Variable Eddington Factor for Mixed Hybrid Finite Element/Linear Discontinuous Galerkin Source Iteration

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Abstract

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1 Introduction

The Variable Eddington Factor (VEF) method, also known as Quasi-Diffusion (QD), was one of the first nonlinear methods for accelerating source iterations in S_n calculations []. It is comparable in effectiveness to both linear and nonlinear forms of Diffusion-Synthetic Acceleration (DSA), but it offers much more flexibility than the DSA. Stability can only be guaranteed with DSA if the diffusion equation is differenced in a manner consistent with that of the S_n equations []. Modern S_n codes often use advanced discretization schemes such as discontinuous Galerkin (DG) since classic discretization schemes such as step and diamond are not suitable for radiative transfer calculations in the high-energy density physics regime or coupled electron-photon calculations. Diffusion discretizations consistent with the DG S_n discretizations cannot actually be expressed in diffusion form, but rather must be expressed in first-order or P_1 form, and are much more difficult to solve than standard diffusion discretizations. Considerable effort has gone into the development of "partially consistent diffusion discretizations that yield a stable DSA algorithm with some degree of degraded effectiveness, but such discretizations are also generally difficult to develop. A great advantage of the VEF method is that the drift-diffusion equation that accelerates the S_n source iterations can be discretized in any valid manner without concern for consistency with the S_n discretization. When the VEF drift-diffusion equation is discretized in a way that is "non-consistent, the S_n and VEF drift-diffusion solutions for the scalar flux do not necessarily become identical when the iterative process converges. However, they do become identical in the limit as the spatial mesh is refined, and the difference between the two solutions is proportional to the spatial truncation errors associated the S_n and drift-diffusion discretizations. In general the order accuracy of the S_n and VEF drift-diffusion solutions will be the lowest order accuracy of their respective independent discretizations. Although the

S_n solution obtained with such a "non-consistent VEF method is not conservative, the VEF drift-diffusion solution is in fact conservative. This is particularly useful in multiphysics calculations where the low-order VEF equation can be coupled to the other physics components rather than the high-order S_n equations. Another advantage of the non-consistent approach is that even if the S_n spatial discretization scheme does not preserve the thick diffusion limit [], that limit will generally be preserved using the VEF method.

The purpose of this paper is to investigate the application of the VEF method with the 1-D S_n equations discretized with the lumped linear-discontinuous method (LDG) and the drift-diffusion equation discretized using the constant-linear mixed finite-element method (MFEM). To our knowledge, this combination has not been previously investigated. Our motivation for this investigation is that MFEM methods are now being used for high-order hydrodynamics calculations at Lawrence Livermore National Laboratory []. A radiation transport method compatible with MFEM methods is clearly desirable for developing a MFEM radiation-hydrodynamics code. Such a code would combine thermal radiation transport with hydrodynamics. However, MFEM methods are inappropriate for the first-order form of the transport equation, and are problematic even for the even-parity form. []. Thus the use of the VEF method with a DG S_n discretization and a MFEM drift-diffusion discretization suggests itself. Here we define a VEF method that should exhibit second-order accuracy since both the transport and drift-diffusion discretizations are second-order accurate in isolation. In addition, our VEF method should preserve the thick diffusion limit [], which is essential for radiative transfer calculations in the High-Energy Density Laboratory Physics (HEDLP) regime. We use the lumped rather than the standard LDG discretization because lumping yields a much more robust scheme, and robustness is essential for radiative transfer calculations in the HEDLP regime. Because this is an initial study, we simplify the investigation by considering only by considering only one-group neutron transport rather than the full radiative transfer equations, which include a material temperature equation as well as the radiation transport equation. The vast majority of relevant properties of a VEF method for radiative transfer can be tested with an analogous method for one-group neutron transport. Furthermore, a high-order DG-MFEM VEF method could be of interest for neutronics in addition to radiative transfer calculations. A full investigation for radiative transfer calculations will be carried out in a future study.

