

# Mixed Hybrid Finite Element Eddington Acceleration of Discrete Ordinates Source Iteration

ANS Student Conference

Mathematics and Computation

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**NUCLEAR ENGINEERING**  
TEXAS A&M UNIVERSITY

1. Motivation
2. Source Iteration Background
3. Eddington Acceleration
4. Results
5. Conclusions

# Motivation

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# Motivation

## Radiation Hydrodynamics

- Describes the effects of emission, absorption, scattering on fluid momentum and energy
- Required in high energy density laboratory experiments (NIF, Z Machine) and astrophysics

## Mixed Hybrid Finite Element Method (MHFEM) hydrodynamics

### Problems

- MHFEM and first-order form of transport are incompatible  $\Rightarrow$  can't use linear acceleration scheme
- Radiation transport is expensive

### Goal

Develop a transport algorithm that

1. Accelerates Discrete Ordinates Source Iteration
2. Bridges Linear Discontinuous Galerkin (LDG) transport and MHFEM multiphysics

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## Source Iteration Background

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# Boltzmann Equation

Steady-state, mono-energetic, isotropically-scattering, fixed-source **Linear Boltzmann Equation** in 1D slab geometry:

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x) \psi(x, \mu) = \frac{\Sigma_s(x)}{2} \int_{-1}^1 \psi(x, \mu') d\mu' + \frac{Q(x)}{2}$$

$\mu = \cos \theta$  the cosine of the angle of flight  $\theta$  relative to the  $x$ -axis

$\Sigma_t(x)$ ,  $\Sigma_s(x)$  total and scattering macroscopic cross sections

$Q(x)$  the isotropic fixed-source

$\psi(x, \mu)$  the angular flux

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**Integro-differential equation**



## Discrete Ordinates ( $S_N$ ) Angular Discretization

Compute angular flux on  $N$  discrete angles

$$\psi(x, \mu) \xrightarrow{S_N} \begin{cases} \psi_1(x), & \mu = \mu_1 \\ \psi_2(x), & \mu = \mu_2 \\ \vdots \\ \psi_N, & \mu = \mu_N \end{cases}$$

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Integrate order  $2N - 1$  polynomials exactly with

$$\phi(x) = \int_{-1}^1 \psi(x, \mu) d\mu \xrightarrow{S_N} \sum_{n=1}^N w_n \psi_n(x)$$

## $S_N$ Equations

$$\mu_n \frac{d\psi_n}{dx}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}, \quad 1 \leq n \leq N$$

$$\phi(x) = \sum_{n=1}^N w_n \psi_n(x)$$

$N$  coupled, ordinary differential equations

All coupling in scattering term

# Source Iteration

Decouple by lagging scattering term

$$\mu_n \frac{d\psi_n^{\ell+1}}{dx}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^\ell(x) + \frac{Q(x)}{2}, 1 \leq n \leq N$$

$N$  independent, first-order, ordinary differential equations

Solve each equation with well-known sweeping process

## Source Iteration

1. Given previous estimate for  $\phi^\ell(x)$ , solve for  $\psi_n^{\ell+1}$
2. Compute  $\phi^{\ell+1}(x) = \sum_{n=1}^N w_n \psi_n^{\ell+1}(x)$
3. Update scattering term with  $\phi^{\ell+1}(x)$  and repeat until:

$$\frac{\|\phi^{\ell+1}(x) - \phi^\ell(x)\|}{\|\phi^{\ell+1}(x)\|} < \epsilon$$

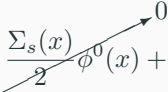
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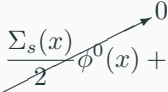
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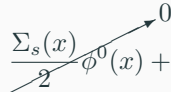
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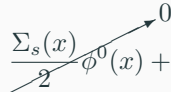
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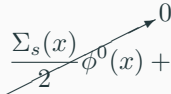
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**Slow to converge in optically thick systems with minimal losses to absorption and leakage**

# Need For Acceleration in Source Iteration

Radiation Hydrodynamics problems often contain highly diffusive regions

$S_N$  is too expensive in these regions

Need an **acceleration scheme** that rapidly increases the rate of convergence of source iteration

# Eddington Acceleration

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# Conservative Form of Boltzmann Equation

Take moments of Boltzmann equation until have enough equations for the number of unknowns

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Zeroth Moment: integrate over all angles

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Use  $J(x) = \int_{-1}^1 \mu \psi(x, \mu) d\mu$ ,  $\phi(x) = \int_{-1}^1 \psi(x, \mu) d\mu$

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$$\frac{d}{dx} J(x) + \Sigma_a(x) \phi(x) = Q(x)$$



