PMATH 764: Assignment 5

Due: Monday, 13 July, 2015.

1. Let $C = V(f)$ be an affine plane curve that is smooth at the point $p \in C$, so that $\mathcal{O}_p(C)$ is a DVR.
   (a) Prove that the maximal ideal $M_p(C)$ of C at p can be generated by the residue class in $\mathcal{O}_p(C)$ of any linear polynomial $h \in k[x, y]$ such that $V(h) \subset \mathbb{A}^2$ is a line passing through p that is *not* tangent to C at p.

      Hint: First apply an affine coordinate change so that $p = (0, 0)$ and $T_p(C) = V(x) \subset \mathbb{A}^2$.
   (b) Prove that the order function $\mathrm{ord}_p^C : \mathcal{O}_p(C) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ does not depend on the choice of local parameter of $\mathcal{O}_p(C)$.
   (c) Consider the extension of the order function $\mathrm{ord}_p^C : \mathcal{O}_p(C) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ to the function field $k(C)$, which is defined as

$$\begin{array}{rcl} \mathrm{ord}_p^C : k(C) & \to & \mathbb{Z} \cup \{\infty\} \\ f = \bar{a}/\bar{b} & \mapsto & \mathrm{ord}_p^C(\bar{a}) - \mathrm{ord}_p^C(\bar{b}). \end{array}$$

      Prove the following:
      (i) ord_p^C is a well-defined, i.e., $\mathrm{ord}_p^C(f)$ does not depend on the presentation $\bar{a}/\bar{b}$ of f.
      (ii) $\mathrm{ord}_p^C(f) = 0$ if and only if f is a unit in $\mathcal{O}_p(C)$.
      (iii) $\mathrm{ord}_p^C(f) = \infty$ if and only if f identically zero on C.
      (iv) $\mathrm{ord}_p^C(f_1 f_2) = \mathrm{ord}_p^C(f_1) + \mathrm{ord}_p^C(f_2)$.
      (v) $\mathrm{ord}_p^C(f_1 + f_2) \geq \min\{\mathrm{ord}_p^C(f_1), \mathrm{ord}_p^C(f_2)\}$.

      Note: Notice that $\mathcal{O}_p(C) = \{f \in k(C) | \, \mathrm{ord}_p^C(f) \geq 0\}$ and $M_p(C) = \{f \in k(C) | \, \mathrm{ord}_p^C(f) > 0\}$. This means in particular that $\mathrm{ord}_p^C(f) < 0$ if and only if p is a pole of f at p.

2. Let C be an affine plane curve and $p \in C$.
   (a) Show that if C is reducible, then C is singular at points that are contained in more than one irreducible component.
   (b) Assume that C is a smooth at p and let L be a line through p in $\mathbb{A}^2$. Prove that $I(p, C \cap L) = 1$ if and only if L is *not* the tangent line to C at p, otherwise, $I(p, C \cap L) \geq 2$.
   (c) Suppose there exist two distinct lines L_1 and L_2 such that $I(p, L_i \cap C) \geq 2$ for $i = 1, 2$. Show that C is singular at p.
   (d) *(Optional)* Find the intersection multiplicity $I(p, C \cap C')$ of the affine plane curves $C = V(x^2 - 1 - y^3) \subset \mathbb{C}^2$ and $C' = V(x^2 - 1 + 2y^4) \subset \mathbb{C}^2$ at the point $p = (1, 0)$.

3. *Divisors.* Let C be a smooth irreducible affine plane curve. A *divisor* D on C is defined to be a formal sum of points

$$D = \sum_{p \in C} n_p p$$

   for which $n_p \in \mathbb{Z}$ and only a finite number of the n_p's are non-zero. Moreover, the *support* of a divisor D is the set of all points $p \in C$ for which $n_p \neq 0$. If the support of D is empty, then D is called the *zero divisor* and is denoted $D = 0$.

   (a) Let $z \in k(C)$. We define the *divisor* $\mathrm{div}(z)$ *of* z to be

$$\mathrm{div}(z) := \sum_{p \in C} \mathrm{ord}_p^C(z) p.$$

Prove that $\operatorname{div}(z)$ is indeed a well-defined divisor on C whose support is the set of all zeroes and poles of z. Note that if $p \in C$ is in the support of $\operatorname{div}(z)$, then p is either a zero or a pole of z and its coefficient n_p is the order of that zero or pole. Show that $\operatorname{div}(z) = 0$ if z is a constant function and that the converse holds when k is algebraically closed.

