Revision

System of Linear Equations $Ax = b$

Direct Methods

1. Gauss Elimination (GE)
2. Gauss Jordon
3. LU Decomposition

Computational | Algorithmic complexity

$$A = LU$$

Unique if l_{ii} or $u_{ii} = 1$

- GE → All steps including scaling and pivoting
- Doolittle } $l_{ii} = 1$
- Crout – $u_{ii} = 1$
- Thomas algorithm – tridiagonal
- Cholesky method → symmetric positive definite

compact

Doolittle | Crout Method

Example

Doolittle

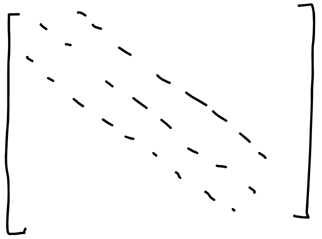
$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 3 & 1 \\ 0 & 4 - \frac{2}{3} \end{bmatrix}}^{U}$$

$= 10/3$

Crout

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 10/3 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$$

Thomas Algorithm - Tridiagonal



$$\begin{bmatrix} d_1 & u_1 & & & \\ l_2 & d_2 & u_2 & & \\ & l_3 & d_3 & u_3 & \\ & & \ddots & \ddots & \ddots \\ & & & l_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Define

$$\alpha_1 = d_1$$

$$\beta_1 = b_1$$

$$\lambda_2 = \frac{l_2}{\alpha_1}$$

$$R_1 - \alpha_1 x_1 + u_1 x_2 = \beta_1$$

$$R_2 - l_2 x_1 + d_2 x_2 + u_2 x_3 = b_2$$

$$R_2 \to R_2 - \lambda_2 R_1$$

$$x_2 \left(d_2 - \lambda_2 u_1 \right) + u_2 x_3 = b_2 - \lambda_2 \beta_1$$

Define

$$\alpha_2 = d_2 - \lambda_2 u_1$$

$$\beta_2 = b_2 - \lambda_2 \beta_1$$

$$\Rightarrow \quad \alpha_2 x_2 + u_2 x_3 = \beta_2$$

$$\lambda_3 = \frac{l_3}{\alpha_2} \qquad l_3 x_2 + d_2 x_3 + u_3 x_4 = b_3$$

$$\lambda_i = \frac{l_i}{\alpha_{i-1}}$$

$$\alpha_i = d_i - \lambda_i u_{i-1}$$

$$\beta_i = b_i - \lambda_i \beta_{i-1}$$

$$x_i = \frac{\beta_i - u_i x_{i+1}}{\alpha_i}, \quad i = n-1 \to 1 \quad \leftarrow \quad \alpha_i x_i + u_i x_{i+1} = \beta_i$$

$\underline{i=n}$

$$\alpha_n x_n = \beta_n \Rightarrow x_n = \beta_n / \alpha_n$$

GE : $O(2/3\, n^3)$

Thomas : $O(n)$

Example

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

$\alpha_1 = d_1 \qquad \beta_1 = b_1$

$\lambda_i = \dfrac{l_i}{\alpha_{i-1}}$

$\alpha_i = d_i - \lambda_i u_{i-1}$

$\beta_i = b_i - \lambda_i \beta_{i-1}$

$x_n = \beta_n / \alpha_n$

$x_i = \dfrac{\beta_i - u_i x_{i+1}}{\alpha_i}$

i	l	d	u	b	λ	α	β	x
1		2	1	4		2	4	$x_1 = 1$
2	1	2	1	8	$\frac{l_2}{\alpha_1} = 0.5$	$2 - 0.5\times 1 = 1.5$	$8 - 0.5\times 4 = 6$	$x_2 = \frac{6 - 1\times 3}{3/2} = 2$
3	1	2		8	$\frac{l_3}{\alpha_2} = \frac{1}{1.5} = 2/3$	$4/3$	4	$x_3 = 4/\tfrac{4}{3} = 3$

