University of Waterloo
Pmath 450 - Summer 2015
Assignment 2

Sina Motevalli 20455091

Problem 1

Part a

Let E_1 and E_2 be measurable sets. We have: $E_1 \cup E_2 = E_1 \cup (E_2 \setminus (E_1 \cap E_2))$.
So we have:

$$\begin{aligned} m(E_1 \cup E_2) &= m(E_1 \cup (E_2 \setminus (E_1 \cap E_2))) \\ &= m(E_1) + m((E_2 \setminus (E_1 \cap E_2))) \quad \textit{since} \quad E_1 \cap (E_2 \setminus (E_1 \cap E_2)) = \emptyset \\ &= m(E_1) + m(E_2) - m(E_1 \cap E_2) \end{aligned}$$

Thus $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Part b

Let $\alpha \in \mathbb{R}$. We have:

$$\{x : \sup f_n \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n \leq \alpha\} \tag{1}$$

$$\{x : \inf f_n < \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n < \alpha\} \tag{2}$$

Since f_n's are measurable and countable union and countable intersection of measurable sets are measurable, by (1) and (2), $\sup f_n$ and $\inf f_n$ are measurable.

Part c

Since $f = g$ a.e, $h = f - g = 0$ a.e. Since f and g are continuous, $h = f - g$ is conitunous. Let $E = \{x : h(x) \neq 0\}$. We know that $m(E) = 0$. Assume for a contradiction that $E \neq \emptyset$. Let $p \in E$. There exist $\delta > 0$ such that $(p - \frac{\delta}{2}, p + \frac{\delta}{2}) \cap E = \{p\}$, otherwise E contains an interval and it's measure cannot be zero. Let $0 < \epsilon < |f(p)|$. Since h is continuous, there exist $\delta' > 0$ such that if $|x - y| < \delta'$, $|f(x) - f(y)| < \epsilon$. Let $\delta'' = \min\{\delta, \delta'\}$. Choose $x \in (p - \frac{\delta''}{2}, p + \frac{\delta''}{2})$. Note that $|p - x| < \delta'' \leq \delta'$, but $|f(p) - f(x)| = |f(p)| < \epsilon < |f(p)|$ which is a contradiction. So $E = \emptyset$
Thus $h = 0$ everywhere implying $f = g$ everywhere.

Problem 2

Part a

Let $\alpha \in \mathbb{R}$. Let $(q_n)_{n=1}^{\infty} \in (-\infty, \alpha)$ be a sequence such that each $q_n \in \mathbb{Q}$ and $q_n \to \alpha$. We have:

$$\{x : f(x) < \alpha\} \quad = \quad \bigcup_{n=1}^{\infty} \{x : f(x) < q_n\}$$

Since each $\{x : f(x) < q_n\}$ and a countable unioun of measurable sets is measurable, $\{x : f(x) < \alpha\}$ is measurable. Thus f is measurable.

Part b

We first define:

$$M_f = \{A \subset \mathbb{R} : f^{-1}(A) \textit{ is measurable}\}$$

Claim: M_f is a σ-algebra.
Proof: Let $A_1, A_2, ... \in M_f$, then we have $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$ is measurable because countable union of measurable sets is measurable.
Also if $A \in M_f$, we have: $f^{-1}(\mathbb{R} \setminus A) = \mathbb{R} \setminus f^{-1}(A)$ which is measurable since $f^{-1}(A)$ is measurable.
Hence M_f is a σ-algebra.
Note that from the problem statement we have, $S \subset M_f$, and the smallest σ-algebra containing S includes all open sets, thus M_f contains all open sets implying f is measurable.

Problem 3

Part a

If $m(E) = \infty$, take $G = \mathbb{R}$ and we are done.
Assume $m(E) < \infty$. For every $n \in \mathbb{N}$, let O_n be an open set with $E \subset O_n$ such that $m(O_n) - m(E) < \frac{1}{n}$.
Let $G = \bigcap_{n=1}^{\infty} O_n$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.
We have $m(G) - m(E) \leq m(O_N) - m(E) < \frac{1}{N} < \epsilon$. Thus $m(G \setminus E) = 0$ and G is a Borel set.

