CHAPTER 1

LITERATURE REVI EW

1.1 General properties of dielectric resonators

Dielectric resonator (DR) is a sample of certain shape made of dielectric (non-magnetic) material, placed into unbounded outer space and able to enhance the amplitude of the electromagnetic field. Most frequent shape of DR is round disk (see Figs. 1.1 and 1.2) however many other shapes exist. Such a sample is able to play the role of resonator thanks to the internal reflections of electromagnetic waves from the dielectric-air boundary. This leads to the concentration of electromagnetic energy inside the sample, although, as the boundary of any dielectric is partially transparent, a fraction of energy always leaks to the outer space.

Fig. 1.1 Dielectric resonators for the filters and oscillators of microwave and millimeter-wave bands [47]

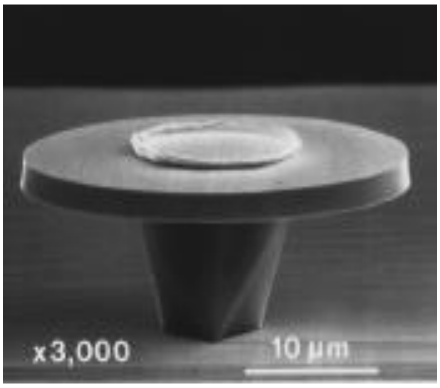


Fig. 1.2 Microdisk laser of the infrared band with optical pumping on a pedestal [48]

A pioneering work, which had demonstrated that dielectric objects could display resonant behavior similarly to the metal-wall cavities, was published by Richtmyer in 1939 [49]. More detailed investigation of the microwave DRs and their areas of application were started much later, in the 1950-1960s. It encompassed theoretical analysis of resonance frequencies of natural modes and their fields, development of the design principles for DRs and DR-based subsystems, and other questions [50-53]. However, the absence of durable low-loss materials had been blocking the microwave applications of DRs for years. Still these pioneering studies were important as they enabled researchers to clarify the properties of DRs and stimulated their efforts towards sophistication of the technologies for manufacturing high-quality dielectric materials.

Advantages of DRs in microwave bands follow from their properties. In fact, they bridge the gap between the waveguides and the printed circuits because they provide, from the one side, almost the same high values of Q-factors and remarkable temperature stability as hollow metal cavities from Ingvar and copper and, from the other side, the same rich integration opportunities as microstrip resonators [47].

The natural modes are the characteristic discrete forms of electromagnetic field in the resonators that have the shape of standing waves and high values of amplitude. Electromagnetic theory explains them as solutions to the time-harmonic Maxwell equations in the absence of sources; such solutions can exist only at certain discrete complex values (called eigenvalues) of the frequency [54]. In microwave applications, DRs usually have dimensions smaller or comparable to the wavelength, and their resonance properties are usually associated to the quasi-TE_{01} mode in cylindrical or disk resonators and quasi-TE_{11} mode in rectangular resonators. For example, the quasi-TE_{01} mode, for certain values of the ratio of the disk diameter to the disk height, has the lowest natural frequency and is called the fundamental or principal mode of the disk DR. In general, however, the nomenclature of the modes in DRs is so well established as in hollow cavities.

Quality factor (Q-factor) is the ratio of the real part of the complex natural frequency of a resonator mode to the doubled imaginary part [54]. Equivalently, it can be defined as the ratio of the power stored in the resonator to the power lost, averagely for the period of oscillations, provided that the frequency coincides with the complex natural frequency.

The ability of the field to radiate from DR is controlled by the contrast between the materials of DR and the host medium, in terms of dielectric permittivities or refractive indices. The permittivity is a macroscopic parameter, which is introduced after the averaging (homogenization) of the microscopic parameters of the medium.

In the case of non-magnetic material, its dielectric permittivity equals to the square of refractive index ν^2, so that the choice between these two quantities is a matter of choice or habit of researcher. It should be noted that, at the high frequencies, the radiation losses also considerably depend on the roughness of the boundary of DR.

Besides of the radiation losses, realistic DRs possess also the losses in the materials they are made of. When building a model of DR, its material losses are characterized with the aid of a macroscopic parameter – imaginary part of the dielectric permittivity, $\mathrm{Im}\,\varepsilon$, or refractive index, $\mathrm{Im}\,\nu$ (or with the tangent of the loss angle, which is the ratio of the real and imaginary parts of the permittivity).

Q-factors of today DRs with fundamental modes reach 10^4. However, in the millimeter, terahertz and shorter waves, resonators with the quasi-TE_{01} mode become prohibitively small to be efficiently used. Therefore in the mentioned wavebands the most frequently used DRs are the large-size disks able to support **whispering gallery** (WG) mode resonances [48]. Additional advantage of such higher-order modes is in the stronger concentration of the field inside DR and, therefore, much higher values of Q-factors. The champion values of Q-factors of the WG modes in real-life DRs reach 10^8; they have been obtained after melting and cooling of the disk rim [55].

Applications of DRs are numerous. In the microwave band, for the frequencies between 2 and 70 GHz, they are most frequently used as various **filters** and their area of implementation is expected to grow further [56]. However, as noted above, a part of electromagnetic energy always leaks out of DR to the outer space and, far from DR, transforms into an outgoing spherical or cylindrical wave.

Therefore every DR can be also viewed as **antenna**, where every natural mode of DR has its specific far-field radiation pattern (RP) and directivity of emission. An attractive property of DR as antenna is its ability to enhance the radiation resistance of elementary small-size radiator (e.g. a probe or a slot) approximately in Q times. This property enables dielectric resonator antennas to compete successfully with microstrip and other printed antennas [57].