(b) Let $C = V(y^4 - x^3 + x) \subset \mathbb{A}^2$. Compute the divisor $\operatorname{div}(z)$ of the rational function $z = \bar{y}/(\bar{x} - 1)$.

(c) Let $\bar{g} \in \Gamma(C)$ and suppose that $\bar{g}$ is not a constant function on C. Let $C' = V(g) \subset \mathbb{A}^2$. Then,

$$\operatorname{div}(\bar{g}) = \sum_{p \in C} I(p, C \cap C')p,$$

implying that $\operatorname{div}(\bar{g})$ consists of the points of intersection of C and C', counting multiplicity, and is called the *intersection divisor of* C *and* C', which we denote $C \cdot C'$.
Find $C \cdot C'$ where $C = V(x + y^3) \subset \mathbb{C}^2$ and $C' = V(y(y^2 - x)) \subset \mathbb{C}^2$.

4. *Resolving singularities.* Consider the Cartesian product $\mathbb{A}^n \times \mathbb{A}^{n-1} = \mathbb{A}^{2n-1}$ and fix $i \in \{1, \ldots, n\}$. Let $(x_1, \ldots, x_n)$ be coordinates in $\mathbb{A}^n$ and $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ be coordinates in $\mathbb{A}^{n-1}$. We can then define a *blow-up* $\widetilde{\mathbb{A}^n}$ *of* $\mathbb{A}^n$ *at* 0 in terms of these coordinates as

$$\widetilde{\mathbb{A}^n} := V(x_1 - u_1 x_i, \ldots, x_{i-1} - u_{i-1} x_i, x_{i+1} - u_{i+1} x_i, \ldots, x_n - u_n x_i) \subset \mathbb{A}^n \times \mathbb{A}^{n-1} = \mathbb{A}^{2n-1}.$$

For example, if $n = 2$ and we choose coordinates (x, y) on $\mathbb{A}^2$ and u on $\mathbb{A}^1$, we can define two blow-ups of $\mathbb{A}^2$, namely,

$$\widetilde{\mathbb{A}^2} := V(y - ux) \subset \mathbb{A}^3$$

or

$$\widetilde{\mathbb{A}^2} := V(x - uy) \subset \mathbb{A}^3.$$

Note that the affine coordinate change $(x, y) \mapsto (y, x)$ maps $V(y - ux)$ isomorphically onto $V(x - uy)$. Moreover, the natural projection map $\pi : \widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^2, (x, y, u) \mapsto (x, y)$, is a birational equivalence in both cases. In all cases, the natural projection map

$$\begin{array}{rcl} \pi : \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{A}^{n-1} & \longrightarrow & \mathbb{A}^n \\ (x_1, \ldots, x_n, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) & \longmapsto & (x_1, \ldots, x_n) \end{array}$$

is a birational equivalence. Let $X \subset \mathbb{A}^n$. We define

$$\widetilde{X} := \overline{\pi^{-1}(X \setminus \{0\})}$$

to be the *blow-up* of X at 0. Then, $\widetilde{X} \sim X$, implying in particular that $\dim \widetilde{X} = \dim X$.

(a) Show that the blow up of the alpha curve $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$ at $(0, 0)$ is a smooth curve in $\mathbb{A}^3$.

(b) *(Optional)* Let Y be the plane curve $V(y^3 - x^5) \subset \mathbb{A}^2$, which has a higher order cusp at $(0, 0)$. Show that $(0, 0)$ is a singular point, and that blowing up Y at $(0, 0)$ gives rise to a curve $\widetilde{Y}$ in $\mathbb{A}^3$ that is singular at $(0, 0, 0)$. Moreover, show that blowing up $\widetilde{Y}$ at $(0, 0, 0)$ resolves the singularity. Hence, by blowing up Y twice, one obtains a smooth curve in $\mathbb{A}^5$.

5. *(Optional)* Let $X \subset \mathbb{A}^n$ be a variety. Moreover, let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial such that $Y = V(f) \cap X$ is a codimension 1 subvariety of X with $I(Y) = \langle \bar{f} \rangle$ in $\Gamma(X)$. We define

$$\mathcal{O}_Y(X) := \{z \in k(X) : z = \frac{\bar{a}}{\bar{b}} \text{ with } b \text{ not constantly zero on } Y\}$$

to be the subring of functions in $k(X)$ whose restrictions to Y are well-defined rational functions on Y (that may have poles on Y). Show that $\mathcal{O}_Y(X)$ is a DVR and describe the valuation map.