The remainder of this paper is organized as follows. First, we describe the VEF method analytically. Then we describe our discretized S_n equations, followed by a description of the discretized VEF drift-diffusion equation. We next give computational results. More specifically, we describe two ways to represent the S_n variable Eddington factor in the MHEM drift-diffusion equation and several ways to construct the S_n scattering source from the drift-diffusion solution for the scalar flux. Each of these options yields a different VEF method. The accuracy of these methods is then compared to that of the standard lumped LDG S_n solution for several test problems, and the iterative convergence rate of these methods is compared to that of the lumped LDG S_n equations with fully-consistent DSA acceleration. Finally, we give conclusions and recommendations for future work.

2 Variable Eddington Method

2.1 Source Iteration

The steady-state, mono-energetic, isotropically-scattering, fixed-source Linear Boltzmann Equation in slab geometry is:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2} \,, \tag{1}$$

where $\mu = \cos\theta$ is the cosine of the angle of flight θ relative to the x–axis, $\Sigma_t(x)$ and $\Sigma_s(x)$ the total and scattering macroscopic cross sections, $Q(x)$ the isotropic fixed-source and $\psi(x, \mu)$ the angular flux [**?**]. Applying the Discrete Ordinates (S$_n$) angular discretization yields the following system of N coupled, ordinary differential equations:

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2} \,, 1 \leq n \leq N \,, \tag{2}$$

where $\psi_n(x) = \psi(x, \mu_n)$ is the discrete angular flux. The scalar flux is computed using an N-point quadrature rule where the quadrature weights, w_n, sum to two. In other words,

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x) \tag{3}$$

The Source Iteration (SI) scheme lags the flux in the scattering term resulting in a system of N independent, first-order, ordinary differential equations:

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^{\ell}(x) + \frac{Q(x)}{2} \,, 1 \leq n \leq N \,, \tag{4}$$

where the superscripts indicate the iteration index.

2.2 Variable Eddington Factor Acceleration

The Eddington equations are found by taking the first two angular moments of Eq. 1:

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x)\,, \tag{5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0\,, \tag{5b}$$

where $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$ is the scalar flux, $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$ the current and

$$\langle \mu^2 \rangle(x) = \frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu} \tag{6}$$

the Eddington factor.

This formulation is beneficial because Eq. 5a is a conservative balance equation and—if $\langle \mu^2 \rangle(x)$ is known—the Eddington equations' system of two first-order, ordinary differential equations can be solved directly with well-established methods. However, computing $\langle \mu^2 \rangle(x)$ requires knowledge of the angular flux.

In VEF, S_n is used to compute the Eddington factor needed to solve the Eddington equations. Source iteration is then:

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve Eq. 4 for $\psi_n^{\ell+1/2}(x)$.

2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$ with Eq. 10.

3. Solve the Eddington equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$.

4. Update the scalar flux estimate on the right side of Eq. 4 with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges.

Acceleration occurs because the angular shape of the angular flux, and thus the Eddington factor, converges much faster than the scalar flux. In addition, the Eddington equations model the contributions of all scattering events at once, reducing the dependence on source iterations to introduce scattering information. The solution from the Eddington equations is then an approximation for the full flux and not the $\ell - 1$ collided flux as it was without acceleration.

In addition to acceleration, this scheme allows the S_n equations and Eddington equations to be solved with different spatial discretization methods. S_n can be spatially discretized using normal methods, such as Linear Discontinuous Galerkin, while the Eddington equations can be solved with MFEM.

2.3 Lumped Linear Discontinuous Galerkin S_n

The LLDG discretization of Eq. 2 is:

$$\mu_n \left(\psi_{n,i} - \psi_{n,i-1/2}\right) + \frac{\Sigma_{t,i} h_i}{2} \psi_{n,i}^L = \frac{\Sigma_{s,i} h_i}{4} \phi_i^L + \frac{h_i}{4} Q_i^L \,, 1 \leq n \leq N \,, 1 \leq i \leq I \,, \tag{7a}$$

$$\mu_n \left(\psi_{n,i+1/2} - \psi_{n,i}\right) + \frac{\Sigma_{t,i} h_i}{2} \psi_{n,i}^R = \frac{\Sigma_{s,i} h_i}{4} \phi_i^R + \frac{h_i}{4} Q_i^R \,, 1 \leq n \leq N \,, 1 \leq i \leq I \,, \tag{7b}$$

where h_i, $\Sigma_{t,i}$, and $\Sigma_{s,i}$ are the cell width, total cross section, and scattering cross section in cell i. The cell-edged angular fluxes are found through upwinding:

