Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

American Nuclear Society Student Conference Presentation

Samuel S. Olivier

March 21, 2017

Department of Nuclear Engineering, Texas A&M University

Table of contents

Introduction

Motivation

Radiation transport simulations are expensive

Multiphysics Coupling

- Operator split requires iteration
- Efficient solution methods are often incompatible

Radiation Hydrodynamics

- Mixed Hybrid Finite Element Method for hydro
- Linear Discontinuous Galerkin for transport

Goal

Develop an acceleration scheme that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Remains compatible with MHFEM multiphysics

Test in 1D slab geometry case

Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^{1} \psi(x, \mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x, \mu)$ the angular flux

Factors of $1/2$ come from

$$\phi(x) = \int_{-1}^{1} \psi(x, \mu)\, \mathrm{d}\mu$$

Integro-differential equation

Discrete Ordinates Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots & \\ \psi_N, & \mu = \mu_N \end{cases}$$

μ_1, μ_2, ..., μ_N defined by N-point Gauss Quadrature Rule

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{\mathsf{S}_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N coupled, ordinary differential equations

Source Iteration

Lag scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^{\ell}(x) + \frac{Q(x)}{2}, 1 \leq n \leq N$$

Source Iteration

1. Given previous estimate for $\phi^{\ell}(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^{\ell}(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

N **independent, first-order, ordinary differential equations**

Need For Acceleration in Source Iteration

If $\phi^0(x) = 0$

$$\mu_n \frac{\mathrm{d}\psi_n^1}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^1(x) = \frac{\Sigma_s(x)}{2}\phi^0(x) + \frac{Q(x)}{2}\,, 1 \le n \le N$$

$\Rightarrow \phi^1(x)$ is the uncollided flux

Each source iteration adds scattering information

$\phi^\ell(x)$ is the scalar flux of particles that have undergone at most $\ell - 1$ collisions

Number of iterations is linked to the number of collisions in a particle's lifetime

Slow to converge in optically thick systems with minimal losses to absorption and leakage

Diffusion Synthetic Acceleration

Large, highly scattering systems $\Rightarrow$ Diffusion Theory is accurate!

Diffusion Synthetic Acceleration

1. Given previous estimate for $\phi^{\ell}(x)$, solve for $\psi_n^{\ell+1/2}$
2. Compute $\phi^{\ell+1/2}(x) = \sum_{n=1}^{N} w_n \psi_n^{\ell+1/2}(x)$
3. **Solve diffusion equation for a correction factor,** $f^{\ell+1}(x)$
4. Update scattering term with $\phi^{\ell+\mathbf{1}}(\mathbf{x}) = \phi^{\ell+\mathbf{1/2}}(\mathbf{x}) + \mathbf{f}^{\ell+\mathbf{1}}(\mathbf{x})$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^{\ell}(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

DSA Problems

Transport and Diffusion steps must be consistenly differenced to prevent non-convergence

Consistently differenced diffusion is much more expensive to solve

Transport and MHFEM are not compatible

A new acceleration scheme is needed!

Eddigton Acceleration

Zeroth Angular Moment

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

Zeroth Angular Moment

$$\int_{-1}^{1} \mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,\mathrm{d}\mu$$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\mu^2\rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0$$

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the moment equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Eddington Acceleration Properties

Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges

Solution to moment equations models all scattering events at once

Reduces dependence on source iterations to introduce scattering information

Benefits

1. Moment Equations are conservative
2. Transport and Acceleration steps can be differenced with arbitrarily different methods
3. Accelerates source iterations

Metropolis titleformats

metropolis supports 4 different titleformats:

- Regular
- Smallcaps
- allsmallcaps
- ALLCAPS

They can either be set at once for every title type or individually.

Small caps

This frame uses the `smallcaps` titleformat.

Potential Problems

Be aware, that not every font supports small caps. If for example you typeset your presentation with pdfTeX and the Computer Modern Sans Serif font, every text in smallcaps will be typeset with the Computer Modern Serif font instead.

This frame uses the `allsmallcaps` titleformat.

Potential problems

As this titleformat also uses smallcaps you face the same problems as with the `smallcaps` titleformat. Additionally this format can cause some other problems. Please refer to the documentation if you consider using it.