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1 equation, 2 unknowns

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First Moment: multiply by  $\mu$  and integrate

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Rearrange derivative

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Angular flux weighted average of  $\mu^2$

# Moment Equations

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$$\frac{d}{dx}J(x) + \Sigma_a(x)\phi(x) = Q(x) \quad (\text{Zeroth Moment})$$

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Numerically: use  $S_N$  to compute estimate of  $\langle \mu^2 \rangle(x)$ , Moment Equations to find  $\phi(x)$

## Eddington Acceleration

1. Given the previous estimate for the scalar flux,  $\phi^\ell(x)$ , solve for  $\psi_n^{\ell+1/2}(x)$
2. Compute  $\langle \mu^2 \rangle^{\ell+1/2}(x)$
3. Solve the Moment Equations for  $\phi^{\ell+1}(x)$  using  $\langle \mu^2 \rangle^{\ell+1/2}(x)$
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Acceleration occurs because

1. Angular shape of the angular flux converges quickly  $\Rightarrow$  Eddington factor quickly converges
2. Moment Equations model all scattering at once  $\Rightarrow$  dependence on source iterations to introduce scattering information is reduced

Produces 2 solutions ( $S_N$  and Moment)

# Eddington Acceleration Properties

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Relaxes consistent differencing requirements important in linear acceleration



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Relaxes consistent differencing requirements important in linear acceleration

Transport can be LDG and Moment can be MHFEM

Moment Equations are conservative and relatively inexpensive to solve

Downside: Which solution is correct?

Difference between  $S_N$  and Moment solutions can be used as a measure of mesh convergence

## Results

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$S_8$  in 1D slab geometry

# Test Problem

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Lumped Linear Discontinuous Galerkin transport

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Mixed Hybrid Finite Element Method Moment

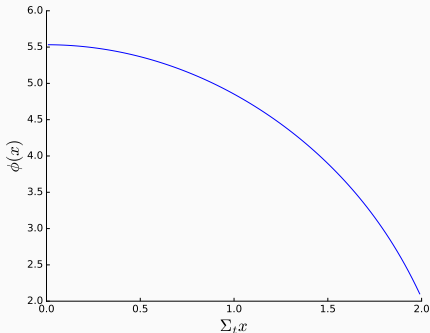


# Test Problem

$S_8$  in 1D slab geometry

Lumped Linear Discontinuous Galerkin transport

Mixed Hybrid Finite Element Method Moment



# Diffusion Limit

Scale cross sections, source

$$\Sigma_t \rightarrow \Sigma_t/\epsilon$$

$$\Sigma_a \rightarrow \epsilon \Sigma_a$$

$$Q \rightarrow \epsilon Q$$

System becomes diffusive as  $\epsilon \rightarrow 0$

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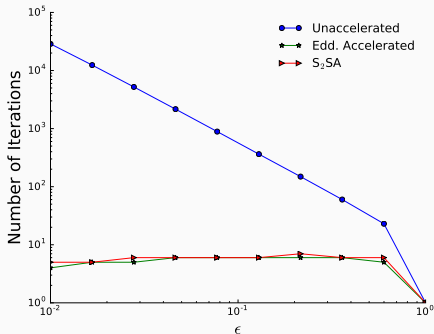
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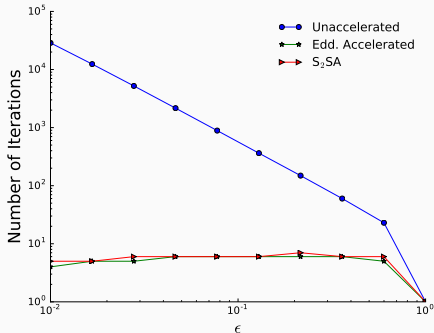
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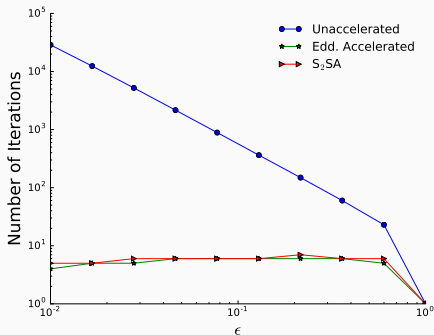
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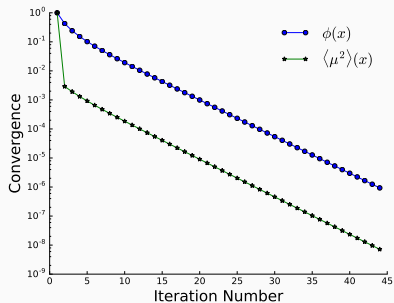


