RECAP

System of linear equations

$$Ax = b$$

- Condition number of the matrix
- Method of iterative refinement
  $Ae = r$
- Indirect or iterative methods for solving $Ax = b$

  - Jacobi
  - Gauss - Seidel
  - Relaxation technique

  Fixed point method for linear equation

Jacobi

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_i, z_i)$$
$$z_{i+1} = g_3(x_i, y_i)$$

Gauss - Seidel

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_{i+1}, z_i)$$
$$z_{i+1} = g_3(x_{i+1}, y_{i+1})$$

Convergence

$$\cancel{\left|\frac{\partial g_1}{\partial x}\right|} + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| < 1$$

same for g_2 & g_3

$$\boxed{|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|}$$

Diagonally dominant system

- Condition is sufficient (not necessary) is an upper bound on convergence criteria
- Convergence rate is linear

Improvement in the convergence rate for Gauss-Seidel method

Relaxation techniques

$$x'_{i+1} = g_1(y_i, z_i)$$

$$\boxed{\begin{aligned} x_{i+1} &= \lambda x'_{i+1} + (1-\lambda) x_i \\ \Rightarrow \quad x_{i+1} &= x_i + \lambda (x'_{i+1} - x_i) \end{aligned}}$$

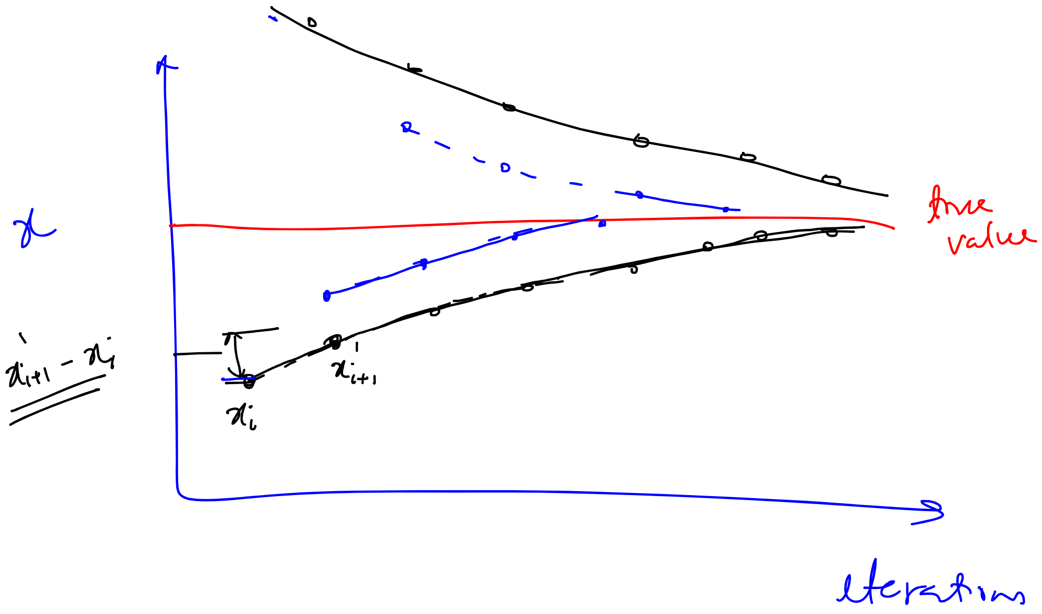
λ is a weighing factor (Relaxation factor) $\boxed{\lambda \in (0,2)}$

