University of Waterloo
Pmath 450 - Summer 2015
Assignment 3

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Problem 1

Assume $f : \mathbb{R} \to \mathbb{R}$ is integrable. Then since f^+ and f^- are non-negative and measurable, so we have:

$$\begin{aligned}
\int_{\mathbb{R}} f(x+y)dm(x) &= \int_{\mathbb{R}} f^+(x+y)dm(x) - \int_{\mathbb{R}} f^-(x+y)dm(x) \\
&= \int_{\mathbb{R}} f^+(x)dm(x) - \int_{\mathbb{R}} f^-(x)dm(x) \\
&= \int_{\mathbb{R}} f(x)dm(x)
\end{aligned}$$

Now if $f : \mathbb{R} \to \mathbb{C}$ is integrable, we have that $Re(f)$ and $Im(f)$ are integrable and therefore we have:

$$\begin{aligned}
\int_{\mathbb{R}} f(x+y)dm(x) &= \int_{\mathbb{R}} Re(f(x+y))dm(x) + i\int_{\mathbb{R}} Im(f(x+y))dm(x) \\
&= \int_{\mathbb{R}} Re(f(x))dm(x) - \int_{\mathbb{R}} Im(f(x))dm(x) \\
&= \int_{\mathbb{R}} f(x)dm(x)
\end{aligned}$$

Problem 2

Part a

We know that $\sup\{|f(x)+g(x)| : x \in A\} \leq \sup\{|f(x)| : x \in A\} + \sup\{|g(x)| : x \in A\}$. This implies that

$$\inf\{\sup|f(x)+g(x)| : x \in A\} \leq \inf\{\sup|f(X)| : x \in A\} + \inf\{\sup|g(x)| : x \in A\}$$

Hence $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

Part b

We know that $m\{x : |h(x)| > ||h||_\infty\} = 0$. Thus

$$\inf\{\alpha \in \mathbb{R} : m\{x : |h(x)| > \alpha\} = 0\} \leq ||h||_\infty$$

Now assume for a contradiction that there exist $\alpha < ||h||_\infty$ such that $m\{x : |h(x)| > \alpha\} = 0$. Since $||h||_\infty = \inf_{m(E\setminus A)=0}\{\sup|h(x)| : x \in A\}$ and $\alpha < ||h||_\infty$, for every set A with $m(E \setminus A) = 0$, we have that $\alpha < \{\sup|h(x)| : x \in A\}$. This means that there exist no set A with $m(E \setminus A) = 0$ such that $A = \{x : |h(x)| \leq \alpha\}$. Thus $m\{x : |h(x)| > \alpha\} > 0$. Hence

$$\inf\{\alpha \in \mathbb{R} : m\{x : |h(x)| > \alpha\} = 0\} = ||h||_\infty$$

Problem 3

By lotus lemma we have:

$$\int \liminf f_n \leq \liminf \int f_n$$

Since $f_n \to f$, we have $\int f \leq \liminf \int f_n$. Thus f is integrable. Now dominated convergence theorem readily implies $\int f = \lim_n \int f_n$.

Problem 4

Part a

Let $f(x) = 0$ when $x < 1$ and $f(x) = \frac{1}{x}$ when $x \geq 1$. We have:

$$\int_{\mathbb{R}} |f|^2 = \int_1^{\infty} \frac{1}{x^2} = 1$$

Thus $f \in L^2(\mathbb{R})$. But $\int_{\mathbb{R}} |f| = \int_1^{\infty} \frac{1}{x}$ does not converge, thus $f \notin L^1(\mathbb{R})$.
Let $g(x) = \frac{1}{\sqrt{x}}$ on $[0, 1]$ and $g(x) = 0$ elsewhere. We have:

$$||g||_1 = \int_{\mathbb{R}} g = \int_0^1 \frac{1}{\sqrt{x}} = 2$$

But $\int_{\mathbb{R}} g^2 = \int_0^1 \frac{1}{x} = \infty$. Thus $g \notin L^2(\mathbb{R})$.

Part b

Let $f^2 \in L^1[0, 1]$. So $\int_0^1 |f^2| = \int_0^1 |f|^2 < \infty$, thus $f \in L^2[0, 1]$. We have:

$$\begin{aligned} \int_0^1 |f| &= \int_0^1 |f|.1 \\ &\leq ||f||_2 ||1||_2 \quad \textit{by holder's inequality} \end{aligned}$$

Since $f \in L^2[0, 1]$, $||f||_2 < \infty$, so $||f||_2||1||_2 < \infty$ which implies $\int_0^1 |f| < \infty$.
Hence $f \in L^1[0, 1]$.

