PMath 450 Assignment #2
due Friday June 5, 2015

1. (a) Suppose E_1, E_2 are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

   (b) Suppose $\{f_n\}_{n=1}^{\infty}$ are measurable functions. Show that $\sup f_n$ and $\inf f_n$ are measurable.

   (c) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f = g$ a.e. Prove that $f = g$ (everywhere).

2. (a) Suppose that for each rational number q the set $\{x : f(x) > q\}$ is measurable. Show that f is measurable.

   (b) More generally, suppose S is a family of subsets of $\mathbb{R}$ such that all open sets belong to the smallest σ-algebra containing S. If $f^{-1}(E)$ is measurable for all $E \in S$, show that f is measurable.

3. (a) Suppose E is a measurable set. Prove there is a Borel set $G \supseteq E$ such that $m(G \backslash E) = 0$.

   (b) Suppose $m(E) < \infty$. Show that for every $\varepsilon > 0$ there is a finite union of open intervals, U, such that $m(U \backslash E) + m(E \backslash U) < \varepsilon$.

4. Let $f : [a, b] \to \mathbb{R}$ be measurable and $\varepsilon > 0$. Prove the following.

   (a) There is an N such that $|f| \leq N$ except on a set of measure less than ε.

   (b) Given M, there is a simple function ϕ such that $|f(x) - \phi(x)| < \varepsilon$ except on the set where $|f(x)| \geq M$. Furthermore, if $m \leq f \leq M$, then we may take ϕ so that $m \leq \phi \leq M$.

   (c) Given a simple function ϕ on $[a, b]$, there is a step function g defined on $[a, b]$ such that $g(x) = \phi(x)$ except on a set of measure less than ε. (Hint: 3b) If $m \leq \phi \leq M$, then we can take g so that $m \leq g \leq M$.

   (d) There is a continuous function h such that $m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$. If, in addition, $m \leq f \leq M$, then we can choose h with $m \leq h \leq M$.