System of Linear Equations

n - equations , n - unknowns

$$E_1 : a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$E_2 : a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$E_n : a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

✓ Coefficient matrix $A_{n \times n}$

✓ Augmented matrix $\tilde{A}_{n \times n+1} = [A \ b]$

Homogeneous $\underline{b} = 0$

Non-homogeneous $\underline{b} \neq 0$

Important Square Matrices

1. Symmetric Matrix

$$a_{ij} = a_{ji}$$

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

2. Diagonal Matrix -

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} \ddots & \\ & & & a_{nn} \end{bmatrix}$$

3. Identity Matrix

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

4. Upper triangular matrix

All elements below the main diagonal are zero

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

4. Lower triangular matrix

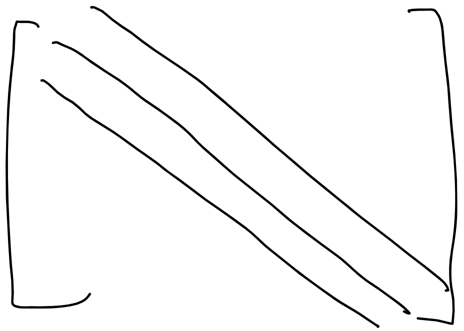
All elements above main diagonal are zero

$$L = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ \vdots & & & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

6. Banded Matrix

All elements are zero except for a band centered on the main diagonal

A =



$$= \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{32} & \\ & a_{32} & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix}$$

Tridiagonal matrix

7. Sparse matrix

Most of the elements are zero

8. Dense matrix

most of the elements are non-zero

9. Positive definite matrix

A symmetric matrix, such that $x^T A x$ is positive for every non zero column vector x of n real numbers

$\underset{1\times n}{x^T} \underset{n\times n}{A} \underset{n\times 1}{x} \Rightarrow$ Scalar

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x = \begin{bmatrix} a \\ b \end{bmatrix}$

$x^T I x = a^2 + b^2$

DIRECT METHOD

1. If the A is I

$$\begin{bmatrix} 1 & 0 & 0 & \cdot & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = b_1$$
$$\vdots$$
$$x_n = b_n$$

2. If A is a diagonal matrix

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$
$$\vdots$$
$$x_n = \frac{b_n}{a_{nn}}$$

3. If A is an Upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & \ddots & & \\ 0 & 0 & 0 & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Back substitution

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}\, x_n}{a_{n-1,n-1}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

4. If A is a Lower triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & & 0 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21} x_1}{a_{22}}$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}}$$

If the coefficient matrix A is "full"

Strategy - To reduce the matrix to

U — Gauss elimination

L & U — L U decomposition

D or I - Gauss Jordan

Gauss Elimination

Gauss elimination is one of the ubiquitous algorithms used in computer methods

E: $ax + by + cz = d$

If we: multiply

Divide

Add

Subtract

Gauss Elimination

objective – To convert A to U

E_1 : $2x_1 + 3x_2 - x_3 = 5$

E_2 : $4x_1 + 4x_2 - 3x_3 = 3$

E_3 : $-2x_1 + 3x_2 - x_3 = 1$

$$\tilde{A} = \left[\begin{array}{ccc:c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$$

Step 1

multiplying factors

$l_{21} = \dfrac{a_{21}}{a_{11}} = \dfrac{4}{2} = 2$ Pivot

$l_{31} = \dfrac{a_{31}}{a_{11}} = \dfrac{-2}{2} = -1$

$R_2 = R_2 - l_{21} R_1$; $R_3 = R_3 - l_{31} R_1$

Pivot equation

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{bmatrix}$$

Step 2

$R_3 = R_3 - l_{32} R_2$

$l_{32} = \dfrac{a_{32}}{a_{22}} = \dfrac{6}{-2} = -3$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

U

Back Substitution

$x_3 = -15/-5 = 3$

$x_2 = \dfrac{-7 - (-1 \times 3)}{-2} = 2$

$x_1 = 1$

Forward Elimination

for $K = 1$ to $n-1$

  for $i = K+1$ to n

    $l_{ik} = \dfrac{a_{ik}}{a_{kk}}$ $a_{i,n+1} = b_i$

    for $j = i$ to $n+1$

      $a_{ij} = a_{ij} - l_{ik} \cdot a_{kj}$

    end

  end

end

Backward Substitution

$x_n = \dfrac{b_n}{a_{nn}}$

For $i = n-1$ to 1

  $sum = a_{i,n+1}$

  for $j = i+1, n$

    $sum = sum - a_{ij} x_j$

  end

  $x_i = \dfrac{sum}{a_{ii}}$

end

1. Division by zero

$E_1 : \checkmark \quad 3x_2 - x_3 = 5$

$E_2 : 4x_1 + 4x_2 - 3x_3 = 3$

$E_3 : -2x_1 + 3x_2 - x_3 - 1$

$$A \quad \left[\begin{array}{ccc|c} 0 & 2 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$$

$$l_{21} = \frac{a_{21}}{a_{11}}$$

Row Pivoting $\Rightarrow$ Partial pivoting

X Column pivoting

Total pivoting

2. Ill conditioned

$x_1 + 2x_2 = 10$

$\cancel{1.1}\, x_1 + 2x_2 = 10.4$

1.05

$x_1 = 4 \quad x_2 = 3$

$x_1 = 8 \quad x_2 = 1$

3. Round off errors

$0.0004\, x_1 + 1.402\, x_2 = 1.406$

$0.4003\, x_1 - 1.502\, x_2 = 2.501$

4 significant digits

$x_1 = 10$

$x_2 = 1.0$

$$\begin{bmatrix} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{bmatrix}$$

$$R_2 = R_2 - l_{21} R_1 \qquad l_{21} = \frac{a_{21}}{a_{11}}$$

$$= \frac{0.4003}{0.0004} = 0.1001 \times 10^4$$

$$-1405 x_2 = 1404$$

$$x_2 = 0.9993$$

$$x_1 = \frac{(1.406 - 1.402 \times 0.9993)}{0.0004}$$

$$= 12.5$$