Revision

Solution of non-linear equations

1. Graphical methods ✓
2. Bracketing methods ✓
   - Bisection
   - False-position
   - Modified false-position

   } Linear but guaranteed convergence
3. Open methods
   - Fixed point —— Linear
   - Newton-Raphson } – Quadratic, – may diverge
   - Secant

   - problem with $f'(x) = 0$

Open method

1. Fixed point

$$f(x) = 0$$

Arrange $\boxed{x = g(x)}$

Error analysis

$$\boxed{\frac{|e_{i+1}|}{|e_i|} = g'(\xi)} \qquad \xi \in (x_i, s)$$

$$\lim_{i \to \infty} \frac{|e_{i+1}|}{|e_i|} = g'(s)$$

$$|g'(\xi)| < 1 \quad \text{for } |e_{i+1}| < e_i$$

$f(x) = e^{-x} - x$

(1) $x = e^{-x}$

$g(x) = e^{-x}$

$g'(x) = -e^{-x}$

$|g'(x)| < 1$

(2) $x = -\log(x)$

$g(x) = -\log(x)$

$\boxed{|g'(x)| = \frac{-1}{x}}$ $0 < x < 1$

> 1

Newton-Raphson

$f(x) = 0$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Error analysis

$\xi_i \in (x_i, s)$

$$\frac{|e_{i+1}|}{|e_i|^2} = \Bigg| \underbrace{\frac{1}{2} \frac{f''(\xi_i)}{f'(x_i)}}_{M} \Bigg|$$

$$M|e_{i+1}| \approx M^2|e_i|^2$$

$$M|e_{i+1}| \approx (M|e_i|)^2 \rightarrow$$

$$M|e_0| = M|x_0 - s| < 1$$

$$\Rightarrow |x_0 - s| < 1/M = \left|\frac{2f'(x)}{f''(\xi)}\right|$$

System of non-linear equations

$$u(x,y) = 0 \qquad x = g_1(x,y)$$

$$v(x,y) = 0 \qquad y = g_2(x,y)$$

$$x_{i+1} = g_1(x_i, y_i)$$

$$y_{i+1} = g_2(x_{i+1}, y_i)$$

Convergence

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

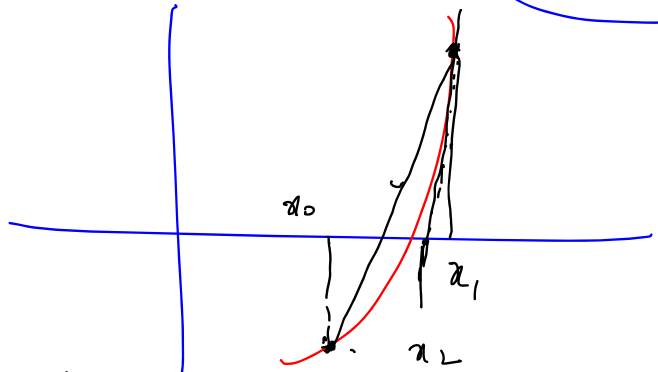
$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$$

Secant Method

$$f(x) = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

$$= \frac{1}{f'(x_i)}$$



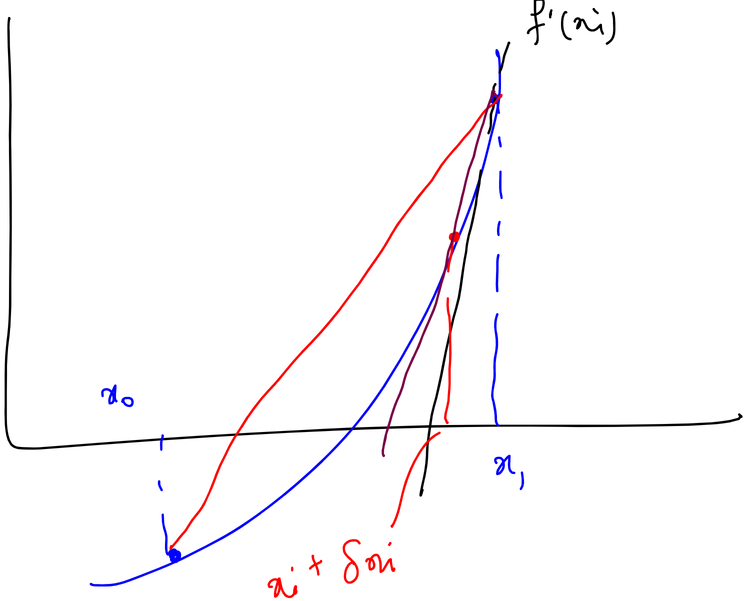
$$O(\Delta x)$$

Modified secant method

small perturbat

$$f'(x_i) = \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

??