Part b

Let $\epsilon > 0$. Let O be an open set with $E \subset O$ such that $m(O \setminus E) < \frac{\epsilon}{2}$. We can express O as a countale union of disjoint open intervals $O = \bigcup_{n=1}^{\infty} I_n$ where I_n's are disjoint open intervals. We have:

$$\begin{aligned}
\sum_{n=1}^{\infty} m(I_n) &= m(\bigcup_{n=1}^{\infty} I_n) \\
&= m(O) \\
&= m(O \setminus E) + m(E) \\
&< \frac{\epsilon}{2} + m(E) < \infty
\end{aligned}$$

Thus $\lim_{k \to \infty} \sum_{n=k}^{\infty} m(I_n) = 0$.
So there exist $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} m(I_n) < \frac{\epsilon}{2}$.
Now let $U = \bigcup_{n=1}^{N} I_n$. U is a finite union of open intervals. We now have $(E \setminus U) \subset (O \setminus U)$. So

$$\begin{aligned}
m(E \setminus U) &\leq m(O \setminus U) \\
&= m((\bigcup_{n=1}^{\infty} I_n) \setminus (\bigcup_{n=1}^{N} I_n)) \\
&= m(\bigcup_{n=N+1}^{\infty} I_n) \\
&= \sum_{n=N+1}^{\infty} m(I_n) \\
&< \frac{\epsilon}{2}
\end{aligned}$$

Also

$$m(U \setminus E) \leq m(O \setminus E) < \frac{\epsilon}{2}$$

Hence $m(U \setminus E) + m(E \setminus U) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Problem 4

Part a

For every $n \in \mathbb{N}$, we define $M_n = \{x : f(x) \leq n\}$. So we have:

$$M_1 \subset M_2 \subset M_3 \subset ..... \subset \bigcup_{n=1}^{\infty} M_n = f^{-1}(\mathbb{R})$$

So by continuity of measure, we have $m(f^{-1}(\mathbb{R})) = \lim_{n\to\infty} m(M_n)$.
So there exist $N \in \mathbb{N}$ such that $m(f^{-1}(\mathbb{R})) - m(M_n) < \epsilon$ for all $n \geq N$.
Observe that $f^{-1}([\mathbb{R}]) = M_N \cup \{x : f(x) > N\}$ and $M_N \cap \{x : f(x) > N\} = \emptyset$, thus we have:

$$m(f^{-1}(\mathbb{R})) = m(M_N) + m(\{x : f(x) > N\}) \quad \Rightarrow \quad m(\{x : f(x) > N\}) = m(f^{-1}(\mathbb{R})) - m(M_N) < \epsilon$$

Hence $|f| \leq N$ except on a set of measure less than ϵ.

Part b

Let $M > 0$. Partition $[0, M]$ into equal intervals each of length $< \epsilon$.
$0 = a_0 < a_1 < a_2 < ... < a_n = M$ Where $a_i - a_{i-1} < \epsilon \ \forall i \in \{1, .., n\}$.
For each $i \in \{1, .., n\}$, let $A_i = f^{-1}([a_{i-1}, a_i))$. Note that A_i's are measurable because f is measurable. For each $i \in \{1, .., n\}$, let $a_i \in [a_{i-1}, a_i)$. We define

$$\phi(x) = \sum_{j=1}^{n} a_j X_{A_j}(x)$$

Choose x such that $|f(x)| < M$. Then $x \in A_k$ and $f(x) \in [a_{k-1}, a_k)$ for some k. We have:

$$\begin{aligned} |f(x) - \phi(x)| &= |f(x) - \sum_{j=1}^{n} a_j X_{A_j}(x)| \\ &= |f(x) - a_k| \\ &\leq L([a_{k-1}, a_k)) \\ &< \epsilon \end{aligned}$$

Thus ϕ is the simple function we wanted.
Note that if $m \leq f \leq M$ we could have made the exact same arguments by partitioning $[m, M]$ and clearly ϕ would have taken values only in $[m, M]$.

Part c

I first prove the statement for a characteristic function because it will be useful for the proof.
Let X_A be a characteristic function where $A \subset [a, b]$ is measurable.
Let $U \supset A$ be an open set such that $m(U \setminus A) < \frac{\epsilon}{2}$.
We can write U as a countable union of disjoint open intervals $U = \bigcup_{n=1} \infty I_n$ where I_n's are open intervals.
For each $k \in \mathbb{N}$, we define $L_k = \bigcup_{n \geq k} I_n$. So $L_1 \subset L_2 \subset L_3 \subset$ .....
By downward continuity of measure we have: $\lim_{k \to \infty} m(L_k) = m(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} I_n = m(\emptyset) = 0$.
So there exist $N \in \mathbb{N}$ such that $m(L_n) < \frac{\epsilon}{2}$ for all $n \geq N$. In particular $m(L_{N+1}) < \frac{\epsilon}{2}$.
Let $g_A(x) = \sum_{n=1}^{N} X_{I_n}$.
I will prove that $X_A(x) = g_A(x)$ except on a set of measure less than ϵ.
If $x \in A \cap (\bigcup_{n=1}^{N} I_n)$, then $g_A(x) = X_A(x) = 1$ since $x \in A$.
if $x \notin U$, then $g_A(x) = X_A(x) = 0$.
So it is suffiecient to prove that $m(U \setminus (A \cap (\bigcup_1^N I_n)) < \epsilon$.
We have: $U \setminus (A \cap (\bigcup_1^N I_n) = (U \setminus A) \cup (U \setminus (\bigcup_1^N I_n)) = (U \setminus A) \cup L_{N+1}$. Thus,

