A Predictor-Corrector Algorithm with an Increased Range of Absolute Stability

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Abstract. A new predictor for use with the Adams-Moulton corrector has been developed. Truncation errors at each step are determined, to first order, solely by the characteristics of the corrector. Likewise, the propagation of error in the evaluation of definite integrals is dependent only on the corrector equation. (The only purpose of the predictor here is to form an error estimate.) The predictor equation and the corrector equation are independently and jointly of the fourth order. The predictor equation developed here is believed to have the largest range of absolute stability (including $\underline{h} = 0$) for the combined predictor-corrector algorithm that is possible. At the same time the method has a range of relative stability which will maintain stable propagation of relative errors when truncation errors of less than one part in one thousand are being incurred. The storage required for previous derivative values is no greater than that for the standard Adams-Moulton method with the Adams-Bashforth predictor.

I. *Introduction*

Predictor-corrector algorithms for the numerical solution of ordinary differential equations are of considerable interest because of their relative economy in terms of the number of derivative evaluations required for a given order algorithm. In the commonly used version where a single application of the predictor formula is followed by a single application of the corrector formula, two derivative evaluations are required for one step forward. This compares with four derivative evaluations in a fourth-order Runge-Kutta algorithm for one step forward, for example.

In applying a predictor-corrector algorithm, spurious solutions of the difference equation may be generated which bear no direct relation to the differential equation being solved. This phenomenon is known as numerical instability. It limits the interval of integration which may be used. Initial efforts to produce algorithms having increased ranges of numerical stability limited their considerations to the corrector formula [1, 2]. While this would be appropriate if the corrector formula were applied iteratively to convergence, substantial differences in behavior occur when the corrector formula is applied only once [3].

In this paper a predictor formula is developed which when used with one application of the Adams-Moulton corrector leads to optimal absolute stability characteristics. The range of absolute stability for a single first order differential equation is nearly doubled. Although in the optimization process considerations were limited to the range of absolute numerical stability for a single differential equation, the resultant algorithm has favorable properties from other points of view as well.

II. *Development of the Algorithm*

As a starting point, a generalized predictor-corrector formula is considered:

$$\begin{aligned} p_{n+1} &= a_1 y_n + b_1 y_{n-1} + c_1 y_{n-2} + d_1 y_{n-3} + h(e_1 y_n' + f_1 y_{n-1}' + g_1 y_{n-2}' + k_1 y_{n-3}'), \\ y_{n+1} &= a_2 y_n + b_2 y_{n-1} + c_2 y_{n-2} + h(d_2 p_{n+1}' + e_2 y_n' + f_2 y_{n-1}' + g_2 y_{n-2}'). \end{aligned} \tag{1}$$

The behavior of this algorithm when applied to the numerical solution of

$$y' = f(x, y) \tag{2}$$

is to be studied. The definitions

$$\begin{aligned} y_n' &= f(x_n, y_n) \\ p_{n+1}' &= f(x_{n+1}, p_{n+1}) \end{aligned} \tag{3}$$

are implicit in the equations (1).

The form of (1) is chosen to involve the storage of the same number of back derivative values as many commonly used fourth order predictor-corrector algorithms such as Adams-Bashforth/Adams-Moulton, for example [4, pp. 192–199]. The parameters of (1) could be varied to construct an algorithm of maximum order. However, such an algorithm would be numerically unstable for all intervals of integration [5]. Instead the algorithm (1) will be specified to be a fourth order algorithm and the additional parameters will be selected to maximize the range of absolute numerical stability.

In fact, it will be specified that the predictor and corrector formulas be independently of fourth order. Then the leading truncation error term of the combined algorithm (1) is identical with that of the corrector alone [4, p. 261]. In addition, $p_n - y_n$ will then furnish a convenient estimate of the truncation error at each step. This leaves five free parameters in the algorithm (1); three in the predictor and two in the corrector. These are selected as d_1, e_1, k_1, d_2 and e_2. The following relations hold for the remaining coefficients:

$$\begin{aligned} a_1 &= 9 - d_1 - 3e_1 + 3k_1, & a_2 &= 9 - 15d_2 - 3e_2, \\ b_1 &= 9 - 9d_1 + 24k_1, & b_2 &= 9 - 24d_2, \\ c_1 &= -17 + 9d_1 + 3e_1 - 27k_1, & c_2 &= -17 + 39d_2 + 3e_2, \\ f_1 &= -18 + 6d_1 + 4e_1 - 17k_1, & f_2 &= -18 + 39d_2 + 4e_2, \\ g_1 &= -6 + 6d_1 + e_1 - 14k_1, & g_2 &= -6 + 14d_2 + e_2. \end{aligned} \tag{4}$$

The error term for the combined predictor-corrector algorithm (1) is given to first order by

$$E_{pc} = \tfrac{1}{30}[9 - 24d_2 - e_2]h^5 y^{(v)}(x). \tag{5}$$

When the characteristic polynomial for (1) is obtained in the usual way [3, pp. 459–460], it is found that

$$\rho^4 + q_3\rho^3 + q_2\rho^2 + q_1\rho + q_0 = 0, \tag{6}$$

where

$$q_3 = (-9 + 15d_2 + 3e_2) + \underline{h}[(-9 + d_1 + 3e_1 - 3k_1)d_2 - e_2] + \underline{h}^2(-d_2e_1),$$

$$q_2 = (-9 + 24d_2) + \underline{h}[(-48 + 9d_1 - 24k_1)d_2 + 18 - 4e_2] + \underline{h}^2[(18 - 6d_1 - 4e_1 + 17k_1)d_2],$$

