Mixed Hybrid Finite Element Method Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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Overview

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

Motivation

Motivation

Radiation Hydrodynamics

- Propogation of thermal radiation through a fluid
- Effects of radiation on fluid momentum and energy
- Required in high energy density laboratory physics (NIF, Z Machine) and astrophysics

Efficient methods developed independently

Want

- Mixed Hybrid Finite Element Method (MHFEM) for hydrodynamics
- Linear Discontinuous Galerkin (LDG) for transport

Need hydrodynamics and transport to be consistently differenced

- Interpolating between spatial grids introduces noise
- Matching grids between methods is not always possible in higher dimensions

Solution:

- Extra work to make differing methods agree (MUSCL-Hancock and LDG)
- Discretize with same method

MHFEM and first-order form of transport are incompatible

Goal

Develop a transport algorithm that

1. Robustly reduces the number of source iterations in Discrete Ordinates calculations
2. Bridges LDG transport and MHFEM multiphysics

Test in 1D slab with lumped LDG transport

Source Iteration Background

Boltzmann Equation

Steady-state, mono-energetic, istropically-scattering, fixed-source Linear Boltzmann Equation in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos\theta$ the cosine of the angle of flight θ relative to the x-axis

$\Sigma_t(x)$, $\Sigma_s(x)$ total and scattering macroscopic cross sections

$Q(x)$ the isotropic fixed-source

$\psi(x,\mu)$ the angular flux

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Integro-differential equation

Discrete Ordinates (S_N) Angular Discretization

Compute angular flux on N discrete angles

$$\psi(x,\mu) \xrightarrow{\mathsf{S}_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

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μ_1, μ_2, ..., μ_N defined by N-point Gauss Quadrature Rule

Integrate order $2N-1$ polynomials exactly with

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu \xrightarrow{S_N} \sum_{n=1}^{N} w_n \psi_n(x)$$

S_N Equations

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,,\ 1 \le n \le N$$

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)$$

N **coupled, ordinary differential equations**

All coupling in scattering term

Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N$$

N **independent, first-order, ordinary differential equations**

Solve each equation with well-known sweeping process

Source Iteration

1. Given previous estimate for $\phi^\ell(x)$, solve for $\psi_n^{\ell+1}$
2. Compute $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with $\phi^{\ell+1}(x)$ and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

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Convergence rate is linked to the number of collisions in a particle's lifetime

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Slow to converge in optically thick systems with minimal losses to absorption and leakage

Eddington Acceleration

Zeroth Angular Moment

Boltzmann Equation

$$\mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}$$

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Integrate over all angles

$$\int_{-1}^{1} \mu \frac{\mathrm{d}\psi}{\mathrm{d}x}(x, \mu)\,\mathrm{d}\mu \; + \int_{-1}^{1} \Sigma_t(x)\psi(x, \mu)\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu \; + \int_{-1}^{1} \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Use $J(x) = \int_{-1}^{1} \mu\psi(x,\mu)\,\mathrm{d}\mu$, $\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu$

Zeroth Angular Moment

$$\frac{\mathrm{d}}{\mathrm{d}x}J(x) + \Sigma_a(x)\phi(x) = Q(x)$$

First Angular Moment

Multiply by μ and integrate

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}\psi}{\mathrm{d}x}(x,\mu)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \Sigma_t(x)\psi(x,\mu)\,\mathrm{d}\mu \;=\; \int_{-1}^{1} \mu \frac{\Sigma_s(x)}{2}\phi(x)\,\mathrm{d}\mu + \int_{-1}^{1} \mu \frac{Q(x)}{2}\,\mathrm{d}\mu$$

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Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

Eddington Factor

Rearrange derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{1} \mu^2 \psi(x,\mu) \, \mathrm{d}\mu$$

