April 27, 2011

The MUSCL-Hancock Method

To: Students in NUEN 627

CC: Dr. J. E. Morel

From: Jarrod Edwards

Date: 4/27/2011

Re: Solving the Euler equations using the MUSCL-Hancock Method

---

Introduction

Hydrodynamic motion, e.g. the fluid motion of a gas, can be described using the Euler equations. A dynamic gas can be characterized by its density, velocity, internal energy, and pressure. In three dimensions, this represents 6 independent variables, since velocity is a vector having three components. However, in a 1-D system, this reduces to 4 unknowns. The Euler equations for a 1-D, incompressible gas consist of 3 conservation equations – one for mass, one for momentum, and one for total energy:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{1.1}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0 \tag{1.2}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + e\right)\right] + \frac{\partial}{\partial x}\left\{\left[\rho\left(\frac{1}{2}u^2 + e\right) + p\right]u\right\} = 0 \tag{1.3}$$

where ρ is the density, u is the velocity, e is the internal energy, and p is the pressure. Note, however, that another equation is required to close the system. This is called an equation of state, which is determined by the properties of the gas in question. For an ideal gas, an appropriate equation of state is:

$$p = (\gamma - 1)\rho e \tag{1.4}$$

where γ is determined by the molecular properties of the gas. Equations (1.1) - (1.3) may be written as a system of equations as follows:

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = 0 \tag{1.5}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho\left(\frac{1}{2}u^2 + e\right) \end{bmatrix} \tag{1.6}$$

and

$$F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \left[\rho\left(\frac{1}{2}u^2 + e\right) + p\right]u \end{bmatrix} \tag{1.7}$$

These notes describe how to solve system (1.5) using the MUSCL-Hancock Method (MHM). MUSCL stands for Monotone Upstream-centered Scheme for Conservation Laws. Here, monotone implies that the scheme resists spurious oscillations, except near strong gradients. The MHM is second-order accurate in both space and time and yields a piecewise constant solution.

MUSCL-Hancock Method

The MHM is a three step method that achieves second-order accuracy through data reconstruction in space and evolution in time. Consider a solution vector U such that u_i^n represents the piecewise constant solution in cell i at time $t = t^n$. (Note the change in notation that, for the purposes of this memo, u no longer represents the velocity, but is generalized to represent one of the three conserved variables given in U.) The first step of the MHM is to construct a linear representation of the solution in space from the piecewise constant data:

$$u_{i,L}^n = u_i^n - \frac{1}{2}\Delta_i \tag{1.8}$$

$$u_{i,R}^n = u_i^n + \frac{1}{2}\Delta_i \tag{1.9}$$

where Δ_i is the slope derived from the piecewise constant data and is computed by:

$$\Delta_i = \frac{1}{2}(1+\omega)\Delta u_{i-1/2} + \frac{1}{2}(1-\omega)\Delta u_{i+1/2} \tag{1.10}$$

where

$$\Delta u_{i-1/2} \equiv u_i - u_{i-1} \tag{1.11}$$

$$\Delta u_{i+1/2} \equiv u_{i+1} - u_i \tag{1.12}$$

and ω is a parameter in the range [-1, 1] chosen to weight the slope toward the left or right difference. If ω = 0 is chosen, then (1.10) becomes the centered difference approximation, multiplied by Δx, to the first derivative in space of the solution at time $t = t^n$.

The second step of the MHM is to evolve the linear representation obtained by (1.8) and (1.9) by a half step in time. This is done using the conservation statement in (1.5) integrated over Δt and Δx, yielding:

$$u_{i,L}^{n+1/2} = u_{i,L}^{n} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[f\left(u_{i,L}^{n}\right) - f\left(u_{i,R}^{n}\right)\right] \tag{1.13}$$

$$u_{i,R}^{n+1/2} = u_{i,R}^{n} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[f\left(u_{i,L}^{n}\right) - f\left(u_{i,R}^{n}\right)\right] \tag{1.14}$$

where f(u) is the flux, given by (1.7), corresponding to the conserved variable u. Now, at each cell interface i+½, there are two evolved variables, $u_{i,R}^{n+1/2}$ and $u_{i+1,L}^{n+1/2}$. In order to find the intercell flux $F_{i+1/2}$, we treat each interface as a Riemann problem:

$$\begin{aligned} &\frac{\partial}{\partial t}u + \frac{\partial}{\partial x} f\left(u\right) = 0 \\ &u\left(x,0\right) = \begin{cases} u_{i,R}^{n+1/2}, & x < 0 \\ u_{i+1,L}^{n+1/2}, & x > 0 \end{cases} \end{aligned} \tag{1.15}$$

