Revision

System of linear equations $Ax = b$

Eigen value problem $AV = \lambda V$

Methods for finding λ and V

1. Characteristic polynomial

$$P_n(\lambda) = \det(A - \lambda I) = 0$$

If a_{ij} are real, there will be n λs $[\lambda_1, \lambda_2, \dots \lambda_n]$

These can be either real or complex or repeating

Example – Forming characteristic polynomial and then solving it is non-trivial

2. Power method
3. QR method

Extremely simple to program

much faster (computational complexity)

and much robust (CA).

Remarks

- Linearly independent vectors
- basis vectors

Linearly Independent Vectors

⇒ If $V_1, V_2, \ldots V_n$ are n-vectors, they are said to be linearly independent if

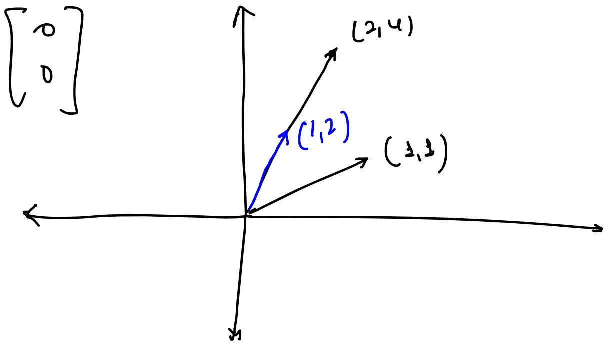
$$\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n = 0$$

iff $\alpha_1 = \alpha_2 = \ldots \alpha_n = 0$

Example

$$V_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\alpha_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



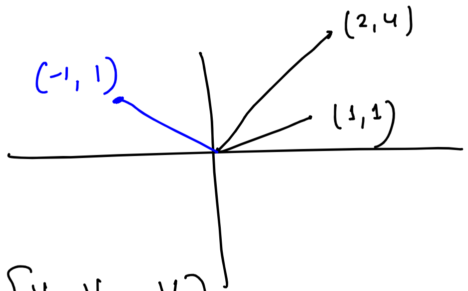
$$V_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\alpha_1 = 1 \qquad \alpha_2 = -2$

⇒ A set of n-linearly independent vectors form a basis for n-space

i.e. any vector X can be uniquely represented as a linear combination of the independent vectors

$$X = \alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha V_n$$



$$\alpha_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

⇒ $V = [V_1, V_2 \ldots V_n]$ ~ non-singular & hence invertible

Eigen Values

$A_{n\times n}$ of real numbers

$\lambda_1, \lambda_2, \ldots \lambda_n$

product of eigen values $=$ det(A)

Sum of eigen values $=$ trace(A)

Power Method

a. Direct power method

To find the largest [in terms of absolute value] eigen value and corresponding eigen vector.

Example

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad \lambda_1 = -6 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \qquad \lambda_2 = -1 \begin{bmatrix} \ \end{bmatrix}$$

Start with a guess vector $X = [1 \quad 1]^T$

$$A X_i = X_{i+1}$$

$$A X_{i+1} = X_{i+2}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Scale so that the largest element is 1

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \qquad S = -3 \qquad S = -5$$

$$A \begin{bmatrix} 1 \\ -0.4 \end{bmatrix} = \begin{bmatrix} -5.8 \\ 2.97 \end{bmatrix} \qquad S = -5.8$$

$$A \begin{bmatrix} 1 \\ -0.497 \end{bmatrix} = \begin{bmatrix} -5.96 \\ 2.99 \end{bmatrix} \qquad S = -5.96$$

$$A \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} \qquad S = -6$$

At convergence
$\lambda_{max} = S$
$X \rightarrow V$ corresponding to λ_{max}

* Later we have used M instead of S

Algorithm $A_{n \times n}$

1. Start with a guess vector $X_{n \times 1}$

2. Multiply $Y = AX$

3. Find Scaling factor

$$M = \text{sign}(y_{i\max}) \max_{1 \le i \le n} |y_i|$$

4. Divide each component of Y by M

$$X = Y / M$$

5. Repeat 2 to 4, unless change in M is negligible

$$\lambda_{max} = M$$

$$V = X$$

- Some books suggest that you pick one component of y and keep making it 1 after every iteration

- It is the correct way, but may result in division by zero

- If the algorithm is converging, the element corresponding to maximum value will not change

Why the power method works?

