Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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Overview

Motivation

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Radiation Hydrodynamics

- Describes the effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory experiments (NIF, Z Machine) and astrophysics

Mixed Hybrid Finite Element Method (MHFEM) hydrodynamics

Problems

- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme
- Radiation transport is expensive

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Develop a transport algorithm that

1. Accelerates Discrete Ordinates Source Iteration
2. Bridges Linear Discontinuous Galerkin (LDG) transport and MHFEM multiphysics

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Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x,\mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

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Integrate order $2N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

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$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S$_N$ is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Conservative Form of Boltzmann Equation

Take moments of Boltzmann equation until have enough equations for the number of unknowns

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Zeroth Moment: integrate over all angles

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

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1 equation, 2 unknowns

Conservative Form of the Boltzmann Equation

First Moment: multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \;=\; \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

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Angular flux weighted average of μ^2

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

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Numerically: use S$_N$ to compute estimate of $\langle \mu^2 \rangle(x)$, Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

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Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Produces 2 solutions (S$_N$ and Moment)

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Difference between S_N and Moment solutions can be used as a measure of mesh convergence

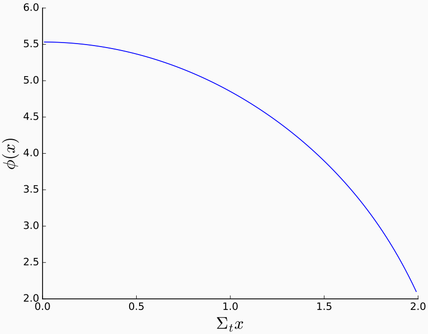
Results

Test Problem

S_8 in 1D slab geometry

Lumped Linear Discontinuous Galerkin transport

Mixed Hybrid Finite Element Method Moment



Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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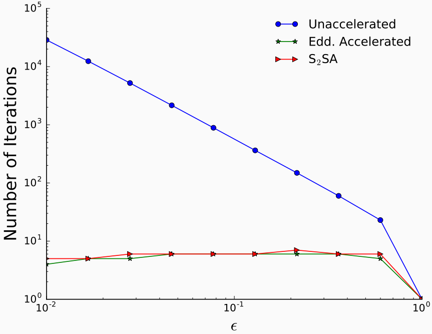

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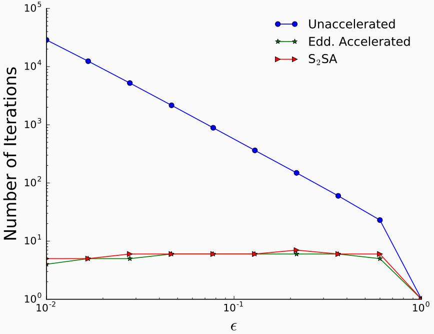

Survives diffusion limit

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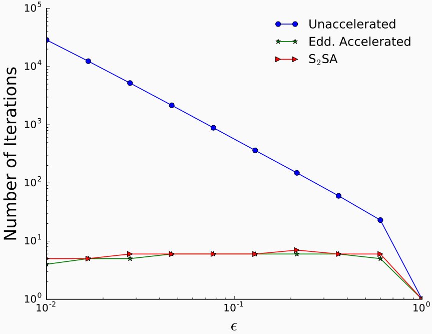
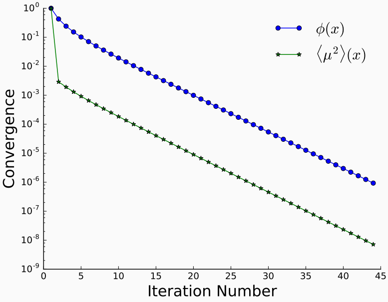


Survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Unaccelerated



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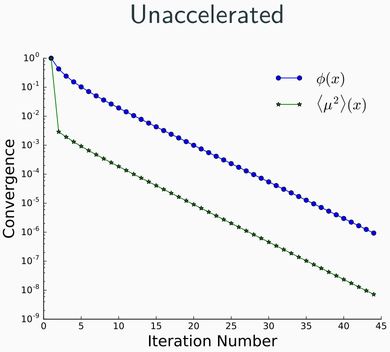
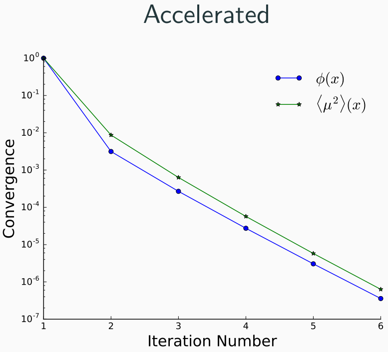
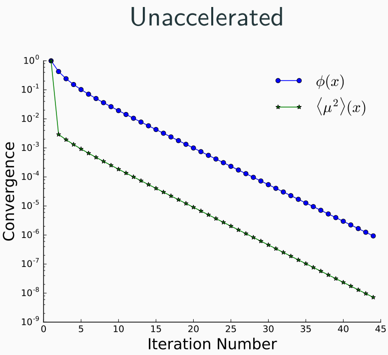
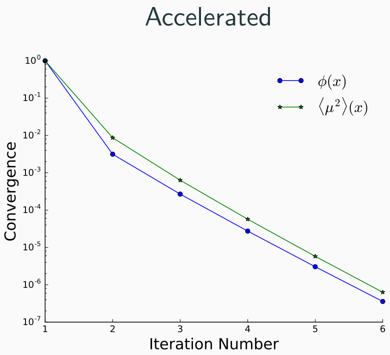




Convergence Rate Comparison





Fast rate of convergence of $\langle\mu^2\rangle(x)$ is transfered to $\phi(x)$

Method of Manufactured Solutions Order of Accuracy

Set $Q(x)$ to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Compare numerical results to MMS solution as cell width is decreased

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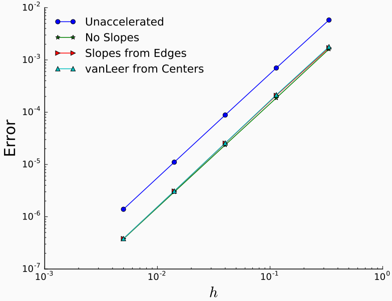

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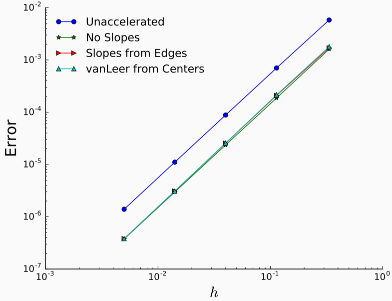

Data reconstruction: recover linear representation from MHFEM $\phi(x)$

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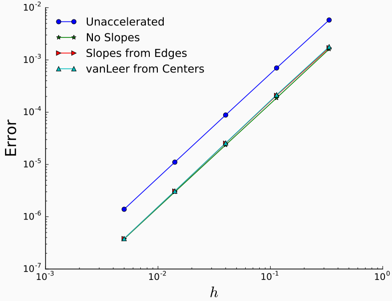

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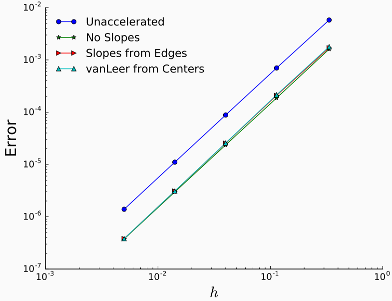

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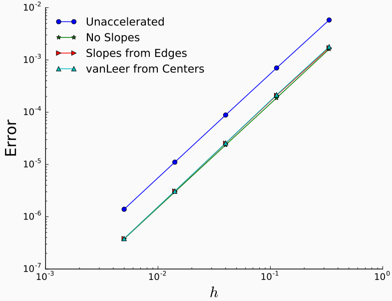

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All slope recovery methods have similar accuracy