Thus, the third step of the MHM is to solve the Riemann problem given by (1.15) at each interface. Many Riemann solution techniques exist that vary in simplicity and accuracy. In these notes, we describe the HLL Riemann solver and the HLLC Riemann solver, which is a modified version of HLL that preserves contact discontinuities. The Riemann solvers are used to compute the similarity solution, $U_{i+1/2}$, and the flux, $F_{i+1/2}$, at the cell interfaces, i.e. for i = 1, .., N-1. Finally, the interface flux values, along with the piecewise constant solution at time $t=t^n$, are then used to compute the piecewise constant solution at time $t=t^{n+1}$ using the conservative equation:

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left(F_{i-1/2} - F_{i+1/2}\right) \tag{1.16}$$

HLL and HLLC Riemann Solvers

The exact solution to the Riemann problem is composed of several regions separated by various waves. The HLL Riemann solver, which is named for Harten, Lax, and van Leer, was developed by lumping all intermediate solution states to the Riemann problem, i.e. those bound by the fastest moving waves on the left and right, into a single solution state U^{hll}. The HLL solution is then characterized by:

$$\tilde{U}\left(x,t\right) = \begin{cases} U_L & \frac{x}{t} \le S_L \\ U^{hll} & S_L \le \frac{x}{t} \le S_R \\ U_R & S_R \le \frac{x}{t} \end{cases} \tag{1.17}$$

where U_L and U_R are the solution states initially to the left and right of the interface, respectively, and U^{hll} is the lumped intermediate state given by:

$$U^{hll} = \frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L} \tag{1.18}$$

The two bounding waves, moving at speeds S_L and S_R, separating these three states are illustrated in figure 1.

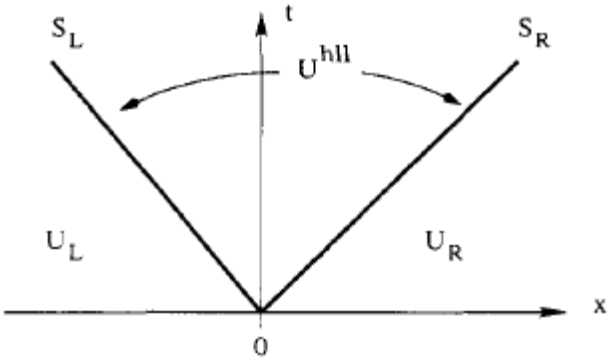

Figure 1. Solution domain of the HLL Riemann solver. [Toro]

In practice, we should therefore be certain that the estimates for S_L and S_R do bound the actual wave speeds present in the true solution. The simplest and most well known method for computing estimates for these wave speeds is:

$$\begin{aligned} S_L &= \min\left\{u_L - a_L, u_R - a_R\right\} \\ S_R &= \max\left\{u_L + a_L, u_R + a_R\right\} \end{aligned} \tag{1.19}$$

where a is the speed of sound. For an ideal gas, this is:

$$a = \sqrt{\frac{\gamma p}{\rho}} \tag{1.20}$$

Additional methods for computing the wave speeds may be found in §10.5 of Toro. The flux at each cell interface is given by:

$$F_{i+1/2} = \begin{cases} F_L & if \quad 0 \le S_L \\ F^{hll} & if \quad S_L \le 0 \le S_R \\ F_R & if \quad S_R \le 0 \end{cases} \tag{1.21}$$

where

$$F^{hll} = \frac{S_R F_L - S_L F_R + S_L S_R \left(U_R - U_L\right)}{S_R - S_L} \tag{1.22}$$

and $F_L = F(U_L)$ and $F_R = F(U_R)$. Derivations for U^{hll} and F^{hll} are in Toro.

The primary drawback to the HLL Riemann solver is numerical smearing due to the two wave assumption. This assumption does not account for certain physical features, such as contact surfaces, material interfaces, and shear waves. As a result, these features become smeared in the computed solution. This phenomenon is worst in the limiting case of contact surfaces moving with the mesh, e.g. if the spatial mesh is fixed in time, then the speed of the limiting case contact wave would be zero.

The HLLC Riemann solver was developed to overcome this deficiency. This is accomplished by restoring the contact and shear waves to the intermediate region of the HLL solution domain, as shown in Figure 2. (Note: the "C" in HLLC stands for contact.)

