Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

Samuel S. Olivier

March 28, 2017

Department of Nuclear Engineering, Texas A&M University

NUCLEAR ENGINEERING
TEXAS A&M UNIVERSITY

Overview

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

Motivation

Motivation

Radiation Hydrodynamics

- Propogation of thermal radiation through a fluid
- Effects of radiation on fluid momentum and energy
- Required in high energy density laboratory physics (NIF, Z Machine) and astrophysics

Spatial Discretizations

- Mixed Hybrid Finite Element Method (MHFEM) for hydrodynamics
- Linear Discontinuous Galerkin (LDG) for transport

Need hydrodynamics and transport to be consistently differenced

- Use the same method or do extra work to make differing methods agree
- Interpolating between spatial grids introduces noise
- Matching grids between methods is not always possible in higher dimensions

Problem: MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme

Goal

Develop a non-linear acceleration scheme that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Bridges LDG transport and MHFEM multiphysics

Test in 1D slab with lumped LDG transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

μ_1, μ_2, $\ldots$, μ_N defined by N-point Gauss Quadrature Rule

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

μ_1, μ_2, $\ldots$, μ_N defined by N-point Gauss Quadrature Rule

Integrate order $2N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S$_N$ Equations

S$_N$ Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \frac{\Sigma_s(x)}{2}\phi^0(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

$\phi^1(x)$ is the uncollided flux

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

$\phi^1(x)$ is the uncollided flux

$\phi^2(x)$ is uncollided and once collided flux

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}, 1 \le n \le N$$

$\phi^1(x)$ is the uncollided flux

$\phi^2(x)$ is uncollided and once collided flux

$\vdots$

$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}, 1 \leq n \leq N$$

$\phi^1(x)$ is the uncollided flux

$\phi^2(x)$ is uncollided and once collided flux

$\vdots$

$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S_N is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Zeroth Angular Moment

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Zeroth Angular Moment

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \ + \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \ + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Zeroth Angular Moment

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \;+ \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \;+ \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \;=\; \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \underbrace{\int_{-1}^{1} \mu \Sigma_t(x) \psi(x,\mu)\,\mathrm{d}\mu}_{\Sigma_t(x) J(x)} = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2} \phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \underbrace{\int_{-1}^{1} \mu\Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu}_{\Sigma_t(x)J(x)} = \underbrace{\int_{-1}^{1} \mu\frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu\frac{Q(x)}{2}\,\mathrm{d}\mu}_{\text{Isotropic}\Rightarrow 0}$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \underbrace{\int_{-1}^{1} \mu \Sigma_t(x) \psi(x,\mu)\,\mathrm{d}\mu}_{\Sigma_t(x) J(x)} = \underbrace{\int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2} \phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu}_{\text{Isotropic} \Rightarrow 0}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x}\underbrace{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}_{\phi(x)}\;\underbrace{\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}}_{\text{Eddington Factor}}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}_{\phi(x)} \underbrace{\frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}}_{\text{Eddington Factor}}$$

Eddington Factor

$$\langle \mu^2 \rangle(x) = \frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}_{\phi(x)} \underbrace{\frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}}_{\text{Eddington Factor}}$$

Eddington Factor

$$\langle \mu^2 \rangle(x) = \frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}$$

Shape function

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \quad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \quad \text{(First Moment)}$$

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle\mu^2\rangle(x)\phi(x)$$

If $\langle\mu^2\rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

If $\langle \mu^2 \rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

Moment Equations = transport informed diffusion

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

If $\langle \mu^2 \rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

Moment Equations = transport informed diffusion

Transport information passed through $\langle \mu^2 \rangle(x)$

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

If $\langle \mu^2 \rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

Moment Equations = transport informed diffusion

Transport information passed through $\langle \mu^2 \rangle(x)$

Just as accurate as S_N

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

If $\langle \mu^2 \rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)} \frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

