Revision

System of linear equations

$$Ax = b$$

1. Direct Methods

- Gauss elimination — $O(2/3\, n^3)$
- Gauss Jordon — $O(n^3)$
- LU Decomposition
  - GE
  - Doolittle
  - Crout
  - Thomson algorithm — $O(n)$ (Tridiagonal)
  - Cholesky decomposition — $O(\frac{1}{3} n^3)$
    (symmetric positive definite)

Forward Error Analysis

Recall $f(x)$ $\qquad x + \Delta x$

$$C_p = \frac{|\Delta f / f|}{|\Delta x / x|}$$

$|x|$

$|\Delta x|$

$$\Rightarrow \left|\frac{\Delta f}{f}\right| = C_p \left|\frac{\Delta x}{x}\right|$$

Linear system

$$Ax = b$$

$$x = A^{-1} b$$

What would be the relative changes in x, ie $\frac{\Delta x}{x}$, due to small perturbations in A, $\frac{\Delta A}{A}$, or b, $\frac{\Delta b}{b}$.

Norms

Vector Norms - For a vector $X = [x_1\ x_2 \dots x_n]^T$

norm is a generalized length of the vector. It is denoted by $\|X\|$ and it satisfies the following properties

$\|X\|$ is a non negative real number

$\|X\| = 0$ iff $X = 0$ $\quad x_i = 0 \quad i = 1, \dots n$

$\|KX\| = |K|\,\|X\|$

$\|X+Y\| \leq \|X\| + \|Y\|$ - Triangular inequality

Examples

$\|X\|_1 = |x_1| + |x_2| + \dots |x_n|$ $\quad l_1$-norm

$\|X\|_2 = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$ $\quad l_2$-norm or Euclidean norm

$\|X\|_p = \left[\sum_{i=1}^{n} |x_i|^p \right]^{1/p}$ $\quad l_p$-norm

$\|X\|_\infty = \max_{1 \leq i \leq n} |x_i|$

Matrix Norm

$A_{n \times n}$

$\underset{n\times n}{A}\ \underset{n\times 1}{X} \longrightarrow$ vector of size $n \times 1$

The matrix norm is the maximum extension ratio achievable

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \quad \text{for all non-zero } X$$

Maximum is taken over all possible non-zero X

Another representation

$$\|A\| = \max_{\|x\|=1} \|AX\|$$

Satisfy the following properties

$\|A\| > 0$

$\|A\| = 0$ iff $A = 0$

$\|kA\| = |k| \|A\|$

$\|A + B\| \leq \|A\| + \|B\|$

$\|Ax\| \leq \|A\| \|x\|$

$\|AB\| \leq \|A\| \|B\|$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \leq \begin{bmatrix} (a_{11}+a_{12})x_2 \\ (a_{21}+a_{22})x_2 \end{bmatrix}$$

$$\|A\|_\infty = \max \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

Example

$$\|A\|_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |a_{ij}| \right) \quad \text{row-sum}$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \quad \text{column sum}$$

Example

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\|A\|_\infty = 9$$

$$\|A\|_1 = 8 \qquad \begin{bmatrix} 6 & 8 \end{bmatrix}$$

Frobenius norm -

$$\|A\|_F = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2}$$

$$= \sqrt{\text{trace}\,(A^T A)}$$

Spectral norm

$$\|A\|_2 = \left(\text{maximum eigen value of } \underline{\underline{A^T A}}\right)^{1/2}$$

Spectral radius

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i|$$

$$\boxed{A x_i = \lambda_i x_i}$$

λ_i - eigen values

x_i - eigen vectors

Spectral radius provides a lower bound on the matrix norm

$$\|A x_i\| = \|\lambda_i x_i\|$$

$$\|A x_i\| = |\lambda_i| \, \|x_i\|$$

But $\|A x_i\| \le \|A\| \, \|x_i\|$

$\Rightarrow \|A\| \, \|x_i\| \ge |\lambda_i| \, \|x_i\|$

$\Rightarrow \boxed{\|A\| \ge |\lambda_i| = \rho(A)}$ $\max |\lambda_i|$

Condition number

$Ax = b$

(a) Perturb A

$$(A + \Delta A)(x + \Delta x) = b$$

$$\cancel{Ax} + \Delta A x + A \Delta x + \Delta A \Delta x = \cancel{b}$$

$$\Rightarrow \quad \Delta x = -A^{-1} \Delta A (x + \Delta x)$$

Norm

$$\|\Delta x\| = \|A^{-1} \Delta A (x + \Delta x)\|$$

$$\leq \|A^{-1}\| \, \|\Delta A (x + \Delta x)\|$$

$$\leq \|A^{-1}\| \, \|\Delta A x\| + \|A^{-1}\| \cancel{\|\Delta A \Delta x\|}$$

If we assume $\|\Delta A \Delta x\| \leq$

$$\|\Delta x\| \leq \|A^{-1}\| \, \|\Delta A\| \, \|x\|$$

$$\Rightarrow \quad \frac{\|\Delta x\|}{\|x\|} \leq \underbrace{\|A^{-1}\| \, \|A\|}_{C(A)} \, \frac{\|\Delta A\|}{\|A\|}$$

$$\boxed{C_A = \|A^{-1}\| \, \|A\|}$$

(b) Perturb b

$$A(x + \Delta x) = (b + \Delta b)$$

$$\Rightarrow \quad \Delta x = A^{-1} \Delta b$$

$$\Rightarrow \quad \|\Delta x\| \leq \|A^{-1}\| \, \|\Delta b\|$$

$$\leq \|A^{-1}\| \, \|b\| \, \frac{\|\Delta b\|}{\|b\|}$$

$$\leq \|A^{-1}\| \, \|A\| \, \|x\| \, \frac{\|\Delta b\|}{\|b\|}$$

$$\Rightarrow \quad \boxed{\frac{\|\Delta x\|}{\|x\|} \leq \|A^{-1}\| \, \|A\| \, \frac{\|\Delta b\|}{\|b\|}}$$