Assume that the matrix $A_{n\times n}$ has n linearly independent eigen vectors $v_1, v_2 \ldots v_n$ and the corresponding eigen values are $\lambda_1, \lambda_2, \ldots \lambda_n$

such

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots |\lambda_n|$$

Any vector X can be represented as a linear combination of V's

$$X = \alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n$$

$$AX = \alpha_1 AV_1 + \alpha_2 AV_2 + \ldots + \alpha_n AV_n$$
$$= \alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2 + \ldots + \alpha_n \lambda_n V_n$$

$$A^2 X = \alpha_1 \lambda_1^2 V_1 + \alpha_2 \lambda_2^2 V_2 + \ldots + \alpha_n \lambda_n^2 V_n$$

$\vdots$

$$A^k X = \alpha_1 \lambda_1^k V_1 + \alpha_2 \lambda_2^k V_2 + \ldots + \alpha_n \lambda_n^k V_n$$

$$= \lambda_1^k \left[\alpha_1 V_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k V_2 + \ldots \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k V_n\right]$$

if $k \to \infty$

$$y^k = A^k X = \lambda_1^k \alpha_1 V_1 \leftarrow \text{———} \quad (1)$$

Thus y^k is a multiple of V_1

if $\lambda_1 < 1$ $\quad y^k \to 0$ $\qquad \lambda > 1$ $\quad y^k \to \infty$

Scaling to avoid it

$$y_k^{(i)} = \lambda_1^k \alpha_1 V_1^{(i)} \qquad \frac{y_{k+1}^{(i)}}{y_k^{(i)}} \approx \lambda_1$$

$$y_{k+1}^{(i)} = \lambda_1^{k+1} \alpha_1 V_1^{(i)} \qquad \text{if } y_k^{(i)} = 1, \; y_{k+1}^{(i)} = \lambda_1$$

i - any element of y

Consequently, scaling a particular component of vector y at each iteration essentially factors λ_1 out. So, the equation (1) attains a finite value as $k \to \infty$, the scaling factor (M) approaches λ.

Remark.

1. The largest eigen value λ_1 is distinct [not repeated]

2. The eigen vectors should be independent

   [If all eigen values are distinct, the eigen vectors will be independent
   - otherwise it is still possible that vectors are independent, but not guaranteed.]

3. The initial guess value should contain a component of V_1,

$$\alpha_1 \neq 0$$

4. The converge rate is proportional to

$$\frac{|\lambda_1|}{|\lambda_2|}$$

where λ_1 is the largest eigen value
λ_2 is the second largest
(in terms of absolute value)

2. Inverse power method

⇒ To find minimum (smallest) eigen value and its vector

$$AV = \lambda V$$

$$\Rightarrow \quad A^{-1}AV = \lambda A^{-1}V$$

$$V = \lambda (A^{-1}V)$$

$$\Rightarrow \quad \boxed{A^{-1}V = \frac{1}{\lambda}V}$$

If (λ, V) are eigen pairs of A
$(1/\lambda, V)$ are eigen pairs of A^{-1}

⇒ To find smallest eigen value of A find largest eigen value of A^{-1} by direct power method.

(a) If A^{-1} is given

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1/3 & -1/3 \\ -1/3 & -5/6 \end{bmatrix}$$

$$X = [1 \quad 1]^T$$

$$A^{-1}X = \begin{bmatrix} -0.6667 \\ -1.667 \end{bmatrix} \qquad M = -1.667$$

$$A^{-1}\begin{bmatrix} 0.5714 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5238 \\ -1.0238 \end{bmatrix} \qquad M = -1.0238$$

$\vdots$

$$\begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1.0 \end{bmatrix} \qquad M = -1.0$$

smallest eigen value of $A = 1/M = -1.0$

(b) If A^{-1} is not given

$A = LU$

$y = A^{-1} X$

$\Rightarrow \boxed{Ay = X}$

$LUy = X$

$Lz = X$ — forward substitution

$Uy = z$ — backward substitution

3. The Shifted power method

If the matrix A is shifted by scalar s

$$(A - sI)X$$

$$= AX - sIX$$

$$= \lambda X - sX$$

$$= (\lambda - s)X$$

Shifting of a matrix, shift the eigen value also.

(a). To find opposite extreme eigen values

$\lambda_1 \quad \lambda_2 \quad - - - \quad \lambda_n$ — (Positive)

λ_1 is the largest

$[A - \lambda_1 I]$

$10, 5, 1$

Direct power method $\lambda_1 = 10$

$(A - 10I) \Rightarrow 0, -5, -9$

$-9 + 10 = 1$

maximum of the shifted matrix

$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad \lambda_1 = -6$

$[A - \lambda_1 I] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad M = 6$

$\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5 \end{bmatrix} \quad M = 5$

$\begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5 \end{bmatrix} \quad M = 5$

$\lambda_n - \lambda_1 = 5$

$\lambda_n = 5 + \lambda_1$

$\boxed{\lambda_n = -1}$

Smallest eigen value of A