Cholesky Decomposition

Remark 1

$$A = LU$$

if l_{ii} or $u_{ii} = 1$ → decomposition is unique

Can we write U such that its diagonal elements are unity

$$\begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & d_2 & u_{23} & \cdots & u_{2n} \\ & & d_3 & \cdots & u_{3n} \\ & & & \ddots & \\ & & & & u_{nn} \end{bmatrix} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots & u_{1n}/d_1 \\ & 1 & u_{23}/d_2 & \cdots & u_{2n}/d_2 \\ & & \ddots & & \\ & & & & 1 \end{bmatrix}$$

$$U = D\tilde{U}$$

$\tilde{U}_{ii} = 1$

$$\boxed{A = LDU}$$

$l_{ii} = u_{ii} = 1$

Remark 2 - Symmetric Positive Definite Matrix

Problem of minimization

$f(x)$

Minimum
necessary — $\frac{df}{dx} = 0$

sufficient $\frac{d^2f}{dx^2} > 0$

$f(x_1, x_2)$, $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$

necessary

positive definite system ⇐ $\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} > 0$

$$(x_1 \;\; x_2) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x^T A x > 0$ — Positive definite matrix

$$a x_1^2 + 2b x_1 x_2 + c x_2^2 > 0$$

It is possible : $a > 0$, $c > 0$
$ac > b^2$

For n dimensional function $f(x_1, x_2, \ldots x_n)$

$$a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

symmetric

A should be positive definite

Symmetric matrix

$$A^T = A$$

$$A = LDU$$

$$A = A^T = U^T D L^T$$

$$\Rightarrow \quad L = U^T$$

$$U = L^T$$

$$\Rightarrow \quad \boxed{A = LDL^T}$$ — symmetric matrix

If A is positive definite $\therefore$ D will be positive

$$A = L D^{1/2} D^{1/2} L^T$$

$$\boxed{A = L_c L_c^T}$$ — Cholesky decomposition

$$\boxed{L_c = L D^{1/2}}$$

Cholesky factor.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ \vdots & & & \ddots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & l_{32} & \cdots \\ & & \ddots & \\ & & & l_{nn} \end{bmatrix}$$

$$a_{ij} = a_{ji}$$

$$l_{11}^2 = a_{11}$$

$$l_{11} = \sqrt{a_{11}}$$

$$l_{j1} = \frac{a_{j1}}{l_{11}}$$

$$l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2} \qquad j = 2, \ldots, n$$

$$l_{kj} = \left(a_{kj} - \sum_{s=1}^{j-1} l_{js} l_{ks}\right) \Big/ l_{jj} \qquad k = j+1, \ldots, n$$

Example

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \quad l_{11}^2 = a_{11} \quad \Rightarrow \quad \boxed{l_{11} = \sqrt{a_{11}}} = \underline{\underline{2}}$$

$$l_{21}\, l_{11} = a_{21}$$

$$\Rightarrow \quad l_{21} = \frac{a_{21}}{l_{11}} \qquad \Rightarrow \quad l_{21} = \frac{2}{2} = \underline{\underline{1}}$$

$$\boxed{l_{j1} = \frac{a_{j1}}{l_{11}}}$$

$$l_{31} = \frac{14}{2} = 7$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \qquad L^T = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$l_{21}^2 + l_{22}^2 = a_{22}$$

$$\Rightarrow \quad l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{17 - 1^2} = 4$$

$$\boxed{l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}}$$

$$l_{31}\, l_{21} + l_{32} \cdot l_{22} = a_{32}$$

$$\Rightarrow \quad l_{32} = \frac{a_{32} - l_{31}\, l_{21}}{l_{22}} = \frac{-5 - 7 \times 1}{4} = -3$$

$$l_{kj} = \frac{a_{kj} - \sum_{s=1}^{j-1} l_{js}\, l_{ks}}{l_{jj}} \qquad k = j+1, \dots, n$$

$$l_{33} = \sqrt{83 - 7^2 - 3^2} = 5$$

Error Analysis

Condition number of the problem

$f(x)$, $x+\delta x$

$$C_p = \frac{|\Delta f / f|}{|\Delta x / x|}$$

System of linear equations

$$A x = b$$

To find x.

$$A + \delta A \rightarrow x + \delta x$$

$$b + \delta b \rightarrow x + \delta x$$

Norms