Revision

$A_{n \times n}$

$AV = \lambda V$

$\lambda_1, \lambda_2, \ldots \lambda_n$
$v_1, v_2, \ldots v_n$

1. Characteristic equation

2. Power method

Direct

a. Power method – find maximum eigen value and corresponding vector

Initial guess – X

1. $Y = AX$

2. $X = Y / M$ $\quad M = \max_{1 \le i \le n} |y_i|$

Repeat 1. and 2. untill convergence

(Abuse of notation)

$M = \text{sign}(y_{i_{max}}) \max_{1 \le i \le n} |y_i|$

Example $\quad Y = \begin{bmatrix} -12 \\ -14 \\ 13 \end{bmatrix} \quad M = ?$

Scaling

$y_k = \lambda_1^k \alpha_1 V_1$

$$\frac{y_{k+1}(i)}{y_k(i)} = \lambda_1$$

i – can be any element

Convergence rate $\propto \dfrac{|\lambda_1|}{|\lambda_2|}$

b. Inverse power method – Minimum eigen value

$A \;—\; (\lambda, V)$

$A^{-1} \;—\; (1/\lambda, V)$

Convergen rate $\propto \dfrac{|\lambda_{n-1}|}{|\lambda_n|}$

(c) Shifted power method

Shift matrix by s $\quad A - sI$

Eigen values shift by s $\quad (\lambda - s)$

(i) Two extremes eigen values

Example - $\quad 2, 4, 12$

Direct power method $\quad \lambda_1 = 12$

$(A - \lambda_1 I) \rightarrow -10, -8, 0$

Example 2 $\quad 2, -4, 12 \quad \lambda_1 = 12$

$(A - \lambda_1 I) \rightarrow -10, -16, 0$

(ii) Intermediate eigen values

Gerschgorins Disk Theorem

Every eigen value of $A_{n \times n}$ lies in at least one of the circles $c_1, c_2, \ldots c_n$

Center of c_i - diagonal element a_{ii}

Radius of c_i - sum of absolute values of rest of the terms in the row

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

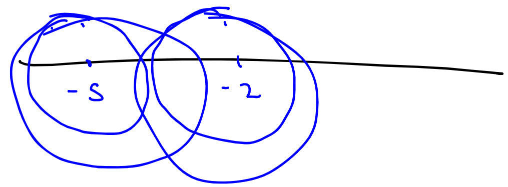
Example $\quad A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$C = [-5, -2]$

$r = [2, 2]$

-5 ± 2

-2 ± 2



Example

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 8.25 \\ 4.42 \\ 2.33 \end{bmatrix}$$

$8 \pm 1 \qquad 4 \pm 2 \qquad 3 \pm 1$

(a) Guess the value of shift 's' close to the eigen value given by Gerschgorins Disk theorem

(b) The corresponding eigen value of the shifted matrix will be close to zero

(c) Inverse power method to estimate it.

(3) Accelerates convergence

- The method can be used in conjunction with inverse power method to improve convergence rate

$$(A - SI)^{-1} X_k = X_{k+1}$$

$$(A - SI)\, X_{k+1} = X_k$$

$4.42 \downarrow 4 \downarrow 4.2$

$(A - 4I) \qquad \lambda' = 0.42$

$\lambda' = 0.22$

$$\text{Convergence rate} \propto \frac{|\lambda_{i+1} - s_k|}{|\lambda_i - s_k|}$$

$(A - sI)^{-1}$ $\quad M = 2.3624 \quad s = 4.0$

$(A - sI) = 1/M = 0.4231$

$A = s + 0.4231 = 4.4231$

QR method

Remarks – (1) The eigen values of a diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \qquad \lambda_i = d_i$$

(2) Upper triangular matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{22} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix} \qquad \lambda_i = u_{ii}$$

Analogous to Gauss Elimination

GE $\quad A \xrightarrow{\text{tranform}} D, L, U$

- The transformation should not change the solution

QR $\quad A \xrightarrow{\text{transfor}} D$ or U

transformation should not change eigen values

Similarity Transformation

Given a matrix $A_{n \times n}$, the similarity transformation is given by

$$B = M^{-1} A M$$

where M is any invertible matrix

Show that A & B have same eigen values

$$B = M^{-1} A M$$

$$M B M^{-1} = M M^{-1} A M M^{-1}$$

$$M B M^{-1} = A \qquad \text{(1)}$$

$$A V = \lambda V$$

From (1)

$$M B M^{-1} V = \lambda V$$

$$\Rightarrow \boxed{B(M^{-1} V) = \lambda (M^{-1} V)}$$

i.e. B has same eigen values as A λ

eigen vector are different $M^{-1} V$

What M will transform A into a diagonal or upper triangular matrix?

