Solution of non-linear equations

Mathematical preliminaries

a. **Intermediate value theorem for continuous functions**

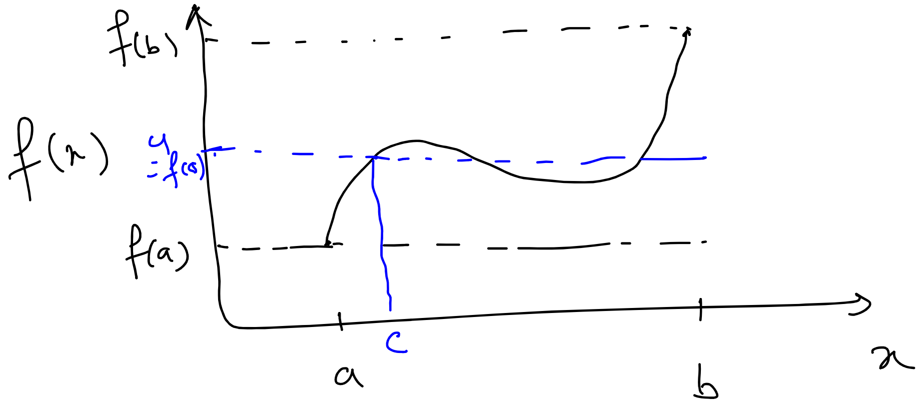
$$I = [a, b] \qquad b > a$$

Continuous function $f : I \to \mathbb{R}$

If u is a number between $f(a)$ and $f(b)$

i.e. $u \in (f(a), f(b))$

Then there is $c \in (a, b)$ such that $f(c) = u$



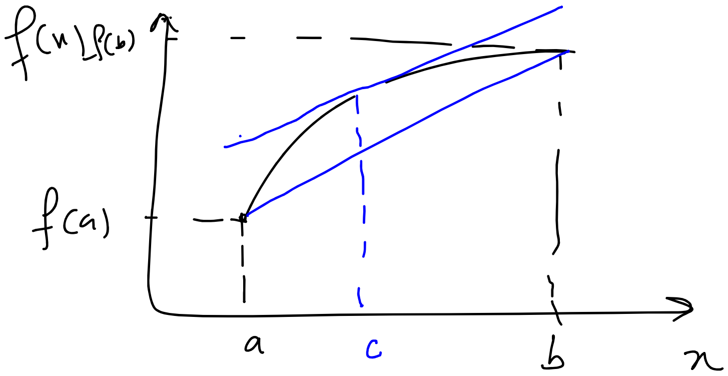
(b) Mean value theorem (MVT)

$I = [a, b] \quad b > a$

$f : I \to \mathbb{R}$

There exists $c \in (a, b)$ such that

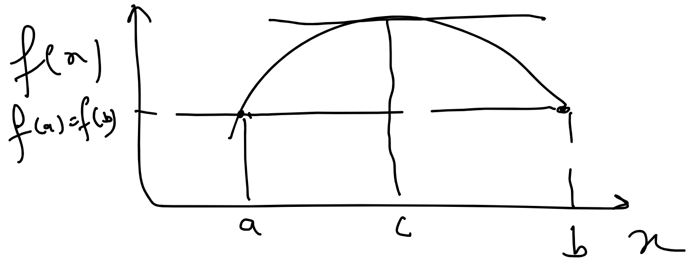
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Rolle's theorem

$$f(a) = f(b)$$

$$f'(c) = 0$$



MVT for integrals

If $g(x)$ be a non-negative or non-positive integrable function

$$\int_a^b f(x)\, \underline{\underline{g(x)}}\, dx = f(c) \int_a^b g(x)\, dx$$

$c \in (a, b)$

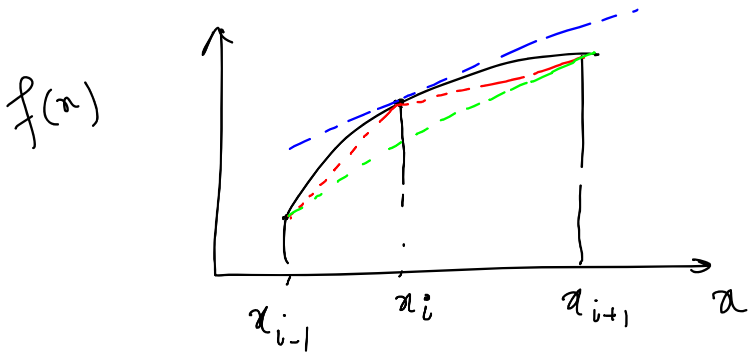
Numerical Differential

$$f(x_{i+1}) = f(x_i) + \underbrace{(x_{i+1} - x_i)}_{\Delta x} f'(x_i)$$

$$(1) \quad + \frac{(x_{i+1}-x_i)^2}{2!} f''(x_i) + \frac{(x_{i+1}-x_i)^3}{3!} f'''(x_i) + \dots$$

$$\Rightarrow \boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)}$$

First forward defference



$$(2) \quad f(x_{i-1}) = f(x_i) + -\Delta x f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) - \frac{\Delta x^3}{3!} f'''(x_i)$$