Moment Equations = transport informed diffusion

Transport information passed through $\langle \mu^2 \rangle(x)$

Just as accurate as S$_N$

Solving the Moment Equations requires knowledge of the angular flux (the solution)

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Non-linear scheme ⇒ produces 2 solutions (S_N and Moment)

Eddington Acceleration Properties

Non-linear scheme $\Rightarrow$ produces 2 solutions (S_N and Moment)

Relaxes consistent differencing requirements

Eddington Acceleration Properties

Non-linear scheme $\Rightarrow$ produces 2 solutions (S_N and Moment)

Relaxes consistent differencing requirements

Solutions converge as cell spacing decreases

Eddington Acceleration Properties

Non-linear scheme $\Rightarrow$ produces 2 solutions (S$_N$ and Moment)

Relaxes consistent differencing requirements

Solutions converge as cell spacing decreases

Benefits

1. Transport can be LDG and Moment can be MHFEM
2. Moment Equations are conservative and relatively inexpensive compared to transport sweep
3. Difference between S$_N$ and Moment solution can be used as a measure of spatial truncation error (measure of mesh convergence)

Results

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

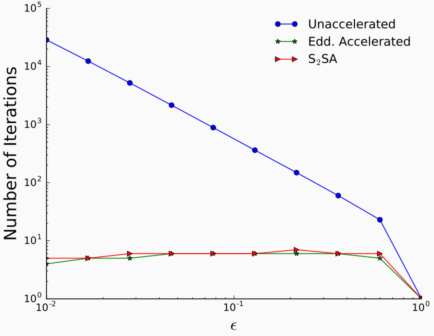
System becomes diffusive as $\epsilon \to 0$

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

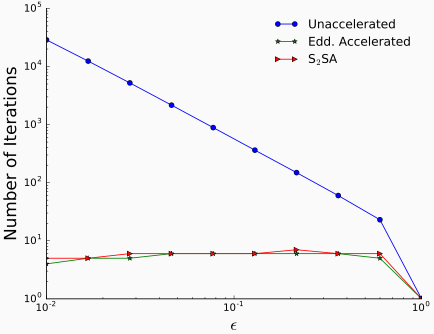


Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$



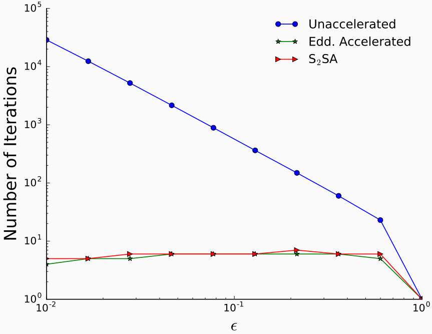
Accelerates source iteration, survives diffusion limit

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

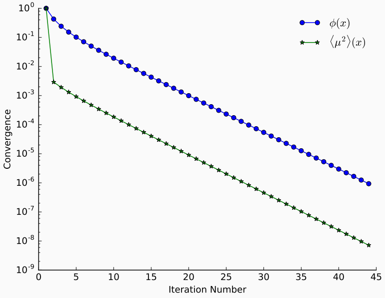


Accelerates source iteration, survives diffusion limit

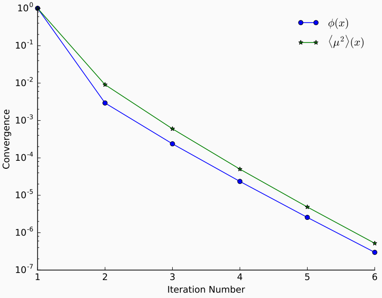
Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Unaccelerated

