University of Waterloo
Pmath 450 - Summer 2015
Assignment 2

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Problem 1

Part a

Let E_1 and E_2 be measurable sets. We have: $E_1 \cup E_2 = E_1 \cup (E_2 \setminus (E_1 \cap E_2))$.
So we have:

$$\begin{aligned} m(E_1 \cup E_2) &= m(E_1 \cup (E_2 \setminus (E_1 \cap E_2))) \\ &= m(E_1) + m((E_2 \setminus (E_1 \cap E_2))) \quad \textit{since} \quad E_1 \cap (E_2 \setminus (E_1 \cap E_2)) = \emptyset \\ &= m(E_1) + m(E_2) - m(E_1 \cap E_2) \end{aligned}$$

Thus $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Part b

Let $\alpha \in \mathbb{R}$. We have:

$$\{x : \sup f_n \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n \leq \alpha\} \tag{1}$$

$$\{x : \inf f_n < \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n < \alpha\} \tag{2}$$

Since f_n's are measurable and countable union and countable intersection of measurable sets are measurable, by (1) and (2), $\sup f_n$ and $\inf f_n$ are measurable.

Part c

Since $f = g$ a.e, $h = f - g = 0$ a.e. Since f and g are continuous, $h = f - g$ is conitunous. Let $E = \{x : h(x) \neq 0\}$. We know that $m(E) = 0$. Assume for a contradiction that $E \neq \emptyset$. Let $p \in E$. There exist $\delta > 0$ such that $(p - \frac{\delta}{2}, p + \frac{\delta}{2}) \cap E = \{p\}$, otherwise E contains an interval and it's measure cannot be zero. Let $0 < \epsilon < |f(p)|$. Since h is continuous, there exist $\delta' > 0$ such that if $|x - y| < \delta'$, $|f(x) - f(y)| < \epsilon$. Let $\delta'' = \min\{\delta, \delta'\}$. Choose $x \in (p - \frac{\delta''}{2}, p + \frac{\delta''}{2})$. Note that $|p - x| < \delta'' \leq \delta'$, but $|f(p) - f(x)| = |f(p)| < \epsilon < |f(p)|$ which is a contradiction. So $E = \emptyset$
Thus $h = 0$ everywhere implying $f = g$ everywhere.

Problem 2

Part a

Let $\alpha \in \mathbb{R}$. Let $(q_n)_{n=1}^{\infty} \in (-\infty, b)$ be a sequence such that each $q_n \in \mathbb{Q}$ and $q_n \to \alpha$. We have:

$$\{x : f(x) < \alpha\} \quad = \quad \bigcup_{n=1}^{\infty} \{x : f(x) < q_n\}$$

Since each $\{x : f(x) < q_n\}$ and a countable unioun of measurable sets is measurable, $\{x : f(x) < \alpha\}$ is measurable. Thus f is measurable.