As a rule of thumb: Just use it for plaintext-only titles.

ALL CAPS

This frame uses the `allcaps` titleformat.

Potential Problems

This titleformat is not as problematic as the `allsmallcaps` format, but basically suffers from the same deficiencies. So please have a look at the documentation if you want to use it.

Results

Typography

```
The theme provides sensible defaults to
\emph{emphasize} text, \alert{accent} parts
or show \textbf{bold} results.
```

becomes

The theme provides sensible defaults to *emphasize* text, accent parts or show **bold** results.

Font feature test

- Regular
- *Italic*
- SmallCaps
- **Bold**
- ***Bold Italic***
- **Bold SmallCaps**
- `Monospace`
- *`Monospace Italic`*
- **`Monospace Bold`**
- ***`Monospace Bold Italic`***

Lists

Items

- Milk
- Eggs
- Potatos

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.

Beamer Yeeeha.

Animation

- This is important

Animation

- This is important
- Now this

Animation

- This is important
- Now this
- And now this

Animation

- This is really important
- Now this
- And now this

Figures

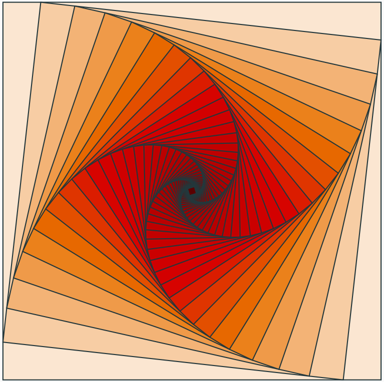


Figure 1: Rotated square from texample.net.

Tables

Table 1: Largest cities in the world (source: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Default

Block content.

Default

Block content.

Alert

Block content.

Alert

Block content.

Example

Block content.

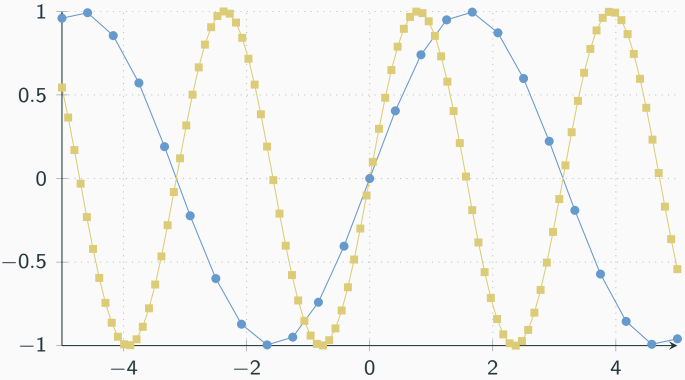
Example

Block content.

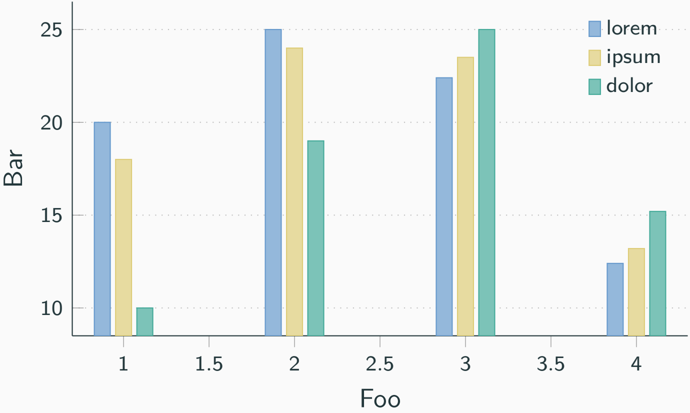
Math

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Line plots



Bar charts



Quotes

Veni, Vidi, Vici

Frame footer

metropolis defines a custom beamer template to add a text to the footer. It can be set via

```
\setbeamertemplate{frame footer}{My custom footer}
```

References

Some references to showcase [allowframebreaks] [**?**, **?**, **?**, **?**, **?**]

Conclusion

Summary

Get the source of this theme and the demo presentation from

`github.com/matze/mtheme`

Questions?

Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix.

References I