University of Waterloo

Pmath 450 - Summer 2015

Assignment 5

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Problem 1

Part a

We write the fourire series of f to find out what A_n and B_n are:

$$\begin{aligned}
f &= \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx} \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} \hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx} \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n)(\cos nx + i\sin nx)] + [\hat{f}(-n)(\cos -nx + i\sin -nx)] \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n)(\cos nx + i\sin nx)] + [\hat{f}(-n)(\cos nx - i\sin nx)] \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} \cos nx(\hat{f}(n) + \hat{f}(-n)) + \sin nx(i\hat{f}(n) - i\hat{f}(-n))
\end{aligned}$$

So we can see that $A_n = \hat{f}(n) + \hat{f}(-n)$ and $B_n = i(\hat{f}(n) - \hat{f}(-n))$. Now we have:

$$\begin{aligned}
A_n &= \hat{f}(n) + \hat{f}(-n) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx \\
&= \frac{1}{\pi}\int_0^{2\pi} f(x)\frac{e^{-inx} + e^{inx}}{2}dx \\
&= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx
\end{aligned}$$

We also have that

$$\begin{aligned}
B_n &= i(\hat{f}(n) - \hat{f}(-n)) \\
&= \frac{i}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx - \frac{i}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx \\
&= \frac{i}{2\pi}\int_0^{2\pi} f(x)(e^{-inx} - e^{inx})dx \\
&= \frac{-1}{2i\pi}\int_0^{2\pi} f(x)(e^{-inx} - e^{inx})dx \\
&= \frac{1}{2i\pi}\int_0^{2\pi} f(x)(e^{inx} - e^{-inx})dx \\
&= \frac{1}{\pi}\int_0^{2\pi} f(x)\frac{e^{inx} - e^{-inx}}{2i}dx \\
&= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx
\end{aligned}$$

Part b

Assume f is even then we have

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx dx
\end{aligned}$$

Since f is even and sin is odd, we have that $f(x)\sin nx$ is an odd function so $\frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx = -\frac{1}{\pi}\int_0^{\pi} f(x)\sin nx dx$. Thus $B_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx dx = 0$.

Problem 2

We have:

$$
\begin{aligned}
||D_N||_p^p &= \frac{1}{2\pi}\int_0^{2\pi} |\sum_{k=-N}^{N} e^{ikx}|^p \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\sum_{k=-N}^{N} e^{ikx}|^{p-1}|\sum_{k=-N}^{N} e^{ikx}| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |\sum_{k=-N}^{N} e^{ikx}|^{p-1}|\sum_{k=-N}^{N} e^{ikx}| \\
&= \frac{1}{2\pi}|||\sum_{k=-N}^{N} e^{ikx}|^{p-1}|\sum_{k=-N}^{N} e^{ikx}|||_1 \\
&\leq \frac{1}{2\pi}||(\sum_{k=-N}^{N} e^{ikx})^{p-1}||_1||\sum_{k=-N}^{N} e^{ikx}|||_\infty \quad by \quad Holder \\
&= c\frac{1}{2\pi}||\sum_{k=-N}^{N} e^{ikx}||_1 \quad c \quad constant \\
&= \frac{c}{2\pi}\int_0^{2\pi} |\sum_{k=-N}^{N} e^{ikx}|^{p-1} \\
&\leq \frac{c}{2\pi}\int_0^{2\pi} (\sum_{k=-N}^{N} |e^{ikx}|)^{p-1} \\
&= \frac{c}{2\pi}\int_0^{2\pi} (\sum_{k=-N}^{N} 1)^{p-1} \\
&= \frac{c}{2\pi}\int_0^{2\pi} (2N+1)^{p-1} \\
&= \frac{c}{2\pi}(2N+1)^{p-1}\int_0^{2\pi} 1 \\
&= (2N+1)^{p-1}
\end{aligned}
$$

So $||D_N||_p \leq (2N+1)^{\frac{p-1}{p}}$.

