Hardness vs. Randomness

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[1] https://www.math.ias.edu/avi/node/780 Noam Nisan and Avi Wigderson

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  - quality of the output – n, ϵ
  - price – r

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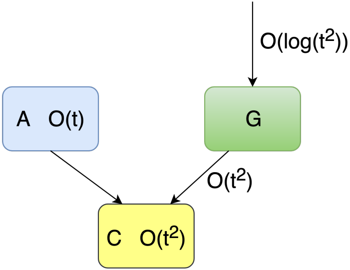

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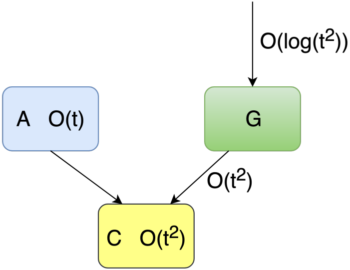

- Simulate A deterministically, by trying all the possible random seeds and taking a majority vote

Applications

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- $\mathsf{BPP} \subset \cap_{\epsilon>0}\mathsf{DTIME}(2^{n^\epsilon})$
- $\mathsf{BPP} \subset \mathsf{DTIME}(2^{(\log n)^c})$
- $\mathsf{BPP} = P$
- $\mathsf{RNC} \subset \cap_{\epsilon>0}\mathsf{DSPACE}(n^\epsilon)$
- $\mathsf{RNC} \subset \mathsf{DSPACE}(polylog)$

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Prior Work

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- Blum and Micali: Intractability of Discrete Logarithm function
- Yao: One-way permutation
  - Require a strong unproven assumption. (the existence of a one-way function, an assumption which is even stronger than $P \neq NP$)
  - sequential

Contribution of Paper

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- Let $f_1, \cdots, f_k$ all be (ϵ, S)-hard. Then for any $\delta > 0$, the function $f(x_1, \cdot, x_k)$ defined by

$$f(x_1, \cdot, x_k) = \sum_{i=i}^{k} f_i(x_i)(mod\ 2)$$

is $(\epsilon^k + \delta, \delta^2(1-\epsilon)^2 S)$-hard

Hardness

- Let $f = \{0,1\}^* \to \{0,1\}$ be a boolean function. We say that f cannot be approximated by circuits of size $s(n)$ if for some constant k, all large enough n, and all circuits C_n of size $s(n)$:

$$\Pr\left[C_n(x) \neq f(x)\right] > n^{-k}$$

  - small circuits attempting to compute f have a non-negligible fraction of error

- Let $f : \{0,1\}^* \to \{0,1\}$ be a boolean function, and let f_m be the restriction of f to strings of length m. The Hardness of f at m, $H_{f(m)}$ is defined to be the maximum integer h_m such that fm is $(1/h_m, h_m)$-hard

Hardness

- Let $s(m)$ be any function such that $m \leq s(m) \leq 2^m$; if there exists a function f in EXPTIME that cannot be approximated by circuits of size $s(m)$, then for some $c > 0$ there exists a function f' in EXPTIME that has hardness $H_{f'(m)} \geq s(m^c)$.

Construction

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- A collection of sets $\{S_1, \cdots, S_n\}$, where $S_i \subset \{1, \cdots, l\}$ is called a (k, m)-design if:
  - $\forall i : \ |S_i| = m$
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  - $G : l \rightarrow n$ given by $G(x) = f_{A(x)}$ is a pseudorandom generator
  - $f_{A(x)}$ the n bit vector of bits computed by applying the function f to the subsets of the $x's$ denoted by the n different rows of A

Proof (Main Idea)

- Assume that G is not pseudorandom generator

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- Construct the circuit that predicts the next bit of $f_{A(x)}$
- This contradicts that $H_{f(m)} \geq n^2$

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  - For some $c > 0$ there exists a function in EXPTIME with hardness $s(l^c)$
  - For some $c > 0$ there exists a quick pseudorandom generator $G : l \rightarrow s(l^c)$

Questions?