Recap

Mathematics Preliminaries

1. Intermediate value theorem

2. Mean value theorem

3. Numerical differentiation

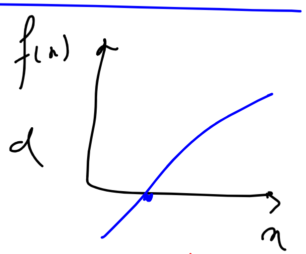
(a) Forward difference $$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(\Delta x)$$

(b) Backward $$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} + O(\Delta x)$$

(c) Central $$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} + O(\Delta x^2)$$

Solution of non-linear equations

1. Graphical method



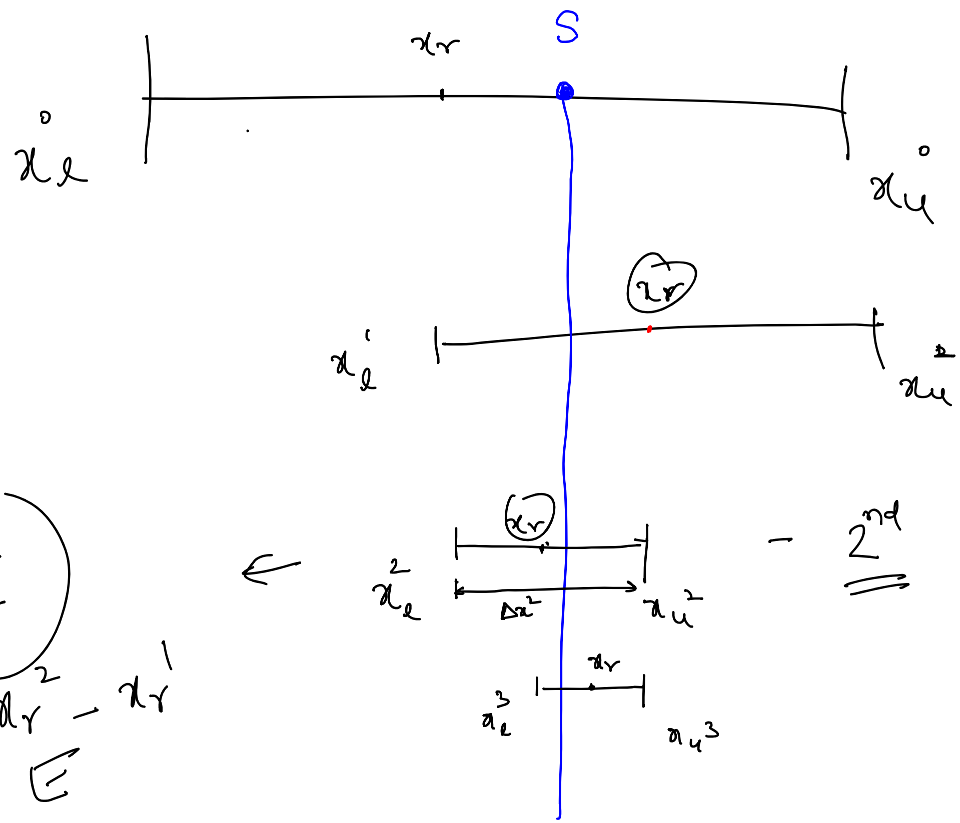
Only in rare cases it is possible to get exact solutions

2. Bracketing methods

(i) Bisection method

Bisection method

$f(x_u)\, f(x_l) < 0$



$E = \dfrac{\Delta x^2}{2}$

$\varepsilon = x_r^2 - x_r^1$

$= E$

Algorithm

1. Start with x_l and x_u
2. $x_r = \dfrac{x_l + x_u}{2}$
3. $f(x_l)\, f(x_r) < 0 \qquad x_u = x_r$

   $f(x_l)\, f(x_r) = 0 \qquad S = x_r$

   else $f(x_l)\, f(x_r) > 0 \qquad x_l = x_r$
4. Stopping criteria

   else step (2)

Error Analysis

$$E^0 = |x_l - x_u| = \Delta x^0$$

$$E^1 = \frac{\Delta x^0}{2}$$

$\vdots$

$$\boxed{E^n = \frac{\Delta x^0}{2^n}}$$

Error bound reduces with iterations

hence the algorithm will converge

$$\frac{|E_{i+1}|}{|E_i|} = \frac{1}{2}$$

$p = 1$

Linear convergence

$c = 1/2$

Rate of convergence for an iterative sequence

If an iterative sequence

$x_r^1, x_r^2, \ldots$ converges to

the solution S, and the true error

$$e^i = S - x_r^i$$

and if

$$\lim_{i \to \infty} \frac{|e_{i+1}|}{|e_i|^p} = c$$

Then p — order of convergence

c — asymptotic error constant

$c > 1$ diverging $\quad c < 1$ converging

Stopping Criteria

(i) maximum number of iterations

(ii) $\varepsilon = x_r^{new} - x_r^{old}$ $\qquad |\varepsilon| \leq \varepsilon_s$

