University of Waterloo
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Assignment 1

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Problem 1

Part a

Part b

The closed unit ball clearly contains the points of the form $x_n = (0, 0, ..., 1, 0, 0, 0, ...)$ (1 is on the nth position and everything else is 0). So the distance between any two of these points is $2^{\frac{1}{p}}$. Thus this sequence has no converging subsequesnce (because no subsequence is couchy). Hence the closed unit ball is not compact in l^p.

Part c

Let $S = \{(q_n)_{n=1}^{\infty} : q_n \in \mathbb{Q} \;\; and \;\; \exists N \in \mathbb{N} \;\; with \;\; q_n = 0, \forall n \geq N \;\; and \;\; ||(q_n)||_p < \infty\}$.
Claim: S is dense in l^p.
Proof: Let $(x_n)_{n=1}^{\infty} \in l^p$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |x_n|^p < \frac{\epsilon}{2}$.
For each $i \in \{1, 2, ..., N-1\}$ find $q_i \in \mathbb{Q}$ such that $|q_i - x_i|^p < \frac{\epsilon}{2(N-1)}$.
Now consider the sequence $(q_1, q_2, ..., q_{N-1}, 0, 0, 0, 0, ...)$. Now we compute the difference between the two sequences in l^p:

$$
\begin{aligned}
(||(q_n) - (x_n)||_p)^p &= \sum_{i=1}^{N-1} |q_i - x_i|^p + \sum_{i=N}^{\infty} |q_i - x_i|^p \\
&= \sum_{i=1}^{N-1} |q_i - x_i|^p + \sum_{i=N}^{\infty} |x_i|^p \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

Thus $||(q_n) - (x_n)||_p < \epsilon^p$, but since ϵ was arbitrary and p is a constant,S is desne in l^p.

Problem 2

Part a

Let $A_n = [a, b - \frac{1}{n}]$. So We have: $A_1 \subset A_2 \subset ... \subset \cup_{n=1}^{\infty} A_n = [a, b)$.
By the continuity of measure we have:

$$\begin{aligned} m([a,b)) &= \lim_{n\to\infty} m(A_n) \\ &= \lim_{n\to\infty} m([a, b - \frac{1}{n}]) \\ &= \lim_{n\to\infty} b - \frac{1}{n} - a \\ &= b - a \end{aligned}$$

Part b

Let $A \subset \mathbb{R}$ be a lebesgue measurable set. Let $t \in \mathbb{R}$. Need to show $A + t$ islebesgue measurabel.
Let $E \subset \mathbb{R}$. Since A is measurable, we have

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) \tag{1}$$

We have:

$$\begin{aligned} x \in E \cap A + t &\iff x \in E \ \ and \ \ x \in A + t \\ &\iff x \in E \ \ and \ \ x - t \in A \\ &\iff x - t \in E - t \ \ and \ \ x - t \in A \\ &\iff x - t \in (E - t) \cap A \\ &\iff x \in (E - t \cap A) + t \end{aligned}$$

So $E \cap A + t = [(E - t) \cap A] + t$. Thus,

$$m^*(E \cap A + t) = m^*([(E - t) \cap A] + t) = m^*((E - t) \cap A)$$

. By a similar argument we get $E \cap (A + t)^c = [(E - t) \cap A^c] + t$. Thus,

$$m^*(E \cap (A + t)^c) = m^*([(E - t) \cap A^c] + t) = m^*((E - t) \cap A^c)$$

. So we have:

$$\begin{aligned} m^*(E \cap (A + t)^c) + m^*(E \cap A + t) &= m^*((E - t) \cap A^c) + m^*((E - t) \cap A) \\ &= m^*(E - t) \\ &= m^*(E) \end{aligned}$$

Hence $A + t$ is lebesgue measurable.

Problem 3

Problem 4

Let $X \subset \mathbb{R}$ be open. Let $X \cap Q = \{p_1, p_2, ...\}$. For every $i \in \mathbb{N}$, let B_i be an open ball containing p_i such that $B_i \subset X$ (possible since X is open).
Claim:$X = \cup_{i=1}^{\infty} B_i$
Proof: It is clear that $\cup_{i=1}^{\infty} B_i \subset X$ because every B_i is inside X. So I prove the other inclusion. Let $x \in X$.

Problem 5

Part a

Let $C_0 = [0, 1]$. We define C_n recursively for $n \in \mathbb{N}$ as follows:
C_n is defined as every interval of C_{n-1} with the open middle third of each interval removed.
Now we have: $C = \cap_{n=0}^{\infty} C_n \subset ... \subset C_2 \subset C_1 \subset C_0$.