$$\boxed{f'(x_i) = \frac{f(x_i) - f'(x_{i-1})}{\Delta x} + O(\Delta x)}$$

first Backward difference

(1) - (2)

$$\boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x^2)}$$

Central defference

Solution of non-linear equations

$$f(x) = 0$$

To find the value of x

$$f(x) = ax^2 + bx + c = 0$$

Analytical solutions may not be available

Four approaches / Methods

1. Graphical method
2. Bracketing method — Bisection, False position
3. Open method — Newton-Raphson, Secant
4. Hybrid

Graphical method

$$f(x) = 0$$

Example (i) $f(x) = e^{-x} - x = 0$

(ii) $f(x) = (1-x)^6 = 0$

$f(x) = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$
$= 0$

In rare cases it is possible to find exact solution

Bracketing Method

Intermediate value theorem

$$a < x < b$$

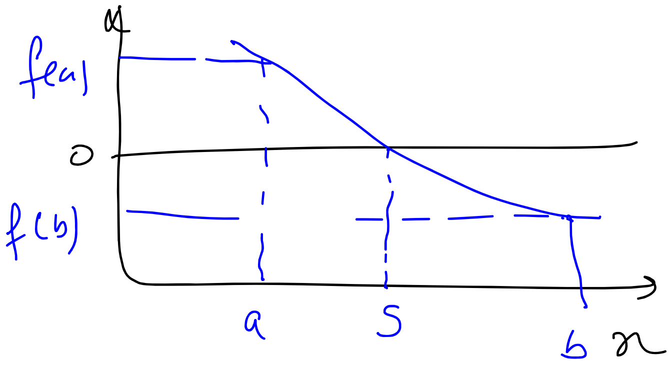
$$f(a)f(b) < 0$$

then there exists $f(s) = 0$

such that $s \in (a, b)$

$f(x)$

$= e^{-x} - x$

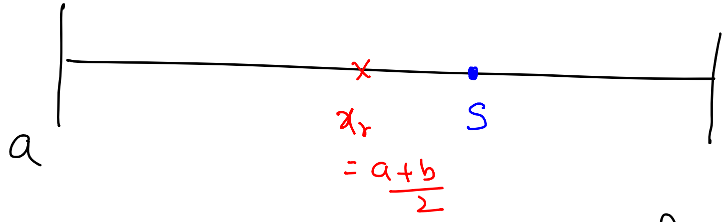


$f(s) = 0$

(a) Bisection method

We know that the solution lies between a and b

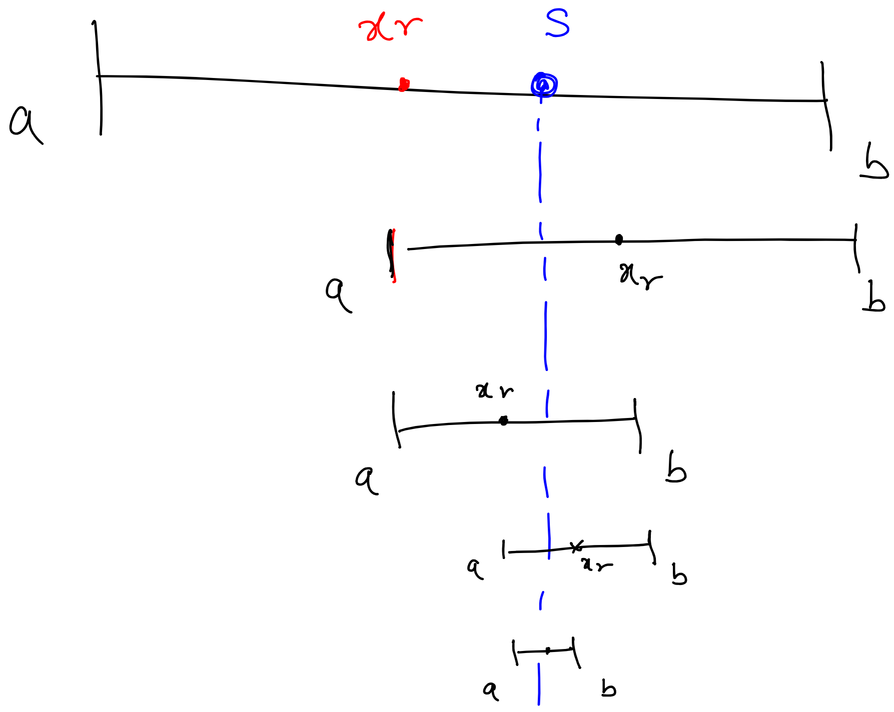
$$x_r = \frac{a+b}{2}$$



$f(a)f(x_r) < 0$

$f(a)f(x_r) > 0$

$f(a)f(x_r) = 0$



Error Analysis

$$E^0 = |a - b| = \Delta x^0$$

$$E^1 = \frac{\Delta x^0}{2}$$

$\vdots$

$$E^n = \frac{\Delta x^0}{2^n}$$