Data Reconstruction Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

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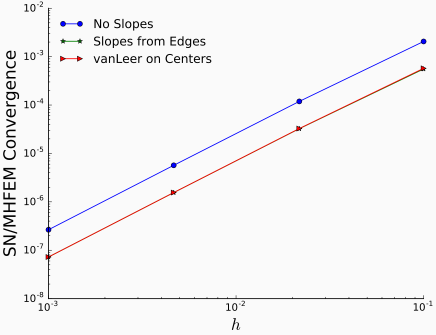

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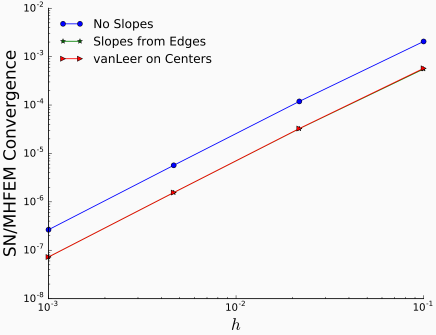

S_N and Moment solutions converge as mesh is refined

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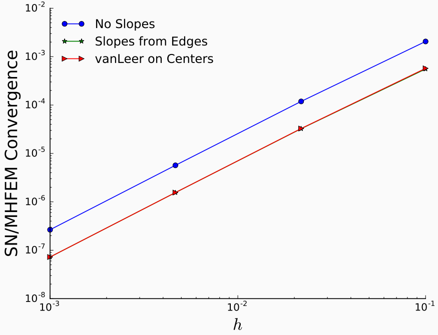
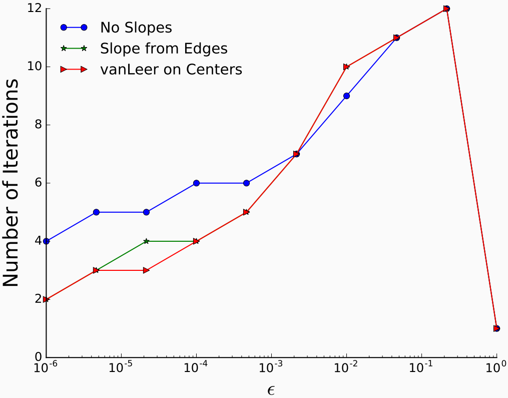


S_N and Moment solutions converge as mesh is refined

Slope recovery effects solution convergence but not accuracy

Data Reconstruction Diffusion Limit



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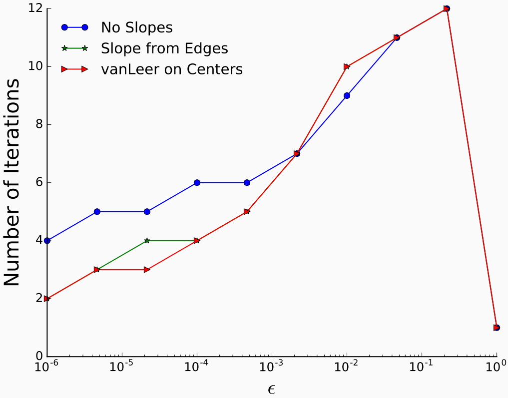
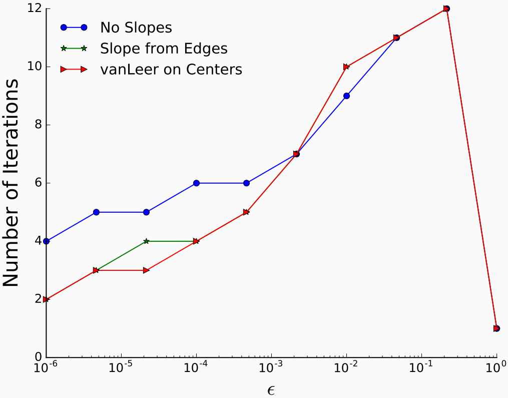


All data reconstruction methods survived diffusion limit

Data Reconstruction Diffusion Limit



All data reconstruction methods survived diffusion limit

Eddington Acceleration is extemely robust

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
  - Transport and acceleration steps can be differenced with different methods
  - Reduces expense of source iteration
  - Provides inexpensive, conservative solution
- Showed MHFEM/LLDG pairing is robust

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Future Work

- Add temperature for radiative transfer
- Higher dimensions
- Develop an efficient rad hydro algorithm that makes use of the inexpensive Moment solution in multiphysics iterations

References

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[8] J. S. WARSA, T. A. WAREING, AND J. E. MOREL, *Fully Consistent Diffusion Synthetic Acceleration of Linear Discontinuous Transport Discretizations on Three–Dimensional Unstructured Meshes*.

