Variable Eddington Factor for Mixed Hybrid Finite Element/Linear Discontinuous Galerkin Source Iteration

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Abstract

Abstract goes here

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1 Introduction

The Variable Eddington Factor (VEF) method, also known as Quasi-Diffusion (QD), was one of the first nonlinear methods for accelerating source iterations in S_n calculations []. It is comparable in effectiveness to both linear and nonlinear forms of Diffusion-Synthetic Acceleration (DSA), but it offers much more flexibility than the DSA. Stability can only be guaranteed with DSA if the diffusion equation is differenced in a manner consistent with that of the S_n equations []. Modern S_n codes often use advanced discretization schemes such as discontinuous Galerkin (DG) since classic discretization schemes such as step and diamond are not suitable for radiative transfer calculations in the high-energy density physics regime or coupled electron-photon calculations. Diffusion discretizations consistent with the DG S_n discretizations cannot actually be expressed in diffusion form, but rather must be expressed in first-order or P_1 form, and are much more difficult to solve than standard diffusion discretizations. Considerable effort has gone into the development of "partially consistent diffusion discretizations that yield a stable DSA algorithm with some degree of degraded effectiveness, but such discretizations are also generally difficult to develop. A great advantage of the VEF method is that the drift-diffusion equation that accelerates the S_n source iterations can be discretized in any valid manner without concern for consistency with the S_n discretization. When the VEF drift-diffusion equation is discretized in a way that is "non-consistent, the S_n and VEF drift-diffusion solutions for the scalar flux do not necessarily become identical when the iterative process converges. However, they do become identical in the limit as the spatial mesh is refined, and the difference between the two solutions is proportional to the spatial truncation errors associated the S_n and drift-diffusion discretizations. In general the order accuracy of the S_n and VEF drift-diffusion solutions will be the lowest order accuracy of their respective independent discretizations. Although the

S_n solution obtained with such a "non-consistent VEF method is not conservative, the VEF drift-diffusion solution is in fact conservative. This is particularly useful in multiphysics calculations where the low-order VEF equation can be coupled to the other physics components rather than the high-order S_n equations. Another advantage of the non-consistent approach is that even if the S_n spatial discretization scheme does not preserve the thick diffusion limit [], that limit will generally be preserved using the VEF method.

The purpose of this paper is to investigate the application of the VEF method with the 1-D S_n equations discretized with the lumped linear-discontinuous method (LDG) and the drift-diffusion equation discretized using the constant-linear mixed finite-element method (MFEM). To our knowledge, this combination has not been previously investigated. Our motivation for this investigation is that MFEM methods are now being used for high-order hydrodynamics calculations at Lawrence Livermore National Laboratory []. A radiation transport method compatible with MFEM methods is clearly desirable for developing a MFEM radiation-hydrodynamics code. Such a code would combine thermal radiation transport with hydrodynamics. However, MFEM methods are inappropriate for the first-order form of the transport equation, and are problematic even for the even-parity form. []. Thus the use of the VEF method with a DG S_n discretization and a MFEM drift-diffusion discretization suggests itself. Here we define a VEF method that should exhibit second-order accuracy since both the transport and drift-diffusion discretizations are second-order accurate in isolation. In addition, our VEF method should preserve the thick diffusion limit [], which is essential for radiative transfer calculations in the High-Energy Density Laboratory Physics (HEDLP) regime. We use the lumped rather than the standard LDG discretization because lumping yields a much more robust scheme, and robustness is essential for radiative transfer calculations in the HEDLP regime. Because this is an initial study, we simplify the investigation by considering only by considering only one-group neutron transport rather than the full radiative transfer equations, which include a material temperature equation as well as the radiation transport equation. The vast majority of relevant properties of a VEF method for radiative transfer can be tested with an analogous method for one-group neutron transport. Furthermore, a high-order DG-MFEM VEF method could be of interest for neutronics in addition to radiative transfer calculations. A full investigation for radiative transfer calculations will be carried out in a future study.

