University of Waterloo
Pmath 450 - Summer 2015
Assignment 1

Sina Motevalli 20455091

Problem 1

Part a

Part b

The closed unit ball clearly contains the points of the form $x_n = (0, 0, ..., 1, 0, 0, 0, ...)$ (1 is on the nth position and everything else is 0). So the distance between any two of these points is $2^{\frac{1}{p}}$. Thus this sequence has no converging subsequesnce (because no subsequence is couchy). Hence the closed unit ball is not compact in l^p.

Part c

Let $S = \{(q_n)_{n=1}^{\infty} : q_n \in \mathbb{Q} \;\; and \;\; \exists N \in \mathbb{N} \;\; with \;\; q_n = 0, \forall n \geq N \;\; and \;\; ||(q_n)||_p < \infty\}$.
Claim: S is dense in l^p.
Proof: Let $(x_n)_{n=1}^{\infty} \in l^p$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |x_n|^p < \frac{\epsilon}{2}$.
For each $i \in \{1, 2, ..., N-1\}$ find $q_i \in \mathbb{Q}$ such that $|q_i - x_i|^p < \frac{\epsilon}{2(N-1)}$.
Now consider the sequence $(q_1, q_2, ..., q_{N-1}, 0, 0, 0, 0, ...)$. Now we compute the difference between the two sequences in l^p:

$$\begin{aligned}
(||(q_n) - (x_n)||_p)^p &= \sum_{i=1}^{N-1} |q_i - x_i|^p + \sum_{i=N}^{\infty} |q_i - x_i|^p \\
&= \sum_{i=1}^{N-1} |q_i - x_i|^p + \sum_{i=N}^{\infty} |x_i|^p \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus $||(q_n) - (x_n)||_p < \epsilon^p$, but since ϵ was arbitrary and p is a constant,S is desne in l^p.

Problem 2

Part a

Let $A_n = [a, b - \frac{1}{n}]$. So We have: $A_1 \subset A_2 \subset ... \subset \cup_{n=1}^{\infty} A_n = [a, b)$.
By the continuity of measure we have:

$$\begin{aligned}
m([a,b)) &= \lim_{n\to\infty} m(A_n) \\
&= \lim_{n\to\infty} m([a, b - \frac{1}{n}]) \\
&= \lim_{n\to\infty} b - \frac{1}{n} - a \\
&= b - a
\end{aligned}$$

Part b

Let $A \subset \mathbb{R}$ be a lebesgue measurable set. Let $t \in \mathbb{R}$. Need to show $A + t$ islebesgue measurabel.
Let $E \subset \mathbb{R}$. Since A is measurable, we have

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) \tag{1}$$

We have:

$$\begin{aligned}
x \in E \cap A + t &\iff x \in E \ \ and \ \ x \in A + t \\
&\iff x \in E \ \ and \ \ x - t \in A \\
&\iff x - t \in E - t \ \ and \ \ x - t \in A \\
&\iff x - t \in (E - t) \cap A \\
&\iff x \in (E - t \cap A) + t
\end{aligned}$$

So $E \cap A + t = [(E - t) \cap A] + t$. Thus,

$$m^*(E \cap A + t) = m^*([(E - t) \cap A] + t) = m^*((E - t) \cap A)$$

. By a similar argument we get $E \cap (A + t)^c = [(E - t) \cap A^c] + t$. Thus,

$$m^*(E \cap (A + t)^c) = m^*([(E - t) \cap A^c] + t) = m^*((E - t) \cap A^c)$$

. So we have:

$$\begin{aligned}
m^*(E \cap (A + t)^c) + m^*(E \cap A + t) &= m^*((E - t) \cap A^c) + m^*((E - t) \cap A) \\
&= m^*(E - t) \\
&= m^*(E)
\end{aligned}$$

Hence $A + t$ is lebesgue measurable.

Problem 3

We have 2 cases:
Case 1: E is bounded.
Let $\epsilon > 0$. Choose an open set U such that $\bar{E} \cap E^c \subset U$ and $m(U) < m(\bar{E} \cap E^c) + \epsilon$.
Let $K = E \cap U^c = \bar{E} \cap U^c$. Since K is closed and bounded, it is compact. We have:

$$\begin{aligned} m(E) = &= m(\bar{E}) - m(\bar{E} \cap E^c) \\ &< m(\bar{E}) - m(U) + \epsilon \\ &\leq m(\bar{E} \cap U^c) + \epsilon \\ &= m(K) + \epsilon \end{aligned}$$

Case 2: E is unbounded.
Write $E = \cup E_j$ where each E_j is bounded and $E_j \subset E_{j+1}$. Let $\epsilon > 0$. By the previous part, there exist a compact set $K_j \subset E_j$ for each j such that $m(E_j) < m(K_j) + \epsilon$. Since $E_j \to E$, the result follows.