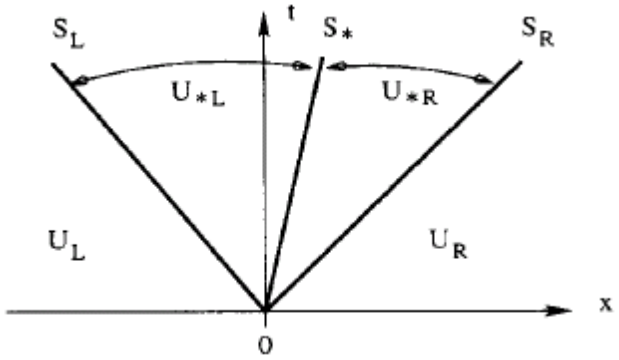

Figure 2. Solution domain for the HLLC Riemann solver. [Toro]

Here, S^* is the speed of the restored contact and shear waves. Because S_L and S_R correspond by definition to the bounding waves of the intermediate states, S^* will always be bound by these values. A useful estimate for S^* is as follows:

$$S^* = \frac{p_R - p_L + \rho_L u_L \left(S_L - u_L\right) - \rho_R u_R \left(S_R - u_R\right)}{\rho_L \left(S_L - u_L\right) - \rho_R \left(S_R - u_R\right)} \tag{1.23}$$

By applying the Rankine-Hugoniot conditions across each of the three waves and setting $u_{L,*} = u_{R,*} = S^*$ and $p_{L,*} = p_{R,*}$, the approximate solution is given by:

$$\tilde{U}(x,t)=\begin{cases} U_L & \frac{x}{t}\le S_L \\ U_{L,*} & S_L\le\frac{x}{t}\le S^* \\ U_{R,*} & S^*\le\frac{x}{t}\le S_R \\ U_R & S_R\le\frac{x}{t} \end{cases} \tag{1.24}$$

where

$$U_{K,^*}=\rho_K\left(\frac{S_K-u_K}{S_K-S^*}\right)\begin{bmatrix} 1 \\ S^* \\ \left(\frac{1}{2}u_K^2+e_K\right)+\left(S^*-u_K\right)\left[S^*+\frac{p_K}{\rho_K\left(S_K-u_K\right)}\right] \end{bmatrix} \tag{1.25}$$

for K = L and K = R. The interface flux values are given by:

$$F_{i+1/2}^{hllc}=\begin{cases} F_L & if \quad 0\le S_L \\ F_{L,*}=F_L+S_L\left(U_{L,*}-U_L\right) & if \quad S_L\le 0\le S^* \\ F_{R,*}=F_R+S_R\left(U_{R,*}-U_R\right) & if \quad S^*\le 0\le S_R \\ F_R & if \quad S_R\le 0 \end{cases} \tag{1.26}$$

Again, derivations for the intermediate states are included in Toro.

Slope Limiter Techniques

Though the MUSCL Hancock method is designed to avoid oscillations, oscillations will still occur in the vicinity of strong gradients, e.g. shear or contact waves. The reason for this may be demonstrated by setting ω = 0 in the slope and noting that the MHM reduces to the Fromm method. The Fromm method is a second order linear method, which according to Gudonov's Theorem (see §13.5.3 of Toro) will produce spurious oscillations near strong gradients.

Therefore, in order to try to eliminate these oscillations, we apply the Total Variation Diminishing (TVD) constraint. The Total Variation of a solution applied to a mesh is defined as:

$$TV\left(u^n\right)=\sum_{i=-\infty}^{\infty}\left|u_{i+1}^n-u_i^n\right| \tag{1.27}$$

where, if the total variation is to be bounded, the solution values must approach some constant as i approaches ±∞. In a TVD scheme, the total variation decreases from one time step to the next:

$$TV\left(u^{n+1}\right) < TV\left(u^{n}\right) \tag{1.28}$$

for all n. Two fundamental properties for a TVD scheme applied to an IVP with finite TV(u^0) are that: (1) no new local extrema in space may be created, and (2) the value of a local minimum will increase over time and a local maximum will decrease over time. Two techniques that are commonly used to construct TVD schemes of the MHM are slope limiters and flux limiters. Here, I will describe the use of slope limiters which can be applied to the MHM to make it TVD.