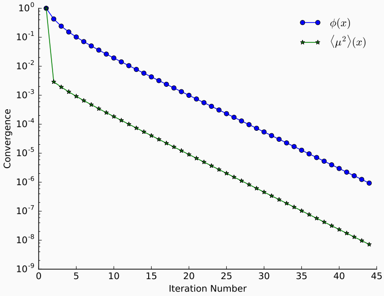


Accelerated

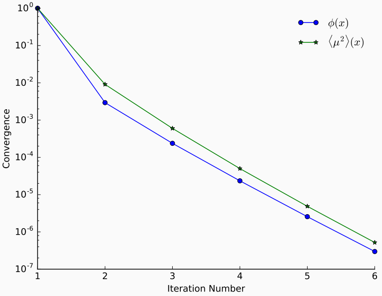


Convergence Rate Comparison

Unaccelerated



Accelerated



Fast rate of convergence of $\langle\mu^2\rangle(x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

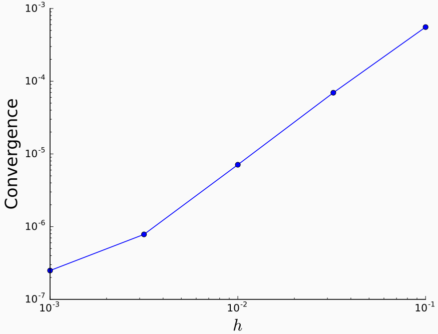
as $h \to 0$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

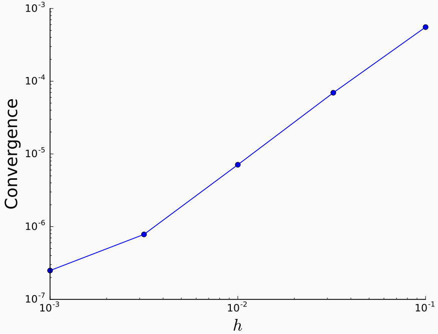


Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$



S_N and Moment solutions converge as mesh is refined

Method of Manufactured Solutions Order of Accuracy

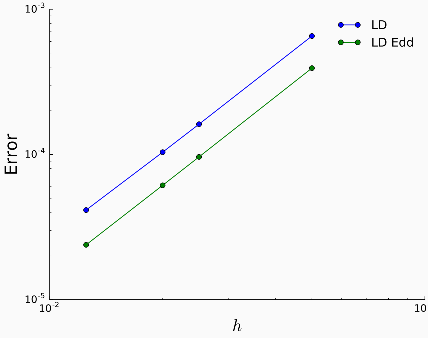
Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

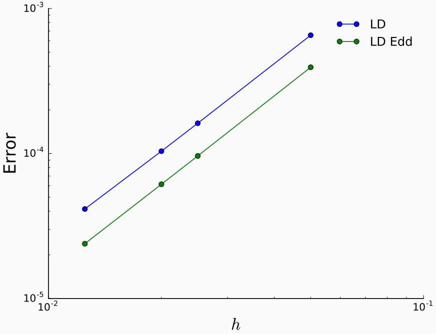
$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$



Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

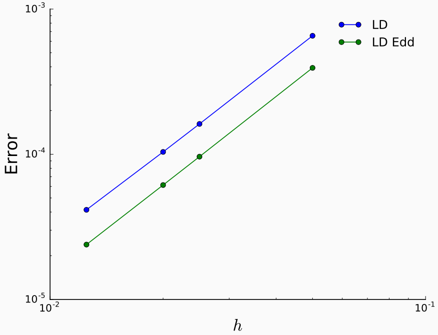


Both second order accurate

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$



Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
    - LDG transport
    - MHFEM hydrodynamics
    - Source iteration acceleration
    - Provides inexpensive, conservative solution
- Proved MHFEM can be used to accelerate lumped LDG transport in 1D slab geometry

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
    - LDG transport
    - MHFEM hydrodynamics
    - Source iteration acceleration
    - Provides inexpensive, conservative solution
- Proved MHFEM can be used to accelerate lumped LDG transport in 1D slab geometry

Future Work

- Develop a rad-hydro algorithm
    - Make use of inexpensive Moment solution in multiphysics iterations
- Add energy, time dependence
- Test in 3D
- Explore other multiphysics applications

References

[1] M. L. ADAMS AND E. W. LARSEN, *Fast Iterative Methods for Discrete–Ordinates Particle Transport Calculations*, vol. 40, Progress in Nuclear Technology, 2002.

