University of Waterloo
Algebraic Geometry - Summer 2015
Assignment 4

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Problem 1

Let $f_1, f_2, ..., f_r \in k[x_1, ..., x_n]$ such that $I =< f_1, ..., f_r >$. For each $i \in \{1, 2, .., r\}$ let $f_i = \prod_j f_{ij}$ be a factorization of f_i into irreducibles.
So $V(f_i) = \cup_j V(f_{ij})$ is a decomposition of $V(f_I)$ into irreducible components.
$V(I) = \cup_{(j_1,...,j_r)}(V(f_{1j_1}) \cap ... \cap V(f_{rj_r}))$ is a decomposition of $V(I)$ into irreducibles.
Fix $(j_1, ..., j_r)$. We have that
$\dim(V(f_{1j_1}) = n - 1$
$\dim(V(f1j_1 \cap V(f_{2j_2})) \geq (n - 1) - 1 = n - 2$
$\dim(V(f1j_1 \cap V(f_{2j_2}) \cap V(f_{3j_3})) \geq (n - 2) - 3 = n - 3$ ...
$\dim(V(f_{1j_1}) \cap ... \cap V(f_{rj_r})) \geq n - r$.
Thus every irreducible component of $V(I)$ has dimension $\geq n - r$.

Problem 2

Let $\varphi : X \to Y$ be a dominant rational map.
Since φ is dominant, the pull-back $\varphi^* : k(Y) \to k(X)$ is well-defined.
Let $f_1, ..., f_s \in k(Y)$ be algebraically independent.
Let $h_i = \varphi^*(f_i) = f_i \circ \varphi$ for all $i \in \{1, 2, ..., s\}$.
Assume for a contradiction that $h_1, ..., h_s \in k[X]$ is algebraically dependent.
So there exist $g \in k[x_1, ..., x_s]$ such that $g(h_1, ..., h_s) = 0$. We have:

$$\begin{aligned} g(h_1, ..., h_s) = 0 &\Rightarrow g(f_1 \circ \varphi, ..., f_s \circ \varphi) = 0 \\ &\Rightarrow g(f_1, ..., f_s) \circ \varphi = 0 \\ &\Rightarrow g(f_1, ..., f_2) = 0 \end{aligned}$$

This contradicts with $f_1, ..., f_s \in k(Y)$ being algebraically independent.
Thus $h_1, ..., h_s \in k[X]$ are algebraically independent.
Hence $\dim Y \leq \dim X$.

Problem 3

Part a

Let I be a prime ideal of $\Gamma(X)$ contained in M_p and let J be a prime ideal of $O_p(X)$.The correspondence is given by

$$\begin{aligned} I &\rightarrow IO_p(X) \\ J &\rightarrow J \cap \Gamma(X) \end{aligned}$$

To prove this is one-to-one we need to argue that $IO_p(X) \cap \Gamma(X) = I$.
Note that since $IO_p(X)$ is the prime ideal in $O_p(X)$ generated by I and every polynomial in I evalutes to 0 at p, if $a, b \in \Gamma(X)$ and $f \in I$ then $f\frac{a}{b}$ is a polynomial iff $b(p) = f(p)$ but since $f(p) = 0$ this implies that $\frac{a}{b} \notin O_p(X)$. So every polynomial in $IO_p(X)$ is in I.
Hence $IO_p(X) \cap \Gamma(X) = I$ and we are done.

Part b

It's easy to see that there is a one-to-one correspondence between subvarieties of X containing p and prime ideals in $\Gamma(X)$ contained in M_p because every subvariety of X containing p corresponds to a prime ideal in $\Gamma(X)$ contained in M_p and every prime ideal in $\Gamma(X)$ contained in M_p corresponds to a subvariety of X. So by part (a), there is a one-to-one correspondence between the prime ideals of $O_p(X)$ and subvareties of X containing p.

Part c

Let $P_0 \subset P_1 \subset ... \subset P_n = O_p(X)$ be the longest chain of prime ideals in $O_p(X)$. By part (b), there is a one-to-one correspondence between P_i's and the subvarieties of X containing p. So this chain corresponds to the longest chain of irreducible closed subsets of X containing p (Because if there is a longer chain of irreducible closed subsets of X containing p, then by the one-to-one correspondence we can get a longer chain of prime ideals in $O_p(X)$). Thus $\dim X = \dim O_p(X)$.