Hybrid Methods

fzero

Combined

- Bracketing method
- Open-method

DEKKER METHOD — Bisection, Secant

Brent Algorithm — Bisection, Open (inverse quadratic)

Multiple roots

A function can have more than one roots of the same value

Example $f(x) = (x-2)^2 = 0$ - Double root

$f(x) = (x+3)^3 (x-4) = 0$ - 4 roots
Triple root
Single root

$$f(x) = (x-2)^2 = 0$$

$x = 2$ $\quad f'(x) = 2(x-2) = 0$

Let s be a solution of the function $f(x)$ which can be factorized as

$$\boxed{f(x) = (x-s)^m h(x)}$$

with integer $m \geq 1$ and continuous function $h(x)$ for which $h(s) \neq 0$

Then, we say that s is a root of $f(x)$ of multiplicity m

If s is a root of multiplicity m, then

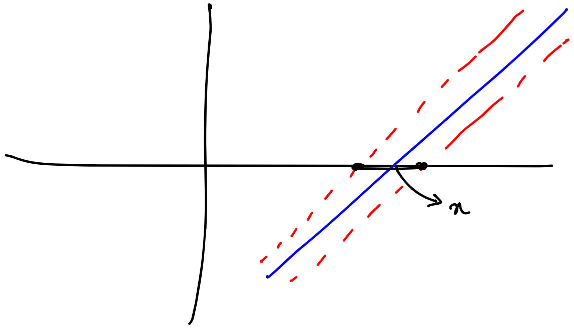
$$f(s) = f'(s) = f''(s) = \cdots = f^{m-1}(s) = 0$$

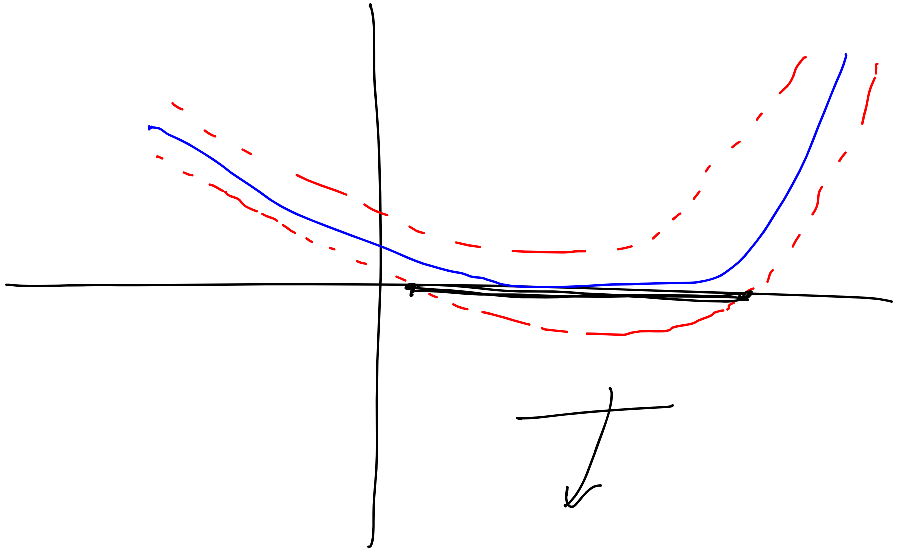
$$f^{m}(s) \neq 0$$

Problems with multiple roots

1. Bracketing methods cannot be used for m - even
2. Newton-Raphson may not work $f'(x) = 0$
3. Large interval of uncertainty

$f(x)$





No solulution

ONE OPTION → Change or reformulate $f(x) = 0$ to $u(x) = 0$ such that $u(x) = 0$ has one solution

1. Newton Raphson for multiple roots

Quadratic convergence becomes linear

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = g(x_i) \qquad s = g(s)$$

$$x_{i+1} = \underbrace{g(s)}_{s} + g'(s)(x_i - s) + \frac{g''(s)}{2!}(x_i - s)^2 + \cdots$$

$$\Rightarrow e_{i+1} = g'(s)\, e_i + \frac{g''(s)}{2!} e_i^2$$

$f'(x) \neq 0 \qquad g'(s) = 0$

If $f'(s) = 0$

$g'(s) \neq 0 \qquad g'(s) = 0$

Two modifications of NR

1. First modification

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

2. Second modification

$f(x) = 0$

$$\Rightarrow u(x) = \frac{f(x)}{f'(x)}$$

Instead of $f(x) = 0$

Solve $u(x) = \dfrac{f(x)}{f'(x)}$

$$f(x) = (x-s)^m h_1(x)$$

$$f'(x) = m(x-s)^{m-1} h_1(x) + (x-s)^m h_1'(x)$$

$$= (x-s)^{m-1} h_2(x)$$

$h_1(s), h_2(s)$

$h_3(s) \neq 0$

$$u(x) = \frac{(x-s)^m h_1(x)}{(x-s)^{m-1} h_2(x)}$$

$$\boxed{u(x) = (x-s) h_3(x)}$$

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$= x_i - \frac{f(x_i) f'(x_i)}{f'(x_i)^2 - f(x_i) f''(x_i)}$$

$f(x_i) f'(x_i) f''(x_i)$

Roots of Polynomials

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$
$$= 0$$

nth order polynomial

If $a_0, a_1, \dots a_n$ are real

- n roots (real or complex)
- if n is odd [at least one real root]
- Complex root {conjugate pairs]

$u + iv$

$u - iv$

Evaluate a polynomial

$$f_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

n^{th} — n - additions

$\frac{n(n+1)}{2}$ - multiplications

$$f_3(x) = ((a_3 x + a_2)x + a_1)x + a_0$$

n^{th} — n - addition

n - multiplication