Problem 3

Part a

I will use problem 1 for this. So $S(f) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$.
First note that since $f(x) = x$ is odd and $\cos nx$ is even, we have that $A_n = 0$ for all $n \neq 0$.
So I need to find B_n's and $A_0 = \hat{f}(0)$.

$$\begin{aligned}
B_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx \\
&= \frac{1}{\pi} \frac{\sin(2\pi n) - 2\pi n \cos(2\pi n)}{n^2} \\
&= \frac{1}{\pi} \frac{-2\pi n}{n^2} \\
&= \frac{-2}{n}
\end{aligned}$$

We also know that $A_0 = \hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} x = \frac{1}{2\pi} 2\pi^2 = \pi$.
So the fouries series is

$$\begin{aligned}
S(f) &= A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx \\
&= A_0 + \sum_{n=1}^{\infty} B_n \sin nx \\
&= \pi + \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx \\
&= \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}
\end{aligned}$$

Part b

First using the method in part (a), we find the fourier series of $f(x) = x^2$.
$S(f) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$.
First note that since $f(x) = x^2$ is even $B_n = 0$ for all n. So I need to find A_n's.

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx \\ &= \frac{1}{\pi}\frac{(4\pi^2 n^2 - 2)\sin(2\pi n) + 4\pi n\cos(2\pi n)}{n^3} \\ &= (-1)^2\frac{4}{n^2} \end{aligned}$$

Also $A_0 = \hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{\pi^2}{3}$.
So we have $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos nx$.
Since $f(\pi) = \pi^2$ we have:

$$\begin{aligned} \pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos n\pi \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n(-1)^n \frac{4}{n^2} \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \end{aligned}$$

Now we have $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}$.

Problem 4

First assume that f is cont (so f is uniformly cont since $[0, 2\pi]$ is compact). Let $\epsilon > 0$. By continuity of f there exist $\delta > 0$ such that if $|t| < \delta$ then $|f_t - f| < \epsilon$. Now we have:

$$\begin{aligned} ||f_t - f||_p^p &= \frac{1}{2\pi}\int_o^{2\pi} |f_t - f| \\ &< \frac{1}{2\pi}\int_o^{2\pi} \epsilon \\ &= \frac{1}{2\pi}\epsilon\int_o^{2\pi} 1 \\ &= \epsilon \end{aligned}$$

Now let $f \in L^p(\mathbb{T})$. Let g be a cont function such that $||f - g||_p < \epsilon$. By continuity of g choose $\delta > 0$ such that if $|t| < \delta$ then $||g - g_t||_p < \epsilon$. We have:

$$\begin{aligned} ||f_t - f||_p &= ||f_t - g_t + g_t - g + g - f||_p \\ &\leq ||f_t - g_t||_p + ||g_t - g||_p + ||g - f||_p \\ &< \epsilon + \epsilon + \epsilon \end{aligned}$$

This fails for $p = \infty$ because for instance look at $f = X_{[0,1]}$. Then f_t converges to f if and only if f is uniformly cont (or cont here since $[0, 2\pi]$ is compact).

Problem 5

Part a

Let $f \in A(\mathbb{T})$. Let $x \in \mathbb{T}$. Let $\epsilon > 0$. We can choose N such that $\sum_{|k|>N} |\hat{f}(k)| < \epsilon$. Then for $n, m > N$ we have:

$$\begin{aligned} |S_n(f) - S_m(f)| &= |\sum_{n<|k|\leq m} \hat{f}(k)e^{ikx}| \\ &\leq \sum_{n<|k|\leq m} |\hat{f}(k)e^{ikx}| \\ &= \sum_{n<|k|\leq m} |\hat{f}(k)| < \epsilon \end{aligned}$$

So $S_n(f)$ is cauchy therefore convergent. It converges to f.

$$\begin{aligned} ||S_n(f) - f||_\infty &= \sup_{x\in\mathbb{T}} |\sum_{k>n} \hat{f}(k)e^{ikx}| \\ &\leq \sum_{k>n} |\hat{f}(k)||e^{ikx}| \\ &= \sum_{k>n} |\hat{f}(k)| \to 0 \quad as \quad n \to \infty \end{aligned}$$

Since $S_n(f)$ is cont and converges to f uniformly, f is cont.

