Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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https://github.com/smsolivier/EddingtonAcceleration.git

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TEXAS A&M UNIVERSITY

Overview

Motivation

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Radiation Hydrodynamics

- Propagation of thermal radiation through a fluid
- Effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory physics (NIF, Z Machine) and astrophysics

Need hydrodynamics and transport to be consistently differenced

- Use the same method or do extra work to make differing methods agree
- Interpolating between spatial grids introduces noise
- Matching grids between methods is not always possible in higher dimensions

Hydrodynamics will be discretized with Mixed Hybrid Finite Element Method (MHFEM)

Want to be able to pair with Linear Discontinuous Galerkin (LDG) transport

Problems

- Radiation transport is expensive
- MHFEM and first-order form of transport are incompatible $\Rightarrow$ can't use linear acceleration scheme

Goal

Develop a transport algorithm that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Bridges LDG transport and MHFEM multiphysics

Show scheme works in 1D slab with lumped LDG transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x,\mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x, \mu) \xrightarrow{\ S_N\ } \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

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$\mu_1, \mu_2, \ldots, \mu_N$ defined by N-point Gauss Quadrature Rule

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μ_1, μ_2, ..., μ_N defined by N-point Gauss Quadrature Rule

Integrate order $2N - 1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

> ### S_N Equations
>
> $$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\, 1 \leq n \leq N$$
>
> $$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}, 1 \le n \le N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

Need For Acceleration in Source Iteration

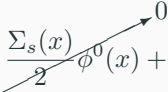Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \frac{\Sigma_s(x)}{2}\phi^0(x) + \frac{Q(x)}{2} \, , 1 \le n \le N$$

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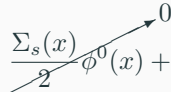
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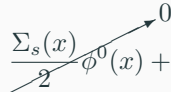
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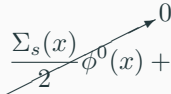$\phi^2(x)$ is uncollided and once collided flux

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$\phi^2(x)$ is uncollided and once collided flux

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$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Need For Acceleration in Source Iteration

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

S_N is too expensive in these regions

Need an acceleration scheme that rapidly increases the rate of convergence of source iteration

Eddington Acceleration

Zeroth Angular Moment

Take angular moments of the Boltzmann equation

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Take angular moments of the Boltzmann equation

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

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Integrate over all angles

$$\int_{-1}^{1}\mu\frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu \; + \int_{-1}^{1}\Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1}\frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \; + \int_{-1}^{1}\frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu \frac{\int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu}{\int_{-1}^{1} \psi(x, \mu) \, \mathrm{d}\mu}$$

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Angular flux weighted average of μ^2

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0 \qquad \text{(First Moment)}$$

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Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

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If $\langle\mu^2\rangle(x) = \frac{1}{3}$

$$J(x) = -\frac{1}{3\Sigma_t(x)}\frac{\mathrm{d}\phi}{\mathrm{d}x} \qquad \text{(Fick's Law)}$$

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Transport information passed through $\langle\mu^2\rangle(x)$ and boundary conditions

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Solving the Moment Equations requires knowledge of the angular flux (the solution)

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Acceleration occurs because

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Moment Equations model all scattering at once $\Rightarrow$ dependence on source iterations to introduce scattering information is reduced

Eddington Acceleration Properties

Non-linear scheme $\Rightarrow$ produces 2 solutions (S_N and Moment)

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Linear schemes require that diffusion and S_N produce one solution

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Relaxes consistent differencing requirements

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Benefits

1. Transport can be LDG and Moment can be MHFEM
2. Moment Equations are conservative and relatively inexpensive compared to transport sweep
3. Can use Moment solution in MHFEM multiphysics iterations without needing a full transport sweep
4. Difference between S_N and Moment solution can be used as a measure of spatial truncation error (measure of mesh convergence)

Results

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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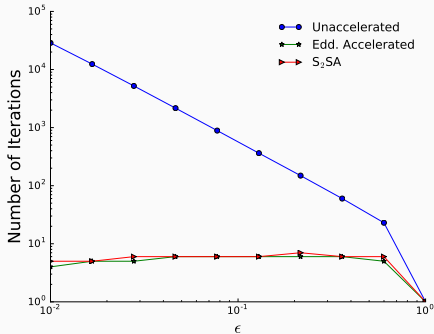
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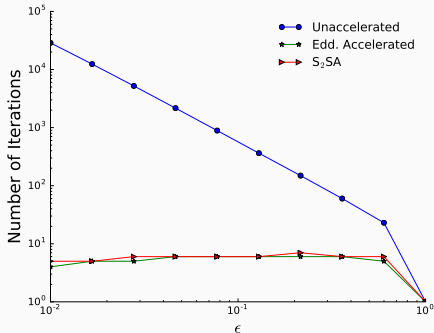
Accelerates source iteration, survives diffusion limit

