Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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March 26, 2017

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Overview

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

Motivation

Motivation

Radiation Hydrodynamics

- Couple radiation transport and fluid dynamics
- Material temperatures so high that fluid momentum and energy is altered by release of thermal radiation
- Required in high energy density physics (National Ignition Facility, astrophysics)

Radiation transport simulations are expensive and difficult to incorporate into hydro

Efficient methods developed independently

Mixed Hybrid Finite Element Method works well for hydrodynamics but is incompatible with radiation transport

Want to use MHFEM for hydrodynamics and Linear Discontinuous Galerkin for transport

Goal

Develop an acceleration scheme that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Increases compatibility with MHFEM multiphysics

Show that MHFEM can be used to accelerate lumped LDG radiation transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates Angular Discretization

Compute angular flux on N discrete angles defined by Gauss Quadrature

$$\psi(x, \mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

μ_1, μ_2, $\ldots$, μ_N defined by N-point Gauss Quadrature Rule

Integrate order $N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x, \mu) \,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S$_N$ Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \frac{\Sigma_s(x)}{2}\phi^0(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

$\Rightarrow \phi^1(x)$ is the uncollided flux

$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Each source iteration adds scattering information

Slow to converge in optically thick systems with minimal losses to absorption and leakage

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Diffusion Synthetic Acceleration

Large, highly scattering systems $\Rightarrow$ Diffusion Theory is accurate!

Diffusion Synthetic Acceleration

1. Given previous estimate for $\phi^{\ell}(x)$, solve for $\psi_n^{\ell+1/2}$
2. Compute $\phi^{\ell+1/2}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1/2}(x)$
3. Solve diffusion equation for a correction factor, $f^{\ell+1}(x)$
4. Update scattering term with $\phi^{\ell+1}(x) = \phi^{\ell+1/2}(x) + f^{\ell+1}(x)$ and repeat until:
$$\frac{\|\phi^{\ell+1}(x) - \phi^{\ell}(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

DSA Problems

Becomes non-convergent in highly scattering media with coarse spatial grids

Transport and Diffusion steps must be consistently differenced

Consistently differenced equations are more expensive to solve

Transport and MHFEM are not compatible

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A new acceleration scheme is needed!

Eddington Acceleration

Zeroth Angular Moment

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \; = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu \frac{\int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu}{\int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu}$$

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Eddington Factor

$$\langle\mu^2\rangle(x)=\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

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Eddington Factor

$$\langle\mu^2\rangle(x)=\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

Shape function

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

Combine Zero and First Moments $\Rightarrow$ Drift Diffusion Equation

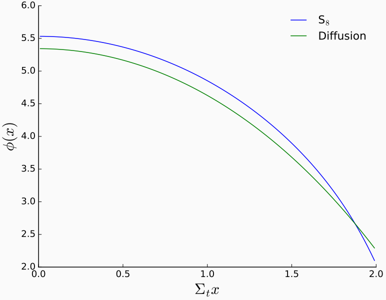
$$-\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_a(x)\phi(x) = Q(x)$$

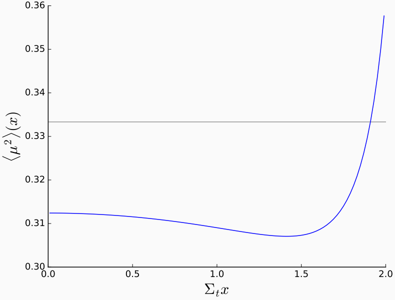
Fick's Law: set $\langle \mu^2 \rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \phi(x)$$

S_8 v. Diffusion

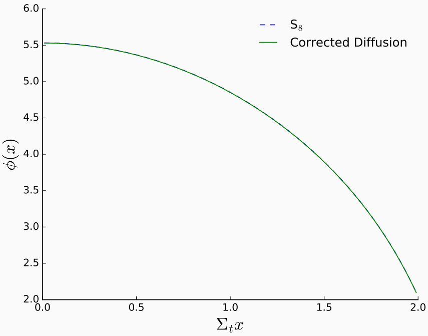
Small system $\Rightarrow$ diffusion not expected to be accurate





S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



Drift Diffusion and S_N match!

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Eddington Acceleration Properties

Acceleration occurs due to:

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Solution to moment equations models all scattering events at once
3. Dependence on source iterations to introduce scattering information is reduced

Downside: produces 2 solutions (S_N and Drift Diffusion)

Benefits

1. No need for consistent differencing $\Rightarrow$ transport and acceleration steps can be differenced with arbitrarily different methods
2. Moment Equations are conservative
3. Accelerates source iterations
4. Difference between S_N and Drift Diffusion solution can be used as a measure of iteration uncertainty

Results

Test Problem

Slab with reflecting left boundary and vacuum right boundary

Thickness of 20 cm

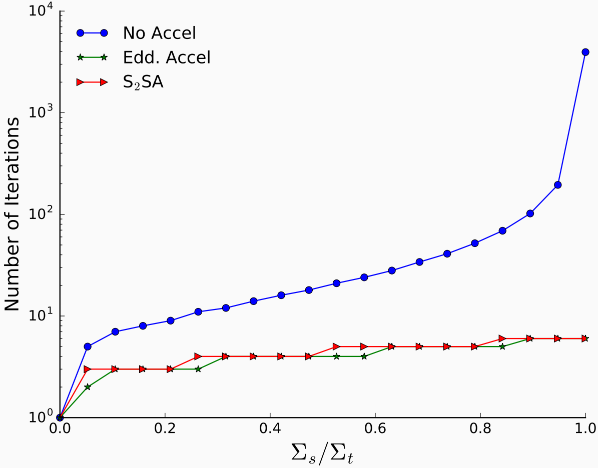
$\Sigma_t(x) = 1\,\text{cm}^{-1}$

100 cells $\Rightarrow$ optical thickness of 20 and optical thickness per cell of 0.2

S_8 solved with lumped Linear Discontinuous Galerkin

Moment Equations solved with Mixed Hybrid Finite Element

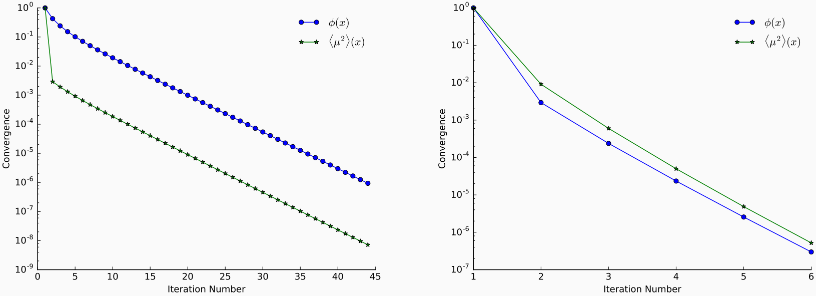
Iterations to Convergence Comparison



Accelerates between 2.5 and 650 times $\Rightarrow$ acceleration is occurring

Performs similarly to consistent acceleration scheme

Convergence Rate Comparison



Fast rate of convergence of $\langle\mu^2\rangle(x)$ is transfered to $\phi(x)$

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Inherently compatible with rad-hydro multiphysics
  - Transport and acceleration steps can be discretized with arbitrarily different methods
  - Avoids consistency issues
  - Provides less expensive, conservative solution
- Proved Mixed Hybrid Finite Element Method can be used to accelerate lumped Linear Discontinuous Galerkin transport

Future Work

- Develop a rad-hydro algorithm
  - Make use of inexpensive Moment solution in operator split iterations
- Add discretization in energy
- Higher dimensions
- Anisotropic scattering

References

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Questions?