$$q_1 = (17 - 39d_2 - 3e_2) + \underline{h}[(3 - 9d_1 - 3e_1 + 27k_1)d_2 + 6 - e_2] + \underline{h}^2[(6 - 6d_1 - e_1 + 14k_1)d_2],$$

$$q_0 = \underline{h}(-d_1d_2) + \underline{h}^2(-d_2k_1).$$

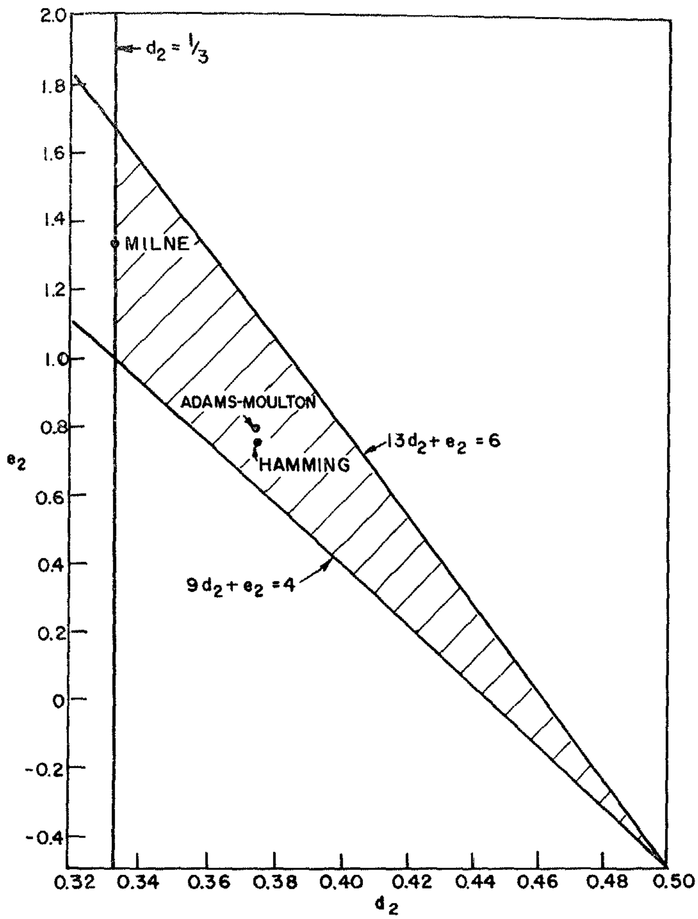

FIG. 1. Range of d_2 and e_2 for $|\rho_i| \leqq 1$, $i = 1, 2, 3, 4$ at $\underline{h} = 0$

The above is derived under the usual assumptions that the truncation error as well as the partial derivative of $f(x, y)$ (equation (2)) with respect to y is constant from point to point. The quantity $\underline{h}$ is defined by

$$\underline{h} = h \frac{\partial f}{\partial y}. \tag{7}$$

When $\underline{h} = 0$, equation (6) factors into

$$\rho(\rho - 1)[\rho^2 + (-8 + 15d_2 + 3e_2)\rho + (-17 + 39d_2 + 3e_2)] = 0. \tag{8}$$

The roots of (6) and (8) determine the numerical stability of the algorithm (1). In this paper, an algorithm will be called *initially stable* if each root of (8) has magnitude less than or equal to one and every root of unit magnitude is simple. An algorithm will be called *absolutely* $\underline{h}^*$ *stable* if $\underline{h}^*$ is the least value for which the roots of (6) satisfy these same conditions for

$$\underline{h}^* < \underline{h} \leqq 0. \tag{9}$$

It is clear that a necessary condition for an algorithm to be absolutely $\underline{h}^*$ stable is that it be initially stable. An algorithm will be called *relatively stable* in an interval of the $\underline{h}$-axis if all of the roots of (6) are equal to or less than exp ($\underline{h}$) in absolute value and roots equal to exp ($\underline{h}$) are simple for every $\underline{h}$ within the interval.

Relative stability is often appropriate when solving differential equations which arise from physical systems with no forcing functions. In such cases, the solution "damps out" (or grows) as x increases. When, however, the physical system has a periodic forcing function, absolute stability is important. In particular, when the physical system is driven by a forcing function with a period that is much longer than the time constants of the system, the absolute stability of the numerical method will limit the solution. As shown below, it is possible to derive a predictor formula for use with an Adams-Moulton corrector formula which gives a numerical method with nearly twice the range of absolute stability while retaining satisfactory relative stability characteristics.

It is desired to find a combination of the five parameters d_1, e_1, k_1, d_2 and e_2 which will provide an algorithm (1) which is absolutely $\underline{h}^*$ stable and such that $\underline{h}^*$ is minimized. It is to be noted that $\underline{h}^*$ cannot be negatively infinite since at least one of the roots of (6) must approach infinity as $\underline{h}$ becomes negatively infinite.

It is first observed that the consideration of initial stability limits the (d_2, e_2) pairs that may be used through equation (8). There will be one root

$$\rho = +1 \qquad \text{when } 9d_2 + e_2 - 4 = 0, \tag{10}$$

and one root

$$\rho = -1 \qquad \text{when } 3d_2 - 1 = 0. \tag{11}$$

When the two roots are a complex pair of unit magnitude,

$$13d_2 + e_2 - 6 = 0. \tag{12}$$

These three straight lines define a region in the (d_2, e_2) plane within which initially stable methods must be contained. This is illustrated in Figure 1. Points show the locations of a number of well-known algorithms. It is to be noted that this restriction is on the corrector coefficients only. The predictor coefficients are not involved as they do not affect the initial stability of the algorithm (1). The truncation error increases as one moves from left to right across this region in accordance with (5).