Survives diffusion limit

Performs similarly to consistently differenced, linear acceleration ( $S_2SA$ )

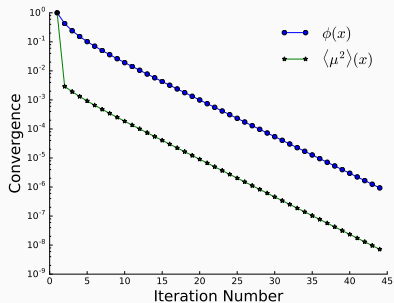
# Convergence Rate Comparison

## Unaccelerated

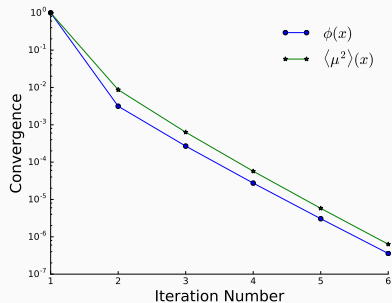


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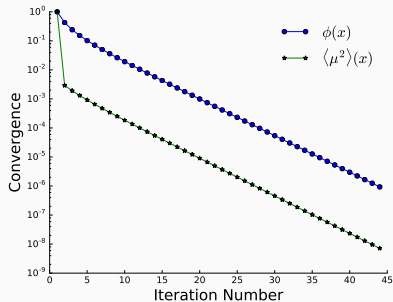


## Accelerated

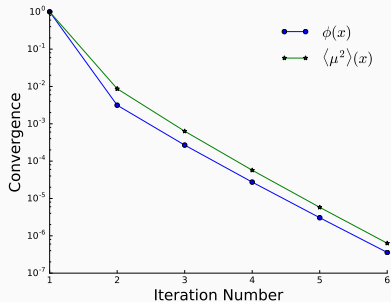


# Convergence Rate Comparison

Unaccelerated



Accelerated



Fast rate of convergence of  $\langle \mu^2 \rangle(x)$  is transferred to  $\phi(x)$



# Method of Manufactured Solutions Order of Accuracy

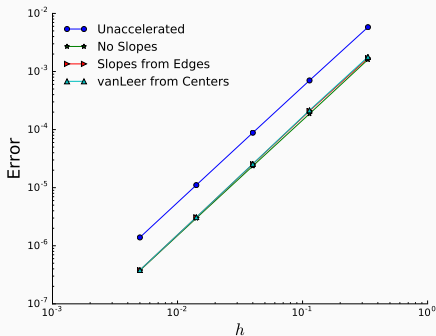
Set  $Q(x)$  to force solution to

$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$

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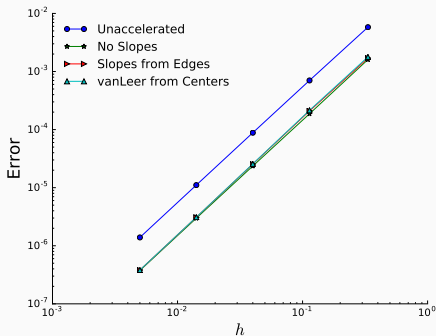
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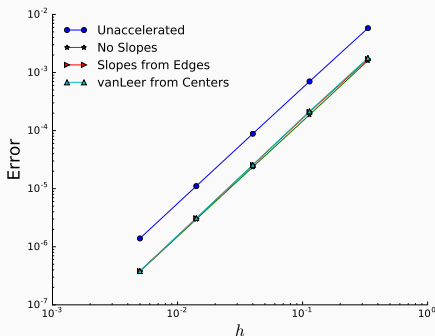


All second order as expected

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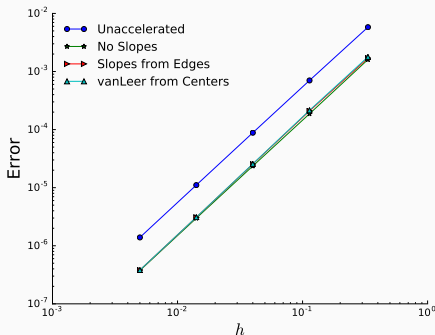
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Eddington Acceleration did not effect the order of accuracy of lumped LDG

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$$\phi(x) = \sin\left(\frac{\pi x}{x_b}\right)$$



All second order as expected

Eddington Acceleration did not effect the order of accuracy of lumped LDG

All slope recovery methods have similar accuracy

# Data Reconstruction Solution Convergence

Compare

$$\frac{\|\phi_{S_N}(x) - \phi_{\text{Moment}}(x)\|}{\|\phi_{\text{Moment}}(x)\|}$$

