Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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Overview

Motivation

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Radiation Hydrodynamics

- Describes the effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory experiments (NIF, Z Machine) and astrophysics

Mixed Hybrid Finite Element Method (MHFEM) hydrodynamics

Problems

- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme
- Radiation transport is expensive

Goal

Develop a transport algorithm that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Bridges Linear Discontinuous Galerkin (LDG) transport and MHFEM multiphysics

Show scheme works in 1D slab with lumped LDG transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{S_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

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4

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μ_1, μ_2, ..., μ_N defined by N-point Gauss Quadrature Rule

Integrate order $2N - 1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

S_N Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}, 1 \leq n \leq N$$

N independent, first-order, ordinary differential equations

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

Need For Acceleration in Source Iteration

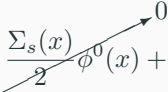Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

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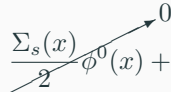
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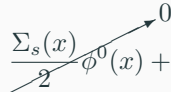
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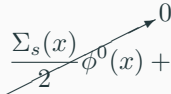
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$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Radiation Hydrodynamics problems often contain highly diffusive regions

S_N is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Zeroth Angular Moment

Find conservative form of Boltzmann equation

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Boltzmann Equation

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Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \;+\; \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \;+\; \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

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Angular flux weighted average of μ^2

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0 \qquad \text{(First Moment)}$$

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Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)}\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x)$$

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If $\langle\mu^2\rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)}\frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

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Transport information passed through $\langle\mu^2\rangle(x)$ and boundary conditions

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Moment Equations = transport informed diffusion

Transport information passed through $\langle\mu^2\rangle(x)$ and boundary conditions

Just as accurate as S_N

Solving the Moment Equations requires knowledge of the angular flux (the solution)

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Non-linear scheme $\Rightarrow$ produces 2 solutions (S_N and Moment)

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Relaxes consistent differencing requirements important in linear acceleration

Eddington Acceleration Properties

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Which solution is correct?

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Relaxes consistent differencing requirements important in linear acceleration

Which solution is correct?

Benefits

1. Transport can be LDG and Moment can be MHFEM
2. Moment Equations are conservative and relatively inexpensive compared to transport sweep
3. Difference between S_N and Moment solution can be used as a measure of spatial truncation error (measure of mesh convergence)

Results

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$
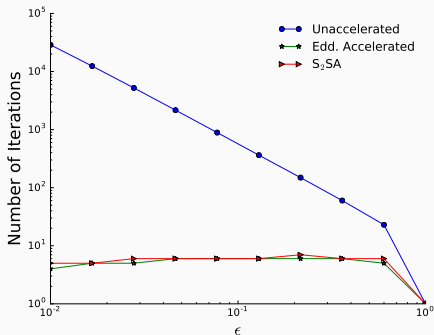
Accelerates source iteration, survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)
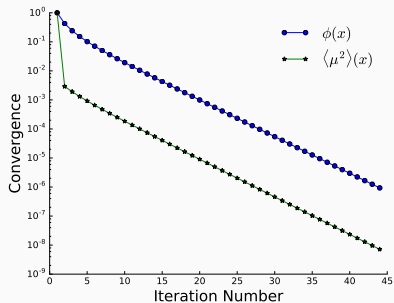
Unaccelerated

Convergence Rate Comparison
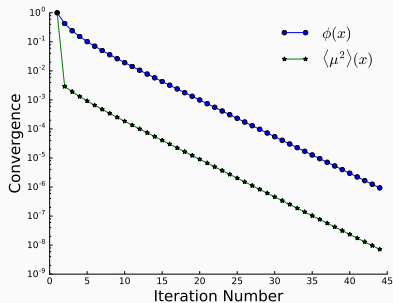
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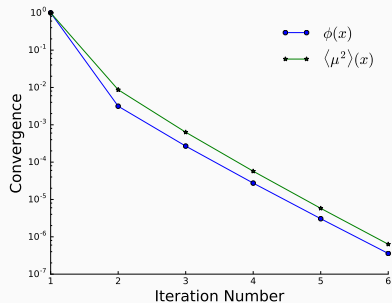
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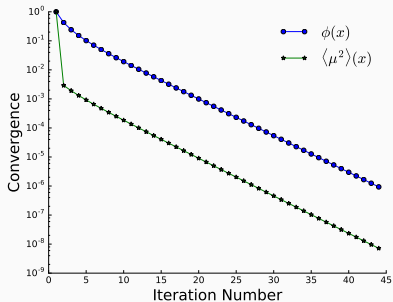
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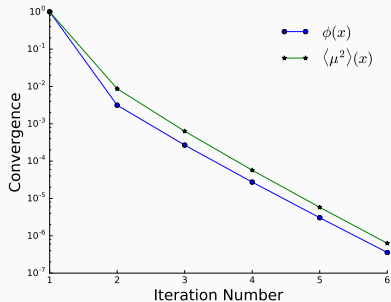
Fast rate of convergence of $\langle \mu^2 \rangle(x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

Solution Convergence

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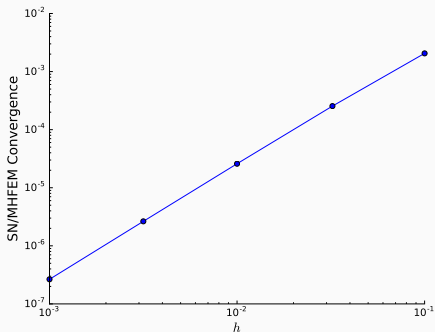
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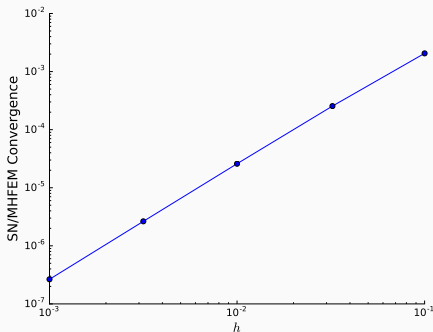
S_N and Moment solutions converge as mesh is refined

