Revision

$$Ax = b$$

Forward Error Analysis

[Condition number of the matrix]

- A measure for the magnitude of a vector or a matrix

NORM

(a) Vector norm - measure of length of the vector

$$x = [x_1 \; x_2 \; . . . \; x_n]^T$$

l_1 norm $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$

l_2 norm $\quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

l_∞ norm $\quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$

Properties of a vector norm

1. $\|x\| \geq 0 \quad ; \quad \|x\| = 0$ iff $x = 0$
2. $\|kx\| = |k| \, \|x\| \qquad k$ is scalar
3. $\|x + y\| \leq \|x\| + \|y\|$

(b) Matrix Norm $\quad A_{n \times n}$

(natural or Induced norm)

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

[longest elongation ratio]

$$= \max_{\|x\| = 1} \|Ax\|$$

[longest elongation of a unit vector]

Properties of Matrix Norm

$\|A\| \geq 0$ $\qquad \|A\| = 0$ iff $A = 0$

$\|kA\| = |k|\,\|A\|$

$\|A + B\| \leq \|A\| + \|B\|$

$\|Ax\| \leq \|A\|\,\|x\|$

l_1-norm $\quad \|A\|_1 = \max\limits_{1 \leq j \leq n} \sum\limits_{i=1}^{n} |a_{ij}|$

$l_\infty \quad \|A\|_\infty = \max\limits_{1 \leq i \leq n} \sum\limits_{i=1}^{n} |a_{ij}|$

$l_2 \quad \|A\|_2$ = square root of the maximum eigen value of the matrix $A^T A$

Spectral norm

Spectral radius

$\rho(A) = \max |\lambda_i|$ $\qquad \lambda \to$ eigen value of A

$Ax = \lambda x$

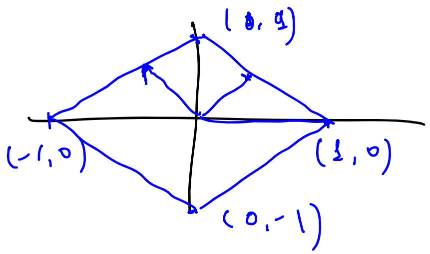
$\|A\| \geq \rho(A)$

Spectral radius provides a lower bound

Animation (2D case)

l_1 norm $\qquad \|A\|_1 = \max \|Ax\|_1$

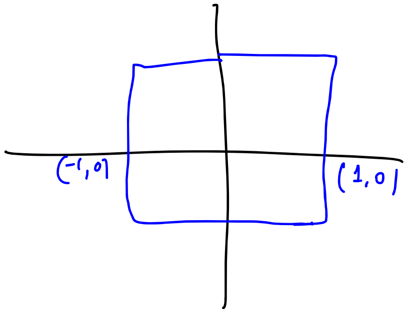
$\|x\|_1 = 1$



$x = [1 \; 0]$

$x = [0.5 \;\; 0.5]$

ℓ_∞ norm



Condition number of the matrix

Perturb $A + \Delta A$

change in solution $x + \Delta x$

$$\Rightarrow \frac{\|\Delta x\|}{\|x\|} \le C(A)\frac{\|\Delta A\|}{\|A\|}$$

$C(A) \geq 1$
$C(A) = 1$ (identity matrix)

$$C(A) = \|A^{-1}\|\,\|A\|$$

Perturb $b + \Delta b$

$$\frac{\|\Delta x\|}{\|x\|} \le C(A)\frac{\|\Delta b\|}{\|b\|}$$

$\Rightarrow C(A)$ is invariant of scaling

Question - Is the residual $r = b - \tilde{b}$ a good measure for $e = x - \tilde{x}$

Answer

$$\frac{\|e\|}{\|x\|} \le C(A)\frac{\|r\|}{\|b\|}$$

Iterative Refinement or Improvement

$$\boxed{Ax = b}$$

$$A\tilde{x} = \tilde{b}$$

$$Ax - A\tilde{x} = b - \tilde{b}$$

$$A(x - \tilde{x}) = r$$

$$\Rightarrow \boxed{Ae = r}$$

Unknown "e" can be estimated by $O(n^2)$

$$e = x - \tilde{x}$$

$$\Rightarrow \boxed{x = \tilde{x} + e}$$

Example

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$A\tilde{x} = \tilde{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$r = b - \tilde{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$Ae = r \longrightarrow e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$x = \tilde{x} + e$$

$$= \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Indirect or Iterative Methods

- Jacobi iteration
- Gauss - Seidel
- Relaxation techniques

All these methods are version of Fixed Point Iteration for linear system of equations

RECAP

1. $f(x) = 0$

Rearrange. $x_{i+1} = g(x_i)$

Convergence $|g'(s)| < 1$

Convergence rate : linear

2. $u(x,y) = 0 \Rightarrow x_{i+1} = g_1(x_i, y_i)$

$v(x,y) = 0 \Rightarrow y_{i+1} = g_2(x_{i+1}, y_i)$

Convergence

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$$

Example

$a_{13}|a_{11}$ $\quad$ $a_{13}|a_{11}$

$E_1 \quad a_{11}x + a_{12}y + a_{13}z - b_1 = 0 \quad -(1)$

$E_2 \quad a_{21}x + a_{22}y + a_{23}z - b_2 = 0$

$E_3 \quad a_{31}x + a_{32}y + a_{33}z - b_3 = 0$

$x = g_1(y,z) \qquad x = \dfrac{b_1 - a_{12}y - a_{13}z}{a_{11}}$ $\quad a_{11} = 1$

$y = g_2(x,z) \qquad y = \dfrac{b_2 - a_{21}x - a_{23}z}{a_{22}}$

$z = g_3(x,y) \qquad z = \dfrac{b_3 - a_{31}x - a_{32}y}{a_{33}}$

Jacobi Iteration

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_i, z_i)$$
$$z_{i+1} = g_3(x_i, y_i)$$

Gauss Seidel

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_{i+1}, z_i)$$
$$z_{i+1} = g_3(x_{i+1}, y_{i+1})$$

- GS method is faster than Jacobi iteration

Relaxation method - A way to improve convergence

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

$$x = \frac{4 - y}{2}$$

$$y = \frac{8 - x - z}{2}$$

$$z = \frac{8 - y}{2}$$

Jacobi

iter	x	y	z
0.	0	0	0
1	2	4	4
2.	0	1	2
3	1.5	3.0	3.5

Gauss Seidel

iter	x	y	z
0	0	0	0
1	2	3	2.5
2	0.5	2.5	2.75
3	0.75	2.25	2.875

Comments

1. Useful when dealing with large sparse system

2. To save computations, divide the equation by its diagonal

3 Convergence is not guranteed [like FP methods]

- linear convergence rate.

Convergence criteria

$$\cancel{\left|\frac{\partial g_1}{\partial x}\right|} + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| < 1$$

$$x = \frac{b_1}{a_{11}} - \frac{a_{12}\,y}{a_{11}} - \frac{a_{13}}{a_{11}}z = g(y,z)$$

$$\frac{|a_{12}|}{|a_{11}|} + \frac{|a_{13}|}{|a_{11}|} < 1$$

$$\Rightarrow |a_{11}| > |a_{12}| + |a_{13}|$$

$$\Rightarrow |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Diagonally dominant system

The magnitude of diagonal element should be greater than sum of absolute values of all off diagonal terms

- The criteria for convergence is sufficient but not necessary, i.e. the method may converge even if the criteria is not met.