Revision

Solution of non-linear equations

1. Graphical method

Only in rare cases it is possible to find exact solution

2. Bracketing methods

- Bisection
- False-position
- Modified false position

Start by bracketing the solution

Guaranteed convergence

Linear convergence rate

- Only one solution

No ONE algorithm is "Universally" superior

3. Open Methods

Distinguishing features

- Only one starting value
- Convergence is not guaranteed
- If algorithm converges, the rate of convergence may be faster

1.

1. Fixed point method

[One point iteration | successive substitution]

$$f(x) = 0$$

Arrange the equation

$$x = g(x)$$

- Start with x_0
- $x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

$x_{i+1} = g(x_i)$

$\vdots$

$S = g(S)$ — solution

Example

$$f(x) = e^{-x} - x = 0$$

$S = 0.5671$

1 — $x = e^{-x}$

2 — $x = -\log(x)$

$x_0 = 0$

$x_1 = e^{0} = 1$

$x_2 = e^{-1} = 0.3678$

$x_3 = e^{-0.36788} = 0.692$

$x_4 = e^{-x_3} = 0.5$

$x_0 = 0.50$

$x_1 = 0.69$

$x_2 = \vdots$

Convergen of Fixed-point

$$x_{i+1} = g(x_i) \quad \text{--- (1)}$$

$$S = g(s) \quad \text{--- (2)}$$

2 − 1

$$S - x_{i+1} = \underbrace{g(s) - g(x_i)}_{\text{MVT}}$$

$$e_{i+1} = g'(\xi)(S - x_i)$$

$$\xi \in (s, x_i)$$

$$\Rightarrow e_{i+1} = g'(\xi)\, e_i$$

$$\Rightarrow \frac{|e_{i+1}|}{|e_i|} = |g'(\xi)|$$

as $i \to \infty$ $\quad \left|\frac{e_{i+1}}{e_i}\right| = |g'(s)|$ ✓

If $|g'(s)| < 1$ algorithm converges

Linear convergence

$$\boxed{\frac{|e_{i+1}|}{|e_i|^p} = C}$$

If $|g'(s)| > 1$ - algorithm diverges

$g'(s) = +$ive

errors will reduce montonically

$g'(s) = -$ive

errors will oscillate

Example

(1) $x = e^{-x}$

$g'(x) = -e^{-x}$

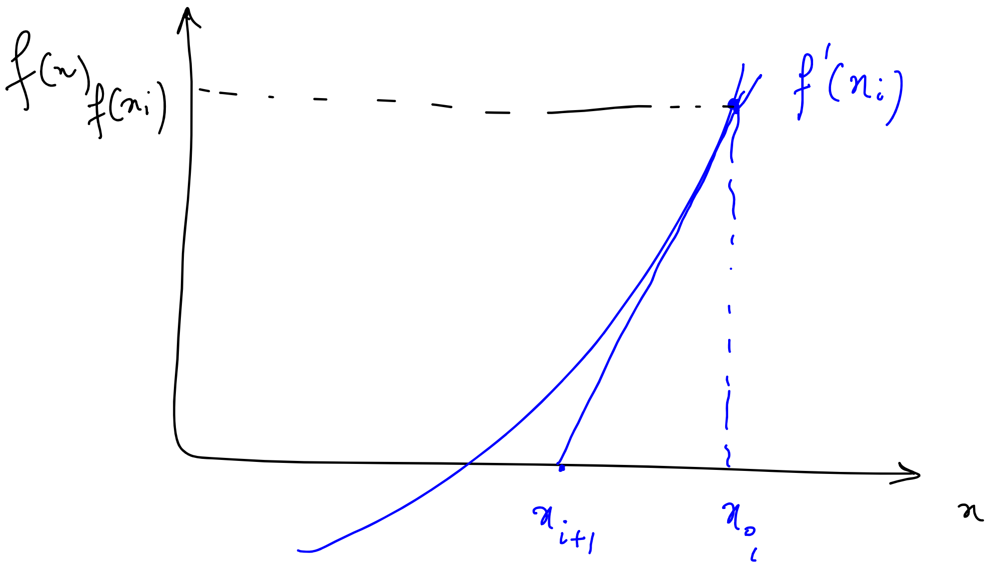
$|g'(s)| < 1$

(2) $x = -\log(x)$

$g'(x) = -\frac{1}{x}$

$s = 0.567$ $\quad |g'(s)| > 1$

2. Newton-Raphson method



$$f'(x_i) = \frac{-f(x_i)}{x_{i+1} - x_i}$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Example

$$x = \sqrt{a}$$

$$f(x) = x^2 - a = 0$$

$$f'(x) = 2x$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - a}{2x_i}$$

$$= \frac{x_i^2 + a}{2x_i}$$

$$\boxed{x_{i+1} = \frac{1}{2}\left[x_i + \frac{a}{x_i}\right]}$$

Convergence of NR method

Taylor series

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Let's assume that at i^{th} step we are just one step away from the true solution

$$f(s) = f(x_i) + (s - x_i) f'(x_i) + \frac{1}{2}(s - x_i)^2 f''(\xi)$$

$$\xi \in (x_i, s)$$

We know S is the solution

$$f(s) = 0$$

$$\Rightarrow \quad 0 = f(x_i) + (s - x_i) f'(x_i) + \frac{1}{2}(s - x_i)^2 f''(\xi)$$