At the same time, dielectric bodies find applications as elements of antennas based on completely different principles than natural-mode excitation – namely, quasi-optical lens antennas of millimeter and THz waves and optical lenses [58]. Here, the beam focusing (in the reception regime) or collimation (in the transmission regime) is achieved due to geometrical-optics properties of the lenses. Nevertheless, because of finite dimensions every lens remains an open resonator.

Further, DRs are also used in the design of very stable **oscillators** of microwaves and millimeter waves. The oscillators whose output frequency is stabilized with the aid of high-Q DRs possess several attractive features such as small dimensions, simplicity of design, low cost, good stability for vibrations, absence of higher harmonics (no frequency multiplication), and very low noise level [59].

The lasers of the THz to visible to ultra-violet bands are, apparently, the latest and very promising area of DR application [8]. Frequently such lasers are shaped as semiconductor or monocrystal or polymeric disks standing on pedestals (see Fig. 1.2). In the laser DR, the losses of the working mode for radiation and for heating its passive components are fully compensated by the electromagnetic power generated in the active region (see below subsection 1.2). Microlasers made as disks with smooth rim demonstrate remarkably low thresholds of lasing and are considered as promising sources of light for the future optical integrated circuits of high density, and also for the quantum computers.

Problems, challenges and trends. As noted, for years the main problem has been the absence of good materials for DRs. Available materials were limited both in terms of dielectric permittivity and in durability as the losses used to increase because of degradation of material homogeneity. Today there is a wide variety of industrially manufactured materials having dielectric permittivities as high as 100-150 and loss-tangent as small as 10^{-4} [47].

A widely recognized and high-technology way to avoid manufacturing many costly prototypes and therefore to shorten the time and the cost of design and development of electromagnetic devices and systems is their preliminary computer-aided design, i.e. modeling and optimization. Note that the tools and techniques of computer design are required to provide simultaneously high efficiency and acceptable accuracy, as without the latter the modeling results cannot be trusted.

In the modeling of any dielectric scatterers, the main challenge is their fundamental property to behave as dielectric resonators. If the real-valued frequency of excitation comes close to a resonance one, then the field both inside and outside of DR changes abruptly and takes the shape close to the field of corresponding natural mode (eigenmode). Therefore, for instance, in every dielectric lens having dimensions of the order of tens of wavelengths the field combines the ray-like features (which provide the focusing off the resonances) and the modal features (which destroy the focusing in the resonances). The interaction of these different mechanisms can be very complicated and its accurate description is a very nontrivial task. Moreover, the correct interpretation of the field behavior is not possible without the analysis of natural modes of dielectric scatterers as open DRs that implies the study of the corresponding natural frequencies and Q-factors. As a result, one must study together the scattering problems and the eigenvalue problems, although in the former problems the frequency is real-valued and in the latter it is complex-valued.

1.2 Microlasers as dielectric resonators with active regions

Microdisk semiconductor lasers first appeared in the 1990s as miniature sources of light in the infrared and later the terahertz, visible and ultraviolet bands [48, 60-67]. At first, these lasers were shaped as disks having diameters of 5-10 micrometers and thickness of 100-300 nm, on a pedestal (see Figs. 1.2 and 1.3).

(a) (b) (c) (d)

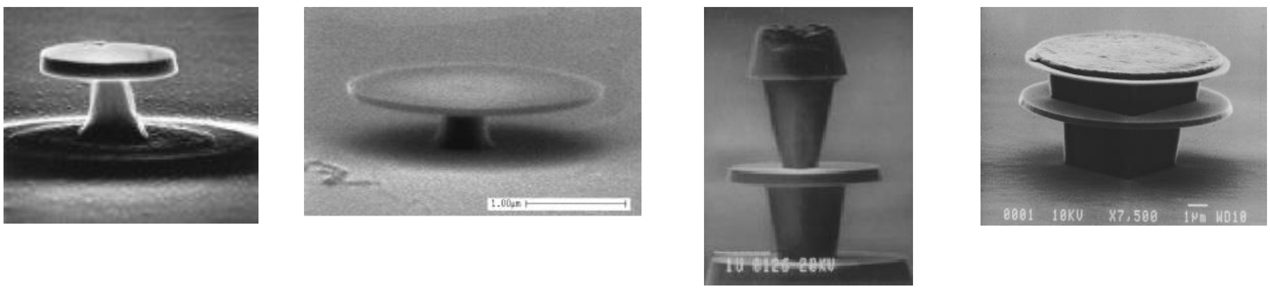


Fig. 1.3 Microphotographs of disk lasers with optical pumping (a) from [65] and (b) [66] and injection-type pumping (c) from [64] and (d) [63]

For manufacturing microdisk lasers, various semiconductor **material systems** are used. Between the semiconductor layers, one or several cascaded active layers are placed [60-67], either homogeneous quantum wells or layers of randomly located quantum dots.

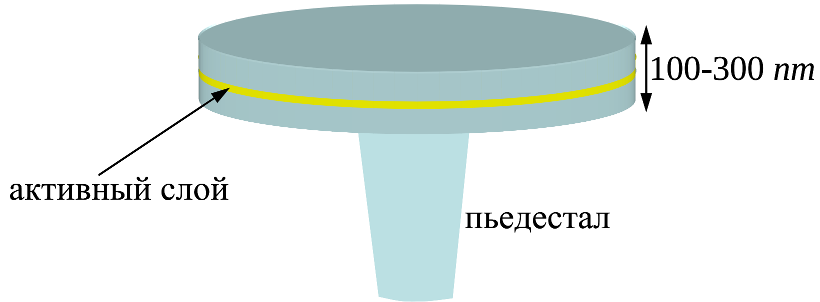


Fig. 1.4. Sketch of the inner composition of a typical microdisk laser standing on a pedestal