Problem 4

Part a

Let $\varphi^* : O_q(Y) \to O_p(X)$ be the extension of pull-back that sends $f \in O_q(Y)$ to $f \circ \varphi$. Note that if $f \in O_q(Y)$, then f is defined on q and since $q = \varphi(p)$, $(f \circ \varphi)(p) = f(\varphi(p)) = f(q)$ is defined therefore $f \circ \varphi \in O_p(X)$. So the pull-back is well-defined
Let $f \in M_q$. Then $(f \circ \varphi)(p) = f(\varphi(p)) = f(q) = 0$, thus $f \circ \varphi \in M_p$.
So $\varphi^*(M_q(Y)) \subset M_p(X)$.
φ^* cannot be extended to all of $k(Y)$ because some elements of $k(Y)$ are not defined on all of Y.
Now assume φ is an isomorphism. Let $f \in M_p(X)$. There exist $a, b \in \Gamma(X)$ with $a(p) = 0$ and $b(p) \neq 0$ and $f = \frac{a}{b}$.
Since φ^* is an isomorphism, $\Gamma(X)$ and $\Gamma(Y)$ are isomorphis. So there exists $a', b' \in \Gamma(Y)$ such that $a = a' \circ \varphi$ and $b = b' \circ \varphi$. We have:
$a'(q) = a'(\varphi(p)) = a(p) = 0$ and $b'(q) = b'(\varphi(p)) = b(p) \neq 0$ so $g = \frac{a}{b} \in M_q(Y)$.
Now note that $\varphi^*(g) = \frac{a'}{b'} \circ \varphi = \frac{a}{b} = f$. Thus $\varphi^*(M_q(Y)) = M_p(X)$.

Part b

Let $\varphi : X \to Y$ be an isomorphism. Let $p \in X$. By part (a), $O_p(X)$ is isomorphic to $O_{\varphi(p)}(Y)$.
X is smooth at p if and only if $O_p(X)$ is regular which happens if and only if $O_{\varphi(p)}(Y)$ is regular because $O_p(X)$ is isomorphic to $O_{\varphi(p)}(Y)$ and $O_{\varphi(p)}(Y)$ is regular if and only if Y is smooth at $\varphi(p)$.

Part c

No. We will show that X is not smooth at $(0, 0, 0)$ but Y is.Therefore by part (b), X and Y are not isomorphic.
First note that $\dim X = \dim Y = 2$ because $x^2 + y^2 - z^2$ and $x^2 + y^2 - z$ are irreducible polynomials.
We have $Jac(x^2 + y^2 - z^2)(0, 0, 0) = (2x, 2y, -2z)(0, 0, 0) = (0, 0, 0)$. So $\dim(T_{\cdot}0(X)) = 3$ but $\dim(X) = 2$ thus $\dim(T_0(X)) \neq \dim(X)$.
We also have $jac(x^2 + y^2 - z)(0, 0, 0) = (2x, 2y, -1)(0, 0, 0) = (0, 0, -1)$. So $\dim(T_{\cdot}0(Y)) = 2$ and $\dim(Y) = 2$. Thus $\dim(Y) = \dim(T_0(Y))$.

Problem 5

Part a

(i) Since k is a field, $(k, +)$ is a group and $m(x, y) = x + y$ and $i(x) = -x$ and $e = 0$ and it can be identified by $\mathbb{A}^1$. So the additive group is an affine algebraic group.

(ii) This is a group with $m(x, y) = xy$ and $i(x) = x^{-1}$ and $e = 1$.
Also since this group can be identified by $V(xy - 1)$, the multiplicative group is an affine algebraic group.

(iii) This is a group where matrix multiplication is given by polynomials in the entries of the matrcies and therefore is a polynomial map. Inversion is also given by polynomials in the entries of a matrix divided by the determinant but with the variable $t_{n^2+1} = \frac{1}{\det A}$, so inversion is also a polynomial map.
Also note that GL_n can be identified by $V(\det(A)t_{n^2+1} - 1)$ where $\det(A)t_{n^2+1} - 1$ is an irreducible polynomial (because $\det A$ is irreducible and t_{n^2+1} is a variable not used in $\det A$). So the general linear group is an affine algebraic group and GL_n is irreducible.

(iv) This is a group where matrix multiplication is given by polynomials in the entries of the matrcies and therefore is a polynomial map. Inversion is also given by polynomials in the entries of a matrix divided by the determinant but determinant is 1, so inversion is also a polynomial map.
Also since this group can be identified with $V(\det(A) - 1)$, special linear group is an affine algebraic group.

(v) Orthogonal group is a group where matrix multiplication is given by polynomials in the entries of the matrcies and therefore is a polynomial map. Inversion is also given by polynomials in the entries of a matrix divided by the determinant but determinant is either 1 or -1, so inversion is also a polynomial map.
O_n can be identified by $V(AA^T - I, \det A - 1) \cup V(AA^T - I, \det A + 1)$,so it is an affine algebraic group.
Since O_n and SL_n are both affine algebraic groups, their intersection SO_n is also an affine algebraic group.