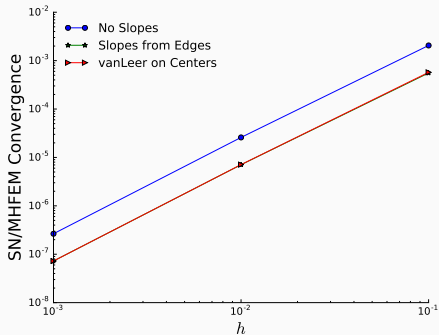
as  $h \rightarrow 0$

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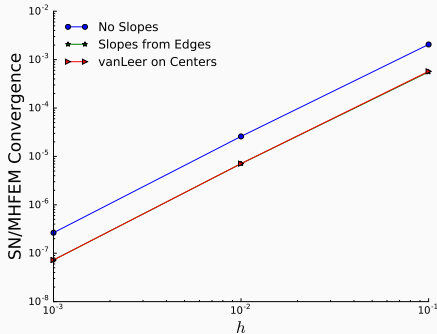


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$$\frac{\|\phi_{S_N}(x) - \phi_{\text{Moment}}(x)\|}{\|\phi_{\text{Moment}}(x)\|}$$

as  $h \rightarrow 0$



$S_N$  and Moment solutions converge as mesh is refined

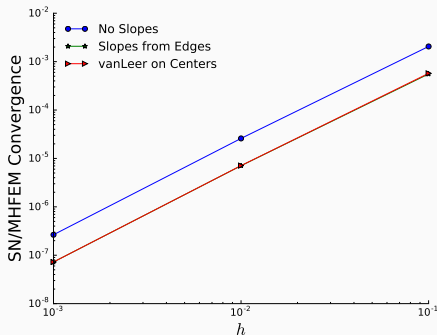


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$$\frac{\|\phi_{S_N}(x) - \phi_{\text{Moment}}(x)\|}{\|\phi_{\text{Moment}}(x)\|}$$

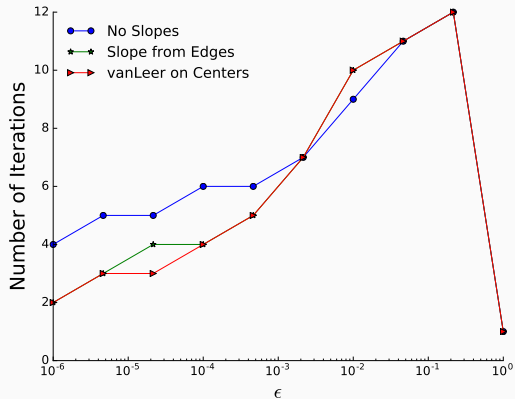
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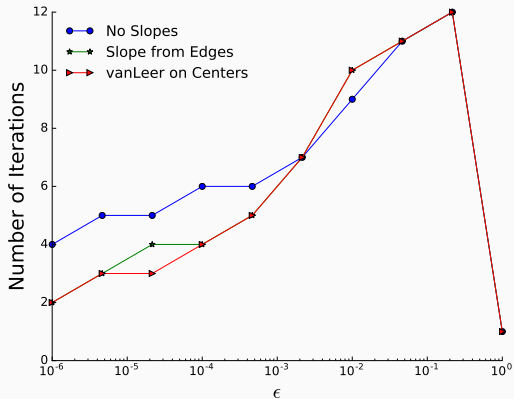
$S_N$  and Moment solutions converge as mesh is refined

Slope recovery effects solution convergence but not accuracy

# Data Reconstruction Diffusion Limit

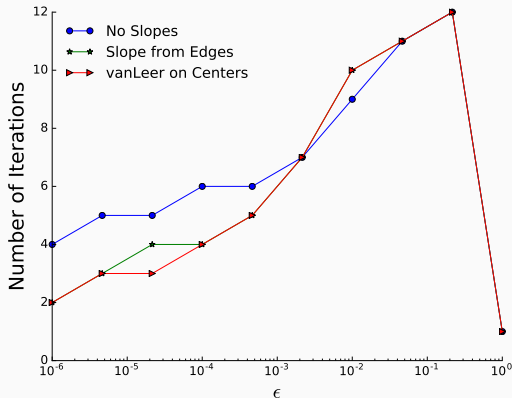


# Data Reconstruction Diffusion Limit



All data reconstruction methods survived diffusion limit

# Data Reconstruction Diffusion Limit



All data reconstruction methods survived diffusion limit

**Eddington Acceleration is extremely robust**

# Conclusions

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## Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
- Eddington Acceleration is uniquely suited for radiation hydrodynamics
  - Transport and acceleration steps can be differenced with different methods
  - Reduces expense of source iteration
  - Provides inexpensive, conservative solution
- Showed MHFEM/LLDG pairing is robust

