Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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NUCLEAR ENGINEERING
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Overview

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

Motivation

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Radiation Hydrodynamics

- Describes the effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory experiments (NIF, Z Machine) and astrophysics

Mixed Hybrid Finite Element Method (MHFEM) hydrodynamics

Problems

- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme
- Radiation transport is expensive

Goal

Develop a transport algorithm that

1. Accelerates Discrete Ordinates Source Iteration
2. Bridges Linear Discontinuous Galerkin (LDG) transport and MHFEM multiphysics

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Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu') d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

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Integrate order $2N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

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$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S_N is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Conservative Form of Boltzmann Equation

Take moments of Boltzmann equation until have enough equations for the number of unknowns

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Zeroth Moment: integrate over all angles

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

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Zeroth Angular Moment

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1 equation, 2 unknowns

Conservative Form of the Boltzmann Equation

First Moment: multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \;=\; \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

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$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_{-1}^{1} \mu^2 \psi(x,\mu) \, \mathrm{d}\mu}_{\text{Second Moment of } \psi(x,\,\mu)}$$

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Angular flux weighted average of μ^2

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

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Numerically: use S_N to compute estimate of $\langle \mu^2 \rangle(x)$, Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

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Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Produces 2 solutions (S_N and Moment)

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Relaxes consistent differencing requirements important in linear acceleration

Transport can be LDG and Moment can be MHFEM

Moment Equations are conservative and relatively inexpensive to solve

Downside: Which solution is correct?

Difference between S$_N$ and Moment solutions can be used as a measure of mesh convergence

Results

Test Problem

S_8 in 1D slab geometry

Test Problem

S_8 in 1D slab geometry

Lumped Linear Discontinuous Galerkin transport

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S_8 in 1D slab geometry

Lumped Linear Discontinuous Galerkin transport

Mixed Hybrid Finite Element Method Moment

Test Problem

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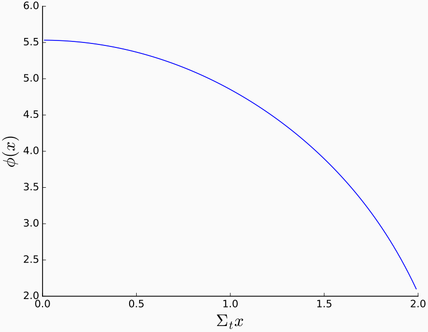

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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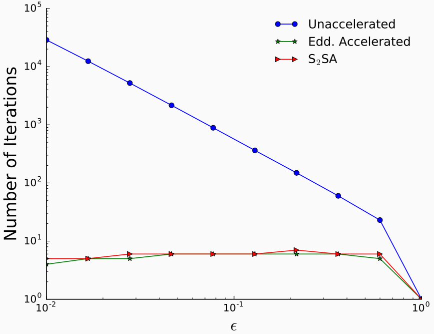

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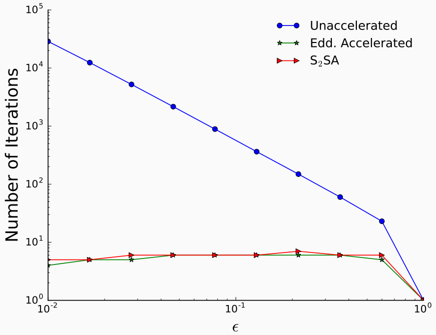

Survives diffusion limit

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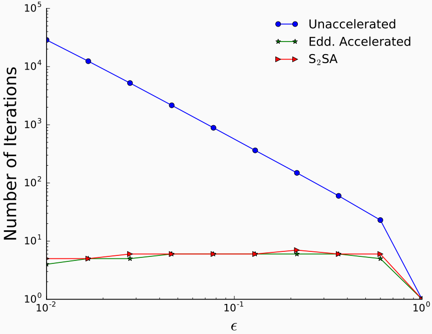
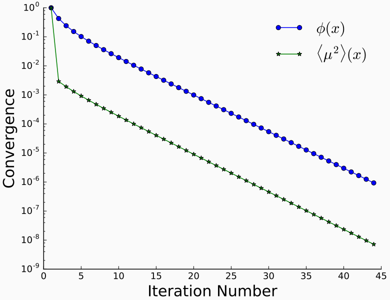


Survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Unaccelerated



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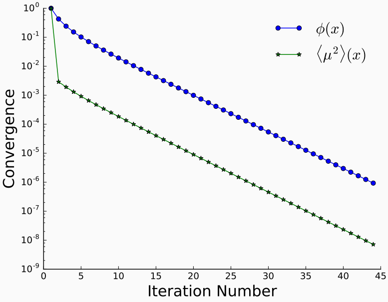
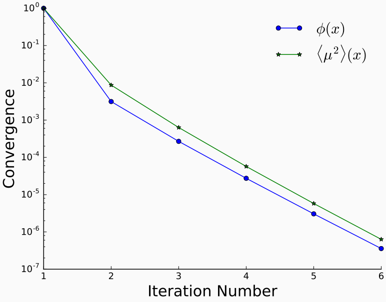
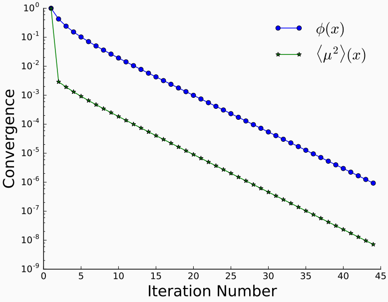


Accelerated

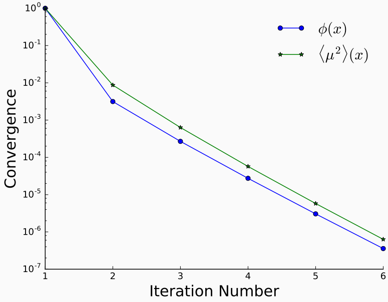


Convergence Rate Comparison

Unaccelerated



Accelerated



Fast rate of convergence of $\langle \mu^2 \rangle(x)$ is transfered to $\phi(x)$

Method of Manufactured Solutions Order of Accuracy

Set $Q(x)$ to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

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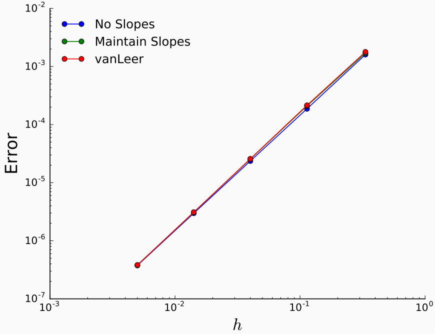

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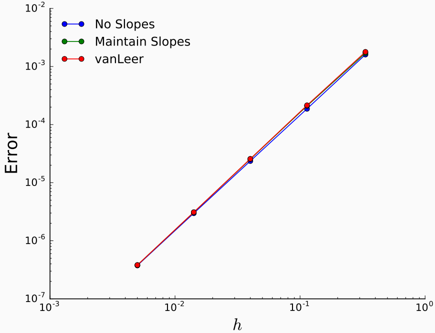

Both second order accurate

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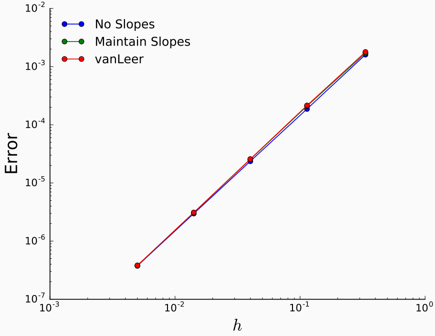

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Eddington Acceleration did not effect the order of accuracy of lumped LDG

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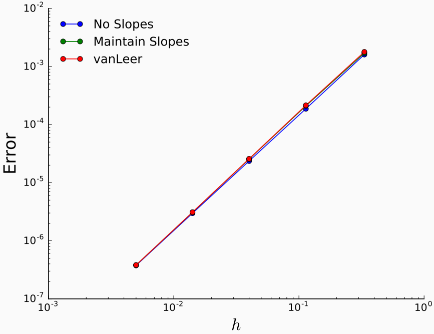

Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

All slope recovery methods have similar accuracy

Data Reconstruction Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

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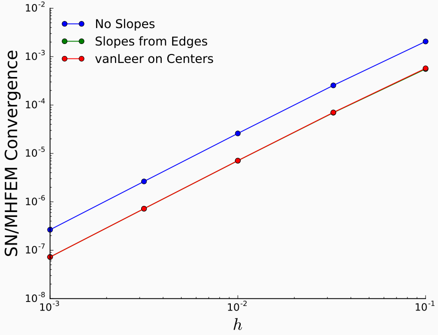

