

**University of Waterloo**  
**Alebraic Geometry - Summer 2015**  
**Assignment 1**

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**Problem 1**

**Problem 2**

**Part a**

I will use problem 1 here. Notice that the points in  $\mathbb{R}^2$  whose polar coordinates satisfy  $r = \theta$  intersect the line  $y = 0$  at every point  $(2k\pi, 0)$  for all  $k \in \mathbb{N}$ . Thus By problem 1 the set is not algebraic.

**Part b**

The set simply represents the circle of radius  $\frac{1}{2}$  centered at  $(\frac{1}{2}, 0)$ . So it is the zero set of the polynomial  $(x - \frac{1}{2})^2 + y^2 - \frac{1}{4}$ . Thus the set is algebraic.

**Part c**

Claim:  $V(x^2 + z^2 - 1, y - 1) = \{(\cos t, 1, \sin t) : t \in \mathbb{R}\} \subset \mathbb{R}^3$

Proof: We have

$$V(x^2 + z^2 - 1, y - 1) = V(x^2 + z^2 - 1) \cap V(y - 1)$$

Also,  $V(y - 1)$  is the set of points with  $y = 1$  and  $V(x^2 + z^2 - 1)$  is the cylinder of radius 1 around around  $y$  axis. So the intersection of  $V(y - 1)$  and  $V(x^2 + z^2 - 1)$  is the circle of radius 1 centered at  $(0, 1, 0)$  which can be parametrized by  $\{(\cos t, 1, \sin t) : t \in \mathbb{R}\}$ . Thus the set is algebraic.

**Part d**

Let  $L = \{(1, t, 0) : t \in \mathbb{R}\}$  be the parametrization of a line. It's easy to see that the set in the question intersects  $L$  at every point of the form  $(1, 2k\pi, 0)$  for all  $k \in \mathbb{Z}$ . So by problem 1, the set is not algebraic.

**Part e**

This is clearly  $V(x^2 + y^2 + z^2 + w^2 - 1)$  because the set is just  $S^3$ . Thus it is algebraic.

## Part f

First note that there is a ring homomorphism  $\phi : \mathbb{R}^4 \rightarrow \mathbb{C}^2$  with  $\phi(x, y, z, w) = (x+iy, z+iw)$ .

## Problem 3

We know from class that the only algebraic sets (Closed sets) on  $\mathbb{R}$  are of one the following:

1.  $\emptyset$
2.  $\mathbb{R}$
3. Sets with finitely many points

Now let  $p, q \in \mathbb{R}$ . Let  $U$  and  $V$  be neighbours of  $p$  and  $q$  respectively. So the complement of each of  $U$  and  $V$  contains only finitely many points. Just pick a point  $r \in \mathbb{R}$  that is not among the finitely many points in  $U^c \cup V^c$ . Then  $r \in U \cap V$ . Thus  $U \cap V \neq \emptyset$ . Hence Zariski topology on  $\mathbb{R}$  is not hausdorff.

Note that we proved a stronger statement than the zariski topology not being hausdorff, namely, that every two non-empty open sets intersect (with respect to zariski topology).

## Problem 4

We know from class that a point  $a \in \mathbb{A}^n$  as a set  $\{a\}$  is an algebraic set. This is because given  $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ , we have  $V(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n) = \{a\}$ .

So if  $k$  is a finite field, every subset of  $\mathbb{A}^n$  is a finite set and is therefore algebraic. This readily implies that every subset of  $\mathbb{A}^n(k)$  is both open and closed (because the complement of every set is finite and therefore closed). We also have that the Zariski topology is hausdorff here because given two distinct points  $p, q \in \mathbb{A}^n$ , the sets  $\{p\}$  and  $\{q\}$  are open sets with no intersection.

## Problem 5

We just need to argue that the complement of the set of  $n \times n$  invertible matrices is an algebraic set. In other words we need to show that the set of non-invertible matrices is an algebraic set. This can be easily achieved because non-invertible matrices are exactly the matrices with 0 determinant. So the set of non-invertible matrices is  $V(\det_n)$  and thus is an algebraic set.

