Lecture 23
Spatially Analytic Sn Solutions

1 Introduction

We derive spatially analytic solutions to the S_n equations that are analogous to the separation-of-variables technique used for the analytic transport equation. However, S_n solutions are much easier to construct than analytic solutions.

2 Application of Separation-of-Variables

We begin with the homogeneous S_N equations with isotropic scattering:

$$\mu_m \frac{\partial \psi}{\partial x} + \sigma_t \psi_m = \sigma_s \phi \,, \quad m = 1, N. \tag{1}$$

where

$$\phi = \sum_{m=1}^{N} \psi_m w_m \,, \tag{2}$$

and the weights are assumed to sum to unity. The cosines are indexed in increasing order, i.e., $\mu_{min} = \mu_1$ and $\mu_{max} = \mu_N$. It is useful at this point to transform the spatial variable from x to $z = \sigma_t x$, i.e., to measure distance in mean-free-paths rather than centimeters.

dividing Eq. (1) by σ_t, and recognizing that $\frac{\partial}{\partial z} = \frac{1}{\sigma_t}\frac{\partial}{\partial x}$, we get

$$\mu \frac{\partial \psi_m}{\partial z} + \psi_m = c\phi \,. \tag{3}$$

where $c = \sigma_s/\sigma_t$. We next assume an exponential spatial dependence for the solution:

$$\psi(z, \mu_m) = \psi_\nu(\mu_m) \exp^{-\frac{z}{\nu}} \,, \tag{4}$$

where μ is a relaxation length measured in mean-free-paths. Substituting from Eq. (4) into Eq. (3), we get

$$-\frac{\mu_m}{\nu}\psi_\nu(\mu_m) \exp^{-\frac{z}{\nu}} + \psi_\nu(\mu_m) \exp^{-\frac{z}{\nu}} = c\phi_\nu \exp^{-\frac{z}{\nu}} \,. \tag{5}$$

Solving Eq. (5) for ψ, we obtain

$$\psi_\nu(\mu_m) = c\frac{\nu\phi_\nu}{\nu - \mu_m} \,. \tag{6}$$

We next integrate Eq. (6) over all directions to obtain a consistency criterion for the values of ν:

$$1 \quad = \quad c\sum_{m=1}^{N} \frac{\nu w_m}{\nu - \mu_m} \,. \tag{7}$$

There are N values of ν that satisfy this consistency criterion. We index these ν-values in increasing order in analogy with the indexing of the quadrature points. The ν-values are symmetric about zero, which reflects the symmetry of the quadrature set. The two ν-values of largest magnitude, ν_1 and ν_N are the analogs of the analytic asymptotic relaxation lengths and lie outside the interval $[-1,+1]$. Furthermore, from symmetry it follows that $\nu_1 = -\nu_N$. The remaining ν-values lie within $[-1,+1]$ and are the analogs of the transient relaxation lengths. More specifically, each negative transient relaxation length lies between two cosine values: ν_2 lies between μ_1 and μ_2, ν_3 lies between μ_2 and μ_3, ..., $\mu_{N/2}$ lies between $\mu_{N/2-1}$ and $\mu_{N/2}$.

Note that if the consistency criterion is satisfied, one can choose any positive value for ϕ_ν. We choose a scalar flux value of 1, which yields the following explicit expression for the angular flux:

$$\psi_{\nu_k}(z,\mu_m) = c\frac{\nu_k}{\nu_k - \mu_m}\exp\left(-z/\nu_k\right), \quad k=1,N,\ m=1,N. \tag{8}$$

Re-expressed in term x, Eq. (8) becomes

$$\psi_{\nu_k}(x,\mu_m) = c\frac{\nu_k}{\nu_k - \mu_m}\exp\left(-\sigma_t x/\nu_k\right), \quad k=1,N,\ m=1,N. \tag{9}$$

Difficulties occur in trying to take the limit of these solutions as $c \to 0$ and as $c \to 1$. Homogeneous solutions for pure absorbers have been previously discussed. In particular, these solutions are trivial and given by

$$\psi_m(x) = \exp\left(-\sigma_t x/\mu_m\right), \quad m=1,N. \tag{10}$$

The difficulty with the $c \to 1$ limit is that the asymptotic relaxation lengths approach infinity. In this case, the asymptotic modal solutions can be assumed to be isotropic in angle and satisfy the diffusion equation.

