PMath 450 Assignment #5
due Friday July 17, 2015

1. Suppose $f \in L^1(\mathbb{T})$ has Fourier series $A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$.

   (a) Show that $A_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt$, $B_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt$

   (b) Show that if f is even then $B_n = 0$ for all n.

2. Show that $\|D_N\|_p = O(N^{1-1/p})$ if $p > 1$.

3. (a) Show that the function $f(x) = x$ defined on $[0, 2\pi)$ has Fourier series

$$S(f) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

   (b) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

4. Suppose $f \in L^p(\mathbb{T})$. Prove that $\lim_{t \to 0} \|f_t - f\|_p = 0$ for $1 \le p < \infty$, but not if $p = \infty$. (Hint: Do it first for f continuous.)

5. Let $A(\mathbb{T}) = \{f \in \mathbb{L}^1(\mathbb{T}) : \sum_n \left|\widehat{f}(n)\right| < \infty\}$

   (a) Prove that $A(\mathbb{T}) \subseteq C(\mathbb{T})$

   (b) If $f, g \in L^2(\mathbb{T})$, prove $f * g \in A(\mathbb{T})$

   (b) Define trigonometric polynomials, P_n, Q_n, inductively as follows: $P_0 = Q_0 = 1$ and

$$\begin{aligned} P_{n+1}(t) &= P_n(t) + e^{i2^n t} Q_n(t) \\ Q_{n+1}(t) &= P_n(t) - e^{i2^n t} Q_n(t). \end{aligned}$$

   (i) Verify that

$$\begin{aligned} |P_{n+1}(t)|^2 + |Q_{n+1}(t)|^2 &= 2\left(|P_n(t)|^2 + |Q_n(t)|^2\right), \\ |P_n(t)|^2 + |Q_n(t)|^2 &= 2^{n+1} \text{ and} \\ \|P_n\|_\infty &\le 2^{(n+1)/2}. \end{aligned}$$

   (ii) Show that for each $|k| < 2^n$, $\widehat{P_{n+1}}(k) = \widehat{P_n}(k)$, and that there is a sequence of $\{r_k\}_{k=0}^{\infty}$, $r_k = \pm 1$, such that each $P_n(t) = \sum_{k=0}^{2^n - 1} r_k e^{ikt}$.

   (iii) Construct a continuous function $P(t)$ with $\sum_{k=-\infty}^{\infty} \left|\widehat{P}(k)\right| = \infty$.