Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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https://github.com/smsolivier/EddingtonAcceleration.git

NUCLEAR ENGINEERING
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Overview

Motivation

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Radiation Hydrodynamics

- Propogation of thermal radiation through a fluid
- Effects of radiation on fluid momentum and energy
- Required in high energy density laboratory physics (NIF, Z Machine) and astrophysics

Need hydrodynamics and transport to be consistently differenced

- Use the same method or do extra work to make differing methods agree
- Interpolating between spatial grids introduces noise
- Matching grids between methods is not always possible in higher dimensions

Hydrodynamics will be discretized with Mixed Hybrid Finite Element Method (MHFEM)

Want to be able to pair with Linear Discontinuous Galerkin (LDG) transport

Problems

- Radiation transport is expensive
- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme

Goal

Develop a transport algorithm that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Bridges LDG transport and MHFEM multiphysics

Show scheme works in 1D slab with lumped LDG transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

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Integrate order $2N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

S_N Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S_N is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Zeroth Angular Moment

Take angular moments of the Boltzmann equation

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Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

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Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \; + \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \; + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \; = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

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$$\langle\mu^2\rangle(x)=\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

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Angular flux weighted average of μ^2

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

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Solve First Moment for $J(x)$

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Transport information passed through $\langle \mu^2 \rangle(x)$ and boundary conditions

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Just as accurate as S$_N$

Solving the Moment Equations requires knowledge of the angular flux (the solution)

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Non-linear scheme $\Rightarrow$ produces 2 solutions (S$_N$ and Moment)

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Relaxes consistent differencing requirements

Benefits

1. Transport can be LDG and Moment can be MHFEM
2. Moment Equations are conservative and relatively inexpensive compared to transport sweep
3. Can use Moment solution in MHFEM multiphysics iterations without needing a full transport sweep
4. Difference between S$_N$ and Moment solution can be used as a measure of spatial truncation error (measure of mesh convergence)

Results

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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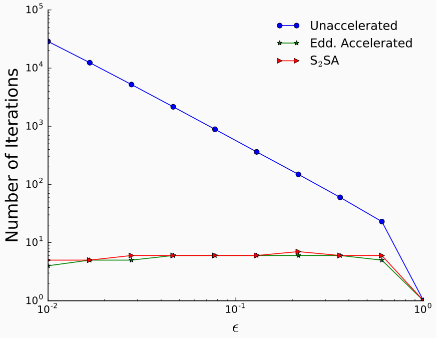

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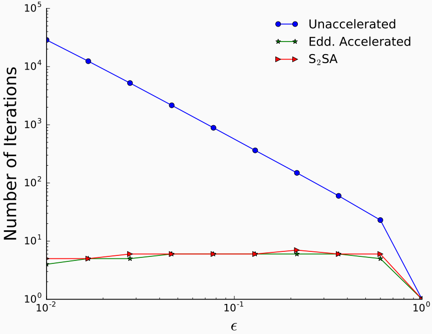

Accelerates source iteration, survives diffusion limit

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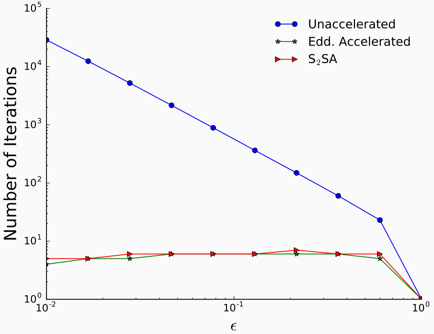
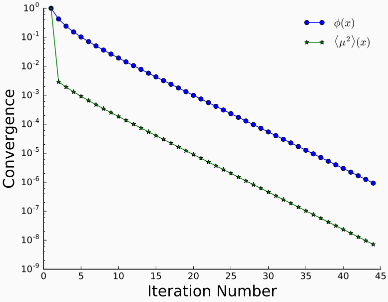


Accelerates source iteration, survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Unaccelerated



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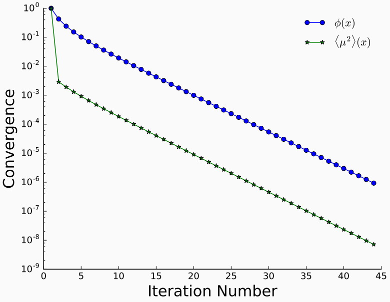

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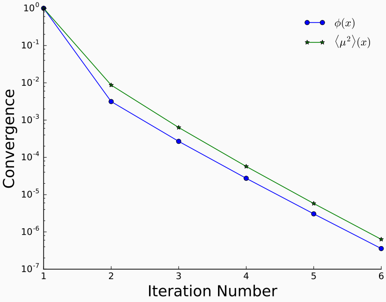

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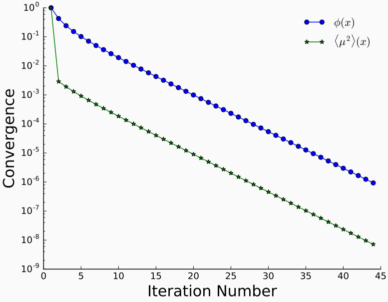

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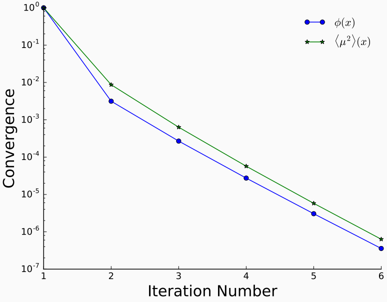

Fast rate of convergence of $\langle\mu^2\rangle(x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