3 Constructing Solutions

3.1 Half-Space Problems

We first consider a half-space problem on the domain $x \in [0, \infty]$ with an incident flux, $f(\mu)$ at $x = 0$. We assume that the angular flux solution is a linear combination of the modal solutions defined by Eq. (9):

$$\psi_m(x) = \sum_{k=1}^{N} a_k \psi_{\nu_k}(x, \mu_m)\,, \tag{11}$$

where each a_k is a constant that must be determined by boundary conditions. Since the solution must be finite as $z \to \infty$, all negative ν components must be set to zero. Thus Eq. (10) can be re-expressed as

$$\psi_m(x) = \sum_{k=N/2+1}^{N} a_k \psi_{\nu_k}(x, \mu_m)\,, \tag{12}$$

The equations for the coefficients of the positive-ν components are obtained simply by setting the incoming fluxes at $x = 0$ to $f(\mu)$. In particular, following Eq. (12), we obtain $N/2$ linear equations for $N/2$ unknowns:

$$\sum_{k=N/2+1}^{N} a_k \psi_{\nu_k}(0, \mu_m) = f(\mu_m)\,, \quad m = N/2+1, N. \tag{13}$$

To ensure that the incoming half-range current is exact, one oftens rescales the discrete values of f, i.e.,

$$\sum_{k=N/2+1}^{N} a_k \psi_{\nu_k}(0,\mu_m) = \alpha f(\mu_m)\,, \quad m = N/2+1, N, \tag{14}$$

where

$$\alpha \sum_{m=N/2+1}^{N} f(\mu_m) w_m = \frac{1}{2}\int_0^1 f(\mu)\mu \, d\mu\,. \tag{15}$$

3.2 A Finite Homogeneous Domain

Let us now consider a finite domain, $x \in [x_L, x_R]$. For numerical purposes, we renormalize the component functions functions as follows:

$$\psi_{\nu_k}(x,\mu_m) = c\frac{\nu_k}{\nu_k - \mu_m} \exp\left\{\sigma_t\left[(x_L + x_R)/2 - x\right]/\nu_k\right\}\,, \quad k = 1, N. \tag{16}$$

On a finite domain, we set the incoming fluxes on the right and left boundaries, respectively, to obtain N equations and N unknowns:

$$\sum_{k=1}^{N} a_k \psi_{\nu_k}(x_R,\mu_m) = \alpha_R f_R(\mu_m)\,, \quad m = 1, N/2, \tag{17}$$

$$\sum_{k=1}^{N} a_k \psi_{\nu_k}(x_L,\mu_m) = \alpha_L f_L(\mu_m)\,, \quad m = N/2+1, N. \tag{18}$$

To set a source condition on the right boundary and a reflective condition on the left boundary, one sets the incoming fluxes on the right boundary to the incident flux values, and sets the incoming fluxes on the left boundary equal to the outgoing fluxes on the right boundary:

$$\sum_{k=1}^{N} a_k \psi_{\nu_k}(x_R, \mu_m) = \alpha_R f_R(\mu_m)\,, \quad m = 1, N/2, \tag{19}$$

$$\sum_{k=1}^{N} a_k \left[\psi_{\nu_k}(x_L, \mu_m) - \psi_{\nu_k}(x_L, -\mu_m)\right] = 0\,, \quad m = N/2+1, N. \tag{20}$$

3.3 Multiple Homogeneous Regions

The fluxes are required to be continuous across the interface between two homogeneous regions:

$$\sum_{k=1}^{N} a_k^L \psi_{\nu_k}^L(x_{int}, \mu_m) = \sum_{k=1}^{N} a_k^R \psi_{\nu_k}^R(x_{int}, \mu_m)\,, \quad m = 1, N, \tag{21}$$

where in a superscript "L" denotes a quantity defined in the cell to the left of the interface and a superscript "R" denotes a quantity defined in the cell to the right of the interface.

3.4 Particular Solutions

The simplest particular solution corresponds to a constant inhomogeneous source:

$$\psi_p(x, \mu_m) = q/\sigma_a\,. \tag{22}$$

As in the analytic case, the sum of the homogeneous and particular solutions is forced to meet the boundary conditions. For instance, Eq. (14) becomes:

$$\sum_{k=N/2+1}^{N} a_k \psi_{\nu_k}(0,\mu_m) + q/\sigma_a = \alpha f(\mu_m)\,, \quad m = N/2+1, N, \tag{23}$$