Recap : Error Analysis

Error = True value − Approx. value

$$e = f(x) - \tilde{f}(x)$$

$$e = x - \tilde{x}$$

Relative error

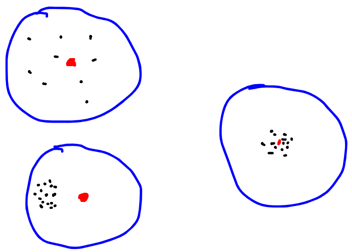
$$e_r = \frac{f(x) - \tilde{f}(x)}{f(x)} \qquad f(x) \neq 0$$

$$e_r = \frac{x - \tilde{x}}{x} \qquad x \neq 0$$

True value is almost never known

Accuracy

Precision



1. Approximate value of error

Example − For iterative algorithms

$$\varepsilon = \text{Current approximation} - \text{Previous approx.}$$

$$\varepsilon_r = \frac{\text{Current approx} - \text{Previous approx.}}{\text{Current approx}}$$

2. Determine error bound

$$E \geq e$$

Truncation error

Taylor Series

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) + \frac{(x_{i+1} - x_i)^2}{2!} f''(x_i) + \dots + \frac{(x_{i+1} - x_i)^n}{n!} f^{n}(x_i) + R$$

$$R = \frac{(x_{i+1} - x_i)^{n+1}}{(n+1)!} f^{n+1}(\xi)$$

$$x_i \leq \xi \leq x_{i+1}$$

Example

$x_0 = 0 \qquad f(x_0) = 1.2$

$x = 1 \qquad f(x) = 0.2$

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Zeroth order

$$\tilde{f}(x_{i+1}) = f(x_i) = 1.2$$

$$e = 0.2 - 1.2 = -1.0$$

$$R = \frac{(x_{i+1} - x_i)}{1!} f'(\xi)$$

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$E \geq |R|$$

$$E = 2.1$$

First order

$$f(x_{i+1}) = 0.95$$

$$e = 0.2 - 0.95 = -0.75$$

$$R = \frac{(x_{i+1} - x_i)^2}{2!} f''(\xi)$$

$$f''(x) = -1.2x^2 - 0.9x - 1$$

$$E = 1.55$$

$$e_a = 0.95 - 1.2 = -0.25$$

Order	$\tilde{f}(x_{i+1})$	$\lvert e \rvert$	E	$\lvert e_a \rvert$
0	1.2	1.0	2.1	–
1st	0.95	0.75	1.55	0.25
2nd	0.45	0.25	0.55	0.50
3rd	0.30	0.10	0.10	0.15
4th	0.20	0.0	0.0	0.10

Data error

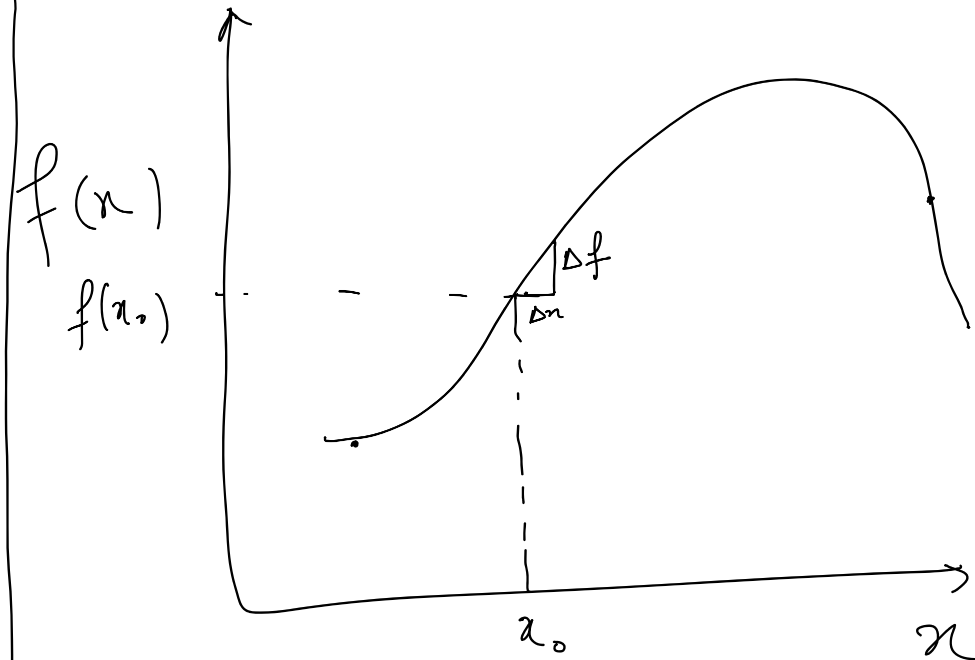
$$f(x+\Delta x) - f(x)$$

$$= \Delta x \, f'(x) + \frac{\Delta x^2}{2!} f''(x) + \ldots$$

propagation

First order error

$$f(x+\Delta x) - f(x) = \Delta x \, f'(x)$$



$$\frac{\Delta f}{\Delta x} = f'(x)$$

$$\Delta f = \Delta x \, f'(x)$$

Example

$$g = \frac{2H}{t^2}$$

$$\Delta g \approx \left|\Delta t \frac{\partial g}{\partial t}\right| + \left|\Delta H \frac{\partial g}{\partial H}\right|$$

$$\Delta g \approx \left|\Delta t \frac{4H}{t^3}\right| + \left|\Delta H \frac{2}{H^2}\right|$$

$H = 660 \pm 0.01$ m

$t = 11.65 \pm 0.01$ s

$$\Delta g = 0.01 \times 1.67 + 0.01 \times 0.00147$$

$$= 0.016998 \text{ m/s}^2$$

Round-off error

Number representation

Integer — unsigned (0, 1, 2), signed (−1, −2, 1, 2, 0)

Fixed-point — $\pi = 3.14$

Floating-point — $\frac{1}{\sqrt{2}} = 0.71$

Floating point numbers

$$x = \pm m b^{p}$$

m - mantissa

b - base — 10

p - exponent

$\_\_ \times 10^{384}$

Example

$$\pi = 0.314 \times 10^{1}$$

$$1/\sqrt{2} = 0.707 \times 10^{0}$$

To store floating point numbers, a "computer word" is divided into 3 parts

16-bit

1 bit	5 bit	10 bit
sign	signed exponent	mantissa

IEEE 754 technical standard

32-bit

1	8 bit	23-bit

binary

64-bit

1	11 bit	52-bit

-383 to 384

binary

System - Decimal

3 decimal places - mantissa

1 place - exponent

$d \} - p$

$$x = \pm \underbrace{0.ddd}_{m} \; \underbrace{10}_{b}$$

Mantissa → $\frac{1}{b} \leq m < 1$

Decimal minimum 0.100

maximum 0.999

Binary b minimum $2^{-1} = 0.5$

maximum $2^{-1} + 2^{-2} + 2^{-3}$

3 - important properties

1. Maximum positive value

x_{max} 0.999×10^{9}

x_{min} -0.999×10^{9}

Over flow error

2. Hole near zero

0.000×10^{-9} } 10^{-9}

0.100×10^{-9} } $- 10^{-12}$

0.101×10^{-9}

0.102×10^{-9}

3. Internal between numbers increase

0.998×10^3

0.999×10^3 — 1

0.100×10^4 — 1

0.101×10^4 — 10

Mathematics

$-\infty$ 0 $\Delta \to 0$ ∞

Computer

hole near zero

overflow ← 0 → overflow

Underflow

quantization error

↓

Round off error