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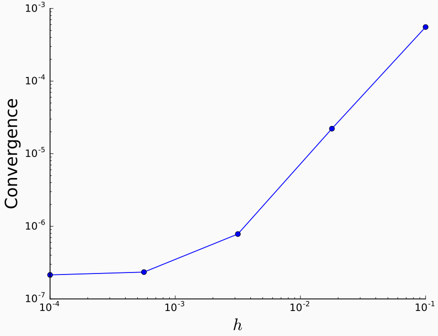

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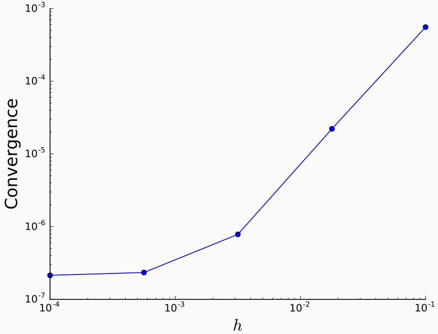
$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\text{Moment}}(x)\|}{\|\phi_{\text{Moment}}(x)\|}$$

as $h \to 0$



S_N and Moment solutions converge as mesh is refined

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

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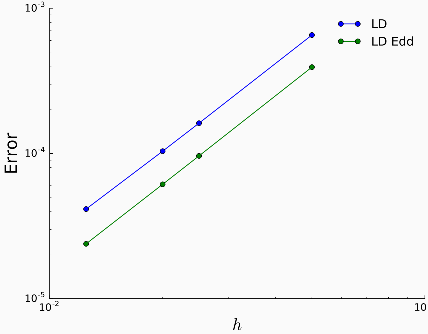

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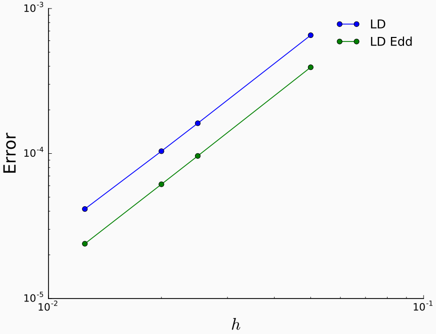

Both second order accurate

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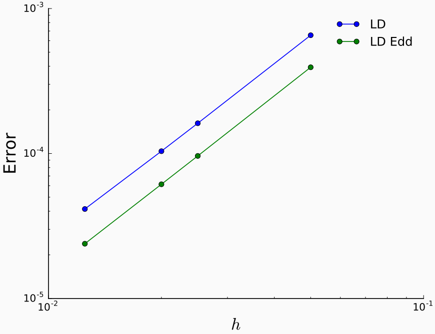

Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
  - Transport and acceleration steps can be differenced with different methods
  - Reduces expense of source iteration
  - Provides inexpensive, conservative solution
- Showed MHFEM and lumped LDG can be paired

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Future Work

- Develop a rad-hydro algorithm
    - Make use of inexpensive Moment solution in multiphysics iterations
- Add temperature
- High order of accuracy
- Explore other multiphysics applications

References

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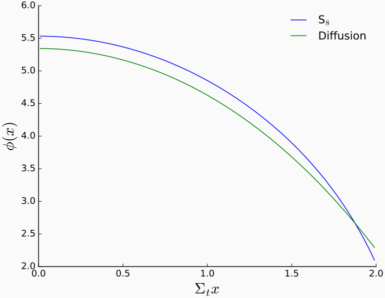
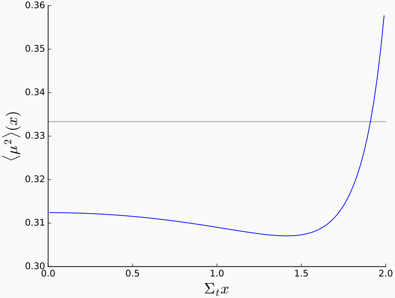
Questions?

S_8 v. Diffusion

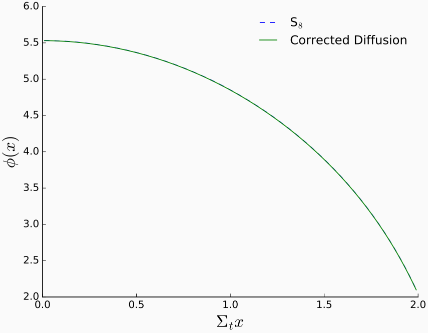
Small system $\Rightarrow$ diffusion not expected to be accurate





S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



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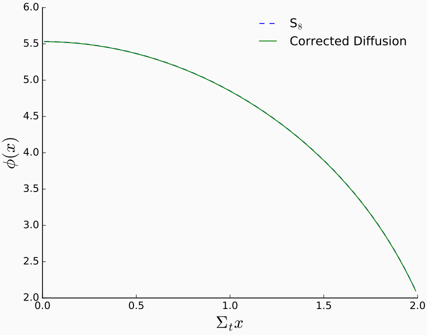

Moment Equations and S_N match!

S_8 v. Drift Diffusion

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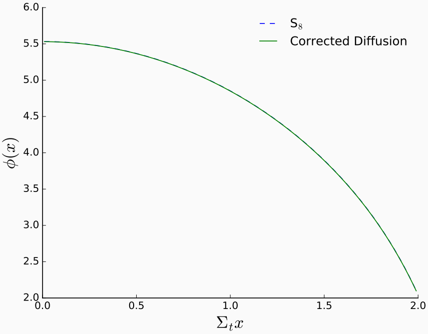

Moment Equations and S_N match!

Requires knowledge of angular flux