Each material system as a bulk medium has its own spontaneous emission band or photoluminescence spectrum [60-62]. Under pumping, in this band the stimulated emission is observed. The emission of light in the bands from infrared to ultraviolet can be obtained using the following systems [63-67] (λ is the wavelength in the center of luminescence band and α is the refractive index):

GaAs/InP - $\lambda = 1.55\ \mu\text{m}$, $\alpha = 3.63$

GaAs-AlAs/InAs - $\lambda = 970$ nm

GInP/InP - $\lambda = 650$ nm

ZnSe/CdS - $\lambda = 510$ nm

ZnO/SiO_2 - $\lambda = 390$ nm

InGaN/GaN - $\lambda = 370$ nm

Later on, microdisk lasers made of monocrystals doped with ions of Erbium appeared [55, 68], and eventually in the recent years dye microlasers on polymeric host materials have been studied.

Thus, a monocrystal, semiconductor or polymeric microcavity laser is a dielectric resonator equipped with active region. The thickness of active layers is usually 5-10 nm, which is much smaller than the disk thickness. As, in its turn, the disk thickness is considerably smaller than the wavelength, one can assume that all the disk material can become active under pumping. In this case the shape of the active region is determined by the pumping.

According to the way of **pumping** all microcavity lasers can be divided to two groups: photopumped and pumped by injection of carriers through the electrodes [60-67]. Optically pumped cavities are manufactured with flat upper face, see Fig. 1.3 (a), (b), which is normally flood-illuminated by a wide external beam of the pumping laser. Then, the whole resonator can be considered as active. In contrast, injection-pumped lasers have a disk resonator sandwiched between an electrode and the substrate or between two electrodes, see Fig. 1.3 (c), (d). Therefore here the size and shape of the active region depend on the shape and location of the electrodes. As a rule, first experimental realizations of lasers based on new semiconductor materials had been demonstrated at cryogenic temperatures and with optical pumping. However, after refining the technologies similar lasers have usually been designed for operation at the room temperatures and with the injection pumping. Today there exists a wide recognition that the proper choice of the electrodes shape and placement plays crucial role in the efficient operation of any injection laser.

The fundamental properties of microdisk lasers are revealed by their measurements and are as follows [60-63]:

(i) almost **equidistant lasing frequencies** in the photoluminescence band,

(ii) **ultra-low thresholds of lasing**,

(iii) **low directionality of emission**, which occurs mainly in the disk plane and displays many similar beams.

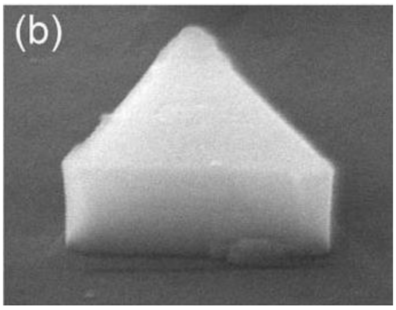
All these properties find their explanation if one assumes that such lasers emit light on the WG modes having high azimuth indices, whose fields are confined at the disk rim due to almost total internal reflections in the disk plane [68].

Indeed, for these modes the disk circumference is approximately multiple to the wavelength in the disk material, their Q-factors in the pump-off regime are record-high due to almost total internal reflections, and emission in the vertical direction is very small because of the media contrast (waveguide effect). The distance between the neighboring lasing frequencies is called **free spectral range**. Its ratio to the width of the mode emission line is called **finesse** of resonator. The radiation pattern for the *m*-th whispering-gallery mode has 2*m* identical beams (lobes) and therefore its directivity is always 2 (for omnidirectional emission this quantity is 1).

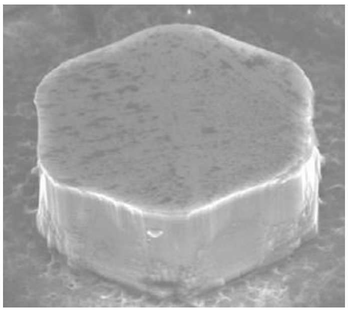
Today trends in the research into microcavity lasers are connected to the search of the ways of lowering the thresholds and improving the directionality of emission. To achieve these goals, researchers try to smooth the disk rim, optimize shape and location of pumped area, integrate disk with an annular Bragg reflector (ABR), collect the disks into a photonic molecule, and find optimal shapes of non-circular microresonators (Fig. 1.5) [8].

Low directionality of radiation is considered as a serious drawback of microdisk lasers. The fields of the WG modes have large number of identical beams (normally several dozens) because of the circular symmetry. Therefore it is clear that for the improvement of directionality one should somehow perturb the circular shape of cavity. There is a large number of works where non-circular microcavity lasers have been fabricated and measured: triangular [69], square [70], hexagonal [71], gear-like [72], stadium [48,73], and elliptic [74].

For such lasers, directivity of emission typically reaches 4 or 5 because their modal radiation patterns have only a few identical intensive lobes whose number depends on the degree of symmetry of the cavity. Principally different is the shape of the **spiral microlaser**, whose cavity contour follows an Archimedean spiral (Fig. 1.5 (e)) [75, 76]. Such a contour does not possess either a rotational symmetry or a mirror symmetry. This laser emits light pattern with one main beam having the width of 40^{o}-60^{o} and directivity around ~10. However, a natural cost of directionality improvement is a higher value of the threshold of lasing.