$$\psi_{n,i-1/2} = \begin{cases} \psi^R_{n,i-1} & \mu_n > 0 \\ \psi^L_{n,i} & \mu_n < 0 \end{cases} \tag{8a}$$

$$\psi_{n,i+1/2} = \begin{cases} \psi^R_{n,i} & \mu_n > 0 \\ \psi^L_{n,i+1} & \mu_n < 0 \end{cases} \tag{8b}$$

and the cell-centered angular flux is computed with:

$$\psi_{n,i} = \frac{1}{2}\left(\psi^L_{n,i} + \psi^R_{n,i}\right) . \tag{9}$$

In S_n, the Eddington factor is

$$\langle\mu^2\rangle(x) = \frac{\sum_{n=1}^N \mu_n^2 w_n \psi_n(x)}{\sum_{n=1}^N w_n \psi_n(x)} . \tag{10}$$

2.4 Data Reconstruction

Data reconstruction methods are required to recover the linear discontinuous representation in LDG from the piecewise constant with discontinuous jumps at the cell edges representation in MFEM. Data reconstruction methods use neighboring cell centers or edges to extrapolate the MFEM flux to determine the discontinuous cell edge values in LDG that maintain the average and slope. Two data reconstruction methods have been implemented: one that uses the slope between the MFEM edge values and one that only uses cell centered values.

The first method sets the LDG discontinuous left and right fluxes in cell i, $\phi_{i,L}$ and $\phi_{i,R}$

to

$$\phi_{i,L/R} = \phi_i \mp \frac{1}{2}\left(\phi_{i+1/2} - \phi_{i-1/2}\right) \tag{11}$$

The second method uses a van Leer slope limiter on the MFEM cell centers.

3 Computational Results

3.1 Diffusion Limit

To test the algorithm in the diffusion limit, the cross sections and source were scaled according to:

$$\Sigma_t(x) \to \Sigma_t(x)/\epsilon\,, \Sigma_s(x) \to \epsilon\Sigma_s(x)\,, Q(x) \to \epsilon Q(x)\,. \tag{12}$$

As $\epsilon \to 0$, the system becomes diffusive.

3.2 Method of Manufactured Solutions

3.3 Solution Convergence

4 Conclusions and Future Work

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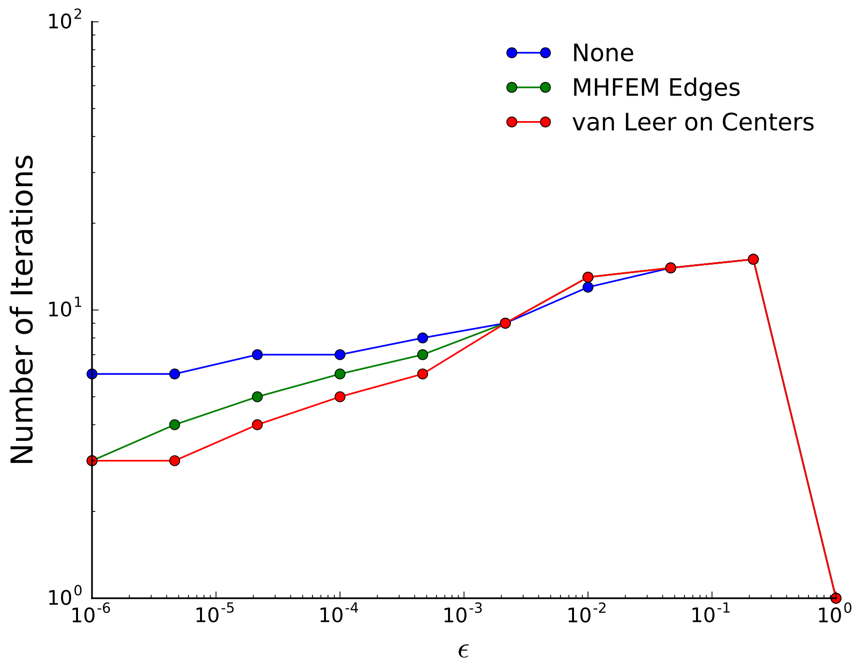

Figure 1: Test