Set $Q(x)$ to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Method of Manufactured Solutions Order of Accuracy

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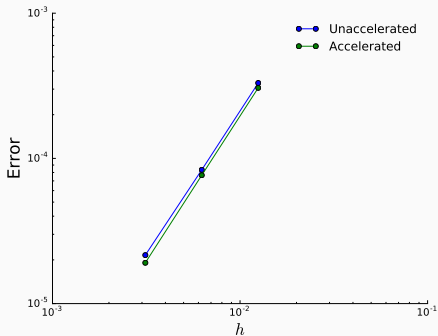
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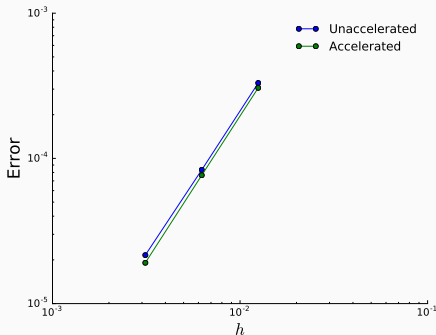
$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Both second order accurate

Method of Manufactured Solutions Order of Accuracy

Set $Q(x)$ to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$
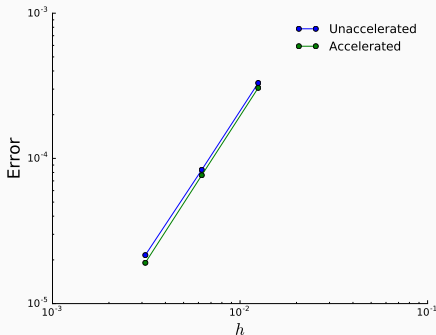
Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

Conclusions

Summary

Conclusions
- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
  - Transport and acceleration steps can be differenced with different methods
  - Reduces expense of source iteration
  - Provides inexpensive, conservative solution
- Showed MHFEM and lumped LDG can be paired

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Future Work

- Develop a rad-hydro algorithm
  - Make use of inexpensive Moment solution in multiphysics iterations
- Add temperature
- Higher order of accuracy
- Explore other multiphysics applications

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[8] J. S. WARSA, T. A. WAREING, AND J. E. MOREL, *Fully Consistent Diffusion Synthetic Acceleration of Linear Discontinuous Transport Discretizations on Three–Dimensional Unstructured Meshes*.
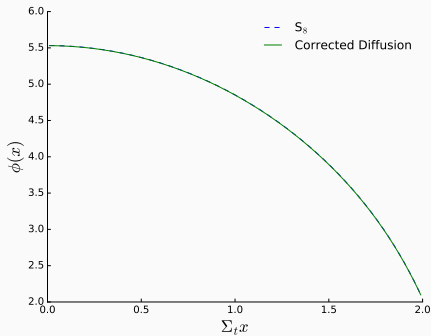
Questions?

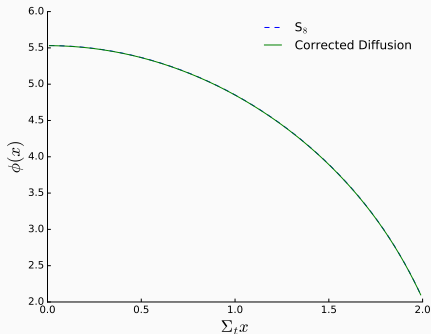Small system $\Rightarrow$ diffusion not expected to be accurate
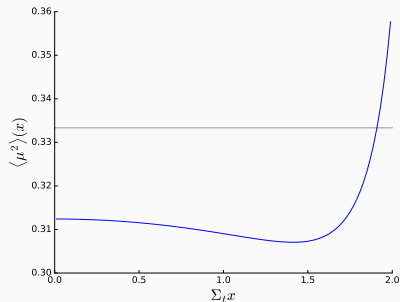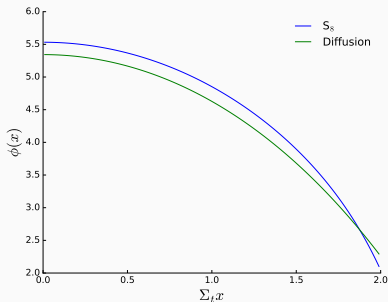
Use $\langle \mu^2 \rangle (x)$ from S_8 in Moment Equations
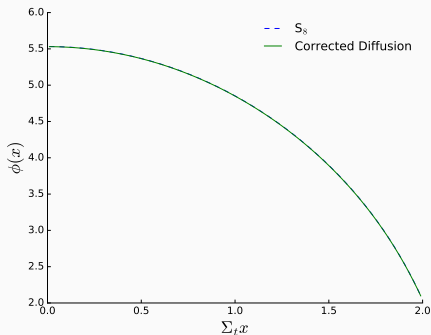
S$_8$ v. Drift Diffusion

Use $\langle \mu^2 \rangle (x)$ from S$_8$ in Moment Equations

Moment Equations and S$_N$ match!

S$_8$ v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S$_8$ in Moment Equations

Moment Equations and S$_N$ match!

Requires knowledge of angular flux