[2] R. E. ALCOUFFE, *Diffusion Synthetic Acceleration Methods for the Diamond–Differenced Discrete–Ordinates Equations*, 1977.

[3] S. BOLDING AND J. HANSEL, *Second-Order Discretization in Space and Time for Radiation-Hydrodynamics*, Journal of Computational Physics, 2017.

[4] F. BREZZI AND M. FORTIN, *Mixed and Hybdrid Finite Element Methods*, Springer, 1991.

[5] J. I. CASTOR, *Radiation Hydrodynamics*, Lawrence Livermore National Lab, 2003.

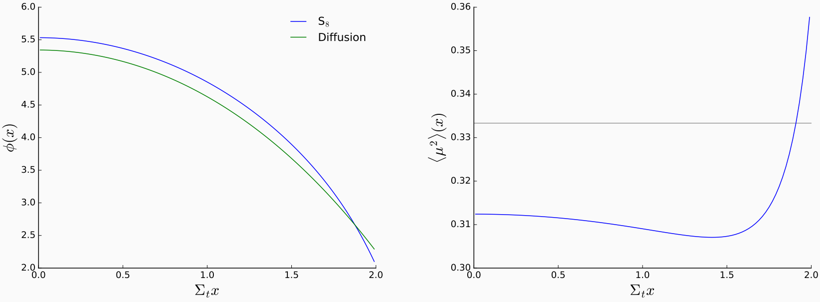
[6] S. N. SHORE, *An Introduction to Astrophysicial Hydrodynamics*, Academic Press, Inc., 1992.

[7] J. S. WARSA, T. A. WAREING, AND J. E. MOREL, *Fully Consistent Diffusion Synthetic Acceleration of Linear Discontinuous Transport Discretizations on Three–Dimensional Unstructured Meshes*.

Questions?

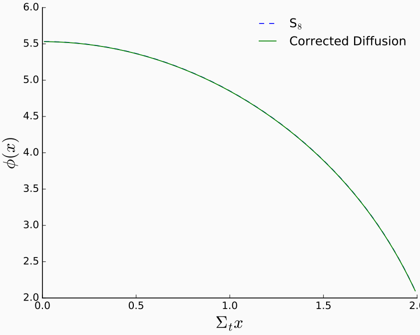
S_8 v. Diffusion

Small system $\Rightarrow$ diffusion not expected to be accurate



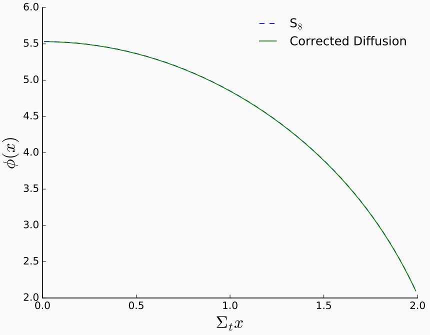
S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



S_8 v. Drift Diffusion

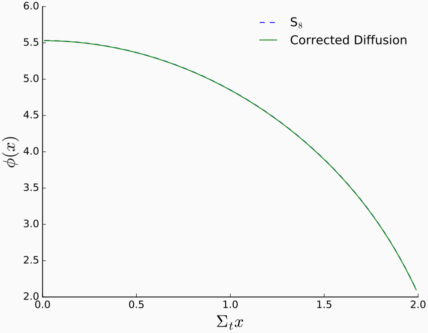
Use $\langle\mu^2\rangle(x)$ from S_8 in Moment Equations



Moment Equations and S_N match!

S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



Moment Equations and S_N match!

Requires knowledge of angular flux