PMath 450 Assignment #4
due Wednesday July 8, 2015

1. Notation: $\ell^\infty = \{(x_n)_{n=1}^\infty : \|(x_n)\|_\infty = \sup |x_n| < \infty\}$; $c_0 = \{(x_n) : x_n \to 0 \text{ as } n \to \infty\}$.

   (a) Show that c_0 is a closed subspace of ℓ^∞.

   (b) Prove that c_0 is separable, but ℓ^∞ is not.

2. Suppose H is a Hilbert space and that $x_n \to x$ and $y_n \to y$ in H. Show that

$$< x_n, y_n > \quad \to \quad < x, y >$$

3. Consider $C[0,1]$ with the inner product $< f, g >= \int_0^1 f\overline{g}$. Show that the subset $A = \{f \in C[0,1] : \|f\|_\infty < 1\}$ is not open in $C[0,1]$ with respect to the norm coming from the inner product.

4. Let H be a separable Hilbert space and $S \subseteq H$. Let

$$S^\perp = \{x \in H : \ < x, s >= 0 \text{ for every } s \in S\}$$

   (a) Show that $S^\perp$ and $\overline{span(S)}$ are closed subspaces of H.

   (b) Show that $S^\perp \cap \overline{span(S)} = (0)$

   (c)Prove that every subset of H is separable.

   (d) Prove that $H = S^\perp \oplus \overline{span(S)}$ meaning that given any $x \in H$, there is a unique choice of $y \in S^\perp$ and $z \in \overline{span(S)}$ such that $x = y + z$.