## Problem 6

Let  $S_1 \subset k[x_1, \dots, x_n]$  and  $S_2 \subset k[y_1, \dots, y_m]$  be finite sets such that  $X = V(S_1)$  and  $Y = V(S_2)$ . (This is possible since  $X$  and  $Y$  are algebraic sets).

For each  $f \in k[x_1, \dots, x_n]$  we define a polynomial  $f' \in k[x_1, \dots, x_n, y_1, \dots, y_m]$  such that

$$f'(a_1, \dots, a_n, b_1, \dots, b_m) = f(a_1, \dots, a_n)$$

for any  $b_1, \dots, b_m \in \mathbb{A}^m$ .

Similarly for each  $g \in k[y_1, \dots, y_m]$  we define a polynomial  $g' \in k[x_1, \dots, x_n, y_1, \dots, y_m]$  such that

$$g'(a_1, \dots, a_n, b_1, \dots, b_m) = g(b_1, \dots, b_m)$$

for any  $a_1, \dots, a_n \in \mathbb{A}^n$ .

Now let  $S'_1$  and  $S'_2$  be  $S_1$  and  $S_2$  with every polynomial in each set extended to a polynomial in  $k[x_1, \dots, x_n, y_1, \dots, y_m]$  in the manner described above.

Claim:  $V(S_1 \cup S_2) = V \times W$ .

Proof: Let  $a \in V(S'_1 \cup S'_2)$ . So for any  $f' \in S'_1$ ,  $f'(a) = 0$ , so  $f(a) = 0$  implying  $a \in V(S_1) = V$ . Also for any  $g' \in S'_2$ ,  $g'(a) = 0$ , so  $g(a) = 0$  implying that  $a \in V(S_2) = W$ . Hence  $a \in V \times W$  proving  $V(S_1 \cup S_2) \subset V \times W$ .

Let  $a = (a_1, \dots, a_n, b_1, \dots, b_m) \in V \times W$ . Let  $f' \in S'_1 \cup S'_2$ . We have the following cases:

( $f' \in S'_1$ ) In this case  $f'(a_1, \dots, a_n, b_1, \dots, b_m) = f(a_1, \dots, a_n) = 0$

( $f' \in S'_2$ ) In this case  $f'(a_1, \dots, a_n, b_1, \dots, b_m) = f(b_1, \dots, b_m) = 0$

Thus  $a \in V(S'_1 \cup S'_2)$  proving  $V \times W \subset V(S'_1 \cup S'_2)$ .

## Problem 7

### Part a

As I mentioned before in this assignment, the closed sets on the affine line are  $\mathbb{A}^1$  or  $\emptyset$  or a finite set of points. Let  $X = \{a_1, \dots, a_k\}$  and  $Y = \{b_1, \dots, b_m\}$  be closed sets in  $\mathbb{A}^1$ . Then

$$A \times B = \{(a, b) : a \in \{a_1, \dots, a_k\}, b \in \{b_1, \dots, b_m\}\}$$

$$A \times \mathbb{A}^1 = \{(a, b) : a \in \{a_1, \dots, a_k\}, b \in \mathbb{A}^1\}$$

$$\mathbb{A}^1 \times A = \{(a, b) : a \in \mathbb{A}^1, b \in \{a_1, \dots, a_n\}\}$$

So the closed sets in the Zariski topology of  $\mathbb{A}^1 \times \mathbb{A}^1$  are

1.  $\emptyset$
2.  $\mathbb{A}^1 \times \mathbb{A}^1$
3. Finite sets
4. Finite collection of vertical lines
5. Finite collection of horizontal lines

### Part b

We have that  $V(x - y)$  is a closed set in the zariski topology of  $\mathbb{A}^2$ . But it is not a closed set in the topology product zariski topology  $\mathbb{A}^1 \times \mathbb{A}^1$  as we can see from the previous part of the problem.