Divid $f'(x_i)$

$$\frac{-f(x_i)}{f'(x_i)} = (s - x_i) + \frac{1}{2}(s - x_i)^2 \frac{f''(\xi)}{f'(x_i)}$$

$$x_{i+1} - \cancel{x_i} - s + \cancel{x_i} = \frac{1}{2}(s - x_i)^2 \frac{f''(\xi)}{f'(x_i)}$$

$$e_{i+1} = -\frac{1}{2} e_i^2 \frac{f''(\xi)}{f'(x_i)}$$

$$\Rightarrow \quad \frac{|e_{i+1}|}{|e_i|^2} = \left| \frac{1}{2} \frac{f''(\xi)}{f'(x_i)} \right|$$

$i \to \infty$

$$\boxed{\frac{|e_{i+1}|}{|e_i|^2} = \left| \frac{1}{2} \frac{f''(s)}{f'(s)} \right|}$$

- Quadratic convergence

Example

$$f(x) = e^{-x} - x$$

$$f'(x) = -e^{-x} - 1$$

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

$$= x_i - \frac{(x_i - e^{-x_i})}{(1 + e^{-x_i})}$$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 0.5663$

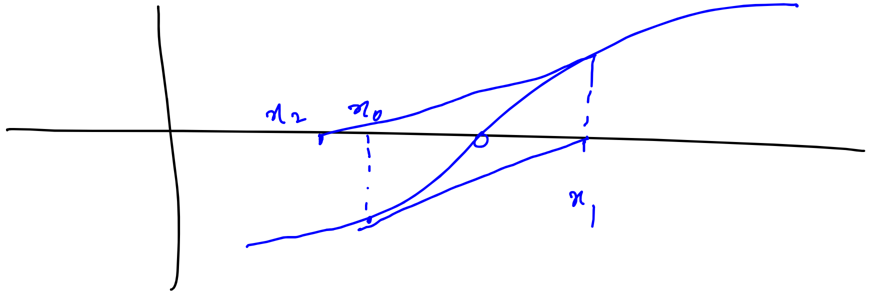
$x_3 = 0.5671$

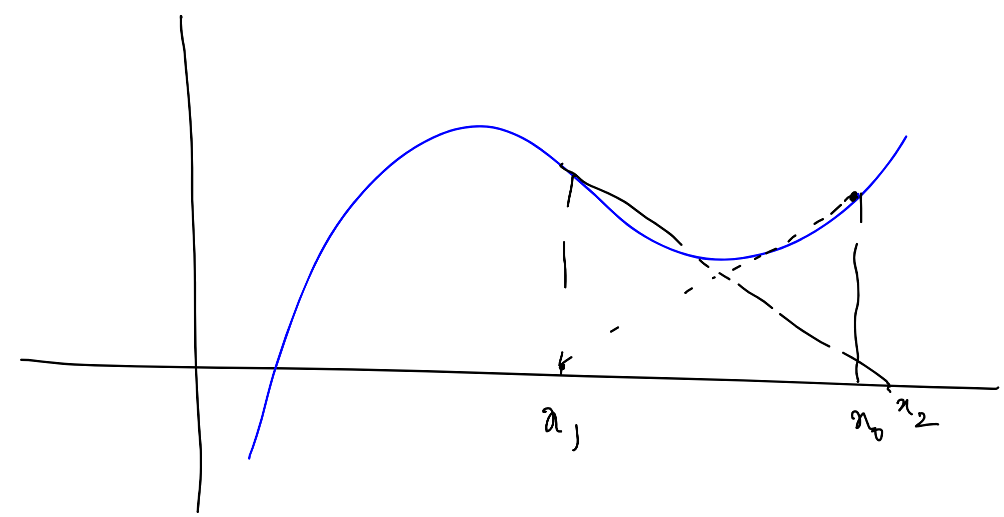
$$C = \left| \frac{1}{2} \frac{f''(s)}{f'(s)} \right|$$