A gradient technique was developed for increasing the range of absolute stability. This technique is sketched below, but some of the more voluminous details are omitted. It is convenient to consider three cases. Let $\mathbf{x}$ denote the vector $(d_1, e_1, k_1, d_2, e_2)$ of five parameters. If the limiting value of ρ is $+1$, then from (6)

$$F_1(\underline{h}, \mathbf{x}) = 1 + q_3 + q_2 + q_1 + q_0 = 0. \tag{13}$$

If the limiting value of ρ is -1, then again from (6)

$$F_2(\underline{h}, \mathbf{x}) = 1 - q_3 + q_2 - q_1 + q_0 = 0. \tag{14}$$

Finally, if the limiting value of ρ is a complex conjugate pair of unit magnitude, division of (6) by the quadratic factor $\rho^2 - 2 \cos \phi\rho + 1$ leads to

$$F_3(\underline{h}, \mathbf{x}) = (q_3 - q_1)^2 + q_3(q_3 - q_1)(q_0 - 1)$$
$$+ (q_2 - q_0 - 1)(q_0 - 1)^2 = 0, \quad \left|\frac{q_3 - q_1}{q_0 - 1}\right| \leqq 2 \tag{15}$$

being a required additional condition. (13) leads to a linear polynomial in $\underline{h}$, (14) is a quadratic polynomial in $\underline{h}$, and (15) is a sixth degree polynomial in $\underline{h}$. For a given point in parameter space, these three polynomials are solved for their roots in $\underline{h}$. The least negative real $\underline{h}$ determines $\underline{h}^*$ which limits the range of absolute stability. Positive real and complex roots are ignored for this purpose. For example, in the case of the Adams-Bashforth predictor and Adams-Moulton corrector, (15) has a negative real root of -1.285 which determines $\underline{h}^*$.

Now the gradient of $\underline{h}^*$ in parameter space can be formed by implicit differentiation of (13–15). Thus,

$$\nabla_j(\underline{h}^*) = \left(\frac{\partial \underline{h}}{\partial d_1}, \frac{\partial \underline{h}}{\partial e_1}, \frac{\partial \underline{h}}{\partial k_1}, \frac{\partial \underline{h}}{\partial d_2}, \frac{\partial \underline{h}}{\partial e_2}\right)\Bigg|_{\underline{h}=\underline{h}^*} \quad j = 1, 2, 3, \tag{16}$$

depending on which condition is limiting in a given case. It should be mentioned that much of the involved algebra necessary in the above can be carried out numerically in a computer program. This results in substantial simplification of the process.

An attempt was first made to use this gradient in the 5-parameter space, starting at the point determined by Hamming's method. The direction of the gradient was such that the constraint $9d_2 + e_2 = 4$ soon limited further progress. One can show that when this constraint is satisfied, a double root $\rho = +1$ exists when $\underline{h} = 0$. One of these roots is approximately equal to $e^{\underline{h}}$. In order to insure

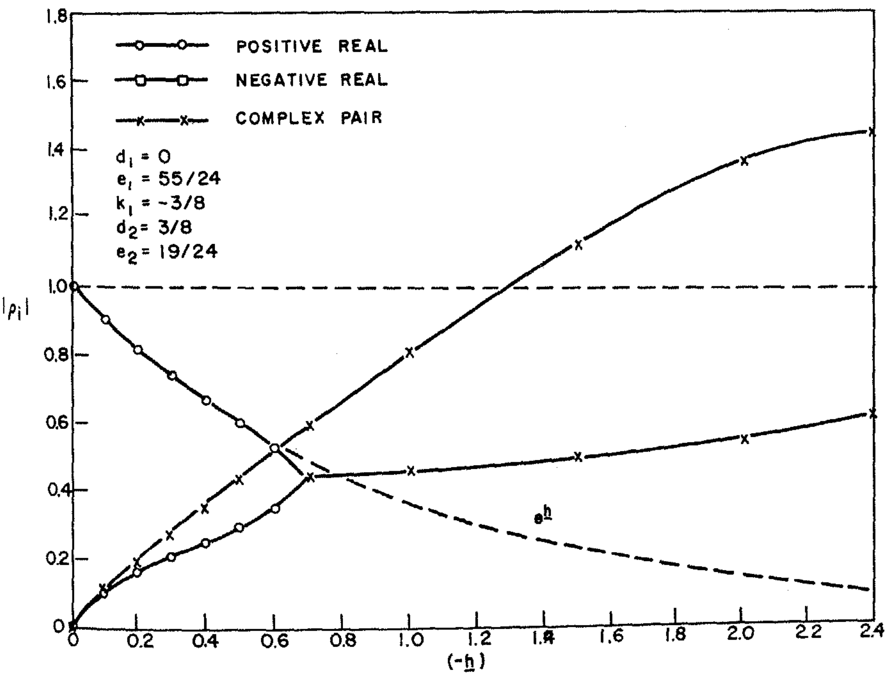

FIG. 2. $|\rho_i|$ vs. $(-\underline{h})$ for Adams-Moulton method

some range of absolute stability, the slope of the second root, $d\rho/d\underline{h}_{\rho=1,\underline{h}=0}$, must be constrained to be non-negative. This results in the condition $4 - 2d_1 - e_1 + 5k_1 \geqq 0$ when $d_2 \neq \frac{1}{2}$. This condition was not satisfied by the point in the parameter space to which use of (16) had led. To satisfy the condition, the point of intersection of the gradient and the hyperplane $4 - 2d_1 - e_1 + 5k_1 = 0$ was taken. Investigation of the effect of requiring that the conditions $9d_2 + e_2 = 4$ and $4 - 2d_1 - e_1 + 5k_1 = 0$ hold simultaneously showed that if there is a root $\rho = 1$ for some $\underline{h} \neq 0$, then there is a constant root $\rho = 1$. This, of course, is undesirable since no range of relative stability could exist.