The remainder of this paper is organized as follows. First, we describe the VEF method analytically. Then we describe our discretized S_n equations, followed by a description of the discretized VEF drift-diffusion equation. We next give computational results. More specifically, we describe two ways to represent the S_n variable Eddington factor in the MHEM drift-diffusion equation and several ways to construct the S_n scattering source from the drift-diffusion solution for the scalar flux. Each of these options yields a different VEF method. The accuracy of these methods is then compared to that of the standard lumped LDG S_n solution for several test problems, and the iterative convergence rate of these methods is compared to that of the lumped LDG S_n equations with fully-consistent DSA acceleration. Finally, we give conclusions and recommendations for future work.

2 Variable Eddington Factor Method

The steady-state, mono-energetic, isotropically-scattering, fixed-source Linear Boltzmann Equation in slab geometry is:

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{\Sigma_s(x)}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{Q(x)}{2}\,, \tag{1}$$

where $\mu = \cos\theta$ is the cosine of the angle of flight θ relative to the x–axis, $\Sigma_t(x)$ and $\Sigma_s(x)$ the total and scattering macroscopic cross sections, $Q(x)$ the isotropic fixed-source and $\psi(x,\mu)$ the angular flux. Applying the Discrete Ordinates angular discretization yields the following set of N coupled, ordinary differential equations:

$$\mu_n \frac{\mathrm{d}\psi_n}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n(x) = \frac{\Sigma_s(x)}{2}\phi(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N\,, \tag{2}$$

where $\psi_n(x) = \psi(x,\mu_n)$ is the angular flux in direction μ_n. The scalar flux, $\phi(x)$, is computed using an N-point Gauss quadrature rule such that

$$\phi(x) = \sum_{n=1}^{N} w_n \psi_n(x)\,. \tag{3}$$

The Source Iteration (SI) scheme decouples the system of equations defined by Eq. 2 by lagging the scattering term. In other words,

$$\mu_n \frac{\mathrm{d}\psi_n^{\ell+1}}{\mathrm{d}x}(x) + \Sigma_t(x)\psi_n^{\ell+1}(x) = \frac{\Sigma_s(x)}{2}\phi^{\ell}(x) + \frac{Q(x)}{2}\,, 1 \leq n \leq N\,, \tag{4}$$

where the superscripts indicate the iteration index. The Lumped Linear Discontinuous Galerkin (LLDG) spatial discretization can now be applied:

$$\mu_n\left(\psi_{n,i}^{\ell+1} - \psi_{n,i-1/2}^{\ell+1}\right) + \frac{\Sigma_{t,i}h_i}{2}\psi_{n,i,L}^{\ell+1} = \frac{\Sigma_{s,i}h_i}{4}\phi_{i,L}^{\ell} + \frac{h_i}{4}Q_{i,L}\,, 1 \leq n \leq N\,, 1 \leq i \leq I\,, \tag{5a}$$

$$\mu_n\left(\psi_{n,i+1/2}^{\ell+1} - \psi_{n,i}^{\ell+1}\right) + \frac{\Sigma_{t,i}h_i}{2}\psi_{n,i,R}^{\ell+1} = \frac{\Sigma_{s,i}h_i}{4}\phi_{i,R}^{\ell} + \frac{h_i}{4}Q_{i,R}\,, 1 \leq n \leq N\,, 1 \leq i \leq I\,, \tag{5b}$$

$$\psi_{n,i}^{\ell+1} = \frac{1}{2}\left(\psi_{n,i,L}^{\ell+1} + \psi_{n,i,R}^{\ell+1}\right)\,, \tag{5c}$$

$$\psi_{n,i-1/2}^{\ell+1} = \begin{cases} \psi_{n,i-1,R}^{\ell+1}\,, & \mu_n > 0 \\ \psi_{n,i,L}^{\ell+1}\,, & \mu_n < 0 \end{cases}\,, \tag{5d}$$