Questions?

Data Reconstruction Methods

MHFEM $\phi(x)$ is piecewise constant with discontinuous cell edges ($\phi_{i-1/2}$, ϕ_i, $\phi_{i+1/2}$)

LLDG is linear discontinuous ($\phi_{i,L}$, $\phi_{i,R}$)

Need a way to recover slope information when S_N scattering term is updated with MHFEM $\phi(x)$

No Slopes:

$$\phi_{i,L/R} = \phi^*_{i\mp1/2}$$

Slopes from Edges:

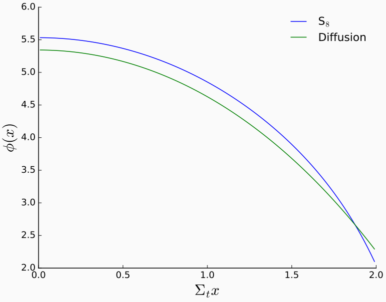
$$\phi_{i,L/R} = \phi^*_i \mp \frac{1}{2}\left(\phi^*_{i+1/2} - \phi^*_{i-1/2}\right)$$

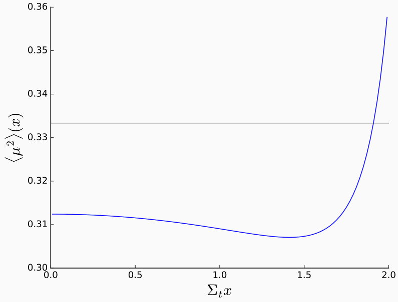
vanLeer on Centers:

$$\phi_{i,L/R} = \phi^*_i \mp \frac{1}{4}\xi_{\text{vanLeer}}\left[\left(\phi^*_{i+1} - \phi^*_i\right) + \left(\phi^*_i - \phi^*_{i-1}\right)\right]$$

S_8 v. Diffusion

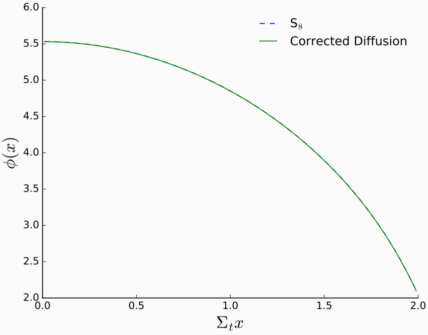
Small system $\Rightarrow$ diffusion not expected to be accurate





S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle (x)$ from S_8 in Moment Equations



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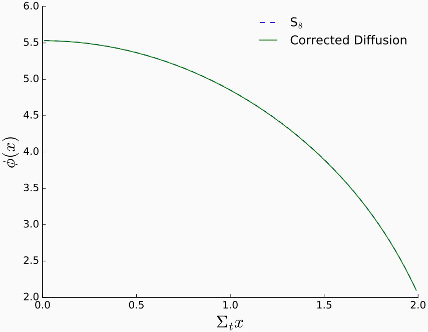

Moment Equations and S_N match!

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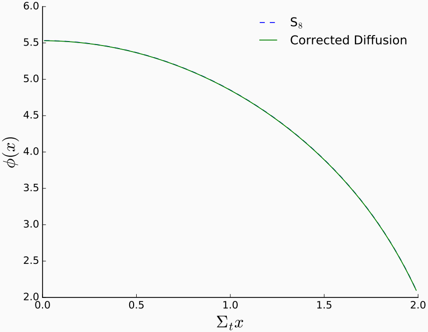

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Requires knowledge of angular flux