To avoid complications such as those just described and to insure that the methods under study would have some range of relative stability, it was decided to fix the corrector formula, thus reducing the number of parameters to three. The Adams-Moulton corrector was selected primarily because, for it, the three extraneous roots of (6) equal zero when $\underline{h} = 0$. Other correctors could, of course, be used. It is doubtful in the opinion of the authors, however, that a significantly better algorithm than the one developed here would result when the factors of absolute stability, relative stability, truncation error and performance for systems of equations are all considered, as is done in this paper.

The starting point for the optimization procedure was chosen as $(d_1, e_1, k_1) = (0, \frac{55}{24}, -\frac{3}{8})$, which corresponds to the Adams-Bashforth predictor [4, pp. 192-194]. The stability characteristics of the Adams-Bashforth, Adams-Moulton algorithm are illustrated in Figure 2. It is absolutely $\underline{h}^*$ stable with $\underline{h}^* = -1.285$ and relatively stable for $\underline{h}$ greater than about -0.6. In all subsequent references

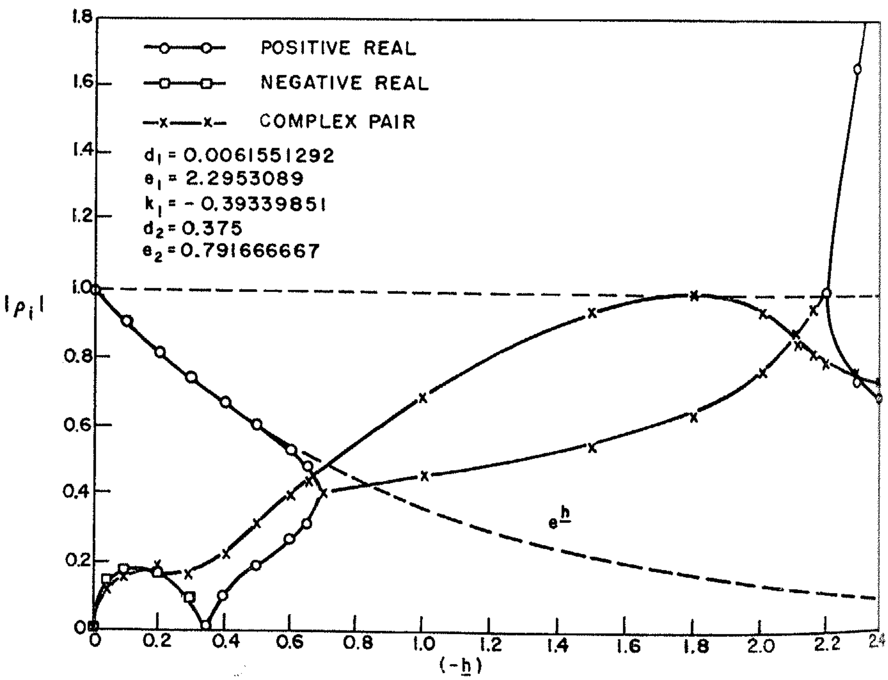

FIG. 3. $|\rho_i|$ vs. $(-\underline{h})$

to the gradient vectors (16) it is assumed that the last two components are se equal to zero.

In the case at hand, the range of absolute stability is limited by a pair of con plex conjugate roots of unit magnitude. In parameter space, one takes

$$\Delta\mathbf{x} = -C\nabla_3(\underline{h}^*), \tag{17}$$

with C a positive real constant. This increases the range of absolute stabilit This process can be continued until some other condition becomes limitin When two conditions are simultaneously limiting, say F_1 and F_2, one ma continue by constructing a vector $\Delta\mathbf{x}$ in the plane of ∇_1 and ∇_2 such that

$$\begin{aligned} \Delta\mathbf{x}\cdot\nabla_1(\underline{h}^*) &= \Delta\mathbf{x}\cdot\nabla_2(\underline{h}^*), \\ \Delta\mathbf{x}\cdot(\nabla_1 - \nabla_2) &= 0. \end{aligned} \tag{18}$$

In this way, one arrives at the situation indicated in Figure 3. In moving $\underline{h}$ out to about -2.19, an intermediate pair of complex roots has become equal t unity in magnitude. Any further attempt to decrease $\underline{h}^*$ will cause them t exceed unity and limit the range of absolute stability. However, the intermediat extremum may be depressed without changing $\underline{h}^*$ by moving in a direction pe pendicular to the plane defined by ∇_1 and ∇_2, that is,

$$\Delta\mathbf{x} = C\nabla_1 \times \nabla_2. \tag{19}$$

Then $\underline{h}^*$ may again be decreased. By continuing this process one is finally led t the situation shown in Figure 4 with $\underline{h}^* = -2.481$. A plot over an expande range of $\underline{h}$ is shown in Figure 5. Here repeated roots of plus one and a tange complex pair of unit magnitude occur simultaneously. No further improveme

