University of Waterloo
Alebraic Geometry - Summer 2015
Assignment 1

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Problem 1

We assume $L = V(x_2 - (a_2x_1 - b_2), x_3 - (a_3x_1 - b_3), ..., x_n - (a_nx_1 - b_n))$. Since $L \not\subset X$, $X \neq \mathbb{A}^n$. So there exist a non-zero polynomial $F \in k[x_1, ..., x_n]$ such that $X \subset V(F)$. Assume for a contradiction that $X \cap L$ is an infinite set. So $L \cap V(F)$ is an infinite set. We have:

$$\begin{aligned} L \cap V(F) &= V(x_2 - (a_2x_1 - b_2), x_3 - (a_3x_1 - b_3), ..., x_n - (a_nx_1 - b_n)) \cap V(F) \\ &= V(F(x_1, a_2x_1 - b_2, ..., a_nx_1 - b_n)) \in k[x_1] \quad \textit{since} \quad x_i - (a_ix_1 - b_i) = 0 \end{aligned}$$

So $L \cap V(F)$ is the zero set of some polynomial in one variable, but there can only be finitely many points in such a set. Contradiction.

Problem 2

Part a

I will use problem 1 here. Notice that the pints in $\mathbb{R}^2$ whose polar coordinates satisfy $r = \Theta$ intersect the line $y = 0$ at every point $(2k\pi, 0)$ for all $k \in \mathbb{N}$. Thus By problem 1 the set is not algebraic.

Part b

The set simply represents the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$. So it is the zero set of the polynomial $(x - \frac{1}{2})^2 + y^2 - \frac{1}{4}$. Thus the set is algebraic.

Part c

Claim: $V(x^2 + z^2 - 1, y - 1) = \{(\cos t, 1, \sin t) : t \in \mathbb{R}\} \subset \mathbb{R}^3$
Proof: We have

$$V(x^2 + z^2 - 1, y - 1) = V(x^2 + z^2 - 1) \cap V(y - 1)$$

Also, $V(y - 1)$ is the set of points with $y = 1$ and $V(x^2 + z^2 - 1)$ is the cylinder of radius 1 around around y axis. So the intersection of $V(y - 1)$ and $V(x^2 + z^2 - 1)$ is the circle of radius 1 centered at $(0, 1, 0)$ which can be paramtrized by $\{(\cos t, 1, \sin t) : t \in \mathbb{R}\}$. Thus the set is algebraic.

Part d

Let $L = \{(1, t, 0) : t \in \mathbb{R}\}$ be the parametrization of a line. It's easy to see that the set in the question intersects L at every point of the form $(1, 2k\pi, 0)$ for all $k \in \mathbb{Z}$. So by problem 1, the set is not algebraic.

Part e

This is clearly $V(x^2 + y^2 + z^2 + w^2 - 1)$ because the set is just S^3 (unit 3-shere). Thus it is algebraic.

Part f

We know that there is a ring homomorphism between $\mathbb{R}^4$ to $\mathbb{C}^2$. This ring homomorphism, by definition, carries zero sets from $\mathbb{R}^4$ to $\mathbb{C}^2$. We proved in the previous part that $\{v \in \mathbb{R}^4 : |v| = 1\}$ is the zero set of the polynomial $x^2 + y^2 + z^2 + w^2 - 1$. But if $z = x + iy$ and $w = z + it$, then $x^2 + y^2 + z^2 + t^2 = 1 \Rightarrow |z|^2 + |w|^2 = 1$. So this must be an algebraic set.

Problem 3

We know from class that the only algebraic sets (Closed sets) on $\mathbb{R}$ are of one the following:

1. $\emptyset$
2. $\mathbb{R}$
3. Sets with finitely many points

Now let $p, q \in \mathbb{R}$. Let U and V be neighbours of p and q respectively. So the complement of each of U and V contains only finitely many points. Just pick a point $r \in \mathbb{R}$ that is not among the finitely many points in $U^c \cup V^c$. Then $r \in U \cup V$. Thus $U \cup V \neq \emptyset$. Hence Zariski topology on $\mathbb{R}$ is not housdorff.
Note that we proved a stronger statement than the zariki topology not being housdorff, namely, that every two non-empty open sets intersect (with respect to zariski topology).

