# Nyström-method Analysis of Active Spiral Subwavelength 2-D Microresonators

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## Lasing eigenvalue problem

Time dependence is assumed as $\sim e^{-ikct}, k=\omega/c,\ U=E_z$ or $H_z$

$$\left(\Delta+k_{i,j}^2\right)U_{i,e}(\vec r)=0,\quad \vec r\in D_i\cup D_e$$

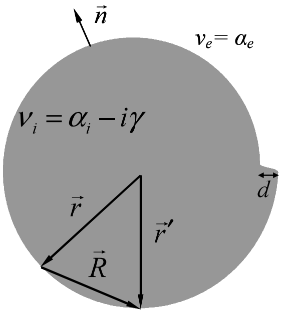
$$U_i\big|_\Gamma=U_e\big|_\Gamma;\quad \eta_i\left.\frac{\partial U_i}{\partial n}\right|_\Gamma=\eta_e\left.\frac{\partial U_e}{\partial n}\right|_\Gamma$$

$U_e$ is satisfied Sommerfeld rad.cond. at $\infty$

$$k_{i,e}=kv_{i,e};\quad v_i=\alpha_i-i\gamma;\quad v_e=\alpha_e$$

$$\eta_{i,e}=1/v_{i,e}^2\ \text{(Hpol) or 1 (Epol)}$$

$(k_s,\ \gamma_s)$ are eigenparameters




## Muller's integral equations

$$\varphi(\vec r)-\int_\Gamma\varphi(\vec r')A(\vec r,\vec r')dl'+\int_\Gamma\psi(\vec r')B(\vec r,\vec r')dl'=0$$

$$\frac{\eta_i+\eta_e}{2\eta_e}\psi(\vec r)-\int_\Gamma\varphi(\vec r')C(\vec r,\vec r')dl'+\int_\Gamma\psi(\vec r')D(\vec r,\vec r')dl'=0$$

Here the unknown functions $\varphi(r)$ and $\psi(r)$ are the values of the field function and its normal derivative, respectively on the contour and kernel are given as

$$A(\vec r,\vec r')=\frac{\partial G_i(\vec r,\vec r')}{\partial n'}-\frac{\partial G_e(\vec r,\vec r')}{\partial n'};\quad B(\vec r,\vec r')=G_i(\vec r,\vec r')-\frac{\eta_i}{\eta_e}G_e(\vec r,\vec r')$$

$$C(\vec r,\vec r')=\frac{\partial^2 G_i(\vec r,\vec r')}{\partial n\partial n'}-\frac{\partial^2 G_e(\vec r,\vec r')}{\partial n\partial n'};\quad D(\vec r,\vec r')=\frac{\partial G_i(\vec r,\vec r')}{\partial n}-\frac{\eta_i}{\eta_e}\frac{\partial G_e(\vec r,\vec r')}{\partial n}$$

$G_j(\vec r,\vec r')$ is the 2-D Green's functions

## Nyström method

Separating the logarithmic parts from kernels as follows:

$$F(\vec r,\vec r')=F_1(\vec r,\vec r')\ln\left[4\sin^2\frac{t-\tau}{2}\right]+F_2(\vec r,\vec r'),\quad F=A,B,C,D$$

and functions are given by

$$A_1(\vec r,\vec r')=(-1/4\pi)\left[k_iJ_1(k_iR)-k_eJ_1(k_eR)\right](\vec R\cdot\vec n')/R$$

$$B_1(\vec r,\vec r')=(-1/4\pi)\left[J_0(k_iR)-(\eta_i/\eta_e)J_0(k_eR)\right]$$

$$C_1(\vec r,\vec r')=(1/4\pi)\left[k_i^2J_2(k_iR)-k_e^2J_2(k_eR)\right](\vec R\cdot\vec n')(\vec R\cdot\vec n)/R^2-(1/4\pi)\left[k_iJ_1(k_iR)-k_eJ_1(k_eR)\right](\vec n'\cdot\vec n)/R$$

$$D_1(\vec r,\vec r')=(1/4\pi)\left[k_iJ_1(k_iR)-(\eta_i/\eta_e)k_eJ_1(k_eR)\right](\vec R\cdot\vec n)/R$$