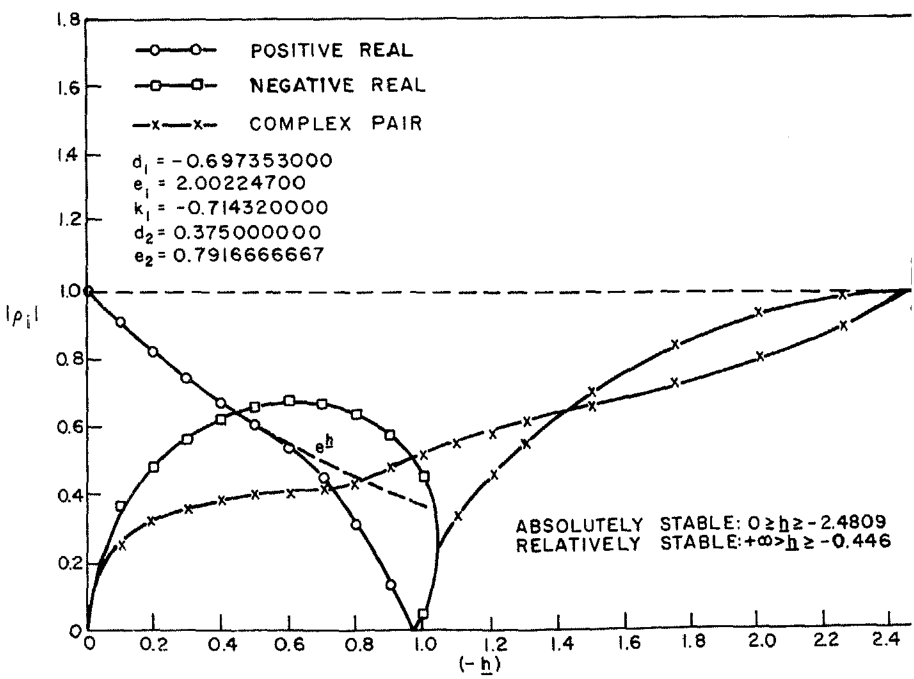

FIG. 4. $|\rho_i|$ vs. $(-\underline{h})$

seems possible, and this is believed to represent the predictor having optimal absolute stability characteristics when used with the Adams-Moulton corrector.

The complete set of coefficients representing this method are

$$\begin{aligned}
a_1 &= 1.54765200, & a_2 &= 1.00000000,\\
b_1 &= -1.86750300, & b_2 &= 0,\\
c_1 &= 2.01720400, & c_2 &= 0,\\
d_1 &= -0.697353000, & d_2 &= 0.375000000,\\
e_1 &= 2.00224700, & e_2 &= 0.791666667,\\
f_1 &= -2.03169000, & f_2 &= -0.208333333,\\
g_1 &= 1.81860900, & g_2 &= 0.0416666667.\\
k_1 &= -0.714320000, & &
\end{aligned} \tag{20}$$

An error estimator is given at each step of the application of (1) through

$$E_n \cong \frac{p_n - y_n}{16.21966}. \tag{21}$$

This can be used for online modification of the interval of integration to retain a prescribed accuracy if desirable. The trailing zeros introduced in the predictor coefficients of (20) are for the purpose of making the predictor as nearly "numerically" fourth order as possible. The relations (4) are satisfied "numerically" by these numbers.

The truncation error of the combined algorithm (1) is given by (5) and is clearly identical with that of the Adams-Moulton corrector formula alone.

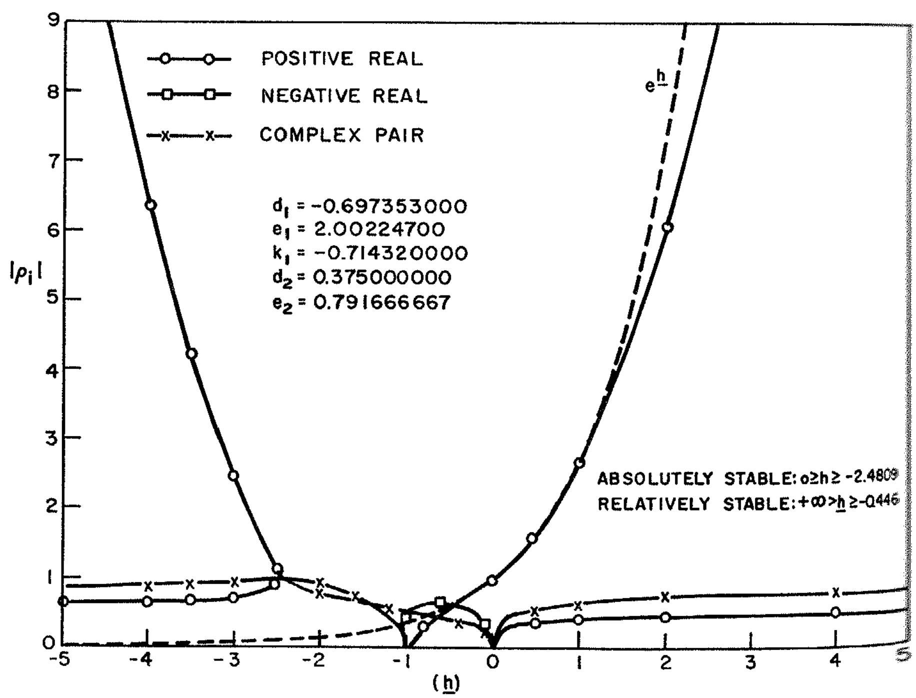

FIG. 5. $|\rho_i|$ vs. $(\underline{h})$

III. *Systems of Differential Equations*

The optimization of the algorithm (1) has been carried out with respect to the solution of a single differential equation (2) only. In point of fact, one is seldom interested in solving a single first order differential equation, but rather systems of many coupled first-order equations (perhaps hundreds!). In order to judge the algorithm derived it is necessary to study its characteristics in solving systems of differential equations. It turns out that there is a reasonably simple relation between the algorithm's performance for a single differential equation and systems of equations [7].