## Conclusions

- Scheme successfully accelerated source iteration in 1D slab geometry
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## Future Work

- Add temperature for radiative transfer
- Show still works in higher dimensions
- Develop an efficient rad hydro algorithm that makes use of the inexpensive Moment solution in multiphysics iterations

# References

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- [8] J. S. WARSA, T. A. WAREING, AND J. E. MOREL, *Fully Consistent Diffusion Synthetic Acceleration of Linear Discontinuous Transport Discretizations on Three-Dimensional Unstructured Meshes*.



**Questions?**

# Data Reconstruction Methods

MHFEM  $\phi(x)$  is piecewise constant with discontinuous cell edges  
( $\phi_{i-1/2}, \phi_i, \phi_{i+1/2}$ )

LLDG is linear discontinuous ( $\phi_{i,L}, \phi_{i,R}$ )

Need a way to recover slope information when  $S_N$  scattering term is updated with MHFEM  $\phi(x)$

No Slopes:

$$\phi_{i,L/R} = \phi_{i \mp 1/2}^*$$

Slopes from Edges:

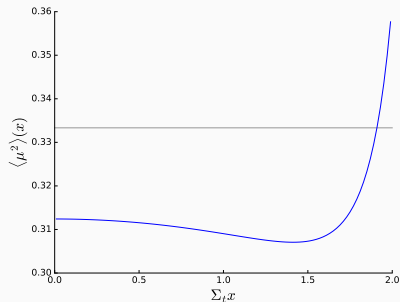
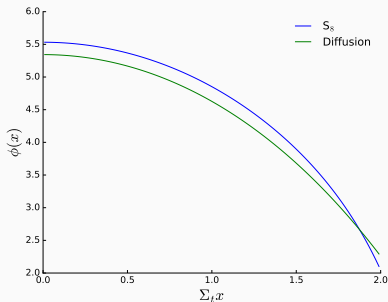
$$\phi_{i,L/R} = \phi_i^* \mp \frac{1}{2} (\phi_{i+1/2}^* - \phi_{i-1/2}^*)$$

vanLeer on Centers:

$$\phi_{i,L/R} = \phi_i^* \mp \frac{1}{2} \xi_{\text{vanLeer}} [(\phi_{i+1}^* - \phi_i^*) + (\phi_i^* - \phi_{i-1}^*)]$$

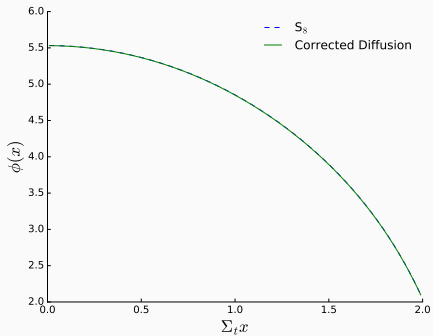
# $S_8$ v. Diffusion

Small system  $\Rightarrow$  diffusion not expected to be accurate



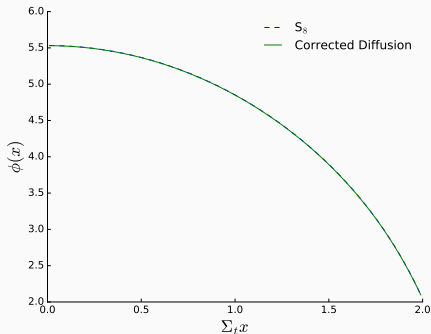
# $S_8$ v. Drift Diffusion

Use  $\langle \mu^2 \rangle(x)$  from  $S_8$  in Moment Equations



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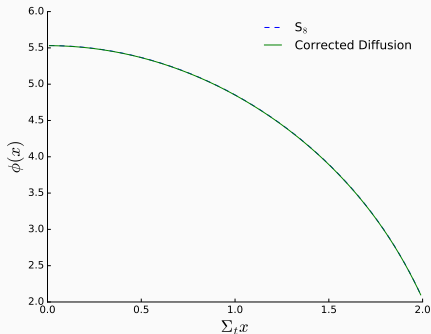
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Moment Equations and  $S_N$  match!

## $S_8$ v. Drift Diffusion

Use  $\langle \mu^2 \rangle(x)$  from  $S_8$  in Moment Equations



Moment Equations and  $S_N$  match!

Requires knowledge of angular flux