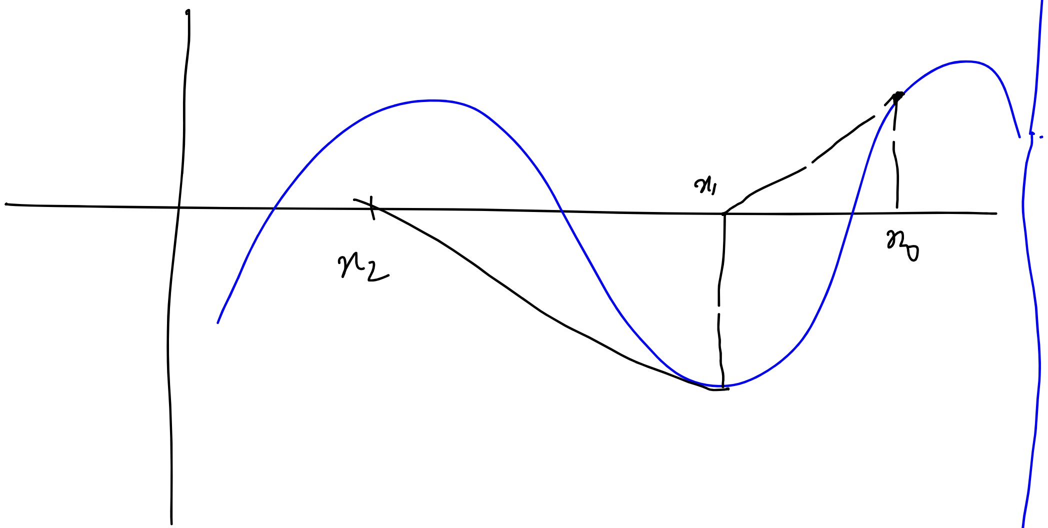
$$= 0.1809$$

Places where Newton-Raphson may not work

(a) Inflection point







- Convergence depends on the function
- Guess is close to the solution
- No substitute for understanding of the problem

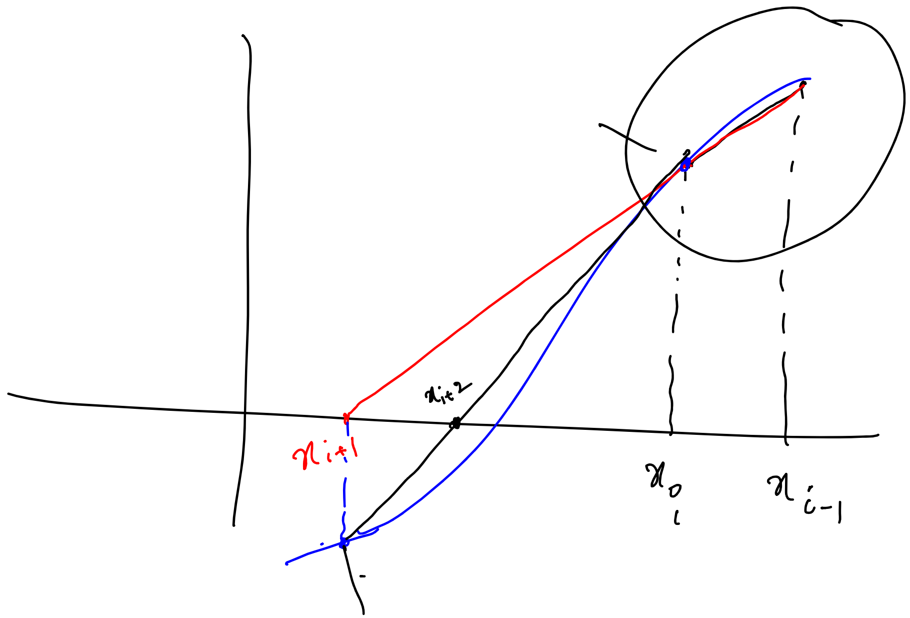
(iii) Secant Method

$f'(x_i)$ is not easy to estimate

Approximate

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)\left[x_{i-1} - x_i\right]}{f(x_{i-1}) - f(x_i)}}$$



Convergence

$$\frac{|e_{i+1}|}{|e_i|^p} = c$$

$$\underline{\underline{p = 1.618}}$$

System of non-linear equations

1. Fixed point

$$\begin{cases} u(x,y) = 0 \\ v(x,y) = 0 \end{cases}$$

$$x_{i+1} = g_1(x_i, y_i)$$

$$y_{i+1} = g_2(x_{i+1}, y_i)$$

2. Newton-Raphson

$$u(x_i, y_i) = u_i$$

$$v(x_i, y_i) = v_i$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i, y_i)} = \frac{\partial u_i}{\partial x}$$

Taylor's series

$$\begin{cases} u_{i+1} = u_i + (x_{i+1} - x_i)\dfrac{\partial u_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial u_i}{\partial y} \\ v_{i+1} = v_i + (x_{i+1} - x_i)\dfrac{\partial v_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial v_i}{\partial y} \end{cases}$$

x_{i+1} $\quad$ y_{i+1}