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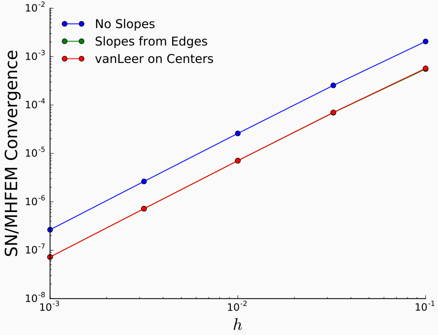

S_N and Moment solutions converge as mesh is refined

Data Reconstruction Solution Convergence

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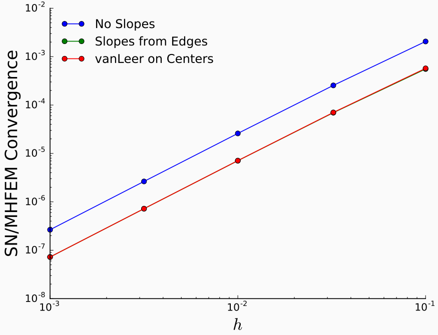
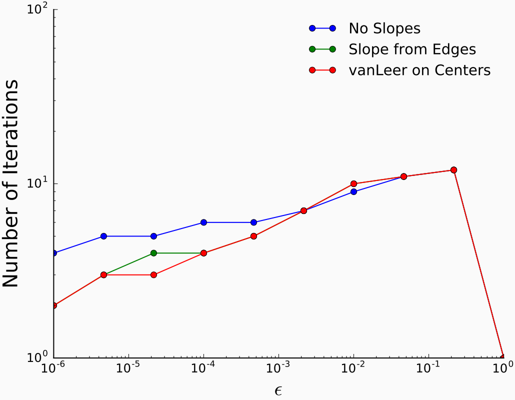


S_N and Moment solutions converge as mesh is refined

Slope recovery effects solution convergence but not accuracy

Data Reconstruction Diffusion Limit



All data reconstruction methods survived diffusion limit

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
    - Transport and acceleration steps can be differenced with different methods
    - Reduces expense of source iteration
    - Provides inexpensive, conservative solution
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Future Work

- Add temperature for radiative transfer
- Show still works in higher dimensions
- Develop an efficient rad hydro algorithm that makes use of the inexpensive Moment solution in multiphysics iterations

References

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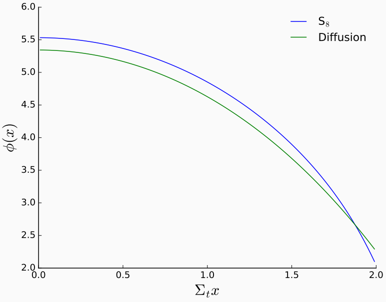
Questions?

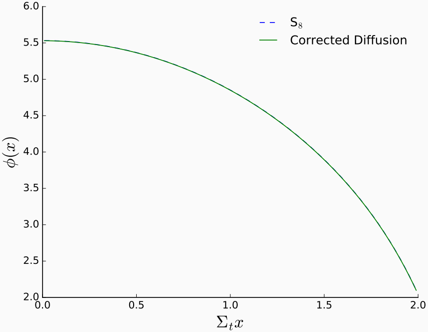
S_8 v. Diffusion

Small system $\Rightarrow$ diffusion not expected to be accurate



S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



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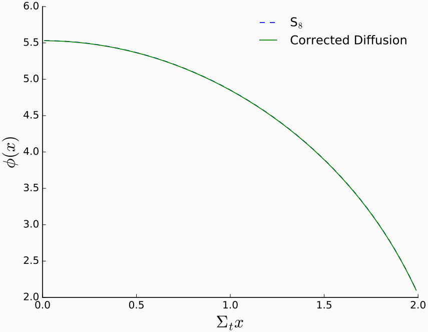

Moment Equations and S_N match!

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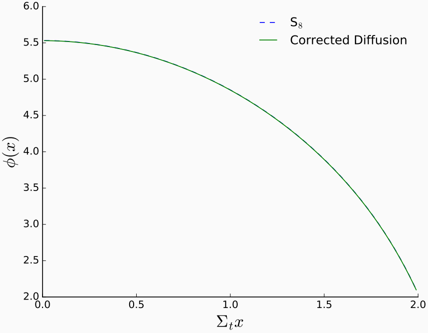

Moment Equations and S_N match!

Requires knowledge of angular flux