Ant Colony Optimization (ACO): Complete Description and Equations

1. Overview

Ant Colony Optimization (ACO) is a **probabilistic metaheuristic** inspired by the foraging behavior of real ants. In nature, ants deposit **pheromones** on paths between nest and food sources. Over time, shorter and better paths accumulate more pheromone, increasing the likelihood they'll be reused.

In ACO, this mechanism is used to solve **combinatorial optimization problems**, especially graph-based ones such as the **Traveling Salesman Problem (TSP)**, **vehicle routing**, **scheduling**, and others.

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2. Core Idea

Each artificial ant builds a candidate solution by moving along a graph. At each step, it uses:

- **Pheromone information** (τ): collective memory of previous good solutions
- **Heuristic information** (η): local problem-specific desirability (e.g., inverse of distance)

After constructing solutions, pheromones are updated:

- **Evaporation** decreases all pheromone values
- **Deposit** increases pheromones on edges used by good solutions

This process iterates until stopping criteria are met (max iterations, convergence, etc.).

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3. Probability of Selecting the Next Node

For an ant $k$currently at node i, the probability of moving to node $j$is:

$$P_{ij}^{(k)} = \frac{[\tau_{ij}]^{\alpha}[\eta_{ij}]^{\beta}}{\sum_{l \in \mathcal{N}_i^{(k)}} [\tau_{il}]^{\alpha}[\eta_{il}]^{\beta}}$$

Where:

- τ_{ij}= pheromone level on edge (i, j)
- η_{ij}= heuristic desirability, often $\eta_{ij} = \frac{1}{d_{ij}}$
- α= relative influence of pheromone
- β= relative influence of heuristic information
- $\mathcal{N}_i^{(k)}$= feasible next nodes for ant k

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4. Pheromone Update Equation

At each iteration, pheromones are first **evaporated** and then **deposited**.

General pheromone update rule:

$$\tau_{ij} \leftarrow (1-\rho)\tau_{ij} + \Delta\tau_{ij}$$

Where:

- $\rho \in (0,1]$= evaporation rate
- $\Delta\tau_{ij}$= total new pheromone added to edge (i,j)

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5. Pheromone Deposit

In the basic Ant System (AS), pheromone deposit is:

$$\Delta\tau_{ij} = \sum_{k=1}^{m} \Delta\tau_{ij}^{(k)}$$

Each ant $k$deposits:

$$\Delta\tau_{ij}^{(k)} = \begin{cases} \frac{Q}{L_k}, & \text{if ant } k \text{ uses edge } (i,j) \\ 0, & \text{otherwise} \end{cases}$$

Where:

- Q= constant
- L_k= total length (cost) of ant k's solution

Better solutions (lower cost) result in **stronger pheromone reinforcement**.

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6. Heuristic Information

The heuristic component typically uses domain-specific knowledge.
For TSP:

$$\eta_{ij} = \frac{1}{d_{ij}}$$

where $d_{ij}$is the distance between cities $i$and j.

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7. Algorithm Flow

Initialization

1. Set parameters α, β, ρ, Q
2. Initialize pheromone levels $\tau_{ij} = \tau_0$

Main Loop

For each iteration:

1. **Solution Construction**
   Each ant builds a complete solution by repeatedly applying the transition probability rule.
2. **Evaluation**
   Compute each solution's cost L_k.
3. **Pheromone Evaporation**
   Apply: $\tau_{ij} \leftarrow (1-\rho)\tau_{ij}$
4. **Pheromone Deposit**
   Add pheromone according to solution quality.
5. **Optional Enhancements**
   - Best-so-far ant updates (ASrank, ACS)
   - Local pheromone updates
   - Candidate lists (restrict neighborhood)

Stopping Criteria

- Max number of iterations
- Convergence threshold
- No improvement for several iterations

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8. Variants

Common extensions include:

Ant Colony System (ACS)

Uses **local updates** and deposits pheromone only from the **best-so-far ant**.

Local update:

$$\tau_{ij} \leftarrow (1-\phi)\tau_{ij} + \phi\tau_0$$

Global update (best ant only):

$$\tau_{ij} \leftarrow (1-\rho)\tau_{ij} + \rho\,\frac{1}{L_{\text{best}}}$$

Max-Min Ant System (MMAS)

Constrains pheromone levels within:

$$\tau_{\min} \leq \tau_{ij} \leq \tau_{\max}$$