Part b

Let $f, g \in L^2(\mathbb{T})$. So $\sum_n |\hat{f}(n)|^2, \sum_n |\hat{g}(n)|^2 < \infty$. So we have that $\sum_n |\hat{f}(n)\hat{g}(n)| \leq (\sum_n |\hat{f}(n)|^2)(\sum_n |\hat{g}(n)|^2) < \infty$. Now we have:

$$\sum_n |f \hat{*} g(n)| = \sum_n |\hat{f}(n)\hat{g}(n)| < \infty$$

Hence $f * g \in A(\mathbb{T})$.

Part b

i Lets show that $|P_{n+1}(t)|^2 + |Q_{n+1}(t)|^2 = 2(|P_n(t)|^2 + |Q_n(t)|^2)$.
$|P_{n+1}(t)|^2 + |Q_{n+1}(t)|^2 = 2(|P_n(t)|^2 + |Q_n(t)|^2)$. We have:

$$\begin{aligned}
LHS &= |P_{n+1}(t)|^2 + |Q_{n+1}(t)|^2 \\
&= P_{n+1}(t)\overline{P_{n+1}(t)} + Q_{n+1}(t)\overline{Q_{n+1}(t)} \\
&= (P_n(t) + e^{i2^n t}Q_n(t))\overline{(P_n(t) + e^{i2^n t}Q_n(t))} + (P_n(t) - e^{i2^n t}Q_n(t))\overline{(P_n(t) - e^{i2^n t}Q_n(t))} \\
&= P_n(t)\overline{P_n(t)} + Q_n(t)\overline{Q_n(t)} + P_n(t)\overline{P_n(t)} + Q_n(t)\overline{Q_n(t)} \\
&= 2(|P_n(t)|^2 + |Q_n(t)|^2)
\end{aligned}$$

Now we show by induction that $|P_n(t)|^2 + |Q_n(t)|^2 = 2^{n+1}$.
Base case: $|P_0(t)|^2 + |Q_o(t)|^2 = 1 + 1 = 2$.
Assume it is true for $0, 1, 2, .., n-1$. We need to show $|P_n(t)|^2 + |Q_n(t)|^2 = 2^{n+1}$.

$$\begin{aligned}
|P_n(t)|^2 + |Q_n(t)|^2 &= 2(|P_{n-1}|^2 + |Q_{n-1}|^2) \\
&= 2(2^n) \\
&= 2^{n+1}
\end{aligned}$$

We also have:

$$\begin{aligned}
|2P_n(t)|^2 &= |P_{n+1}(t) + Q_{n+1}(t)|^2 \\
&\leq |P_{n+1}(t)|^2 + |Q_{n+1}(t)|^2 \\
&= 2^{n+2}
\end{aligned}$$

So $||P_n(t)||_\infty \leq 2^{\frac{n+1}{2}}$.

(ii) Let $|k| < 2^n$. We have:

$$\begin{aligned}
\hat{P_n}(k) &= \frac{1}{2\pi}\int_0^{2\pi} P_n(x)e^{ikx} \\
\hat{P_{n+1}}(k) &= \frac{1}{2\pi}\int_0^{2\pi} (P_n(x) + e^{i2^n x}Q_n(X))e^{ikx} \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_n(x)e^{ikx} + e^{i2^n x}Q_n(X)e^{ikx} \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_n(x)e^{ikx} + \int_0^{2\pi} Q_n(X)e^{ikx}e^{i2^n x} \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_n(x)e^{ikx} \\
&= \hat{P_n}(k)
\end{aligned}$$