^2 + d_1X_1 + e_1y + f_1$. (It's very tedious to calculate these constants but they clearly exist).
Let $X_2 = X_1 + \frac{d_1}{2}$. Then there exist constant $f_2$ such that $F = X_2^2 + b_1y^2 + e_1y + f_2$.
We have 2 subcases to consider here:

Subcase 1: $b_1 = 0$.
So $F = X_2^2 + e_1y + f_2$. Note that if $e_1 = 0$ we get $F = (X_2 - \sqrt{f_2})(X + \sqrt{f_2})$ which is reducible so $e_1 \neq 0$.
Now let $Y = -e_1y - f_2$ and we have $F = X_2^2 - Y$.

Subcase 2: $b_1 \neq 0$.
So $F = X_2^2 + b_1y^2 + e_1y + f_2 = X_2^2 + y^2 + \frac{e_1}{b_1}y + \frac{f_2}{b_1}$.
Let $Y_1 = y + \frac{e_1}{2b_1}$. Then there exist constant $f_3$ such that $F = X_2^2 + Y_1^2 - f_3$. Note that if $f_3 = 0$ then $F = (X_2 - iY_1)(X_2 + iY_1)$ which is reducible. So $f_3 \neq 0$.
Now let $X_3 = \sqrt{f_3}X_2$ and $Y_2 = \sqrt{f_3}Y_1$. Then $F = f_3(X_3^2 + Y_2^2 - 1)$. Therefore $V(F)$ can be written in the form $X_3^2 + Y_2^2 - 1 = 0$.

Case 2: $a = b = 0$.
So $F = cxy + dx + ey + f$. Note that $c \neq 0$ because otherwise thepolynomial would have degree 1. We have:
$F = c(xy + \frac{d}{c}x + \frac{e}{c}y) + f$. So there exist constant $c_1$ such that $F = c(x + \frac{e}{c})(y + \frac{d}{c}) + f_1$.
Where $f_1 \neq 0$ otherwise the polynomial would be reducible.

Let $X = \frac{-\sqrt{c}}{f_1}(\frac{1}{2}x + \frac{1}{2}y + \frac{e}{2c} + \frac{d}{2c})$ and $Y = \frac{-i\sqrt{c}}{f_1}(\frac{1}{2}x - \frac{1}{2}y + \frac{e}{2c} - \frac{d}{2c})$.
Then $X^2 + Y^2 - 1 = (X - iY)(X + iY) - 1 = \frac{-c}{f_1}(x + \frac{e}{c})(y + \frac{d}{c}) - 1$.
So $V(F)$ can be written in the form $X^2 + Y^2 - 1 = 0$.

Now we consider the case that $char(k) = 2$.
Let $F(x, y) = ax^2 + by^2 + cxy + dx + ey + f \in k[x, y]$ be irreducible.

Case 1: $c = 0$
Let $X = \sqrt{a}x + \sqrt{b}y$ and let $Y = -dx - ey - f$. Then $F = X^2 - Y$. Thus $V(F)$ is isomorphic to $V(Y - X^2)$.

Case 2: $c \neq 0$
We have 2 subcases

Subcase 1: $a = b = 0$
Let $X = cx + e$ and let $Y = y + \frac{d}{c}$. Then $F = XY - f_1$ for some constant $f_1$. Thus $V(F)$ is isomorphic to $V(XY - 1)$ and I will prove in part (b) of this problem that $V(XY - 1)$ is isomorphic to $V(X^2 + Y^2 - 1)$.

Subcase 2: Either $a$ or $b$ is nonzero. WLOG assume $a \neq 0$.

## Part b

The polynomial map $\phi : \mathbb{A}^1 \to V(y - x^2)$ which sends $t$ to $(t^2, t)$ is clearly an isomorphism because it's inverse simply sends $(t^2, t)$ to $t$ which is polynomial. So $V(y - x^2)$ is isomorphic to $\mathbb{A}^1$.
Claim: $V(x^2 + y^2 - 1)$ is isomorphic to $V(xy - 1)$.
Proof: Let $\phi : V(x^2 + y^2 - 1) \to V(xy - 1)$ be a polynomial map that sends $(a, b)$ to $(a - ib, a + ib)$.
It's inverse $\phi^{-1} : V(xy - 1) \to V(x^2 + y^2 - 1)$ sends $(a, b)$ to $(\frac{1}{2}(a + b), i\frac{1}{2}(a - b))$.
Hence $V(x^2 + y^2 - 1)$ is isomorphic to $V(xy - 1)$.
We proved in class that $V(xy-1)$ is not isomorphic to the affine line, thus $V(x^2+y^2-1)$ is not isomorphic to $\mathbb{A}^1$.

## Part c

Given a point $(a, b) \in V(x^2 + y^2 - 1)$, we need to verify that the intersection of $y = 0$ and the line containing $(a, b)$ and $(0, 1)$ is the point $(\frac{a}{1-b}, 0)$.
The line passing through $(a, b)$ and $(0, 1)$ is $y - 1 = \frac{b-1}{a}x$. $y = 0$, so $-1 = \frac{b-1}{a}x$. Thus $x = \frac{a}{1-b}$.
Hence the rational map sending $(x, y)$ to $\frac{x}{1-y}$ is stereographic projection.
The inverse $\Theta^{-1} : \mathbb{A}^1 \to V(x^2+y^2-1)$ is also a rational map that sends $t$ to $(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$.
Also note that both $\Theta^{-1}$ and $\Theta$ are dominant because
$cl(\Theta^{-1}(\mathbb{A}^1)) = cl(X \setminus \{(0, 1)\}) = X$ and $cl(\Theta(X)) = \mathbb{A}^1$.
Thus $\Theta$ is a birational equivalence.