$$\psi_{n,i+1/2}^{\ell+1} = \begin{cases} \psi_{n,i,R}^{\ell+1}\,, & \mu_n > 0 \\ \psi_{n,i+1,L}^{\ell+1}\,, & \mu_n < 0 \end{cases}\,, \tag{5e}$$

where h_i, $\Sigma_{t,i}$, and $\Sigma_{s,i}$ are the cell width, total cross section and scattering cross section in cell i. The i, L and i, R subscripts indicate the the subscripted value is the left and right discontinuous edge value. Equation 5 can be rewritten as

$$\begin{bmatrix} \mu_n + \Sigma_{t,i} h_i & \mu_n \\ -\mu_n & \Sigma_{t,i} + \mu_n \end{bmatrix} \begin{bmatrix} \psi_{n,i,L}^{\ell+1} \\ \psi_{n,i,R}^{\ell+1} \end{bmatrix} = \begin{bmatrix} \frac{\Sigma_{s,i} h_i}{2} \phi_{i,L}^{\ell} + \frac{h_i}{2} Q_{i,L} + 2\mu_n \psi_{n,i-1,R}^{\ell+1} \\ \frac{\Sigma_{s,i} h_i}{2} \phi_{i,R}^{\ell} + \frac{h_i}{2} Q_{i,R} \end{bmatrix}\,, \tag{6}$$

for sweeping from left to right ($\mu_n > 0$) and

$$\begin{bmatrix} -\mu_n + \Sigma_{t,i} h_i & \mu_n \\ -\mu_n & -\mu_n + \Sigma_{t,i} h_i \end{bmatrix} \begin{bmatrix} \psi_{n,i,L}^{\ell+1} \\ \psi_{n,i,R}^{\ell+1} \end{bmatrix} = \begin{bmatrix} \frac{\Sigma_{s,i} h_i}{2} \phi_{i,L}^{\ell} + \frac{h_i}{2} Q_{i,L} \\ \frac{\Sigma_{s,i} h_i}{2} \phi_{i,R}^{\ell} + \frac{h_i}{2} Q_{i,R} - 2\mu_n \psi_{n,i+1,L}^{\ell+1} \end{bmatrix}\,, \tag{7}$$

for sweeping from right to left ($\mu_n < 0$). The right hand sides of Eqs. 6 and 7 are known as the scalar flux from the previous iteration, the fixed source, and the angular flux entering from the previous cell are all known values.

SI is then: conduct a transport sweep using Eqs. 6 and 7 to find $\{\psi_{n,i}\}$, compute $\{\phi_i\}$ using Gauss quadrature, update the scalar flux on the right hand side of Eqs. 6 and 7, and repeat until both $\phi_{i,L}$ and $\phi_{i,R}$ converge.

The VEF acceleration method alters the above SI scheme by adding a drift diffusion solve. The VEF drift diffusion equations are found by taking the first two moments of Eq.

1:

$$\frac{\mathrm{d}}{\mathrm{d}x} J(x) + \Sigma_a(x)\phi(x) = Q(x)\,, \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle(x)\phi(x) + \Sigma_t(x)J(x) = 0\,, \tag{8b}$$

where $J(x) = \int_{-1}^{1} \mu\psi_(x,\mu)\,\mathrm{d}\mu$ and

$$\langle \mu^2 \rangle(x) = \frac{\int_{-1}^{1} \mu^2 \psi(x,\mu)\,\mathrm{d}\mu}{\int_{-1}^{1} \psi(x,\mu)\,\mathrm{d}\mu} \tag{9}$$

the Eddington factor. In the context of VEF acceleration, the drift diffusion equations are

$$\frac{\mathrm{d}}{\mathrm{d}x} J^{\ell+1}(x) + \Sigma_a(x)\phi^{\ell+1}(x) = Q(x)\,, \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \langle \mu^2 \rangle^{\ell+1/2}(x)\phi^{\ell+1}(x) + \Sigma_t(x)J^{\ell+1}(x) = 0\,, \tag{10b}$$

