System of Linear Equations

n - equations

n - unknowns

$$Ax = b$$

1. Direct Methods

(i) Diagonal

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_i = \frac{b_i}{a_{ii}}$$

(ii) Upper triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{nn} \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

Back substitution

(iii) Lower triangular

Forward substitution

(iii) Full Matrix (A)

Transfor or convert A to

Gauss elimination — U

Gauss Jordan — D

LU Decomposition — L & U

Gauss Elimination

Forward elimination + Backward substitution

- Pivot equation k
- Pivot element a_{kk}
- Multiplication factor $l_{ik} = \frac{a_{ik}}{a_{kk}}$
- Row reduction $a_{ij} = a_{ij} - l_{ik} a_{kj}$

Issues with GE

1. Pivot element is zero [Division by zero]
2. Round-off errors
3. Ill-conditioned

$$x_1 + 2x_2 = 10$$
$$1.1 x_1 + 2x_2 = 10.4$$

$\left.\right\}$ $x_1 = 4$, $x_2 = 3$

$S_1 = -\frac{1}{2} = -0.5$

$S_2 = -\frac{1.1}{2} = 0.55$

1.05 — $\left\{ \begin{array}{l} x_1 = 8 \\ x_2 = 1 \end{array} \right.$

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$x_2 = \boxed{-\frac{a_{11}}{a_{12}} x_1} + \frac{b_1}{a_{12}}$$
$$x_2 = -\frac{a_{21}}{a_{22}} x_1 + \frac{b_2}{b_1}$$

$$\frac{a_{11}}{a_{12}} \approx \frac{a_{21}}{a_{22}} \quad - \text{ILL conditioning}$$

$$\boxed{a_{11} a_{22} - a_{12} a_{21} \approx 0}$$

If determinant is close to zero
— ILL conditioning

If determinant is zero
— Singular

- Can we use determinant as a measure of ill conditioning?

1. $\begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix} \Rightarrow D = 2 - 2.2 = -0.2$

2. $\begin{bmatrix} 10 & 20 \\ 11 & 20 \end{bmatrix} \Rightarrow D = -20$

Avoiding 3 issues with GE

1. Use more significant digits

2. Pivoting

Row or Partial pivoting - exchange rows of the augmented matrix

Exchange rows which will result is largest magnitude of pivot element

Total pivoting – exchange rows & columns

Example

$$0.0004\, x_1 + 1.402\, x_2 = 1.406$$
$$0.4003\, x_1 - 1.502\, x_2 = 2.501$$

Exact – $x_1 = 10$
$x_2 = 1$

If we solved the problem by a 4-digit

– $x_1 = 0.9993$
– $x_2 = 12.5$

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{array}\right]$$

Pivoting

$$\begin{bmatrix} 0.4003 & -1.502 & 2.501 \\ 0.0004 & 1.402 & 1.406 \end{bmatrix}$$

$$l_{21} = \frac{0.0004}{0.4003} = 0.9993 \times 10^{-3}$$

$$\left[\begin{array}{cc|c} 0.4003 & -1.502 & 2.501 \\ 0 & 1.404 & 1.404 \end{array}\right]$$

Back sub.

$$x_2 = 1.0$$

$$x_1 = \frac{2.501 - (-1.502 \times x_2)}{0.4003} = \frac{4.003}{0.4003} = \underline{\underline{10}}$$

Total pivoting

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{array}\right]$$

Total pivoting

$$\left[\begin{array}{cc|c} -1.502 & 0.4003 & 2.502 \\ 1.402 & 0.0004 & 1.406 \end{array}\right]$$

$$l_{21} = -\frac{1.402}{1.502} = -0.9334$$

$$x_1 = 1.0$$

$$x_2 = 10.0$$

Why pivoting has worked?

$$10^{m} \times 0.0004\, x_1 + 10^{m} \times 1.402\, x_2 = 10^{m} \times 1.406$$

$$0.4003\, x_1 - 1.502\, x_2 = 2.501$$

$$l_{21} = \frac{0.4003}{0.0004 \times 10^{m}} = 1001 \times 10^{-m}$$

$$a_{22} = -1.502 - (1.402 \times \cancel{10^{m}} \times 1001 \times \cancel{10^{-m}})$$

$$= -1405$$

$$b_2 = 1404$$

$$x_1 = 0.9993$$

$$x_2 = 12.5$$

So, it is not the magnitude of pivot element that avoided round-off errors

$$|a_{12}| > |a_{11}|$$

$$x_1 = \frac{1.406 \times 10^m - 1.402 \times 10^m \times 0.9993}{0.0004}$$

x_2

If we use eq. (2)

$$x_1 = \frac{2.501 + 1.502 \quad 0.9993}{0.4003}$$

$$= \frac{4.002}{0.4003} = 9.998$$

Scaling of elements of 'A' ignores the round off errors

Scaling

$$\left[\begin{array}{cc|c} \boxed{2 \quad 10^5} & & 10^5 \\ 1 & 1 & 2 \end{array}\right]$$

Exact

$x_1 = 1.00002$

$x_2 = 0.99998$

If we use 3-digit

$l_{21} = 1/2 = 0.5$

$x_2 = 1.0$

$x_1 = 0.0$

Scaling - Divide the $\left|\frac{a_{ij}}{A_i}\right|$

A_j is the maximum value in a row

$$\left[\begin{array}{cc|c} 2\times10^{-5} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

partial pivoting

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2\times10^{-5} & 1 & 1 \end{array}\right]$$

multiplication factor $\quad a_{21} = \dfrac{2\times10^{-5}}{1}$

$$x_1 = x_2 = 1.0$$

Perform pivoting by using scaled coefficient matrix but perform computations (GE) by using original coefficients

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 10^5 & 10^5 \end{array}\right]$$

$$x_1 = x_2 = 1.0$$

Most common implementation of GE

1. Use scaled values of the coefficient as a criterion to decide pivoting

2. Retain original coefficients for actual elimination and substitutions

No General pivoting strategy that will work for all linear systems

Example – If coefficient matrix is a positive definite matrix, the BEST strategy is No Interchange

3. If you know, any special characteristic of the system, use it to decide the pivoting strategy.

Gauss Jordan

In this method the coefficient matrix is reduced to an Identity matrix.

- Requires a minor modification in GE

- At each step, first the pivot element is made unity by dividing pivot equation by the pivot element

- In addition to sub-diagonal elements, the above diagonal elements are also made zero.

Example

① $$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$$

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$

$R_1 = \frac{R_1}{a_{11}}$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 2.5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 2.5 \\ 0 & \underline{\underline{-2}} & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right]$$

$R_2 = R_2 / a_{22}$

$$\left[\begin{array}{cccc} 1 & 1.5 & -0.6 & 2.5 \\ 0 & 1 & 0.5 & 3.5 \\ 0 & 6 & -2 & 6 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 0 & -1.25 & -2.75 \\ 0 & 1 & 0.5 & 3.5 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$