Problem 4

We know from class that a point $a \in \mathbb{A}^n$ as a set $\{a\}$ is an algebraic set. This is because given $a = (a_1, ..., a_n) \in \mathbb{A}^n$, we have $V(x_1 - a_1, x_2 - a_2, ..., x_n - a_n) = \{a\}$.
So if k is a finite field, every subset of $\mathbb{A}^n$ is a finite set and is therefore algebraic. This readily implies that every subset of $\mathbb{A}^n(k)$ is both open and closed (because the complement of every set is finite and therefore closed). We also have that the Zariski topology is housdorff here because given two distinct points $p, q \in \mathbb{A}^n$, the sets $\{p\}$ and $\{q\}$ are open sets with no intersection.

Problem 5

We just need to argue that the complement of the set of $n \times n$ invertible matrcies is an algebraic set. In other words we need to show that the set of non-invertible matrcies is an alebraic set. This can be easily achieved because non-invertible matrcies are exactly the matrcies with 0 determinent. So the set of non-invertible matrcies is $V(\det_n)$ and thus is an algebraic set.

Problem 6

Let $S_1 \subset k[x_1, ...x_n]$ and $S_2 \subset k[y_1, .., y_m]$ be finite sets such that $X = V(S_1)$ and $Y = V(S_2)$. (This is possible since X and Y are algebraic sets).
For each $f \in k[x_1, ...x_n]$ we define a polynomial $f' \in k[x_1, ..., x_n, y_1, ..., y_m]$ such that

$$f'(a_1, ..., a_n, b_1, ..., b_m) = f(a_1, ..., a_n)$$

for any $b_1, ..., b_m \in \mathbb{A}^m$.
Similarly for each $g \in k[y_1, ...y_m]$ we define a polynomial $g' \in k[x_1, ..., x_n, y_1, ..., y_m]$ such that

$$g'(a_1, ..., a_n, b_1, ..., b_m) = g(b_1, ..., b_m)$$

for any $a_1, ..., a_n \in \mathbb{A}^n$.
Now let S_1' and S_2' be S_1 and S_2 with every polynomial in each set extended to a polynomial in $k[x_1, ..., x_n, y_1, ..., y_m]$ in the manner described above.
Claim: $V(S_1 \cup S_2) = V \times W$.
Proof: Let $a \in V(S_1' \cup S_2')$. So for any $f' \in S_1'$, $f'(a) = 0$, so $f(a) = 0$ implying $a \in V(S_1) = V$. Also for any $g' \in S_2'$, $g'(a) = 0$, so $g(a) = 0$ implying that $a \in V(S_2) = W$. Hence $a \in V \times W$ proving $V(S_1 \cup S_2) \subset V \times W$.
Let $a = (a_1, ..., a_n, b_1, ..., b_m) \in V \times W$. Let $f' \in S_1' \cup S_2'$. We have the following cases:

($f' \in S_1'$) In this case $f'(a_1, ..., a_n, b_1, ..., b_m) = f(a_1, ..., a_n) = 0$

($f' \in S_2'$) In this case $f'(a_1, ..., a_n, b_1, ..., b_m) = f(b_1, ..., b_m) = 0$

Thus $a \in V(S_1' \cup S_2')$ proving $V \times W \subset V(S_1' \cup S_2')$.

Problem 7

Part a

As I mentioned before in this assignment, the closed sets on the affine line are $\mathbb{A}^1$ or $\emptyset$ or a finite set of points. Let $X = \{a_1, ..., a_k\}$ and $Y = \{b_1, ..., b_m\}$ be closed sets in $\mathbb{A}^1$. Then

$$A \times B = \{(a, b) : a \in \{a_1, ..., a_k\}, b \in \{b_1, ...b_m\}\}$$

$$A \times \mathbb{A}^1 = \{(a, b) : a \in \{a_1, ..., a_k\}, b \in \mathbb{A}^1\}$$

$$\mathbb{A}^1 \times A = \{(a, b) : a \in \mathbb{A}, b \in \{a_1, ..., a_n\}\}$$

So the closed sets in the Zariski topology of $\mathbb{A}^1 \times \mathbb{A}^1$ are

1. $\emptyset$
2. $\mathbb{A}^1 \times \mathbb{A}^1$
3. Finite sets
4. Finite collection of vertical lines
5. Finite collection of horizontal lines

Part b

We have that $V(x - y)$ is a closed set in the zariski topology of $\mathbb{A}^2$. But it is not a closed set in the product zariski topology $\mathbb{A}^1 \times \mathbb{A}^1$ as we can see from the previous part of the problem.