(a)



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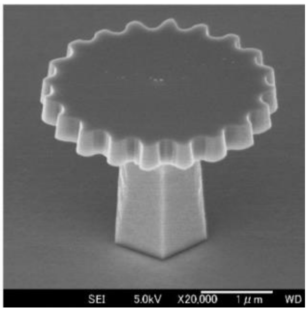

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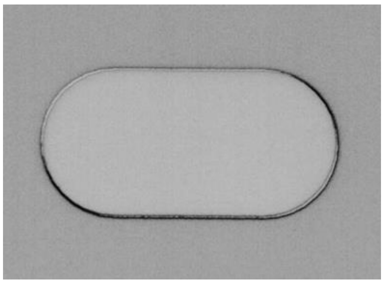

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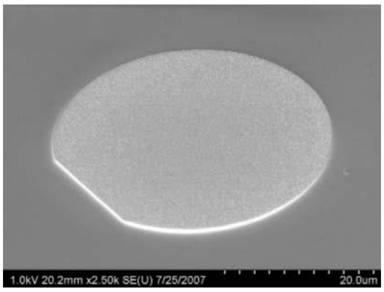

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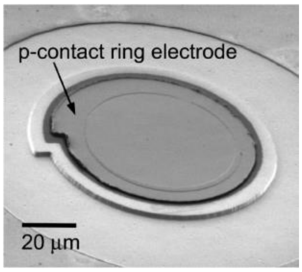


(e)

Fig. 1.5. Laser microcavities of various shapes: (a) triangular [69], (b) hexagonal [71], (c) gear-like [72], (d) stadium [73], (e) cut-off disk [77], (f) spiral [75]

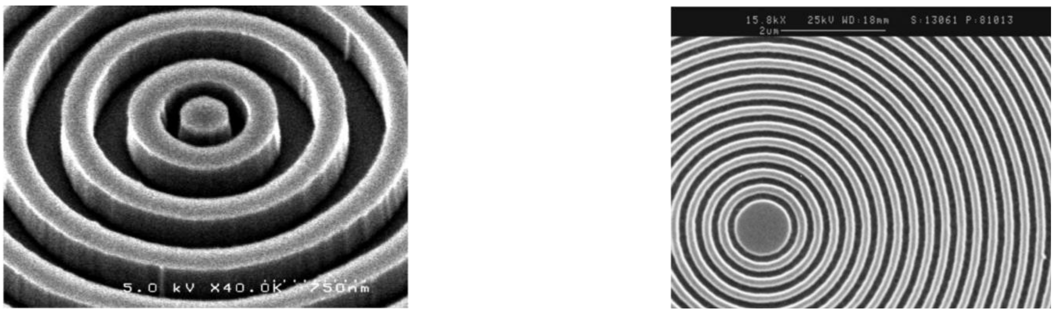
The smoothness of the rim surface of a microdisk laser has a double effect on its characteristics. First, deviation from the circular shape leads to the splitting of modes to doublets [77] and to their shift in frequency.

Second, if the scale of roughness on the side surface is comparable to the wavelength in the cavity material, this results in the high radiation losses of each mode of the doublet. This drastically limits the growth of Q-factor for a passive disk (pump off) or reduction of threshold of lasing for an active disk (pump on) when the frequency or the disk radius or the WG mode azimuth index increases.

The measured values of Q-factors of microdisk resonators usually have the values of 10^3-10^4 [55,78]. This is much lower than the 10^8-10^9 quality of the WG modes in the spherical resonators fabricated by melting a tip of an optical fiber [79], where the surface tension forces reduce the roughness to “atomic” scale (1-2 nm). Recently the melting and cooling has been applied to the monocrystal disk cavities as well that has led to the increase in the Q-factor up to 10^7-10^8 [80].

Until recently, the experiments with optical pumping used the pump lasers producing wide beams to provide a uniform pumping of microdisks (flood pumping). However, as the WG mode field is contracted around the disk rim, it is enough to pump the ring-like area around the rim. This idea was implemented for the first time in [81] by using a hollow pump beam due to the use of an axicon and immediately led to the reduction of the threshold pump power. Similar effects are observed for the injection-type microdisk lasers as well. Such lasers with electrodes attached to the disk center, like in Fig. 1 (c), (d), have much higher thresholds of emission than the lasers with ring electrodes [64]. Today it is widely recognized that proper placement of electrodes is crucially important in the design of any injection-type lasers [82].

For the development of single-mode lasers, it is necessary to increase the free spectral range. This can be achieved by reducing the disk resonator radius however it leads to the lower order modes as working modes instead of the WG modes. The thresholds of lasing of the lower modes are high because of greater radiation losses. To reduce the threshold one has to reduce the emission in all directions and especially in the plane of the disk. This can be achieved by placing the disk into an annual Bragg reflector (see Fig.1.6) [83-86].



(a) (б)

Fig. 1.6 Microdisk resonators inside annular Bragg reflectors etched in the substrate: (a) from [85], (b) from [83]

Such reflectors are usually shaped as concentric circular grooves etched in the same material layer as the disk. However, the pump is applied only to the central disk so that the rings of ABR remain passive.

Alternative approach to the reduction of emission threshold suggests collecting several elementary microdisks in the optically coupled assembly (see Fig. 1.7). By analogy with photonic crystals (infinite-periodic structures having dielectric particles in the nods of regular mesh), the stand-alone microcavities are also called “photonic atoms” and the ordered configurations of finite number of microcavities are called “**photonic molecules**.” As elementary cavities here are optically coupled, their modes are not independent and therefore they are called **supermodes**. One can expect that the radiation losses of supermodes in photonic molecules can be reduced because of partial cancellation of waves emitted by different elements, for instance, due to the symmetry properties.