QR method

Q — orthogonal matrix

It is a square matrix whose columns are orthonormal vectors

- Orthogonal vectors - The vectors that are perpendicular to one another

$x \quad y \qquad x^T y = 0$

- Normal - Vector of unit length

$u = \frac{x}{\|x\|_2} \qquad u^T u = 1$

We denote orthonormal vectors by q

$q_1, q_2, \ldots q_n$

$\Rightarrow$ Orthogonal matrix

$$Q = \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_2 & \cdots & q_n \\ | & | & | & & | \end{bmatrix} \qquad q_i^T q_j = 0 \quad i \neq j \quad \text{else} = 1$$

$$Q Q^T = Q^T Q = I$$

$$\Rightarrow \boxed{Q^{-1} = Q^T}$$

R — Upper triangular matrix

QR method for finding eigen values

$\rightarrow \quad A_0 = A$

Decomposition $\quad A_0 = Q_0 R_0$

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Gram-Schmidt process

$$A_1 = R_0 Q_0$$

Is A_1 similar to A_0

$$B = Q_0^{-1} A_0 Q_0$$
$$= Q_0^{-1} (Q_0 R_0) Q_0$$
$$= R_0 Q_0 = A_1$$

A_k as $k \rightarrow \infty$ will be

Diagonal matrix — A is symmetric

Upper triangular matrix - otherwise

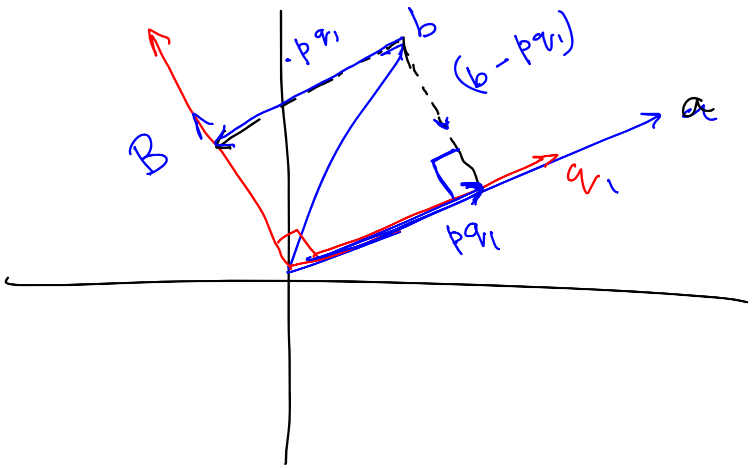
Schur's Lemma $\rightarrow$

How to decompose A into QR?

Gram-Schmidt process

A process for orthogonalizing vectors

Two independent vectors a & b



Gram Schmidt process

- It will make the vectors a and b perpendicular
- make the vectors length unity

$A = [a, b]$

$Q = [q_1\ q_2]$

Steps

1. $q_1 = \frac{a}{\|a\|}$

2. Projection of b on q_1 or component of b along the q_1 is subtracted from b

$(b - pq_1) \perp q_1$

$\Rightarrow q_1^T(b - pq_1) = 0$

$\Rightarrow p = \frac{q_1^T b}{q_1^T q_1} = q_1^T b$

$$p\,q_1 = (q_1^T b)\, q_1$$

$$B = b - (q_1^T b)\, q_1$$

$$q_2 = \frac{q_2}{\|q_2\|}$$

In general - The Gram-schmidt process starts with $q_1, q_2, \ldots a_n$ and ends up with $q_1, q_2 \ldots q_n$

At j^{th} step, it subtract the component of a_j from $q_1, q_2, \ldots q_{j-1}$

$$A_j = a_j - (q_1^T a_j)\, q_1 - (q_2^T a_j)\, q_2 - \ldots - (q_{j-1}^T a_j)\, q_{j-1}$$

column vector

$$[A] = [Q]\,[R]$$

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} = \begin{bmatrix} | & | & & \\ q_1 & q_2 & \ldots & q_n \\ | & | & & \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & \ldots & q_1^T a_n \\ & q_2^T a_2 & \ldots & \\ & & \ddots & \end{bmatrix}$$