University of Waterloo
Algebraic geometry - Summer 2015
Assignment 5

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Problem 1

Part a

Write $p = (a, b)$. Set $X = x - a$ and $Y = y - b$ so that p corresponds to $(0, 0)$ in the (X, Y)-plane. Then $f(X, Y)$ has no constant since $p = (0, 0,) \in V(f)$ and it has a non-zero linear part since X is smooth at $p = (0, 0)$.
Write $f = aX + bY + (\deg \geq 2 \ \ terms)$. We know that $T_p(X) = V(aX + bY)$.
Let $h = cX+dY \in k[X, Y]$ (no constant term so that $h(p) = 0$) such that $cX+dY \neq aX+bY$ (so that $V(h)$ is not tangent to C at p).
I need to prove $M_p(X) =< \bar{h} >$.
Since $aX + bY \neq 0$ we can assume WLOG that $b \neq 0$. Then we know that $M_p(X) =< \bar{x} >$.
So we can write $\bar{Y} = g\bar{X}$ since $\bar{Y} \in M_p(X)$. We have:

$$c\bar{X} + d\bar{Y} = c\bar{X} + dg\bar{X} = \bar{X}(c + dg)$$

Thus $M_p(X) =< \bar{h} >$.

Part b

Let $t = \bar{h}$ and $t' = \bar{g}$ be local parameters (g and h are homogeneous linear polynomials such that $h = 0$ and $g = 0$ are lines through p that are not tangent to C at p).
Let $0 \neq z \in O_p(C)$. We have: $z = t^n u = t'^m v$ for some n, m and u, v units.
I need to prove $n = m$.
Since t and t' both generate $M_p(C)$ we have that $t' = tu'$ for unit u'.
We have $z = t^n u = (tu')^m v = t^m u'^m v$. Since we are in a PID we can cancel, so
$t^{n-m} = u^{-1}u'^m v$ which is a unit, thus $n = m$.

Part c

(i) Let $f = \frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$. We can write:
$\frac{\bar{a}}{\bar{b}} = \frac{t^n u}{t^m v}$ and $\frac{\bar{c}}{\bar{d}} = \frac{t^{n'} u'}{t^{m'} v'}$. We have:

$$\frac{t^n u}{t^m v} = \frac{t^{n'} u'}{t^{m'} v'} \quad \Rightarrow \quad t^{n-m} u v^{-1} = t^{n'-m'} u' v'^{-1}$$

We can cancel in PID's, so we have: $t^{(n-m)-(n'-m')} = u' v'^{-1} v u^{-1}$ which is a unit, thus $n - m = n' - m'$. Hence ord_p^C is well-defined.

(ii) If f is a unit then obviously $f = t^0 f$ and therefore $ord_p^C = 0$.
If $ord_p^C(f) = 0$, it means that $f = \frac{\bar{a}}{\bar{b}}$ such that $\bar{a} = t^n u$ and $\bar{b} = t^n v$. So $f = \frac{t^n u}{t^n v} = \frac{u}{v}$ which is a unit.

(iii) If $f = 0$ then we have that $ord_p^C(f) = \infty$.
If $ord_p^C(f) = \infty$ then $f = \frac{\bar{a}}{\bar{b}}$ such that $ord_p^C(\bar{a}) = \infty$ which implies $\bar{a} = 0$ therefore $f = 0$.

(iv) First we prove it for when $f_1, f_2 \in O_p(C)$. Then $f_1 = t^n u$ and $f_2 = t^m v$.
So $f_1 f_2 = t^n u t^m v = t^{n+m} uv$ where uv is a unit since u and v are units. Thus $ord_p^C(f_1 f_2) = n + m = ord_p^C(f_1) + ord_p^C(f_2)$.
Now let $f_1, f_2 \in K(C)$.
We can write $f_1 = \frac{\bar{a}}{\bar{b}}$ and $f_2 = \frac{\bar{c}}{\bar{d}}$.
We have $\frac{\bar{a}}{\bar{b}} = \frac{t^n u}{t^m v}$ and $\frac{\bar{c}}{\bar{d}} = \frac{t^{n'} u'}{t^{m'} v'}$.
$f_1 f_2 = \frac{\bar{a}\bar{c}}{\bar{b}\bar{d}} = \frac{t^n u t^{n'} u'}{t^m v t^{m'} v'}$.
So $ord_p^C(f_1 f_2) = (n + n') - (m + m') = (n - m) + (n' - m') = ord_p^C(f_1) + ord_p^C(f_2)$.