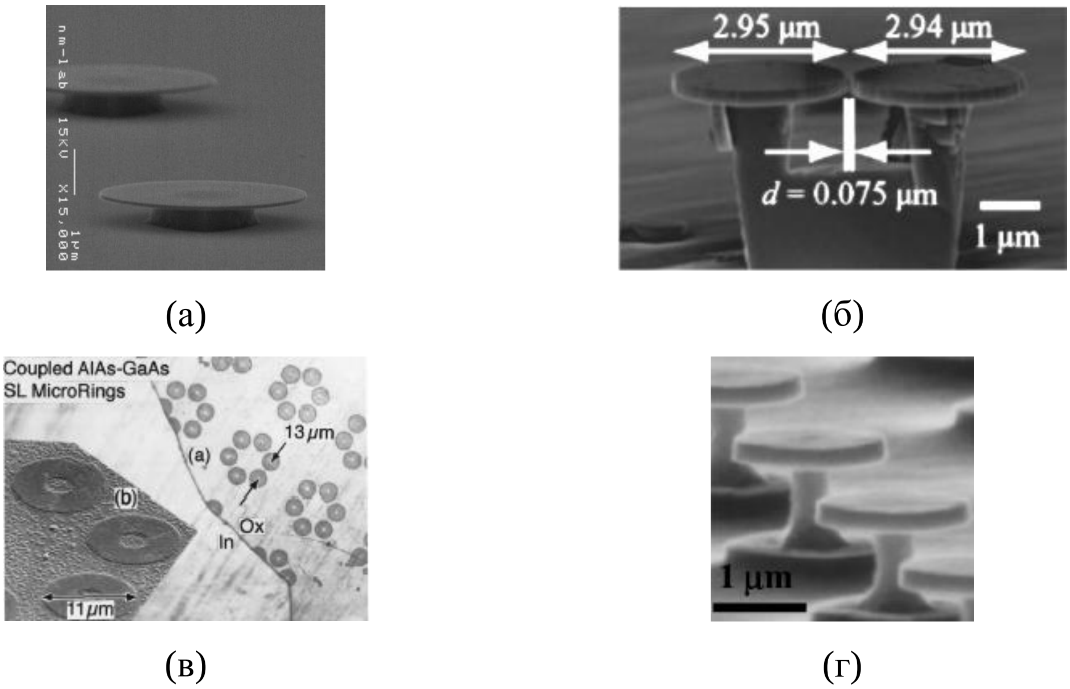


Fig. 1.7. Examples of photonic-molecule lasers: (a), (b) twin-disk [87,88], (c) cyclic array of rings [89], (d) linear array of disks [90].

As a result, the total radiation losses, and hence the thresholds of lasing, of the supermodes in an active photonic molecule can be considerably lower than of the corresponding modes in a stand-alone active cavity. Such lasers have been investigated in [87-90], although no reduction of threshold has been reported.

1.3 Methods of analysis of passive dielectric resonators

For the accurate electromagnetic analysis of DRs, and even more so for their optimization, one has to have a reliable and efficient computational tool, which takes full account of their specific properties. For example, such a tool should be able to handle arbitrarily curved boundaries of DRs, use accurate boundary conditions on them, and take correct account of infinite host medium. This short list immediately suggests that the Geometrical Optics and Finite-Difference Time-Domain techniques are not good candidates and other, more adequate, methods should be considered. From the viewpoint of mathematics, the most promising are the methods based on Integral Equations (IEs) that is visible from the growth of corresponding publications. They should be used, however, in combination with discretization methods, which provide convergence of approximate solutions to the accurate ones, when the order of discretization increases. Unfortunately, this is not always taken into account.

Dimensionality reduction. Modern technologies of dry and wet etching and molecular-beam epitaxy enable manufacturing and control of thin microcavities. The 3-D problem for the electromagnetic field in the presence of a disk whose thickness is smaller than the wavelength can be approximately reduced to a 2-D problem in the disk plane with the aid of the effective refractive index of the thin dielectric slab [91,92]. This approximation is in good agreement with predominantly in-plane light emission and leads to separate consideration of the mode of H and E type. There is, however, one aspect of this approximation that almost always escapes from the attention of researchers. Effective refractive index is determined in non-unique manner and can take several discrete values depending on the type of natural guided wave in the dielectric slab that is taken for the field in the disk. Besides, the effective refractive index is dispersive, i.e. depends on the frequency [2, 92].

In 2-D, the simplest resonator is a circle. **Circular DR** having refractive index equal to the effective refractive index of the slab of the same thickness and material is the model of a thin 3-D circular-disk. This is one of the shapes that support WG modes, and it can be studied analytically by the separation of variables. However, even in this simple resonator it is much easier to determine the natural mode frequencies then their Q-factors. A good real-valued approximation for the frequencies can be obtained if one neglects the leakage and assumes that the circle contour is metallized [60,63]. Calculation of the Q-factor (using the imaginary part of the frequency) needs a much finer technique using the Debye asymptotics for cylindrical functions. These difficulties become even more significant for the modes of more complicated circular-concentric resonators such as a disk inside an ABR (Fig. 1.6). Therefore it is not a surprise that there are no analytical formulas for the Q-factors of the WG modes in such resonators [93-95].

Geometrical Optics (GO) can be successfully applied to the characterization of the WG modes in a large-radius circle. Moreover, GO in the form of the billiards theory has played an important role in the understanding of the modes in non-circular cavities [96-98]. In particular, it has led to the discovery of the so-called “bow-tie” modes in the stadium-shape resonator [73]. The billiards theory is able to predict the natural-mode frequency however fails to characterize its Q-factor and emission pattern. The attempts to improve this approach by combining it with Snell’s law and Fresnel coefficients have limited sense as they are based on the assumption that the field is a locally plane wave and the cavity boundary is a straight line. The both assumptions are very far from reality for the cavities comparable to the wavelength. Even more serious drawback is that the billiards theory is not able to reproduce the discreteness of the complex natural frequencies of the resonator. The same is valid with respect to the more accurate version of this approach known as “Gaussian beam optics” [99] and using approximate substitution of the Helmholtz equation with the parabolic equation.

