1. $X \subseteq \mathbb{A}^n$ algebraic set and $L$ a line in $\mathbb{A}^n$ such that $L \not\subseteq X$. Then $X = V(f_1, \dots, f_r)$, for some $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. Moreover, if $\vec{a} = (a_1, \dots, a_n)$ is a direction vector and $b = (b_1, \dots, b_n)$ is a point on $L$, then $L$ is given parametrically by

$$L = \{ (a_1 t + b_1, \dots, a_n t + b_n) \mid t \in k \}.$$

The points in $X \cap L$ are the solutions of the system:

$$\begin{cases} f_1(a_1 t + b_1, \dots, a_n t + b_n) = 0 \\ \vdots \\ f_r(a_1 t + b_1, \dots, a_n t + b_n) = 0 \end{cases}$$

Since $X \not\subseteq L$, at least one of the polynomials

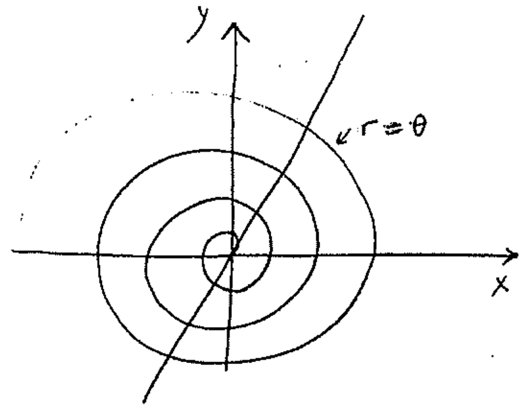
$$g_\ell(t) := f_\ell(a_1 t + b_1, \dots, a_n t + b_n),$$

$\ell = 1, \dots, r$. Suppose that $g_s \neq 0$. Then, $g_s \in k[t]$ and $V(g_s)$ is at most a finite set of points. In addition,

$$X \cap L \subseteq V(g_s) = \left( \begin{array}{c} \text{at most a finite} \\ \text{set of points} \end{array} \right),$$

proving that $X \cap L$ is at most a finite set of points. ∎

2. (a)

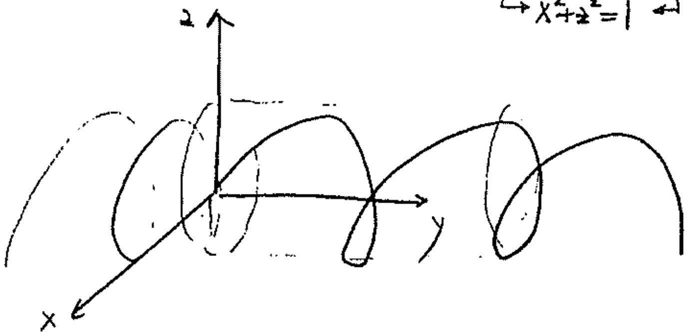



Since any line passing through the origin intersects the spiral in an infinite number of points, it is NOT algebraic.

(b) Note that $\{r = \cos\theta\} \iff V\left(\left(x-\frac{1}{2}\right)^2 + y^2 - \frac{1}{4}\right)$, so the set is algebraic. ②

(c) $\{(\cos t, 1, \sin t) \mid t \in \mathbb{R}\} = V(x^2+y^2-1, z-1) \subseteq \mathbb{R}^3$, which is algebraic

(d) The helix $\{(\underbrace{\cos t, t, \sin t}_{x^2+z^2=1}) \mid t \in \sin t\} \subseteq \mathbb{R}^3$




intersects the $y$-axis an infinite set of points. It is therefore NOT algebraic.

(e) $\{v \in \mathbb{R}^4 \mid \|v\| = 1\} = V(x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1)$, which is algebraic.

(f) $X = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$.
Consider the complex line $L = \{z = 0\}$. Then $L \not\subseteq X$ since $(0,0) \in L$ but $(0,0) \notin X$. Moreover,

$$L \cap X = \{w \in \mathbb{C} \mid |w|^2 = 1\},$$

which is an infinite set of points. The set $X \subseteq \mathbb{C}^2$ is therefore NOT algebraic.

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3- Let $p, q \in \mathbb{R}$ and consider two Zariski open neighbourhoods $\mathcal{U}_p$ and $\mathcal{U}_q$ of $p$ and $q$, respectively. Then $\mathcal{U}_p = \mathbb{R} - \{a_1, \dots, a_r\}$, for some $a_1, \dots, a_r \in \mathbb{R} \setminus \{p\}$, and $\mathcal{U}_q = \mathbb{R} - \{b_1, \dots, b_s\}$, for some $b_1, \dots, b_s \in \mathbb{R} \setminus \{q\}$. Moreover,

$$\mathcal{U}_p \cap \mathcal{U}_q = \mathbb{R} - \{a_1, \dots, a_r, b_1, \dots, b_s\} \neq \emptyset.$$

$\Rightarrow$ The Zariski topology on $\mathbb{R}$ is not Hausdorff.

