Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

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Mathematics and Computation

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Overview

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

Motivation

Motivation

Radiation transport simulations are expensive

Multiphysics Coupling

- Operator split requires iteration
- Efficient solution methods are often incompatible

Radiation Hydrodynamics

- Mixed Hybrid Finite Element Method for hydro
- Linear Discontinuous Galerkin for transport

Goal

Develop an acceleration scheme that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Increases compatibility with MHFEM multiphysics

Test in 1D slab geometry case

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x,\mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates Angular Discretization

Compute angular flux on N discrete angles defined by Gauss Quadrature

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

μ_1, μ_2, $\ldots$, μ_N defined by N-point Gauss Quadrature Rule

Integrate order $N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \leq n \leq N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

N **independent, first-order, ordinary differential equations**

Need For Acceleration in Source Iteration

Convergence rate is linked to the number of collisions in a particle's lifetime

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \cancelto{0}{\frac{\Sigma_s(x)}{2}\phi^0(x)} + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

$\Rightarrow \phi^1(x)$ is the uncollided flux

Each source iteration adds scattering information

$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Slow to converge in optically thick systems with minimal losses to absorption and leakage

Diffusion Synthetic Acceleration

Large, highly scattering systems $\Rightarrow$ Diffusion Theory is accurate!

Diffusion Synthetic Acceleration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}$
2. Compute $\phi^{\ell+1/2}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1/2}(x)$
3. Solve diffusion equation for a correction factor, $f^{\ell+1}(x)$
4. Update scattering term with $\phi^{\ell+1}(x) = \phi^{\ell+1/2}(x) + f^{\ell+1}(x)$ and repeat until:
$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

DSA Problems

Becomes non-convergent in highly scattering media with coarse spatial grids

Transport and Diffusion steps must be consistently differenced to prevent non-convergence

Consistently differenced diffusion is more expensive to solve

Transport and MHFEM are not compatible

DSA Problems

Becomes non-convergent in highly scattering media with coarse spatial grids

Transport and Diffusion steps must be consistently differenced to prevent non-convergence

Consistently differenced diffusion is more expensive to solve

Transport and MHFEM are not compatible

A new acceleration scheme is needed!

Eddington Acceleration

Zeroth Angular Moment

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Zeroth Angular Moment

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Zeroth Angular Moment

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$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

First Angular Moment

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \; = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

First Angular Moment

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

First Angular Moment

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x, \mu)\,\mathrm{d}\mu$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x, \mu) \, \mathrm{d}\mu$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu} \int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}_{\phi(x)} \underbrace{\frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu}}_{\langle \mu^2 \rangle (x)}$$

First Angular Moment

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x, \mu)\,\mathrm{d}\mu \to \frac{\mathrm{d}}{\mathrm{d}x}\langle \mu^2 \rangle(x)\phi(x)$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0$$

$$\Rightarrow J(x) = -\frac{1}{\Sigma_t(x)}\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x)$$

Drift Diffusion Equation

Combining the zeroth and first moments

Drift Diffusion Equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\Sigma_t(x)}\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_a(x)\phi(x) = Q(x)$$

Diffusion Equation is recovered if $\langle\mu^2\rangle(x) = \frac{1}{3}$

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle (x)$ and Drift Diffusion to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^{\ell}(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Drift Diffusion Equation for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Eddington Acceleration Properties

Acceleration occurs due to:

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Solution to moment equations models all scattering events at once
3. Dependence on source iterations to introduce scattering information is reduced

Downside: produces 2 solutions (S_N and Drift Diffusion)

Benefits

1. Moment Equations are conservative
2. Transport and Acceleration steps can be differenced with arbitrarily different methods
3. Accelerates source iterations
4. Difference between S_N and Drift Diffusion solution can be used as a measure of iteration uncertainty

Results

Test Problem

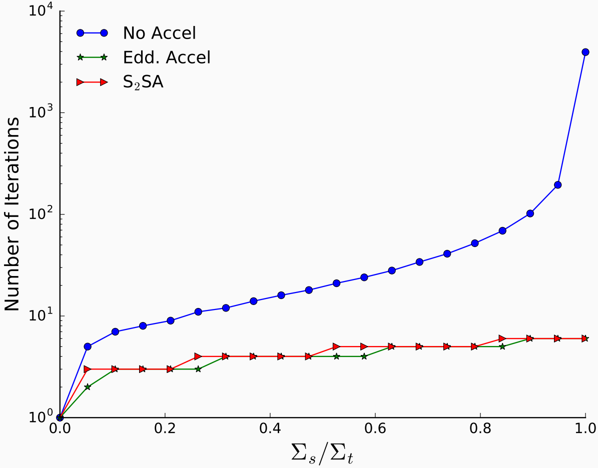
Slab with reflecting left boundary and vacuum right boundary

Thickness of 20 cm

$\Sigma_t(x) = 1\,\mathrm{cm}^{-1}$

100 cells $\Rightarrow$ optical thickness of 20 and optical thickness per cell of 0.2

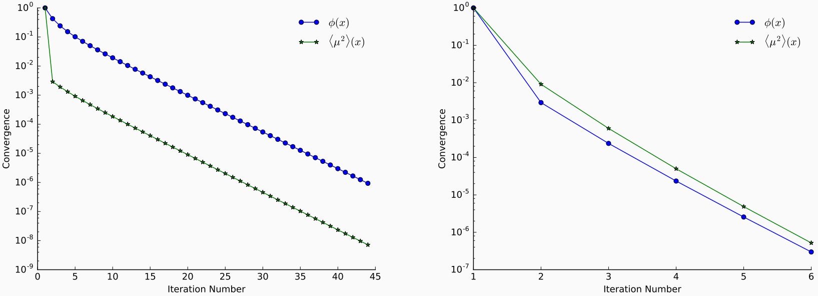
Iterations to Convergence Comparison



Accelerates between 2.5 and 650 times $\Rightarrow$ acceleration is occurring

Performs similarly to consistent acceleration scheme

Convergence Rate Comparison



Fast rate of convergence of $\langle \mu^2 \rangle(x)$ is transfered to $\phi(x)$

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Inherently compatible with rad-hydro multiphysics
    - Transport and acceleration steps can be discretized with arbitrarily different methods
    - Avoids consistency issues
    - Provides less expensive, conservative solution
- Proved Mixed Hybrid Finite Element Method can be used to accelerate lumped Linear Discontinuous Galerkin transport

Future Work

- Develop a rad-hydro algorithm
    - Make use of inexpensive Drift Diffusion solution in operator split iterations
- Add discretization in energy
- Higher dimensions
- Anisotropic scattering

References

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Questions?