Applying the MFEM and enforcing continuity of current yields:

$$-\frac{6}{\Sigma_{t,i}h_i}\langle \mu^2 \rangle_{i-1/2}\phi_{i-1/2} + \left(\frac{12}{\Sigma_{t,i}h_i}\langle \mu^2 \rangle_i + \Sigma_{a,i}h_i\right)\phi_i - \frac{6}{\Sigma_{t,i}h_i}\langle \mu^2 \rangle_{i+1/2}\phi_{i+1/2} = Q_i h_i \tag{11a}$$

$$\begin{aligned} -\frac{2}{\Sigma_{t,i}h_i}\langle \mu^2 \rangle_{i-1/2}\phi_{i-1/2} + \frac{6}{\Sigma_{t,i}h_i}\langle \mu^2 \rangle_i\phi_i - 4\left(\frac{1}{\Sigma_{t,i}h_i} + \frac{1}{\Sigma_{t,i+1}h_{i+1}}\right)\langle \mu^2 \rangle_{i+1/2}\phi_{i+1/2} \\ + \frac{6}{\Sigma_{t,i+1}h_{i+1}}\langle \mu^2 \rangle_{i+1}\phi_{i+1} - \frac{2}{\Sigma_{t,i+1}h_{i+1}}\langle \mu^2 \rangle_{i+3/2}\phi_{i+3/2} = 0\,. \end{aligned} \tag{11b}$$

Here, the Eddington factor has been assumed to be constant in each cell with discontinuous jumps at the edges. A more consistent representation will be presented later.

3 Computational Results

3.1 Diffusion Limit

To test the algorithm in the diffusion limit, the cross sections and source were scaled according to:

$$\Sigma_t(x) \rightarrow \Sigma_t(x)/\epsilon\,, \Sigma_s(x) \rightarrow \epsilon\Sigma_s(x)\,, Q(x) \rightarrow \epsilon Q(x)\,. \tag{12}$$

As $\epsilon \rightarrow 0$, the system becomes diffusive.

3.2 Method of Manufactured Solutions

3.3 Solution Convergence

4 Conclusions and Future Work

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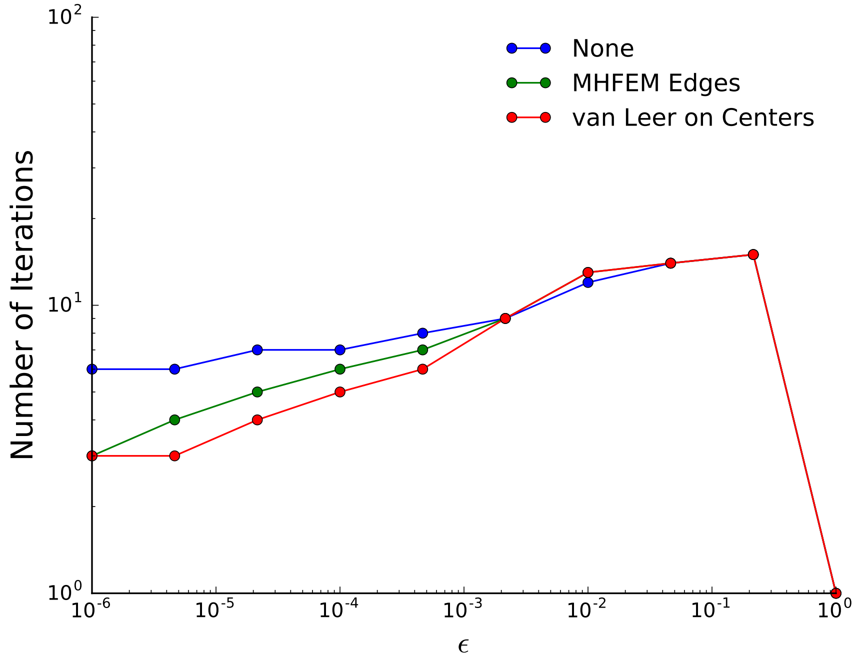

Figure 1: Test