Multiply and divide by $\int_{-1}^{1} \psi(x,\mu) \, \mathrm{d}\mu$

$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_{-1}^{1} \psi(x,\mu) \, \mathrm{d}\mu}_{\phi(x)} \underbrace{\frac{\int_{-1}^{1} \mu^2 \psi(x,\mu) \, \mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu) \, \mathrm{d}\mu}}_{\text{Eddington Factor}}$$

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Eddington Factor

$$\langle\mu^2\rangle(x)=\frac{\int_{-1}^{1}\mu^2\psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu}$$

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Shape function

Moment Equations

Moment Equations

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x) \qquad \text{(Zeroth Moment)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x) J(x) = 0 \qquad \text{(First Moment)}$$

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Solve First Moment for $J(x)$

$$J(x) = -\frac{1}{\Sigma_t(x)} \frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x)$$

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Just as accurate as S$_N$

Solving the Moment Equations requires knowledge of the angular flux

Eddington Acceleration

Use S_N to compute $\langle \mu^2 \rangle(x)$ and Moment Equations to find $\phi(x)$

Eddington Acceleration

1. Given the previous estimate for the scalar flux, $\phi^\ell(x)$, solve for $\psi_n^{\ell+1/2}(x)$
2. Compute $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for $\phi^{\ell+1}(x)$ using $\langle \mu^2 \rangle^{\ell+1/2}(x)$
4. Update the scalar flux estimate with $\phi^{\ell+1}(x)$ and repeat the iteration process until the scalar flux converges

Eddington Acceleration Properties

Acceleration occurs due to:

1. Angular shape of the angular flux converges quickly $\Rightarrow$ Eddington factor quickly converges
2. Solution to moment equations models all scattering events at once
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Benefits

1. No need for consistent differencing $\Rightarrow$ transport can be LDG and acceleration can be MHFEM
2. Accelerates source iterations
3. Moment Equations are conservative and relatively inexpensive compared to transport sweep
4. Difference between S$_N$ and Moment solution can be used as a measure of spatial truncation error (measure of mesh convergence)

Results

Diffusion Limit

Scale cross sections, source

$$\Sigma_t \to \Sigma_t/\epsilon$$
$$\Sigma_a \to \epsilon\Sigma_a$$
$$Q \to \epsilon Q$$

System becomes diffusive as $\epsilon \to 0$

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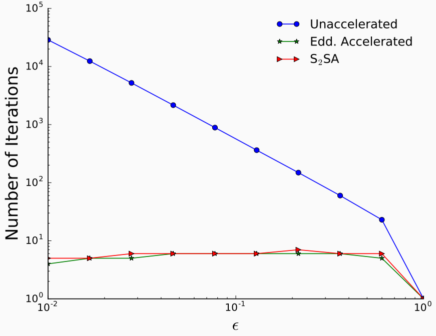

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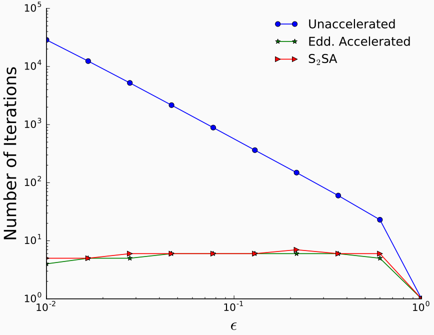

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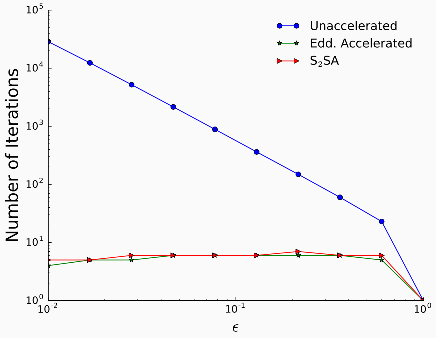
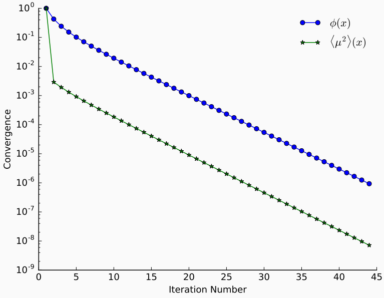