$C(A) = \|A^{-1}\| \, \|A\|$

Example

$$x_1 + 2x_2 = 10 \quad (4\ 4\ 3)$$

$$1.1\, x_1 + 2x_2 = 10.4 \quad \rightarrow 8\ 4\ 1$$

1.05

$$\|\Delta x\|_\infty = 4$$

$$A = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix}$$

$$\|A\|_\infty = 3.1$$

$$A^{-1} = \begin{bmatrix} -10 & 10 \\ 5.5 & -5 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = 20$$

$$C(A) = \|A^{-1}\|_\infty \|A\|_\infty$$

$$= 3.1 \times 20 = 62$$

ill-conditioned

$$\boxed{\text{Smallest condition number} \quad A = I \quad C(A) = 1}$$

$$\frac{\|\Delta x\|}{\|x\|} \le C_p \frac{\|\Delta A\|}{\|A\|} \qquad x = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\le 62 \cdot \frac{0.05}{3.1}$$

$$\frac{\|\Delta x\|}{\|x\|} \le 1$$

$$\|\Delta x\| \le \|x\|_\infty$$

$$\|\Delta x\| \le 4$$

Determinant is not a good measure of the ill or well conditioning of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix} \qquad \det(A) = -0.2$$

$$A = \begin{bmatrix} 1000 & 2000 \\ 1100 & 2000 \end{bmatrix} \qquad \det(A) = 2\times10^{5}$$

$C(A) = 62$

$$\|A\|_\infty = 3100$$

$$A^{-1} = \begin{bmatrix} -0.01 & 0.01 \\ 0.0055 & -0.005 \end{bmatrix} \qquad \|A^{-1}\|_\infty = 0.02$$

$$C(A) = \|A^{-1}\|_\infty \|A\|_\infty = \underline{\underline{62}}$$

Invariant

Q. Recommendation that after estimating x, substitute it in the equation and see whether the equation is satisfied or not

$$Ax = b$$

Instead of x we estimated $\tilde{x}$

$$A\tilde{x} = \tilde{b}$$

$r = b - \tilde{b}$ is small, then the estimate $\tilde{x}$ is good?

$$e = x - \tilde{x}$$
$$r = b - \tilde{b}$$
$$Ax - A\tilde{x} = r$$
$$\Rightarrow A(x - \tilde{x}) = r$$
$$\Rightarrow e = A^{-1} r$$
$$\Rightarrow \|e\| \le \|A^{-1}\| \, \|r\| \quad \text{—— (1)}$$

$$Ax = b$$
$$\|b\| = \|Ax\|$$
$$\|b\| \le \|A\| \, \|x\|$$
$$\Rightarrow \|x\| \ge \frac{\|b\|}{\|A\|} \quad \text{—— (2)}$$

$$\frac{\|e\|}{\|x\|} \le \frac{\|A^{-1}\| \, \|r\|}{\|b\| / \|A\|}$$
$$\le \|A^{-1}\| \, \|A\| \, \frac{\|r\|}{\|b\|}$$
$$\Rightarrow \boxed{\frac{\|e\|}{\|x\|} \le C(A) \, \frac{\|r\|}{\|b\|}}$$

Example

$$x_1 + 2x_2 = 10 \rightarrow 10$$
$$1.05\, x_1 + 2x_2 = 10.4 \rightarrow 10.2$$

Tm
84 1

1.1

4, 3

Matrix Norm

The l_∞ norm of a matrix can be easily computed from the entries of the matrix.

If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Proof First we show that $\|A\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. Let $\mathbf{x}$ be an n-dimensional vector with $1 = \|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$. Since $A\mathbf{x}$ is also an n-dimensional vector,

$$\|A\mathbf{x}\|_\infty = \max_{1\le i\le n} |(A\mathbf{x})_i| = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}x_j\right| \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \max_{1\le j\le n} |x_j|.$$

But $\max_{1\le j\le n} |x_j| = \|\mathbf{x}\|_\infty = 1$, so

$$\|A\mathbf{x}\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Consequently,

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|. \tag{7.3}$$

Now we will show the opposite inequality, that $\|A\|_\infty \ge \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. Let p be an integer with

$$\sum_{j=1}^{n} |a_{pj}| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|,$$

and $\mathbf{x}$ be the vector with components

$$x_j = \begin{cases} 1, & \text{if } a_{pj} \ge 0, \\ -1, & \text{if } a_{pj} < 0. \end{cases}$$

Then $\|\mathbf{x}\|_\infty = 1$ and $a_{pj}x_j = |a_{pj}|$, for all $j = 1, 2, \ldots, n$, so

$$\|A\mathbf{x}\|_\infty = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}x_j\right| \ge \left|\sum_{j=1}^{n} a_{pj}x_j\right| = \left|\sum_{j=1}^{n} |a_{pj}|\right| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

This result implies that

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \ge \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|,$$

which, together with Inequality (7.3), gives

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

From: Numerical Analysis by
Burden & Faires