(v) First assume $f_1, f_2 \in O_p(C)$. Look at the taylor expansion of f_1, f_2.
$f_1 = a_m t^m + a_{m+1} t^{m+1} + ... = t^m(a_m + a_{m+1} t + ...)$ where a_m is the lowest non-zero term and therefore $(a_m + a_{m+1} t + ...)$ is a unit.
Similarly $f_1 = a_n t^n + a_{n+1} t^{n+1} + ... = t^n(a_n + a_{n+1} t + ...)$ where $(a_n + a_{n+1} t + ...)$ is a unit. So order of a polynomial is the lowest power of t in polynomial expansion. So $ord_p^C(f_1 + f_2) \geq \min\{ord_p^C(f_1), ord_p^C(f_2)\}$.
Now let $f_1, f_2 \in K(C)$. Write $f_1 = \frac{\bar{a}}{\bar{b}}$ and $f_2 = \frac{\bar{c}}{\bar{d}}$. We have: (I will ommit the bars for residue classes)

$$\begin{aligned}
ord_p^C(f_1 + f_2) &= ord_p^C\left(\frac{ad + bc}{bd}\right) \\
&= ord_p^C(ad + bc) - ord_p^C(bd) \\
&\geq \min\{ord_p^C(ad), ord_p^C(bc)\} - ord_p^C(bd) \\
&= \min\{ord_p^C(a) + ord_p^C(d), ord_p^C(b) + ord_p^C(c)\} - (ord_p^C(b) + ord_p^C(d)) \\
&\geq \min\{ord_p^C(f_1), ord_p^C(f_2)\}
\end{aligned}$$

Problem 2

Part a

Assume $C = V(f)$ where $f = gh$ for non-constant $g, h \in k[x, y]$. So $C = V(g) \cup V(h)$.
Let $p \in V(g) \cap V(h)$. Now we have:
$\nabla(f)(p) = (f_x(p), f_y(p)) = (g_x(p)h(p) + h_x(p)g(p), g_y(p)h(p) + h_y(p)g(p)) = 0$ because $h(p) = g(p) = 0$. Thus C is singular at p.

Part b

By propert (vi), $I(p, C \cap L) = 1$ if and only if C and L intersect transversely at p which happens if and only if L is not the tangent line to C at p, otherwise $I(p, C \cap L) \geq m_p(C)m_p(L)$ and we have that $m_p(C) = m_p(L) = 1$ since L is linear and C is smooth at p and also $I(p, C \cap L) \neq 1$. Thus $I(p, C \cap L) \geq 2$.

Part c

Since $I(p, L_1 \cap C) \geq 2$ and $I(p, L_2 \cap C) \geq 2$, by part (b) we have that both L_1 and L_2 are tangent lines to C at p and since L_1 and L_2 are distinct tangent lines to C at p, C does not have a linear part and therefore C is singular at p.

Part d

Let $X = x - 1$ and $Y = y$. Then $C = V((X + 1)^2 - 1 - Y^3)) = V(X^2 + 2X + Y^3)$ and $C' = V((X + 1)^2 - 1 + 2Y^4) = V(X^2 + 2X + 2Y^4)$. Now $p = (0, 0)$ and we need to find $I(p, C \cap C')$.
Since $X = 0$ is tangent to C at $p = (0, 0)$, $Y \in O_p(C)$ is a local parameter.
Now note that $\bar{X^2} + 2\bar{X} = \bar{Y^3}$ in $O_p(C)$, so $\bar{X^2} + 2\bar{X} + 2\bar{Y^4} = \bar{Y^3} + 2\bar{Y^4} = \bar{Y^3}(1 + 2\bar{Y})$ where $(1 + 2\bar{Y})$ is a unit, thus $ord_p^C(X^2 + 2X + 2Y^4) = 3$. Hence $I(p, C \cap C') = 3$.

Problem 3

part a

We have that $ord_p^C(z) = 0$ if and only if z is a unit in $O_p(C)$. So $ord_p^C(z) \neq 0$ if and only if z is not a unit in $O_p(C)$ which hapens if and only if p is either a zero of z or a poleof z. So the divisor is well-defined and it's support is the set of all zeroes and poles of z.
Assume $div(z) = 0$. So z has no poles which means $z \in \Gamma(C)$ and z has no zeroes which forces $z = c$ for some constant c.

