University of Waterloo

Pmath 450 - Summer 2015
Assignment 4

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Problem 1

Part a

I will first prove that C_0 is a subspace of l^∞.
Let $(x_n) \in C_0$. If $x_n = 0$ for all n then $||(X_n)||_\infty = 0$ and we are done. So we assume that the sequence (x_n) has non-zero terms. Let a be the first non-zero term.
Let $\epsilon < |a|$. Since $x_n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $x_n < \epsilon < |a|$ for all $n \geq N$. So we have:

$$\begin{aligned} ||(x_n)||_\infty &= \sup |X_n| \\ &= \max\{x_1, ..., x_N\} < \infty \end{aligned}$$

Thus $(x_n) \in l^\infty$.
Now I need to prove that C_0 is closed.
Let $(x_n)_m$ be a sequence in C_0 meaning $(x_n)_i \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$. So $(x_n)_m \to 0$ as $n, m \to \infty$ and we are done.

Part b

We first show that l^∞ is not separable.
Consider the set of sequences whose elements are made up of only zeroes and ones. This is clearly a subset of l^∞. Note that there is a one-to-one correspondence between each of these sequences and the binary representation of numbers in the interval $(0, 1)$ $(0.x_1, x_2, ...$ is the binary representation of a number in $(0, 1))$.
So our set is uncountable. Note that any two distinct elements are one distance apart (with respect to our norm). Now if we put a ball of radius $\frac{1}{4}$ around these points, non of these balls intersect. Since every dense subset of l^∞ must have an element in each of these balls, any dense subset must be uncountable. Hence l^∞ is separable.
I now prove that C_0 is separable.
Consider the set of sequences $\{(x_n) : x_i \in \mathbb{Q}\ \forall i, x_n \to 0 \quad as \quad n \to \infty\}$.
This is a countable subset of c_0 and it is clearly dense in c_0 as $\mathbb{Q}$ is dense in $\mathbb{R}$.
Hence C_0 is separable.

Problem 2

We have:

$$\begin{aligned}
| < x_n, y_n > - < x, y > | &= | < x_n, y_n > - < x_n, y > + < x_n, y > - < x, y > | \\
&\leq | < x_n, y_n > - < x_n, y > | + | < x_n, y > - < x, y > | \\
&\leq | < x_n, y_n - y > | + | < x_n - x, y > | \\
&\leq ||x_n|| ||y_n - y|| + ||x_n - x|| ||y|| \quad By \quad C.S
\end{aligned}$$

Now since $||x_n||, ||y|| < \infty$ and $||x_n - x||, ||y_n - y| \to 0$ as $n \to \infty$ we have that $| < x_n, y_n > - < x, y > | \to \infty$ as $n \to \infty$.

Problem 3

Problem 4

Part a

Note that $S_\perp$ the intersection of inverse images of the closedset $\{0\}$ of the maps $i_s : x \to < x, s >$ for every element of S.

$$S_\perp = \bigcap_{s \in S} i_s^{-1}(\{0\})$$

Since i_s is continious, $S_\perp$ is the intersection of closed sets and is therefore closed.
$\overline{span}(S)$ closed by definition.

Part b

If $x \in \ span(S)$ then $x = \sum_{k=1}^{\infty} < x, s_i > s_i$ for some s_i's in S.
Now if $x \in S_\perp$ then $< x, s_i >= 0$ for all i and therefore $x = \sum_{k=1}^{\infty} < x, s_i > s_i = 0$.
Hence $S_\perp \cap \overline{span}(S) = \{0\}$.

Part c

Since H is separable, it is second countable and second countability passes to susets and a hilbert space is separable if and only if it is second countable. Hence every subset of H is separable.

Part d

Let $\{e_n\}$ be a basis for S. We can extend this basis to get $\{e_n\} \cup \{f_k\}$ a basis for H. (Note that these are countable sets because H is separable).
Let $x \in H$. We can write $x = \sum_n < x, e_n > e_n + \sum_k < x, f_k > f_k$.
Let $z = \sum_n < x, e_n > e_n$ and $y = \sum_k < x, f_k > f_k$.
Clearly $z \in \overline{span}(S)$. I need to show that $y \in S^\perp$.
Let $s = \sum_n a_n e_n \in S$. We have:

$$< y, s >=< \sum_k < x, f_k > f_k, \sum_n a_n e_n >= \sum_k < x, f_k > \sum_n a_n < e_n, f_k >= 0$$

Thus $y \in S^\perp$.
Now assume $x = y' + z'$ where $y' \in S^\perp$ and $z' \in \overline{span}(S)$. We have:
$y - y' = z - z' \in S^\perp \cap \overline{span}(S) = \{0\}$. So $y = y'$ and $z = z'$.
Thus y and z are unique.

Bonus

Assume such a measurable set exists. Let $0 < \epsilon < m(A)$.
Let O be an open set with $A \subset O$ such that $m(O \setminus A) < \epsilon$. Then $m(A \cap O) = m(A) = \frac{m(O)}{2}$.
but $m(O \setminus A) = m(O) - m(A) = \frac{m(O)}{2} = m(A) < \epsilon$.
This is a contradiction because $\epsilon < m(A)$.