PMATH 764: Assignment 4

Due: Monday, 29 June, 2015.

1. Let $I \subset k[x_1, \cdots, x_n]$ be an ideal that can be generated by r elements. Show that every irreducible component of $V(I)$ has dimension $\geq n - r$.

2. *(Optional)* Let X and Y be affine varieties. Show that if there is a dominant rational map from X to Y, then $\dim Y \leq \dim X$.

3. Let X be an affine variety and let $p \in X$. Moreover, let M_p be the maximal ideal of p in $\Gamma(X)$.

   (a) Prove that there is a one-to-one correspondence between prime ideals in $\Gamma(X)$ contained in M_p and prime ideals in $\mathcal{O}_p(X)$.

   (b) Use (a) to show that there is a one-to-one correspondence between the prime ideals of the local ring $\mathcal{O}_p(X)$ and the subvarieties of X containing p.

   (c) Use (b) to prove that $\dim \mathcal{O}_p(X) = \dim X$, where $\dim \mathcal{O}_p(X)$ denotes the Krull dimension of $\mathcal{O}_p(X)$.
   Note: The *Krull dimension* of a Noetherian ring R is the number n of strict inclusions in the longest chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ in R.

4. Let X and Y be affine varieties.

   (a) Let $\varphi : X \to Y$ be a polynomial map. Let $p \in X$ and set $q = \varphi(p)$. Show that the pullback map $\varphi^* : \Gamma(Y) \to \Gamma(X)$ extends uniquely to a ring homomorphism (also written φ^*) from $\mathcal{O}_q(Y)$ to $\mathcal{O}_p(X)$ such that $\varphi^*(M_q(Y)) \subset M_p(X)$. Note, however, that φ^* may not extend to all of $k(Y)$: explain why that is true. Furthermore, prove that if φ is an isomorphism, then $\mathcal{O}_q(Y)$ and $\mathcal{O}_p(X)$ are isomorphic as local rings so that $\varphi^*(M_q(Y)) = M_p(X)$.

   (b) Prove that smoothness is invariant under isomorphism. In other words, show that if $\varphi : X \to Y$ is an isomorphism, then X is smooth at p if and only if Y is smooth at $\varphi(p)$.

   (c) Let X be the cone $z^2 = x^2 + y^2$ in $\mathbb{A}^3$ and Y be the paraboloid $z = x^2 + y^2$ in $\mathbb{A}^3$. Are X and Y isomorphic? Justify your answer.

5. *Affine algebraic groups.* An *affine algebraic group* is an affine algebraic set G endowed with a group structure whose multiplication $m : G \times G \to G$ and inverse $i : G \to G$ maps are both polynomial (as in the irreducible case, a map $\varphi : X \to Y$ between algebraic sets $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ is called *polynomial* if there exist polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in X$).

   (a) Prove that the following are affine algebraic groups:

      i. The *additive group* $\mathbb{G}_a = (k, +)$, which can be identified with $\mathbb{A}^1$ under addition.

      ii. The *multiplicative group* $\mathbb{G}_m = (k^\times, \times)$, which can be identified with the hyperbola $V(xy-1) \subset \mathbb{A}^2$ under multiplication.

      iii. The *general linear group* GL_n, which can be identified with
      $$\{(A, t_{n^2+1}) \in M_{n\times n}(k) \times k : \det(A)t_{n^2+1} = 1\} \subset \mathbb{A}^{n^2} \times \mathbb{A}.$$
      Show, in particular, that GL_n is irreducible.

      iv. The *special linear group* $SL_n = \{A \in M_{n\times n}(k) : \det(A) = 1\}$.

      v. The *orthogonal group* $O_n = \{A \in M_{n\times n}(k) : AA^T = I_{n\times n}\}$ and the *special orthogonal group* $SO_n = O_n \cap SL_n$, when k has $\mathrm{char}(k) \neq 2$.

An affine algebraic group is called a *linear algebraic group* if it can be expressed as a Zariski closed subset of GL_n that is closed under multiplication, for some $n \geq 1$. For instance, Gl_n, SL_n, O_n and SO_n are all linear. A theorem of Chevalley states that, in fact, all affine algebraic groups are linear.

(b) Let G be an affine algebraic group and denote by G° the irreducible component of G containing the identity element e.

i. Show that the irreducible components $G_1, \ldots, G_r$ of G are pairwise disjoint and isomorphic to G°. (*Hint:* Consider the map $g : G \to G, h \mapsto m(g, h)$, for fixed $g \in G$.)

ii. Let $g \in G$ and G_{i_0} be the irreducible component of G containing g. Prove that $T_g G_{i_0}$ and $T_e G^\circ$ are isomorphic as k-vector spaces, and use this fact to show that all affine algebraic groups are smooth.

iii. The *Lie algebra* of G is defined to be $\mathfrak{g} := T_e G^\circ$. Show that

$$\mathfrak{sl}_n = \{A \in M_{n\times n}(k) : \mathrm{Tr}(A) = 0\}$$

and

$$\mathfrak{so}_n = \mathfrak{o}_n = \{A \in M_{n\times n}(k) : A + A^T = 0\},$$

where k is assumed to have $\mathrm{char}(k) \neq 2$ for $\mathfrak{so}_n$ and $\mathfrak{o}_n$.