Part b

(i) Let $g \in G$. Let $g : G \to G$ be the map that sends h to $m(g, h)$.
Claim 1: g is an isomorphism.
Proof of claim 1: g is clearly a polynomial map (because m is a polynomial map) and $g^{-1} : G \to G$ which sends h to $m(i(g), h)$ is also a polynomial map because both m and i are polynomail maps.
Let G_1 be an irreducible component of G.
Claim 2: $g(G_1)$ is an irreducible component of G.
Proof of claim 2: Since g is an isomorphism, we know that $g(G_1)$ is a closed subset of G. Now assume for a contradiction that $g(G_1) = A_1 \cup A_2$ where A_1, A_2 are distinct closed subsets of G. Then we have $G_1 = g^{-1}(A_1) \cup g^{-1}(A_2)$ is a decomposition of G_1 into two distict closed subsets contradicting the irreducubility of G_1.
Let G_1 and G_2 be irreducible components of G.
Assume $g \in G_1 \cap G_2$.
Claim 3: $G_1 = G_2$.
Proof of claim 3: By claim 2 $g^{-1}G_1$ is irreducible (Consider the map $g^{-1} : G \to G$ that sends h to $m(i(g), h)$). Also $g^{-1}G_2$ is irreducible.
Note that $e = g^{-1}g \in g^{-1}G_1$ and $e = g^{-1}g \in g^{-1}G_2$ since $g \in G_1 \cap G - 2$. So $e \in g^{-1}G_1 \cap g^{-1}G_2$.
Claim 4: G_1 is isomorphic to G°.
Proof of claim 4: Let $g : G \to G$ be the map that sends h to $m(i(g), h)$. Note that $g(g) = e$. So g restricted to G_{i_0} is an isomorphism between G_{i_0} and G° because g sends irreducible components to irreducible components and we know from part (a) that irreducible components are pairwise disjoint.

(ii) Let $g : G \to G$ be the map that sends h to $m(i(g), h)$. Note that $g(g) = e$. So g restricted to G_{i_0} is an isomorphism between G_{i_0} and G° because g sends irreducible components to irreducible components and we know from part (a) that irreducible components are pairwise disjoint.
Since there is an isomorphism between G_{i_0} and G° that sends g to e, we have that T_eG° is isomorphic to $T_gG_{i_0}$.

(iii) We compute the Lie algebras for $n = 2$.

Note that SL_2 is irreducible because $\det A - 1$ is an irreducible polynomial. We have:

$$\begin{aligned}
T_{I_{2\times 2}}SL_2 &= \{(a,b,c,d):\ jac(\det \begin{bmatrix} x & y \\ z & w \end{bmatrix} - 1)(I_{2\times 2}).(a,b,c,d) = 0\} \\
&= \{(a,b,c,d):\ jac(xw - yz - 1)(I_{2\times 2}).(a,b,c,d) = 0\} \\
&= \{(a,b,c,d) : (w, -z, -y, x)(I_{2\times 2}).(a,b,c,d) = 0\} \\
&= \{(a,b,c,d) : (0, -1, -1, 0).(a,b,c,d) = 0\} \\
&= \{(a,b,,c,d) : -b - c = 0\} \\
&= \{A \in M_{2\times 2} :\ Tr(A) = 0\}
\end{aligned}$$

Given $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ We have $SO_2 = V(xw - yz - 1, x^2 + y^2, xz + yw - 1, z^2 + w^2)$.
$T_{I_{2\times 2}}SO_2$ is the set of vectors (a,b,c,d) satisfying

$$\begin{aligned}
(w, -z, -y, x)(I_{2\times 2}).(a,b,c,d) &= 0 \\
(2x, 2y, 0, 0)(I_{2\times 2}).(a,b,c,d) &= 0 \\
(z, w, x, y)(I_{2\times 2}).(a,b,c,d) &= 0 \\
(0, 0, 2z, 2w)(I_{2\times 2}).(a,b,c,d) &= 0
\end{aligned}$$

So

$$\begin{aligned}
-b - c &= 0 \\
2a &= 0 \\
a + d &= 0 \\
2d &= 0
\end{aligned}$$

So

$$\begin{aligned}
T_{I_{2\times 2}}SO_2 &= \{(a,b,c,d) : b + c = 2b = a + d = 2c = 0\} \\
&= \{A \in M_{2\times 2} : A + A^T = 0\}
\end{aligned}$$