Since recently, commercial codes for the calculation of electromagnetic fields with the **Finite-Difference Time-Domain (FDTD)** method have become widely used. It has been applied to the analysis of modes in passive microcavities in [100-102]. However, FDTD is not applicable to solving the eigenvalue problems directly. Instead, one has to place a pulsing source inside the resonator and find the temporal response to it at some other point. Then one has to apply inverse Fourier transform to transform that signal to the frequency domain and eventually find the mode frequencies Q-factors from the locations and widths of the peaks of that function. This complicated way is strongly dependent on the choice of the source and the observation points, the size of the time interval in the numerical Fourier transform, and the size and dimensions of the computational window together with the type of “non-reflecting” conditions on its boundary. All this results in the fact that the FDTD technique accuracy in the mode analysis is low, and the modes with Q-factors higher than 10^4 are practically inaccessible at all [103-105].

In view of the above mentioned circumstances it is not a surprise that last years the progress in the modeling of microresonators is linked to departure from rough analytical and numerical methods and switch to the full-wave methods based on the integral equations (IEs).

Integral equations of electromagnetic theory make, as known, two large classes: volume IEs and boundary IEs [107]. The first have the advantage of being applicable to the resonators with both homogeneous and heterogeneous material. However, all 3-D volume IEs and also 2-D volume IEs in the H-polarization case are hyper-singular that makes conventional discretization schemes non-convergent and limit the use of these IEs. To overcome this demerit, in [108] there was proposed a regularization method based on the inversion of the part of IE operator corresponding to the elementary circular cell, for the H-polarization IE. In contrast, in the case of the E-polarization volume IEs have only logarithmic singularities and can be easily computed with a convergent [109]. It should be also noted that the method of volume IE combined with the perturbation method, has been used in [78] for the study of the splitting of the WG modes to doublets and spoiling of their Q-factors because of the disk rim roughness.

Boundary IEs are more attractive in view of their lower dimensionality. Although applicable only to the DRs with homogeneous (constant) refractive index, they can be easily reduced to the form free of strong singularities. A good example of the power of the boundary IEs can be recently published [110], where the true nature of the so-called bowtie resonances in the stadium cavity has been revealed.

At the same time, boundary IEs may suffer of a considerable drowback: many types of such IEs possess a spectrum of defect frequencies (spurious eigenvalues) [111], which are the natural frequencies of the inner electromagnetic problem where the boundary is assumed to be perfectly electrically conducting (PEC) and dielectric resonator is filled in with material of the outer medium (e.g. air). As a result, the eigenfrequency problem spectrum contains not only the complex-valued frequencies of the true modes of dielectric cavity but also the real-valued spurious frequencies. The latter do not correspond to any modes but are the poles of the condition number of IE. Such a method was proposed in [112] and was further used for the computation of modes in the hexagonal [113], stadium-like [114], smoothed triangular [115] and spiral dielectric resonators [116].

However, the mentioned above defect has enabled researchers to study only the modes with low and medium values of Q-factors. Even more, in the work [112] it was erroneously stated that the modes with high Q-factors are out of interest when modeling the lasers. Other forms of the boundary IEs, also having defect frequencies, were used in [117,118]. The negative effect of spurious eigenvalues can be decreased - but cannot be eliminated – if the discretization scheme of such IE is built using the analytical regularization procedure. Such an algorithm has been applied in [119,120] for systematic analysis of high-Q whispering-gallery modes in the ring-like and elliptic dielectric resonators placed into a layered host medium.

In line with all mentioned above, from the mathematical point of view the most attractive form of the boundary IEs are the so-called **Muller equations** [121]. This is a set of two coupled IEs in 2-D case and four coupled IEs in 3-D case. Muller IEs are, first, fully equivalent to the original boundary-value problem for the Maxwell equations and therefore have no spurious eigenvalues (defect frequencies), second, they contain only integrable and smooth kernel functions and hence are the Fredholm equations, and, third, they are the second-kind equations. They can be discretized by the method of collocations [122], meshing the contour of integration and using the locally-constant basis functions. One can also build a Galerkin scheme with projection onto global expansion functions, for instance, on the trigonometric polynomials [123]. The Fredholm second-kind nature of the Muller IEs guarantees the convergence of both numerical algorithms; however the rate of actual convergence depends on the smoothness of the resonator contour. According to the results of [123], the order of truncation of the matrix equation is determined, in almost equal manner, by the three parameters.

This is the maximum size of dielectric resonator, in terms of the wavelength in the dielectric material, the peak curvature of the contour, and the desired accuracy (the number of correct digits in the solution) with a coefficient that is inverse proportional to the smoothness of the contour. Therefore the published papers where the authors neglect the last two parameters and blindly rely on the empirical rule of having "ten points per lambda" contain *a priori* inaccurate numerical results. Besides of the mentioned, Muller IEs can be also discretized using interpolation Nystrom-type schemes [124], however their rate of convergence also depends on the contour smoothness [125].

It should be noted that the method of Muller IEs can be viewed as related to the **methods of analytical regularization** where explicitly inverted part is the part corresponding to the small contrast between the materials of the resonator and the host medium. This method has been successfully applied to the analysis of perturbed whispering-gallery modes in elliptic dielectric resonators [123], circular resonators with periodic corrugations [126] and with a notch [127], and rounded triangular and square dielectric resonators [128].

