PMath 450 Assignment #3
due Friday June 19, 2015

1. (a) Show that every simple function has a representation of the form $\sum_{i=1}^{N} a_i \chi_{E_i}$ where the real numbers a_i are distinct, the sets E_i are disjoint and measurable, and $\bigcup_{i=1}^{N} E_i = \mathbb{R}$. Show also that the representation of this form is unique.

   (b) In class we proved that if f is non-negative and measurable, then $\int_{\mathbb{R}} f(x+y)dm(x) = \int_{\mathbb{R}} f(x)dm(x)$ for all $y \in \mathbb{R}$. Prove that the same identity holds for f integrable.

2. (a) Show that for all measurable functions f, g, $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

   (b) Prove that $\|h\|_\infty = \inf\{\alpha \in \mathbb{R} : m\{x : |h(x)| > \alpha\} = 0\}$.

3. Let (f_n) be a sequence of non-negative, measurable functions that converge to f. Assume $f_n \leq f$ for all n. Prove that $\int f = \lim_n \int f_n$

4. (a) Show that $L^2(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$ and that $L^1(\mathbb{R})$ is not contained in $L^2(\mathbb{R})$.

   (b) Suppose $f^2 \in L^1[0,1]$. Show that $f \in L^1[0,1]$.

5. Let $f \in L^\infty[a,b]$. Show that $\|f\|_{L^p[a,b]} \to \|f\|_{L^\infty[a,b]}$ as $p \to \infty$.

6. Suppose that $f \in L^2[0,1]$ and that $\int_0^1 fg = 0$ for all continuous functions g that satisfy $g(0) = g(1) = 0$. Prove that $f = 0$ a.e. Hint: Show that the set of all such continuous functions g is dense in $L^2[0,1]$.

7. Suppose $f \geq 0$ and Lebesgue integrable on $[0,1]$. Suppose that for every positive integer $n = 1, 2, 3, ...$, we have

$$\int_0^1 f^n(x) = \int_0^1 f(x).$$

   Prove that there is some measurable set E such that $f = \chi_E$ a.e.