PMath 450 Assignment #1
due Friday May 22, 2015

1. Recall that for $1 \leq p < \infty$,

$$\ell^p = \{(x_n)_{n=1}^\infty : \|(x_n)\|_p = \left(\sum |x_n|^p\right)^{1/p} < \infty\}.$$

   (a) Prove that $\|(x_n)\|_2 = \sup\{|\sum x_n y_n| : \|(y_n)\|_2 \leq 1\}$.

   (b) Show that the closed unit ball $B = \{(x_n)_{n=1}^\infty : \|(x_n)\|_p \leq 1\}$ in ℓ^p is not compact.

   (c) Prove that ℓ^p is separable, i.e., has a countable dense subset.

2. (a) Show that $m^*([a,b)) = b - a$

   (b) Show that if $A \subseteq \mathbb{R}$ is Lebesgue measurable, then $A + t$ is measurable for all $t \in \mathbb{R}$.

3. Suppose $E \subseteq \mathbb{R}$ is a Lebesgue measurable set. Show that

$$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}.$$

   Hint: Do this first for sets with $m(E) < \infty$.

4. Prove that every open set in $\mathbb{R}$ is a countable union of open intervals.

5. (a) Determine the Lebesgue measure of the Cantor set.

   (b) Find the cardinality of the set of Lebesgue measurable sets.

   (c) Show that for every $0 < \alpha < 1$ there is a perfect, totally disconnected subset of $[0,1]$ whose Lebesgue measure is equal to α.