Problem 4

Let $X \subset \mathbb{R}$ be open. Let $X \cap Q = \{p_1, p_2, ...\}$. For every $i \in \mathbb{N}$, we define

$$B_i = \bigcup_{p_i \in I \subset X}^{I \text{ is open interval}} I$$

Claim:$X = \cup_{i=1}^{\infty} B_i$
Proof: It is clear that $\cup_{i=1}^{\infty} B_i \subset X$ because every B_i is inside X. So it remains to prove the other inclusion. Let $x \in X$. Since X is open, there exist $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset X$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can pick a $p_j \in (x - \epsilon, x + \epsilon)$ where p_j is a rational number. Then by definition of B_j we have $p_j \in (x - \epsilon, x + \epsilon) \subset B_j \subset \cup_{i=1}^{\infty} B_i$. Thus $X \subset \cup_{i=1}^{\infty} B_i$ and we are done.
Note that $\cup_{i=1}^{\infty} B_i$ is a countable union of open intervals because each B_i is a union of open intervals containing a common point, so it is an open interval.

Problem 5

Part a

Let $C_0 = [0, 1]$. We define C_n recursively for $n \in \mathbb{N}$ as follows:
C_n is defined as every interval of C_{n-1} with the open middle third of each interval removed.
So C_n contains 2^n intervals each of length $\frac{1}{3^n}$.
Now we have: $C = \cap_{n=0}^{\infty} C_n \subset ... \subset C_2 \subset C_1 \subset C_0$.
By downward continuity of measure, we have:

$$\begin{aligned} m(C) &= \lim_{n\to\infty} m(C_n) \\ &= \lim_{n\to\infty} \frac{2^n}{3^n} \\ &= 0 \end{aligned}$$

Thus the lebesgue measure of Cantor set is zero.

Part b

Since measure of the cantor set is 0, every subset of cantor set is measurable with measure 0, and since the cardinality of the set of subsets of cantor set is $2^{\mathbb{R}}$, the cardinality of the set of lebesgue measurable sets is also $2^{\mathbb{R}}$.

Part c

Let $\alpha \in (0,1)$.
Let $(x_n)_{n=0}^{\infty} \in [0,1]$ be a decreasing sequence such that $\sum_{n=0}^{\infty} 2^n x_n \to 1-\alpha$.
Let $C_0 = [0,1]$. We define C_n recursively for $n \in \mathbb{N}$ as follows:
C_n is defined as every interval of C_{n-1} with the middle open interval of length x_n of each interval removed.
Then we define $C = \cap_{n=0}^{\infty} C_n$. First I should prove that this construction is possible. Note that to construct the set C_n we are making 2^n intervals each of length $\frac{1-\sum_{k=0}^{n-1} 2^k x_k}{2^n}$. So it suffices to prove that $x_n < \frac{1-\sum_{k=0}^{n-1} 2^k x_k}{2^n}$. Assume for a contradiction that $x_n \geq \frac{1-\sum_{k=0}^{n-1} 2^k x_k}{2^n}$, we have:

$$\begin{aligned} x_n \geq \frac{1-\sum_{k=0}^{n-1} 2^k x_k}{2^n} \quad &\Rightarrow \quad 2^n x^n \geq 1 - \sum_{k=0}^{n-1} 2^k x_k \\ &\Rightarrow \quad \sum_{k=0}^{n} 2^k x_k \geq 1 \end{aligned}$$

But this is a contradiction because $\sum_{n=1}^{\infty} 2^n x_n \to 1-\alpha < 1$.
Now I claim that $m(C) = \alpha$ because:

$$\begin{aligned} m(C) &= m([0,1]) - m(C^c \cap [0,1]) \\ &= 1 - L(intervals\ removed\ in\ the\ construction\ of\ C) \\ &= 1 - \sum_{n=0}^{\infty} 2^n x_n \\ &= 1 - (1-\alpha) \\ &= \alpha \end{aligned}$$