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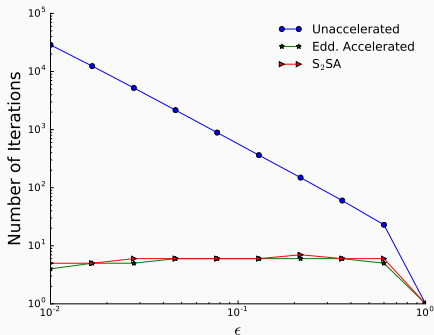
Accelerates source iteration, survives diffusion limit

Performs similarly to consistently differenced, linear acceleration (S_2SA)
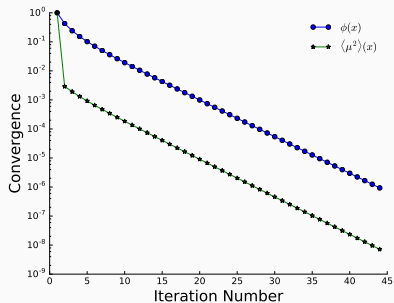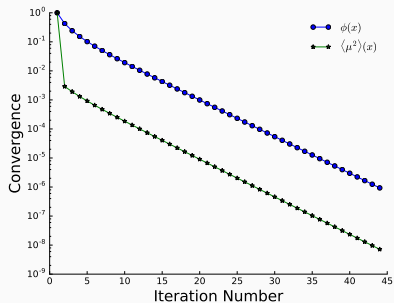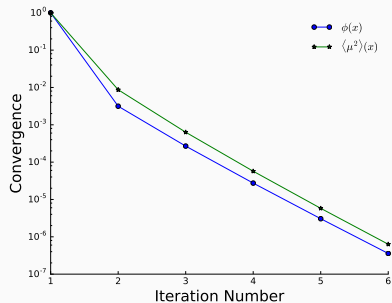
Unaccelerated

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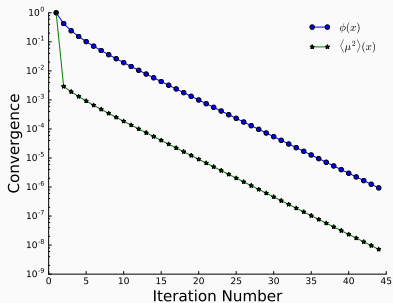
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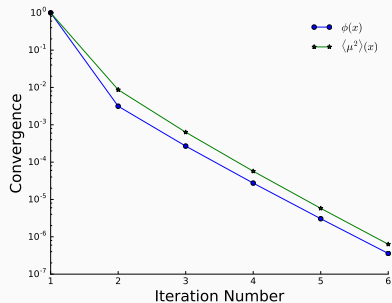
Fast rate of convergence of $\langle\mu^2\rangle(x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{S_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

Solution Convergence

Compare

$$\frac{\|\phi_{S_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

17
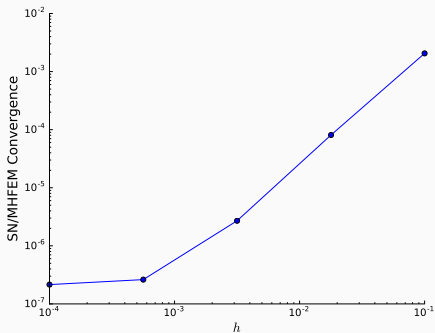
Solution Convergence

Compare

$$\frac{\|\phi_{S_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$
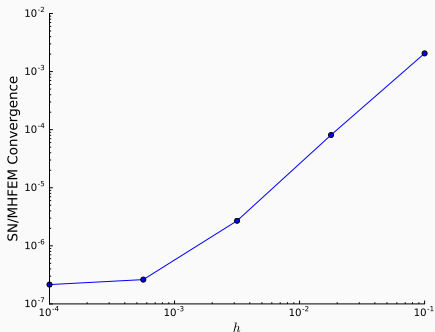
S_N and Moment solutions converge as mesh is refined

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to
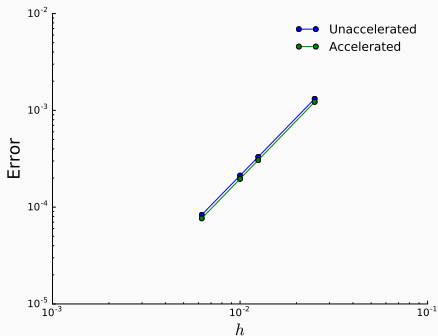
$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Method of Manufactured Solutions Order of Accuracy

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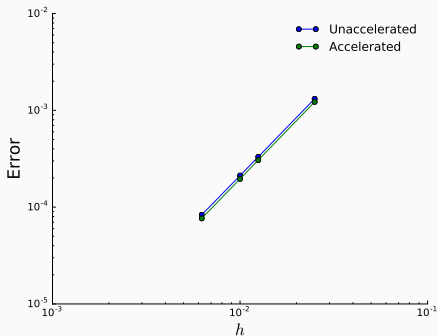
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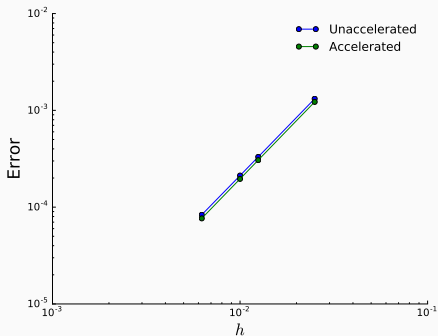
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Both second order accurate

