University of Waterloo
Algebraic Geometry - Summer 2015
Assignment 3

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Problem

Part a

We know from linear algebra that $O(n,k) = V(\det -1) \cup V(\det +1)$. So the set is reducible and therefore is not a variety.

Part b

Part c

The polynomial map $\phi : \mathbb{A}^1 \to V(xz - y^2, yz - x^3, z^2 - x^y)$ that sends t to (t^3, t^4, t^5) is surjective. Therefore since $\mathbb{A}^1$ is irreducible, so is $X = \phi(\mathbb{A}^1)$.

Problem 2

Solution 1

$\phi : \mathbb{A}^1 \to V(y^2 - x^3) : t \to (t^2, t^3)$ does not have a polynomial inverse.
Assume for a contradiction that $\phi^{-1} : V(y^2 - x^3) :\to \mathbb{A}^1$ is polynomial. Then ϕ^{-1} is a polynomial function on $X = V(y^2 - x^3)$, so it is an element of the coordinate ring of X. We have $A(X) = k[x_1, ..., x_n]/I(X)$. Since $\bar{y}^2 = \bar{x}^3$ in $A(X)$, any polynomial in $A(X)$ can be written as $p(\bar{x}) + \bar{y}q(\bar{x})$. Therefore $\phi^{-1}(x, y) = p(x) + yq(x)$ for some $q, p \in k[\bar{x}]$. So $t \to (t^2, t^3) \to p(t^2) + t^3 q(t^2) \neq t$ since its at least power of 2 in t. Hence ϕ^{-1} is not polynomial.

Solution 2

It is sufficient to show that $k(\mathbb{A}^1)$ is not isomorphic to $k(V(y^2 - x^3))$.
We have that $k(V(y^2 - x^3))$ is isomorphic to $k[t^2, t^3]$ and $k(\mathbb{A}^1)$ is isomorphic to $k[t]$.
Note that $k[t]$ is a UFD, but $k[t^2, t^3]$ is not because $t^8 = t^3 t^3 t^2 = (t^2)^3$ has two factorizations.
Hence $k(\mathbb{A}^1)$ is not isomorphic to $k(V(y^2 - x^3))$.

Problem 3

Part a

Note that ϕ^* sends $g + I(Y)$ to $g \circ \phi + I(X)$.
So ϕ^* is injective means $g \in I(Y) \iff g \circ \phi \in I(X)$.
Since $g \circ \phi \in I(X) \iff g \in I(\phi(X))$, we have that ϕ^* is injective if and only if

$$I(Y) = I(\phi(X))$$

which is equivalent to image of ϕ under X being dense in Y.

Part b

Assume ϕ has a polynomial left-inverse ψ.
Let $p + I(X)$ be an arbitrary element of the coordinate ring of X.
Note that $p \circ \psi \in k[y_1, ..., y_m]$.
We have $\phi^*(p \circ \psi + I(Y)) = p \circ \psi \circ \phi + I(x) = p(\psi \circ \phi) + I(X) = p + I(X)$.
Thus ϕ^* is surjective.
Conversely assume ϕ^* is surjective.
Since ϕ^* is surjective, there exist $g_i \in k[y_1, ..., y_m]$ such that $\phi(g_i + I(Y)) = x_i + I(X)$ for every $i \in \{1, 2, ..., n\}$.
So $(g_i \circ \phi) + I(X) = x_i + I(X) \rightarrow (g_i \circ \phi)(x) = x_i(x)\ \forall x \in X$.
Let $\psi = (g_1, g_2, ..., g_n)$, then we clearly have $\psi \circ \phi = id_X$.

Problem 4

Problem 5

$I(y^2 - x^2(x+1)) =< y^2 - x^2(x+1) >$ since $y^2 - x^2(x+1)$ is irreducible.
So we have $\bar{y}^2 = \bar{x}^2(\bar{x}+\bar{1})$ in $\Gamma(V(y^2 - x^2(x+1)))$. Thus

$$z = \frac{\bar{y}}{\bar{x}} = \frac{\bar{x}^2(\bar{x}+\bar{1})}{\bar{x}\bar{y}} = \frac{\bar{x}(\bar{x}+\bar{1})}{\bar{y}}$$

and

$$z^2 = \frac{\bar{y}^2}{\bar{x}^2} = \frac{\bar{x}^2(\bar{x}+\bar{1})}{\bar{x}^2} = \bar{x}+\bar{1}$$

So z^2 has no poles since it has a polynomial representation and also $z^2 \in \Gamma(X)$.
We can also see that z has no pole at (a,b) if $a \neq 0$ or $b \neq 0$. So the only possible pole for z is $(0,0)$.
Assume for a contradiction that z is defined on $(0,0)$. So there exist $g, h \in \Gamma(X)$ such that $z = \frac{g}{h}$ and $h(0,0) \neq 0$. Equivalently $h\bar{y} = g\bar{x}$.
Because of the relation $\bar{y}^2 = \bar{x}^2(\bar{x}+\bar{1})$ any element of $\Gamma(X)$ can be written uniquely in the form $a(\bar{x}) + b(\bar{x})\bar{y}$.
So we can write $h\bar{y} = g\bar{x}$ as:

$$(h_1(\bar{x}) + h_2(\bar{x})\bar{y})\bar{y} = (g_1(\bar{x}) + g_2(\bar{x})\bar{y})\bar{x}$$

where $h_1(0) \neq 0$. So

$$h_1(\bar{x})\bar{y} + h_2(\bar{x})\bar{x}^2(\bar{x}+\bar{1}) = g_1(\bar{x})\bar{x} + g_2(\bar{x})\bar{x}\bar{y}$$

Now by uniqueness we have $h_1(\bar{x}) = g_2(\bar{x})\bar{x}$. Thus $h_1(0) = 0$ which is a contradiction.
Hence $(0,0)$ is the only pole of z and $z \notin \Gamma(X)$.