4- If $k$ is finite, then $\mathbb{A}^n(k)$ has only finitely many points. Consequently, any subset $X \subseteq \mathbb{A}^n(k)$ is finite and therefore algebraic, so that it is closed in the Zariski topology. Moreover, $X = \mathbb{A}^n(k) - (\mathbb{A}^n(k) - X)$, where $(\mathbb{A}^n(k) - X)$ a closed subset since it is finite. Thus, $X$ is also open in the Zariski topology.

Finally, the Zariski topology is Hausdorff in this case because given any 2 distinct points $p, q \in \mathbb{A}^n(k)$, $U_p = \{p\}$ and $U_q = \{q\}$ are disjoint Zariski open neighbourhoods of $p$ and $q$, respectively.

5- Let $A \in M_{n\times n}(k)$. Then, if we write

$$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & & & \vdots \\ \vdots & & & \vdots \\ x_{n^2-n+1} & \cdots & & x_{n^2} \end{bmatrix},$$

we can identify $A$ with the point $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n^2}) \in \mathbb{A}^{n^2}(k)$. Conversely, any point in $\mathbb{A}^{n^2}(k)$ determines an $n\times n$ matrix. Note that $A$ is invertible $\iff \det A \neq 0$ and that $\det A$ is a polynomial of degree $n$ in the variables $x_1, x_2, \dots, x_{n^2}$. Then,

$$GL(n,k) = \mathbb{A}^{n^2}(k) - V(\det),$$

which is open in the Zariski topology.

6- Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be algebraic sets. Then, $V = V(f_1, \dots, f_r)$ and $W = V(g_1, \dots, g_s)$, for some polynomials $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ and $g_1, \dots, g_s \in k[y_1, \dots, y_m]$.

Set $F_i(x_1, \dots, x_n, y_1, \dots, y_m) = f_i(x_1, \dots, x_n)$

and

$$G_j(x_1, \dots, x_n, y_1, \dots, y_m) = g_j(y_1, \dots, y_m),$$

for $i = 1, \dots, r$ and $j = 1, \dots, s$. I.e., $F_i$ and $G_i$ are simply $f_i$ and $g_j$ viewed as polynomials in $k[x_1, \dots, x_n, y_1, \dots, y_m]$.

Then, $\quad V \times W = V(F_1, \dots, F_r, G_1, \dots, G_s)$.

Indeed, let $p = (a_1, \dots, a_n, b_1, \dots, b_m)$. If $p \in V \times W$, then

$F_i(p) = f_i(a_1, \dots, a_n) = 0$, for all $i$, since $(a_1, \dots, a_n) \in V$

and

$G_j(p) = g_j(b_1, \dots, b_m) = 0$, for all $j$, since $(b_1, \dots, b_m) \in W$.

Thus, $p \in V(F_1, \dots, F_r, G_1, \dots, G_s)$, so that $V \times W \subseteq V(F_1, \dots, F_r, G_1, \dots, G_s)$.

Conversely, if $p \in V(F_1, \dots, F_r, G_1, \dots, G_s)$, then

$f_i(a_1, \dots, a_n) = F_i(p) = 0$, for all $i$, implying that $(a_1, \dots, a_n) \in V$

and

$g_j(b_1, \dots, b_m) = G_j(p) = 0$, for all $j$, implying that $(b_1, \dots, b_m) \in W$.

Hence, $p \in V \times W$ so that $V \times W = V(F_1, \dots, F_r, G_1, \dots, G_s)$, **and** $V \times W$ is an algebraic set in $\mathbb{A}^{n+m}$.

7. (a) The closed sets of the Zariski topology on $\mathbb{A}^1$ are: $\phi$, finite sets of points, and $\mathbb{A}^1$. Consequently, $\phi$, points, horizontal lines ($\mathbb{A}^1 \times \{pt\}$), vertical lines ($\{pt\} \times \mathbb{A}^1$), and $\mathbb{A}^1 \times \mathbb{A}^1$ are closed sets in the product topology.

In fact, any closed set in the product topology is a finite union or a (possibly infinite) intersection of such closed sets, i.e., finite sets of points, horizontal and vertical lines, and $\mathbb{A}^1 \times \mathbb{A}^1$.

Note: If $k$ is finite, then $\mathbb{A}^1 \times \mathbb{A}^1$ consists of a finite set of points, so that any subset of $\mathbb{A}^1 \times \mathbb{A}^1$ is closed. Thus, the product and Zariski topologies on $\mathbb{A}^2 \simeq \mathbb{A}^1 \times \mathbb{A}^1$ coincide.

(b) Suppose that $k$ is infinite. Then, the diagonal line $V(y-x)$ is an irreducible closed subset of $\mathbb{A}^2$ in the Zariski topology (because $V(y-x)$ is infinite and $(y-x)$ is irreducible). Hence, $V(y-x)$ cannot be expressed as a finite union of points or horizontal and vertical lines. Nor is it an intersection of such sets. Hence, $V(y-x)$ is not closed in the product topology, proving that the two topologies are not equal