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

Both second order accurate

Eddington Acceleration did not effect the order of accuracy of lumped LDG

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
  - Transport and acceleration steps can be differenced with different methods
  - Reduces expense of source iteration
  - Provides inexpensive, conservative solution
- Showed MHFEM and lumped LDG can be paired

Summary

Conclusions

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Future Work

- Develop a rad-hydro algorithm
  - Make use of inexpensive Moment solution in multiphysics iterations
- Add temperature
- Higher order of accuracy
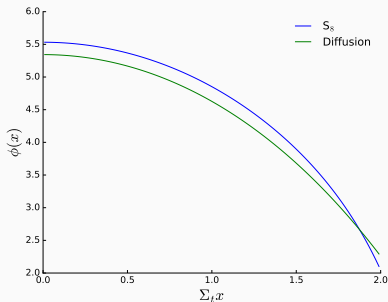- Explore other multiphysics applications

References

[1] M. L. ADAMS AND E. W. LARSEN, *Fast Iterative Methods for Discrete–Ordinates Particle Transport Calculations*, vol. 40, Progress in Nuclear Technology, 2002.

[2] R. E. ALCOUFFE, *Diffusion Synthetic Acceleration Methods for the Diamond–Differenced Discrete–Ordinates Equations*, 1977.

[3] S. BOLDING AND J. HANSEL, *Second-Order Discretization in Space and Time for Radiation-Hydrodynamics*, Journal of Computational Physics, 2017.

[4] F. BREZZI AND M. FORTIN, *Mixed and Hybdrid Finite Element Methods*, Springer, 1991.

[5] J. I. CASTOR, *Radiation Hydrodynamics*, Lawrence Livermore National Laboratory, 2003.

[6] C. NEWMAN, D. KNOLL, AND R. PARK, *Nonlinear Acceleration of Transport Criticality Problems*, Los Alamos National Laboratory, 2011.

[7] S. N. SHORE, *An Introduction to Astrophysicial Hydrodynamics*, Academic Press, Inc., 1992.

[8] J. S. WARSA, T. A. WAREING, AND J. E. MOREL, *Fully Consistent Diffusion Synthetic Acceleration of Linear Discontinuous Transport Discretizations on Three–Dimensional Unstructured Meshes*.
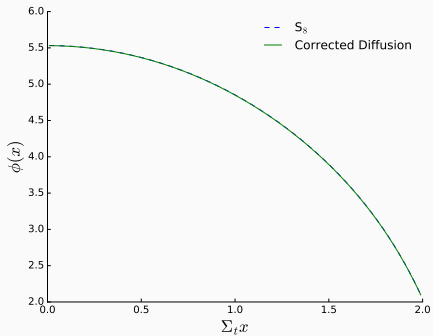
Questions?

Small system $\Rightarrow$ diffusion not expected to be accurate

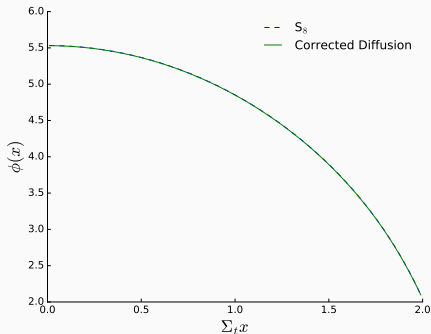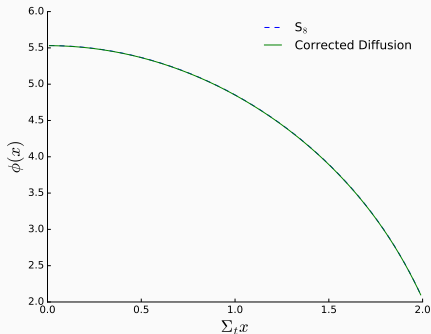Use $\langle \mu^2 \rangle(x)$ from S$_8$ in Moment Equations

Use $\langle\mu^2\rangle(x)$ from S_8 in Moment Equations

Moment Equations and S_N match!
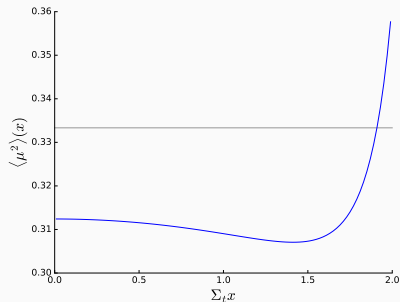
S$_8$ v. Drift Diffusion

Use $\langle \mu^2 \rangle (x)$ from S$_8$ in Moment Equations

Moment Equations and S$_N$ match!

Requires knowledge of angular flux