Accelerates source iteration, survives diffusion limit

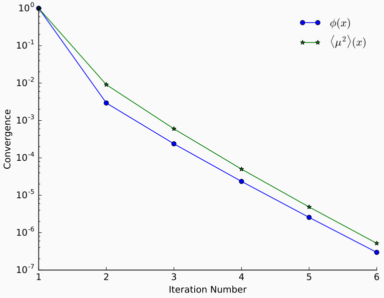
Performs similarly to consistently differenced, linear acceleration (S_2SA)

Convergence Rate Comparison

Unaccelerated



Accelerated



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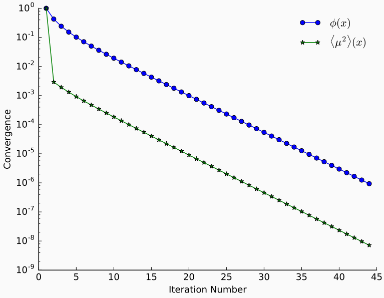
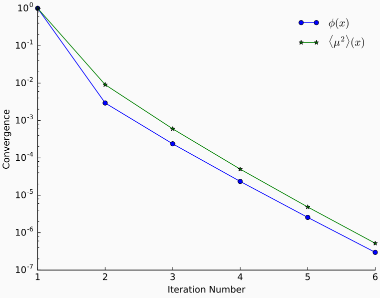


Accelerated



Fast rate of convergence of $\langle \mu^2 \rangle (x)$ is transfered to $\phi(x)$

Solution Convergence

Compare

$$\frac{\|\phi_{\mathsf{S}_N}(x) - \phi_{\mathsf{Moment}}(x)\|}{\|\phi_{\mathsf{Moment}}(x)\|}$$

as $h \to 0$

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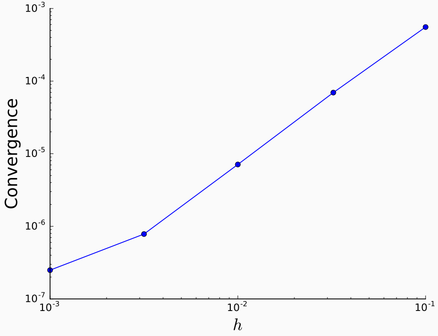

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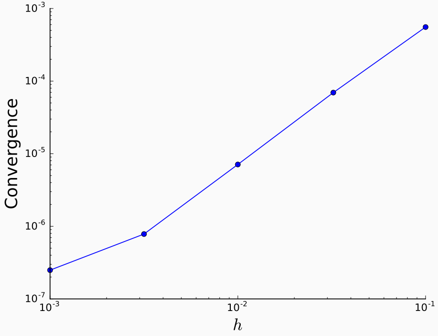

S_N and Moment solutions converge as mesh if refined

Method of Manufactured Solutions Order of Accuracy

Set source term to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

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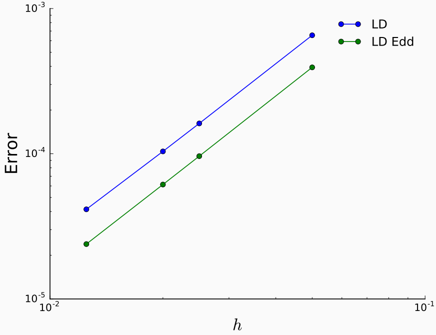

Method of Manufactured Solutions Order of Accuracy

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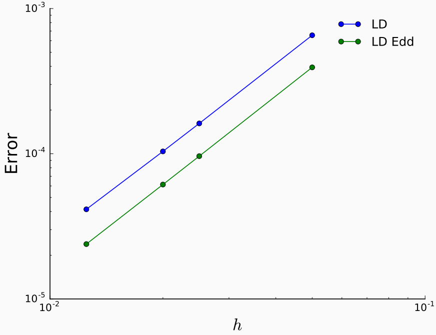