Consider the system of differential equations

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}), \tag{22}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ and $\mathbf{f} = (f_1, f_2, \cdots, f_n)$.

If the incremental variables

$$\xi = x - x_o, \qquad \eta_i = y_i - y_{io} \tag{23}$$

are introduced and the equations are linearized, the system (22) may be written as

$$\frac{d\mathbf{n}}{d\xi} = \mathbf{c} + \xi\mathbf{b} + A\mathbf{n}, \tag{24}$$

where the elements of the matrix A and the vectors $\mathbf{b}$ and $\mathbf{c}$ are

$$a_{ij} = \frac{\partial f_i}{\partial y_j}, \qquad b_i = \frac{\partial f_i}{\partial x}, \qquad c_i = f_i(x_o, \mathbf{y}_o).$$

This system of linear differential equations can now be diagonalized through a change of dependent variable

$$\mathbf{z} = P\mathbf{n}. \tag{25}$$

The system (24) then becomes

$$\begin{aligned} \frac{d\mathbf{z}}{d\xi} &= P\mathbf{c} + \xi P\mathbf{b} + PAP^{-1}\mathbf{z}, \\ &= P\mathbf{c} + \xi P\mathbf{b} + \Lambda\mathbf{z}, \end{aligned} \tag{26}$$

where the matrix Λ is the diagonal matrix of the eigenvalues of A. The exceptional cases where complete diagonalization is not possible are not considered for present purposes. The system of equations (24) has been decoupled by the transformation and now consists of a system of n uncoupled first-order equations.

Therefore, with a system of n coupled equations we consider a set of $\underline{h}_i$ given by

$$\underline{h}_i = \lambda_i h, \tag{27}$$

where the λ_i are the eigenvalues of the Jacobian matrix of the derivative functions with respect to the dependent variables. Ordinarily, there will be one of the eigenvalues which will limit the stability of the numerical solution.

In general, however, it is to be expected that these eigenvalues may be complex numbers so that in order to evaluate the algorithm (20) it will be necessary to investigate the behavior of the roots of (6) for complex $\underline{h}$ as well as real $\underline{h}$. The region in the complex $\underline{h}$ plane where all of the roots are less than or equal to one in absolute value and roots of unit magnitude are simple is the region within which the algorithm is absolutely stable.

This was done for the algorithm (20) and the results are displayed in Figure 6. The cross-hatching indicates the region of absolute stability. Figures 7, 8, 9 and 10 indicate similar results for four other well-known algorithms for solving differential equations. The characteristics for fourth-order Runge-Kutta algorithms are exhibited in Figure 10 even though such methods are not included in the class of algorithms considered in this paper. They were excluded from considera-

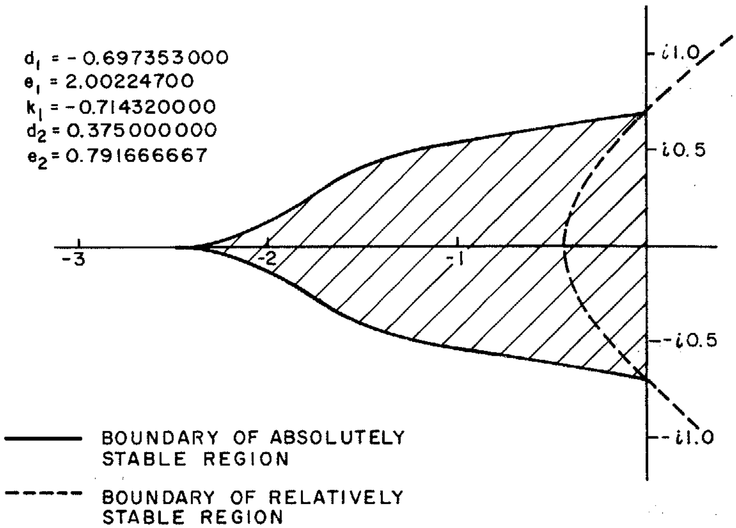

FIG. 6. Region of stability in complex $\underline{h}$ plane for new algorithm (20)

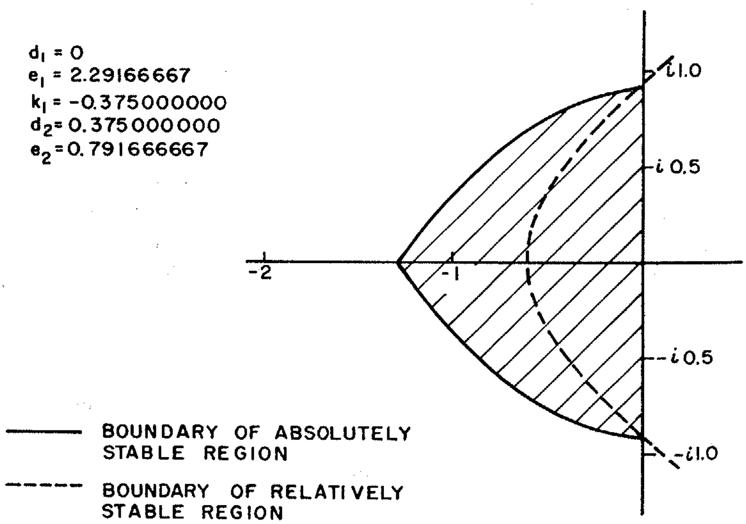

FIG. 7. Region of stability in complex $\underline{h}$ plane for Adams-Moulton method