Part b

Zeroes of z are points with $y = 0$. If $y = 0$ then $x^3 = x$. Thus the zero set of z is $\{(0,0), (-1,0), (1,0)\}$. The pole set of z could possibly be $(1,0)$ but $z = \frac{\bar{y}}{\bar{x}-1} = \frac{\bar{y}}{x^3-y^4}$ so $(1,0)$ is defined for z.
We now need to compute $ord_p^C(z)$ for $p \in \{(0,0), (-1,0), (1,0)\}$.
We first find it for $p = (0,0)$. Since $y^4 - x^3 + x$ has linear part x, we have that $M_p(C) =< \bar{y} >$.
We have that $z = \bar{y}(\frac{1}{\bar{x}-1})$ where $(\frac{1}{\bar{x}-1})$ is a unit. So $ord_p^C(z) = 1$.
Now we compute the order for $p = (1,0)$. For this we find an affine coordinate change so that $p = (0,0)$.
Let $X = x - 1$ and $Y = y$. $\bar{Y}$ can still generate $M_p(C)$. We have $z = \bar{Y}(\frac{1}{\bar{X}-2})$ where $(\frac{1}{\bar{X}-2})$ is a unit. So $or_p^C(z) = 1$ again.
Now we compute the order for $p = (-1,0)$.
Let $X = X+1$ and $Y = y$. Now $ord_p^C(z) = ord_p^C(\bar{Y}) - ord_p^C(\bar{X})$. We know that $ord_p^C(\bar{Y}) = 1$.
We have that $C = V(Y^4 - (X+1)^3 + X + 1) = V(Y^4 - X^3 - 3X^2 - 2X)$. So $\bar{Y}^4(\frac{1}{\bar{X}^2+3\bar{X}+2}) = \bar{X}$ where $(\frac{1}{\bar{X}^2+3\bar{X}+2})$ is a unit. So $ord_p^C(\bar{X}) = 4$. Thus $ord_p^C(z) = -3$.
Hence $div(z) = (0,0) + (1,0) + -3(-1,0)$.

part c

First note that $C \cap C' = \{(1,-1)\}$.Let $p = (1,-1)$. I need to find $I(p, C \cap C')$.
We first find an affine coordinate change so that $p = (0,0)$.
Let $X = x - 1$ and $Y = y + 1$. Then $p = (0,0)$.
$C = V(X + 1 + (Y-1)^3) = V(X + Y^3 - 3Y^2 + 3Y)$.
$C' = V((Y-1)^3 - (Y-1)(X+1)) = V(Y^3 - 3Y^2 + 4Y + XY - X - 2)$.
Note that $Y^3 - 3Y^2 + 4Y + XY - X - 2$ is a unit in $O_p(C)$ so $I(p, C \cap C') = 0$ and hence the divisor is a zero divisor.

Problem 4

Part a

We have that $X = V(y^2 - x^3 - x^2)$ is singular at the origin because $y^2 - x^3 - x^2$ does not have a linear part.
We need to show the blow up of X is a smooth curve in $\mathbb{A}^3$.
Blow up of X is $V(y^2 - x^3 - x^2, y - xu) = V(u^2 - x - 1, y - ux)$.
Let $f = u^2 - x - 1$ and $g = y - ux$. We have:

$$jac(f,g) = \begin{bmatrix} -1 & 0 & 2u \\ -u & 1 & -x \end{bmatrix}$$

This matrix has Rank 2 everywhere, so the blow up is a smooth curve.

Part b

Note that $y^3 - x^5$ has no linear term therefore C is singular at $(0,0)$.
The blow up of Y is $V(y^3 - x^5, y - xu) = V(u^3 - x^2, y - xu)$.
Let $f = u^3 - x^2$ and $g = y - xu$. We have:

$$\begin{aligned} jac(f,g)(0,0,0) &= \begin{bmatrix} -2x & 0 & 3u^2 \\ -u & 1 & -x \end{bmatrix}(0,0,0) \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

This matrix has Rank 1 so the blow-up is not singular at $(0,0,0)$.
We blow this up again to get $V(u^3 - x^2, y - xu, x - t_1 y, u - t_2 y) \subset \mathbb{A}^5$.