$$\vec r(t)=\{x(t),y(t)\}\quad t\in[0,2\pi]\quad t_p=\pi p/N,\ p=0,1,\ldots,2N-1$$

$$\int_0^{2\pi}\ln\left[4\sin^2\frac{t-\tau}{2}\right]F_1(\vec r,\vec r')f(\vec r')dl'=\sum_{p=0}^{2N-1}P_p^{(N)}(t)F_1(t,t_p)f(t_p)L(t_p)$$

$$P_p^{(N)}(t)=-(2\pi/N)\sum_{m=1}^{2N-1}\cos\left[m(t-t_p)\right]/m-\pi\cos\left[N(t-t_p)\right]/N^2$$

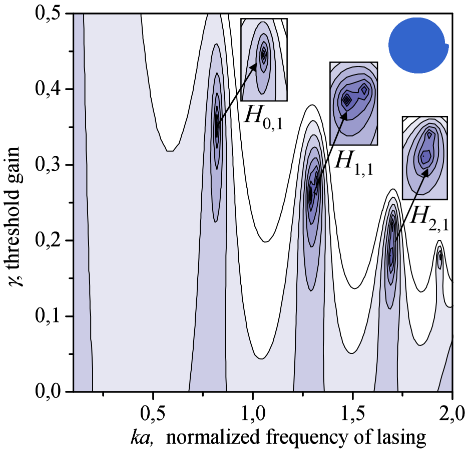
$$\int_0^{2\pi}F_2(\vec r,\vec r')f(\vec r')dl'=(\pi/N)\sum_{p=1}^{2N-2}F_2(t,t_p)f(t_p)L(t_p)+(\pi/2N)\left[F_2(t,t_0)f(t_0)L(t_0)+F_2(t,t_N)f(t_N)L(t_N)\right]$$

$$L(t)=\sqrt{(dx/dt)^2+(dy/dt)^2};\quad f=\varphi,\psi$$

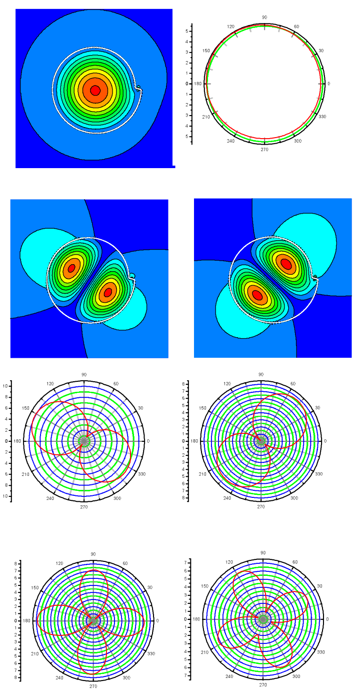
## Numerical results

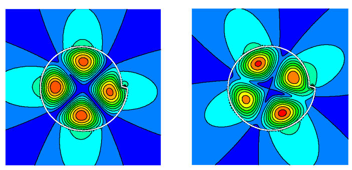
Spiral-shaped contour parameterization with a smooth function ($\beta$ is the step tilt angle):

$$r(t)=\begin{cases}1-\delta/4\pi\left[(2\pi-\beta)/\beta t-\pi/\beta^2t^2-\pi\right],t\in[0,\beta), & \delta=d/a\\ 1+\delta/4\pi t,\quad t\in[\beta,2\pi-\beta] & \\ 1+\delta/4\pi\left[(2\pi-\beta)/\beta(2\pi-t)-\pi/\beta^2(2\pi-t)^2+\pi\right],t\in(2\pi-\beta,2\pi] & \end{cases}$$




*Determinant absolute value relief on the plane* ($ka$, $\gamma$): $d=0.3a$, $\beta=0.03$, $\alpha_i=2.63$, $\alpha_e=1$, $N=50$.





*Near- and far-field patterns for the lowest modes in a spiral-shape cavity*

## Conclusions

Equivalence ➢ no spurious eigenvalues
Smooth or integrable kernels ➢ reliable discretization
Controllable accuracy
Smooth parameterization for a spiral-shaped contour
Modes in a spiral cavity split into doublets

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