Threshold

or $\varepsilon_r = \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}} \times 100$

(iii) Maximum number of iterations can be estimated a priori

$$E^n \leq \alpha \qquad \alpha = 0.01$$

$$\Rightarrow \frac{\Delta x^0}{2^n} \leq \alpha$$

$$\Rightarrow \boxed{n \geq \frac{1}{\log(2)} \log\left(\frac{\Delta x^0}{\alpha}\right)}$$

(iv) Approximate error is always greater than true error

Approximate error is an exact upper bound for the true error

Example

$$f(x) = \exp(-x) - x = 0$$

$$S = 0.5671$$

$$e^i = S - x_r^i$$
$$\varepsilon^i = x_r^i - x_r^{i-1}$$

$$\left.\begin{aligned} x_l &= 0 \\ x_u &= 1 \end{aligned}\right\} \Delta x^0 = 1$$

iterations	x_l	x_u	x_r	e	e_r	ε	ε_r
1.	0	1	0.5	0.0671	11.8	-	—
2.	0.5	1.0	0.75	-0.183	32.24	0.25	33.33
3.							

$$E \leq \alpha \qquad \alpha = 0.01$$

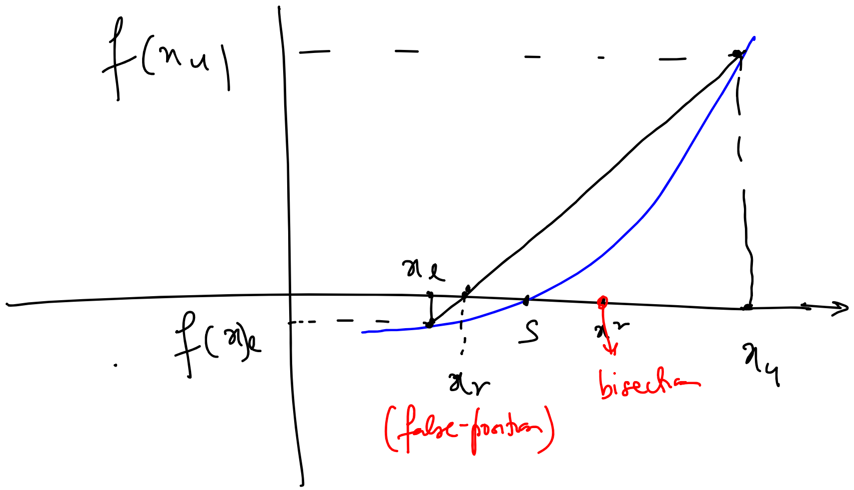
$$n \geq \frac{1}{\log(2)} \log\left(\frac{1.0}{0.01}\right)$$

$\alpha = 0.1 \qquad n = 4$

$\alpha = 0.01 \qquad n = 7$

2. The false position method

(linear interpolation)



Bisection method does not use the values $f(x_l)$ and $f(x_u)$

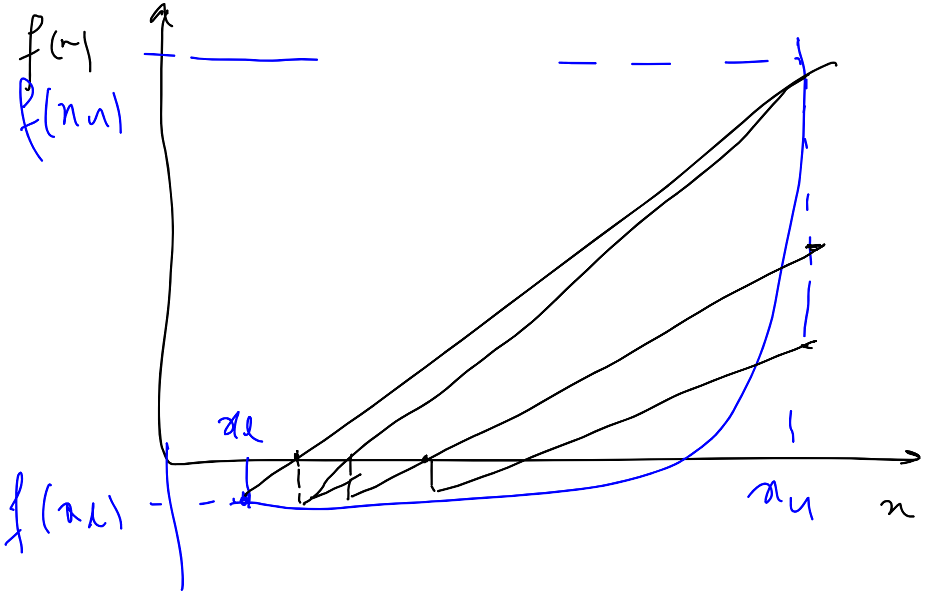
linear convergence

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$\Rightarrow \quad x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$$f(x_l) f(x_r) < 0$$
$$> 0$$



Example

$$f(x) = x^{10} - 1 = 0$$

No one algorithm can be claimed to be universally superior than others

No free lunch theorem

Modified false-position method