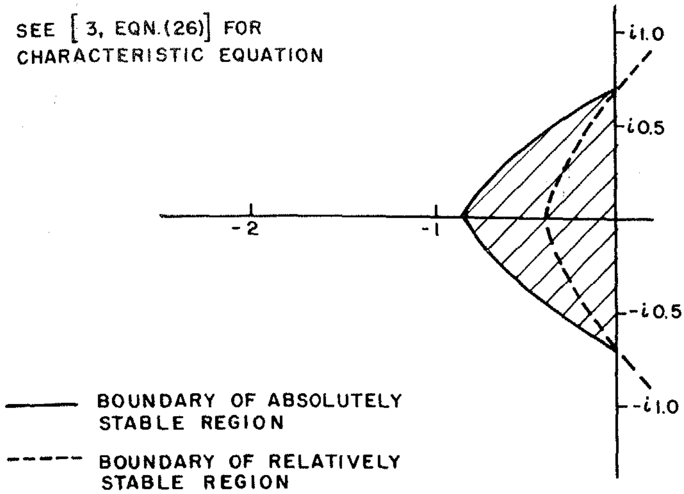

FIG. 8. Region of stability in complex $\underline{h}$ plane for Hamming modified method

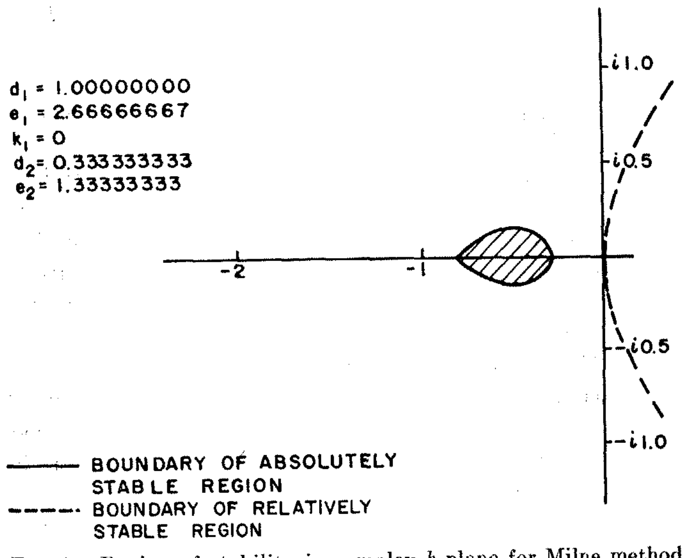

FIG. 9. Region of stability in complex $\underline{h}$ plane for Milne method

ion because they require twice as many derivative evaluations for the same order of accuracy and three times as many if an online error estimator is to be developed at each integration step. However, they do have acceptable stability characteristics as indicated in Figure 10. The imaginary axis is excluded from the region of absolute stability in Figures 6, 7 and 8. However, the boundary is so close to the imaginary axis that this cannot be shown to the scale plotted. It is to be noted that the regions of stability for both the Hamming method [1] and the Milne method [6] are completely contained in the stable regions for both the Adams-Moulton method and the present method.

As is to be expected, a certain amount of trade-off has occurred in optimizing the predictor for the Adams-Moulton corrector on the negative real axis. However, there has been surprisingly little loss of stable region in the complex plane.

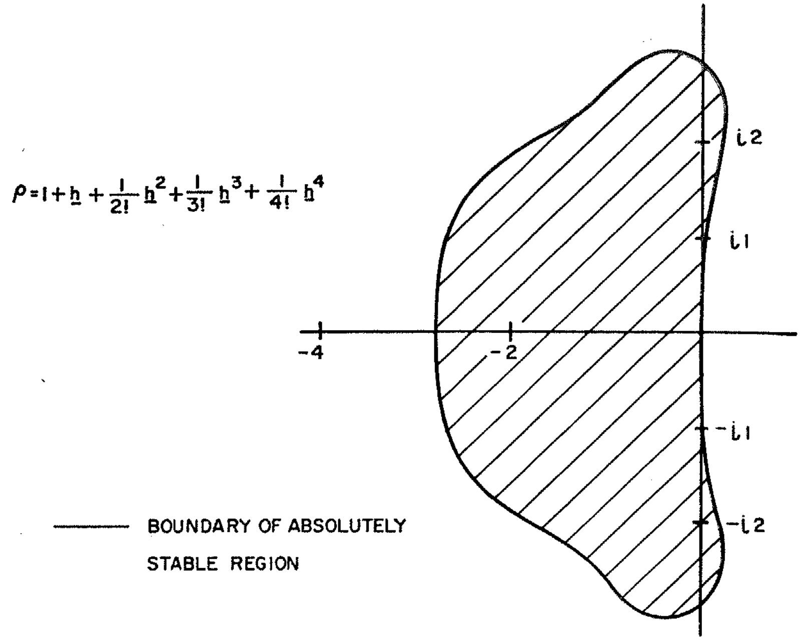

FIG. 10. Region of stability in complex $\underline{h}$ plane for Runge-Kutta fourth order method

On the imaginary axis, the limiting $\underline{h}$ has been reduced in magnitude from about 0.92 to 0.70. This 24 percent reduction on the imaginary axis compares to 92 percent increase on the negative real axis. It is worth noting that on the imaginary axis, the truncation error will become limiting before either algorithm is limited by numerical instability so that the slight reduction there is of no consequence. The boundaries of the absolutely stable regions for the two algorithms intersect at about $\underline{h} = -0.76 \pm i0.57$. In general, when the limiting eigenvalue has a phase angle less than 37 degrees removed from the negative real axis, the present algorithm will be superior to the standard Adams-Moulton method with the Adams-Bashforth predictor. When the limiting eigenvalue is near the negative real axis, the improvement will be substantial.

