Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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NUCLEAR ENGINEERING
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Overview

Motivation

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Radiation Hydrodynamics

- Describes the effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory experiments (NIF, Z Machine) and astrophysics

Mixed Hybrid Finite Element Method (MHFEM) hydrodynamics

Problems

- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme
- Radiation transport is expensive

Goal

Develop a transport algorithm that

1. Accelerates Discrete Ordinates Source Iteration
2. Bridges Linear Discontinuous Galerkin (LDG) transport and MHFEM multiphysics

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Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x,\mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x, \mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

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μ_1, μ_2, ..., μ_N defined by N-point Gauss Quadrature Rule

Integrate order $2N - 1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S$_N$ Equations

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$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}, \; 1 \leq n \leq N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S$_N$ is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Conservative Form of Boltzmann Equation

Zeroth Moment: integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \;+ \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \;+ \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

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Zeroth Angular Moment

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1 equation, 2 unknowns

Conservative Form of the Boltzmann Equation

First Moment: multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \;=\; \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

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Angular flux weighted average of μ^2

Moment Equations

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$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

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Numerically: use S_N to compute $\langle \mu^2 \rangle(x)$, Moment Equations to find $\phi(x)$

Eddington Acceleration

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1. Given the previous estimate for the scalar flux, $\phi^{\ell}(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle\mu^2\rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle\mu^2\rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

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Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Produces 2 solutions (S_N and Moment)

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Difference between S_N and Moment solutions can be used as a measure of mesh convergence

Results

Test Problem

S_N: lumped Linear Discontinuous Galerkin

Test Problem

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Moment: Mixed Hybrid Finite Element

Test Problem

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S_8

Test Problem

S$_N$: lumped Linear Discontinuous Galerkin

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S$_8$

Reflecting left boundary, vacuum right boundary

Test Problem

S_N: lumped Linear Discontinuous Galerkin

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Isotropic source, constant cross sections

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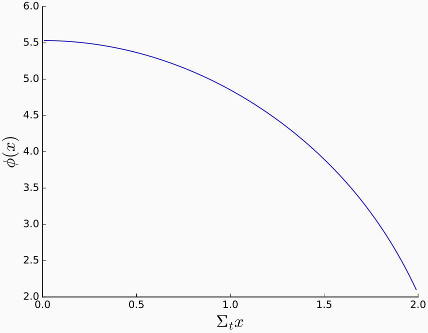

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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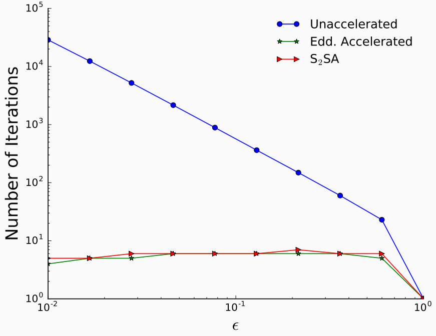

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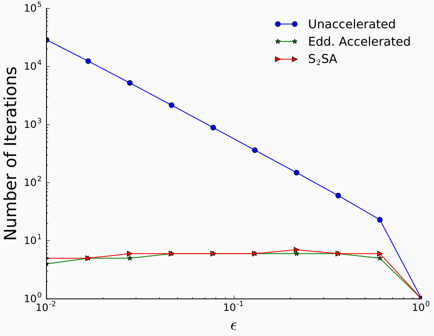

Survives diffusion limit

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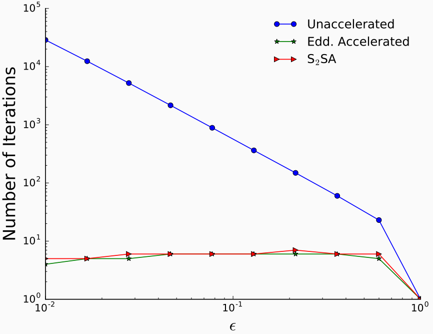


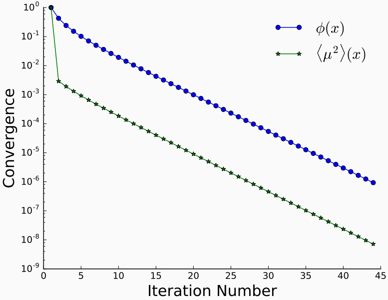
Survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Compare convergence rate

Unaccelerated



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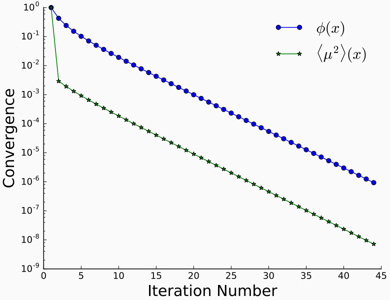

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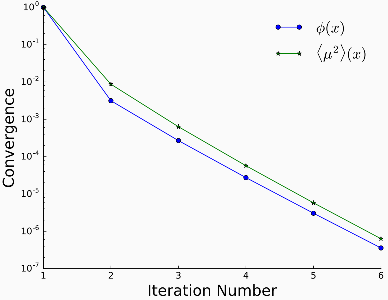

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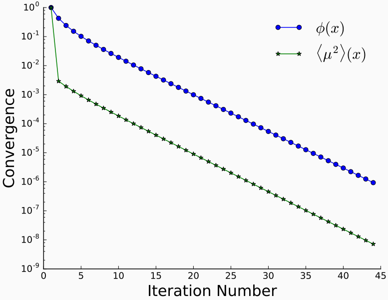
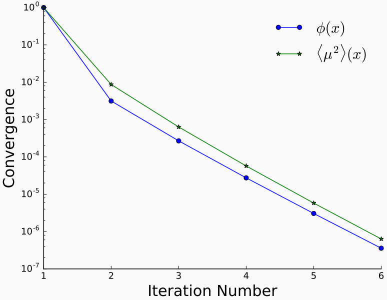