1.4 Modeling of open resonators with active regions

Any laser is a complicated device whose work is based on several physical principles and phenomena, the main of which are three. This is the dynamics of charge carriers and photons, the heating effects, and the electromagnetic field confinement [129]. All these mechanisms are linked together in nonlinear and non-stationary manner. Therefore a development of models and algorithms able to make comprehensive analysis of a laser is a formidable task. Still at the very onset of laser research the scientists have come to understanding that large amount of useful and interesting information can be obtained if one neglects all non-electromagnetic phenomena and consider the problem of existence of time-harmonic natural electromagnetic field in the open resonator of laser.

In the overwhelming majority of publications, such consideration has been reduced to the analysis of natural modes of *passive* dielectric resonators, see subsection 1.3. From the mathematical point of view, this implied solution of eigenvalue problems for the natural frequencies, which can be only complex-valued in an open resonator. Within this approach, the modes with the largest values of Q-factors have been usually associated with lasing.

However, it is easy to see that the model of the passive dielectric resonator fails to provide an adequate description of the effect of emission of non-attenuated waves by a laser. Although it is able to provide certain understanding of the natural mode frequencies, a well-known from the experiments fact of existence of lasing threshold remains beyond its framework. The consequences of such inadequacy are more than considerable: it is enough to remind that the passive-cavity model is not able to explain why a stadium-shaped cavity starts lasing on the bowtie modes [73] instead of the whispering-gallery modes that have much higher Q-factors. To explain this fact, researchers usually dwell on the development of nonlinear “hot” laser models [130]. Intuitively, it is clear that inadequacy is connected to the lack, in the passive-cavity model, of such fundamental feature of laser as *active region*, where the external pumping creates inversed population of carriers able to emit the light.

Meanwhile, the presence of active material in the whole laser resonator or in its part can be characterized in terms of the macroscopic parameters of material: to this end, it is enough to introduce into consideration a complex refractive index (or dielectric permittivity) with the sign of the imaginary part that corresponds to “negative absorption” or material gain [54]. Mathematically, each of the natural eigenfrequencies in arbitrary open dielectric resonator is an analytic or piece-analytic function of the complex refractive index. This follows from the possibility of reducing the corresponding boundary-value problems to the Fredholm second-kind IE (foe example, of the Muller type [121]) and using the operator extensions of the Fredholm theorems [131]. However, in this case the complex natural frequencies of dielectric resonator are allowed to take purely real values at the certain values of the negative imaginary part of refractive index or dielectric permittivity, i.e. gain.

In Fig. 1.8, it is shown a migration of a natural frequency on the complex plane and its crossing of the real axis when the absolute value of the gain or imaginary part of "active" refractive index grows. The frequency is marked with the index j to emphasize the discreteness of the natural frequencies. Here, the symmetry of the natural frequencies with respect to the imaginary axis is taken into account (this is the consequence of the identity of the positive and negative directions in time dependence, for the harmonic processes), as well as the possibility of the brunch-cut in the case of 2-D problems. If $\mathrm{Im}\,\nu = \mathrm{Im}\sqrt{\varepsilon} \geq 0$ and dependence on time is : $e^{-i\omega t}$, then all complex natural frequencies are located strictly in the lower halfplane.

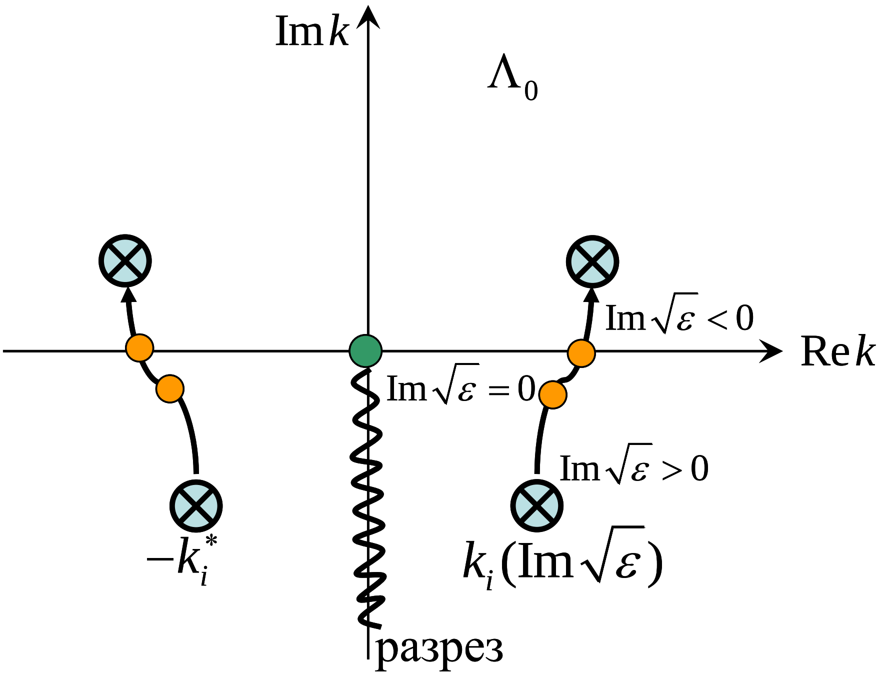


Fig. 1.8. Trajectory of migration of a natural frequency on the complex plane under the variation of the imaginary part of refractive index, $\mathrm{Im}\,\nu$ (or, equivalently, of the dielectric permittivity) from positive to negative values.

