Eddington Acceleration

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INTRODUCTION

One of the most challenging computational tasks is to simulate the interaction of radiation with matter. The steady–state, one–group, isotropically–scattering, fixed–source Linear Boltzmann Equation in planar geometry is:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu') d\mu' + \frac{Q(x)}{2} \quad (1)$$

where $\mu = \cos\theta$ is the cosine of the angle of flight θ relative to the x–axis, $\Sigma_t(x)$ and $\Sigma_s(x)$ the total and scattering cross sections, $Q(x)$ the isotropic fixed–source and $\psi(x, \mu)$ the angular flux [1]. In the Discrete–Ordinates (S_N) angular discretization, μ takes values from Gauss Legendre quadrature. The scalar flux, $\phi(x)$, is then

$$\phi(x) = \int_{-1}^{1} \psi(x, \mu)\, \mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x) \quad (2)$$

where $\psi_n(x) = \psi(x, \mu_n)$ and the w_n the quadrature weights corresponding to the μ_n [2]. The S_N equations are then

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2} \sum_{n=1}^{N} w_n \psi_n(x) + \frac{Q(x)}{2} \quad (3)$$

where $n = 1, 2, \ldots, N$.

In the Source Iteration (SI) scheme, the right hand side of Eq. 3 is lagged. In other words,

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2} \sum_{n=1}^{N} w_n \psi_n^{\ell}(x) + \frac{Q(x)}{2}. \quad (4)$$

Equation 4 is iterated until the flux converges. If $\phi^0(x) = 0$ then ϕ^ℓ is the scalar flux of particles that have undergone $\ell - 1$ collisions [1]. Thus, the number of iterations until convergence is directly linked to the number of collisions in a particle's lifetime. Typically, SI becomes increasingly slow to converge as the ratio of Σ_s to Σ_t approaches unity.

Fortunately, the regime where SI is slow to converge is also the regime where Diffusion Theory is most accurate. A popular method for accelerating SI is Diffusion Synthetic Acceleration (DSA) where a transport sweep is conducted and then a diffusion solve is used to generate a correction factor. Standard DSA requires correction schemes such as the Source Correction, Diffusion Coefficient, and Removal Correction schemes presented in [3] to prevent instability in highly scattering regimes with coarse spatial grids.

Lawrence Livermore National Laboratory (LLNL) is developing a high–order radiation–hydrodynamics code. The hydrodynamics portion is discretized using the Mixed–Hybrid Finite Element Method (MHFEM), where values are taken to be constant within a cell with discontinuous jumps at both cell edges [4]. MHFEM is particularly suited for hydrodynamics but not for radiation transport. This work seeks to efficiently accelerate S_N calculations with a scheme that is both robust and compatible with MHFEM hydrodynamics.

EDDINGTON ACCELERATION

The zeroth and first angular moments of Eq. 1 are

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \quad (5a)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t J = 0 \quad (5b)$$

where $J = \int_{-1}^{1} \mu\, \psi(x, \mu)\, \mathrm{d}\mu$ is the current and

$$\langle \mu^2 \rangle(x) = \frac{\int_{-1}^{1} \mu^2 \psi(x, \mu)\, \mathrm{d}\mu}{\int_{-1}^{1} \psi(x, \mu)\, \mathrm{d}\mu} \quad (6)$$

the Eddington factor. When $\langle \mu^2 \rangle(x) = \frac{1}{3}$, Eqs. 5a and 5b are equivalent to Diffusion Theory.

The proposed acceleration scheme is:

1. Compute ψ_n with S_N and an arbitrary spatial discretization
2. Compute $\langle \mu^2 \rangle$
3. Interpolate $\langle \mu^2 \rangle$ onto the MHFEM grid
4. Solve the moment equations with the preconditioned $\langle \mu^2 \rangle$ using MHFEM.

This scheme allows the S_N equations and moment equations to be solved with different spatial discretizations. This means S_N can be discretized using normal methods such as Linear Discontinuous Galerkin or Diamond Differencing while the moment equations can be solved on the same grid as the hydrodynamics.

This method differs from DSA in that two solutions are generated: one from S_N and one from the moment equations and that the S_N and acceleration steps do not have to be consistently differenced. The solution of the moment equations will be used because the moment equations are conservative while S_N is not.

RESULTS

As a proof of concept for Eddington Acceleration, a Diamond Differenced S_N code was created in addition to an MHFEM solver for Eqs. 5a and 5b. The test problem of steady–state, one–group, isotropically–scattering, fixed–source radiation transport in slab geometry with a reflecting left boundary and vacuum right boundary was used to compare Eddington Accelerated and unaccelerated S_8. Figure 1 shows the number of iterations until the L2 norm of the flux converged to within a tolerance of 1×10^{-6} for varying ratios of Σ_s to Σ_t.

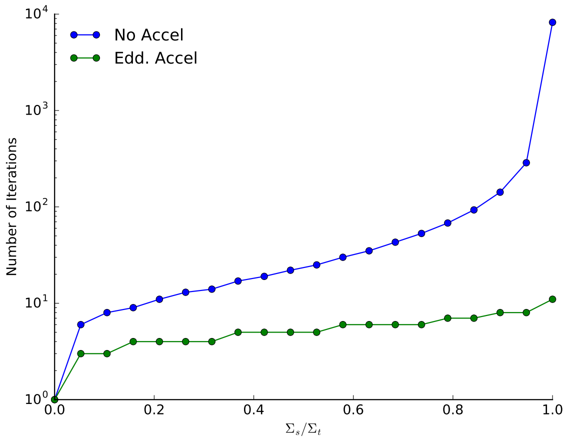

Fig. 1. A comparison of unaccelerated and Eddington accelerated S_8.

CONCLUSIONS

Figure 1 suggests that Eddington Acceleration is a valid method for accelerating S_N SI calculations. The required iterations until convergence is significantly lower for Eddington accelerated S_8. This is especially evident for the pure scattering regime ($\Sigma_s = \Sigma_t$) where S_8 was accelerated by a factor of 750. This scheme produces a conservative solution and does not require the S_N and acceleration steps to be consistently differenced.

ACKNOWLEDGMENTS

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