Accelerated



Fast rate of convergence of $\langle \mu^2 \rangle(x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

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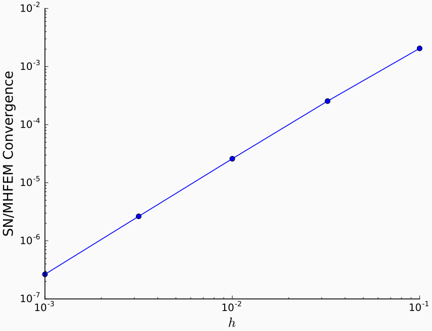

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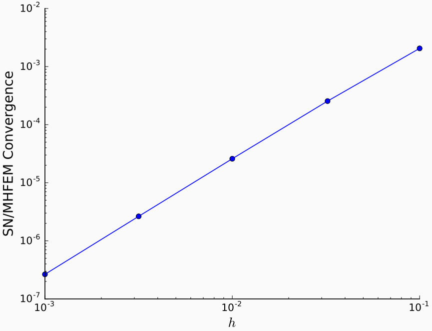

S_N and Moment solutions converge as mesh is refined

Method of Manufactured Solutions Order of Accuracy

Set $Q(x)$ to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

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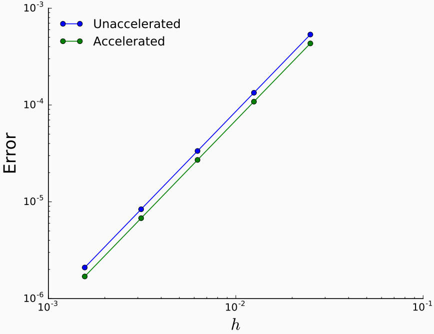

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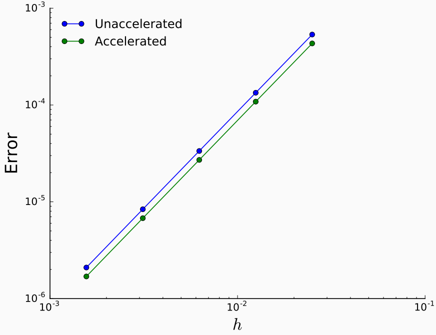

Both second order accurate

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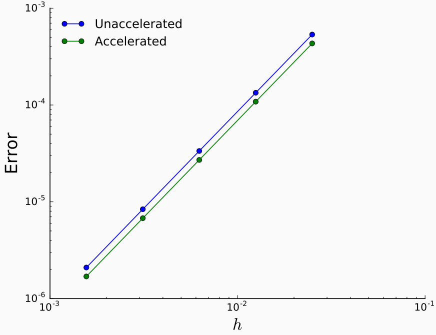

Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
    - Transport and acceleration steps can be differenced with different methods
    - Reduces expense of source iteration
    - Provides inexpensive, conservative solution
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Future Work

- Add temperature for radiative transfer
- Show still works in higher dimensions
- Develop an efficient rad hydro algorithm that makes use of the inexpensive Moment solution in multiphysics iterations

References

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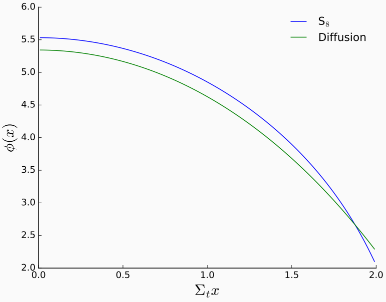
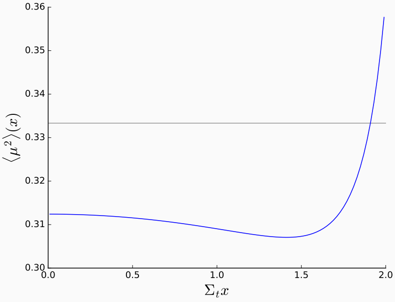
Questions?

S_8 v. Diffusion

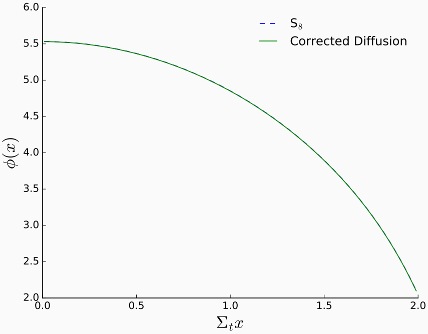
Small system $\Rightarrow$ diffusion not expected to be accurate





S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



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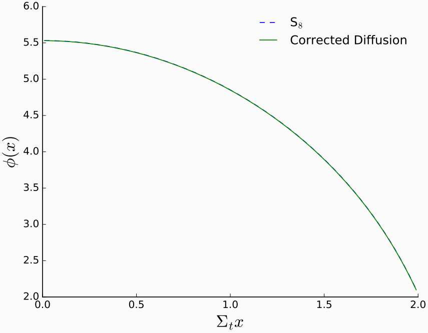

Moment Equations and S_N match!

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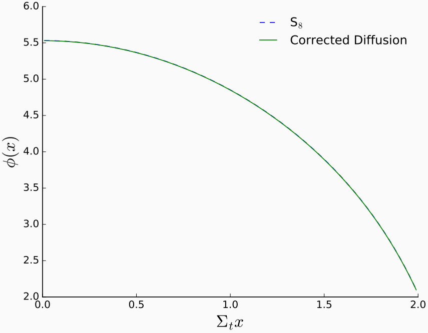

Moment Equations and S_N match!

Requires knowledge of angular flux