The lowering of the material losses given by $\mathrm{Im}\,\nu > 0$ to zero ($\mathrm{Im}\,\nu = 0$) lifts up every natural frequency to the position corresponding to the purely radiation losses of this mode. If the refractive index varies further and obtains a negative imaginary part, $\mathrm{Im}\,\nu < 0$, then the natural frequency is allowed to raise further and even cross the real axis (see Fig. 1. 8)

Thus, one may look for the certain (real) value of the averaged material gain per unite volume, given by the imaginary part of refractive index, $\mathrm{Im}\,\nu$ (or dielectric permittivity, $\mathrm{Im}\,\varepsilon = \mathrm{Im}(\nu^2) = 2\,\mathrm{Re}\,\nu\,\mathrm{Im}\,\nu$), which provides the real value of the considered natural frequency of the open resonator. Such a value can be viewed as material-gain threshold. Here, it is clear that generally speaking this value of $\mathrm{Im}\,\nu$ will be different for different natural frequencies. Therefore this procedure can be applied systematically to all complex natural frequencies by modifying the eigenvalue problem in such a sense that the mentioned value of $\mathrm{Im}\,\nu$ (the gain threshold) becomes a part of every modal eigenvalue together with the frequency.

One can see certain similarity of such a modified eigenvalue problem to the so-called method of generalized natural oscillations [132], where the frequency was a fixed parameter and the eigenvalues were sought in terms of the complex dielectric permittivity.

It is necessary to note that the connection of the complex refractive index (in particular, its imaginary part) with the pump power and the density of injected carriers can be determined using the two-liquid quantum model of active medium – see [133].

One should be reminded that the idea of description of the active material or medium as those able not to absorb the waves but, contrary, emit them with the aid of the properly assigned sign of the imaginary part of refractive index or dielectric permittivity has existed since long ago (see, for instance, page 525 of [54]). In the end of the 1970s, an intensive discussion emerged on the anomalous effects at the electromagnetic wave scattering by the active particles have negative absorption. However, it terminated quickly after it had been revealed that such a formulation might lead to the loss of uniqueness of solution to the scattering problem caused by the coincidence of the incident wave frequency (real value) with the natural frequency of the active particle [134].

Shortly before that, the authors of the paper [135] found, at first, the complex natural frequency of an open resonator containing a dielectric rod and then proposed to look for the value of $\mathrm{Im}\,\varepsilon$, which could bring its imaginary part to zero. This value was associated to the threshold of self-excitation. Within the next 30 years the same idea has been applied several times to the investigation of several different laser configurations [136-141]. However, the formulation of the problem on the electromagnetic field in open resonator has been remaining traditional (i.e. the complex eigenfrequencies were sought) and the threshold of emission has never been considered as an element of the eigenvalue.

This situation can be apparently explained by the observation that for a very long time the laser resonators remained the devices having dimensions measured in the thousands of wavelengths of emitted radiation. Therefore the dominant theory for their design was Geometrical Optics reduced to the calculation of the optical path of the ray along a closed trajectory, normally in the 1-D model of Fabry-Perot type.

However, starting from the 1980s, an active research into and then the development and manufacturing of the semiconductor light-emitting diodes and lasers with the Bragg resonators started. They had dimensions measured in the hundreds of wavelengths. Eventually, in the 1990s there appeared the first semiconductor microdisk lasers whose diameter counted just several wavelengths. These lasers have the shape of circular or non-circular thin disks standing on pedestals or laying on the less optically dense substrates. Their diameters can be from several units to dozens of micrometers. The disk contains an active region: one or several cascaded quantum wells or layers of quantum dots. The pumping of active region is performed either optically or by injection of current directly from metal electrodes. It was the task of reliable modeling of such miniature sources of electromagnetic waves that has called to re-consider the formulation of the eigenvalue problem for the field in an open resonator with an active region.

Such a modified formulation of the field eigenvalue problem able to take into account adequately the presence of the active region and, as a consequence, enable one to determine the thresholds of lasing and link them to the properties of the discrete spectrum of the natural modes of the open resonators, was published by us in [1] and presented in greater details in [2]. In this formulation, the active region is a part of the open resonator volume filled with the dielectric material having negative losses (i.e. the gain), characterized with the imaginary part of refractive index, $\mathrm{Im}\nu = -\gamma, \quad \gamma > 0$ under the time dependence $e^{-i\omega t}, \quad k = \omega / c$, where c is the free-space light velocity. At the boundary of the active region, one has to introduce additional set of conditions demanding the continuity of the field tangential components. Contrary to the complex-frequency problem, here the condition at infinity can be taken as the Sommerfeld condition of radiation (if the problem has been reduced to 2-D one after using the effective refractive index) or the silver-Muller condition (in 3-D). The aim is to find such ordered pairs of real-valued numbers (k,γ), which generate non-zero functions $\{\vec{E},\vec{H}\}$ satisfying the Maxwell equations with the mentioned above conditions.

Important is to emphasize that the modified in this manner eigenvalue problem remains a linear problem in the sense that it does not involve non-linear dependences of material parameters of resonator on the field amplitudes. Nevertheless, such a modification happens to be sufficient for determination of not only the frequencies but also the thresholds of emission. Moreover, it enables one to bring into consideration the effect of several new factors such as location, size, and shape of the active region, on the natural-mode fields, their frequencies and thresholds. In the frames of the passive-cavity model this is not possible. As it has been pointed out to in subsection 1.2, for the injection lasers this directly opens the way of modeling the shape and location of the electrodes that is crucially important for the design of sources with low thresholds of pump power.

The presented above Lasing Eigenvalue Problem (LEP) has been implemented by us for systematic research into the lasing frequencies and thresholds of natural modes in 2-D models of several types of lasers: microdisk laser with uniform [1] and radially inhomogeneous active region [2], pair of coupled active disks [3], cyclic photonic molecules composed of active disks [4,5], active disk inside a passive ring or a Bragg reflector [6], and active spiral microcavity laser [7]. A part of these studies has been also discussed in the invited review paper [8].