Problem 5

Let $||f||_\infty > \epsilon > 0$. Let $A_\epsilon = \{x : |f(x)| \geq ||f||_\infty - \epsilon\}$. So by definition of maximum norm we get that $m(A_\epsilon) > 0$, so we have:

$$||f||_p \geq \left(\int_{A_\epsilon} (||f||_\infty - \epsilon)^p \right)^{\frac{1}{p}} = (||f||_\infty - \epsilon)(m(A_\epsilon))^{\frac{1}{p}} \to ||f||_\infty - \epsilon \quad as \quad p \to \infty$$

Thus, $\lim_{p\to\infty} \inf ||f||_p \geq ||f||_\infty$.
We also have:

$$\begin{aligned} ||f||_p &= \left(\int |f|^{p-1}|f| \right)^{\frac{1}{p}} \\ &\leq ||f||_\infty^{\frac{p-1}{p}} ||f||_1^{\frac{1}{p}} \quad by \quad holder's \quad inequality \\ &\to ||f||_\infty \quad as\ p \to \infty \end{aligned}$$

Thus, $\lim_{p\to\infty} \sup ||f||_p \leq ||f||_\infty$.
Hence $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Problem 6

Claim: S is dense in $C[0,1]$ with respect to $L^2[0,1]$ norm.
Proof:
Let $\epsilon > 0$. Let $f \in C[0,1]$.
WLOG we can assume that f is real-valued. (Otherwise approximate real and imaginary part and put them back together).
First assume that f is bounded. Say $|f(x)| \leq N \ \forall x \in [0,1]$.
We define a new function $g : [0,1] \to \mathbb{R}$ as follows:
$g(x) = f(x)$ for all $x \in (\epsilon, 1-\epsilon)$.
On $[0,\epsilon]$, g is the line from 0 to $f(\epsilon)$ ($g(0) = 0$ and $g(\epsilon) = f(\epsilon)$).
On $[1-\epsilon, 1]$, g is the line from $f(1-\epsilon)$ to 0 ($g(1-\epsilon) = f(1-\epsilon)$ and $g(1) = 0$).
Note that $g \in S$. We have:

$$\begin{aligned}
||f-g||_2^2 &= \int_0^1 |f-g|^2 \\
&= \int_{[0,\epsilon]} |f-g|^2 + \int_{(\epsilon,1-\epsilon)} |f-g|^2 + \int_{[1-\epsilon,1]} |f-g|^2 \\
&\leq N^2\epsilon + 0 + N^2\epsilon \\
&= 2N^2\epsilon
\end{aligned}$$

This concludes the proof for f being bounded.
Now suppose $f \in C[0,1]$ is arbitrary.
Define $f_N(x) = f(x)$ if $|f(x)| \leq N$ and $f_N(x) = 0$ otherwise.
We have $f_N \to f$ pointwise a.e.
So $|f_N - f|^2 \to 0$ pointwise a.e.
Since $|f - f_N|^2 \leq |f|^2$ and $|f|^2$ is integrable, by dominated convergence theorem, we have:

$$\int_{[0,1]} |f - f_N|^2 \to \int_{[0,1]} 0 = 0$$

So $||f - F_N||_2 \to 0$.
Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $||f - f_N||_2 < \frac{\epsilon}{2}$.
Get $h \in S$ with $||h - f_N||_2 < \frac{\epsilon}{2}$. We have:

$$||h - f||_2 \leq ||h - f_N||_2 + ||f_N - f||_2 < \epsilon$$

Hence S is dense in $C[0,1]$ with respect to $L^2[0,1]$ norm.
Let $(g_n)_{n=1}^{\infty}$ be a sequence in S such that $||g_n||_2 \to ||f||_2$ and $g_n \leq f$ for all n.
By the dominated convergence theorem, $\int_0^1 f^2 = \int_0^1 \lim_{n\to\infty} fg_n = \lim_{n\to\infty} \int_0^1 fg_n = 0$ since $\int_0^1 fg = 0$ for all $g \in S$.
Thus $||f||_2 = 0$. Hence $f = 0$ a.e.

Problem 7

Since $f \geq 0$, $||f^n||_1 = \int_0^1 f^n(x) = \int_0^1 f(x) = ||f||_1$ for all $n \in \mathbb{N}$. Now since $||f^n||_1 = ||f||_1$ we have that $f^n(x) = f(x)$ a.e for all $n \in \mathbb{N}$.
So $f(x) = 1$ a.e.
Let $E = \{x : f(x) = 1\}$. We just need to prove that E is measurable. We have:

$$E = \left(\bigcap_{n=1}^{\infty}\{x : f(x) \leq 1 + \frac{1}{n}\}\right) \cap \left(\bigcap_{n=1}^{\infty}\{x : f(x) \geq 1 - \frac{1}{n}\}\right)$$

Since countable intersection of measurable sets is measurable, E is measurable and $f = X_E$ a.e.