Problem 6

Part a

Let $F(x,y) = ax^2 + by^2 + cxy + dx + ey + f \in k[x,y]$ be irreducible. We break the problem down to a few cases and subcases:

Case 1: Either a or b is nonzero. WLOG assume $a \neq 0$.
Let $X_1 = \sqrt{a}(x + \frac{c}{2a}y)$. There exists b_1 such that $F = X_1^2 + b_1y^2 + dx + ey + f$.
There exist constants d_1, e_1, f_1 such that $F = X_1^2 + b_1y^2 + d_1X_1 + e_1y + f_1$. (It's very tedious to calculate these constants but they clearly exist).
Let $X_2 = X_1 + \frac{d_1}{2}$. Then there exist constant f_2 such that $F = X_2^2 + b_1y^2 + e_1y + f_2$.
We have 2 subcases to consider here:

Subcase 1: $b_1 = 0$.
So $F = X_2^2 + e_1y + f_2$. Note that if $e_1 = 0$ we get $F = (X_2 - \sqrt{f_2})(X + \sqrt{f_2})$ which is reducible so $e_1 \neq 0$.
Now let $Y = -e_1y - f_2$ and we have $F = X_2^2 - Y$.

Subcase 2: $b_1 \neq 0$.
So $F = X_2^2 + b_1y^2 + e_1y + f_2 = X_2^2 + y^2 + \frac{e_1}{b_1}y + \frac{f_2}{b_1}$.
Let $Y_1 = y + \frac{e_1}{2b_1}$. Then there exist constant f_3 such that $F = X_2^2 + Y_1^2 - f_3$. Note that if $f_3 = 0$ then $F = (X_2 - iY_1)(X_2 + iY_1)$ which is reducible. So $f_3 \neq 0$.
Now let $X_3 = \sqrt{f_3}X_2$ and $Y_2 = \sqrt{f_3}Y_1$. Then $F = f_3(X_3^2 + Y_2^2 - 1)$. Therefore $V(F)$ can be written in the form $X_3^2 + Y_2^2 - 1 = 0$.

Case 2: $a = b = 0$.
So $F = cxy + dx + ey + f$. Note that $c \neq 0$ because otherwise thepolynomial would have degree 1. We have:
$F = c(xy + \frac{d}{c}x + \frac{e}{c}y) + f$. So there exist constant c_1 such that $F = c(x + \frac{e}{c})(y + \frac{d}{c}) + f_1$.
Where $f_1 \neq 0$ otherwise the polynomial would be reducible.

Let $X = \frac{-\sqrt{c}}{f_1}(\frac{1}{2}x + \frac{1}{2}y + \frac{e}{2c} + \frac{d}{2c})$ and $Y = \frac{-i\sqrt{c}}{f_1}(\frac{1}{2}x - \frac{1}{2}y + \frac{e}{2c} - \frac{d}{2c})$.
Then $X^2 + Y^2 - 1 = (X - iY)(X + iY) - 1 = \frac{-c}{f_1}(x + \frac{e}{c})(y + \frac{d}{c}) - 1$.
So $V(F)$ can be written in the form $X^2 + Y^2 - 1 = 0$.

Part b

The polynomial map $\phi : \mathbb{A}^1 \to V(y - x^2)$ which sends t to (t^2, t) is clearly an isomorphism because it's inverse simply sends (t^2, t) to t which is polynomial. So $V(y - x^2)$ is isomorphic to $\mathbb{A}^1$.
Claim: $V(x^2 + y^2 - 1)$ is isomorphic to $V(xy - 1)$.
Proof: Let $\phi : V(x^2 + y^2 - 1) \to V(xy - 1)$ be a polynomial map that sends (a, b) to $(a - ib, x + ib)$.
It's inverse $\phi^{-1} : V(xy - 1) \to V(x^2 + y^2 - 1)$ sends (a, b) to $(\frac{1}{2}(a + b), i\frac{1}{2}(a - b))$.
Hence $V(x^2 + y^2 - 1)$ is isomorphic to $V(xy - 1)$.
We proved in class that $V(xy-1)$ is not isomorphic to the affine line, thus $V(x^2+y^2-1)$ is not isomorphic to $\mathbb{A}^1$.

Part c

Given a point $(a, b) \in V(x^2 + y^2 - 1)$, we need to verify that the intersection of $y = 0$ and the line containing (a, b) and $(0, 1)$ is the point $(\frac{a}{1-b}, 0)$.
The line passing through (a, b) and $(0, 1)$ is $y - 1 = \frac{b-1}{a}x$. $y = 0$, so $-1 = \frac{b-1}{a}x$. Thus $x = \frac{a}{1-b}$.
Hence the rational map sending (x, y) to $\frac{x}{1-y}$ is stereographic projection.
The inverse $\Theta^{-1} : \mathbb{A}^1 \to V(x^2+y^2-1)$ is also a rational map that sends t to $(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$.
Thus Θ is a birational equivalence.