If $\lambda = 1$ $\quad x_{i+1} = x'_{i+1}$

Gauss-Seidel method

if $\underline{\lambda > 1}$ - more weightage to the present term x'_{i+1}

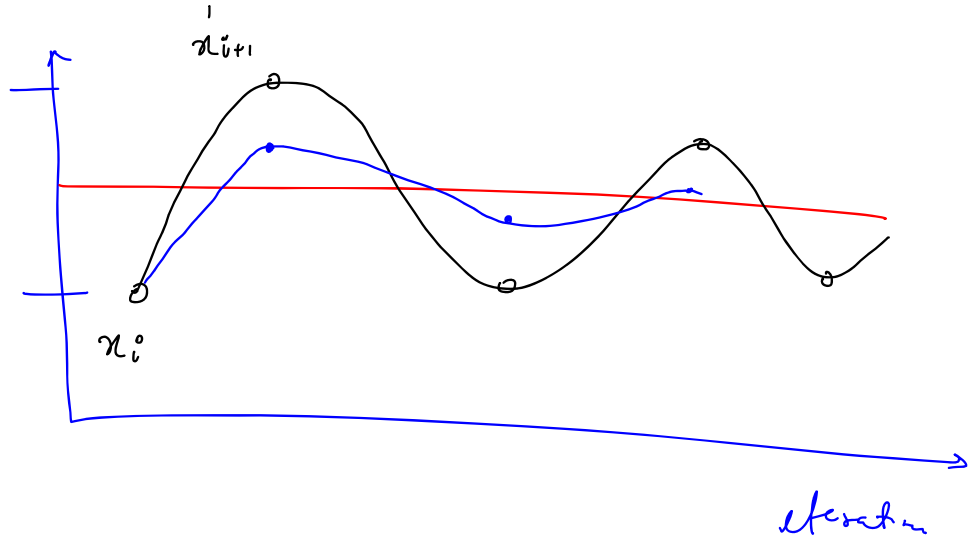
<u>Over-relaxation</u> - Improves convergence



Successive over relaxation
(SOR)

<u>if $\lambda < 1$</u>

Under relaxation - dampens the oscillation



Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

Gauss-Seidel

$$A = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$x = 2 - 0.5y$$

$$y = 4 - 0.5x - 0.5z$$

$$z = 4 - 0.5y$$

$$x_{i+1} = g_1(y_i, z_i)$$

iteration	x	y	z	e_a
0	0	0	0	
1	2	3	2.5	100 %
2.	0.5	2.5	2.75	.

Over relaxation $\lambda = 1.2$

iteration	x'	x	y'	y	z'	z
0		0		0		0
1	2	2.4	2.8	3.36	2.32	2.784
2	0.32	-0.096				

How to get optimal λ

- Problem specific
- The usual procedure is to do empirical evaluation
  - Useful when the system has to be solved number of lines

GAPS

- Why GS is faster than Jacobi
- The convergence criterion is sufficient (not necessary)
- Why the relaxation technique works?
- Why the value of $\lambda \in (0,2)$

Computer Assignment 2

- Permutation matrix

Gauss elimination (Row -exchange)

no exchange $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Eigen Values and Eigen Vectors

Refresher

$$AV = \lambda V$$

(λ, V) - eigen pairs of A

eigen value, eigen vector

$$(A - \lambda I) V = 0 \quad \text{--- (1)}$$

Homogenean equation

For a non-trivial solution ($V \neq 0$)

$$\det(A - \lambda I) = 0$$

Example

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$$= (-5-\lambda)(-2-\lambda) - 4 = 0$$

Polynomial

A $n \times n$ characteristic polynomial will be of order 'n'

$$\Rightarrow \boxed{\lambda^2 + 7\lambda + 6 = 0}$$

$$\lambda_1 = -1$$

$$\lambda_2 = -6$$

Eigen vectors

$$\begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) $\lambda = -1$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4v_1 + 2v_2 = 0$$
$$2v_1 - v_2 = 0$$

$$v_1 = \frac{v_2}{2} \qquad v_1 = 1, \; v_2 = 2$$

$$\rightarrow V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\frac{V}{\|V\|_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Big/ \sqrt{5}$$

$$= \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.89 \end{bmatrix}$$

(b) $\lambda = -6$

$$V = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \frac{V}{\|V\|} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$$

1. The vectors $v_1, v_2, \ldots v_k$ are called linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_k v_k = 0$$

iff $\alpha_1 = \alpha_2 = \alpha_3 \ldots \alpha_k = 0$

else, linearly dependent

2. Any n linearly independent vectors $v_1, v_2 \ldots v_n$ are a "basis" for n-space, i.e.

a vector X is n-space it can be expressed uniquely as a linear combination of basis vectors

$$X = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$$

α's - unique, components of X w.r.t. basis $\{v_1, v_2 \ldots v_n\}$

3. A $n \times n$ matrix V is non-singular if its columns

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

are linearly independent.