It is difficult to judge the class of "typical problems" with sufficient precision to make definitive statements about the overall efficacy of the two algorithms. However, it would appear that the present algorithm has substantial advantages over the standard Adams-Moulton method. The additional multiplications required in this algorithm are of little consequence on high-speed computers where derivative evaluations typically dominate the computation time.

IV. *Relative Stability and Accuracy*

In many applications it is relative errors and relative numerical stability which are the quantities of interest. This occurs typically in physical systems (linear or nonlinear) which are excited and allowed to decay to rest. On the real $\underline{h}$ axis, an algorithm is relatively stable when all of the roots of (6) are equal to or less than exp $(\underline{h})$ in magnitude and roots equal to exp $(\underline{h})$ in magnitude are simple. The algorithm (20) is relatively stable for real $\underline{h} \geqq -0.446$.

One might feel that it would be worthwhile to have a somewhat reduced range of absolute stability in order to increase the range of relative stability exhibited by the algorithm (20). The following discussion is included to show that such a procedure is not well motivated. Consider, for example, the numerical integration of the prototype differential equation

$$y' = -Ay, \tag{28}$$

with $y(0) = 1$, which has the closed form solution

$$y = \exp(-Ax). \tag{29}$$

From (5) the truncation error for a given interval of integration is given to first order by

$$\begin{aligned} E_{pc} &= -Kh^5A^5 \exp(-A\eta), \\ &= +K\underline{h}^5 \exp(-A\eta) \end{aligned} \tag{30}$$

where $K = -19/720$ for the Adams-Moulton corrector and η is an average value of x. For the purposes of the present discussion η is taken as the midpoint of the range of application of the algorithm, i.e., $\eta = x - h$. The relative error is given by

$$E_{pc}/y = K\underline{h}^5 \exp[-A(\eta - x)], \tag{31}$$

$$= K\underline{h}^5 \exp(-\underline{h}). \tag{32}$$

Thus, for $\underline{h} = -0.446$ the relative error will be about one part in a 1000. In other words, the algorithm (20) developed here will be relatively stable so long as intervals of integration are used which restrict the relative error introduced at each step to about one part in a 1000 or less. One would be unlikely to use a larger step size in problems for which the relative error is the quantity of principal interest. Thus, a further extension of the range of relative stability for the algorithm (20) would not be useful.

It may be well to indicate that no similar argument can be applied in regard to a suitable range of absolute stability of a numerical algorithm for the solution of ordinary differential equations. A suitable example to consider for this purpose is

$$y' = A - Ay, \tag{33}$$

with $y(0) = 1$, which has the closed form solution

$$y = 1 - \exp(-Ax). \tag{34}$$

Here it is the absolute error which is the quantity of interest. This absolute error is the same as that given by (30). As the solution proceeds the truncation error per step will decrease exponentially so that larger and larger intervals of integration would be permitted while maintaining a given absolute error per step. The absolute stability of the algorithm will ultimately be limiting, and the larger the range of absolute stability the more efficient the integration process will be.

For systems of ordinary differential equations, the values of $\underline{h}$ will in general be complex. Solutions corresponding to complex $\underline{h}$ in the left-half plane correspond to damped oscillatory functions. It is sensible in this case to consider errors relative to the envelope of such functions. It is also sensible to exempt from consideration that characteristic root which is equal to exp $[\underline{h}]$ through terms of order equal to the order of the method. Error growth due to this root will be at most comparable to the truncation error incurred at each step and, hence, not catastrophic. Thus, for complex $\underline{h}$ an algorithm is defined to be relatively stable if all of the roots of (6) except that one most nearly equal to exp $[\underline{h}]$ are less than or equal to exp $[R(\underline{h})]$ in magnitude where the symbol "R" denotes the real part of the complex number. In addition, roots equal to the limiting value in magnitude must be simple. The regions of relative stability in this sense are indicated by the dashed curves in Figures 6, 7, 8 and 9. The region consists of all of the area to the right of the dashed curves. An argument similar to that given above shows the region of relative stability for the algorithm (20) to be more than adequate for systems of ordinary differential equations.

V. *Conclusions*

Calculations with pilot differential equations have been carried out which verify all of the properties of the algorithm (20) which are described here. This has been done both for a single differential equation and for systems of differential equations for which the eigenvalues of the Jacobian matrix are complex. In addition, the algorithm has been productively used by the Applied Mathematics Group at RCA Laboratories for more than three years and found satisfactory in every way. It is currently incorporated in subroutines for a number of computers including the RCA 601 Computer System.

In summary, a new predictor for use with the Adams-Moulton corrector has been developed. Truncation errors at each step are determined, to first order, solely by the characteristics of the corrector. Likewise, the propagation of error in the evaluation of definite integrals is dependent only on the corrector equation. (The only purpose of the predictor here is to form an error estimate.) The predictor equation and the corrector equation are independently and jointly of the fourth order. The predictor equation developed here is believed to have the largest range of absolute stability (including $\underline{h} = 0$) for the combined predictor-corrector algorithm that is possible. At the same time the method has a range of relative stability which will maintain stable propagation of relative errors when truncation errors of less than one part in 1000 are being incurred. The storage required for previous derivative values is no greater than that for the standard Adams-Moulton method with the Adams-Bashforth predictor.

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