Using the slope limiter technique, the idea is to replace the slope in the data reconstruction with a "limited slope" that constrains the variation in the solution from one time step to the next. Note, however, that in order to achieve this, it is not simply the boundary extrapolated values, $u_{i,L}$ and $u_{i,R}$, that must be constrained, but the slope limiting must be performed for the *evolved* boundary extrapolated values. Hence,

$$\begin{aligned} \min\left\{u_{i-1}^{n}, u_{i}^{n}\right\} \le u_{i,L}^{n+1/2} \le \max\left\{u_{i-1}^{n}, u_{i}^{n}\right\} \quad \forall i \\ \min\left\{u_{i}^{n}, u_{i+1}^{n}\right\} \le u_{i,R}^{n+1/2} \le \max\left\{u_{i}^{n}, u_{i+1}^{n}\right\} \quad \forall i \end{aligned} \tag{1.29}$$

If this condition is met, then the following statement is true, and the resulting method is TVD.

$$\min_{k}\left\{u_{k}^{n}\right\}_{k=i-2}^{i+2} \le u_{i}^{n+1} \le \max\left\{u_{k}^{n}\right\}_{k=i-1}^{i+2} \tag{1.30}$$

The limited slope may be constructed as:

$$\overline{\Delta}_{i} = \xi_{i}\Delta_{i} \tag{1.31}$$

with Δ_i given by (1.10). Several different slope limiters may be constructed that have analogous behavior to some popular flux limiters. The following is a vanLeer type slope limiter:

$$\xi\left(r\right) = \begin{cases} 0 & r \le 0 \\ \min\left\{\dfrac{2r}{1+r}, \xi_{R}\left(r\right)\right\} & r \ge 0 \end{cases} \tag{1.32}$$

where

$$\xi_{R}\left(r\right) = \frac{2\beta_{i+1/2}}{1-\omega+\left(1+\omega\right)r} \tag{1.33}$$

$$r = \frac{\Delta_{i-1/2}}{\Delta_{i+1/2}} \tag{1.34}$$

In order to determine $\beta_{i+1/2}$, the Courant number and correlative upwind information may be used; however, a simple, conservative choice is $\beta_{i+1/2} = 1$. A MINBEE type slope limiter is:

$$\xi(r)=\begin{cases} 0 & r\le 0 \\ r & 0\le r\le 1 \\ \min\left\{1,\xi_R(r)\right\} & r\ge 1 \end{cases} \tag{1.35}$$

A SUPERBEE type slope limiter is:

$$\xi(r)=\begin{cases} 0 & r\le 0 \\ 2r & 0\le r\le \frac{1}{2} \\ 1 & \frac{1}{2}\le r\le 1 \\ \min\left\{r,\xi_R(r),2\right\} & r\ge 1 \end{cases} \tag{1.36}$$

The SUPERBEE slope limiter gives good results for discontinuous solutions; however, it is overly compressive for smooth solutions. The MINBEE and vanLeer type slope limiters are recommended for general use.

Time Step Control

The easiest method of determining the size of the time step for the MHM is to choose an initial time step size and hold that step size constant throughout the calculation. However, this can either be very inefficient if the time step is too small, or it can cause the calculation to become unstable if the time step is too large. A better approach is to change the time step size from one time step to the next based on the "current" physical conditions.

In computing a given time step size, one important limitation arises from the MUSCL Hancock treatment of the cell interface fluxes as solutions to explicit, local Riemann problems (step 3 of the MHM). Because of this, in order to ensure stability, we must prevent the flux at one interface from contributing to the flux at another interface in a given time step by setting the time step size so that the fastest wave in the Riemann solution doesn't travel more than one cell width over that step. This is characterized by the Courant criterion:

$$C_{cfl}=\frac{\left|S^n_{\max}\right|\Delta t}{\Delta x} \tag{1.37}$$

where $S^n_{\max}$ is the maximum signal velocity, i.e. the characteristic along which the fastest changes to the solution travel, and the dimensionless Courant coefficient, C_{cfl}, is such that:

$$0<C_{cfl}\le 1 \tag{1.38}$$

Note that we can also write (1.37) as:

$$C_{cfl}=\frac{\left|S^n_{\max}\right|}{\Delta x/\Delta t} \tag{1.39}$$

In (1.39), we see that C_{cfl} is just the ratio of the fastest signal speed to the "grid speed," which is the fastest speed that solution information can travel while remaining within a cell over a given time step. In practice, then, between steps 1 and 2 of the MHM, we determine the step size for a given time step by choosing C_{cfl} between 0 and 1, computing the maximum signal velocity over the problem domain using (1.19), and setting the time step using:

$$\Delta t = \frac{C_{cfl}\Delta x}{\left|S^n_{\max}\right|} \tag{1.40}$$

References

E. F. Toro, Riemann Solvers and Numerical Methods for Fluid Dynamics – A Practical Introduction , Springer-Verlag.