Lecture 10: May 27

Corollary.

1. If $f_n \geq 0$ are measurable and $f_n \to F$, then
$$\int f = \int \liminf f_n \leq \liminf \int f_n.$$
2. If f_n are measurable and $f_n \to f$ with $\int |f_n| \leq 1 \ \forall n$. Then $|f_n| \to |f|$ and
$$\int |f| \leq \liminf \int |f_n| \leq 1,$$
so f is integrable.

Theorem (Dominated Convergence Theorem)**.** Suppose f_n are measurable and $f_n \to f$ pointwise and there exists an integrable function g such that $|f_n(x)| \leq g(x) \ \forall n, x$. Then $\int f_n \to \int f$. In fact, $\int |f_n - f| \to 0$.

Proof. Look at $2g - |f_n - f|$, which is measurable. Also $g \geq |f_n| \to |f|$, so $|f_n - f| \leq |f_n| + |f| \leq 2g$. Thus $2g - |f_n - f| \geq 0$, so we can apply Fatou's lemma:
$$\int \liminf(2g - |f_n - f|) \leq \liminf \int (2g - |f_n - f|)$$
Notice that $\liminf(2g - |f_n - f|) = 2g$ since $f_n \to f$. Thus
$$\int 2g \leq \liminf \int (2g - |f_n - f|) = \liminf \left(\int 2g - \int |f_n - f| \right) = \int 2g - \limsup \int |f_n - f|$$
Since g is integrable, $\int 2g < \infty$, so we can subtract it off both sides to get $\limsup \int |f_n - f| \leq 0$. Then
$$0 \leq \liminf \int |f_n - f| \leq \limsup \int |f_n - f| \leq 0$$
Thus $\lim \int |f_n - f| = 0$. To see that $\int f_n \to \int f$, first note that f_n and f are indeed integrable, so we can consider
$$\left| \int (f_n - f) \right| \leq \int |f_n - f| \to 0$$
so $\int (f_n - f) \to 0$ and thus $\int f_n \to \int f$. □

Example.

1. Let $f_n(x)$ be n for $x \in (0, \frac{1}{n})$ and 0 otherwise. That is $f_n = n\chi_{(0,\frac{1}{n})}$. Note that $f_n \to 0$ pointwise, but $\int_{[0,1]} f_n = 1$ for all n. Any $g \geq f_n$ for all n will have to be n on $(0, \frac{1}{n})$, i.e. $g \geq n$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right)$ for all n. Then $g \geq G = \sum_{n=1}^{\infty} n\chi_{[\frac{1}{n+1}, \frac{1}{n})}$. Let S_k be the kth partial sum. Each S_k is simple and positive, and $S_k \nearrow G$ pointwise. By MCT,
$$\int G = \lim \int S_k = \lim \int \sum_{n=1}^{k} n\chi_{[\frac{1}{n+1}, \frac{1}{n})} = \lim \sum_{n=1}^{k} \int n\chi_{[\frac{1}{n+1}, \frac{1}{n})} = \lim \sum_{n=1}^{k} n \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim \sum_{n=1}^{k} \frac{1}{n+1} = \infty$$
So G is not integrable, and hence neither is g.

2. Let $f_n = 1$ on $[n, n+1]$ and 0 otherwise. Then $f_n \to 0$, but $\int_{\mathbb{R}} f_n = 1$ for all n. This fails the DCT because the smallest function that dominates f_n will be 1 on $[1, \infty]$, and thus not integrable.

Note. If $f_n \geq 0$ and measurable and $g = \sum_{n=1}^{\infty} f_n$, then by MCT, since $\sum_{n=1}^{k} f_n \nearrow g$,

$$\int g = \lim_{k\to\infty} \int \sum_{n=1}^{k} f_n = \lim_{k\to\infty} \sum_{n=1}^{k} \int f_n = \sum_{n=1}^{\infty} \int f_n.$$

Thus the integral is countably additive for nonnegative functions.

Cantor Set.

- Defined recursively by starting with $C_0 = [0,1]$ and removing the middle third of each closed interval in C_n to get C_{n+1}.
- C_n consists of 2^n closed intervals of length 3^{-n}.
- The Cantor set C is defined as $C = \bigcap_{n=0}^{\infty} C_n$.
- C is closed and bounded, and therefore compact.
- Each endpoint of a Cantor interval is in C.
- Consider the ternary expansion $x \in [0,1]$: $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i \in \{0,1,2\}$. The Cantor set consists of all numbers whose ternary expansion consists of only 0s and 2s. (Step n removes all numbers with a 1 in the nth place after the decimal). This shows that the cantor set it uncountable.
- Every point in the Cantor set is an accumulation point, and the Cantor set is closed, so the Cantor set is perfect.
  - Proof: Let $x \in C$. Then $x \in C_n \ \forall n$. Take an endpoint $y \neq x$ such that $|y - x| < 3^{-n}$ and note that $y \in C$. This can be done for any n so each $x \in C$ is an accumulation point.
- The Cantor set has empty interior (totally disconnected): If $I \subseteq C$ was an interval of length δ, then pick n such that $3^{-n} < \delta$, so $I \nsubseteq C_n$.
- $m(C) = 0$, since $C \subseteq C_n$ and $m(C_n) = 2^n 3^{-n} \to 0$. Alternatively, $C_n \searrow C$ so by downward continuity of measure, $m(C) = \lim_{n\to\infty} m(C_n) = 0$.
- All subsets of C are measurable, having outer measure 0, so the cardinality of Lebesgue measurable sets will be at least $|\mathcal{P}(C)| = |\mathcal{P}(\mathbb{R})|$, which is clearly also an upper bound.
- If instead of keeping outer thirds, we keep outer intervals of length $r\times$ parent interval length, where $r < \frac{1}{2}$, we get a homeomorphic set of measure 0.
- If you vary the lengths in each step, you can get a Cantor-like set with any measure $\alpha \in (0,1)$.