Both second order accurate

Conclusions

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Inherently compatible with rad-hydro multiphysics
    - Transport and acceleration steps can be discretized with arbitrarily different methods
    - Avoids consistency issues
    - Provides less expensive, conservative solution
- Proved Mixed Hybrid Finite Element Method can be used to accelerate lumped Linear Discontinuous Galerkin transport

Summary

Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Inherently compatible with rad-hydro multiphysics
  - Transport and acceleration steps can be discretized with arbitrarily different methods
  - Avoids consistency issues
  - Provides less expensive, conservative solution
- Proved Mixed Hybrid Finite Element Method can be used to accelerate lumped Linear Discontinuous Galerkin transport

Future Work

- Develop a rad-hydro algorithm
  - Make use of inexpensive Moment solution in multiphysics iterations
- Add discretization in energy
- Higher dimensions
- Anisotropic scattering
- Implement LDG basis functions in MHFEM

References

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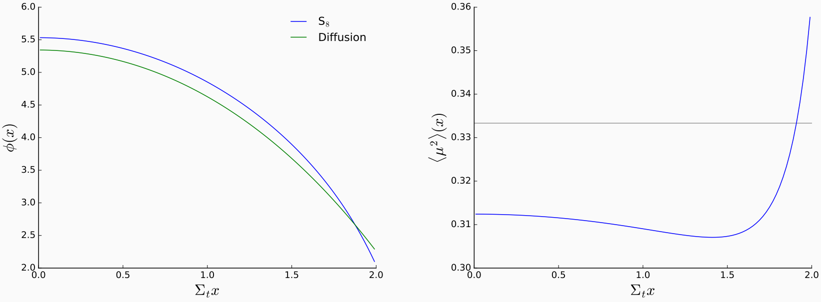
Questions?

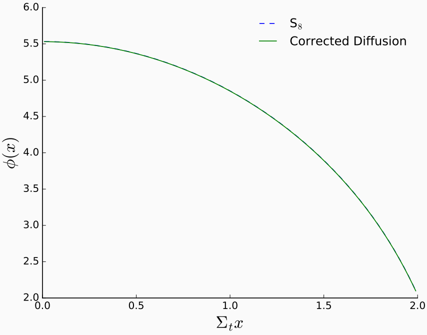
S_8 v. Diffusion

Small system $\Rightarrow$ diffusion not expected to be accurate



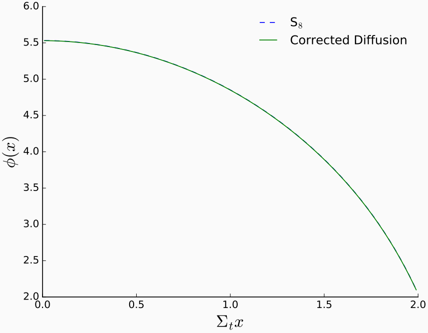
S_8 v. Drift Diffusion

Use $\langle\mu^2\rangle(x)$ from S_8 in Moment Equations



S_8 v. Drift Diffusion

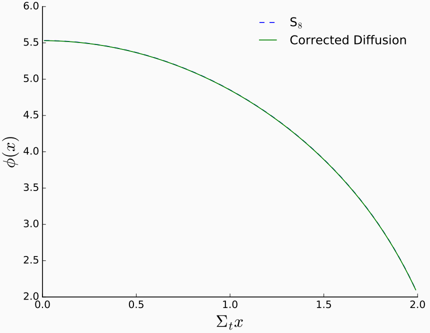
Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



Moment Equations and S_N match!

S_8 v. Drift Diffusion

Use $\langle \mu^2 \rangle(x)$ from S_8 in Moment Equations



Moment Equations and S_N match!

Requires knowledge of angular flux