$$\begin{aligned} m(U \setminus (A \cap (\bigcup_1^N I_n)) &= m((U \setminus A) \cup L_{N+1}) \\ &\leq m(U \setminus A) + m(L_{N+1}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Now we can prove the general case.
Let $\phi(x) = \sum_{n=1}^{k} a_n X_{A_n}$ be a simple function.
For each $n \in \{1, 2, ..., k\}$, we define $g_n(x) = g_{A_n}(x)$ as was defined in the previous part such that $a_n X_{A_n}(x) = a_n g_n(x)$ except on a set E_n with $m(E_n) < \frac{\epsilon}{k}$.
Let $g(x) = \sum_{n=1}^{k} a_n g_n(x)$. It's now easy to see that $g(x)$ is the step function that we want.
Note that if $x \in (\bigcup_{n=1}^{k} E_k)^c$, then $\sum_{n=1}^{k} a_n X_{A_n} = \sum_{n=1}^{k} a_n g_n(x) = g(x)$.
Also note that $m(\bigcup_{n=1}^{k} E_k) \leq m(E_1) + ... + m(E_k) < \frac{\epsilon}{k} + ... + \frac{\epsilon}{k} < \epsilon$.
So we have proven that $g(x) = \phi(x)$ except on a set of measure less than ϵ.
If $m \leq \phi < M$, we have that $m \leq \phi \leq M$ already because $\max \phi = \max\{a_1, ..., a_k\} = \max g$ and $\min \phi = \min\{a_1, ..., a_k\} = \min g$

Part d

We first prove the following lemma:
Lemma: Let $g : [a, b] \to \mathbb{R}$ be a step function. Then there is a continuous function h such that $g(x) = h(x)$ except on a set of measure less than ϵ. If $m \leq g \leq M$, we can choose h with $m \leq h \leq M$.
Proof of lemma: Since g is a step function, there exist a disjoint partitioning of $[a, b]$ into intervals $\{I_n\}_{n=1}^k$ and constants $a_1, a_2, ..., a_k \in \mathbb{R}$ such that

$$g(x) = \sum_{n=1}^{k} a_n x_{I_n}$$

For each $n \in \{1, ..., k\}$, let c_k and d_k be the endpoints of I_k ($c_k \leq d_k$).
Let $p = \min_{n=1}^k \frac{d_n - c_n}{2}$. Let $\delta = \min\{\frac{\epsilon}{2}, p\}$. Now for each $n \in \{1, ..., k\}$, look at the interval

$$A_n = (c_k + \frac{\delta}{2n}, d_k - \frac{\delta}{2n})$$

Notice that the intervals A_n are disjoint and $A_n \subset I_n$ for every n. Here is how we define h: if $x \in A_n$, $h(x) = g(x) = a_n$, between the intervals A_i and A_{i+1} we consider the line from the right end-point of A_i to the left end-point of A_{i+1}. So h is a cont function and $h = g$ on $\bigcup_{n=1}^k A_n$.
It remains to prove that $m((\bigcup_{n=1}^k A_n)^c) < \epsilon$. Just note that $(\bigcup_{n=1}^k A_n)^c$ is a union of k intervals each of length less than $\frac{\epsilon}{k}$, So by sub-additivity we have:

$$m((\bigcup_{n=1}^{k} A_n)^c) < \sum_{n=1}^{k} \frac{\epsilon}{k} = \epsilon$$

$\square$
Now assume $m \leq g \leq M$. Since h was defined by linear interpolation between subintervals in g, we have that $m \leq h \leq M$.
Now we are ready to solve the problem.
By part (a), There exist $M > 0$ such that $|f| \leq M$ except on a set E_1 with $m(E_1) < \frac{\epsilon}{3}$. By part (b), we get a simple function φ such that $|f - \varphi| < \epsilon$ except on E_1. By part (c), we have a step function g such that $g = \varphi$ except on a set E_2 with $m(E_2) < \frac{\epsilon}{3}$. By the lemma, we have a cont function h so that $g = h$ except on a set E_3 with $m(E_3) < \frac{\epsilon}{3}$. We have:

$$|f(x) - h(x)| = |f(x) - g(x)| = |f(x) - \varphi(x)| < \epsilon$$

except when $x \in E_1 \cup E_2 \cup E_3$. By sub-additivity we have

$$m(E_1 \cup E_2 \cup E_3) \leq m(E_1) + m(E_2) + m(E_3) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

If $m \leq f \leq M$, each of the parts and the lemma imply that $m \leq h \leq M$.